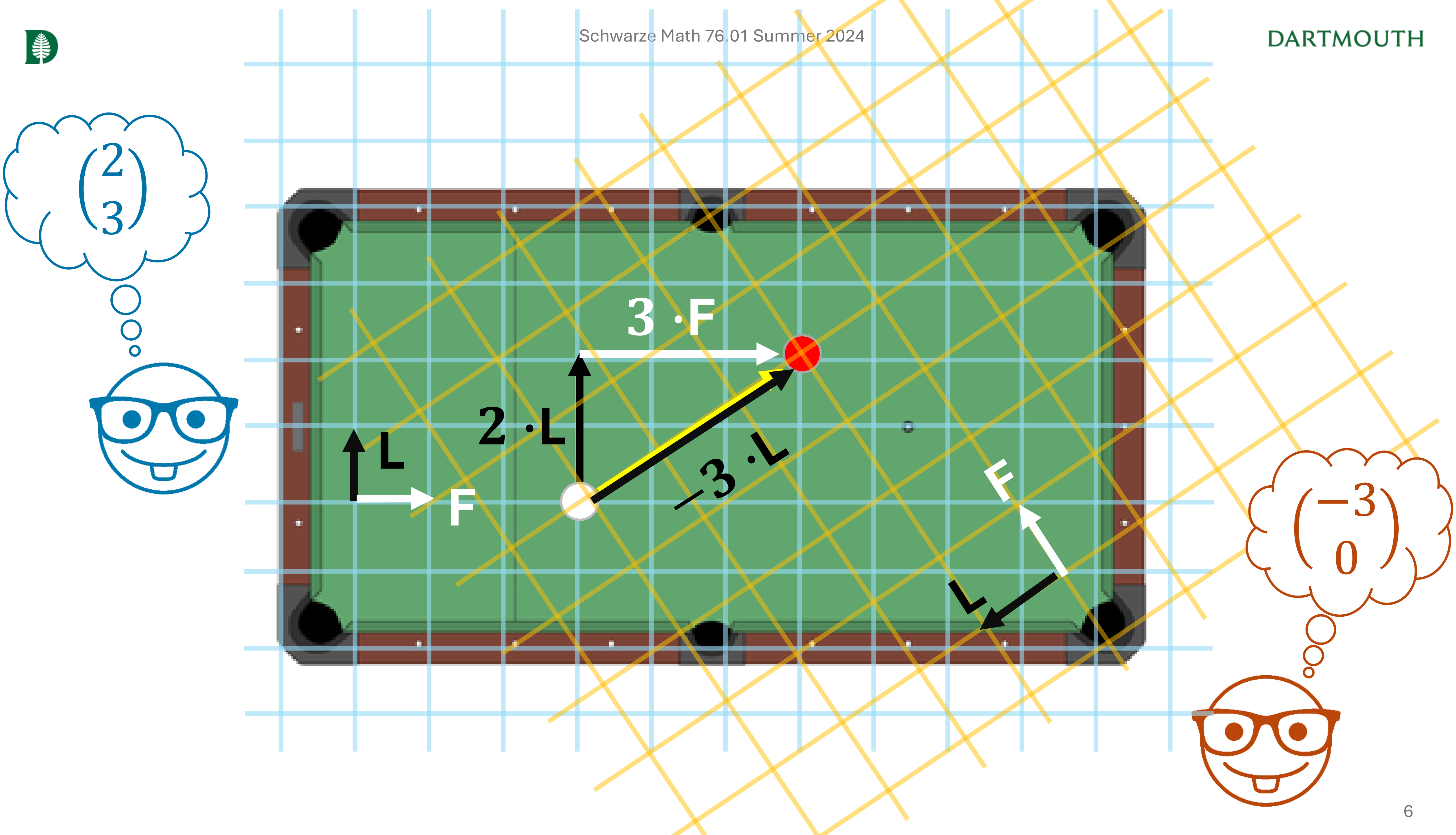
Review of linear algebra

Lecture 6 of “Mathematics and AI”

Outline

1. Matrices as linear transformations
2. Change of basis
3. Eigenvectors and eigenvalues
4. Singular value decomposition

Matrices as linear transformations



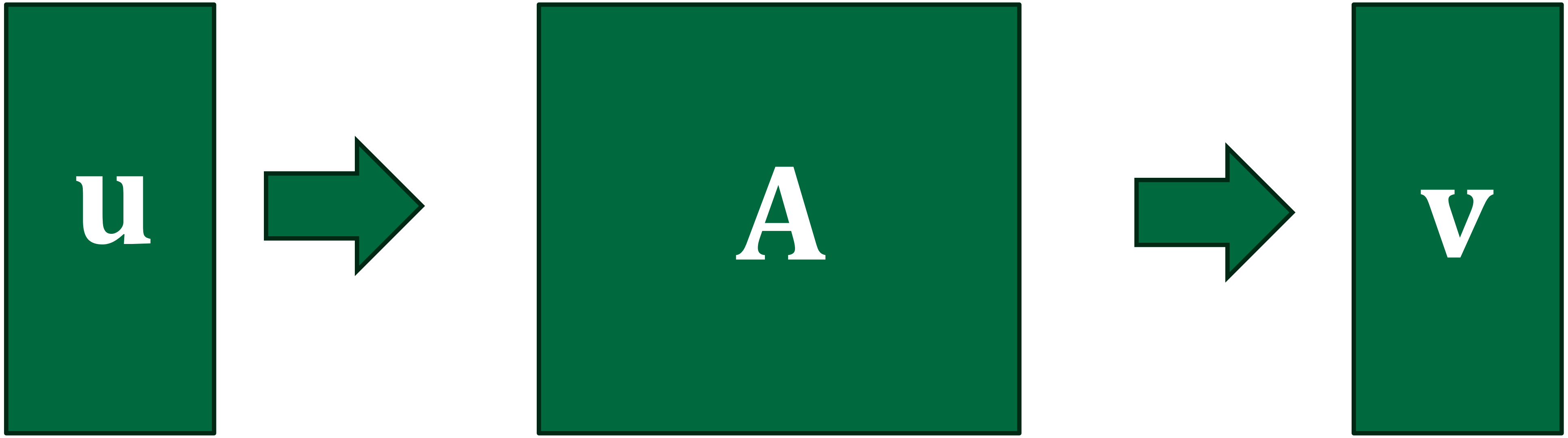
$\begin{pmatrix}2\\3\end{pmatrix}$
L
F
$2 \cdot$L
$3 \cdot$F
$-3 \cdot$L
F
L
$\begin{pmatrix}-3\\0\end{pmatrix}$

A vector and its representation in a basis

- Vector is an object that can describe (among other things!) directed distances
- Given a coordinate system (i.e., a set of basis vectors), vectors can be represented by an ordered set of numbers

A matrix as a linear transformation of vectors

$$\mathbf{A}\mathbf{u} = \mathbf{v}$$

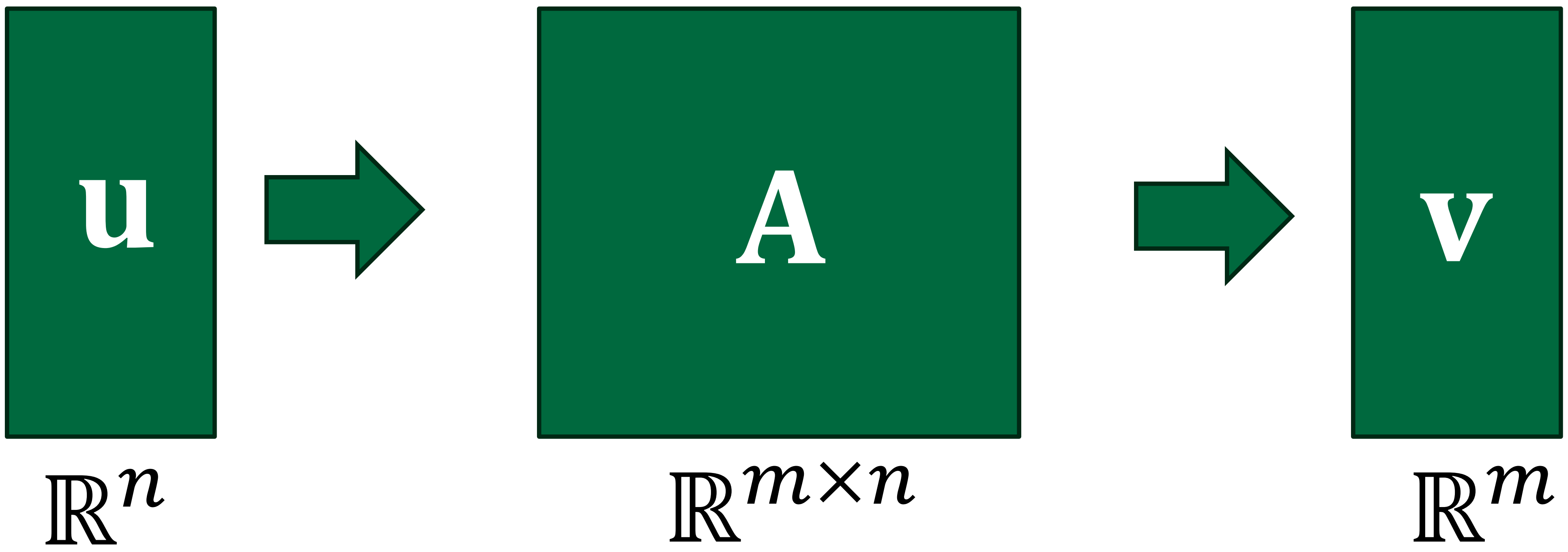


(a representation of) a vector

A corresponding (representation of) a vector

Rectangular matrices as a linear transformations

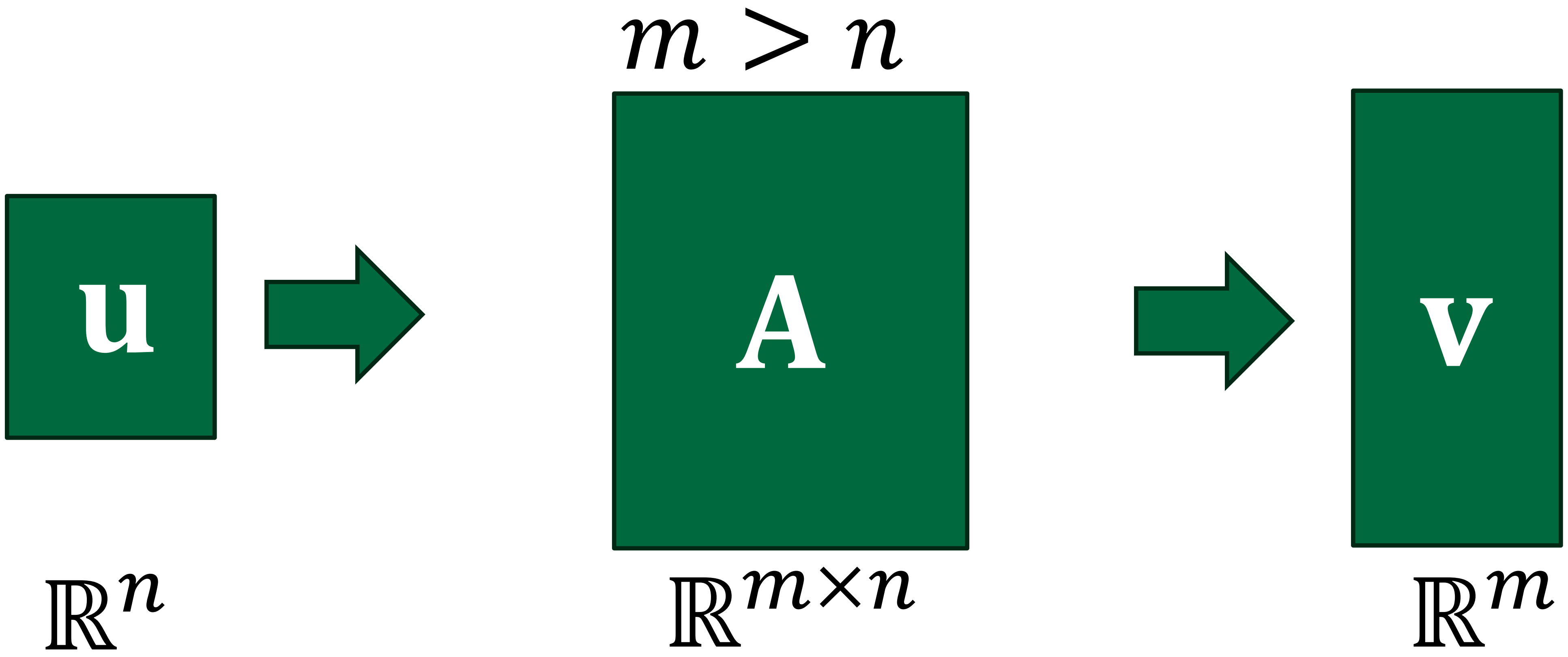
$$\mathbf{A}\mathbf{u} = \mathbf{v}$$



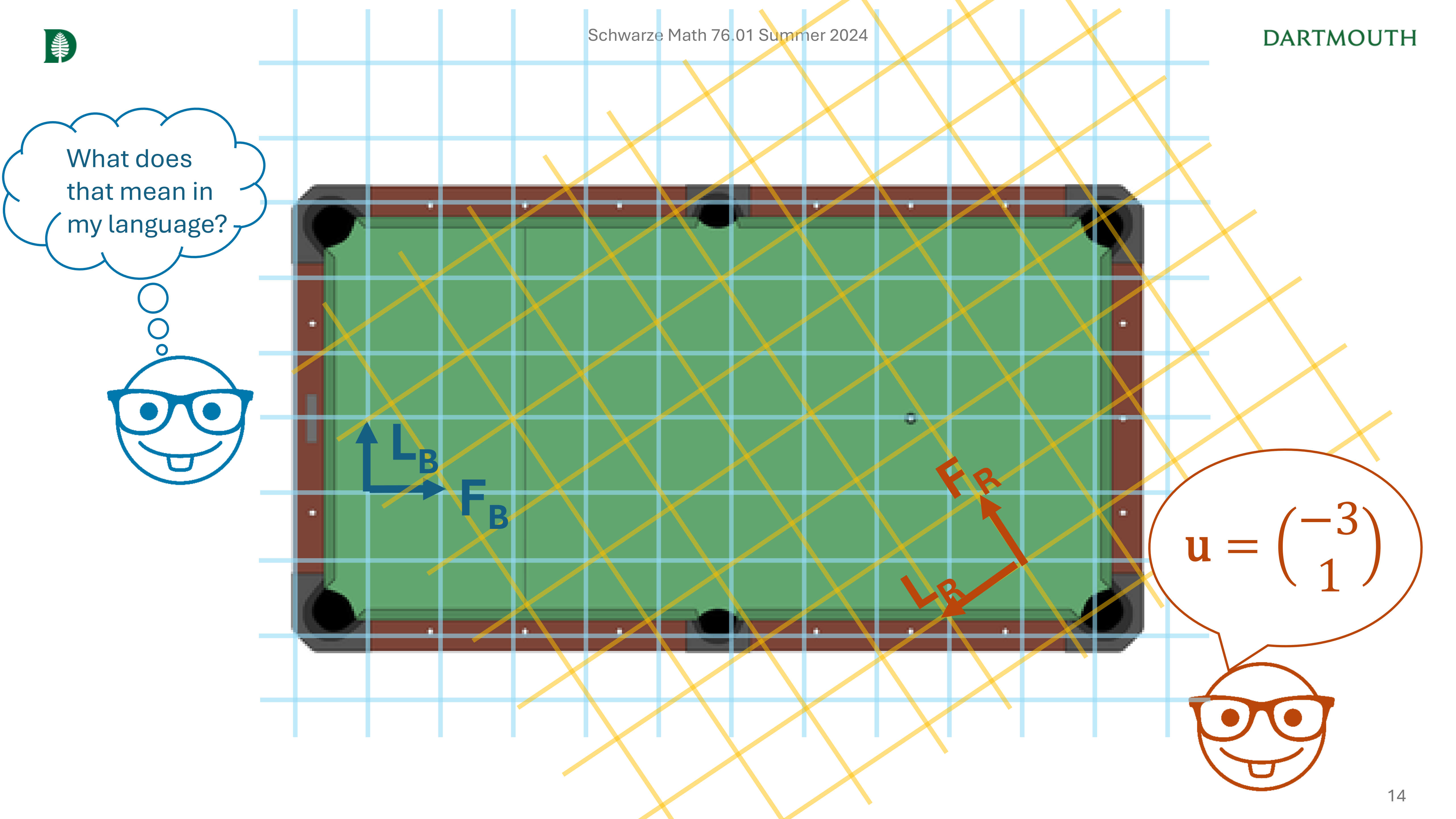
Rectangular matrices as a linear transformations

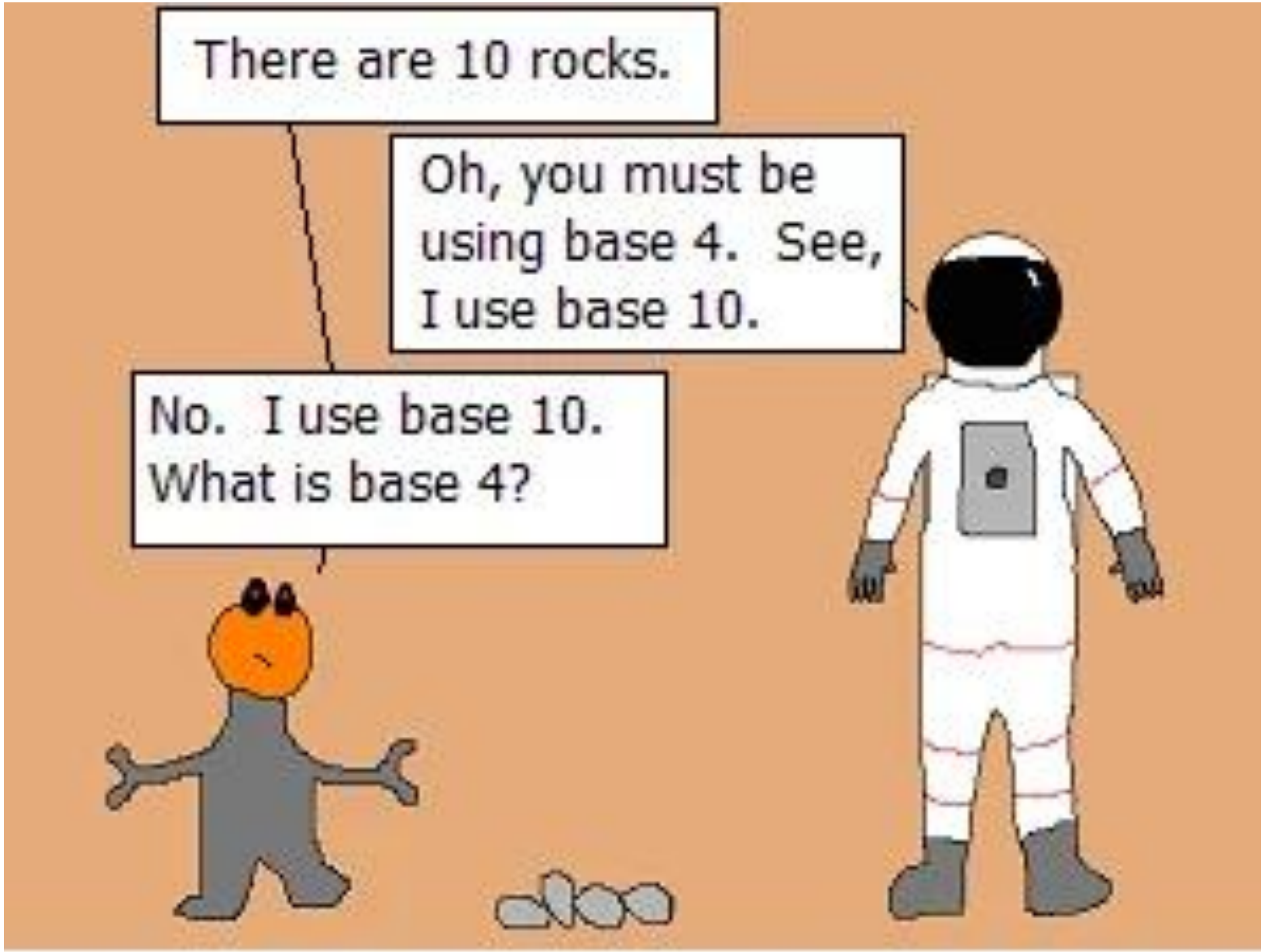
$$m < n$$

Rectangular matrices as a linear transformations



Change of basis


What does that mean in my language?
L_B
F_B
F_R
L_R
$\mathbf{u} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$



Every base is base 10.

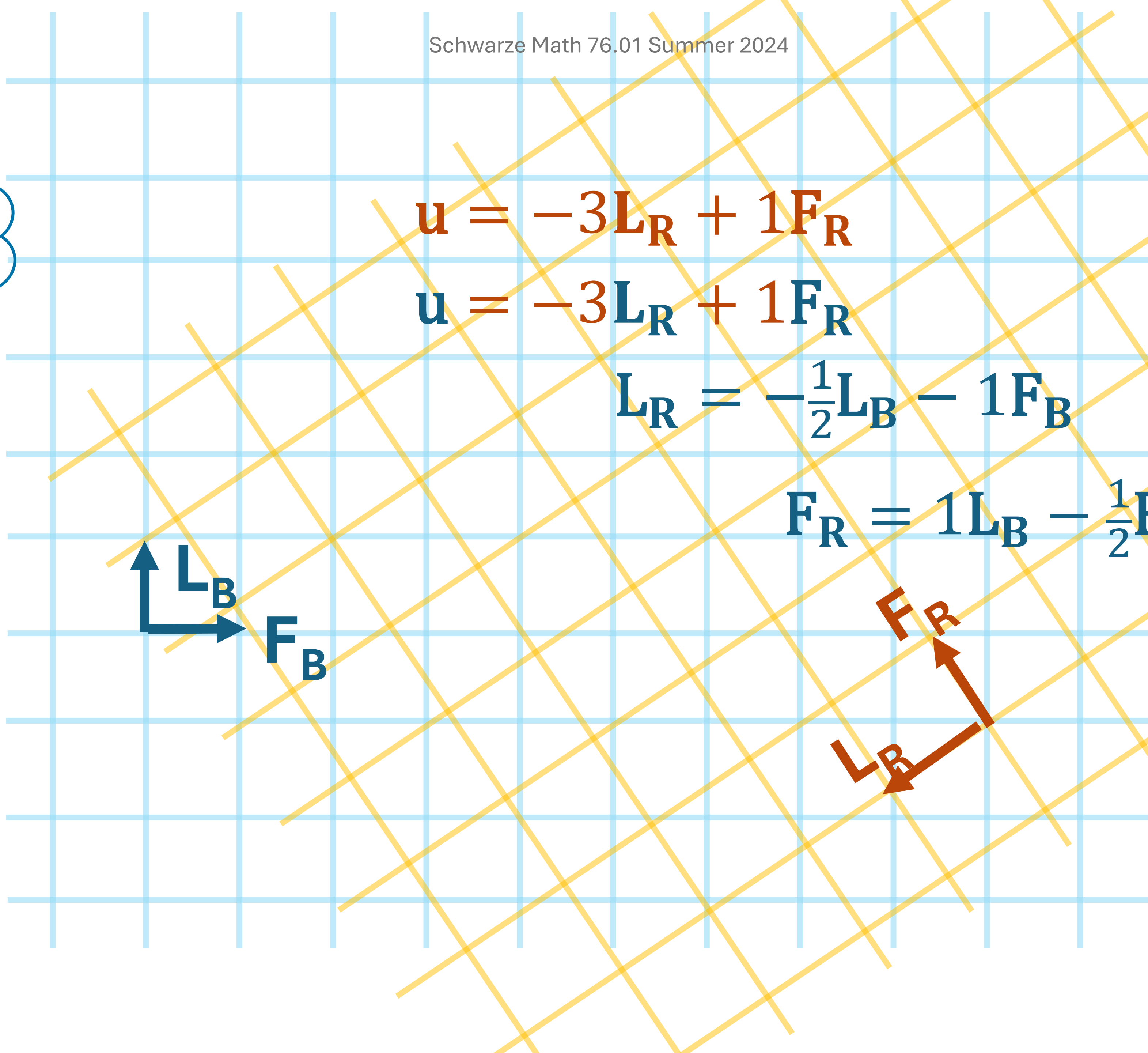
DARTMOUTH

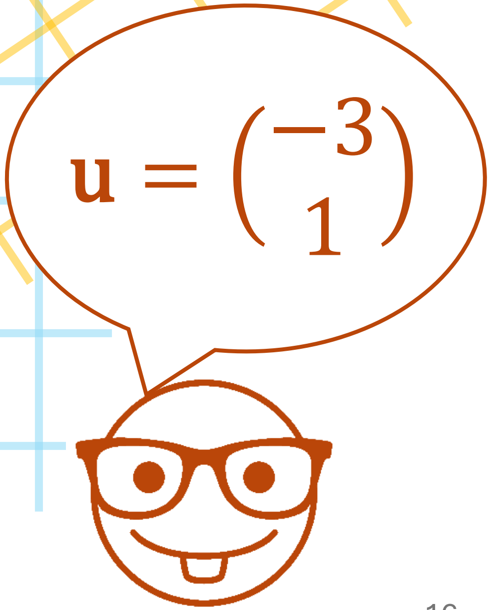
$$\mathbf{u} = -3\mathbf{L_R} + 1\mathbf{F_R}$$

$$\mathbf{u} = -3\mathbf{L_R} + 1\mathbf{F_R}$$

$$\mathbf{L_R} = -\tfrac{1}{2}\mathbf{L_B} - 1\mathbf{F_B}$$

$$\mathbf{F_R} = 1\mathbf{L_B} - \tfrac{1}{2}\mathbf{F_B}$$





What does that mean in my language?

$$\mathbf{u} = \mathbf{u} = \begin{pmatrix} -7/2 \\ 1/2 \end{pmatrix}$$

$$\mathbf{u} = -3\mathbf{L_R} + 1\mathbf{F_R}$$

$$\mathbf{u} = -3\mathbf{L_R} + 1\mathbf{F_R}$$

$$\mathbf{L_R} = -\tfrac{1}{2}\mathbf{L_B} - 1\mathbf{F_B}$$

$$\mathbf{F_R} = 1\mathbf{L_B} - \tfrac{1}{2}\mathbf{F_B}$$

$$\mathbf{u} = -3\left(1\mathbf{L_B} - \tfrac{1}{2}\mathbf{F_B}\right) + 1\left(-\tfrac{1}{2}\mathbf{L_B} - 1\mathbf{F_B}\right)$$

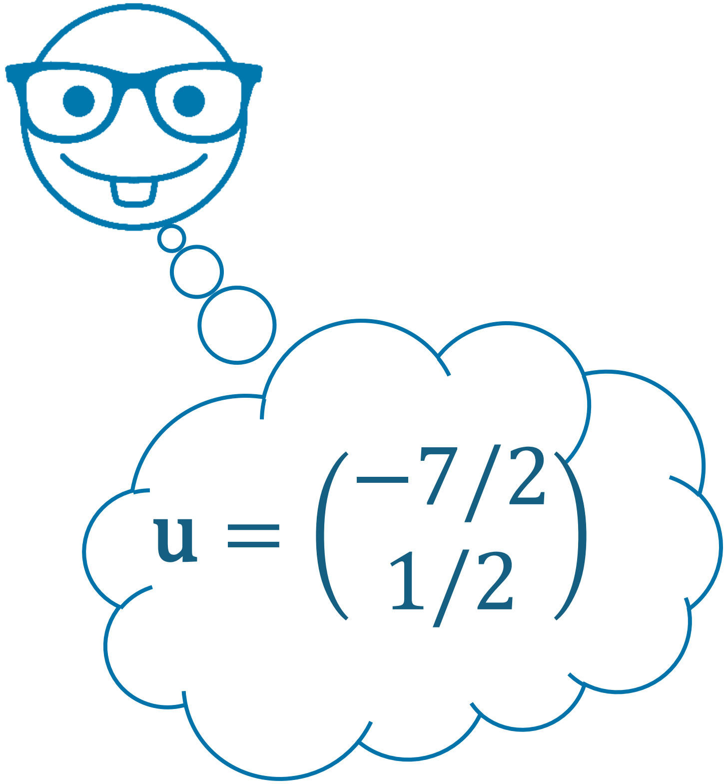
$$\mathbf{u} = -\tfrac{7}{2}\mathbf{L_B} + \tfrac{1}{2}\mathbf{F_B}$$

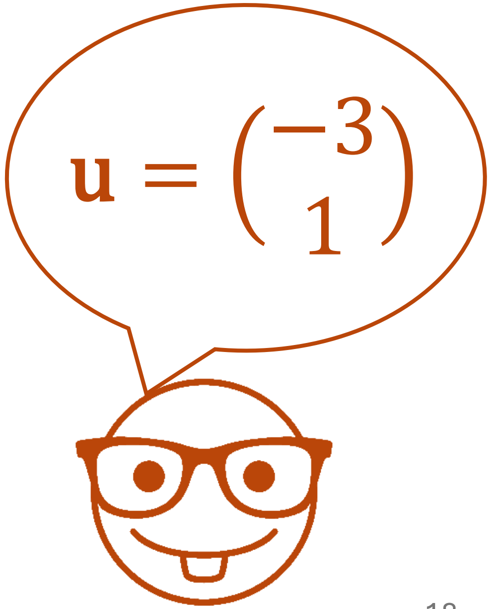
What does the matrix look like that maps every vector $\mathbf{u}$ (in Red's systems) to the corresponding vector $\mathbf{u}$ (in Blue's system)?

Changing the basis of a vector

$$\mathbf{u} = \mathbf{P}\mathbf{u}$$

$$\mathbf{P} = (\mathbf{L}_\mathbf{R}, \mathbf{F}_\mathbf{R}) \in \mathbb{R}^{2\times 2}$$





Changing the basis of a matrix

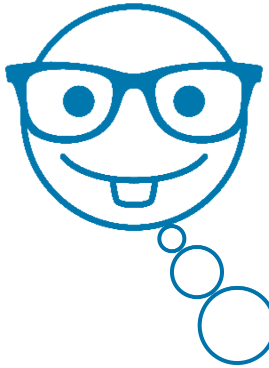
Same operation, but in different representations

$$\mathbf{v} = \mathbf{A}\mathbf{u}$$

$$\mathbf{v} = \mathbf{P}\mathbf{v} = \mathbf{P}\mathbf{A}\mathbf{u}$$

$$\mathbf{v} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{u}$$

$$\mathbf{A} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$$



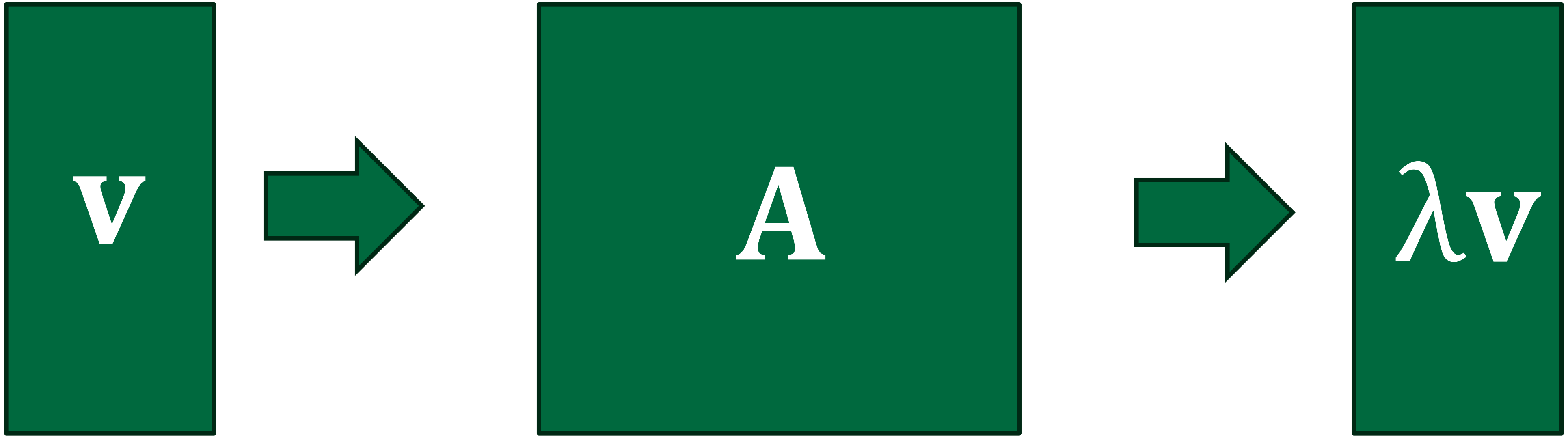
$$\mathbf{A} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

Eigenvectors and eigenvalues

Eigenvalues and eigenvectors

$$\mathbf{Av} = \lambda\mathbf{v}$$



Diagonalization

Assume I have an $n \times n$ matrix $\mathbf{A}$ with n non-zero eigenvalues and n eigenvectors that span $\mathbb{R}^n$ (i.e., $\mathbf{A}$ has eigenspace $\mathbb{R}^n$).

What is the representation of $\mathbf{A}$ in the basis that uses $\mathbf{A}$'s eigenvectors as basis vectors?

What is the corresponding transformation matrix $\mathbf{P}$?

Diagonalization

Some full rank square matrix

Diagonal matrix

Eigendecomposition: $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$

with diagonal matrix of eigenvalues $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & 0 \\ \vdots & 0 & \ddots \end{pmatrix}$

and transformation matrix of eigenvectors

$$\mathbf{P} = (v_1, v_2, \dots)$$

Relevance of the largest eigenvalue(s)

Consider a vector of small values ε

$$\begin{pmatrix}\varepsilon\\ \varepsilon\\ \varepsilon\end{pmatrix} \Rightarrow \mathbf{D} \Rightarrow \begin{pmatrix}\lambda_1\varepsilon\\ \lambda_2\varepsilon\\ \lambda_3\varepsilon\end{pmatrix}$$

New value in direction of basis vector i is proportional to λ_i!

Largest eigenvalues indicate directions of greatest variation.

Singular value decomposition

Rectangular matrices as a linear transformations

$$m > n$$

Rectangular matrices as a linear transformations

$$\mathbf{v} \neq \lambda \mathbf{u}$$

$\mathbf{u}$

$\mathbb{R}^n$

But maybe

$$v_i = \lambda u_i$$

for $i \leq m$?

$\mathbf{v}$

$\mathbb{R}^m$

Singular value decomposition

SVD: $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{T}$

with rectangular diagonal matrix of singular values

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & 0 \\ \vdots & 0 & \ddots \end{pmatrix}$$

and transformation matrices of left and right singular vectors

$$\mathbf{U} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots) \qquad \mathbf{V} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots)$$