Finite Element Solution of Helmholtz Equation for Inhomogeneously Filled Cylindrical Waveguide ---TM$_z$ Solution

D. R. Wilton

University of Houston

Important Cylindrical Waveguide Properties

Homogeneously filled guides:

- Frequency independent transverse modal fields with exp($-jk_z z$) dependence
- Independent TE, TM modes

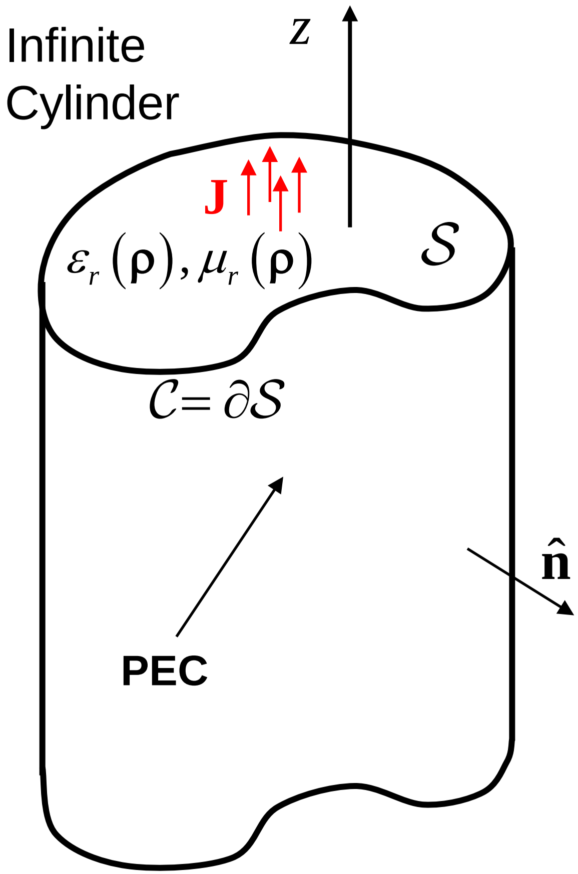
Inhomogeneously filled guides:

- Frequency dependent transverse modal fields
- Coupled TE, TM (hybrid) modes
- But modes *decouple* at cutoff frequency k_z=0
  => no *z*-dependence

Here we consider

- Inhomogeneous (or piecewise homogeneously filled) guides
- At cutoff frequency (no z-dependence)
- TM modes

Helmholtz Equation for Inhomogeneously Filled Cylindrical Waveguide



$$\mathbf{E}_{\tan} = -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}) = \mathbf{0}$$

Obtain the Helmholtz wave equation by eliminating the magnetic field between Maxwell's curl equations:

$$\nabla \times \mathbf{E} = -j\omega \mu_0 \mu_r(\boldsymbol{\rho}) \mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega \varepsilon_0 \varepsilon_r(\boldsymbol{\rho}) \mathbf{E} + \mathbf{J}$$

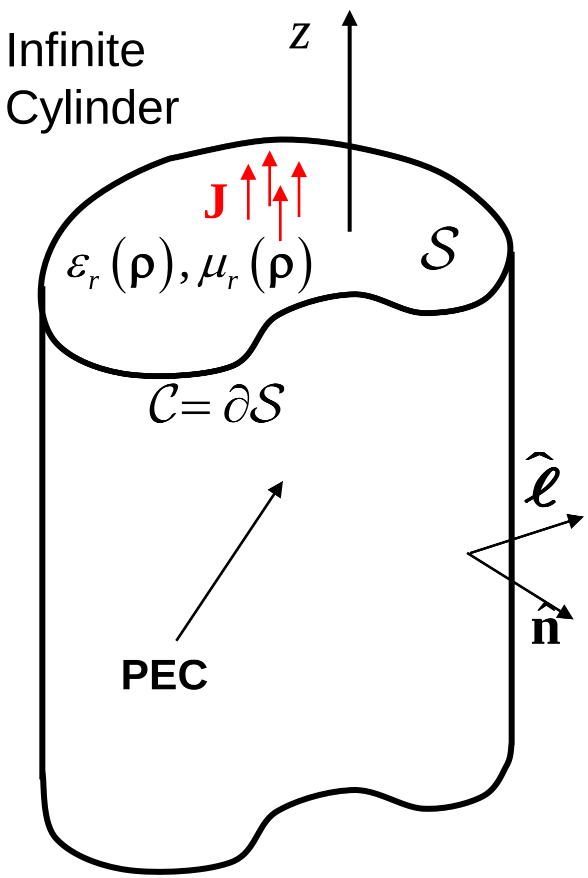
$$\boldsymbol{\rho} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

$$\Rightarrow \nabla \times \left(\mu_r^{-1} \nabla \times \mathbf{E}\right) - k_0^2 \varepsilon_r \mathbf{E} = -j\omega \mu_0 \mathbf{J}$$

or

$$-\frac{1}{j\omega \mu_0} \nabla \times \left(\mu_r^{-1} \nabla \times \mathbf{E}\right) - j\omega \varepsilon_0 \varepsilon_r \mathbf{E} = \mathbf{J}$$

Specialize to TM, *z*-Independent Case



Assume only *z* - components of $\mathbf{E}, \mathbf{J}$ with no *z* - dependence $(\partial/\partial z = 0)$:

$$\mathbf{E} = E_z(\boldsymbol{\rho})\hat{\mathbf{z}},\ \mathbf{J} = J_z(\boldsymbol{\rho})\hat{\mathbf{z}} \ \Rightarrow\ \nabla\times\mathbf{E} = \nabla E_z \times \hat{\mathbf{z}}$$

$$\Rightarrow \nabla\times\left(\mu_r^{-1}\nabla\times\mathbf{E}\right) = \nabla\times\left(\mu_r^{-1}\nabla E_z\times\hat{\mathbf{z}}\right)$$

$$= \mu_r^{-1}\nabla E_z\left(\cancel{\nabla\cdot\hat{\mathbf{z}}}\right) - \hat{\mathbf{z}}\ \nabla\cdot\left(\mu_r^{-1}\nabla E_z\right)$$

$$+\ \cancel{(\hat{\mathbf{z}}\cdot\nabla)\left(\mu_r^{-1}\nabla E_z\right)} - \cancel{\left(\mu_r^{-1}\nabla E_z\cdot\nabla\right)}\hat{\mathbf{z}}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,\nabla\cdot\mathbf{B} - \mathbf{B}\,\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\Rightarrow \nabla\cdot\left(\mu_r^{-1}\nabla E_z\right) + k_0^2\varepsilon_r E_z = j\omega\mu_0 J_z$$

or $$\boxed{\frac{1}{j\omega\mu_0}\nabla\cdot\left(\mu_r^{-1}\nabla E_z\right) - j\omega\varepsilon_0\varepsilon_r E_z = J_z}$$

Helmholtz Equation for E_z

Strong **form of TM Helmholtz equation:**

$$\frac{1}{j\omega\mu_0}\nabla\cdot\left(\mu_r^{-1}\nabla E_z\right)-j\omega\varepsilon_0\varepsilon_r E_z=J_z,\quad \boldsymbol{\rho}\in\mathcal{S}$$

Note Poisson's equation is a special case:

$$\nabla\cdot\left(\varepsilon_r^{-1}\nabla\Phi\right)=-\frac{q}{\varepsilon_0},\quad \boldsymbol{\rho}\in\mathcal{S}$$

- **Test above with $\Lambda_m(\boldsymbol{\rho})$ to obtain**

$$\frac{1}{j\omega\mu_0}<\Lambda_m,\nabla\cdot\left(\mu_r^{-1}\nabla E_z\right)> -j\omega\varepsilon_0<\Lambda_m,\varepsilon_r E_z> \;=<\Lambda_m,J_z>,\quad \boldsymbol{\rho}\in\mathcal{S}$$

where $<A,B> \;\equiv\int_{\mathcal{S}} A\,B\,d\mathcal{S}.$

- **Reduce differentiability requirement on E_z using**

$\nabla\cdot\left(\psi\mathbf{A}\right)=\mathbf{A}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{A}$ **and divergence theorem,**

yielding the ***weak*** **form of the Helmholtz equation:**

$$\boxed{\frac{1}{j\omega\mu_0}<\nabla\Lambda_m;\mu_r^{-1}\nabla E_z>+\;j\omega\varepsilon_0<\Lambda_m,\varepsilon_r E_z> \;=-<\Lambda_m,J_z>+\oint_{\mathcal{C}}\Lambda_m\mathbf{H}\cdot\hat{\ell}\,d\mathcal{C},\quad \boldsymbol{\rho}\in\mathcal{S}}$$

$\oint_{\mathcal{C}}\Lambda_m\mathbf{H}\cdot\hat{\ell}\,d\mathcal{C}=0$ if

on $\mathcal{C}=\partial\mathcal{S}$ either

1) $\mathbf{H}\cdot\hat{\ell}=0$ (*natural BC*)

2) $\Lambda_m=0$ (*essential BC*)

System Matrix

The boundary integral vanishes if $\Lambda_m(\rho)$ are also interpolatory basis functions for E_z,

$$\boxed{E_z = \sum_{n=1}^{N} V_n \Lambda_n(\rho),}$$

since $E_z = \sum_{n=1}^{N} V_n \Lambda_n(\rho) = 0$ on the boundary $\Rightarrow \Lambda_n(\rho) = 0$ on $\mathcal{C}$.

Substituting E_z into the weak form yields $\boxed{[Y_{mn}][V_n] = [I_m]}$ where

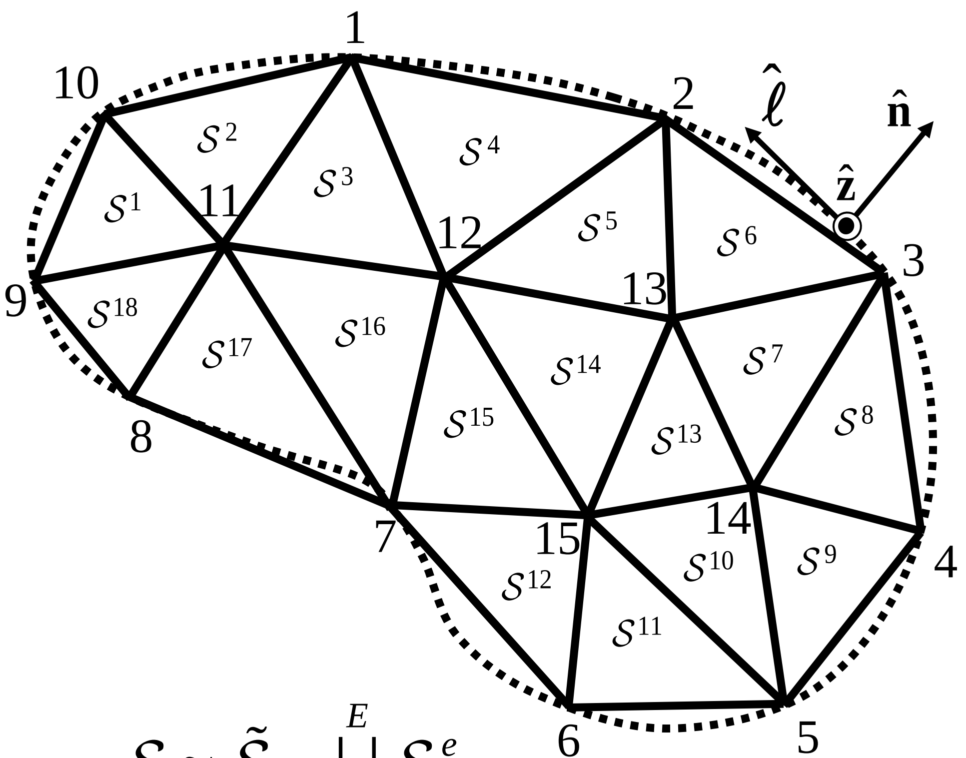
$$[Y_{mn}] = \frac{1}{j\omega}[\Gamma_{mn}] + j\omega[C_{mn}],$$ **(admittance or system matrix)**

$$[\Gamma_{mn}] = \frac{1}{\mu_0}\left[< \nabla\Lambda_m ; \mu_r^{-1} \nabla\Lambda_n >\right],$$ **(reciprocal inductance matrix)**

$$[C_{mn}] = \varepsilon_0\left[< \Lambda_m , \varepsilon_r \Lambda_n >\right],$$ **(capacitance matrix)**

$$[I_m] = \left[- < \Lambda_m , J_z >\right]$$ **(excitation vector)**

Discretize the Guide Cross Section --- Nodal Data

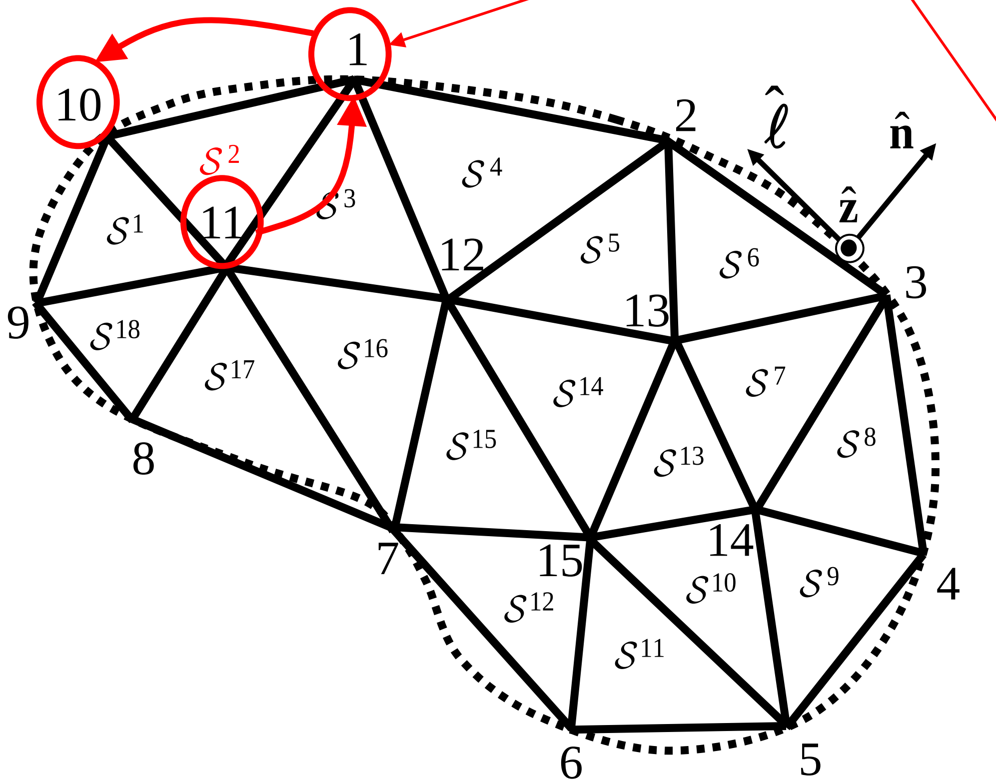


$$\mathcal{S} \approx \tilde{\mathcal{S}} = \bigcup_{e=1}^{E} \mathcal{S}^e$$

Global Node Index v	Coordinates	
	x_v	y_v
1	-0.500	1.100
2	1.100	0.700
⋮	⋮	⋮
12	0.000	0.000
⋮	⋮	⋮
15	0.700	-1.100

Element Connection Data

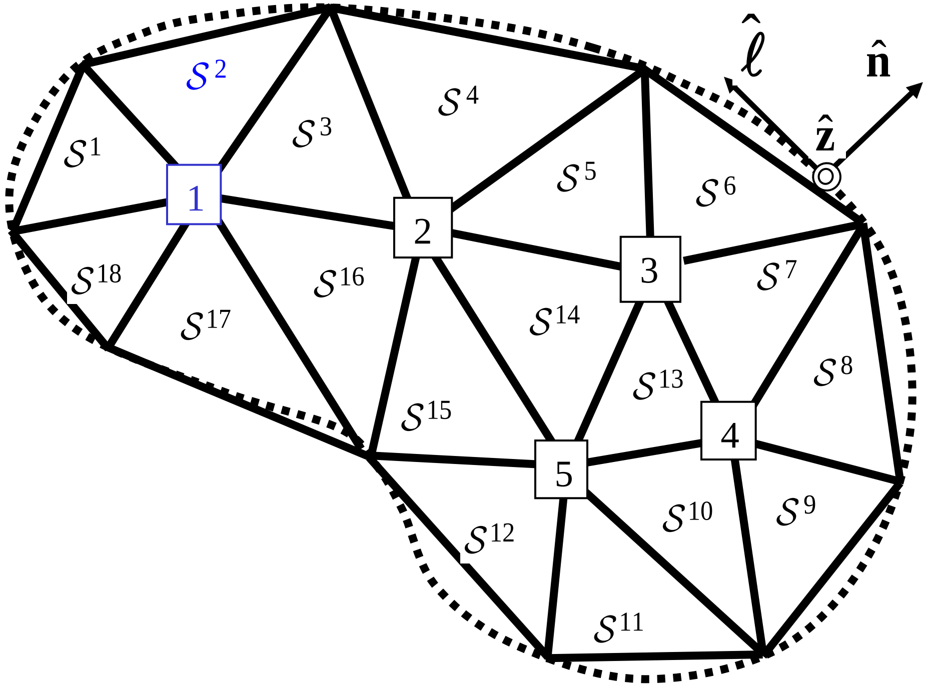
Counterclockwise listing



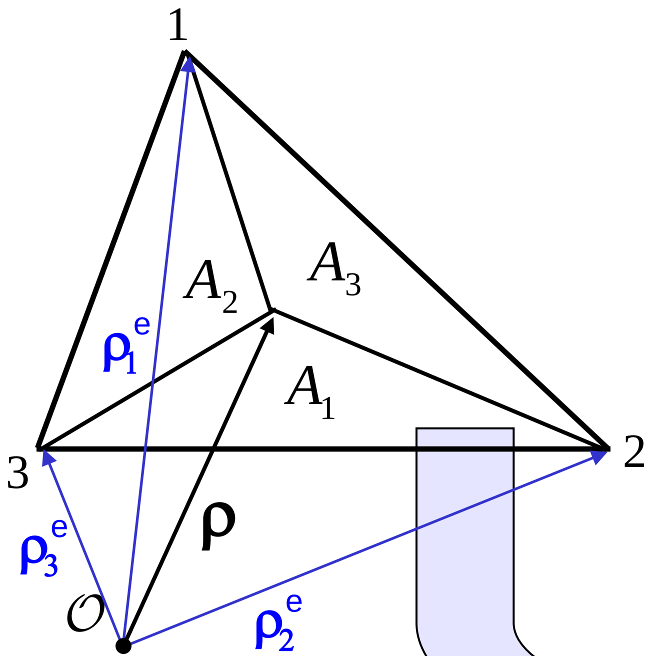
Local Node	1	2	3
e	Global Node No.	Global Node No.	Global Node No.
1	9	11	10
2	11	1	10
⋮	⋮	⋮	⋮
14	15	13	12
⋮	⋮	⋮	⋮
18	8	11	9

Element DoF Data

Local DoF #	1	2	3
e	Global DoF #	Global DoF #	Global DoF #
1	0	1	0
2	1	0	0
⋮	⋮	⋮	⋮



Area Coordinates Used to Represent Bases, Parameterize Element Geometry

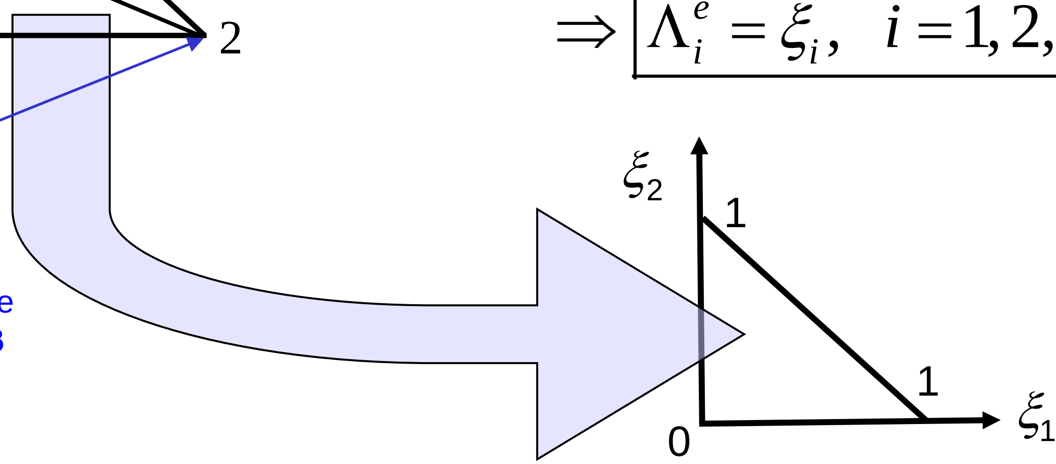


$$\boldsymbol{\rho} = \xi_1 \boldsymbol{\rho}_1^e + \xi_2 \boldsymbol{\rho}_2^e + \xi_3 \boldsymbol{\rho}_3^e$$

$$\xi_i = \frac{A_i}{A^e}, \quad i = 1, 2, 3$$

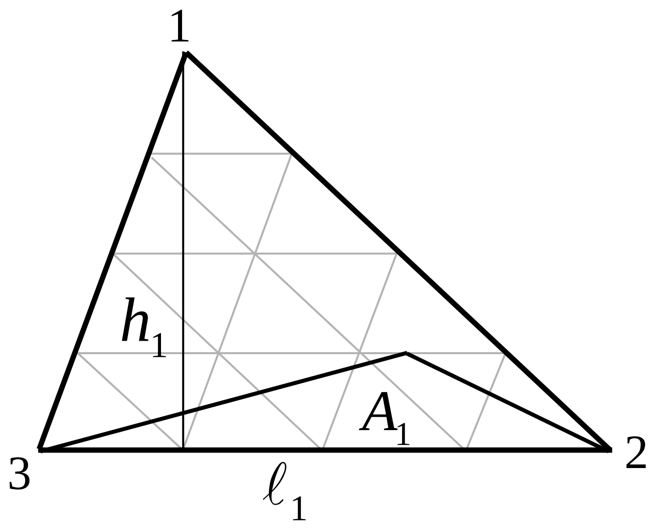
$$\Rightarrow \quad \xi_1 + \xi_2 + \xi_3 = 1$$

$$\Rightarrow \boxed{\Lambda_i^e = \xi_i, \quad i = 1, 2, 3}$$

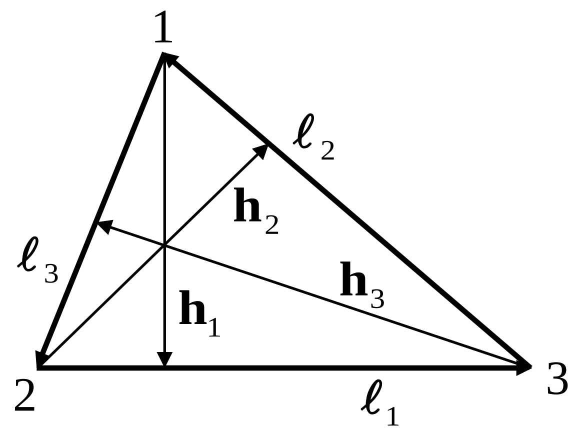


All elements mapped to “parent element”

An Area Coordinate Is Also the Fractional Distance from an Edge to the Opposite Vertex

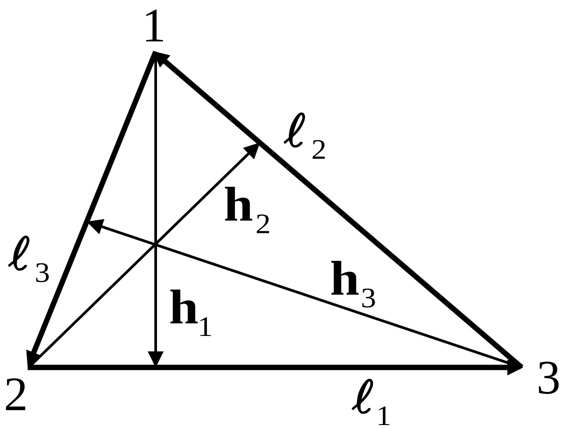


$$\xi_1 = \frac{\frac{1}{2}\ell_1 \times \text{height of } A_1}{\frac{1}{2}\ell_1 h_1} = \frac{\text{height of } A_1}{h_1}$$



It is convenient to define edge vectors associated with each edge and height vectors associated with each vertex.

Recall Local Geometry Definitions

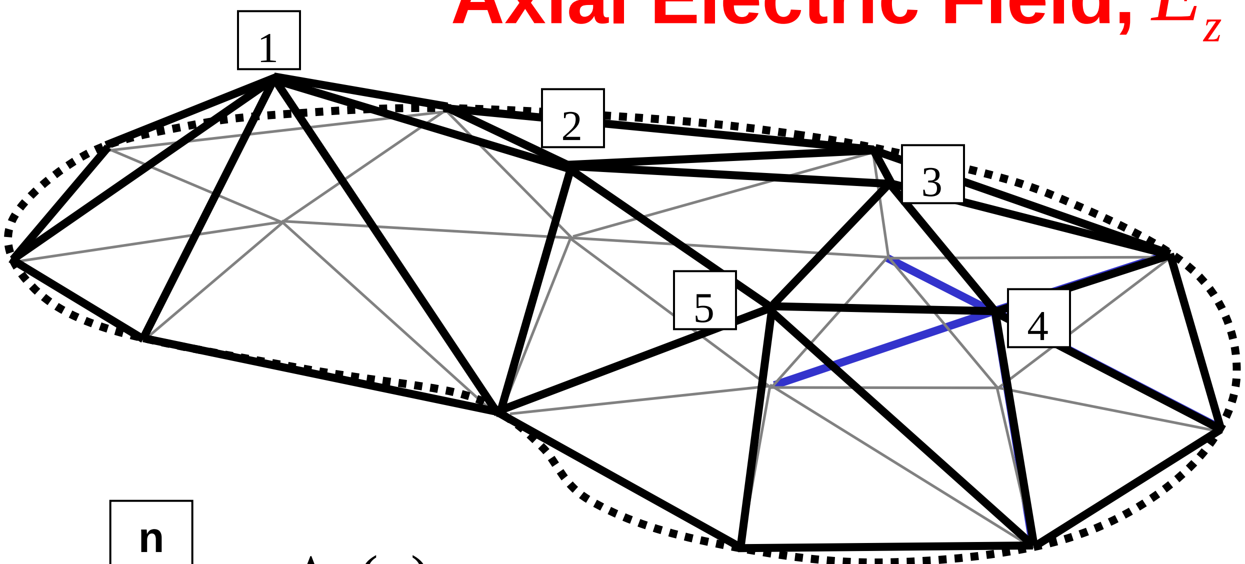


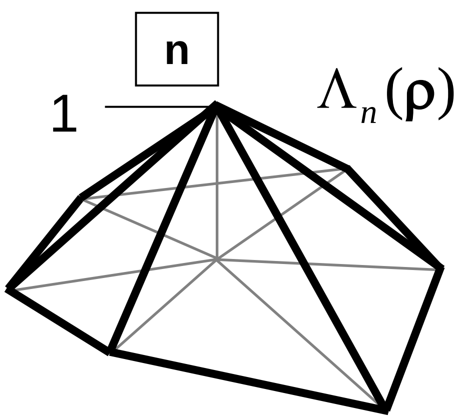
$$\hat{\mathbf{n}} = \frac{\boldsymbol{\ell}_{i-1} \times \boldsymbol{\ell}_{i+1}}{2A^e} \quad (= \hat{\mathbf{z}})$$

Table 8 Geometrical quantities defined on triangular elements.

Edge vectors	$\boldsymbol{\ell}_i = \boldsymbol{\rho}^e_{i-1} - \boldsymbol{\rho}^e_{i+1}$; $\ell_i = \lvert\boldsymbol{\ell}_i\rvert$; $\hat{\boldsymbol{\ell}}_i = \frac{\boldsymbol{\ell}_i}{\ell_i}$, $i = 1, 2, 3$
Area	$A^e = \frac{\lvert\boldsymbol{\ell}_{i-1} \times \boldsymbol{\ell}_{i+1}\rvert}{2}$, $i = 1, 2,$ or 3
Height vectors	$h_i = \frac{2A^e}{\ell_i}$; $\hat{\boldsymbol{h}}_i = -\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\ell}}_i$; $\boldsymbol{h}_i = h_i \hat{\boldsymbol{h}}_i$, $i = 1, 2, 3$
Coordinate gradients	$\boldsymbol{\nabla}\xi_i = -\frac{\hat{\boldsymbol{h}}_i}{h_i}$, $i = 1, 2, 3$

Piecewise Linear Model of Axial Electric Field, E_z





Global basis function associated with DoF n

$$E_z(\boldsymbol{\rho}) \approx \sum_{n=1}^{5} V_n \Lambda_n(\boldsymbol{\rho}), \quad \boldsymbol{\rho} \in \tilde{\mathcal{S}}$$

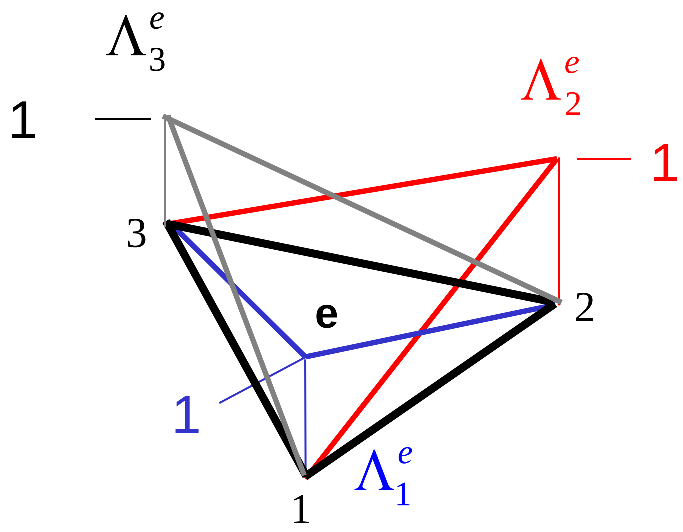
Global Scalar Representation

$$\hat{\mathbf{z}}\, E_z(\boldsymbol{\rho}) \approx \sum_{n=1}^{5} V_n \boldsymbol{\Omega}_n(\boldsymbol{\rho}), \quad \boldsymbol{\Omega}_n \equiv \hat{\mathbf{z}}\, \Lambda_n(\boldsymbol{\rho})$$

Global Vector Representation

Local Representations of E_z

$$\boxed{\Lambda_i^e = \xi_i, \;\; i = 1,2,3}$$



$$E_z(\boldsymbol{\rho}) \approx \sum_{i=1}^{3} V_i^e \Lambda_i^e(\boldsymbol{\rho}), \;\; \boldsymbol{\rho} \in \mathcal{S}^e$$

Local Scalar Representation

$$\hat{\mathbf{z}}\, E_z(\boldsymbol{\rho}) \approx \sum_{i=1}^{3} V_i^e \underbrace{\left[\hat{\mathbf{z}}\, \Lambda_i^e(\boldsymbol{\rho})\right]}_{\text{vector basis } \boldsymbol{\Omega}_i^e(\boldsymbol{\rho})} ,$$

and $\nabla \times \underbrace{\left[\hat{\mathbf{z}}\, \Lambda_i^e(\boldsymbol{\rho})\right]}_{\boldsymbol{\Omega}_i^e(\boldsymbol{\rho})} = \nabla \xi_i \times \hat{\mathbf{z}}, \; \boldsymbol{\rho} \in \mathcal{S}^e,$

Local Vector Representation

Local bases and triangle parameterization can be easily expressed in area coordinates

Summary of Vectorized Bases and Field Representation

Global representation, $\mathbf{E} \approx \sum_{n=1}^{N} V_n \, \boldsymbol{\Omega}_n(\boldsymbol{\rho}), \ \boldsymbol{\rho} \in \mathcal{S}$ **:**

where $\boldsymbol{\Omega}_n(\boldsymbol{\rho}) \equiv \hat{\mathbf{z}} \Lambda_n(\boldsymbol{\rho}).$

Though indexed by vertices, these vector-valued bases should be viewed as edge-based, curl-conforming bases.

Local representation, $\mathbf{E} \approx \sum_{i=1}^{3} V_i^e \boldsymbol{\Omega}_i^e(\boldsymbol{\rho}), \ \boldsymbol{\rho} \in \mathcal{S}^e$ **:**

$$\left.\begin{array}{l} \boldsymbol{\Omega}_1^e(\boldsymbol{\rho}) = \hat{\mathbf{z}}\xi_1 \\ \boldsymbol{\Omega}_2^e(\boldsymbol{\rho}) = \hat{\mathbf{z}}\xi_2 \\ \boldsymbol{\Omega}_3^e(\boldsymbol{\rho}) = \hat{\mathbf{z}}\xi_3 \end{array}\right\} \text{vertex-based DoFs: } \quad \boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \hat{\mathbf{z}}\xi_i$$

$$\nabla \times \boldsymbol{\Omega}_i^e = \nabla \xi_i \times \hat{\mathbf{z}}, \qquad i = 1, 2, 3$$

Element Matrix and Excitation Vector in *Vector* Form

Local admittance matrices and current column vectors corresponding to $[Y_{mn}][V_n] = \frac{1}{j\omega}\left[\Gamma_{mn}\right][V_n] + j\omega\left[C_{mn}\right][V_n] = [I_m]$**:**

$$\left[Y_{ij}^e\right] = \frac{1}{j\omega}\left[\Gamma_{ij}^e\right] + j\omega\left[C_{ij}^e\right],$$ **(admittance element matrix)**

$$\left[\Gamma_{ij}^e\right] = \frac{1}{\mu_0}\left[< \nabla\times\mathbf{\Omega}_i^e ; \mu_r^{-1}\nabla\times\mathbf{\Omega}_j^e >\right],$$ **(reciprocal inductance element matrix)**

$$\left[C_{ij}^e\right] = \varepsilon_0\left[< \mathbf{\Omega}_i^e ; \varepsilon_r\mathbf{\Omega}_j^e >\right],$$ **(capacitance element matrix)**

$$\left[I_i^e\right] = \left[- < \mathbf{\Omega}_i^e ; \tilde{\mathbf{J}} >\right]$$ **(excitation current element vector)**

Add $\sigma_i^e \sigma_j^e Y_{ij}^e$ **to system matrix using matrix assembly rule ! (TM polarization** $\Rightarrow$ $\sigma_i^e = \sigma_j^f = 1$ **)**

Integration over Triangles Using Area Coordinates

$$\int_{A^e} f(\boldsymbol{\rho})\, d\mathcal{S}$$

$$= 2A^e \int_0^1 \int_0^{1-\xi_2} f(\xi_1 \boldsymbol{\rho}_1^e + \xi_2 \boldsymbol{\rho}_2^e + \xi_3 \boldsymbol{\rho}_3^e)\, d\xi_1\, d\xi_2$$

$$\underbrace{\approx 2A^e \sum_{k=1}^{K} w_k\, f(\xi_1^{(k)} \boldsymbol{\rho}_1^e + \xi_2^{(k)} \boldsymbol{\rho}_2^e + \xi_3^{(k)} \boldsymbol{\rho}_3^e)}_{\text{Numerical integration}}$$

Or evaluate analytically using

$$\int_0^1 \int_0^{1-\xi_2} \xi_1^{\alpha} \xi_2^{\beta} \xi_3^{\gamma}\, d\xi_1\, d\xi_2$$

$$= \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!}$$

See scattering notes for efficient, closed form element matrix evaluation

Table 9 Sample points and weighting coefficients for K-point quadrature on triangles.

Sample Points, $\left(\xi_1^{(k)}, \xi_2^{(k)}\right)$ $(\xi_3^{(k)} = 1 - \xi_1^{(k)} - \xi_2^{(k)})$	**Weights,** w_k
K=1, error $\mathcal{O}(\xi_i^2)$**:**	
(0.33333333333333, 0.33333333333333)	0.50000000000000
K=3, error $\mathcal{O}(\xi_i^3)$**:**	
(0.66666666666667, 0.16666666666667)	0.16666666666667
(0.16666666666667, 0.66666666666667)	0.16666666666667
(0.16666666666667, 0.16666666666667)	0.16666666666667
K=7, error $\mathcal{O}(\xi_i^6)$**:**	
(0.33333333333333, 0.33333333333333)	0.11250000000000
(0.79742698535309, 0.10128650732346)	0.06296959027241
(0.10128650732346, 0.79742698535309)	0.06296959027241
(0.10128650732346, 0.10128650732346)	0.06296959027241
(0.47014206410512, 0.47014206410512)	0.06619707639425
(0.47014206410512, 0.05971587178977)	0.06619707639425
(0.05971587178977, 0.47014206410512)	0.06619707639425

Analytical Element Matrix Evaluation

Capacitance element matrix :

$$\left[C_{ij}^{e}\right] = \varepsilon_0\left[<\boldsymbol{\Omega}_i^e;\varepsilon_r\boldsymbol{\Omega}_j^e>\right] = \varepsilon_0\varepsilon_r\left[\int_{\mathcal{S}^e}(\hat{\mathbf{z}}\xi_i)\cdot(\hat{\mathbf{z}}\xi_j)\,d\mathcal{S}\right]$$

$$= \varepsilon_0\varepsilon_r\left[2A^e\int_0^1\int_0^{1-\xi_j}\xi_i\,\xi_j\,d\xi_i\,d\xi_j\right]$$

$$= \frac{\varepsilon_0\varepsilon_r A^e}{12}\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$$

Reciprocal inductance element matrix :

$$\left[\Gamma_{ij}^{e}\right] = \frac{1}{\mu_0}\left[<\nabla\times\boldsymbol{\Omega}_i^e;\mu_r^{-1}\nabla\times\boldsymbol{\Omega}_j^e>\right]$$

$$= \frac{1}{\mu_0\mu_r}\left[2A^e\int_0^1\int_0^{1-\xi_j}(\nabla\xi_i\times\hat{\mathbf{z}})\cdot(\nabla\xi_j\times\hat{\mathbf{z}})\,d\xi_i\,d\xi_j\right] = \frac{\left[\boldsymbol{\ell}_i\cdot\boldsymbol{\ell}_j\right]}{4A^e\mu_0\mu_r}$$

$$\nabla\xi_i = -\frac{\hat{\mathbf{h}}_i}{h_i} = \frac{\hat{\mathbf{z}}\times\hat{\boldsymbol{\ell}}_i}{h_i}\times\frac{\ell_i}{\ell_i} = \frac{\hat{\mathbf{z}}\times\boldsymbol{\ell}_i}{2A^e}$$

Source-Free Problems—Waveguide Cutoff Frequencies and Dispersion Data

- $J_z = 0 \Rightarrow \left[V_n\right] = 0$ ***except*** **for eigenfrequencies** $= \omega_p^2$ **:**

$$\left[\Gamma_{mn}\right]\left[V_n^p\right] = \omega_p^2\left[C_{mn}\right]\left[V_n^p\right], \quad p = 1, 2, \ldots$$

Generalized eigenvalue problem of the form
[*A*] [*x*^p] = λ_p[*B*][*x*^p]

where

$$\left[\Gamma_{mn}\right] = \frac{1}{\mu_0}\left[< \nabla\times\mathbf{\Omega}_m ; \mu_r^{-1}\nabla\times\mathbf{\Omega}_n >\right]$$

$$\left[C_{mn}\right] = \varepsilon_0\left[< \mathbf{\Omega}_m ; \varepsilon_r\mathbf{\Omega}_n >\right]$$

Note: 1×1 matrix case reduces to $\omega = \sqrt{\frac{\Gamma}{C}} = \frac{1}{\sqrt{LC}}$!

- **The values** $\omega_p = 2\pi f_p$ **are the guide cutoff frequencies**
- **The** p **th eigenvector is** $\left[V_n^p\right]$
- **The electric field distribution for mode** p ***at cutoff*** **is** $\mathbf{E}_p = \sum_n V_n^p \mathbf{\Omega}_n$

Integral Equation vs. PDE (FEM) Formulations (modified from V. Jandhyala)

Method	Surface or Volume	Is background modeled	Background needs to be truncated	PDE or IE
FEM/FDTD	Volume	Yes	Yes	PDE
MoM	Surface (typically)	No	No	IE

Method	Best Suited For	Best Suited For
FEM/FDTD	Inhomogeneous media	Volume dominated problem
MoM	Homogeneous or piecewise homogeneous media	Surface dominated problem

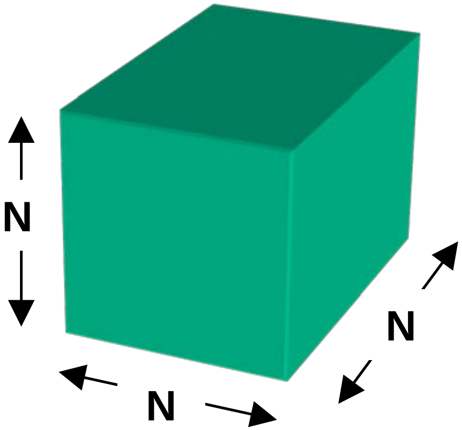
Integral Equation vs. PDE (FEM) Formulations, cont'd

Method	Frequency domain	Time domain
Finite Difference	FDFD (rarely used)	FDTD
FEM	FEM (used regularly, mature field)	TD-FEM (Research/emerging)
Method of Moments (IE- Integral Equation)	MoM (mature field)	TDIE (Emerging field)

Integral Equation vs. PDE (FEM) Formulations, cont'd

Method	Equivalent Matrix System Size	Equivalent Matrix System Density
Finite Difference	Large (volume)	Very sparse (can be done matrix-free i.e. with no explicit matrix)
FEM	Large (volume)	Very sparse
Method of Moments (IE-Integral Equation)	Small (surface)	Full

Integral Equation vs. PDE (FEM) Formulations, cont'd



Assumptions for box example:

- **Cube (all dims., discretization equal)**
- **FEM discretization, interior only**
- **IE on surface only**
- **Full matrix storage O(#unks^2)**
- **Sparse matrix storage O(#unks)**
- **Direct sol'n time O(#unks^3)**
- **Iterative sol'n time O(#unks^2)**
- **Fast method storage & sol'n time, O(#unks ln(#unkns))**

Direct solutions:

- Gauss elimination, etc.

Iterative solutions:

- Conjugate gradient
- Biconjugate gradient
- GMRES, QMR, etc.

Method	Matrix storage	Sol'n time, direct	Sol'n time, per iteration
FEM, (sparse!) $\left[\mathcal{O}\left(N^3\right)\text{unkns}\right]$	$\mathcal{O}\left(N^3\right)$	$\le \mathcal{O}\left(N^9\right)$	$\mathcal{O}\left(N^6\right) \xrightarrow{\text{Use sparsity}} \mathcal{O}\left(N^3\right)$
Int. Eq. (full!) $\left[\mathcal{O}\left(N^2\right)\text{unkns}\right]$	$\mathcal{O}\left(N^4\right) \xrightarrow{\text{Fast Methods}} \mathcal{O}\left(N^2 \ln N\right)$	$\mathcal{O}\left(N^6\right)$	$\mathcal{O}\left(N^4\right) \xrightarrow{\text{Fast Methods}} \mathcal{O}\left(N^2 \ln N\right)$

The End