The 3-D Magnetic Field Integral Equation (MFIE)

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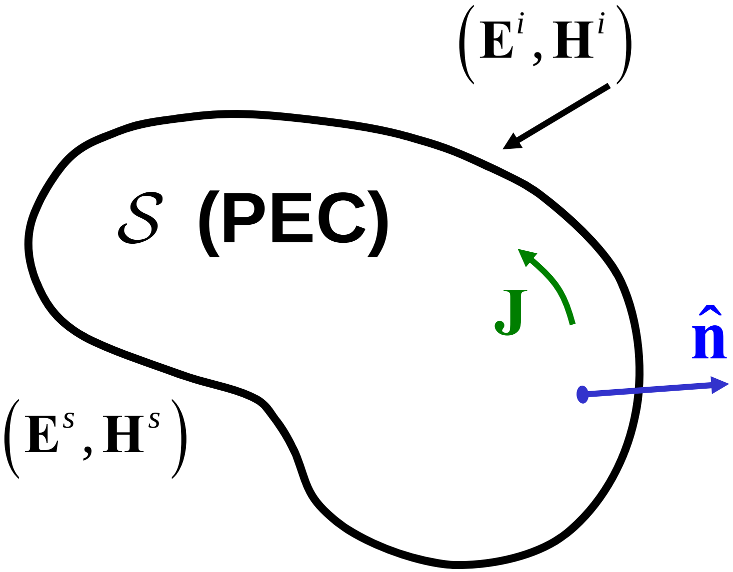
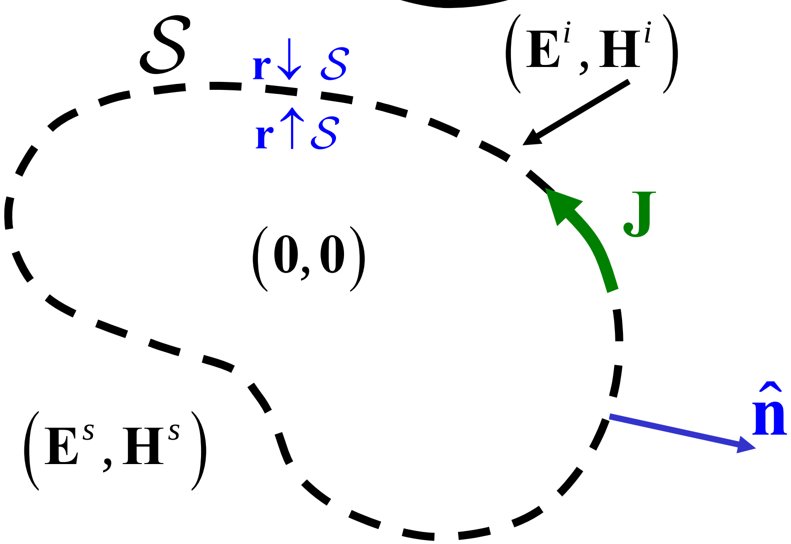
3D MFIE Formulation





EFIE: $-\mathbf{E}^{\mathrm{s}}_{\tan} = \mathbf{E}^{\mathrm{i}}_{\tan}, \quad \mathbf{r} \in \mathcal{S}$

MFIE (two approaches):

1) $\mathbf{J} = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}} + \lim\limits_{\mathbf{r}\downarrow\mathcal{S}^{\dagger}} \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{s}},$

(eq. source condition)

2) $\hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}} + \lim\limits_{\mathbf{r}\uparrow\mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{s}} = \mathbf{0},$

(null field condition)

Since $\mathbf{J} = \lim\limits_{\mathbf{r}\downarrow\mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{s}} - \lim\limits_{\mathbf{r}\uparrow\mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{s}}$

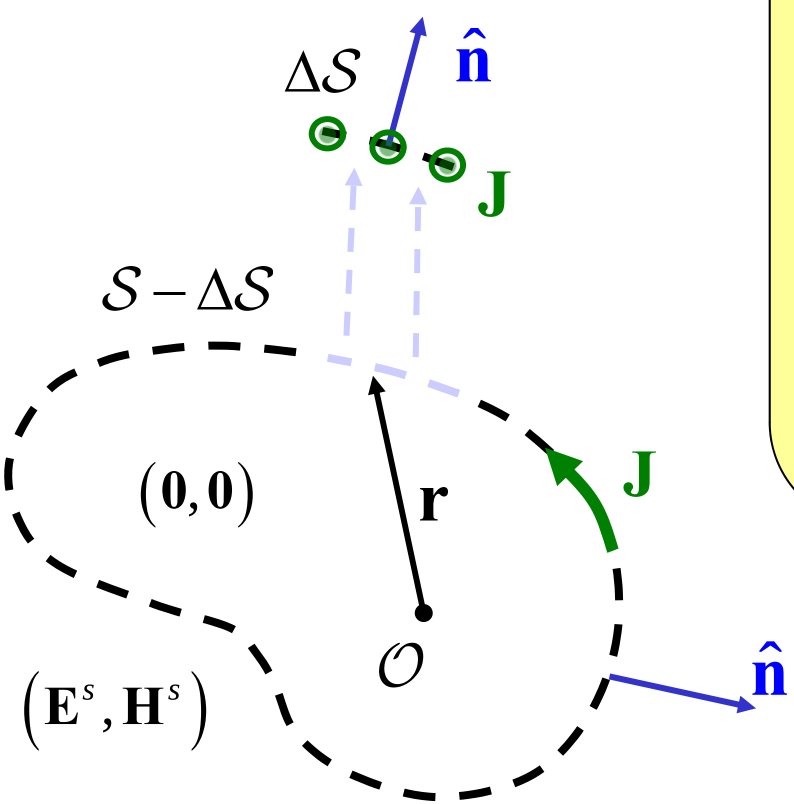
the approaches are equivalent!

$^{\dagger}\,\mathbf{r}\!\downarrow \mathcal{S} \Rightarrow \mathbf{r}$ approaches $\mathcal{S}$ from the *exterior*,

$\mathbf{r}\!\uparrow \mathcal{S} \Rightarrow \mathbf{r}$ approaches $\mathcal{S}$ from the *interior*

Null Field MFIE Formulation, Limiting Process

$\Delta\mathcal{S}$ is a *very* small, flat circular disk of radius a removed from $\mathcal{S}$



MFIE: $\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{i}}+\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{s}}=\mathbf{0},$

where

$$\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{s}}=\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\frac{1}{\mu}\nabla\times\mathbf{A}$$

$$=\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\nabla\times\int_{\mathcal{S}}\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}'$$

$$=\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\int_{\mathcal{S}}\nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}'$$

$$=\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\left(\int_{\Delta\mathcal{S}}+\int_{\mathcal{S}-\Delta\mathcal{S}}\right)\nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}'$$

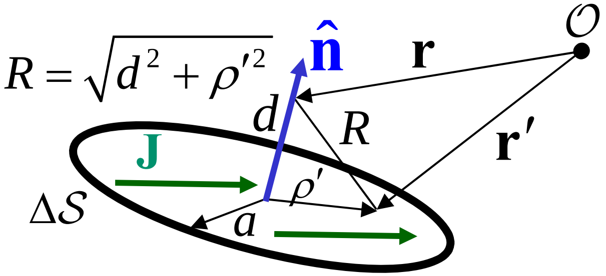
Recall that in homogeneous media,

$$\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')=G(\mathbf{r},\mathbf{r}')\mathcal{I}=\frac{e^{-jkR}}{4\pi R}\mathcal{I}\leftarrow$$ **identity dyad**

$$\Rightarrow\hat{\mathbf{n}}\times\nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')$$ **(layered media)**

$$\xrightarrow{\text{homogeneous media}}\hat{\mathbf{n}}\times\mathbf{J}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')$$

Evaluation of $\hat{\mathbf{n}}\times\mathbf{H}_{\Delta}^{-} \equiv \lim\limits_{\mathbf{r}\uparrow\mathcal{S}} \hat{\mathbf{n}}\times\int_{\Delta\mathcal{S}} \nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}'$



Dominant integrand behavior for small R:

$$\hat{\mathbf{n}}\times\nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')$$

$$= \hat{\mathbf{n}}\times\nabla\times\left[G(\mathbf{r},\mathbf{r}')\mathcal{I}\cdot\mathbf{J}(\mathbf{r}')\right] = \hat{\mathbf{n}}\times\left[\nabla G\times\mathbf{J}(\mathbf{r}')\right] = -\hat{\mathbf{n}}\times\left[(1+jkR)\frac{e^{-jkR}(\mathbf{r}-\mathbf{r}')}{4\pi R^3}\times\mathbf{J}(\mathbf{r}')\right]$$

$$\xrightarrow[kR\to 0]{} -\underbrace{\hat{\mathbf{n}}\times\left[(\mathbf{r}-\mathbf{r}')\times\mathbf{J}(\mathbf{r}')\right]}_{(\mathbf{r}-\mathbf{r}')\cancel{\hat{\mathbf{n}}\cdot\mathbf{J}}-\mathbf{J}\left[(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{n}}\right]}\frac{1}{4\pi R^3} = \mathbf{J}(\mathbf{r}')\frac{\hat{\mathbf{n}}\cdot(\mathbf{r}-\mathbf{r}')}{4\pi R^3} = \mathbf{J}(\mathbf{r}')\frac{d}{4\pi R^3}, \quad R^2=d^2+\rho'^2$$

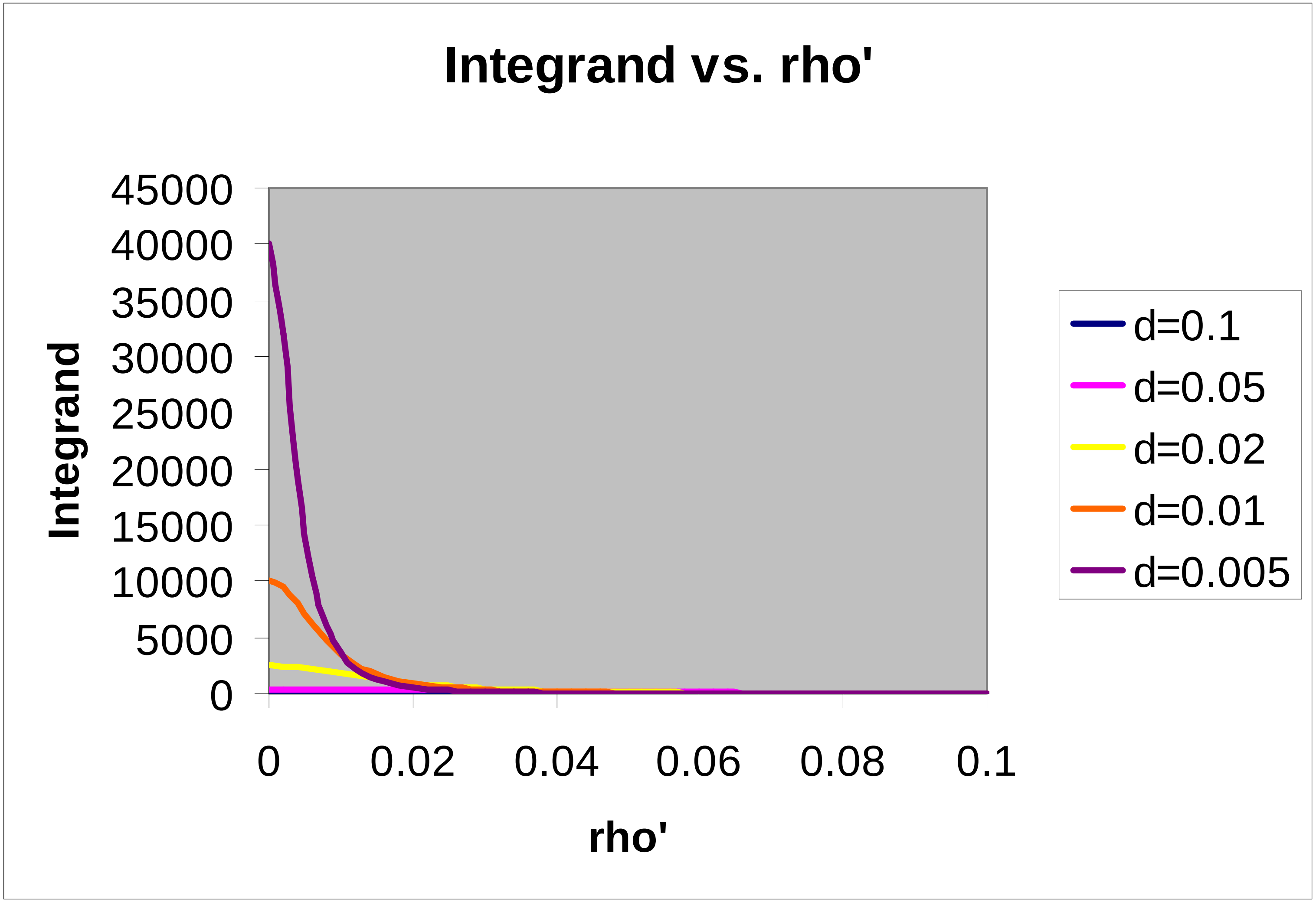
Asymptotic evaluation of integral:

$$\hat{\mathbf{n}}\times\int_{\Delta\mathcal{S}}\nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}' \xrightarrow[kR\to 0]{} \mathbf{J}(\mathbf{r})\int_{0}^{2\pi}\int_{|d|}^{\sqrt{d^2+a^2}}\frac{d}{4\pi R^3}R\,dR\,d\phi' \quad \left(\begin{array}{c}\textbf{since } \rho' d\rho' = R\,dR, \\ \textbf{and } \mathbf{J}(\mathbf{r})\approx\mathbf{J}(\mathbf{r}')\end{array}\right)$$

$$=\frac{\mathbf{J}(\mathbf{r})}{2}\left[\frac{-d}{R}\right]_{R=|d|}^{\sqrt{d^2+a^2}} = \frac{\mathbf{J}(\mathbf{r})}{2}\left[\frac{d}{|d|}-\frac{d}{\sqrt{d^2+a^2}}\right]\xrightarrow[d\to 0]{}\frac{\mathbf{J}(\mathbf{r})}{2}\operatorname{sgn}(d)$$

$$\Rightarrow\quad \hat{\mathbf{n}}\times\mathbf{H}_{\Delta}^{-}\equiv\lim_{\mathbf{r}\uparrow\mathcal{S}}\hat{\mathbf{n}}\times\int_{\Delta\mathcal{S}}\nabla\times\mathcal{G}^{A}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}' = \lim_{d\to 0^-}\frac{\mathbf{J}(\mathbf{r})}{2}\operatorname{sgn}(d) = -\frac{\mathbf{J}(\mathbf{r})}{2}$$

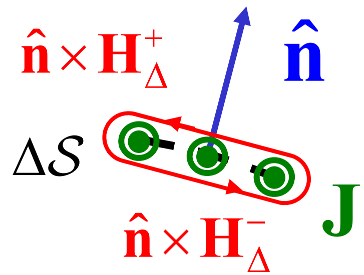
Integrand Approaches a Delta Function in the Limit d→0



Simple Interpretation

Current jump condition:

$$\mathbf{J} = \hat{\mathbf{n}} \times \left(\mathbf{H}_\Delta^+ - \mathbf{H}_\Delta^- \right)$$



By symmetry,

$$\hat{\mathbf{n}} \times \mathbf{H}_\Delta^- = -\hat{\mathbf{n}} \times \mathbf{H}_\Delta^+$$

$$\Rightarrow \hat{\mathbf{n}} \times \mathbf{H}_\Delta^\pm = \frac{\pm \mathbf{J}}{2}$$

Hence $\lim_{\mathbf{r} \uparrow \mathcal{S}} \hat{\mathbf{n}} \times \mathbf{H}_\Delta^- = -\frac{\mathbf{J}(\mathbf{r})}{2}$ **and MFIE is**

$$\boxed{\frac{\mathbf{J}(\mathbf{r})}{2} - \hat{\mathbf{n}} \times \int_{\mathcal{S}} \nabla \times \mathcal{G}^A(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, d\mathcal{S}' = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}}, \quad \mathbf{r} \in \mathcal{S}}$$

After removing the singular contribution, the integral is no longer (strongly-)singular and is sometimes written

$$\mathrm{PV}\int_{\mathcal{S}} d\mathcal{S}' \quad \text{or} \quad ⨍_{\mathcal{S}} d\mathcal{S}'$$

Recall that in homogeneous media, this reduces to

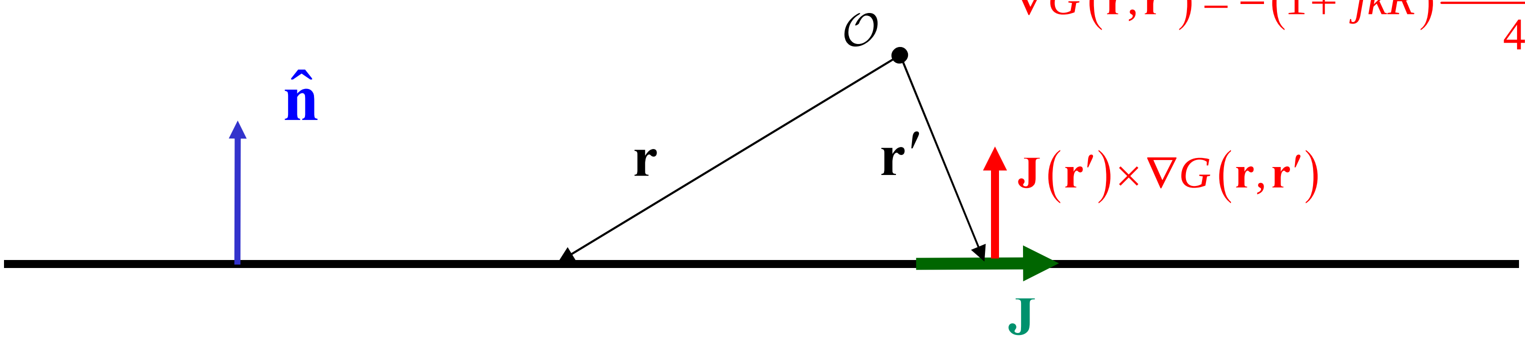
$$\boxed{\frac{\mathbf{J}(\mathbf{r})}{2} + \hat{\mathbf{n}} \times \int_{\mathcal{S}} \mathbf{J}(\mathbf{r}') \times \nabla G(\mathbf{r},\mathbf{r}')\, d\mathcal{S}' = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}}, \quad \mathbf{r} \in \mathcal{S}}$$

$\hat{\mathbf{n}} \times \nabla \times \mathcal{G}^A(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')$ **(layered media)**

homogeneous media $\rightarrow$ $\hat{\mathbf{n}} \times \mathbf{J}(\mathbf{r}') \times \nabla G(\mathbf{r},\mathbf{r}')$

Specialization to Flat Surfaces

$$\nabla G(\mathbf{r},\mathbf{r}') = -(1 + jkR)\frac{e^{-jkR}(\mathbf{r}-\mathbf{r}')}{4\pi R^3}$$



If $\mathcal{S}$ is a flat surface, then $\mathbf{J}$ and ∇G are in the same plane, and $\mathbf{J}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')$ is parallel to $\hat{\mathbf{n}}$; hence

- $\boxed{\hat{\mathbf{n}}\times\int_{\mathcal{S}}\mathbf{J}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}')\,d\mathcal{S}' = \mathbf{0}, \quad \mathbf{r}' \neq \mathbf{r} \in \mathcal{S}}$**, and**
- $\boxed{\mathbf{J}(\mathbf{r}) = 2\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{i}},}$

Choose Surface Divergence-Conforming Bases for Expanding the Current and Testing the MFIE

$$\frac{\mathbf{J}(\mathbf{r})}{2} - \hat{\mathbf{n}} \times \int_{\mathcal{S}} \nabla \times \mathcal{G}^{A}(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, d\mathcal{S}' = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}}$$

$$\mathbf{J}(\mathbf{r}) \approx \sum_{n=1}^{N} I_n \Lambda_n(\mathbf{r})$$

$\Rightarrow \quad [\beta_{mn}][I_n] = [I_m^{\mathrm{i}}]$, **where**

$$\beta_{mn} = \frac{1}{2} < \Lambda_m ; \Lambda_n > - < \Lambda_m ; \hat{\mathbf{n}} \times \nabla \times \mathcal{G}^{A}(\mathbf{r},\mathbf{r}') ; \Lambda_n >$$

$$I_m^{\mathrm{i}} = < \Lambda_m ; \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}} >$$

Note other basis choices are possible, even desirable!

with corresponding element matrix and element vector

$$\beta_{ij}^{ef} = \begin{cases} \frac{1}{2} < \Lambda_i^e ; \Lambda_j^f >, \; e = f \\ - < \Lambda_i^e ; \hat{\mathbf{n}} \times \nabla \times \mathcal{G}^{A}(\mathbf{r},\mathbf{r}') ; \Lambda_j^f >, \; e \neq f \end{cases}$$

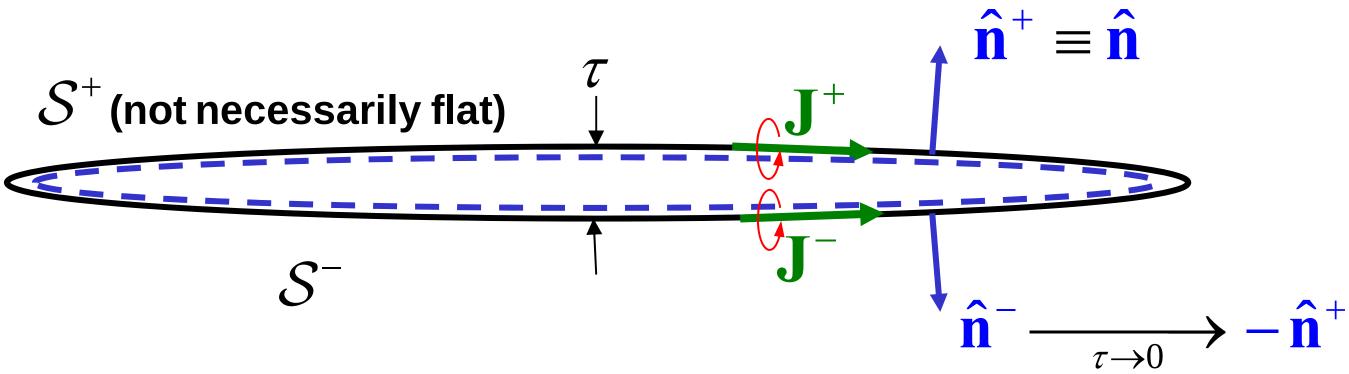
(no integral contribution from flat subdomains!)

$$I_i^{\mathrm{i}e} = < \Lambda_i^e ; \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}} >$$

Features of the MFIE

- **Applies *only* to closed bodies**
- **The contribution from the integral term *vanishes on flat surfaces*, r in the surface plane**
- **MFIE is usually better conditioned than the EFIE (since J appears outside the integral, it is a *2nd kind integral equation*)**
- **It appears possible to use *either* div- or curl-conforming bases**
- **MFIE is sometimes slow to converge compared to EFIE**
- **The MFIE *operator* is important since it appears in both combined field integral equations (CFIE) and in dielectric formulations (PMCHWT)**

Why Does the MFIE Apply to Closed Bodies Only?

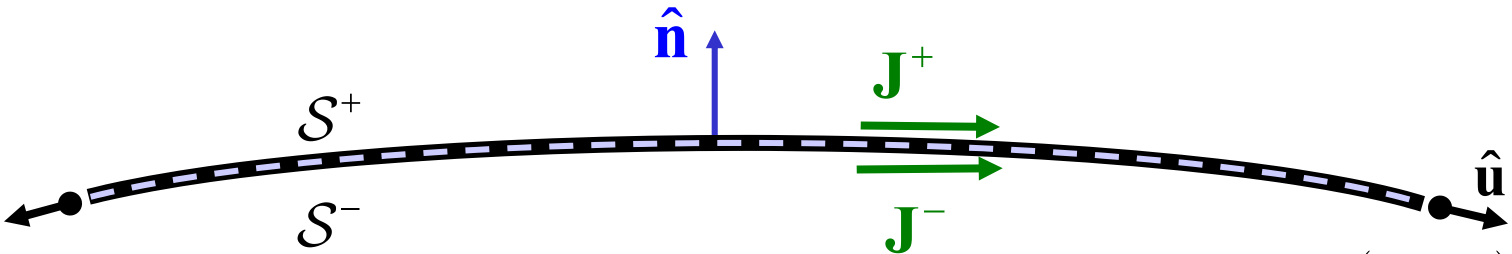


In the limit as $\tau \to 0$, null field surfaces (dashed lines) degenerate to a single surface $\mathcal{S}^- \to \mathcal{S}^+$ with one magnetic field; effect of surface currents $\mathbf{J}^\pm$ at $\mathbf{r}'$ may be *added* in the surface integral for $\mathbf{r}' \neq \mathbf{r}$, however $\mathbf{r}$ is *below* $\mathbf{J}^+(\mathbf{r})$ (as before), but *above* $\mathbf{J}^-(\mathbf{r})$ so there's a sign difference in the singular contributions:

$$\frac{\mathbf{J}^+(\mathbf{r})}{2} - \frac{\mathbf{J}^-(\mathbf{r})}{2} - \hat{\mathbf{n}} \times \int_{\mathcal{S}^+} \nabla \times \mathcal{G}^A(\mathbf{r},\mathbf{r}') \cdot \left[\mathbf{J}^+(\mathbf{r}') + \mathbf{J}^-(\mathbf{r}')\right] d\mathcal{S}' = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{i}}, \quad \mathbf{r} \in \mathcal{S}^+$$

This identity cannot be solved alone for *two* unknowns, $\mathbf{J}^+(\mathbf{r})$, $\mathbf{J}^-(\mathbf{r})$.

Identity Can be Combined with EFIE to Obtain Opposite Side Currents Independently



$$\hat{\mathbf{u}}\cdot\left(\mathbf{J}^+ + \mathbf{J}^-\right) = 0$$

EFIE :

$$j\omega\mu \int_{\mathcal{S}^+} \mathcal{G}^A(\mathbf{r},\mathbf{r}')\cdot\left[\overbrace{\mathbf{J}^+(\mathbf{r}')+\mathbf{J}^-(\mathbf{r}')}^{\triangleq\,\mathbf{J}(\mathbf{r}')}\right] d\mathcal{S}' - \frac{\nabla}{j\omega\varepsilon}\int_{\mathcal{S}^+} K^\Phi(\mathbf{r},\mathbf{r}')\nabla'\cdot\left[\overbrace{\mathbf{J}^+(\mathbf{r}')+\mathbf{J}^-(\mathbf{r}')}^{\triangleq\,\mathbf{J}(\mathbf{r}')}\right] d\mathcal{S}' = \mathbf{E}^i_{\tan}$$

Magnetic field identity :

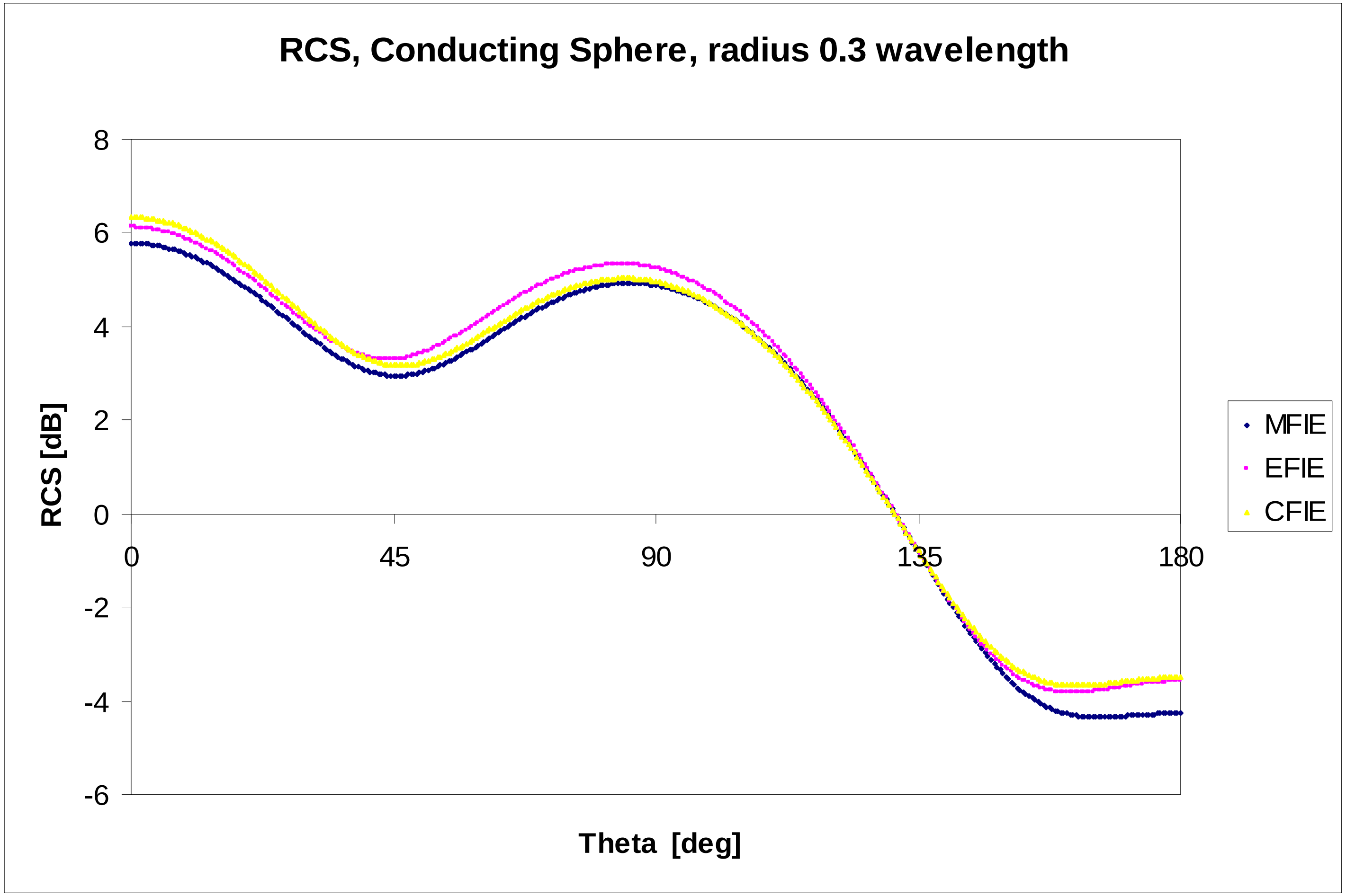
$$\frac{\mathbf{J}^+(\mathbf{r})}{2} - \frac{\mathbf{J}^-(\mathbf{r})}{2} - \hat{\mathbf{n}}\times\int_{\mathcal{S}^+}\nabla\times\mathcal{G}^A(\mathbf{r},\mathbf{r}')\cdot\left[\mathbf{J}^+(\mathbf{r}')+\mathbf{J}^-(\mathbf{r}')\right]d\mathcal{S}' = \hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{i}}, \quad \mathbf{r}\in\mathcal{S}^+$$

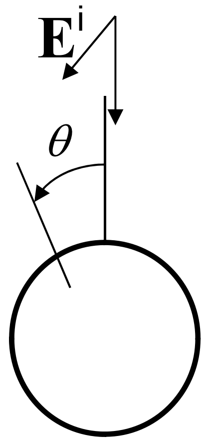
- **Solve EFIE for** $\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$**, use result in identity to obtain** $\mathbf{J}^\pm$ **:**

$$\mathbf{J}^\pm(\mathbf{r}) = \frac{\mathbf{J}(\mathbf{r})}{2} \pm \hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{i}} \pm \hat{\mathbf{n}}\times\int_{\mathcal{S}^+}\nabla\times\mathcal{G}^A(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\mathcal{S}', \quad \mathbf{r}\in\mathcal{S}^\pm$$

- **Or a) solve eqs. simultaneously or b) add and subtract them to get two equations in two unknowns,** $\mathbf{J}^+(\mathbf{r}')$, $\mathbf{J}^-(\mathbf{r}')$.

Scattering by Conducting Sphere Modeled Using 552 Triangles, 828 Unknowns





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