ECE 6350

2D Poisson's Equation

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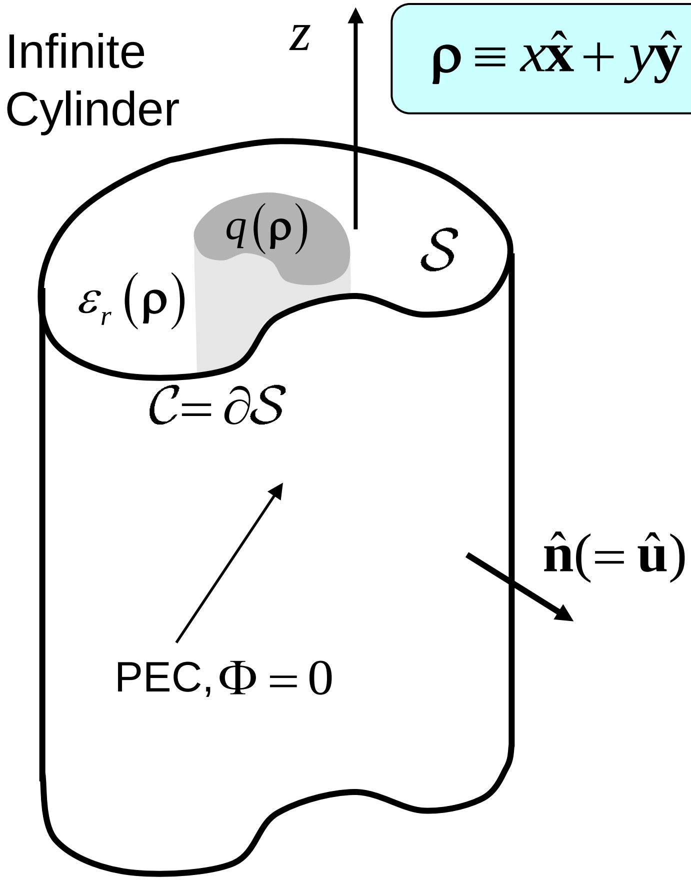
Poisson's Equation for Cylindrical Conducting Tube with *z*-Independent Charge Density



$$\boldsymbol{\rho} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

Poissons' Eq. in 2D:

$$\textbf{2D} \Rightarrow \begin{cases} \partial/\partial z = 0, \\ \mathbf{r} \to \boldsymbol{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} = (x, y) \end{cases}$$

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E} = -\varepsilon_0\varepsilon_r\nabla\Phi$$

$$\nabla\cdot\mathbf{D} = q(\boldsymbol{\rho}) \leftarrow \textbf{(vol. charge density)}$$

$$\Rightarrow \boxed{\begin{array}{c} \nabla\cdot\left(\varepsilon_r(\boldsymbol{\rho})\nabla\Phi\right) = -\dfrac{q(\boldsymbol{\rho})}{\varepsilon_0}, \;\; \boldsymbol{\rho}\in\mathcal{S} \\ \Phi = 0, \;\; \boldsymbol{\rho}\in\partial\mathcal{S} = \mathcal{C} \end{array}}$$

Procedure and New Features for Static 2D Potential Integral Equation

- Model cross section using triangular elements
  - Data structure

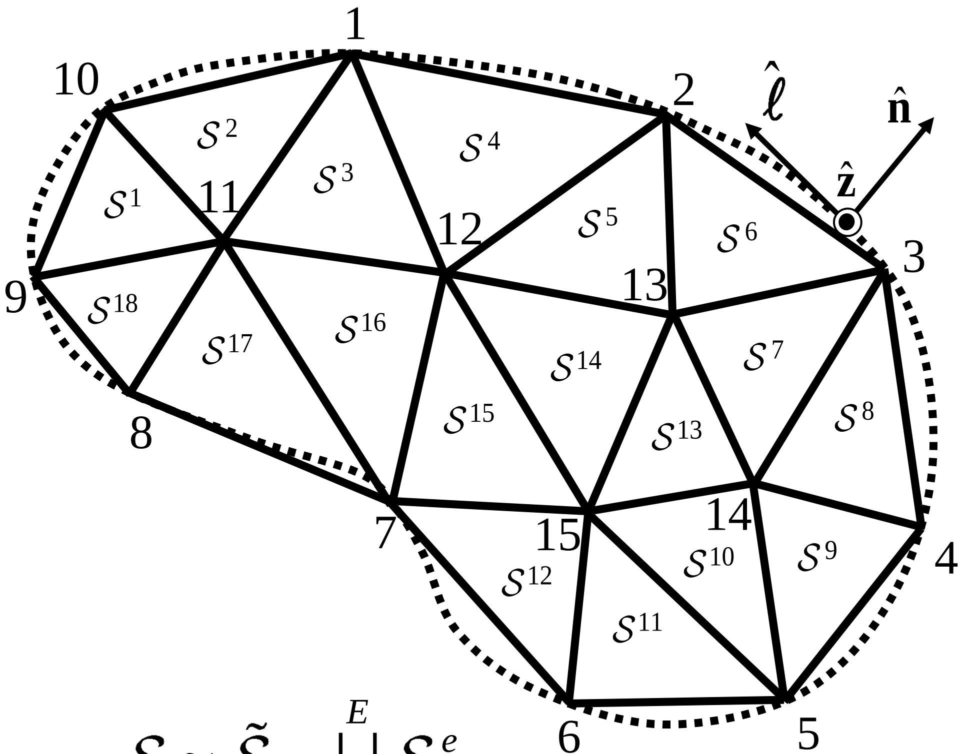
- Obtain weak form of Poisson's equation

- Model potential using scalar triangular bases (piecewise linear representation)

- Obtain a global linear system of equations in terms of global DoFs and bases

- Fill global matrix via local matrix evaluations and matrix assembly

Discretize the Cylinder Cross Section --- Nodal Data

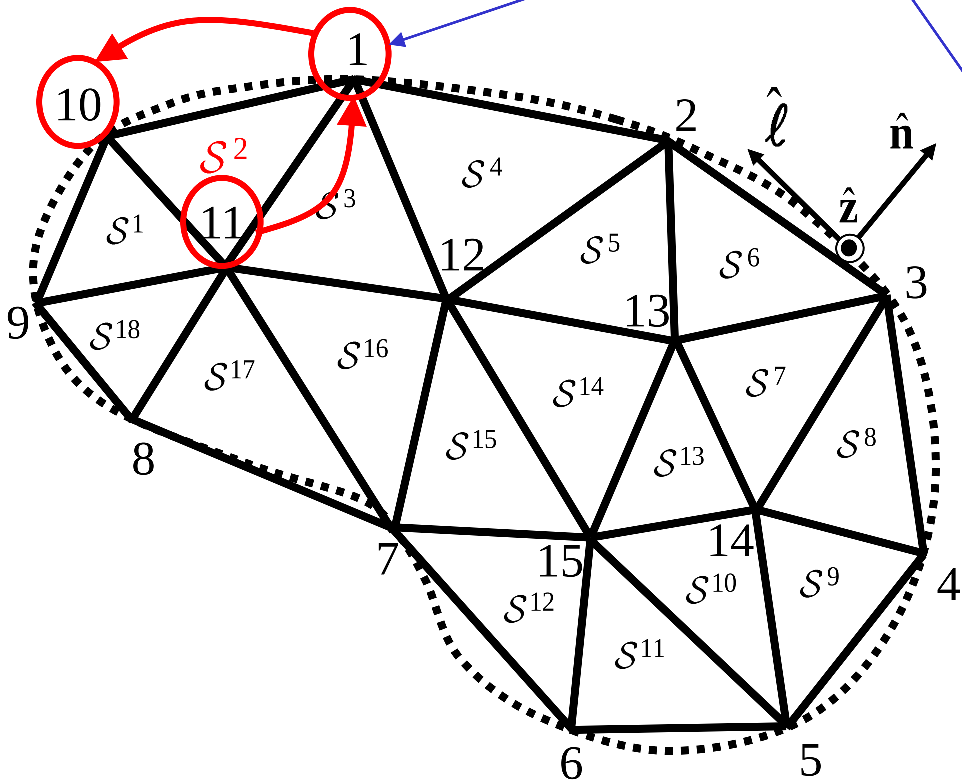


$$\mathcal{S} \approx \tilde{\mathcal{S}} = \bigcup_{e=1}^{E} \mathcal{S}^e$$

Global Node Index v	Coordinates x_v	y_v	
1	-0.500	1.100	$\boldsymbol{\rho}_1$
2	1.100	0.700	$\boldsymbol{\rho}_2$
⋮	⋮	⋮	⋮
12	0.000	0.000	$\boldsymbol{\rho}_{12}$
⋮	⋮	⋮	⋮
15	0.700	-1.100	$\boldsymbol{\rho}_{15}$

Element Connection Data

Counterclockwise listing

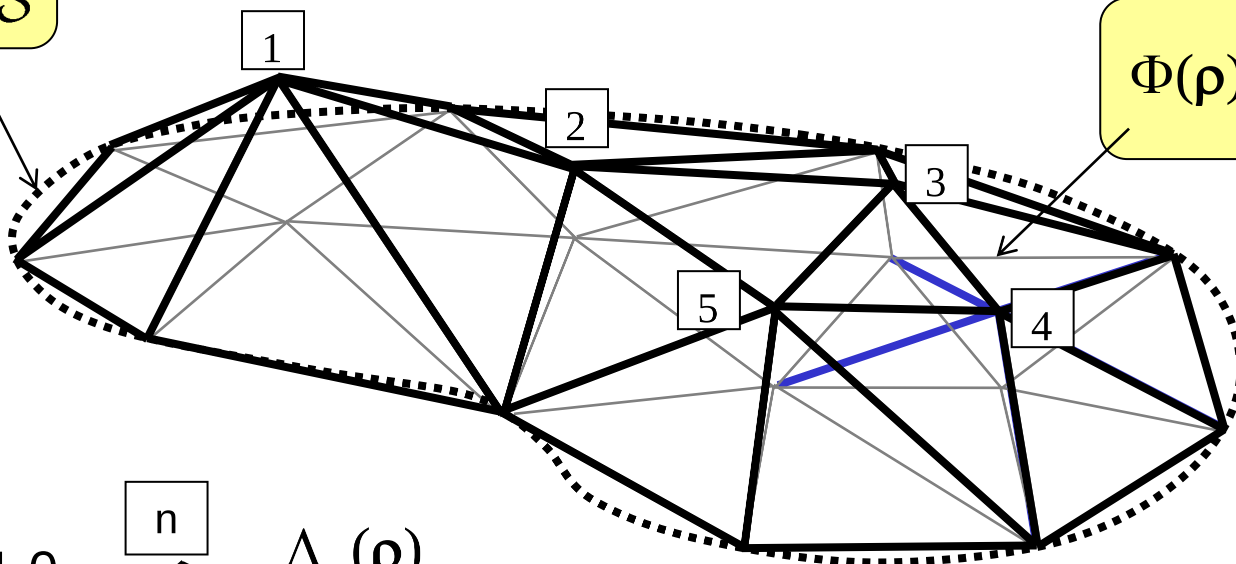


Local Node	1	2	3
e	Global Node No.	Global Node No.	Global Node No.
1	9	11	10
2	11	1	10
⋮	⋮	⋮	⋮
14	15	13	12
⋮	⋮	⋮	⋮
18	8	11	9

Piecewise Linear Model of Potential Φ

$\Phi = 0$
on $\partial \tilde{\mathcal{S}}$

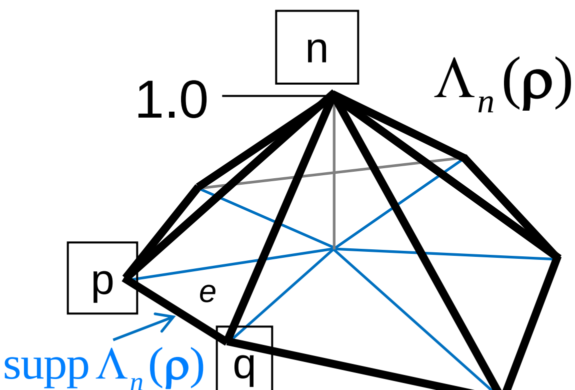
$$\Phi(\boldsymbol{\rho}) \approx \sum_{n=1}^{N} \Phi_n \Lambda_n(\boldsymbol{\rho})$$



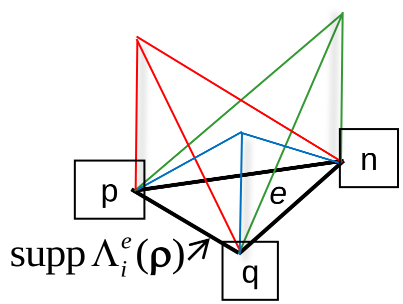
Triangle functions generalized to 2D

$$\Phi(\boldsymbol{\rho}) \approx \sum_{n=1}^{5} \Phi_n \Lambda_n(\boldsymbol{\rho}), \;\; \boldsymbol{\rho} \in \tilde{\mathcal{S}}$$

Global Scalar Representation



Global basis function associated with DoF n



Local bases, element e

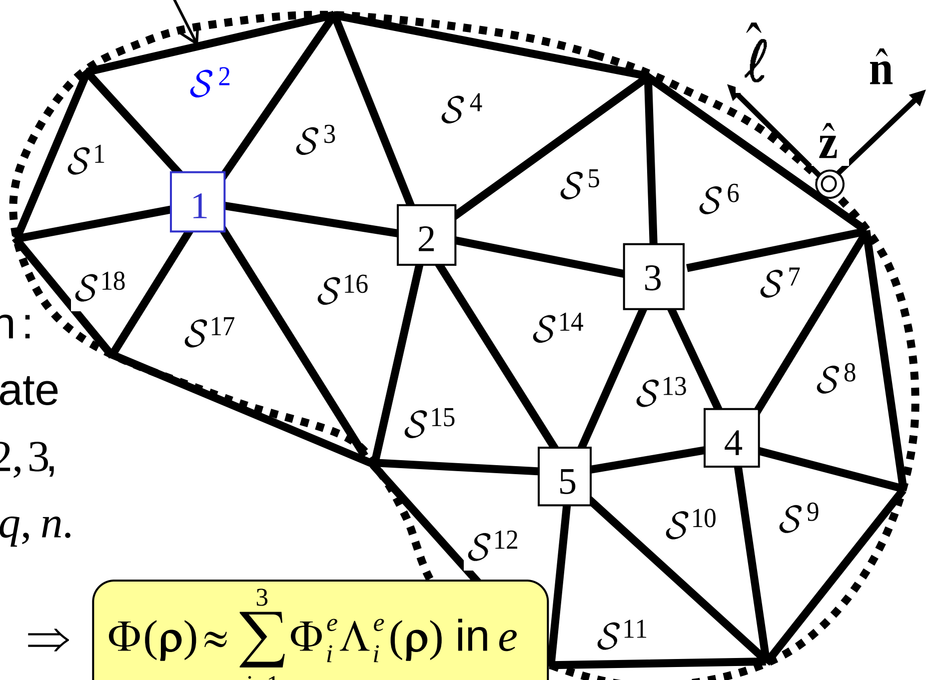
Also use $\Lambda_m(\boldsymbol{\rho})$ as scalar testing functions, m =1,2,…,N (= 5)

Element DoF Data

Local DoF #	1	2	3
e	Global DoF #	Global DoF #	Global DoF #
1	0	1	0
2	1	0	0
⋮	⋮	⋮	⋮

$\Phi = 0$

on $\partial \tilde{\mathcal{S}}$



$$\Phi(\boldsymbol{\rho}) \approx \sum_{n=1}^{N} \Phi_n \Lambda_n(\boldsymbol{\rho})$$

$\max n = N (= 5)$

Local Representation:

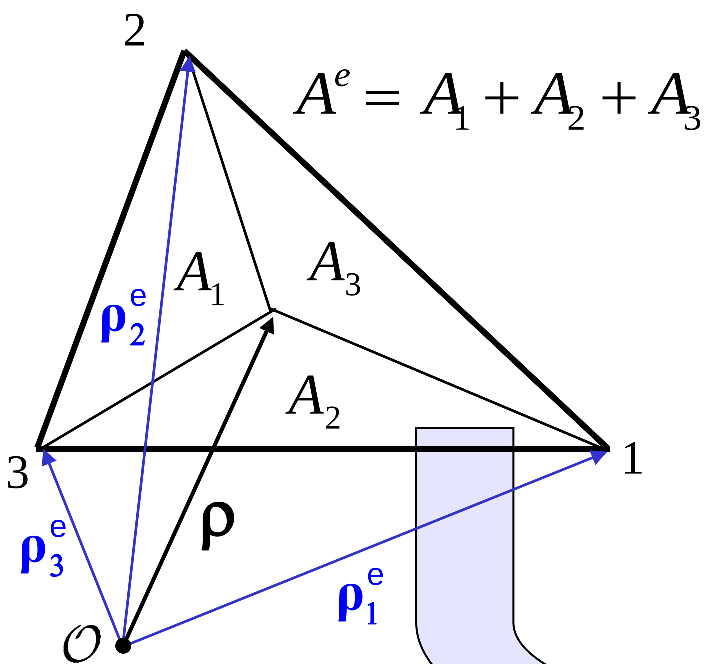
In element e, associate

i th local node, $i = 1, 2, 3,$

with global DoFs p, q, n.

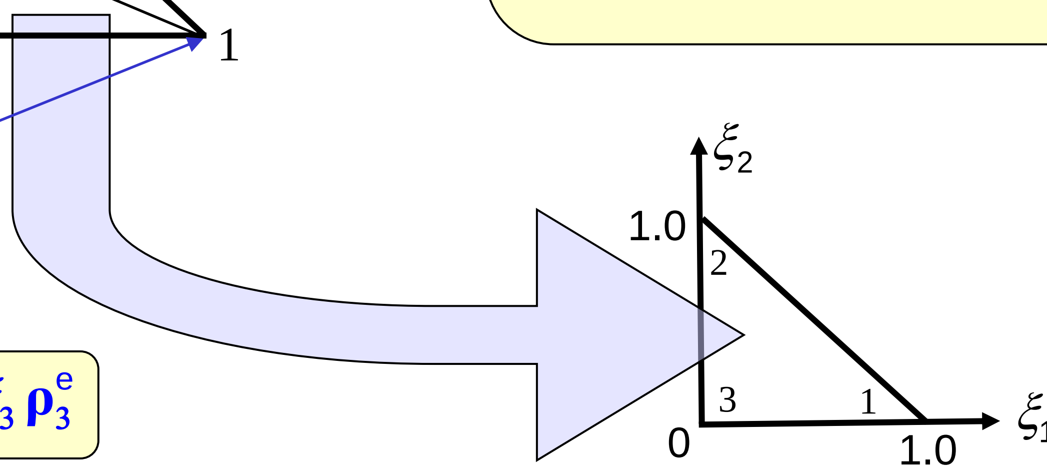
$$\Lambda_p = \Lambda_1^e, \quad \Phi_p = \Phi_1^e,$$
$$\Lambda_q = \Lambda_2^e, \quad \Phi_q = \Phi_2^e, \quad \Rightarrow \quad \Phi(\boldsymbol{\rho}) \approx \sum_{i=1}^{3} \Phi_i^e \Lambda_i^e(\boldsymbol{\rho}) \text{ in } e$$
$$\Lambda_n = \Lambda_3^e, \quad \Phi_n = \Phi_3^e,$$

Area ("Barycentric") Coordinates Are Used to Represent Bases and Parameterize Element Geometry



$$\boxed{\xi_i = \frac{A_i}{A^e}, \quad i = 1, 2, 3}$$

$$\Rightarrow \boxed{\xi_1 + \xi_2 + \xi_3 = 1}$$



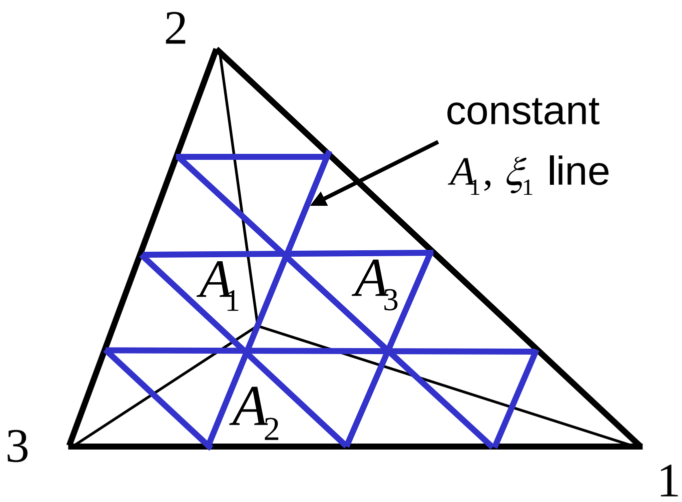
$$\boldsymbol{\rho} = \xi_1 \boldsymbol{\rho}_1^e + \xi_2 \boldsymbol{\rho}_2^e + \xi_3 \boldsymbol{\rho}_3^e$$

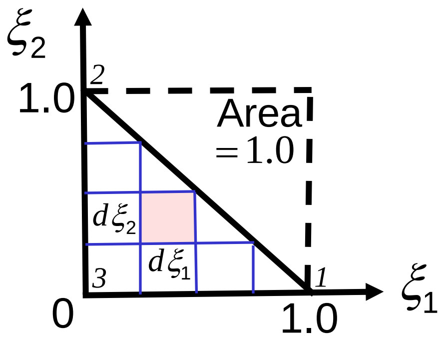
$\boldsymbol{\rho}_i^e = \boldsymbol{\rho}_n$ for some $e, i,$ and n

local global

All elements mapped to "parent element"

Integration in Area Coordinates

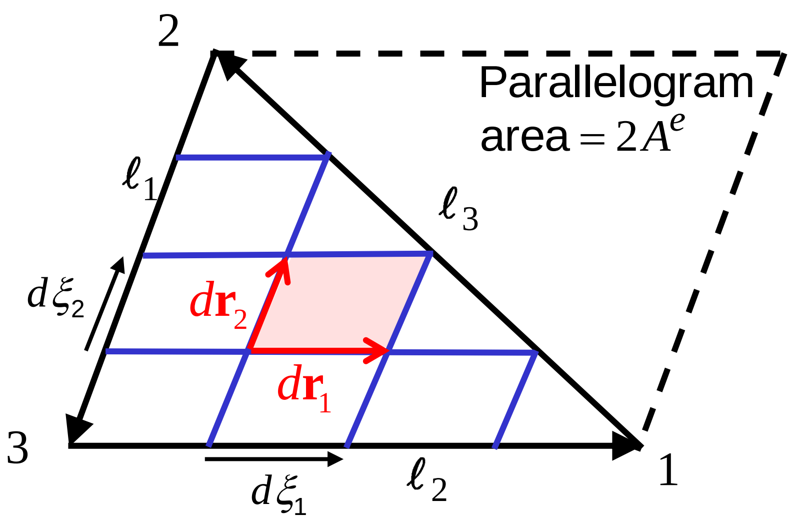




$$\mathbf{dr}_1 = \ell_2 d\xi_1, \qquad \mathbf{dr}_2 = -\ell_1 d\xi_2,$$

$$dS = \left|\mathbf{dr}_1 \times \mathbf{dr}_2\right| = \left|\ell_1 \times \ell_2\right| d\xi_1 d\xi_2$$

$$\Rightarrow \boxed{dS = 2A^e d\xi_1 d\xi_2}$$



$$\ell_i = \boldsymbol{\rho}^e_{i-1} - \boldsymbol{\rho}^e_{i+1}, \quad i = 1,2,3$$

$$\int_{A^e} f(\boldsymbol{\rho})\, dS = 2A^e \int_0^1 \int_0^{1-\xi_2} f(\boldsymbol{\rho}_1^e \xi_1 + \boldsymbol{\rho}_2^e \xi_2 + \boldsymbol{\rho}_3^e \xi_3)\, d\xi_1 d\xi_2$$

$$\approx 2A^e \sum_k w_k f(\boldsymbol{\rho}_1^e \xi_1^{(k)} + \boldsymbol{\rho}_2^e \xi_2^{(k)} + \boldsymbol{\rho}_3^e \xi_3^{(k)})$$

Exact for $f(\boldsymbol{\rho})$ a polynomial; approximate for $f(\boldsymbol{\rho})$ smooth!

Derivation of Weak form of Poisson's Equation

- $$\boxed{\nabla\cdot\left(\varepsilon_r(\boldsymbol{\rho})\nabla\Phi\right) = -\frac{q(\boldsymbol{\rho})}{\varepsilon_0},\ \boldsymbol{\rho}\in\mathcal{S}, \qquad \Phi = 0,\ \ \boldsymbol{\rho}\in\partial\mathcal{S}=\mathcal{C}}$$
- Test equation with scalar testing functions $\Lambda_m(\boldsymbol{\rho})$:

$$<\Lambda_m, \nabla\cdot\left(\varepsilon_r\nabla\Phi\right)> = -\frac{1}{\varepsilon_0}<\Lambda_m, q>$$

$$<A,B> = \int_{\mathcal{S}} A(\boldsymbol{\rho})B(\boldsymbol{\rho})\,d\mathcal{S}$$

- Use the identity $\nabla\cdot\left(\Psi\mathbf{A}\right) = \nabla\Psi\cdot\mathbf{A} + \Psi\nabla\cdot\mathbf{A}$ to integrate by parts:

$$\int_{\mathcal{S}} \nabla\cdot\left(\Lambda_m\left(\varepsilon_r\nabla\Phi\right)\right)d\mathcal{S} = \int_{\mathcal{S}} \nabla\Lambda_m\cdot\left(\varepsilon_r\nabla\Phi\right)d\mathcal{S} + \overbrace{\int_{\mathcal{S}} \Lambda_m\nabla\cdot\left(\varepsilon_r\nabla\Phi\right)d\mathcal{S}}^{<\Lambda_m,\nabla\cdot(\varepsilon_r\nabla\Phi)>}$$

$\Downarrow$ div thm.

$$= \cancelto{0}{\int_{\partial\mathcal{S}} \Lambda_m \overbrace{\left(\varepsilon_r\nabla\Phi\right)\cdot\hat{\mathbf{n}}}^{-\hat{\mathbf{n}}\cdot\mathbf{D}/\varepsilon_0}\,d\mathcal{C}} \quad \Rightarrow \quad <\Lambda_m, \nabla\cdot\left(\varepsilon_r\nabla\Phi\right)> = -<\nabla\Lambda_m;\varepsilon_r\nabla\Phi>$$

Contribution over interior edges vanishes due to contintuity of $\hat{\mathbf{n}}\cdot\mathbf{D}$; boundary contribution vanishes since Λ_m=0 there.

$$\boxed{\Rightarrow \quad <\nabla\Lambda_m;\varepsilon_r\nabla\Phi> = \frac{1}{\varepsilon_0}<\Lambda_m, q>}$$

weak form

$$<\mathbf{A};\mathbf{B}> = \int_{\mathcal{S}} \mathbf{A}(\boldsymbol{\rho})\cdot\mathbf{B}(\boldsymbol{\rho})\,d\mathcal{S}$$

Substitute Basis Representation of Φ into Poisson's Equation

- $$\boxed{< \nabla\Lambda_m;\varepsilon_r\nabla\Phi > = \frac{1}{\varepsilon_0} < \Lambda_m, q >, \; m = 1, 2, \ldots, N}$$ (weak form)

- $$\Phi(\boldsymbol{\rho}) \approx \sum_{n=1}^{N} \Phi_n \Lambda_n(\boldsymbol{\rho})$$

$$\Rightarrow \sum_{n=1}^{N} \Phi_n < \nabla\Lambda_m;\varepsilon_r\nabla\Lambda_n > = \frac{1}{\varepsilon_0} < \Lambda_m, q >, \; m = 1, 2, \ldots, N$$

- Write in matrix form:

$$\boxed{[C_{mn}][\Phi_n] = [Q_m]}$$

where

$$\overbrace{\boxed{C_{mn} = \varepsilon_0 < \nabla\Lambda_m;\varepsilon_r\nabla\Lambda_n >}}^{\text{"capacitance matrix"}}, \quad \boxed{Q_m = < \Lambda_m, q >}$$

- Solving the linear system yields the coefficients of the potential representation: $\Phi(\boldsymbol{\rho}) \approx \sum_{n=1}^{N} \Phi_n \Lambda_n(\boldsymbol{\rho}) = [\Phi_n]^{\mathrm{t}} [\Lambda_n(\boldsymbol{\rho})]$,

where $[\Phi_n] = [C_{mn}]^{-1} [Q_m]$

Element Matrix Evaluation

- $$\left[C_{mn}\right]\left[\Phi_n\right]=\left[Q_m\right]$$

  where

  $$C_{mn} = \varepsilon_0 < \nabla\Lambda_m;\varepsilon_r\nabla\Lambda_n >, \quad Q_m = < \Lambda_m, q >$$

- The *support* of $\nabla\Lambda_n$ consists of all the triangles surrounding the node corresponding to DoF n; for a given triangular element e, however, at most only 3 linear "wedge" portions of global bases overlap onto the element e; denoting them in the local indexing scheme as Λ_i^e, $i=1,2,3$, we see that the corresponding local element matrix is $\left[C_{ij}^e\right]$ where

  $$C_{ij}^e = \varepsilon_0\varepsilon^e < \nabla\Lambda_i^e;\nabla\Lambda_j^e >, \quad i,j=1,2,3$$

  and the corresponding local element column vector is $\left[Q_i^e\right]$ where

  $$Q_i^e = < \Lambda_i^e, q >, \quad i=1,2,3.$$

  Above, we have assumed $\varepsilon_r(\boldsymbol{\rho})=\varepsilon^e$ is constant within element e.

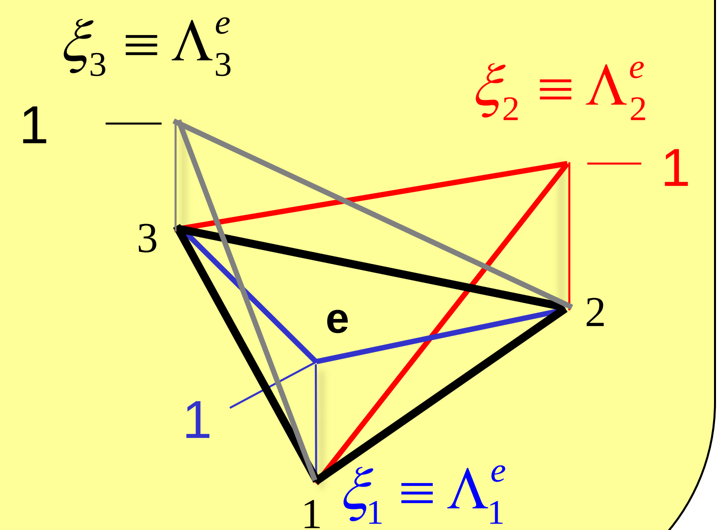
Element Matrix Evaluation, cont'd

- $\left[C^e_{ij}\right] = \varepsilon_0\varepsilon^e\left[<\nabla\Lambda_i^e;\nabla\Lambda_j^e>\right],\ i,j=1,2,3,$ and $\left[Q_i^e\right] = \left[<\Lambda_i^e, q>\right],\ i=1,2,3.$
- Note that $\nabla\Lambda_i^e = \nabla\xi_i = -\frac{1}{h_i}\hat{\mathbf{h}}_i$ and hence

$$\left[C^e_{ij}\right] = \varepsilon_0\varepsilon^e\left[<\nabla\Lambda_i^e;\nabla\Lambda_j^e>\right] = \varepsilon_0\varepsilon^e\left[<-\frac{1}{h_i}\hat{\mathbf{h}}_i;-\frac{1}{h_j}\hat{\mathbf{h}}_j>\right] = \varepsilon_0\varepsilon^e\left[<\frac{\hat{\mathbf{h}}_i}{h_i};\frac{\hat{\mathbf{h}}_j}{h_j}>\right]$$

$$= \varepsilon_0\varepsilon^e A^e\left[\frac{\hat{\mathbf{h}}_i\cdot\hat{\mathbf{h}}_j}{h_i h_j}\right] = \frac{\varepsilon_0\varepsilon^e}{4A^e}\left[\boldsymbol{\ell}_i\cdot\boldsymbol{\ell}_j\right]$$ since $\hat{\mathbf{h}}_i\cdot\hat{\mathbf{h}}_j = \left(\boldsymbol{\ell}_i\cdot\boldsymbol{\ell}_j\right)/(\ell_i\ell_j)$

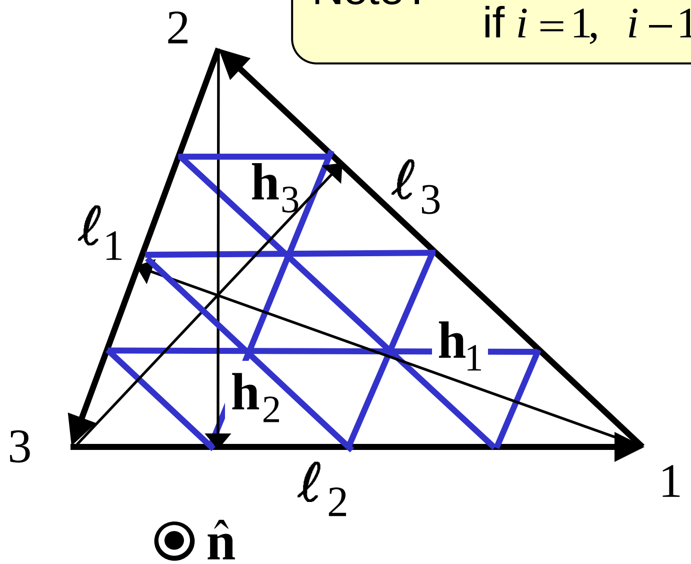
- $$\left[Q_i^e\right] \approx \left[<\Lambda_i^e, \overbrace{q^e}^{\text{assume constant}}>\right] = \left[<\xi_i, q^e>\right] = \frac{q^e A^e}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$



Summary of Triangle Geometrical Parameters

Note : If $i=3,\ i+1=1;$

if $i=1,\ i-1=3;$



- $\boldsymbol{\ell}_i = \mathbf{r}_{i-1}^e - \mathbf{r}_{i+1}^e,\ i=1,2,3$

  (or $\boldsymbol{\ell}_i = \boldsymbol{\rho}_{i-1}^e - \boldsymbol{\rho}_{i+1}^e$ in 2D)
- $\ell_i = |\boldsymbol{\ell}_i|$
- $2A^e = |\boldsymbol{\ell}_{i+1} \times \boldsymbol{\ell}_{i-1}| = \ell_i h_i,\ \ i=1,2,$ or 3
- $\hat{\mathbf{n}} = \dfrac{\boldsymbol{\ell}_{i+1} \times \boldsymbol{\ell}_{i-1}}{2A^e},\ \ i=1,2,$ or 3
- $\hat{\mathbf{h}}_i = \hat{\boldsymbol{\ell}}_i \times \hat{\mathbf{n}} = \boldsymbol{\ell}_i \times \hat{\mathbf{n}}/\ell_i,\ i=1,2,3$

  (or $\hat{\mathbf{h}}_i = \hat{\boldsymbol{\ell}}_i \times \hat{\mathbf{z}} = \boldsymbol{\ell}_i \times \hat{\mathbf{z}}/\ell_i$ in 2D)
- $|\mathbf{h}_i| = h_i = 2A^e/\ell_i = |\boldsymbol{\ell}_{i\pm1} \cdot \hat{\mathbf{h}}_i|$
- $\mathbf{h}_i = h_i \hat{\mathbf{h}}_i = 2A^e(\boldsymbol{\ell}_i \times \hat{\mathbf{n}})/(\boldsymbol{\ell}_i \cdot \boldsymbol{\ell}_i)$

  $= \dfrac{\boldsymbol{\ell}_i \times (\boldsymbol{\ell}_{i+1} \times \boldsymbol{\ell}_{i-1})}{\boldsymbol{\ell}_i \cdot \boldsymbol{\ell}_i}$
- $\nabla \xi_i = \nabla \Lambda_i^e = -\dfrac{1}{h_i}\hat{\mathbf{h}}_i = -\dfrac{\mathbf{h}_i}{h_i^2}$

Since $\nabla\Phi(\boldsymbol{\rho}) \approx \sum_{i=1}^{3} \Phi_i^e \nabla\Lambda_i^e(\boldsymbol{\rho}) = -\sum_{i=1}^{3} \Phi_i^e \dfrac{\mathbf{h}_i}{h_i^2}$,

the $\mathbf{h}_i$ serve as *basis vectors* for grad Φ on e .

But they are not independent since $\sum_{i=1}^{3} \xi_i = 1 \Rightarrow$

$$\sum_{i=1}^{3} \nabla \xi_i = -\sum_{i=1}^{3} \frac{\mathbf{h}_i}{h_i^2} = 0$$

Matrix Assembly

Loop over all elements :

$\vdots$

- **Element #** 2 :

  add C^2_{11} to C_{11};

  discard all other C^2_{ij};

  add Q^2_1 to Q_1;

  discard all other Q^2_i

  $\vdots$

- **Element #** 10 :

  add C^{10}_{11} to C_{55},

  add C^{10}_{13} to C_{54},

  add C^{10}_{31} to C_{45},

  add C^{10}_{33} to C_{44};

  discard all other C^{10}_{ij}

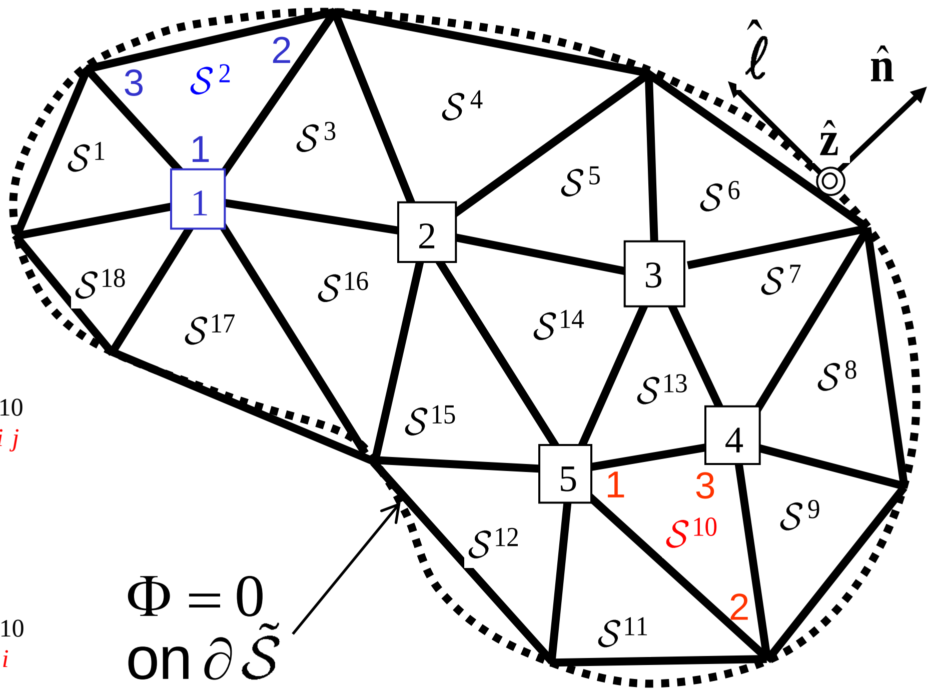
  add Q^{10}_1 to Q_5,

  add Q^{10}_3 to Q_4;

  discard all other Q^{10}_i

  $\vdots$

Local DoF #	1	2	3
e	Global DoF #	Global DoF #	Global DoF #
1	0	1	0
2	1	0	0
⋮	⋮	⋮	⋮



$$\Phi(\boldsymbol{\rho}) \approx \sum_{n=1}^{N} \Phi_n \Lambda_n(\boldsymbol{\rho})$$

$$\max n = N (= 5)$$