ECE 6350

Review of Field Representation Via Potential Integrals

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Ref: Scattering Notes, p. 2

Maxwell’s Equations

Maxwell's equations in frequency domain with $e^{j\omega t}$ time convention factor assumed and suppressed :

- $\nabla \times \mathbf{E} = -j\omega\mathbf{B}$ (Faraday's Law)
- $\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J}$ (Ampere's Law)
- $\nabla \cdot \mathbf{D} = q$ (Electric Form of Gauss's Law)
- $\nabla \cdot \mathbf{B} = 0$ (Magnetic Form of Gauss's Law)

Supplementary Equations

Continuity Equation:

- $\nabla \cdot \mathbf{J} = -j\omega q$ (q is volume charge density)

Constitutive Relations for Linear, Isotropic, Homogeneous Materials:

- $\mathbf{B} = \mu \mathbf{H}$ $(\mu_0 \equiv 4\pi \times 10^{-7} \text{[H/m]})$

- $\mathbf{D} = \varepsilon \mathbf{E}$ $\left(\varepsilon_0 \equiv \dfrac{1}{\mu_0 c_0^2},\ c_0 \equiv 2.99792458 \times 10^{8} \text{[m/s]}\right)$

Potential Representations

Identities:

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla \Phi \;\; = \mathbf{0}$$

Magnetic Vector Potential:

$$\nabla \cdot \mathbf{B} \;=\; 0$$

$$\Rightarrow \boxed{\mathbf{B} \;=\; \nabla \times \mathbf{A}}$$

Electric Scalar Potential:

- $\nabla \times \mathbf{E} + j\omega\mathbf{B} \;= \nabla \times \left(\mathbf{E} + j\omega\mathbf{A}\right) = \mathbf{0} \equiv -\nabla \times \nabla \Phi$

$$\Rightarrow \boxed{\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi}$$

Vector Potential Helmholtz Equation

- Multiply Ampere's Law by μ, obtaining $\nabla\times\left(\mu\mathbf{H}\right) = \quad j\omega\varepsilon\mu\mathbf{E} + \mu\mathbf{J}$,

  and substitute potential representations for $\mathbf{E}$, $\mathbf{H}$:

$$\nabla\times\left(\nabla\times\mathbf{A}\right) - k^2\mathbf{A} + j\omega\varepsilon\mu\nabla\Phi = \mu\mathbf{J},$$

  where $k = \omega\sqrt{\varepsilon\mu}$ is the *wavenumber*

Ludwig Lorenz

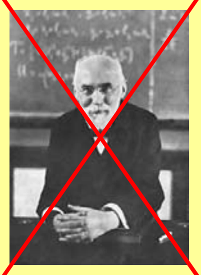
- Use identity $\nabla\times\left(\nabla\times\mathbf{A}\right) \equiv \nabla\left(\nabla\cdot\mathbf{A}\right) - \nabla^2\mathbf{A}$ to write

$$\nabla^2\mathbf{A} + k^2\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A} + j\omega\varepsilon\mu\Phi\right) = -\mu\mathbf{J}$$

- Choose the *Loren~~t~~z gauge* condition, $\boxed{\nabla\cdot\mathbf{A} = -j\omega\varepsilon\mu\Phi}$

  to obtain the *vector (Helmholtz) wave equation*

$$\boxed{\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}}$$



Hendrik Lorentz

Solution for Vector Potential Wave Equation

The *outgoing* solution for the wave equation,

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

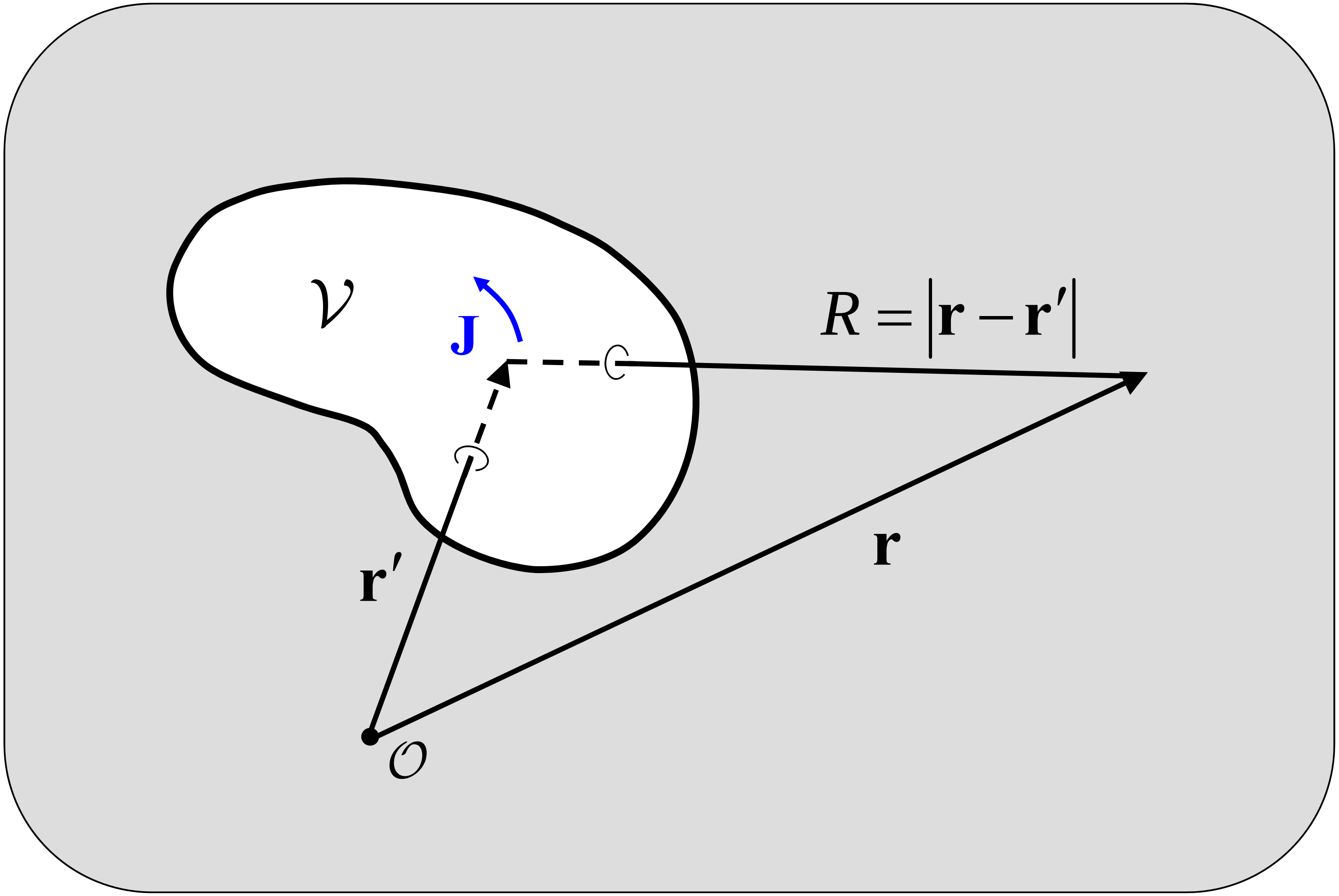
in infinite, linear, homogeneous, isotropic media is

$$\boxed{\mathbf{A}(\mathbf{r}) = \mu \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') \, d\mathcal{V}'}$$

where $G(\mathbf{r}, \mathbf{r}') \equiv \dfrac{e^{-jkR}}{4\pi R}, \quad R \equiv |\mathbf{r} - \mathbf{r}'|.$

$G(\mathbf{r}, \mathbf{r}')$ is the *free space* or *homogeneous medium Green's function*

Definitions of Geometrical Quantities



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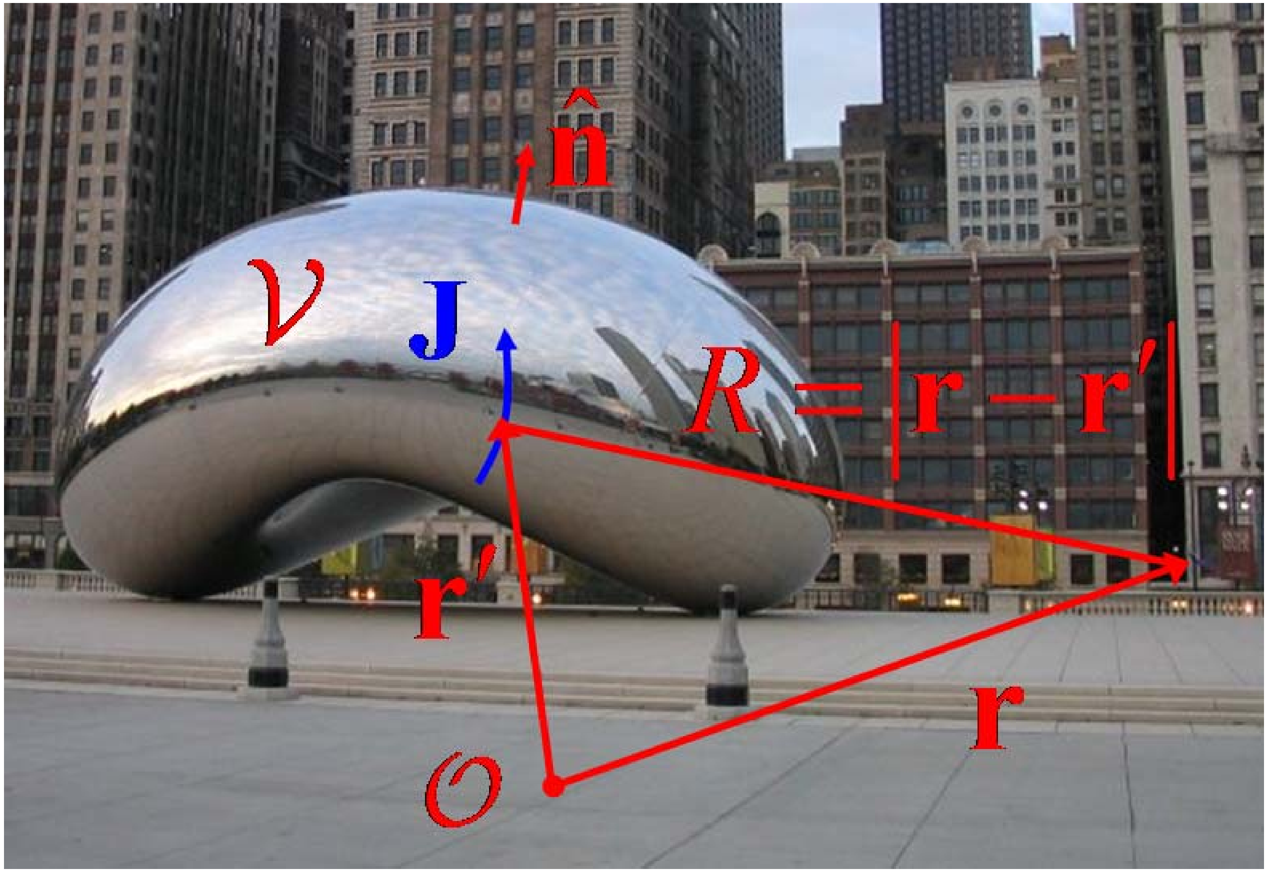


Scalar Potential Wave Equation

- Substitute both the constitutive equation and potential representation into Gauss's Law :

$$\nabla \cdot \mathbf{D} \ = \ \nabla \cdot \left[\varepsilon \underbrace{\left(-j\omega \mathbf{A} - \nabla \Phi\right)}_{\mathbf{E}} \right] = q$$

- Use the Loren~~t~~z gauge condition, $\nabla \cdot \mathbf{A} = -j\omega\varepsilon\mu\Phi$, and $\nabla \cdot \nabla \equiv \nabla^2$ to obtain the *scalar wave (Helmholtz) equation*

$$\boxed{\nabla^2 \Phi + k^2 \Phi \ = \ -\frac{q}{\varepsilon}, \quad \left(k^2 = \omega^2 \varepsilon \mu\right)}$$

Solution for Scalar Potential Wave Equation

The outgoing solution of the scalar wave equation,

$$\nabla^2 \Phi + k^2 \Phi = -\frac{q}{\varepsilon} \ ,$$

is

$$\boxed{\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{\varepsilon} \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')\, q(\mathbf{r}')\, d\mathcal{V}' \\ &= -\frac{1}{j\omega\varepsilon} \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')\, \nabla' \cdot \mathbf{J}(\mathbf{r}')\, d\mathcal{V}' \end{aligned}}$$

where $G(\mathbf{r},\mathbf{r}') \equiv \dfrac{e^{-jkR}}{4\pi R}$ is the *homogeneous medium scalar Green's function* and $R \equiv \left|\mathbf{r} - \mathbf{r}'\right|$

Summary of Mixed Potential Representation for Electric and Magnetic Fields Produced by Electric Current Sources

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$$

where the *magnetic vector potential* is

$$\boxed{\mathbf{A}(\mathbf{r}) = \mu\int_{\mathcal{V}} \frac{e^{-jkR}}{4\pi R}\,\mathbf{J}(\mathbf{r}')\,d\mathcal{V}' = \mu\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d\mathcal{V}'}$$

and the *electric scalar potential* is

$$\boxed{\Phi(\mathbf{r}) = \int_{\mathcal{V}} \frac{e^{-jkR}}{4\pi\varepsilon R}\,q(\mathbf{r}')\,d\mathcal{V}' = -\frac{1}{j\omega\varepsilon}\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')\,\nabla'\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{V}'}$$

The Mixed Potential Representation Applies to More General Electric Current Density Types

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$$

- **The potential representation also applies to other current density distributions. E.g. one can write**

$$\boxed{\mathbf{A}(\mathbf{r}) = \mu\int_{\mathcal{D}} \frac{e^{-jkR}}{4\pi R}\,\mathbf{J}_{\mathcal{D}}(\mathbf{r}')\,d\mathcal{D}' = \mu\int_{\mathcal{D}} G(\mathbf{r},\mathbf{r}')\,\mathbf{J}_{\mathcal{D}}(\mathbf{r}')\,d\mathcal{D}'}$$

where $\mathcal{D}=\mathcal{V},\mathcal{S},$ **or** $\mathcal{C}$ **and** $\mathbf{J}_{\mathcal{D}}(\mathbf{r}')\,d\mathcal{D}'$ **is an elemental dipole,** $\mathbf{J}_{\mathcal{V}}(\mathbf{r}')d\mathcal{V}'$**,** $\mathbf{J}_{\mathcal{S}}(\mathbf{r}')d\mathcal{S}'$**, or** $\mathbf{J}_{\mathcal{C}}(\mathbf{r}')d\mathcal{C}' \equiv I\hat{\ell}\,d\mathcal{C}'$**, respectively.**

- **Similarly, the scalar potential is**

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{\varepsilon}\int_{\mathcal{D}} \frac{e^{-jkR}}{4\pi R}\,q_{\mathcal{D}}(\mathbf{r}')\,d\mathcal{D}' = \frac{1}{\varepsilon}\int_{\mathcal{D}} G(\mathbf{r},\mathbf{r}')\,q_{\mathcal{D}}(\mathbf{r}')\,d\mathcal{D}'}$$

- **For mixed density types, superposition applies**

Note: Alternative Potential Representations for Electric Field Are Not as Convenient as *Mixed Potentials* for Numerical Work

- $\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$ (mixed potential) (1)

  Using the Loren~~t~~z gauge,

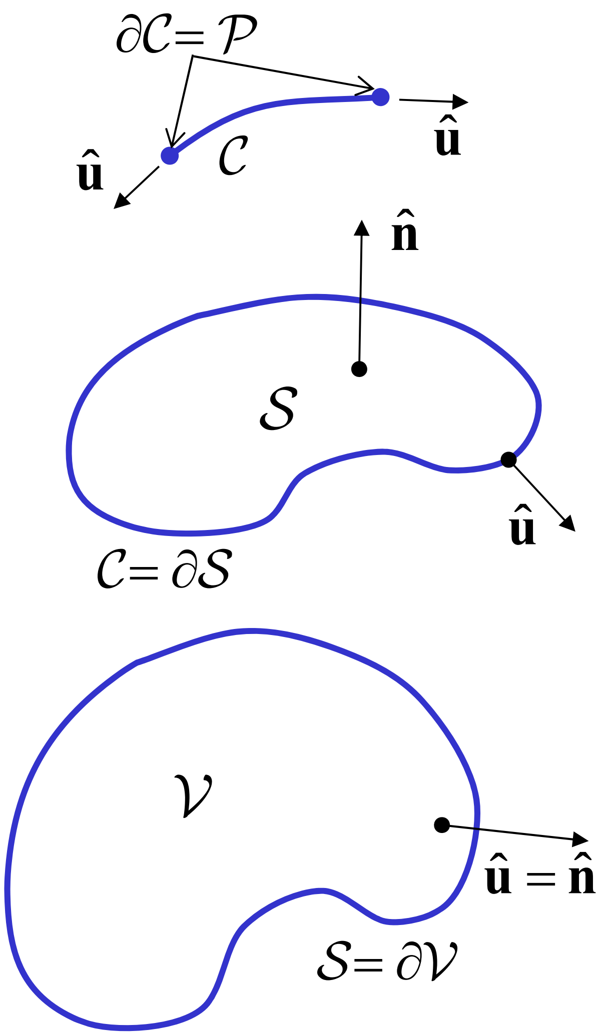
  $$\Phi = -\frac{1}{j\omega\varepsilon\mu}\nabla\cdot\mathbf{A},$$

  $$\Rightarrow$$

  $$\mathbf{E} = \frac{-j\omega}{k^2}\left(k^2\mathbf{A} + \nabla\nabla\cdot\mathbf{A}\right)$$ (vector potential) (2)

- (1) and (2) are *analytically* equivalent, but (2) is more difficult to evaluate numerically!

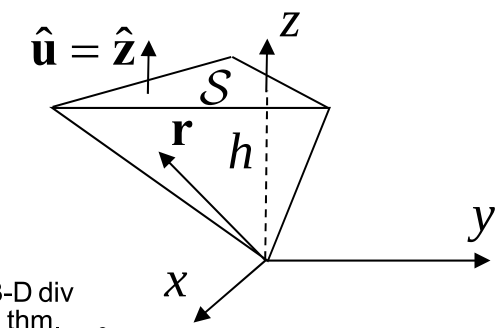
Dimension-Independent Divergence Definition and Theorem



Definitions:

- $\mathbf{J}, \mathbf{D}, \mathbf{B}$ are "flux" ("flow") vectors
- Domain $\mathcal{D} = \mathcal{P}, \mathcal{C}, \mathcal{S}, \mathcal{V}$ (point, curve, surface, volume)
- "Boundary of $\mathcal{D}$" $= \partial\mathcal{D} \equiv \mathcal{B} = \begin{cases} \mathcal{P} \text{ if } \mathcal{D} = \mathcal{C} \text{ (open)}, \\ \mathcal{C}\text{(closed) if } \mathcal{D} = \mathcal{S}\text{(open)}, \\ \mathcal{S}\text{(closed) if } \mathcal{D} = \mathcal{V} \end{cases}$
- "Measure of $\mathcal{D}$" $\equiv \operatorname{meas}\mathcal{D} = \begin{cases} \text{length of } \mathcal{C} \\ \text{area of } \mathcal{S} \\ \text{volume of } \mathcal{V} \end{cases}$
- $\hat{\mathbf{u}}$ is normal to $\mathcal{B}$ and "tangent" to $\mathcal{D}$
- Flux of a vector $\mathbf{F} \equiv \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{u}}\, d\mathcal{B}$
- "Divergence of $\mathbf{F}$" $\equiv \nabla \cdot \mathbf{F} \equiv \lim_{\operatorname{meas}\mathcal{D} \to 0} \dfrac{1}{\operatorname{meas}\mathcal{D}} \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{u}}\, d\mathcal{B}$
- Divergence Thm: $\int_{\mathcal{D}} \nabla \cdot \mathbf{F}\, d\mathcal{D} = \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{u}}\, d\mathcal{B}$

Examples of Divergence Theorem Application



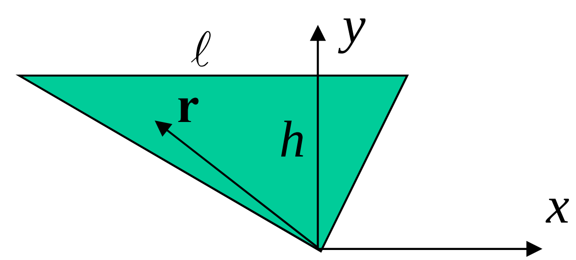
- In 3-D, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

$$\nabla \cdot \mathbf{r} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \right) = 1 + 1 + 1 = 3$$

Applying to tetrahedron shown $\Rightarrow \int_{\mathcal{V}} \nabla \cdot \mathbf{r}\, d\mathcal{V} = 3 \int_{\mathcal{V}} d\mathcal{V} = 3\mathcal{V} \overset{\text{3-D div thm.}}{=} \oint_{\mathcal{S}} \mathbf{r} \cdot \hat{\mathbf{u}}\, d\mathcal{S}$

$$= \oint_{\mathcal{S}} \mathbf{r} \cdot \hat{\mathbf{z}}\, d\mathcal{S} = \oint_{\mathcal{S}} h\, d\mathcal{S} = h\mathcal{S}$$

$$\Rightarrow \boxed{\mathcal{V} = \frac{1}{3} h\mathcal{S}}$$



- In 2-D, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$

$$\nabla \cdot \mathbf{r} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} \right) \cdot \left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \right) = 1 + 1 = 2$$

Applying to triangle shown, $\Rightarrow \int_{\mathcal{S}} \nabla \cdot \mathbf{r}\, d\mathcal{S} = 2 \int_{\mathcal{S}} d\mathcal{S} = 2\mathcal{S} \overset{\text{2-D div thm.}}{=} \oint_{\mathcal{C}} \mathbf{r} \cdot \hat{\mathbf{u}}\, d\mathcal{C} = \oint_{\mathcal{C}} h\, d\mathcal{C} = h\ell$

$$\Rightarrow \boxed{\mathcal{S} = \frac{1}{2} h\ell}$$

Use *Superposition* and *Duality* to Include Magnetic Current Sources

Maxwell's Equations: $\mathbf{E} = \mathbf{E} + \mathbf{E}, \ \mathbf{H} = \mathbf{H} + \mathbf{H}$

- $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{M}$ (Faraday's Law)
- $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J}$ (Ampere's Law)
- $\nabla \cdot \mathbf{D} = q = \dfrac{\nabla \cdot \mathbf{J}}{-j\omega}$ (Electric Form of Gauss's Law)
- $\nabla \cdot \mathbf{B} = m = \dfrac{\nabla \cdot \mathbf{M}}{-j\omega}$ (Magnetic Form of Gauss's Law)

$$\boxed{\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi - \frac{1}{\varepsilon}\nabla \times \mathbf{F}}$$

$$\boxed{\mathbf{H} = -j\omega\mathbf{F} - \nabla\Psi + \frac{1}{\mu}\nabla \times \mathbf{A}}$$

where

Magnetic vector potential

$$\mathbf{A}(\mathbf{r}) = \mu\int_{\mathcal{D}} G(\mathbf{r},\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d\mathcal{D}'$$

Electric vector potential

$$\mathbf{F}(\mathbf{r}) = \varepsilon\int_{\mathcal{D}} G(\mathbf{r},\mathbf{r}')\,\mathbf{M}(\mathbf{r}')\,d\mathcal{D}'$$

Electric scalar potential

$$\Phi(\mathbf{r}) = -\frac{1}{j\omega\varepsilon}\int_{\mathcal{D}} G(\mathbf{r},\mathbf{r}')\,\nabla'\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{D}'$$

Magnetic scalar potential

$$\Psi(\mathbf{r}) = -\frac{1}{j\omega\mu}\int_{\mathcal{D}} G(\mathbf{r},\mathbf{r}')\,\nabla'\cdot\mathbf{M}(\mathbf{r}')\,d\mathcal{D}'$$

Duality:

Blue source terms only	Red source terms only
$\mathbf{E}$	$\rightarrow \mathbf{H}$
$\mathbf{H}$	$\rightarrow -\mathbf{E}$
$\mathbf{J}$	$\rightarrow \mathbf{M}$
$\mathbf{A}$	$\rightarrow \mathbf{F}$
q	$\rightarrow m$
Φ	$\rightarrow \Psi$
ε	$\rightarrow \mu$
μ	$\rightarrow \varepsilon$

Electrostatic and Magnetostatic Problems May Be Treated as Special Case with ω=0

Electrostatics

- $\nabla \times \mathbf{E} = \mathbf{0}$
- $\nabla \cdot \mathbf{D} = q$

$\Rightarrow \mathbf{E} = -\nabla\Phi$

$$\nabla^2\Phi = -\frac{q}{\varepsilon}$$

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon}\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')\, q(\mathbf{r}')\, d\mathcal{V}'$$

Magnetostatics

- $\nabla \times \mathbf{H} = \mathbf{J} \quad (\nabla \cdot \mathbf{J} = 0)$
- $\nabla \cdot \mathbf{B} = 0$

$\Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$

$$\nabla^2\mathbf{A} = -\mu\mathbf{J}, \quad (\nabla \cdot \mathbf{A} = 0)$$

$$\mathbf{A}(\mathbf{r}) = \mu\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')\, \mathbf{J}(\mathbf{r}')\, d\mathcal{V}'$$

where $G(\mathbf{r},\mathbf{r}') = \dfrac{1}{4\pi R}$, $R = |\mathbf{r} - \mathbf{r}'|$

$G(\mathbf{r},\mathbf{r}')$ is the *static Green's function*

Electrostatics in Conducting Media

Electrostatics in Conducting Media

- $\nabla \times \mathbf{E} = \mathbf{0} \quad \Rightarrow \mathbf{E} = -\nabla\Phi$
- $\mathbf{J} = \sigma \mathbf{E}$
- $\nabla \cdot \mathbf{J} = 0 = \nabla \cdot (\sigma \mathbf{E}) = -\nabla \cdot (\sigma \nabla \Phi)$

$$\Rightarrow \boxed{\nabla \cdot (\sigma \nabla \Phi) = 0}$$

The End