COSC 290 Discrete Structures

Lecture 24: Recurrence Relations & Relations

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Plan for today

1. Solving recurrence relations
2. Relations

Solving recurrence relations

Recall: recurrence relations

To analyze runtime of recursive algorithm, we can express runtime *recursively*.

A recurrence relation is a function $T(n)$ that is defined (for some n) in terms of the values $T(k)$ for input values $k < n$.

We will express runtime using a recurrence relation $T(n)$ (where n typically captures some measure of input size).

Solving a recurrence relation

Suppose we have the following recurrence

- $T(0) = c$
- $T(n) = T(\lfloor n/2 \rfloor) + c$

How do we *solve* it? I.e., express it is a (non-recursive) function of n.

Two methods:

1. Guess. Then verify (using proof by induction). ($\leftarrow$ cosc290)
2. Master method. ($\leftarrow$ cosc302)

Guess + verify

Recurrence for RecBinarySearch:

- $T(0) = c$
- $T(n) = T(\lfloor n/2 \rfloor) + c$

Guess: "Iterate" the recurrence starting at $T(0)$ and look for a pattern. $T(n) = c \cdot (\lfloor \log_2 n \rfloor + 2)$.

Verify: We will prove that $\forall n \geq 1 : T(n) = c \cdot (\lfloor \log_2 n \rfloor + 2)$

Base case: $n = 1$, $T(1) = T(0) + c = 2c$ and $c \cdot (\lfloor \log_2 1 \rfloor + 2) = 2c$.

Inductive case: assume true for $1 \leq m \leq n - 1$. Show it's true for n.

$$\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + c && \text{def of recurrence} \\
&= c \cdot (\lfloor \log_2 \lfloor n/2 \rfloor \rfloor + 2) + c && \text{inductive hypothesis} \\
&= c \cdot (\lfloor \log_2 n \rfloor - 1 + 2) + c && \text{math tricks with logs and floors} \\
&= c \cdot (\lfloor \log_2 n \rfloor + 2) && \text{simplify}
\end{aligned}$$

Poll: guess for stack search

Recurrence for StackSearch:

- $T(0) = c$
- $T(n) = T(n - 1) + c$

What is your guess for the solution to $T(n)$?

1. $T(n) = c$
2. $T(n) = c \cdot n$
3. $T(n) = c \cdot (n + 1)$
4. $T(n) = c \cdot (n + 2)$
5. None of above / More than one

Poll: guess for tree search

Recurrence for TreeSearch:

- $T(-1) = c$
- $T(h) = 2T(h - 1) + c$

What is your guess for *asymptotic* solution to $T(h)$?

1. $T(h) = O(1)$
2. $T(h) = O(h)$
3. $T(h) = O(h^2)$
4. $T(h) = O(2^h)$
5. None of the above / more than one

Relations

Relations

A (binary) relation on $A \times B$ is a subset of $A \times B$.

Sometimes interested in relations on $A \times A$ which is sometimes simply called a relation on A.

Examples

- *EnrolledAt* is a relation on *Persons* $\times$ *College*: $\langle p, c \rangle \in$ *EnrolledAt* if person p attends college c.
- *FacebookFriends* is a relation on *FacebookUsers*: $\langle u, v \rangle \in$ *FacebookFriends* if u has friended v on Facebook.
- $\leq$ is a relation on $\mathbb{R}$: $\langle x, y \rangle \in \leq$ if x is less than or equal to y. (We often write using *infix* notation: $x \leq y$.)
- *abs* is a relation on $\mathbb{R} \times \mathbb{R}^{\geq 0}$: $\langle x, y \rangle \in$ *abs* if $|x| = y$.

Poll: Interpreting formal definitions

Here is the formal definition of the inverse of a relation:

Definition (Inverse)
Let R be a relation on $A \times B$. The inverse R^{-1} of R is a relation on $B \times A$ defined by $R^{-1} := \{ \langle b, a \rangle \in B \times A : \langle a, b \rangle \in R \}$

Let relation S on $\{ 1, 2, 3 \} \times \{ a, b \}$ be $S := \{ \langle 1, a \rangle, \langle 1, b \rangle, \langle 3, b \rangle \}$. Which of the following is S^{-1}?

A) $S^{-1} = \{ \langle 2, a \rangle, \langle 2, b \rangle, \langle 3, a \rangle \}$
B) $S^{-1} = \{ \langle a, 2 \rangle, \langle b, 2 \rangle, \langle a, 3 \rangle \}$
C) $S^{-1} = \{ \langle a, 1 \rangle, \langle b, 1 \rangle, \langle b, 3 \rangle \}$
D) More than one / none of the above

Composition

Let R be a relation on $A \times B$ and S be a relation on $B \times C$.

Definition (Composition)

The composition of R and S is a relation on $A \times C$, denoted $S \circ R$, where $\langle a, c \rangle \in S \circ R$ iff there exists a $b \in B$ such that $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in S$.

Example 1: let relation R on $\{ x, y, z \} \times \{ 1, 2, 3 \}$ be $R := \{ \langle x, 1 \rangle, \langle y, 1 \rangle, \langle z, 2 \rangle \}$. Let relation S on $\{ 1, 2, 3 \} \times \{ a, b, c \}$ be $S := \{ \langle 1, a \rangle, \langle 1, b \rangle, \langle 2, b \rangle, \langle 3, b \rangle \}$.

Then, $S \circ R$ is

$$\{ \langle x, a \rangle, \langle x, b \rangle, \langle y, a \rangle, \langle y, b \rangle, \langle z, b \rangle \}$$

Example

Suppose we have the following three relations:

- $taughtIn \subseteq Classes \times Rooms$
- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$

Let's derive a new relation R from the above relations plus the inverse and composition operators.

Example: $R \subseteq Students \times Times$ where $\langle s, t \rangle \in R$ indicates that student s is taking a class at time t.

How do we express R?

$$R = at \circ taking$$

Poll: deriving new relations, 1

Suppose we have the following three relations:

- $taughtIn \subseteq Classes \times Rooms$
- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$

Let's derive a new relation R from the above relations plus the inverse and composition operators: $R \subseteq Students \times Students$ where $\langle s, s' \rangle \in R$ indicates that students s and s' are taking at least one class together.

1. $taking \circ taking$
2. $taking \circ taking^{-1}$
3. $taking^{-1} \circ taking$
4. None of the above / More than one

Poll: deriving new relations, 2

Suppose we have the following three relations:

- $taughtIn \subseteq Classes \times Rooms$
- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$

Let's derive a new relation R from the above relations plus the inverse and composition operators: $R \subseteq Students \times Students$ where $\langle s, s' \rangle \in R$ indicates that students s and s' sit in the same room (but not necessarily for the same class).

A) $(taughtIn \circ taking) \circ (taughtIn \circ taking)^{-1}$
B) $taking^{-1} \circ taughtIn^{-1} \circ taughtIn \circ taking$
C) $(taughtIn \circ taking)^{-1} \circ (taughtIn \circ taking)$
D) $((taughtIn \circ taking) \circ (taughtIn \circ taking))^{-1}$
E) None of the above / More than one

Poll: Cardinality

Suppose that sets A, B, C have cardinalities n_A, n_B, n_C respectively. Let R be a relation on $A \times B$ and S a relation on $B \times C$. What is the *maximum* cardinality of $S \circ R$? (In discussion, justify your answer.)

1. n_B
2. $n_A + n_C$
3. $n_A \cdot n_C$
4. $\min\{ n_A, n_C \}$
5. $\min\{ n_A, n_B, n_C \}$