# COSC 290 Discrete Structures

## Lecture 25: Relations, II

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# Plan for today

1. Relations
2. Graphical representations
3. Properties of relations
4. Closures

# Relations

# Recall: Relations

A (binary) relation on $A \times B$ is a subset of $A \times B$.

Sometimes interested in relations on $A \times A$ which is sometimes simply called a relation on $A$.

# Recall: inverse of a relation

**Definition (Inverse)**

Let $R$ be a relation on $A \times B$. The inverse $R^{-1}$ of $R$ is a relation on $B \times A$ defined by $R^{-1} := \{ \langle b, a \rangle \in B \times A : \langle a, b \rangle \in R \}$

*Intuition for inverse*: think of $R$ a table with columns $A$, $B$, inverse reorders the columns $B$, $A$.

# Recall: composing two relations

**Definition (Composition)**

The composition of $R$ and $S$ is a relation on $A \times C$, denoted $S \circ R$, where $\langle a, c\rangle \in S \circ R$ iff there exists a $b \in B$ such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S$.

*Intuition for composition*: think of $R$ a table with columns $A$, $B$ and think of $S$ a table with columns $B$, $C$. Composition creates new table with columns $A$,$C$ by matching rows from $R$ and $S$ that having match $B$ values.

*(write on the board for later use)*

# Example relations

Suppose we have the following three relations:

- $taughtIn \subseteq Classes \times Rooms$
- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$

What is $at^{-1}$?

What is $at \circ taking$?

# Poll: deriving new relations, part 1

Suppose we have the following three relations:

- $\mathit{taughtIn} \subseteq \mathit{Classes} \times \mathit{Rooms}$
- $\mathit{taking} \subseteq \mathit{Students} \times \mathit{Classes}$
- $\mathit{at} \subseteq \mathit{Classes} \times \mathit{Times}$

Let's derive a new relation $R$ from the above relations plus the inverse and composition operators: $R \subseteq \mathit{Students} \times \mathit{Students}$ where $\langle s, s' \rangle \in R$ indicates that students $s$ and $s'$ are taking at least one class together.

1. $\mathit{taking} \circ \mathit{taking}$
2. $\mathit{taking} \circ \mathit{taking}^{-1}$
3. $\mathit{taking}^{-1} \circ \mathit{taking}$
4. None of the above / More than one

# Poll: deriving new relations, part 2

Suppose we have the following three relations:

- $taughtIn \subseteq Classes \times Rooms$
- $taking \subseteq Students \times Classes$
- $at \subseteq Classes \times Times$

Let's derive a new relation $R$ from the above relations plus the inverse and composition operators: $R \subseteq Students \times Students$ where $\langle s, s' \rangle \in R$ indicates that students $s$ and $s'$ sit in the same room (but not necessarily for the same class).

A) $(taughtIn \circ taking) \circ (taughtIn \circ taking)^{-1}$
B) $taking^{-1} \circ taughtIn^{-1} \circ taughtIn \circ taking$
C) $(taughtIn \circ taking)^{-1} \circ (taughtIn \circ taking)$
D) $((taughtIn \circ taking) \circ (taughtIn \circ taking))^{-1}$
E) None of the above / More than one

# Poll: Cardinality

Suppose that sets $A$, $B$, $C$ have cardinalities $n_A$, $n_B$, $n_C$ respectively. Let $R$ be a relation on $A \times B$ and $S$ a relation on $B \times C$. What is the *maximum* cardinality of $S \circ R$? (In discussion, justify your answer.)

1. $n_B$
2. $n_A + n_C$
3. $n_A \cdot n_C$
4. $\min\{ n_A, n_C \}$
5. $\min\{ n_A, n_B, n_C \}$

# Graphical representations

# Graphical representations of relations

Let $A \coloneqq \{ a, b, c \}$. And consider relation $R$ on $A$ defined as

$$R \coloneqq \{ \langle a, b \rangle , \langle b, b \rangle , \langle b, c \rangle , \langle b, a \rangle \}$$

We can represent this graphically several ways (shown on board).

# Properties of relations

# Reflexivity

A relation $R$ on $A$ is reflexive if for every $a \in A$, $\langle a, a \rangle \in R$.

A relation $R$ on $A$ is irreflexive if for every $a \in A$, $\langle a, a \rangle \notin R$.

A relation can be reflexive, irreflexive, or neither.

examples drawn on board

# Symmetry

A relation $R$ on $A$ is symmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$ too.

A relation $R$ on $A$ is antisymmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

A relation $R$ on $A$ is asymmetric if for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

A relation can be none of the above, or more than one of the above.

examples drawn on board

# Transitive

A relation $R$ on $A$ is transitive if for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$ too.

A relation can be transitive, or not.

examples drawn on board

# Poll: ancestorOf

**R** *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

**IR** *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

**S** *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

**antiS** *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

**AS** *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

**T** *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the *ancestorOf* relation on persons where $\langle a, p \rangle \in$ *ancestorOf* if person $a$ is an ancestor of person $p$. Which properties does this relation have? (You can choose more than one.)

A) R

B) IR

C) S

D) antiS

E) AS

F) T

# Poll: implies

**R** *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

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**antiS** *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

**AS** *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

**T** *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the *implies* relation on all possible propositions expressed in the English language where $\langle p, q \rangle \in$ *implies* if $p \implies q$ is true. Which properties does this relation have? (You can choose more than one.)

A) R

B) IR

C) S

D) antiS

E) AS

F) T

# Poll: unequal sets

**R** *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.

**IR** *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.

**S** *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.

**antiS** *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.

**AS** *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.

**T** *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Let $X$ be an arbitrary set. Consider the relation *diffSize* on $\mathcal{P}(X)$ where $\langle S_1, S_2 \rangle \in \mathit{diffSize}$ if $|S_1| \neq |S_2|$. Which properties does this relation have? (You can choose more than one.)

A) R

B) IR

C) S

D) antiS

E) AS

F) T

# Poll: even divider

- **R** *reflexive*: for every $a \in A$, $\langle a, a \rangle \in R$.
- **IR** *irreflexive*: for every $a \in A$, $\langle a, a \rangle \notin R$.
- **S** *symmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$.
- **antiS** *antisymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$.
- **AS** *asymmetric*: for every $a, b \in A$, if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \notin R$.
- **T** *transitive*: for every $a, b, c \in A$, if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$.

Consider the relation $R$ on $\mathbb{Z}$ where $\langle x, y \rangle \in R$ if $x \mod 2 = 0$ and $y \mod x = 0$. Which properties does this relation have? (You can choose more than one.)

A) R

B) IR

C) S

D) antiS

E) AS

F) T

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$$R' = R \cup \{\, \langle a, a \rangle : a \in A \,\}$$

- symmetric closure:

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- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:

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$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:
  (*hint*: what does $R \circ R$ give you?)

# Poll: towards transitive closure

Consider the *parentOf* relation on persons where $\langle p, c\rangle \in \mathit{parentOf}$ if $p$ is the parent of $c$. What is $\mathit{parentOf} \circ \mathit{parentOf}$?

A) ancestorOf
B) grandParentOf
C) parentOf
D) childOf
E) grandChildOf
F) descendantOf

# Closures

A closure of a relation $R$ on $A$ is a smallest $R' \supseteq R$ that satisfies a desired property.

- reflexive closure:

$$R' = R \cup \{ \langle a, a \rangle : a \in A \}$$

- symmetric closure:

$$R' = R \cup R^{-1}$$

- transitive closure:

$$R' = R \cup (R \circ R) \cup ((R \circ R) \circ R) \cup \cdots$$

# Poll: transitive closure

Consider the *parentOf* relation on persons where $\langle p, c \rangle \in \mathit{parentOf}$ if $p$ is the parent of $c$. What is the transitive closure of $\mathit{parentOf}^{-1}$?

A) ancestorOf

B) parentOf

C) childOf

D) descendantOf

E) siblingOf