COSC 290 Discrete Structures

Lecture 19: Proof review

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Plan for today

1. Proofs about binary trees
2. Structural induction on propositions
3. Discuss mid-semester feedback

Proofs about binary trees

Recall: Binary Tree

A binary tree is either:

a) (base case) an empty tree, denoted *null*
b) (inductive case) a root node x, a left subtree T_ℓ, and a right subtree T_r where x is an arbitrary value and T_ℓ and T_r are both *binary trees*.

Height of a tree

The level of a node in T is the length of the path from it to the root of T. The height of a tree is the max level of any (leaf) node in T. (If a tree has zero nodes, we say the height is -1.)

We can also define height *recursively*: let $h(T)$ denote the height of tree T.

- Base case: tree T is empty, $h(T) = -1$.
- Inductive case: T is non-empty, thus it consists a root node x, a left subtree T_ℓ, and a right subtree T_r. Then, $h(T) = 1 + \max\{h(T_\ell), h(T_r)\}$.

Poll: Lower bound?

Last time we proved an upper bound on the number of nodes in T: $nodes(T) \leq 2^{h(T)+1} - 1$.

Can we use a similar proof to show a matching lower bound?

Possible Claim: $nodes(T) \geq 2^{h(T)+1} - 1$

Poll: Lower bound?

- **False Claim**: $nodes(T) \geq 2^{h(T)+1} - 1$
- **Faulty proof by structural induction**:
  - **Base cases:** T is empty, height is -1 and $nodes(T) \geq 2^{-1+1} - 1 = 0$.
  - **Inductive case:** T is a non-empty tree of height h, consisting of node x and left and right subtrees T_ℓ and T_r.

$$\begin{aligned} nodes(T) &= 1 + nodes(T_\ell) + nodes(T_r) && \text{(a. +1 for root)} \\ &\geq 1 + (2^{h(T_\ell)+1} - 1) + (2^{h(T_r)+1} - 1) && \text{(b. ind. hypothesis)} \\ &\geq 1 + (2^{(h-1)+1} - 1) + (2^{(h-1)+1} - 1) && \text{(c. subtree heights)} \\ &= 2^{h+1} - 1 = 2^{h(T)+1} - 1 && \text{(d. algebra)} \end{aligned}$$

Where's the flaw? A) Inductive case, first sentence; B) Inductive case, line a; C) Inductive case, line b; D) Inductive case, line c; E) Inductive case, line d.

Poll: number of leaves

We just showed an upper bound on the number of nodes in T: $nodes(T) \leq 2^{h(T)+1} - 1$. What can we say about $leaves(T)$, the number of leaves?

Give the *smallest* upper bound you can. (Hint: try some examples... then start sketching out a proof!)

Claim: $leaves(T) \leq$ what goes here?

A) 0

B) $2^{h(T)-1}$

C) $2^{h(T)}$

D) $2^{h(T)+1}$

E) $2^{h(T)+1} - 1$

Proof of claim

- **Claim**: $leaves(T) \leq 2^{h(T)}$
- **Proof by structural induction**:
  - **Base cases:** T is empty, $h(T) = -1$ and $leaves(T) = 0$. Indeed $leaves(T) \leq 2^{-1} = \frac{1}{2}$.
  - **Inductive case:** T is non-empty: root node x, subtrees T_ℓ and T_r.

$$\begin{aligned} leaves(T) &= leaves(T_\ell) + leaves(T_r) && \text{(root is not a leaf)} \\ &\leq (2^{h(T_\ell)}) + (2^{h(T_r)}) && \text{(ind. hypothesis)} \\ &\leq (2^{h(T)-1}) + (2^{h(T)-1}) && \text{Why is this okay?} \\ &= 2 \times 2^{h(T)-1} = 2^{h(T)} && \text{(algebra)} \end{aligned}$$

A) It's not okay, this is a flaw in the proof.
B) Because $h(T_\ell) = h(T) - 1$ (and same holds for T_r)
C) Because $h(T_\ell) \leq h(T) - 1$ (and same holds for T_r)
D) Because $h(T_\ell) \geq h(T) - 1$ (and same holds for T_r)
E) Because of the definition of height.

Structural induction on propositions

Propositions, recursively defined

A proposition φ is a well-formed formula (wff) over the variables in the set $P := \{ p_1, \ldots, p_n \}$, is one of the following:

- (base case) $\varphi := p$ for some $p \in P$
- (inductive cases)
  - $\varphi := \alpha \lor \beta$
  - $\varphi := \alpha \land \beta$
  - $\varphi := \alpha \implies \beta$
  - $\varphi := \neg\alpha$

  where α and β are well-formed formulas.

Negation Normal Form

Claim: For any wff φ, there exists a proposition φ' that is in negation normal form (NNF) and is logically equivalent to φ.

Recall: a proposition φ is in *negation normal form* if the negation connective is applied only to variables and not to more complex expressions, and furthermore, the only connectives allowed are in the set $\{\land, \lor, \neg\}$.

Restating claim

Claim: For any well-formed formula φ, there exists a proposition φ' that is in negation normal form and is logically equivalent to φ.

Notation:

- *isNNF*(φ) denotes the predicate: φ is in NNF.
- *hasNNF*(φ) denotes the predicate: there exists a proposition φ' that is in NNF and $\varphi' \equiv \varphi$.
- $\mathcal{W}$ denotes the set of all well-formed formulae.

Thus, our claim can be restated as $\forall \varphi \in \mathcal{W}$: *hasNNF*(φ).

Proof

Claim A: $\forall \varphi \in \mathcal{W}$: *hasNNF*(φ).

We will instead prove the *stronger* claim:

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

How is this "stronger?"

Poll: base case

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

We will do a proof by structural induction.

How should we structure the base case(s)?

A) Two base cases: $\varphi := p$ and $\varphi := \neg p$. In each, want to show *hasNNF*(φ).

B) One base cases: $\varphi := p$, want to show: *hasNNF*$(p) \wedge$ *hasNNF*$(\neg p)$

C) Either of above is acceptable.

D) Base cases? We don't need no stinkin' base cases.

Inductive cases

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

We will do a proof by structural induction. How many inductive cases? One case for each case in the recursive definition of WFF:

1. AND: $\varphi := \alpha \wedge \beta$
2. OR: $\varphi := \alpha \vee \beta$
3. NOT: $\varphi := \neg\alpha$
4. IMPLIES: $\varphi := \alpha \implies \beta$.

Poll: Inductive case 1

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Proof continued...

Inductive cases: We focus on case 1: $\varphi \coloneqq \alpha \wedge \beta$. What do we want to show?

A) *hasNNF*(α)
B) *hasNNF*$(\alpha \wedge \beta)$
C) *hasNNF*$(\neg(\alpha \wedge \beta))$
D) *hasNNF*$(\neg\alpha \vee \neg\beta)$
E) More than one / None of the above

Poll: Inductive case 1

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Proof continued...

Inductive cases: We focus on case 1: $\varphi \coloneqq \alpha \wedge \beta$.

Want to show: *hasNNF*$(\alpha \wedge \beta) \wedge$ *hasNNF*$(\neg(\alpha \wedge \beta))$. Which of the following can we assume is true (by the inductive hypothesis)?

A) *hasNNF*(α)
B) *hasNNF*$(\neg\alpha)$
C) *isNNF*(α) ... recall this means that α is in NNF.
D) A and B
E) A, B, and C

Proof for inductive case 1

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Inductive cases: We focus on case 1: $\varphi \coloneqq \alpha \wedge \beta$.

Want to show: *hasNNF*$(\alpha \wedge \beta) \wedge$ *hasNNF*$(\neg(\alpha \wedge \beta))$.

Assume by inductive hypothesis:

- *hasNNF*(α), *hasNNF*(β), *hasNNF*$(\neg\alpha)$, *hasNNF*$(\neg\beta)$

Part 1: Since *hasNNF*(α), there exists α' such that $\alpha' \equiv \alpha$ and *isNNF*(α'). Similarly for β. Let $\varphi' \coloneqq \alpha' \wedge \beta'$. We have *isNNF*$(\varphi')$ and $\varphi' \equiv \alpha \wedge \beta$. Thus *hasNNF*$(\alpha \wedge \beta)$.

Part 2: $\neg\varphi = \neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$ by DeMorgan's law. Since *hasNNF*$(\neg\alpha)$, there exists $\bar{\alpha}$ such that $\bar{\alpha} \equiv \neg\alpha$ and *isNNF*$(\bar{\alpha})$. Similarly for β. Thus, let $\bar{\varphi} \coloneqq \bar{\alpha} \vee \bar{\beta}$. We have *isNNF*$(\bar{\varphi})$ and $\bar{\varphi} \equiv \neg(\alpha \wedge \beta)$. Thus *hasNNF*$(\neg(\alpha \wedge \beta))$.

Poll: proof for inductive case 3

Claim B: $\forall \varphi \in \mathcal{W}$: *hasNNF*$(\varphi) \wedge$ *hasNNF*$(\neg\varphi)$.

Inductive cases: Case 3: $\varphi \coloneqq \neg\alpha$.

What do we want to show?

A) *hasNNF*(α)
B) *hasNNF*$(\neg\alpha)$
C) *hasNNF*$(\neg\neg\alpha)$
D) A and B
E) A, B, and C

Proof for inductive case 3

Claim B: $\forall \varphi \in \mathcal{W} : \textit{hasNNF}(\varphi) \wedge \textit{hasNNF}(\neg\varphi)$.

Inductive cases: Case 3: $\varphi \coloneqq \neg\alpha$.

Want to show: $\textit{hasNNF}(\neg\alpha) \wedge \textit{hasNNF}(\neg\neg\alpha)$.

Assume by inductive hypothesis:

- $\textit{hasNNF}(\alpha)$, $\textit{hasNNF}(\neg\alpha)$

Still need to show: $\textit{hasNNF}(\neg\neg\alpha)$.

Since $\neg\neg\alpha \equiv \alpha$ and $\textit{hasNNF}(\alpha)$, then let α' be such that $\alpha' \equiv \alpha$ and $\textit{isNNF}(\alpha')$. Let $\bar{\varphi} \coloneqq \alpha'$. Since $\bar{\varphi} \equiv \neg\neg\alpha$ and $\textit{isNNF}(\bar{\varphi})$, thus $\textit{hasNNF}(\neg\neg\alpha)$.

Discuss mid-semester feedback