ECE 6350

Solution For Surface Currents Induced on PEC Infinite Cylinder with TM Excitation

D. R. Wilton

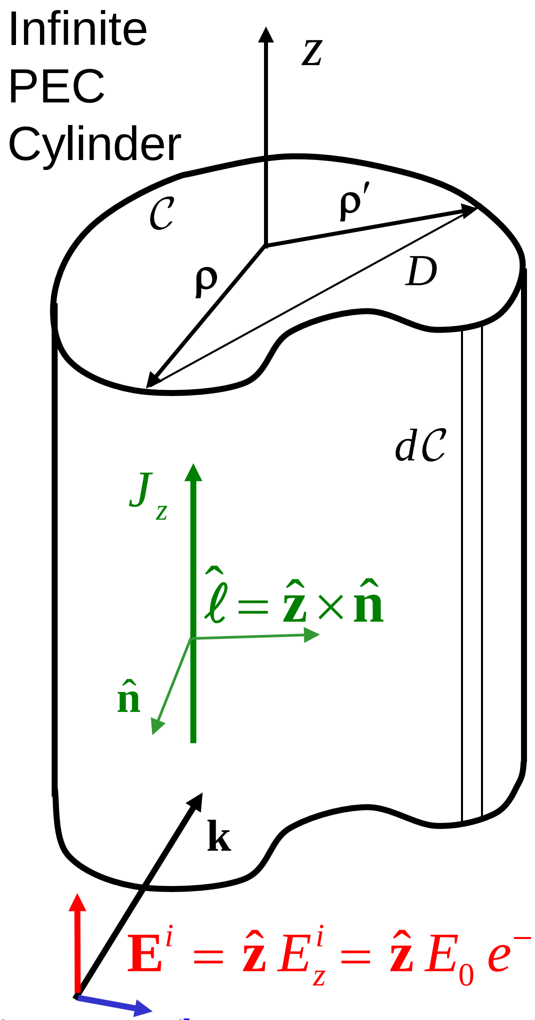
University of Houston

Ref: Scattering Notes, pp.5 -8

Features of TM EFIE vis-à-vis the Electrostatic Charged Cylinder Problem

- The unknown is *current* density rather than *charge* density
- Involves vector---not scalar---potential
- Green’s function is Hankel function, but does have logarithmic singularity
- Time-harmonic vs. static problem requires complex arithmetic
- Right hand side (excitation) is non-constant

Normally Incident, TM Polarized Plane Wave Illumination of PEC Cylinder



- If illumination has no z **variation**, there is no z variation of surface currents, scattered fields, potentials, etc.
- If illumination is also z polarized (TM_z), there is only a z component of surface current.

$$\Rightarrow \quad \bullet\ \mathbf{J}(\mathbf{r}) = J_z(\boldsymbol{\rho})\,\hat{\mathbf{z}}, \quad \boldsymbol{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

$$\Rightarrow \quad \bullet\ \nabla\cdot\mathbf{J}(\mathbf{r}) = \frac{\partial J_z}{\partial z} = 0, \ \Rightarrow \Phi = 0$$

$$\Rightarrow \quad \bullet\ \mathbf{E}^s(\mathbf{r}) = -j\omega\mathbf{A} - \cancel{\nabla\Phi}$$

$$= -j\omega\mu\,\hat{\mathbf{z}}\int_{\mathcal{C}}\int_{-\infty}^{\infty} G(\mathbf{r},\mathbf{r}')J_z(\boldsymbol{\rho}')dz'd\mathcal{C}'$$

$$= \hat{\mathbf{z}}\,E_z^s$$

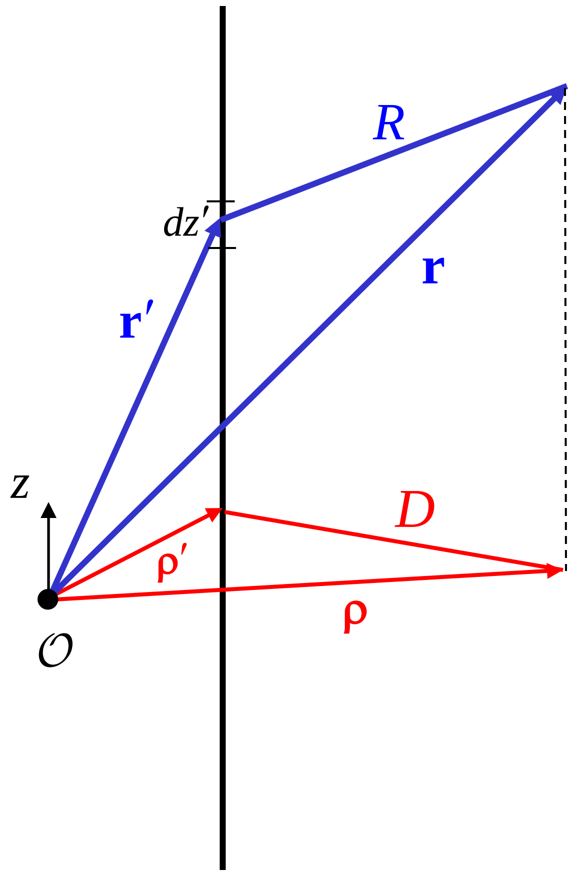
$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-jkR}}{4\pi R}$$

$$\mathbf{E}^i = \hat{\mathbf{z}}\,E_z^i = \hat{\mathbf{z}}\,E_0\,e^{-j\mathbf{k}\cdot\mathbf{r}},$$

$$\mathbf{H}^i = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{k} \equiv k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} = -k\cos\phi^i\hat{\mathbf{x}} - k\sin\phi^i\hat{\mathbf{y}}$$

An Identity Relating Potentials of 2- and 3-D Sources



- Identity :

$$\int_{-\infty}^{\infty} G(\mathbf{r},\mathbf{r}')\,dz' = \int_{-\infty}^{\infty} \frac{e^{-jkR}}{4\pi R}\,dz' = \frac{H_0^{(2)}(k\,D)}{4\,j}$$

where

$$R \equiv \left|\mathbf{r}-\mathbf{r}'\right|, \qquad D \equiv \left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|,$$

$$\boxed{G(\boldsymbol{\rho},\boldsymbol{\rho}') \equiv \frac{H_0^{(2)}(k\,D)}{4\,j} = \frac{J_0(k\,D) - jY_0(k\,D)}{4\,j}},$$

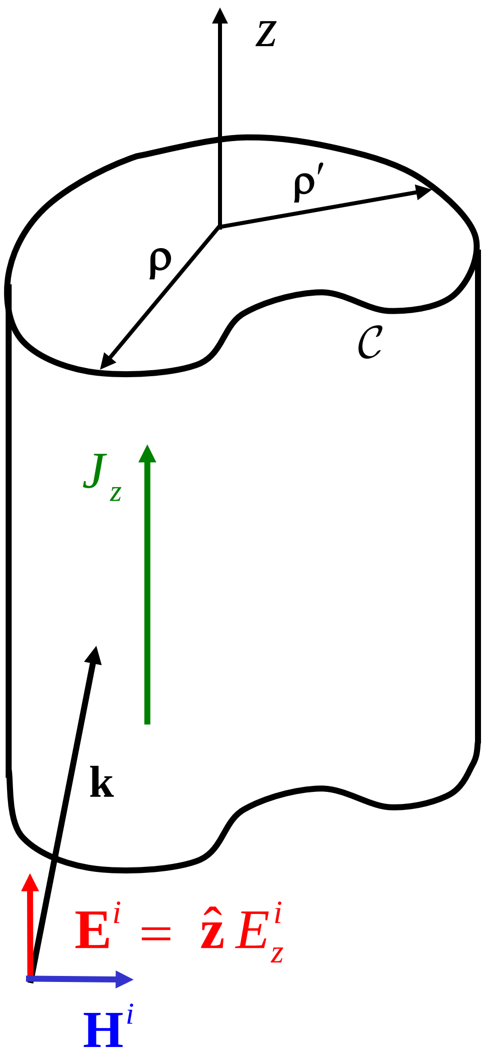
Small and Large Argument Approximations :

- $H_0^{(2)}(x) \xrightarrow[x\to 0]{} 1 - \dfrac{j2}{\pi}\left(\ln\dfrac{x}{2} + \gamma\right),$

  where $\gamma = 0.577215664...$ (Euler's constant)

- $H_0^{(2)}(x) \xrightarrow[x\to\infty]{} \sqrt{\dfrac{2}{\pi x}}\, e^{-j\left(x-\frac{\pi}{4}\right)}$

Application of PEC Boundary Condition Yields Electric Field Integral Equation (EFIE)



- $E_z^s = -j\omega A_z = -j\omega\mu \int_{\mathcal{C}} \left[\int_{-\infty}^{\infty} G(\mathbf{r},\mathbf{r}') dz' \right] J_z(\boldsymbol{\rho}') d\mathcal{C}'$

$$= -j\omega\mu \int_{\mathcal{C}} G(\boldsymbol{\rho},\boldsymbol{\rho}') J_z(\boldsymbol{\rho}') d\mathcal{C}'$$

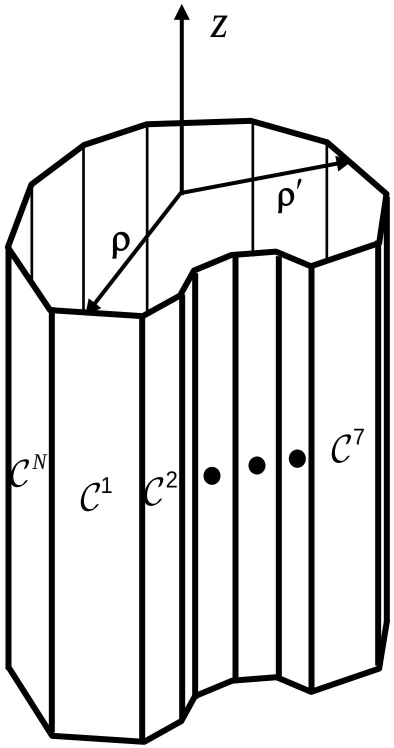
Boundary Condition:

$$\mathbf{E}_{\tan} = \hat{\mathbf{z}}\left(E_z^s + E_z^i\right) = \mathbf{0}, \quad \boldsymbol{\rho} \in \mathcal{C}$$

$$\Rightarrow \boxed{j\omega\mu \int_{\mathcal{C}} G\left(\boldsymbol{\rho},\boldsymbol{\rho}'\right) J_z(\boldsymbol{\rho}') d\mathcal{C}' = E_z^i\left(\boldsymbol{\rho}\right), \ \ \boldsymbol{\rho} \in \mathcal{C}}$$

- **This so-called "strong" form of EFIE holds at every point of $\mathcal{C}$**
- **Must solve for vector-valued current $\mathbf{J}$ at each point $\boldsymbol{\rho}$ of $\mathcal{C}$**

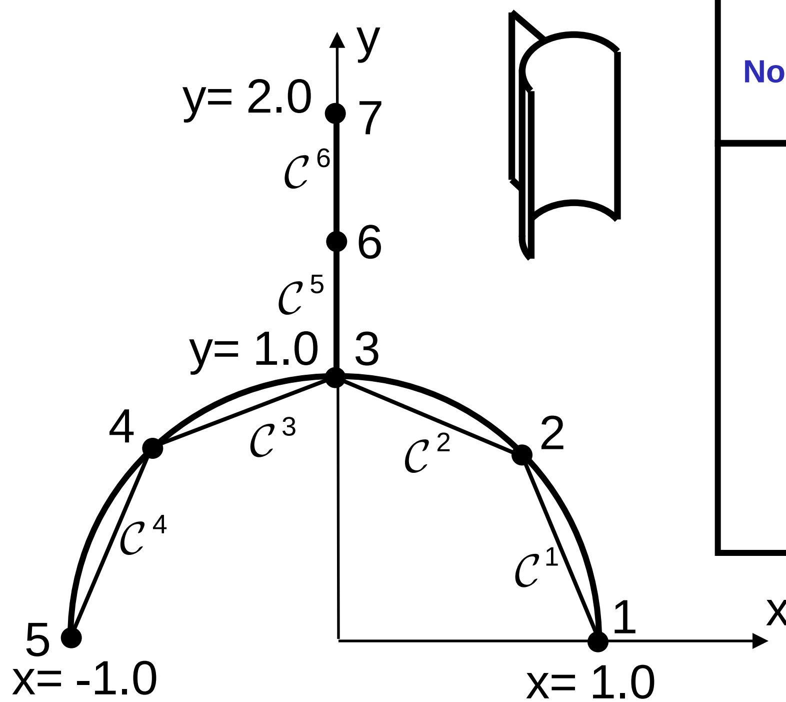
Discretization and Geometry Data Structure



Piecewise linear discretization of geometry

$$\mathcal{C} \approx \tilde{\mathcal{C}} = \bigcup_{n=1}^{N} \mathcal{C}^n$$

Example: Cross section of hemicylinder with fin

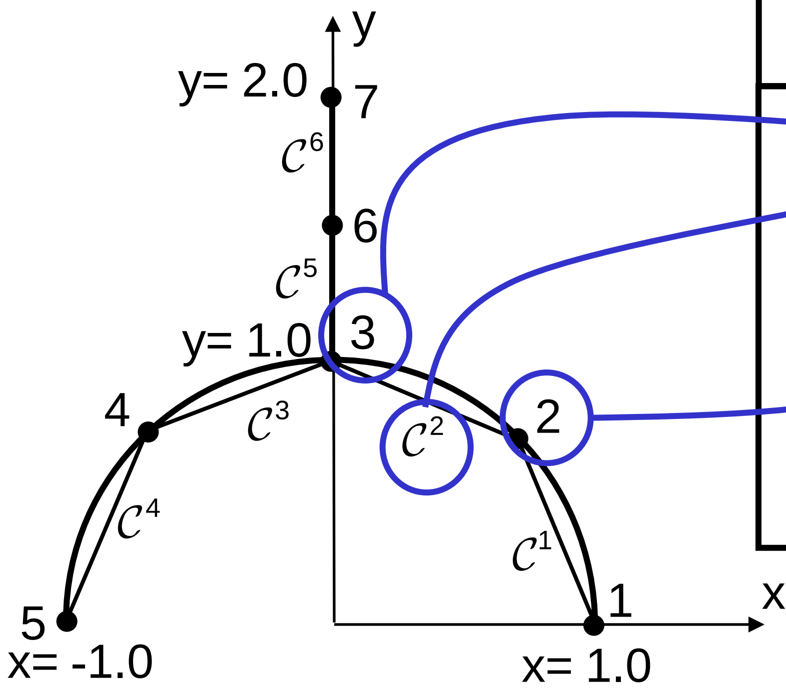


Data structure for element nodes

Global Node Number	Coordinates (z=0) x	Coordinates (z=0) y
1	1.0000	0.0000
2	0.7071	0.7071
3	0.0000	1.0000
4	-0.7071	0.7071
5	-1.0000	0.0000
6	0.0000	1.5000
7	0.0000	2.0000

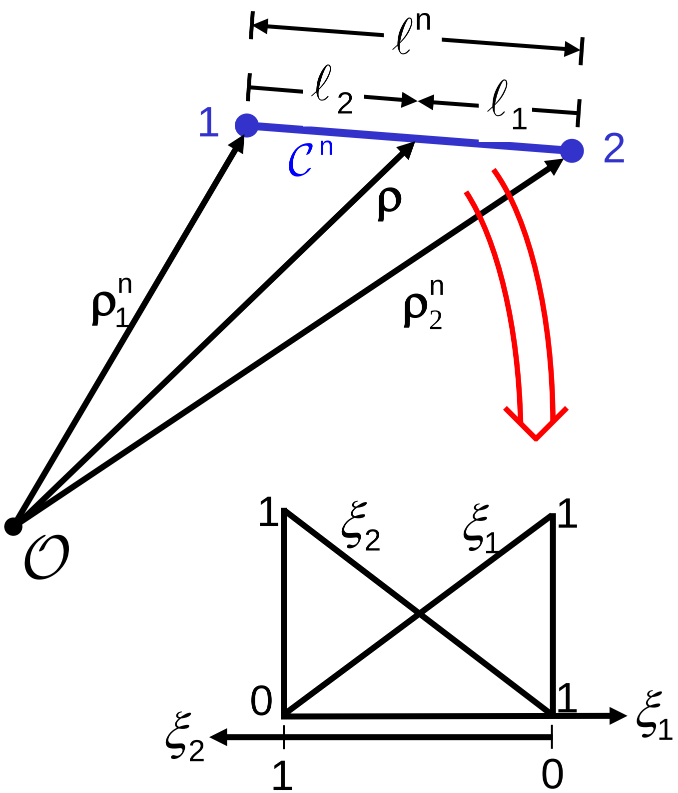
Element Connectivity Data Structure

Element to node mapping



Element Number	Global Node Numbers	
	Local Node 1	Local Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	3	6
6	6	7

Element Parameterization for Integration



All elements are mapped to this unit "parent element"

Parameterization of element geometry:

$$\ell^n = \left| \boldsymbol{\rho}_2^n - \boldsymbol{\rho}_1^n \right|,$$

$$\xi_1 \equiv \frac{\ell_1}{\ell^n}, \quad \xi_2 \equiv \frac{\ell_2}{\ell^n} \qquad (\Rightarrow \xi_1 + \xi_2 = 1)$$

$$\boxed{\boldsymbol{\rho} = \xi_1 \boldsymbol{\rho}_1^n + \xi_2 \boldsymbol{\rho}_2^n \ \text{ on } \mathcal{C}^n}$$

Approximate integration:

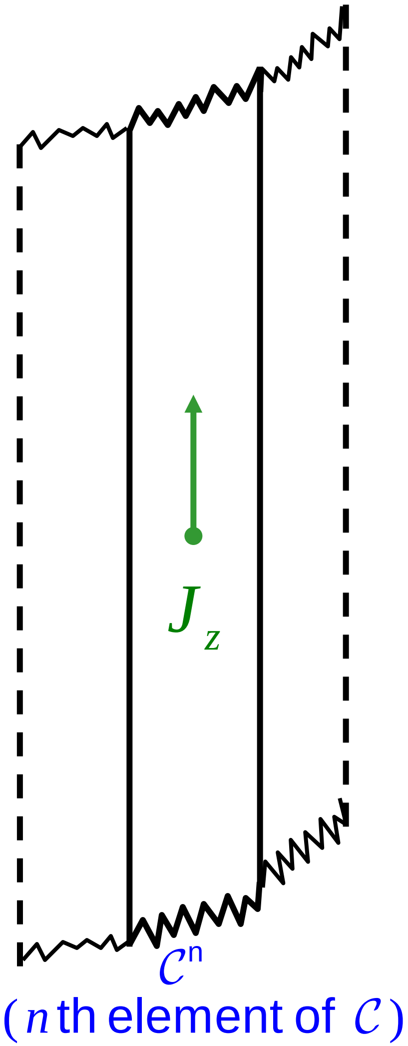
$$\int_{\mathcal{C}^n} f(\boldsymbol{\rho})\, d\mathcal{C} = \ell^n \int_0^1 f(\xi_1 \boldsymbol{\rho}_1^n + \xi_2 \boldsymbol{\rho}_2^n)\, d\xi_{1,2}$$

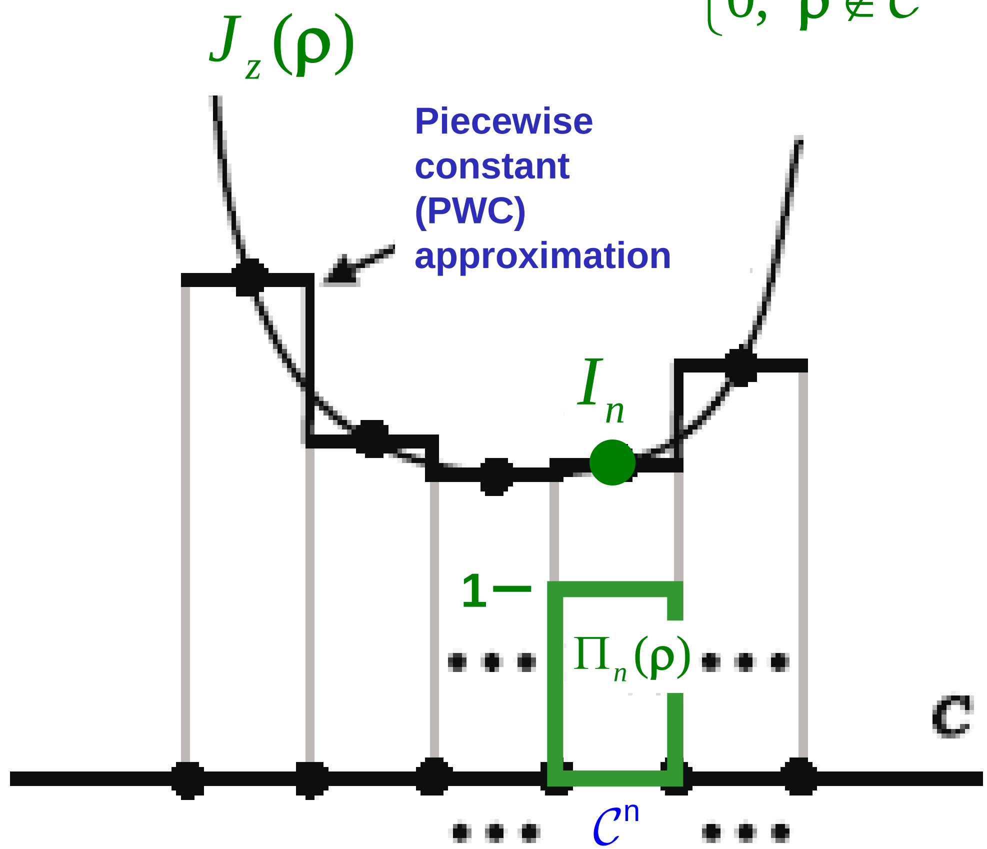
$$\approx \ell^n \sum_{k=1}^{K} w_k\, f(\xi_1^{(k)} \boldsymbol{\rho}_1^n + \xi_2^{(k)} \boldsymbol{\rho}_2^n),$$

$\left(w_k, \xi_1^{(k)}\right)$ **are weights and samples of an appropriate quadrature rule**

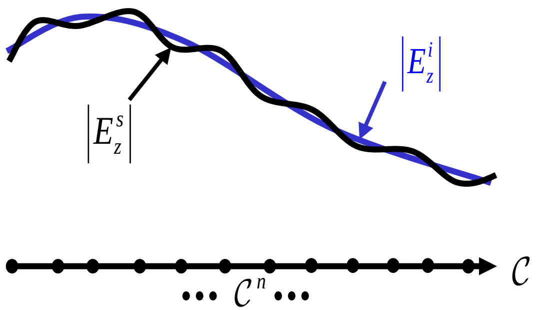
Approximation of TM Current

- $J_z(\rho) \approx \sum_n \underbrace{I_n}_{\text{unknown coeffs.}} \underbrace{\Pi_n(\rho)}_{\text{known basis fn's}}$
- $\Pi_n(\rho) = \begin{cases} 1, & \rho \in \mathcal{C}^n \\ 0, & \rho \notin \mathcal{C}^n \end{cases}$





Formation of Moment Equations



Since $J_z(\boldsymbol{\rho}) \approx \sum_n I_n \Pi_n(\boldsymbol{\rho}), \quad \mathcal{C} \approx \tilde{\mathcal{C}}$

the EFIE becomes

$$\overbrace{\sum_{n=1}^{N} I_n \left[j\omega\mu \int_{\tilde{\mathcal{C}}} G\left(\boldsymbol{\rho},\boldsymbol{\rho}'\right) \Pi_n(\boldsymbol{\rho}') d\mathcal{C}' \right]}^{E_Z^S} \approx E_z^i, \quad \boldsymbol{\rho} \in \tilde{\mathcal{C}}$$

Possible subdomain "equality" constraints :

- **Point match at centroid $\boldsymbol{\rho}_c^m$:**

$$-\int_{\mathcal{C}} \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_c^m) E_z^s \, d\mathcal{C} = \int_{\mathcal{C}} \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_c^m) E_z^i \, d\mathcal{C}, \quad \left(\Rightarrow -E_z^s(\boldsymbol{\rho}_c^m) = E_z^i(\boldsymbol{\rho}_c^m) \right)$$

- **Equate average field over a subdomain :**

$$-\int_{\mathcal{C}} \Pi_m(\boldsymbol{\rho}) E_z^s(\boldsymbol{\rho}) \, d\mathcal{C} = \int_{\mathcal{C}} \Pi_m(\boldsymbol{\rho}) E_z^i(\boldsymbol{\rho}) \, d\mathcal{C}$$

- **Weighted average with *testing function* $T_m(\boldsymbol{\rho})$ as weight :**

$$-\int_{\mathcal{C}} T_m(\boldsymbol{\rho}) E_z^s(\boldsymbol{\rho}) \, d\mathcal{C} = \int_{\mathcal{C}} T_m(\boldsymbol{\rho}) E_z^i(\boldsymbol{\rho}) \, d\mathcal{C}$$

(Notation : $- \langle T_m, E_z^s \rangle = \langle T_m E_z^i \rangle$**)**

($T_m(\boldsymbol{\rho}) = \Pi_m(\boldsymbol{\rho}) \Leftrightarrow$ **Galerkin's method)**

Testing the Equation

To apply the *point - matching* method, sample both sides at subdomain centroids. The result is the system of linear equations,

$$\sum_{n=1}^{N}\left[j\omega\mu \int_{\tilde{\mathcal{C}}} G\left(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}'\right)\Pi_n(\boldsymbol{\rho}')\, d\mathcal{C}' \right] I_n = E_z^i\left(\boldsymbol{\rho}_c^m\right), \quad m = 1, 2, \ldots, N$$

Note that with a more general testing function, $T_m(\boldsymbol{\rho})$, we would write

$$\sum_{n=1}^{N}\left[j\omega\mu \int_{\tilde{\mathcal{C}}} T_m(\boldsymbol{\rho}) \int_{\tilde{\mathcal{C}}} G\left(\boldsymbol{\rho}, \boldsymbol{\rho}'\right)\Pi_n(\boldsymbol{\rho}')\, d\mathcal{C}' d\mathcal{C} \right] I_n = \int_{\tilde{\mathcal{C}}} T_m(\boldsymbol{\rho}) E_z^i\left(\boldsymbol{\rho}\right) d\mathcal{C},$$

$$m = 1, 2, \ldots, N$$

or more succinctly,

$$\left[< T_m, G, \Pi_n > \right]\left[I_n \right] = \left[< T_m, E_z^i > \right]$$

Matrix Form of Moment Equations

In matrix format,

$$\sum_{n=1}^{N} Z_{mn} I_n = V_m, \quad m = 1, 2, \ldots, N \text{ or } [Z_{mn}][I_n] = [V_m]$$

Global impedance matrix :

$$[Z_{mn}] = j\omega [L_{mn}]$$

Global *inductance matrix* :

$$L_{mn} = \mu \int_{\tilde{\mathcal{C}}} G\left(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}'\right) \Pi_n\left(\boldsymbol{\rho}'\right) d\mathcal{C}' = \mu \int_{\mathcal{C}^n} G\left(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}'\right) d\mathcal{C}'$$

Global *excitation voltage vector* :

$$V_m = E_z^i\left(\boldsymbol{\rho}_c^m\right)$$

Element Matrix Calculation—Non-Self Terms

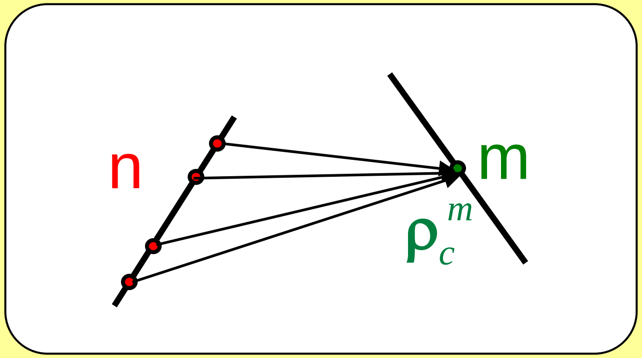
For $m \neq n$, compute the inductance matrix elements as

$$\boxed{L_{mn} = \mu \int_{\mathcal{C}^n} G(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}')\, d\mathcal{C}' \approx \mu \ell^n \sum_{k'=1}^{K'} w_{k'}\, G\left(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}^{(k')}\right)}$$

where

$$\boldsymbol{\rho}^{(k')} = \xi_1^{(k')} \boldsymbol{\rho}_1^n + \xi_2^{(k')} \boldsymbol{\rho}_2^n \,, \qquad \boldsymbol{\rho}_c^m = \frac{\boldsymbol{\rho}_1^m + \boldsymbol{\rho}_2^m}{2} \,,$$

and $\left(\xi_1^{(k')}, w_{k'}\right)$ are Gauss Legendre quadrature samples and weights.



Element Matrix Calculation—Self Terms

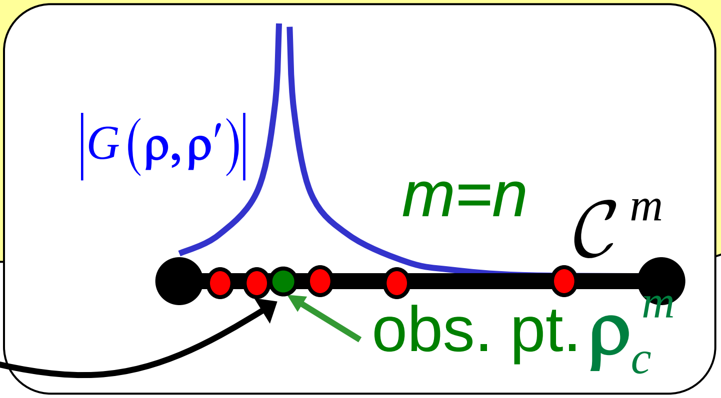
- For $m = n$, kernel is logarithmically singular at $\boldsymbol{\rho} = \boldsymbol{\rho}'$

- Hence, integrate intervals on either side of observation point separately using special (MRW[*]) quadrature rule designed for log singularity:

$$\boxed{L_{mm} = \mu \int_{\mathcal{C}^n} G(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}')\, d\mathcal{C}' \approx \mu \ell^m \sum_{k'=1}^{K'} w_{k'} G\left(\boldsymbol{\rho}_c^m, \boldsymbol{\rho}^{(k')}\right)}$$

[*]Ma, Rokhlin, Wandzura *SIAM J. Numer. Anal.* 33, 1996

$w_{k'}$, $\boldsymbol{\rho}^{(k')} = \boldsymbol{\rho}\left(\xi^{(k')}\right)$ **are functions of** $\boldsymbol{\rho}_c^m$ **!**



Tables of Sample Points and Weights for Gauss-Legendre and MRW Quadrature

Table 3 Sample points and weighting coefficients for K-point Gauss-Legendre quadrature.

Sample Points, $\xi_1^{(k)}$	Weights, w_k
K=1:	
0.500000000000000	1.000000000000000
K=2:	
0.211324865405187	0.500000000000000
0.788675134594813	0.500000000000000
K=4:	
0.069431844202974	0.173927422568727
0.330009478207572	0.326072577431273
0.669990521792428	0.326072577431273
0.930568155797027	0.173927422568727

* **Table 4** Sample points and weighting coefficients for K-point quadratures of form $\int_0^1 f(\xi_1)\, d\xi_1 \approx \sum_{k=1}^{K} w_k f(\xi_1^{(k)})$ where $f(\xi_1)$ has a logarithmic singularity at $\xi_1 = 0$.

Sample Points, $\xi_1^{(k)}$	Weights, w_k
K=1:	
0.367879441171442	1.000000000000000
K=2:	
0.882968651376531 $\times 10^{-1}$	0.298499893705525
0.675186490909887	0.701500106294475
K=3:	
0.288116625309523 $\times 10^{-1}$	0.103330707964930
0.304063729612140	0.454636525970100
0.811669225344079	0.442032766064970
K=5:	
0.565222820508010 $\times 10^{-2}$	0.210469457918546 $\times 10^{-1}$
0.734303717426523 $\times 10^{-1}$	0.130705540744447
0.284957404462558	0.289702301671314
0.619482264084778	0.350220370120399
0.915758083004698	0.208324841671986

*Ma, Rokhlin, Wandzura *SIAM J. Numer. Anal.* 33, 1996

Computation of Voltage Excitation Vector

The RHS voltage excitation column vector simply consists of sampled values of the incident field:

$$\left[V_m \right] \equiv \left[E_z^i (\boldsymbol{\rho}_c^m) \right], \quad m = 1, 2, \ldots, N$$

Summary of Moment Method Solution Procedure

- **Discretize and parameterize the geometry** ⟵ Approx.
- **Choose the formulation (EFIE, etc.)**
- **Choose basis functions** ⟵ Approx.
- **Choose testing functions and test equation to enforce equality** ⟵ Approx.
- **Select quadrature rules for self and non-self terms** ⟵ Approx.

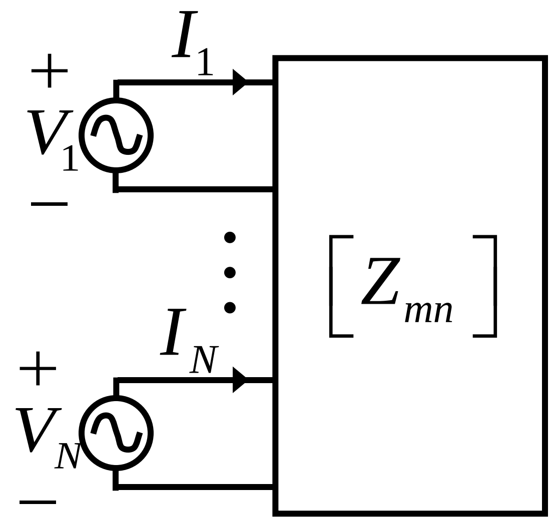
Summary of Moment Method Solution Procedure, Cont'd

- Form matrix and assemble matrix contributions to system matrix
- Form excitation vector (RHS) and assemble contributions to global system vector
- Solve resulting linear system of equations for unknown coefficients
- Use equivalent currents as sources to compute other quantities of interest

Interpretation and Results

$$\left[Z_{mn}\right]\left[I_n\right]=\left[V_m\right]$$

$$\Rightarrow Z_{mn}=\left.\frac{V_m}{I_n}\right|_{I_p=0,\ p\neq n}=$$ **Open circuit impedance matrix**



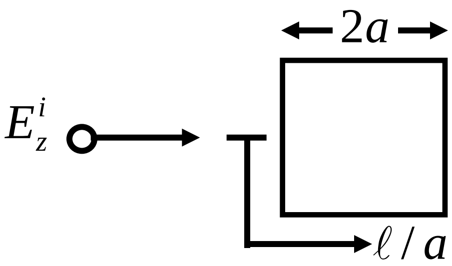
$$\left[I_n\right]=\left[Z_{mn}\right]^{-1}\left[V_m\right]=\left[Y_{nm}\right]\left[V_m\right]$$

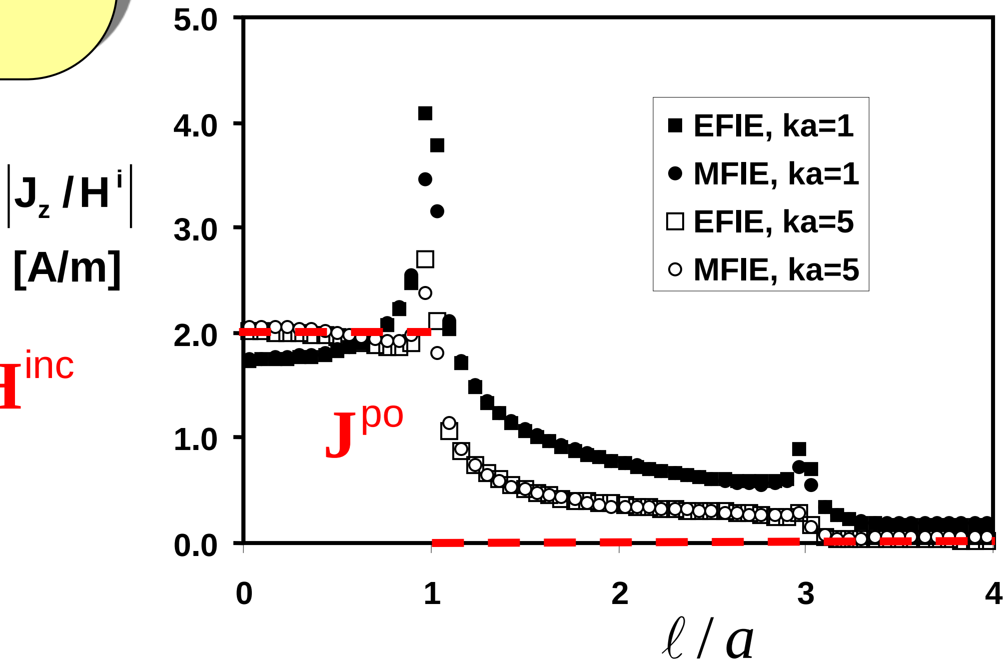
$$\Rightarrow Y_{nm}=\left.\frac{I_n}{V_m}\right|_{V_p=0,\ p\neq n}=$$ **Short circuit admittance matrix**

For an antenna fed at port *m* by a *unit* voltage source, Y_{nm} is the current at element *n*, and Y_{mm} is the *input admittance* : $Y_{\text{in}}=Y_{mm}$.

TM Scattering by a Square Cylinder

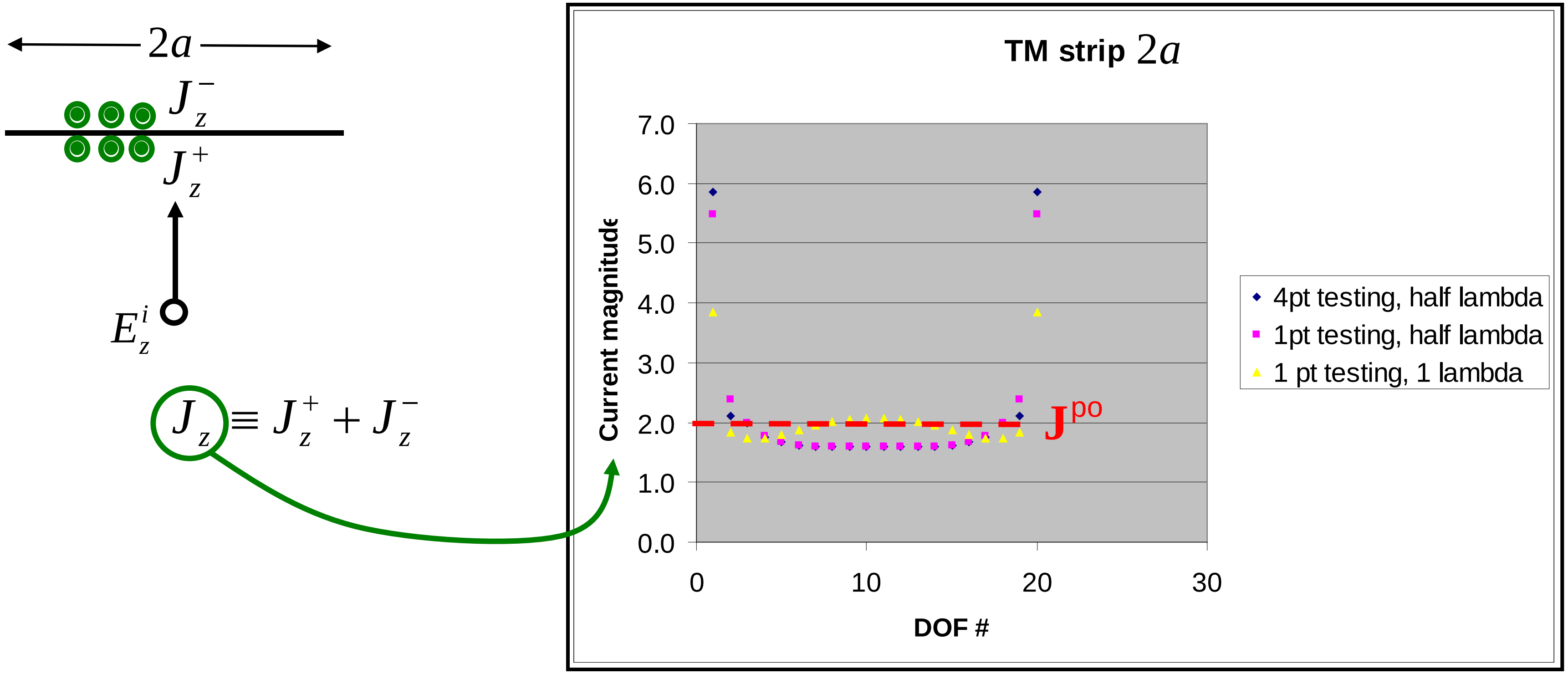
- Current parallel to edges is singular
- At high frequencies, surface current approaches *physical optics* result, J^{po}





$$\mathbf{J} \xrightarrow{\omega \to \infty} \mathbf{J}^{\mathrm{po}} = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{inc}}$$

TM Scattering by Conducting Strip

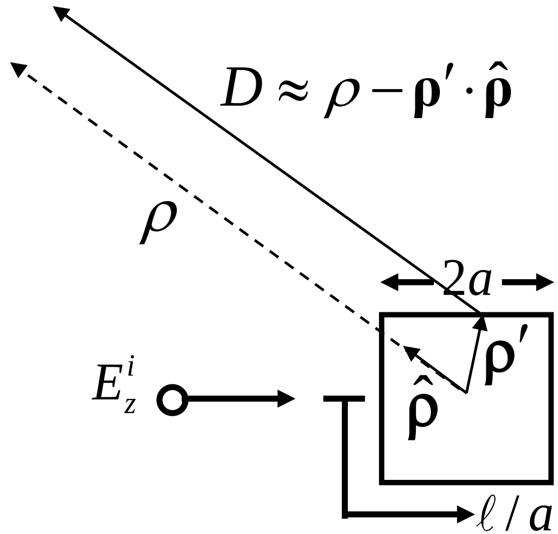


TM Far Scattered Field for a Square Cylinder

Using

$$H_0^{(2)}(kD) \xrightarrow[\rho\to\infty]{} \sqrt{\frac{2}{\pi k D}}\, e^{-j\left(kD-\frac{\pi}{4}\right)}$$

$$\approx \sqrt{\frac{2}{\pi k \rho}}\, e^{-j\left(k\rho-\frac{\pi}{4}\right)} e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'},$$



$$\boxed{\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}}$$

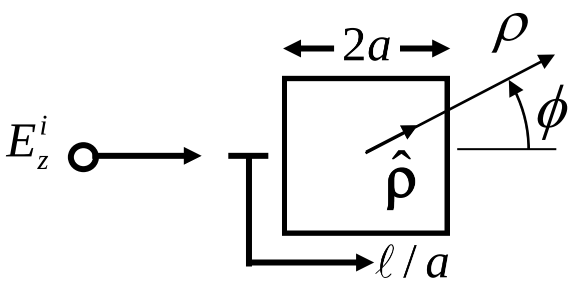
the far electric field is given by

$$E_z^s = -j\omega A_z \xrightarrow[\rho\to\infty]{} \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j\left(k\rho+\frac{\pi}{4}\right)} \int_{\tilde{\mathcal{C}}} J_z(\boldsymbol{\rho}') e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'} d\mathcal{C}'$$

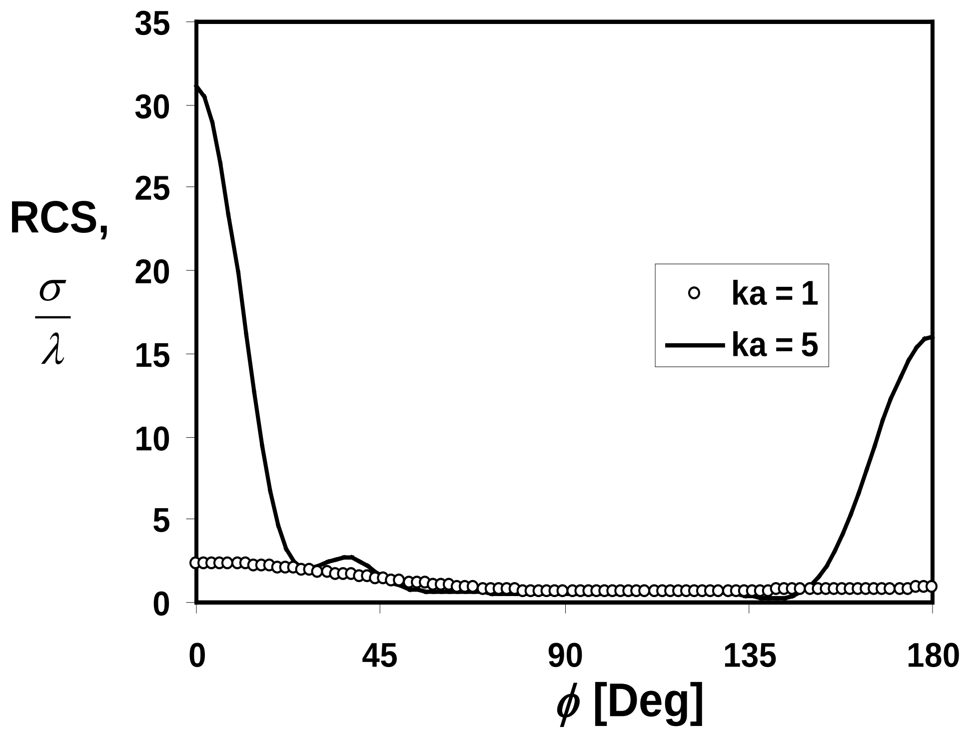
$$= \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j\left(k\rho+\frac{\pi}{4}\right)} \left[\int_{\tilde{\mathcal{C}}} \Pi_n(\boldsymbol{\rho}') e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'} d\mathcal{C}'\right]^t [I_n]$$

$$= \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j\left(k\rho+\frac{\pi}{4}\right)} \left[\int_{\mathcal{C}^n} e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'} d\mathcal{C}'\right]^t [I_n] \equiv \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j\left(k\rho+\frac{\pi}{4}\right)} \left[\tilde{\Pi}_n(k\hat{\boldsymbol{\rho}})\right]^t [I_n]$$

TM Far Scattered Field for a Square Cylinder



$$\sigma \equiv \lim_{\rho\to\infty} 2\pi\rho \frac{\left|\mathbf{E}^s\right|^2}{\left|\mathbf{E}^i\right|^2}$$



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