# Layered Media Green's Function

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**With acknowledgements to**

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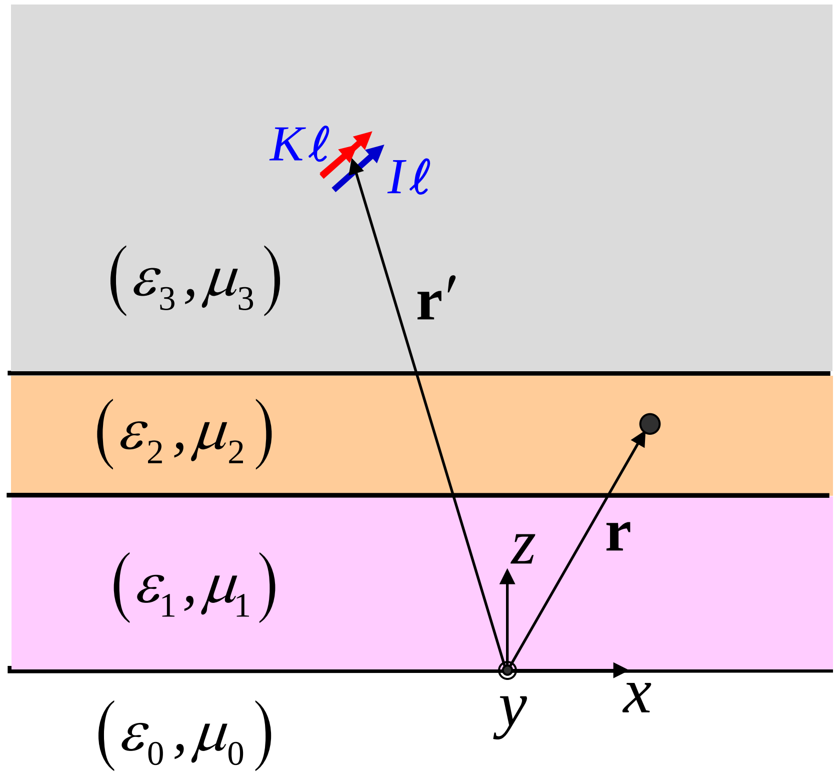
# Preview of Key Points




- The principal problem associated with layered media is the computation of an appropriate *Green's function*, i.e. the field at an observation point **r** due to a unit strength ($K\ell=1$, $I\ell=1$) point (dipole) source located at a source point **r'**.

# Key Points, cont'd

- In a layered medium, the simplest source is a *phased current sheet* that simply launches plane waves.

- A point current source can be expressed as a superposition (Fourier transform) over *phased current sheets* parallel to layer boundaries.

  – Phased current sheets naturally reflect the translational invariance of the Green's function in a layered medium's transverse dimensions, $x$ and $y$.

- Transverse components of current in each constituent sheet may be decomposed into components parallel and perpendicular to the transverse wavevector (phasing vector).

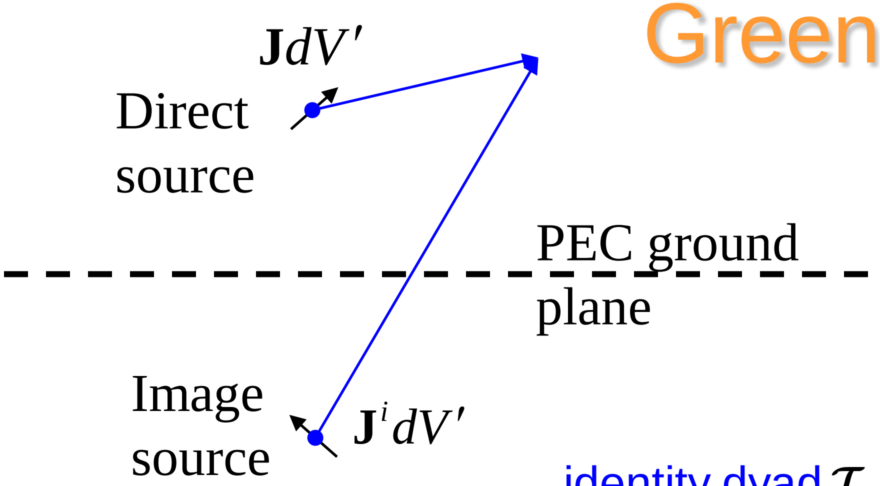
# Key Points, cont'd

- Fields produced by the sheet current components are $TM_z$ and $TE_z$ fields, respectively, and propagate independently through layered media; they are conveniently analyzed by transmission line theory.

- The relationship between vector sources and vector fields is most directly expressed using dyadic Green's functions. The mixed potential representation of Michalski, however, is more appropriate for computational work. The resulting Green's potentials (dyadic vector potential and scalar potentials) can themselves be expressed in terms of scalar transmission line voltage and current Green's functions.

- For efficiency, acceleration of the resulting Green's function representation is still required.

# Green's Dyads




- Direct source $\mathbf{J}dV' = \overbrace{\left(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}\right)}^{\text{identity dyad } \mathcal{I}} \cdot \mathbf{J}dV'$ at $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$
- Image source $\mathbf{J}^{\mathrm{i}}dV' = \overbrace{\left(-\hat{\mathbf{x}}\hat{\mathbf{x}} - \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}\right)}^{\text{reflection dyad } \mathcal{R}_z} \cdot \mathbf{J}dV'$ at $\mathbf{r}^{\mathrm{i}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - z\hat{\mathbf{z}} = -\mathcal{R}_z \cdot \mathbf{r}$
- Total vector potential: $\mathbf{A} = \int_V \mathcal{G}^A \cdot \mathbf{J}dV'$

$\mathcal{I}$ is the *identity dyad* such that
$\mathcal{I} \cdot \mathbf{J} = \mathbf{J}$

$$\mathcal{G}^A(\mathbf{r},\mathbf{r}') = \mu\left[\mathcal{I}\, G(\mathbf{r},\mathbf{r}') + \mathcal{R}_z G(\mathbf{r}, -\mathcal{R}_z \cdot \mathbf{r}')\right]$$

$$= \mu \begin{bmatrix} G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},-\mathcal{R}_z\cdot\mathbf{r}') & 0 & 0 \\ 0 & G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},-\mathcal{R}_z\cdot\mathbf{r}') & 0 \\ 0 & 0 & G(\mathbf{r},\mathbf{r}') + G(\mathbf{r},-\mathcal{R}_z\cdot\mathbf{r}') \end{bmatrix}$$

# The Green's Function Problem

To solve, for example,

(0) $\nabla \times \mu_r^{-1} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = -j\omega\mu_0 \mathbf{J} - \nabla \times \left( \mu_r^{-1} \mathbf{M} \right)$ + boundary conditions,

first find dyads $\mathcal{G}^{\mathrm{EJ}}(\mathbf{r},\mathbf{r}')$, $\mathcal{G}^{\mathrm{EM}}(\mathbf{r},\mathbf{r}')$ such that

(1) $\nabla \times \mu_r^{-1} \nabla \times \mathcal{G}^{\mathrm{EJ}} - k_0^2 \varepsilon_r \mathcal{G}^{\mathrm{EJ}} = -j\omega\mu_0 \mathcal{I} \delta(\mathbf{r} - \mathbf{r}')$ + boundary conditions

(2) $\nabla \times \mu_r^{-1} \nabla \times \mathcal{G}^{\mathrm{EM}} - k_0^2 \varepsilon_r \mathcal{G}^{\mathrm{EM}} = -\nabla \times \left( \mu_r^{-1} \mathcal{I} \delta(\mathbf{r} - \mathbf{r}') \right)$ + boundary conditions

Then dot (1) from the right with $\mathbf{J}(\mathbf{r}')$, (2) with $\mathbf{M}(\mathbf{r}')$, add results, and integrate over $V$ to obtain the RHS of (0) and identify the solution of (0) as

$$\mathbf{E}(\mathbf{r}) = \int_V \mathcal{G}^{\mathrm{EJ}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, dV' + \int_V \mathcal{G}^{\mathrm{EM}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')\, dV'$$

# The Green's Function Problem, cont'd

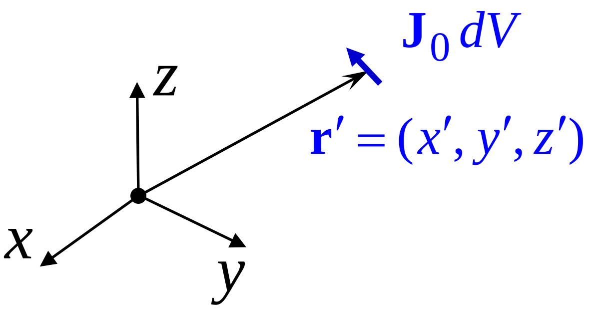
$\mathcal{G}^{\mathrm{EJ}}(\mathbf{r},\mathbf{r}'),\ \mathcal{G}^{\mathrm{EM}}(\mathbf{r},\mathbf{r}')$ can either be represented as a *matrix,*

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = \begin{bmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{yx} & G_{yy} & G_{yz} \\ G_{zx} & G_{zy} & G_{zz} \end{bmatrix}$$

or as a *dyad*,

$$\begin{aligned} \mathcal{G}(\mathbf{r},\mathbf{r}') \quad = \quad & G_{xx}\hat{\mathbf{x}}\hat{\mathbf{x}} + G_{xy}\hat{\mathbf{x}}\hat{\mathbf{y}} + G_{xz}\hat{\mathbf{x}}\hat{\mathbf{z}} \\ & + G_{yx}\hat{\mathbf{y}}\hat{\mathbf{x}} + G_{yy}\hat{\mathbf{y}}\hat{\mathbf{y}} + G_{yz}\hat{\mathbf{y}}\hat{\mathbf{z}} \\ & + G_{zx}\hat{\mathbf{z}}\hat{\mathbf{x}} + G_{zy}\hat{\mathbf{z}}\hat{\mathbf{y}} + G_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} \end{aligned}$$

# Review: Delta-Function Representation of Point Dipole(s)




- Volume current density for an elemental current dipole :

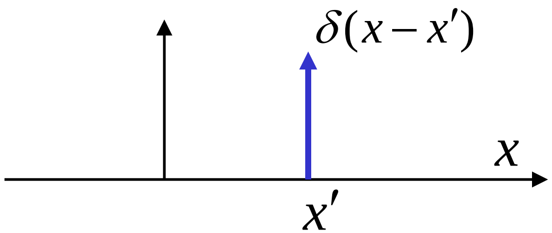
$$\mathbf{J}_0 dV \,\delta(x - x')\delta(y - y')\delta(z - z')$$

# Review: Transform Representation of a Delta-Function

Recall:

$$\delta(x-x') = \int_{-\infty}^{\infty} \tilde{\delta}(k_x)\, e^{-jk_x x} dk_x$$

$$\tilde{\delta}(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-x')\, e^{jk_x x} dx = \frac{e^{jk_x x'}}{2\pi}$$




$$\Rightarrow \quad \boxed{\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk_x (x-x')} dk_x}$$

Exponential phase factor

$$\Rightarrow \quad \boxed{\begin{aligned} \delta(x-x')\delta(y-y') &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-jk_x (x-x')} e^{-jk_y (y-y')} dk_x dk_x \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\mathbf{k}_t \cdot (\boldsymbol{\rho}-\boldsymbol{\rho}')} dk_x\, dk_y, \quad \boldsymbol{\rho} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \end{aligned}}$$

# Review: A Point Dipole(s) as a Superposition of Phased Current Sheets

- Elemental current dipole :

$$
\begin{aligned}
&\mathbf{J}_0 dV \delta(x-x')\delta(y-y')\delta(z-z') \\
&= \mathbf{J}_0 dV \delta(z-z') \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} dk_x\, dk_y,
\end{aligned}
$$

- Note the integral is just a superposition of *phased current sheets :*

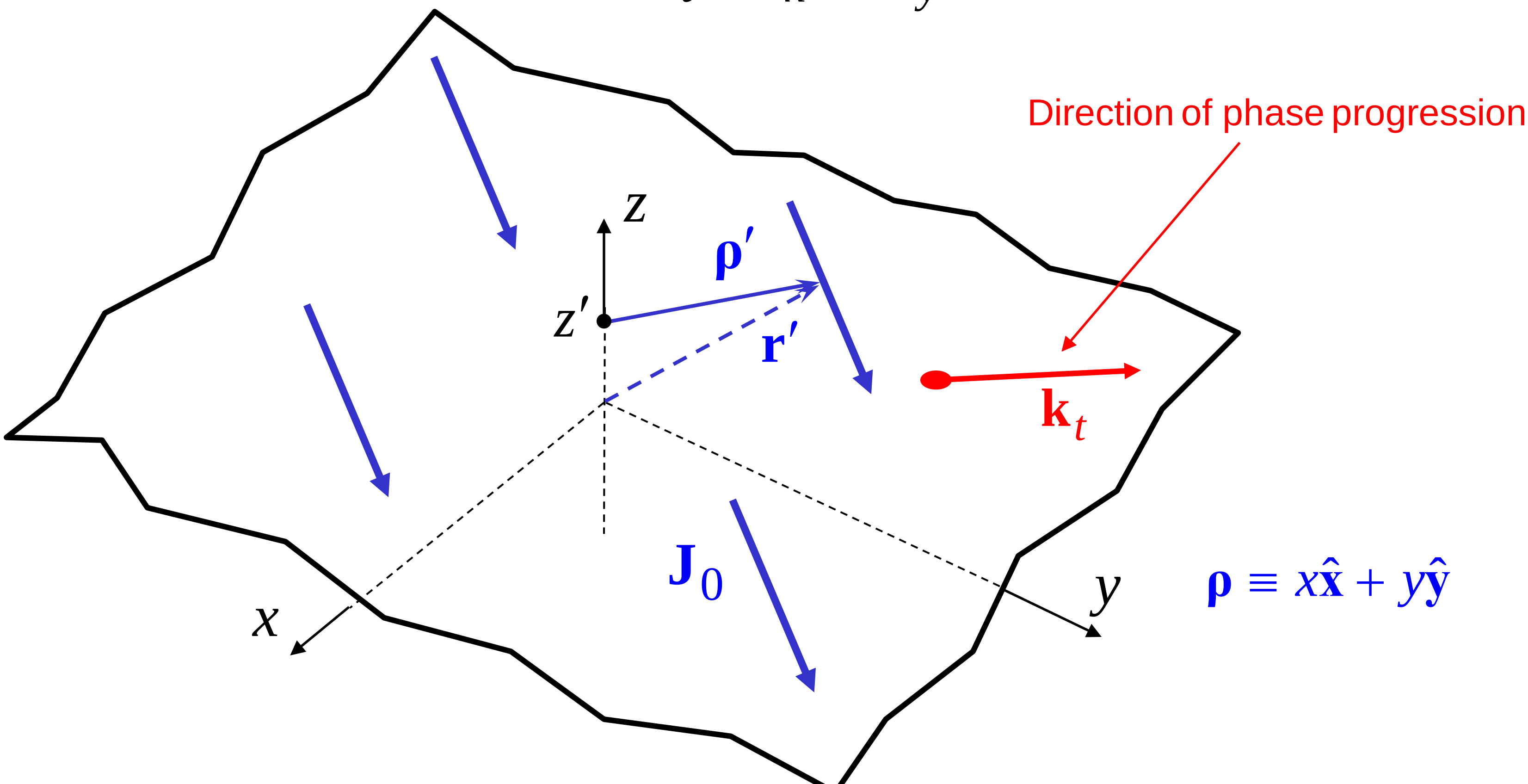
$$
\mathbf{J}_0 dV \delta(z-z') e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}, \qquad \mathbf{k}_t \equiv k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}
$$

# Phased Current Sheet

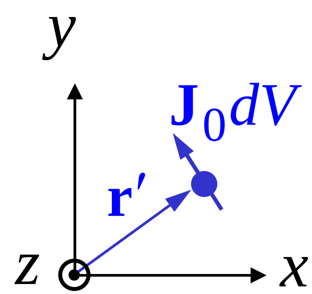
- Surface current : $\mathbf{J}_0\ \delta(z-z')e^{-jk_x(x-x')}e^{-jk_y(y-y')}$
  $= \mathbf{J}_0\,\delta(z-z')e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}$

Exponential phase factor

- Phase gradient : $\mathbf{k}_t = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$



# A Single Vector Point Current Source as a Superposition of Current Sheets




Exponential phase factor

$$\boxed{\mathbf{J}_0 dV\, \delta(z-z')\,\delta(y-y')\,\delta(x-x') = \underbrace{\mathbf{J}_0 dV\, \frac{\delta(z-z')}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} dk_x\, dk_y}_{\text{superposition of a collection of phased current sheets}}}\ ,$$

$$\mathbf{k}_t \equiv k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$$

# Fourier Transform Maxwell's Curl Equations

- Beginning with Maxwell's curl equations,

$$\nabla \times \mathbf{E} = -jk\eta \mathbf{H} - \mathbf{M}, \quad \nabla \times \mathbf{H} = j\frac{k}{\eta}\mathbf{E} + \mathbf{J}, \quad \left(k\eta = \omega\mu,\, k/\eta = \omega\varepsilon\right)$$

define the Fourier transform pair,

$$\tilde{\mathbf{F}}\left(\mathbf{k}_t\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{F}\left(\boldsymbol{\rho}\right) e^{j\mathbf{k}_t \cdot \boldsymbol{\rho}}\, dx\, dy, \quad \mathbf{F}\left(\boldsymbol{\rho}\right) = \frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\mathbf{F}}\left(\mathbf{k}_t\right) e^{-j\mathbf{k}_t \cdot \boldsymbol{\rho}}\, dk_x\, dk_y$$

$$\mathbf{k}_t = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}, \qquad \boldsymbol{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}},$$

and transform the two curl equations :

$$\Rightarrow \boxed{\tilde{\nabla} \times \tilde{\mathbf{E}} = -jk\eta \tilde{\mathbf{H}} - \tilde{\mathbf{M}}, \quad \tilde{\nabla} \times \tilde{\mathbf{H}} = j\frac{k}{\eta}\tilde{\mathbf{E}} + \tilde{\mathbf{J}}}$$

where $\tilde{\nabla} \equiv -j\mathbf{k}_t + \hat{\mathbf{z}}\dfrac{d}{dz}, \quad \tilde{\mathbf{E}} = \tilde{\mathbf{E}}(\mathbf{k}_t, z)$, etc.

# Decompose Fields, Sources into Transverse and Axial Components; Equate Curl Equation Components

Let $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_t + \hat{\mathbf{z}}\,\tilde{E}_z$, $\tilde{\mathbf{J}} = \tilde{\mathbf{J}}_t + \hat{\mathbf{z}}\,\tilde{J}_z$ etc.

$\Rightarrow$

(1) $\dfrac{-d\tilde{\mathbf{E}}_t}{dz} - j\mathbf{k}_t\tilde{E}_z = -jk\eta\left(\hat{\mathbf{z}}\times\tilde{\mathbf{H}}_t\right) - \hat{\mathbf{z}}\times\tilde{\mathbf{M}}_t,$ (2) $j\mathbf{k}_t\cdot\left(\hat{\mathbf{z}}\times\tilde{\mathbf{E}}_t\right) = -jk\eta\tilde{H}_z - \tilde{M}_z$

(3) $\dfrac{-d\tilde{\mathbf{H}}_t}{dz} - j\mathbf{k}_t\tilde{H}_z = j\dfrac{k}{\eta}\left(\hat{\mathbf{z}}\times\tilde{\mathbf{E}}_t\right) + \hat{\mathbf{z}}\times\tilde{\mathbf{J}}_t,$ (4) $j\mathbf{k}_t\cdot\left(\hat{\mathbf{z}}\times\tilde{\mathbf{H}}_t\right) = j\dfrac{k}{\eta}\tilde{E}_z + \tilde{J}_z$

Substitute $(4)\rightarrow(1),(2)\rightarrow(3)$ to obtain *vector* transmission line equations,

$$\frac{-d\tilde{\mathbf{E}}_t}{dz} = -j\eta\frac{k^2-\left(\mathbf{k}_t\mathbf{k}_t\,\cdot\right)}{k}\left(\hat{\mathbf{z}}\times\tilde{\mathbf{H}}_t\right) - \mathbf{k}_t\frac{\eta}{k}\tilde{J}_z - \hat{\mathbf{z}}\times\tilde{\mathbf{M}}_t$$

$$\frac{-d\tilde{\mathbf{H}}_t}{dz} = j\frac{k^2-\left(\mathbf{k}_t\mathbf{k}_t\cdot\right)}{k\eta}\left(\hat{\mathbf{z}}\times\tilde{\mathbf{E}}_t\right) - \frac{\mathbf{k}_t}{k\eta}\tilde{M}_z + \hat{\mathbf{z}}\times\tilde{\mathbf{J}}_t$$

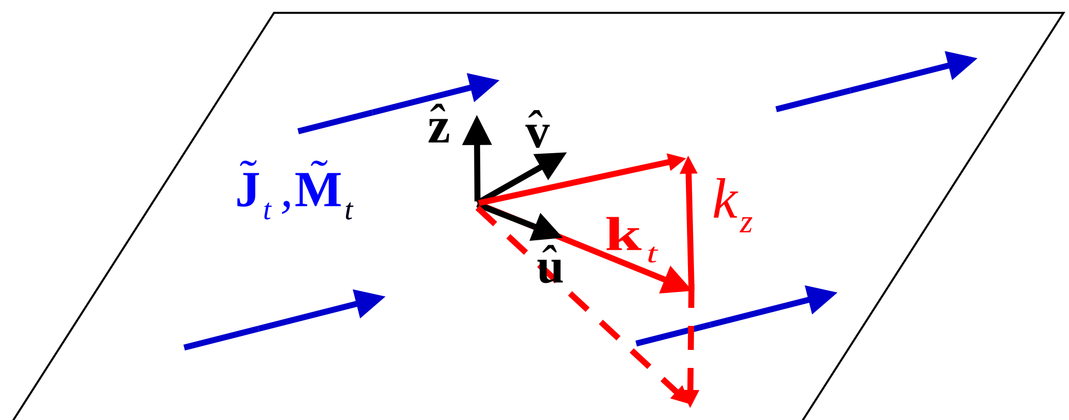
# Split Transverse Fields and Currents into Components Parallel and Perpendicular to the Transverse Wave Vector (Spectral Domain Immittance (SDI) Method)

Let $\hat{\mathbf{u}} = \dfrac{\mathbf{k}_t}{k_t}$, $\hat{\mathbf{v}} = \hat{\mathbf{z}} \times \hat{\mathbf{u}}$, where $k_t \equiv |\mathbf{k}_t|$, define $k_z = \sqrt{k^2 - k_t^2}$, and decompose sources and currents as $\tilde{\mathbf{E}}_t = \hat{\mathbf{u}}\,\tilde{E}_u + \hat{\mathbf{v}}\tilde{E}_v$, $\tilde{\mathbf{J}}_t = \hat{\mathbf{u}}\,\tilde{J}_u + \hat{\mathbf{v}}\,\tilde{J}_v$, etc. Substituting into vector transmission line equations and equating components yields

- $\dfrac{-d\tilde{E}_u}{dz} = j\eta \dfrac{k_z^2}{k} \tilde{H}_v - k_t \dfrac{\eta}{k} \tilde{J}_z + \tilde{M}_v$
- $\dfrac{-d\tilde{H}_v}{dz} = j \dfrac{k}{\eta} \tilde{E}_u + \tilde{J}_u$

and

- $\dfrac{-d\tilde{E}_v}{dz} = -j\eta k \tilde{H}_u - \tilde{M}_u$
- $\dfrac{-d\tilde{H}_u}{dz} = -j \dfrac{k_z^2}{k\eta} \tilde{E}_v - \dfrac{k_t}{k\eta} \tilde{M}_z - \tilde{J}_v$




$k_t < k$

# Decomposed Equations Reduce to Transmission-Line Equations

TM$_z$ Fields $\left(Z^e \equiv 1/Y^e = \eta k_z / k\right)$:

$$\boxed{\begin{aligned} \frac{-d\tilde{E}_u}{dz} &= jk_z Z^e \tilde{H}_v - \frac{k_\rho}{k_z} Z^e \tilde{J}_z + \tilde{M}_v \\ \frac{-d\tilde{H}_v}{dz} &= jk_z Y^e \tilde{E}_u + \tilde{J}_u \end{aligned}}$$

$$\boxed{\tilde{E}_z = -\frac{Z^e}{jk_z}\left(\tilde{J}_z + jk_\rho \tilde{H}_v\right)}$$

Let $V^e = \tilde{E}_u,\ I^e = \tilde{H}_v,$

$$v^e = \frac{k_\rho}{k_z} Z^e \tilde{J}_z - \tilde{M}_v,$$

$$i^e = -\tilde{J}_u$$

TE$_z$ Fields $\left(Z^h \equiv 1/Y^h = \eta k / k_z\right)$:

$$\boxed{\begin{aligned} \frac{-d\tilde{E}_v}{dz} &= -jk_z Z^h \tilde{H}_u - \tilde{M}_u \\ \frac{-d\tilde{H}_u}{dz} &= -jk_z Y^h \tilde{E}_v - \frac{k_\rho}{k_z} Y^h \tilde{M}_z - \tilde{J}_v \end{aligned}}$$

$$\boxed{\tilde{H}_z = -\frac{Y^h}{jk_z}\left(\tilde{M}_z - jk_\rho \tilde{E}_v\right)}$$

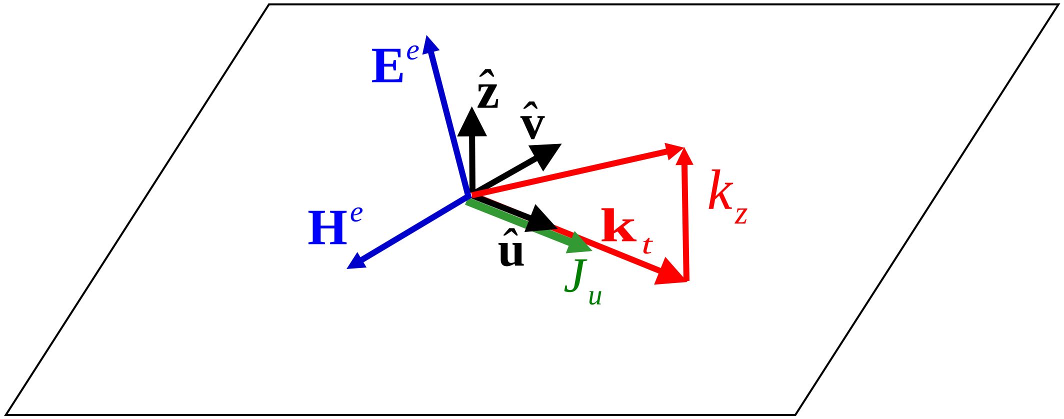
Let $V^h = \tilde{E}_v,\ I^h = -\tilde{H}_u,$

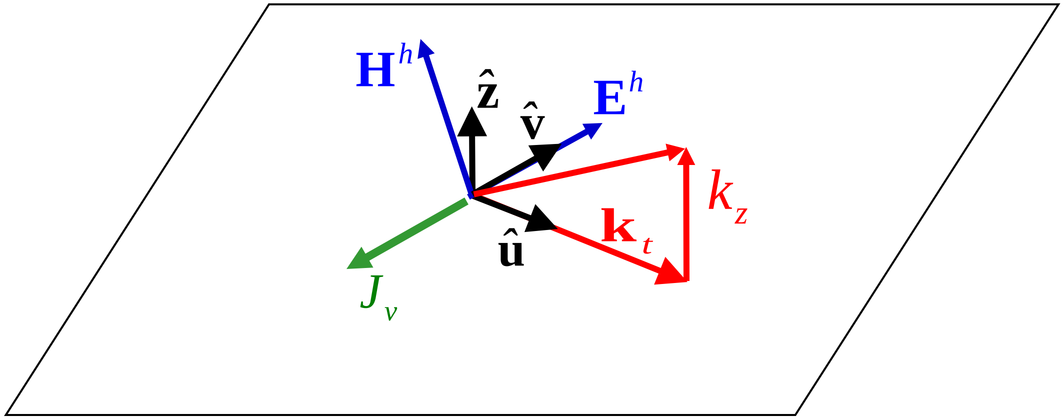
$$v^h = \tilde{M}_u$$

$$i^h = -\frac{k_\rho}{k_z} Y^h \tilde{M}_z - \tilde{J}_v$$

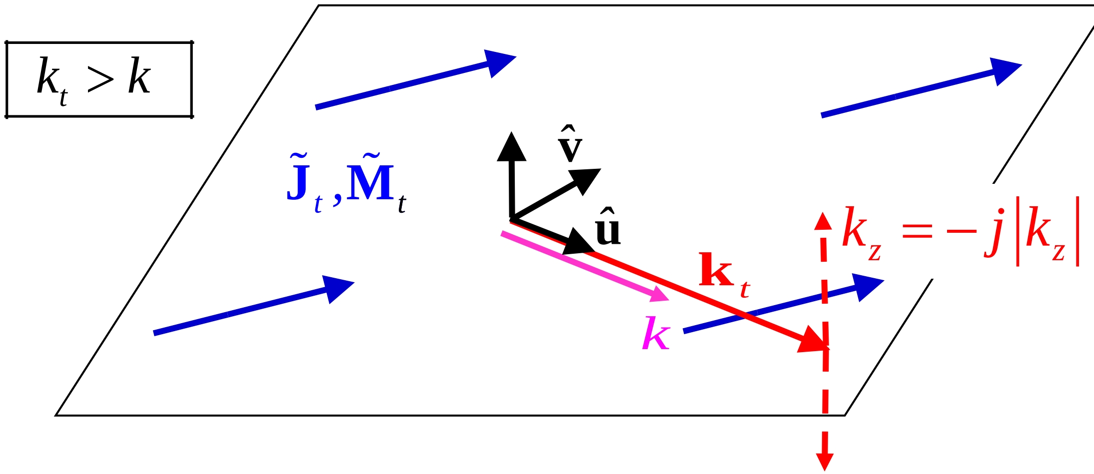
$$\boxed{\begin{aligned} -\frac{dV^\alpha}{dz} &= jk_z Z^\alpha I^\alpha - v^\alpha, \\ -\frac{dI^\alpha}{dz} &= jk_z Y^\alpha V^\alpha - i^\alpha, \end{aligned} \qquad \alpha = e, h}$$

# Current Components $J_u$ and $J_v$ Launch TM$_z$ ($e$) and TE$_z$ ($h$) Waves, Respectively



# Observation: Large Transverse Spectral Values Produce Exponentially Decaying (Inhomogeneous) Plane Waves




- For $k_t > k$, $k_z = \sqrt{k^2 - k_t^2} \equiv -j\sqrt{k_t^2 - k^2}$; hence all transmission line (and field) quantities decay exponentially away from a source at $z = z'$ as

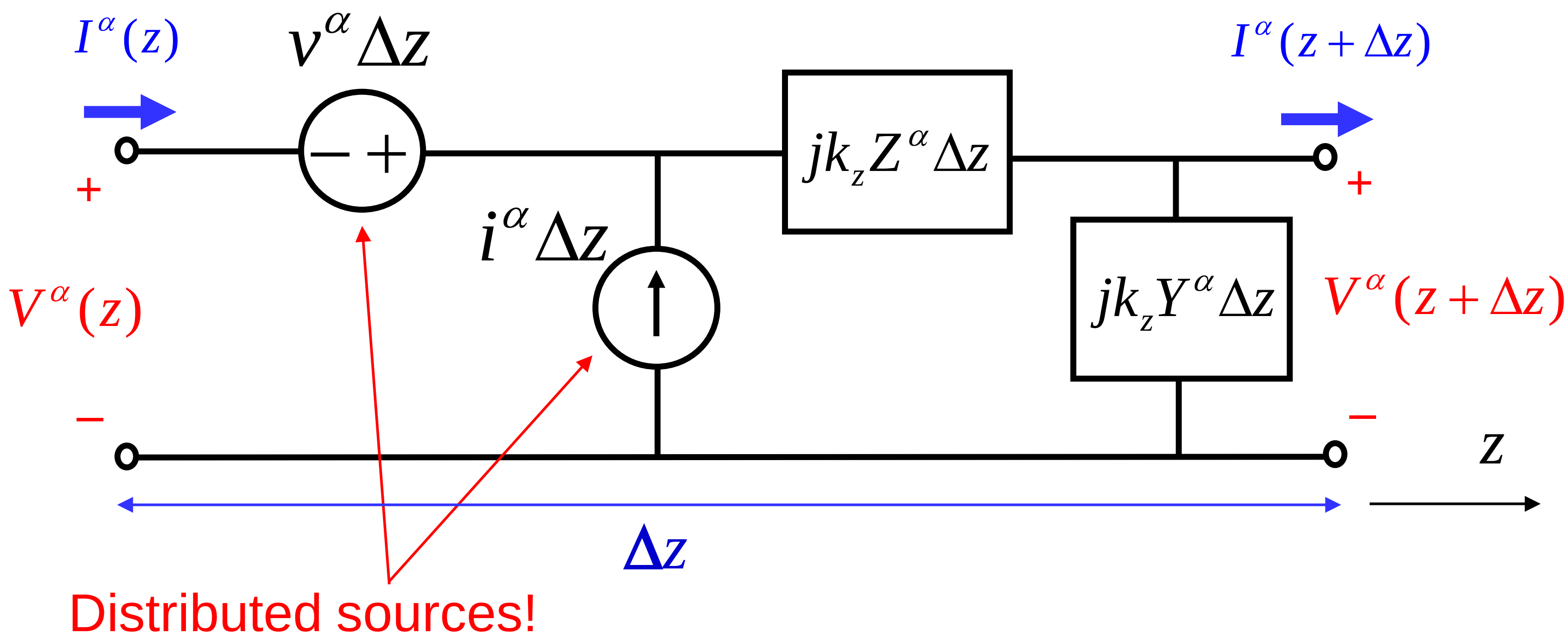
$$e^{-|k_z||z-z'|} \sim e^{-k_t|z-z'|}$$

$\Rightarrow$ Summations or integrals over spectral variables converge exponentially if there is any separation between source and observation planes!

- For multilayered structures, $k_t$ is layer independent; for large $k_t$

$k_{z\ell} = \sqrt{k_\ell^2 - k_t^2} \sim -jk_t$ is *asymptotically* independent of the layer index, $\ell$

# Differential Network Representing Telegrapher's Equations

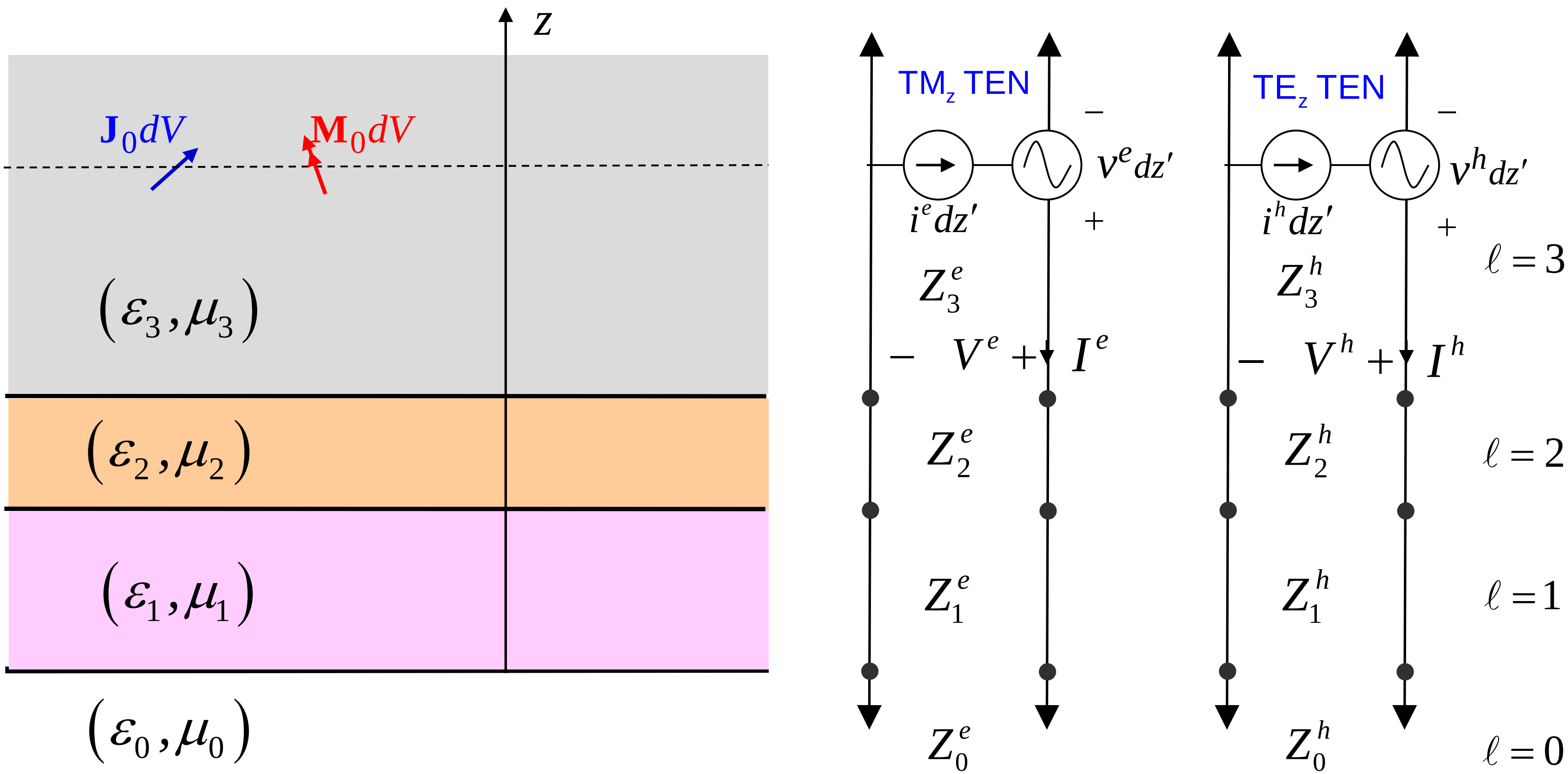



$$
-\frac{dV^{\alpha}}{dz} = jk_z Z^{\alpha} I^{\alpha} - v^{\alpha},
$$

$$
-\frac{dI^{\alpha}}{dz} = jk_z Y^{\alpha} V^{\alpha} - i^{\alpha},
$$

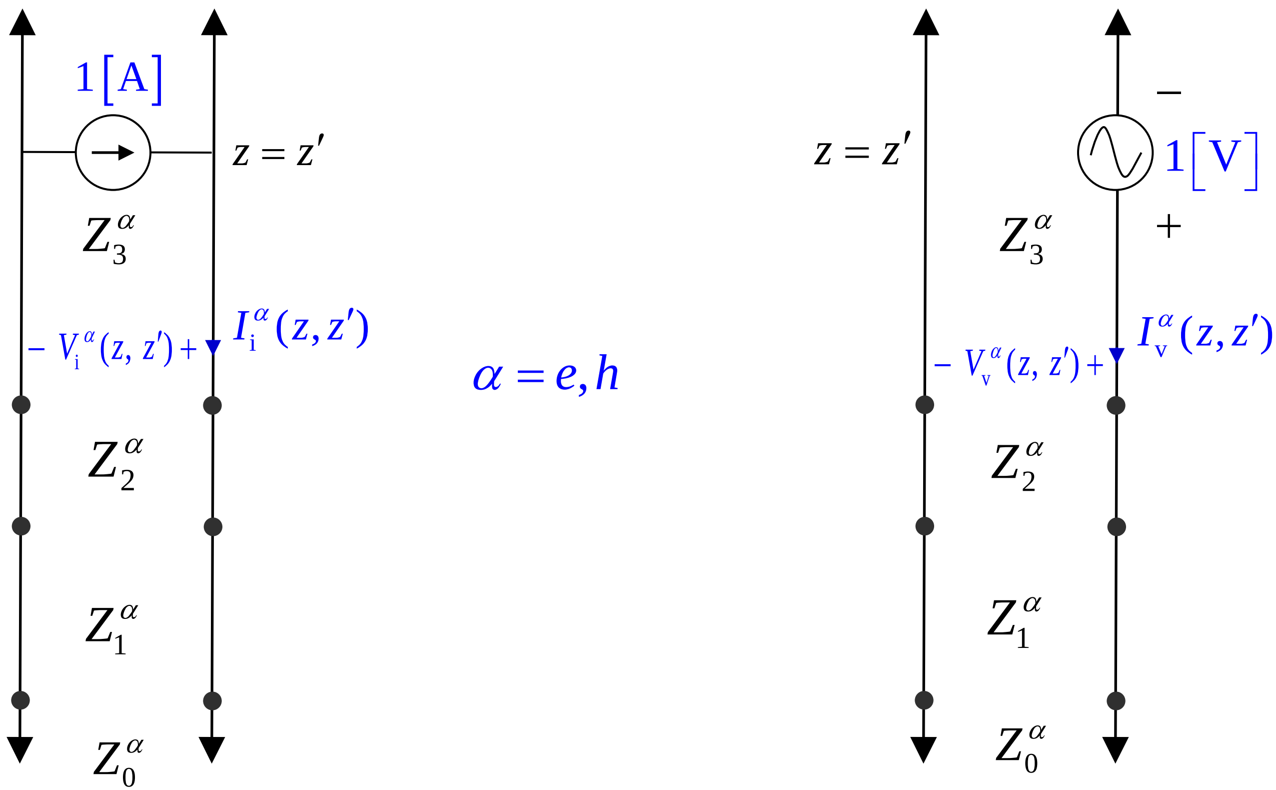
$$
\alpha = e, h
$$

# A Transverse Equivalent Network (TEN) May Be Used to Calculate Transformed Fields




- The $(z, z')$ - dependence is determined by the TEN network
- The transverse $(\boldsymbol{\rho}, \boldsymbol{\rho}')$ - dependence is determined by exponential factors $e^{-j\mathbf{k}_t \cdot \boldsymbol{\rho}'}$ and $e^{j\mathbf{k}_t \cdot \boldsymbol{\rho}}$ in the transforms $i^{e,h}, v^{e,h}$ and inversion integrals, respectively.

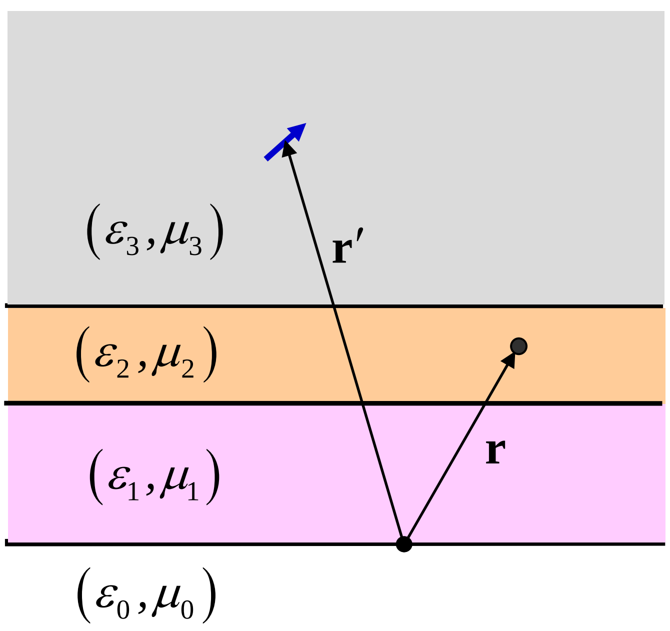
# TEN Quantities May Be Found in Terms of Voltage and Current Source Green's Functions




$$V^\alpha(z) = \int_{\text{TX Line}} V_i^\alpha(z, z') i^\alpha(z') dz' + \int_{\text{TX Line}} V_v^\alpha(z, z') v^\alpha(z') dz', \qquad V_v^\alpha(z, z') = -\frac{Z^\alpha}{jk_z} \frac{dI_v^\alpha}{dz}$$

$$I^\alpha(z) = \int_{\text{TX Line}} I_i^\alpha(z, z') i^\alpha(z') dz' + \int_{\text{TX Line}} I_v^\alpha(z, z') v^\alpha(z') dz', \qquad I_i^\alpha(z, z') = -\frac{Y^\alpha}{jk_z} \frac{dV_i^\alpha}{dz}$$

# Non-Periodic Layered Media Green's Functions



# In Layered Media, the Green's Functions Relating Currents to Fields are *Dyads*

- The field transforms can be written in terms of dyadic Green's functions; e.g., the inverse transform of $\tilde{\mathbf{E}}\left[\tilde{\mathbf{J}},\tilde{\mathbf{M}}\right] = \tilde{\mathcal{G}}^{EJ}\cdot\tilde{\mathbf{J}} + \tilde{\mathcal{G}}^{EM}\cdot\tilde{\mathbf{M}}$ is

$$\mathcal{G}^{EJ}(\mathbf{r},\mathbf{r}') = \begin{bmatrix} \mathcal{G}_{xx}^{EJ} & \mathcal{G}_{xy}^{EJ} & \mathcal{G}_{xz}^{EJ} \\ \mathcal{G}_{yx}^{EJ} & \mathcal{G}_{yy}^{EJ} & \mathcal{G}_{yz}^{EJ} \\ \mathcal{G}_{zx}^{EJ} & \mathcal{G}_{zy}^{EJ} & \mathcal{G}_{zz}^{EJ} \end{bmatrix}$$

$$\mathbf{E}[\mathbf{J},\mathbf{M}] = \iiint \overbrace{\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\mathcal{G}}^{EJ}(\mathbf{k}_t,z,z')\,e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}dk_x\,dk_y\right]}^{\mathcal{G}^{EJ}(\mathbf{r},\mathbf{r}')}\cdot\mathbf{J}(\mathbf{r}')\,dx'dy'dz'$$

$$+\iiint \underbrace{\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\mathcal{G}}^{EM}(\mathbf{k}_t,z,z')\,e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}dk_x\,dk_y\right]}_{\mathcal{G}^{EM}(\mathbf{r},\mathbf{r}')}\cdot\mathbf{M}(\mathbf{r}')\,dx'dy'dz'$$

etc., where the elements of the Green's function dyads (matrices) are expressed in terms of scalar TEN Green's functions.

- $\mathcal{G}^{EJ}$ and $\mathcal{G}^{EM}$ are "too singular" for convenient numerical processing. What is needed is a *mixed potential* representation analogous to that for homogeneous media.

# Michalski's Mixed Potential Representation

Michalski showed that in layered media, one can write

$$\mathbf{E}[\mathbf{J},\mathbf{M}] = -j\omega\mathbf{A} - \nabla\Phi - \frac{1}{\varepsilon}\nabla\times\mathbf{F},$$

$$\mathbf{H}[\mathbf{J},\mathbf{M}] = -j\omega\mathbf{F} - \nabla\Psi + \frac{1}{\mu}\nabla\times\mathbf{A},$$

where

$$\mathbf{A}(\mathbf{r}) = \int_S \mathcal{G}^A(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dS', \qquad \mathbf{F}(\mathbf{r}) = \int_S \mathcal{G}^F(\mathbf{r},\mathbf{r}')\cdot\mathbf{M}(\mathbf{r}')\,dS',$$

$$\Phi(\mathbf{r}) = \int_S \nabla'\cdot\mathbf{J}(\mathbf{r}')\, K^\Phi(\mathbf{r},\mathbf{r}')\,dS' + \int_S \hat{\mathbf{z}}\cdot\mathbf{J}(\mathbf{r}')\, P_z(\mathbf{r},\mathbf{r}')\,dS'$$

$$\Psi(\mathbf{r}) = \int_S \nabla'\cdot\mathbf{M}(\mathbf{r}')K^\Psi(\mathbf{r},\mathbf{r}')\,dS' + \int_S \hat{\mathbf{z}}\cdot\mathbf{M}(\mathbf{r}')Q_z(\mathbf{r},\mathbf{r}')\,dS'$$

and where, with $dk_x dk_y = k_\rho dk_\rho d\varphi$,

$$\mathbf{f}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty}\tilde{\mathbf{f}}(\mathbf{k}_t, z, z')e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}k_\rho dk_\rho d\varphi$$

**K. A. Michalski, D. Zheng, "Electromagnetic Scattering and Radiation by Surfaces of Arbitrary Shape in Layered Media, Part I: Theory," *IEEE Trans. Antennas and Propagat.*, 38, Mar. 1990.**

# The Corresponding Transform Domain Potentials Are Expressed in Terms of TEN Variables

- Vector potential Green's dyad:

$$\tilde{\mathcal{G}}^{A}\left(\mathbf{k}_t, z, z'\right) = \begin{bmatrix} \frac{1}{j\omega} V_{\mathrm{i}}^{h} & 0 & 0 \\ 0 & \frac{1}{j\omega} V_{\mathrm{i}}^{h} & 0 \\ \frac{\mu k_x}{jk_t^2}\left(I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right) & \frac{\mu k_y}{jk_t^2}\left(I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right) & \frac{\mu}{j\omega\varepsilon'} I_{\mathrm{v}}^{e} \end{bmatrix}$$

- Scalar potential kernel:

$$\tilde{K}^{\Phi}\left(\mathbf{k}_t, z, z'\right) = \frac{V_{\mathrm{i}}^{h} - V_{\mathrm{i}}^{e}}{k_t^2},$$

- Scalar potential correction term:

$$\tilde{P}_z\left(\mathbf{k}_t, z, z'\right) = \frac{j\omega\mu'}{k_t^2}\left(V_{\mathrm{v}}^{h} - V_{\mathrm{v}}^{e}\right)$$

Dependence of

$$V_{\mathrm{i}}^{\alpha},\ I_{\mathrm{i}}^{\alpha},\ V_{\mathrm{v}}^{\alpha},\ I_{\mathrm{v}}^{\alpha}$$

on $\mathbf{k}_t, z, z'$ is suppressed

# Transform Domain Potentials (cont'd)

- Curl of vector potential dyad:

$$\nabla \times \tilde{\boldsymbol{\mathcal{G}}}^{A}\left(\mathbf{k}_t, z, z'\right) = \begin{bmatrix} -\dfrac{\mu k_x k_y}{k_t^2}\left[I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right] & -\dfrac{\mu k_y^2}{k_t^2}\left[I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right] + \mu I_{\mathrm{i}}^{h} & -\dfrac{\mu k_y}{\omega\varepsilon'} I_{\mathrm{v}}^{e} \\ \dfrac{\mu k_x^2}{k_t^2}\left[I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right] - \mu I_{\mathrm{i}}^{h} & \dfrac{\mu k_x k_y}{k_t^2}\left[I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right] & \dfrac{\mu k_x}{\omega\varepsilon'} I_{\mathrm{v}}^{e} \\ \dfrac{k_y}{\omega} V_{\mathrm{i}}^{h} & -\dfrac{k_x}{\omega} V_{\mathrm{i}}^{h} & 0 \end{bmatrix}$$

Dependence of

$V_{\mathrm{i}}^{\alpha}, I_{\mathrm{i}}^{\alpha}, V_{\mathrm{v}}^{\alpha}, I_{\mathrm{v}}^{\alpha}$

on $\mathbf{k}_t, z, z'$ is suppressed

# Magnetic Current Potential Quantities Follow by Duality

E.g., since

$$\tilde{\mathcal{G}}^{A}\left(\mathbf{k}_t, z, z'\right) = \begin{bmatrix} \frac{1}{j\omega} V_{\mathrm{i}}^{h} & 0 & 0 \\ 0 & \frac{1}{j\omega} V_{\mathrm{i}}^{h} & 0 \\ \frac{\mu k_x}{jk_t^2}\left(I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right) & \frac{\mu k_y}{jk_t^2}\left(I_{\mathrm{i}}^{h} - I_{\mathrm{i}}^{e}\right) & \frac{\mu}{j\omega\varepsilon'} I_{\mathrm{v}}^{e} \end{bmatrix}$$

$\Rightarrow$

$$\tilde{\mathcal{G}}^{F}\left(\mathbf{k}_t, z, z'\right) = \begin{bmatrix} \frac{1}{j\omega} I_{\mathrm{v}}^{e} & 0 & 0 \\ 0 & \frac{1}{j\omega} I_{\mathrm{v}}^{e} & 0 \\ \frac{\varepsilon k_x}{jk_t^2}\left(V_{\mathrm{v}}^{e} - V_{\mathrm{v}}^{h}\right) & \frac{\varepsilon k_y}{jk_t^2}\left(V_{\mathrm{v}}^{e} - V_{\mathrm{v}}^{h}\right) & \frac{\varepsilon}{j\omega\mu'} V_{\mathrm{i}}^{h} \end{bmatrix}$$

etc.

Duality Replace - ment Table

$\mathcal{G}^{A} \rightarrow \mathcal{G}^{F}$

$K^{\Phi} \rightarrow K^{\Psi}$

$P_z \rightarrow Q_z$

$V \rightarrow I$

$I \rightarrow V$

$\mathrm{v} \rightarrow \mathrm{i}$

$\mathrm{i} \rightarrow \mathrm{v}$

$e \rightarrow h$

$h \rightarrow e$

$\mu \rightarrow \varepsilon$

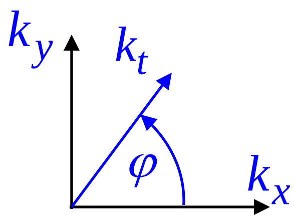
$\varepsilon \rightarrow \mu$

# In Inverting the Transforms, We Make Use of Hankel Transform Notation

- Noting that

$$\frac{k_x}{k_t} = \cos\underbrace{\left(\tan^{-1}\frac{k_y}{k_x}\right)}_{\equiv\,\varphi}, \quad \frac{k_y}{k_t} = \sin\left(\tan^{-1}\frac{k_y}{k_x}\right), \left(\frac{k_x}{k_t}\right)^2 = \cos^2\varphi = \frac{1}{2}+\frac{1}{2}\cos 2\varphi,$$

$$\left(\frac{k_y}{k_t}\right)^2 = \sin^2\varphi = \frac{1}{2}-\frac{1}{2}\cos 2\varphi, \quad \left(\frac{k_x k_y}{k_t^2}\right) = \cos\varphi\,\sin\varphi = \frac{1}{2}\sin 2\varphi \text{ ,etc.,}$$



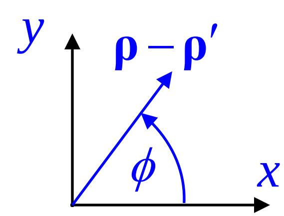

we can write the inverse transforms as Hankel transforms:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{matrix} \sin n \tan^{-1}\dfrac{k_y}{k_x} \\ \underbrace{\cos n \tan^{-1}\dfrac{k_y}{k_x}}_{\varphi} \end{matrix} \; \tilde{f}\left(k_t\right) e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} \underbrace{dk_x dk_y}_{k_t dk_t d\varphi} = \left(-j\right)^n \begin{matrix} \sin n\left(\tan^{-1}\dfrac{y-y'}{x-x'}\right) \\ \cos n\underbrace{\left(\tan^{-1}\dfrac{y-y'}{x-x'}\right)}_{\phi} \end{matrix} \; \mathcal{S}_n\left\{\tilde{f}\left(k_t\right)\right\},$$

where

Usually evaluated numerically!

$$\mathcal{S}_n\left\{\tilde{f}\left(k_t\right)\right\} = \frac{1}{2\pi}\int_0^{\infty}\tilde{f}\left(k_t\right)J_n\left(k_t\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)k_t\,dk_t\,, \quad n = 0,1,2 \text{ (Hankel transform)}$$



# Inverse Transforms (cont'd)

- Magnetic vector potential Green's dyad:

$$\mathcal{G}^A\left(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z'\right) = \begin{bmatrix} \frac{1}{j\omega}\mathcal{S}_0\left\{V_\mathrm{i}^h\right\} & 0 & 0 \\ 0 & \frac{1}{j\omega}\mathcal{S}_0\left\{V_\mathrm{i}^h\right\} & 0 \\ -\mu\frac{(x-x')}{\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|}\mathcal{S}_1\left\{\frac{I_\mathrm{i}^h-I_\mathrm{i}^e}{k_t}\right\} & -\mu\frac{(y-y')}{\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|}\mathcal{S}_1\left\{\frac{I_\mathrm{i}^h-I_\mathrm{i}^e}{k_t}\right\} & \frac{\mu}{j\omega\varepsilon'}\mathcal{S}_0\left\{I_\mathrm{v}^e\right\} \end{bmatrix}$$

- Electric scalar potential kernel:

$$K^\Phi\left(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z'\right) = \mathcal{S}_0\left\{\frac{V_\mathrm{i}^h-V_\mathrm{i}^e}{k_t^2}\right\},$$

- Scalar potential correction term:

$$P_z\left(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z'\right) = j\omega\mu'\mathcal{S}_0\left\{\frac{V_\mathrm{v}^h-V_\mathrm{v}^e}{k_t^2}\right\}$$

# Inverse Transforms (cont'd)

- The Hankel transform integrals must be computed numerically and are exponentially convergent when $z$ and $z'$ are well separated; when this is *not* the case, i.e., $z$ and $z'$ are in the same or adjacent layers, they can be accelerated by removing spectral asymptotic forms of the transmission line Green's functions and adding back their *closed - form Hankel inverses* :

- Vector potential: $\Delta\mathcal{G}^{A}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z') \equiv \mathcal{G}^{A}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')-\mathcal{G}^{A,\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')$

  where $$\mathcal{G}^{A,\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')=\begin{bmatrix} \frac{1}{j\omega}\mathcal{S}_0\left\{V_{\mathrm{i}}^{h,\infty}\right\} & 0 & 0 \\ 0 & \frac{1}{j\omega}\mathcal{S}_0\left\{V_{\mathrm{i}}^{h,\infty}\right\} & 0 \\ -\frac{\mu(x-x')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\mathcal{S}_1\left\{\frac{I_{\mathrm{i}}^{h,\infty}-I_{\mathrm{i}}^{e,\infty}}{k_t}\right\} & -\frac{\mu(y-y')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\mathcal{S}_1\left\{\frac{I_{\mathrm{i}}^{h,\infty}-I_{\mathrm{i}}^{e,\infty}}{k_t}\right\} & \frac{\mu}{j\omega\varepsilon'}\mathcal{S}_0\left\{I_{\mathrm{v}}^{e,\infty}\right\} \end{bmatrix}$$

- Scalar potential kernel: $\Delta K^{\Phi}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z') = K^{\Phi}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')-K^{\Phi,\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')$

  where $K^{\Phi,\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')= \mathcal{S}_0\left\{\frac{V_{\mathrm{i}}^{h,\infty}-V_{\mathrm{i}}^{e,\infty}}{k_t^2}\right\}$

- Scalar potential correction term: $\Delta P_z(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z') = P_z(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')-P_z^{\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')$

  where

$$P_z^{\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')= j\omega\mu'\,\mathcal{S}_0\left\{\frac{V_{\mathrm{v}}^{h,\infty}-V_{\mathrm{v}}^{e,\infty}}{k_t^2}\right\}$$

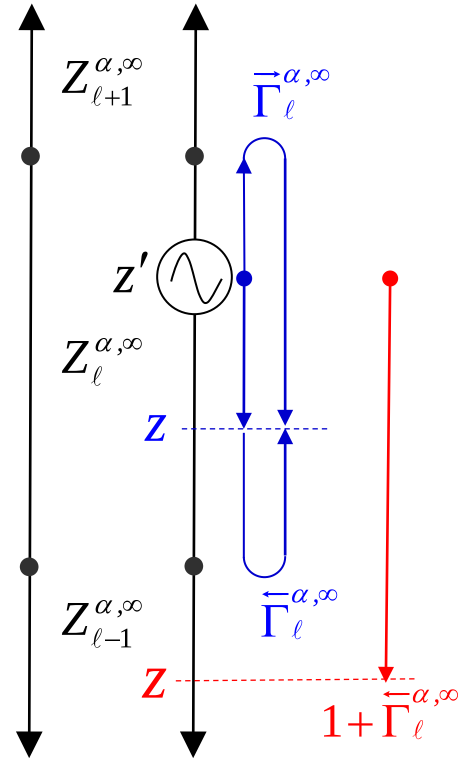
# Asymptotic Forms of Transmission Line Green's Functions

$$V_{\mathrm{i}}^{\alpha,\infty}=\begin{cases}\dfrac{Z_\ell^{\alpha,\infty}}{2}\left[e^{-jk_{z\ell}|z-z'|}+\overleftarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(z+z'-2z_\ell)}+\overrightarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(2z_{\ell+1}-z-z')}\right], & \ell=\ell' \\ \dfrac{Z_\ell^{\alpha,\infty}}{2}\left[e^{-jk_{z\ell}|z-z'|}\left(1+\overset{\rightleftarrows}{\Gamma}_\ell^{\alpha,\infty}\right)\right], & \ell=\ell'\pm1\end{cases}$$

$$I_{\mathrm{i}}^{\alpha,\infty}=\begin{cases}\dfrac{1}{2}\left[\operatorname{sgn}(z-z')e^{-jk_{z\ell}|z-z'|}+\overleftarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(z+z'-2z_\ell)}-\overrightarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(2z_{\ell+1}-z-z')}\right], & \ell=\ell' \\ \dfrac{1}{2}\left[\pm e^{-jk_{z\ell}|z-z'|}\left(1\mp\overset{\rightleftarrows}{\Gamma}_\ell^{\alpha,\infty}\right)\right], & \ell=\ell'\pm1\end{cases}$$

$$V_{\mathrm{v}}^{\alpha,\infty}=\begin{cases}\dfrac{1}{2}\left[\operatorname{sgn}(z-z')e^{-jk_{z\ell}|z-z'|}-\overleftarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(z+z'-2z_\ell)}+\overrightarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(2z_{\ell+1}-z-z')}\right], & \ell=\ell' \\ \dfrac{1}{2}\left[\pm e^{-jk_{z\ell}|z-z'|}\left(1\pm\overset{\rightleftarrows}{\Gamma}_\ell^{\alpha,\infty}\right)\right], & \ell=\ell'\pm1\end{cases}$$

$$I_{\mathrm{v}}^{\alpha,\infty}=\begin{cases}\dfrac{1}{2Z_\ell^{\alpha,\infty}}\left[e^{-jk_{z\ell}|z-z'|}-\overleftarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(z+z'-2z_\ell)}-\overrightarrow{\Gamma}_\ell^{\alpha,\infty}e^{-jk_{z\ell}(2z_{\ell+1}-z-z')}\right], & \ell=\ell' \\ \dfrac{1}{2Z_\ell^{\alpha,\infty}}\left[e^{-jk_{z\ell}|z-z'|}\left(1-\overset{\rightleftarrows}{\Gamma}_\ell^{\alpha,\infty}\right)\right], & \ell=\ell'\pm1\end{cases}$$




$$Z_\ell^{e,\infty}=\frac{k_t}{j\omega\varepsilon_\ell},\qquad Z_\ell^{h,\infty}=\frac{j\omega\mu_\ell}{k_t},$$

$$\overset{\rightleftarrows}{\Gamma}_\ell^{e,\infty}=\frac{\varepsilon_\ell-\varepsilon_{\ell\pm1}}{\varepsilon_\ell+\varepsilon_{\ell\pm1}},\qquad \overset{\rightleftarrows}{\Gamma}_\ell^{h,\infty}=\frac{\mu_{\ell\pm1}-\mu_\ell}{\mu_{\ell\pm1}+\mu_\ell}$$

# Evaluating Hankel Transforms of Asymptotic Forms

The terms $\mathcal{G}^{A,\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z'), K^{\Phi,\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')$, and $P_z^{\infty}(\boldsymbol{\rho}-\boldsymbol{\rho}',z,z')$ may be evaluated in closed form using the asymptotic forms for the transmission line Green's functions and the following Sommerfeld-type integrals:

- $$\int_0^\infty \frac{e^{-jk_z|z-z'|}}{k_z} J_0(k_t D) k_t dk_t = j\frac{e^{-kR}}{R}$$
- $$\begin{pmatrix} \dfrac{x-x'}{D} \\ \dfrac{y-y'}{D} \end{pmatrix} \int_0^\infty \frac{e^{-jk_z|z-z'|}}{k_z} J_1(k_t D) k_t^2\, dk_t = -j \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \frac{e^{-kR}}{R}$$
- $$\int_0^\infty e^{-jk_z|z-z'|} J_0(k_t D) k_t dk_t = -\operatorname{sgn}(z-z') \frac{\partial}{\partial z}\left(\frac{e^{-kR}}{R}\right)$$
- $$\int_0^\infty e^{-jk_z|z-z'|} J_1(k_t D) dk_t = \frac{Re^{-jk|z-z'|} - |z-z'| e^{-kR}}{DR}$$
- $$\int_0^\infty e^{-jk_z|z-z'|} J_2(k_t D) k_t\, dk_t = \frac{2}{D^2}\left(\frac{Re^{-jk|z-z'|} - |z-z'| e^{-kR}}{R}\right) - \frac{(z-z') e^{-kR}(1+jkR)}{R^3}$$
- $$\int_0^\infty \frac{e^{-jk_z|z-z'|}}{k_z} J_n(k_t D) k_t^{n+1}\, dk_t = \sqrt{\frac{2}{\pi}} \left(\frac{-jk}{R}\right)^{n+\frac{1}{2}} D^n K_{n+\frac{1}{2}}(-jkR)$$

where $K(z)$ is the modified Bessel function of the first kind, $D = |\boldsymbol{\rho}-\boldsymbol{\rho}'|$, $R = |\mathbf{r}-\mathbf{r}'|$

# Similarly for the Curl Dyads …

- Curl of vector potential dyad :

$$\nabla\times\tilde{\mathcal{G}}^{A}\left(\mathbf{k}_t,z,z'\right)$$

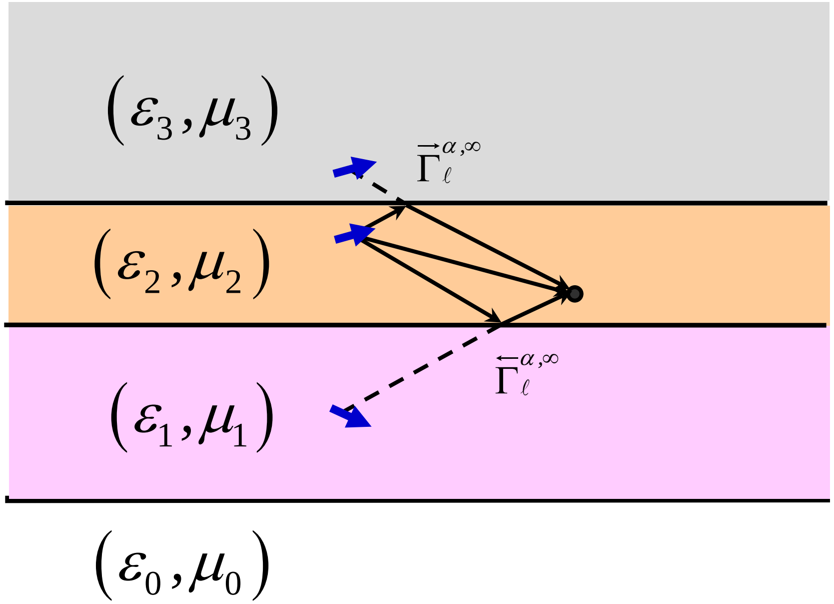
$$=\begin{bmatrix} \mu\dfrac{(x-x')(y-y')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}\mathcal{S}_2\left\{I_\mathrm{i}^h-I_\mathrm{i}^e\right\} & -\mathcal{F}^{-1}\left\{\dfrac{\mu k_y^2}{k_t^2}\left[I_\mathrm{i}^h-I_\mathrm{i}^e\right]+\mu I_\mathrm{i}^h\right\} & -\dfrac{\mu}{j\omega\varepsilon'}\dfrac{(y-y')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\mathcal{S}_0\left\{k_t I_\mathrm{v}^e\right\} \\ \mathcal{F}^{-1}\left\{\dfrac{\mu k_y^2}{k_t^2}\left[I_\mathrm{i}^h-I_\mathrm{i}^e\right]+\mu I_\mathrm{i}^h\right\} & -\mu\dfrac{(x-x')(y-y')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}\mathcal{S}_2\left\{I_\mathrm{i}^h-I_\mathrm{i}^e\right\} & \dfrac{\mu}{j\omega\varepsilon'}\dfrac{(x-x')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\mathcal{S}_0\left\{k_t I_\mathrm{v}^e\right\} \\ \dfrac{1}{j\omega}\dfrac{(y-y')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\mathcal{S}_0\left\{k_t V_\mathrm{i}^h\right\} & -\dfrac{1}{j\omega}\dfrac{(x-x')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\mathcal{S}_0\left\{k_t V_\mathrm{i}^h\right\} & 0 \end{bmatrix}$$

where

$$\mathcal{F}^{-1}\left\{-\frac{\mu k_y^2}{k_t^2}\left[I_\mathrm{i}^h-I_\mathrm{i}^e\right]+\mu I_\mathrm{i}^h\right\}= -\frac{\mu}{2}\mathcal{S}_0\left\{I_\mathrm{i}^h-I_\mathrm{i}^e\right\}+\left[\frac{1}{2}+\frac{(x-x')^2}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}\right]\mathcal{S}_2\left\{I_\mathrm{i}^h-I_\mathrm{i}^e\right\}+\mu\mathcal{S}_0\left\{I_\mathrm{i}^h\right\}$$

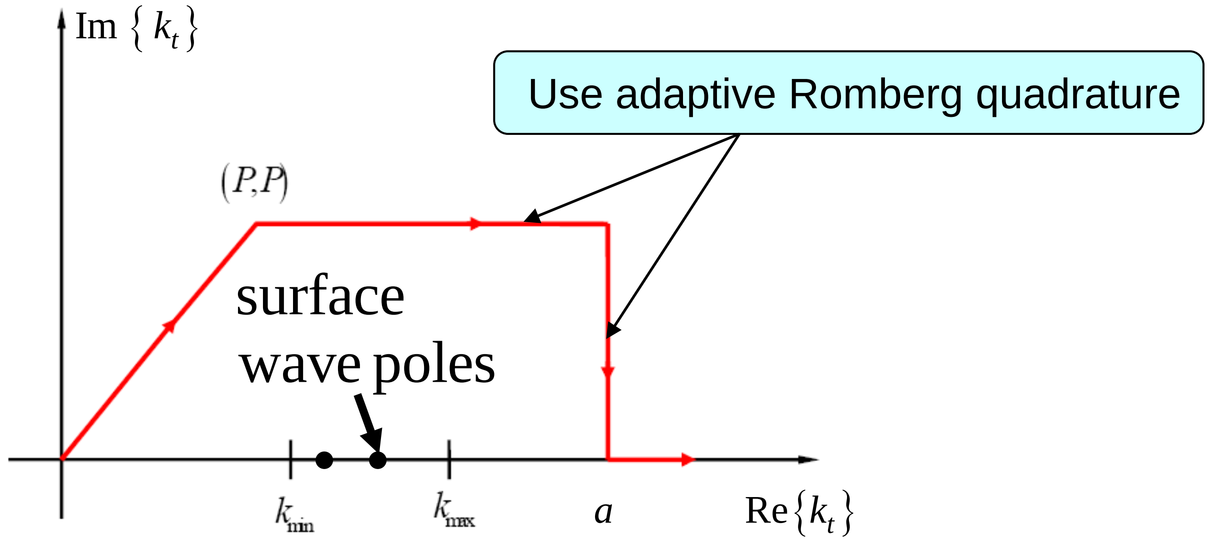
# Removal of Direct and Quasi-Static Images Accelerates Computation of Sommerfeld Integrals

$$\mathcal{G}^{\mathrm{A}}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \underbrace{\left[\tilde{\mathcal{G}}^{\mathrm{A}}(\mathbf{k}_t, z, z') - \sum_i \tilde{\mathcal{G}}^{\mathrm{A},i}(\mathbf{k}_t, z, z')\right]}_{\substack{\Delta\tilde{\mathcal{G}}^{\mathrm{A}}\text{, spectral representation}\\ \text{with asymptotic terms repre-}\\ \text{senting direct and quasi-static}\\ \text{image terms removed}}} e^{-j\mathbf{k}_t\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} k_t\, dk_t\, d\varphi$$




$$+ \underbrace{\sum_i \mathcal{G}^{\mathrm{A},i}(\mathbf{r},\mathbf{r}')}_{\substack{\text{direct and quasi- static}\\ \text{image terms are evaluated}\\ \text{in closed form}}}$$

# …The Integral is Deformed Around Poles, and the Tail of the Integral is Accelerated




$$\int_0^{\infty} \Delta\tilde{\mathcal{G}}^{\mathrm{A}} e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}\; k_t dk_t = \int_{\text{path around poles}} \Delta\tilde{\mathcal{G}}^{\mathrm{A}} e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}\; k_t dk_t$$

$$+ \underbrace{\int_a^{\infty} \Delta\tilde{\mathcal{G}}^{\mathrm{A}} e^{-j\mathbf{k}_t\cdot\boldsymbol{\rho}}\; k_t dk_t}_{\text{tail integral with oscillatory integrand is accelerated using the "method of averages"}}$$

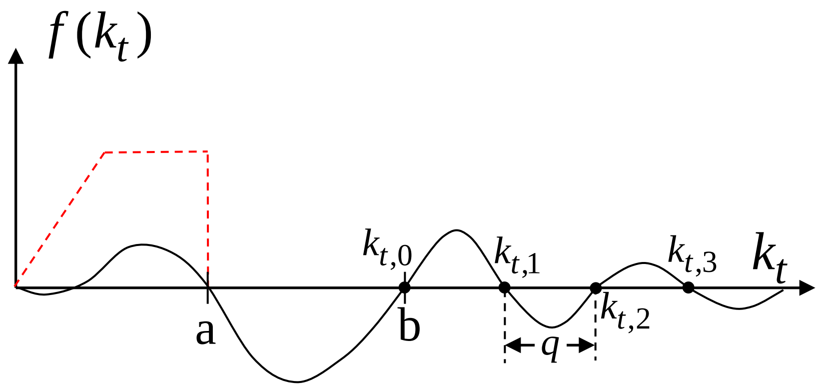
# Method of Weighted Averages

To compute

$$S = \int_a^\infty f(k_t)\,dk_t$$

where $f(k_t) = g(k_t)\,p(k_t)$ with

$$g(k_t) \sim \frac{e^{-k_t|z-z'|}}{k_t^\alpha} \sum_{i=0}^{\infty} \frac{c_i}{k_t^i}$$




and $p(k_t + q) = -p(k_t)$ is an oscillating periodic function

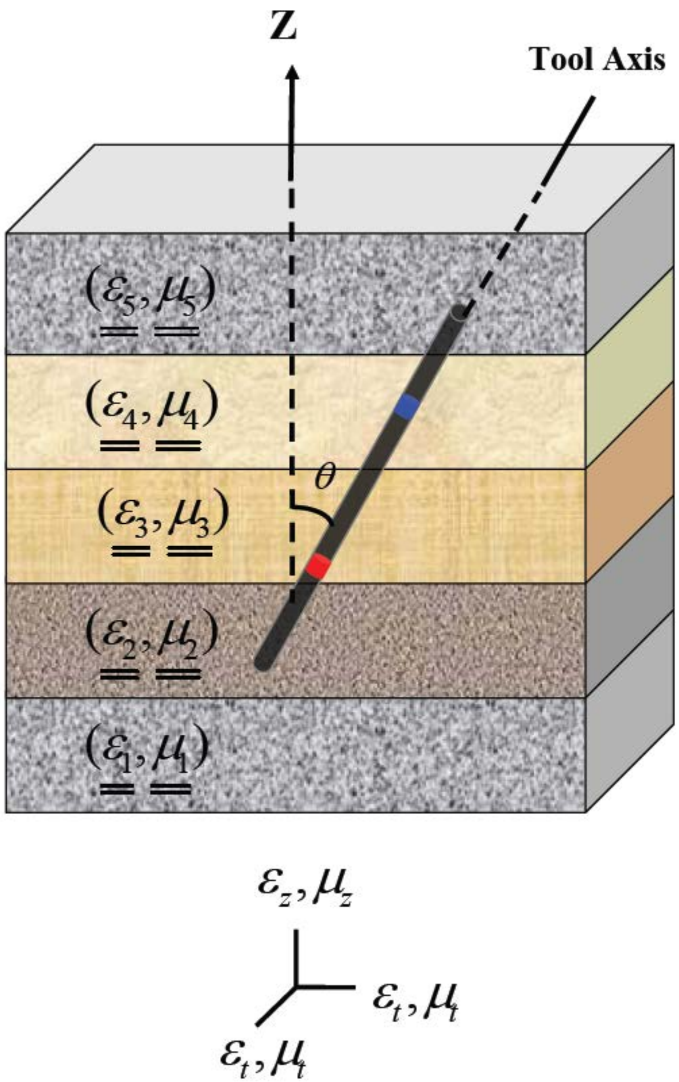
($q$ is the half period, usually $\dfrac{\pi}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|}$), define the partial integral

sequence $S_n = \int_a^{k_{t,n} \equiv b + nq} f(k_t)\,dk_t$ and update estimates of the limit as

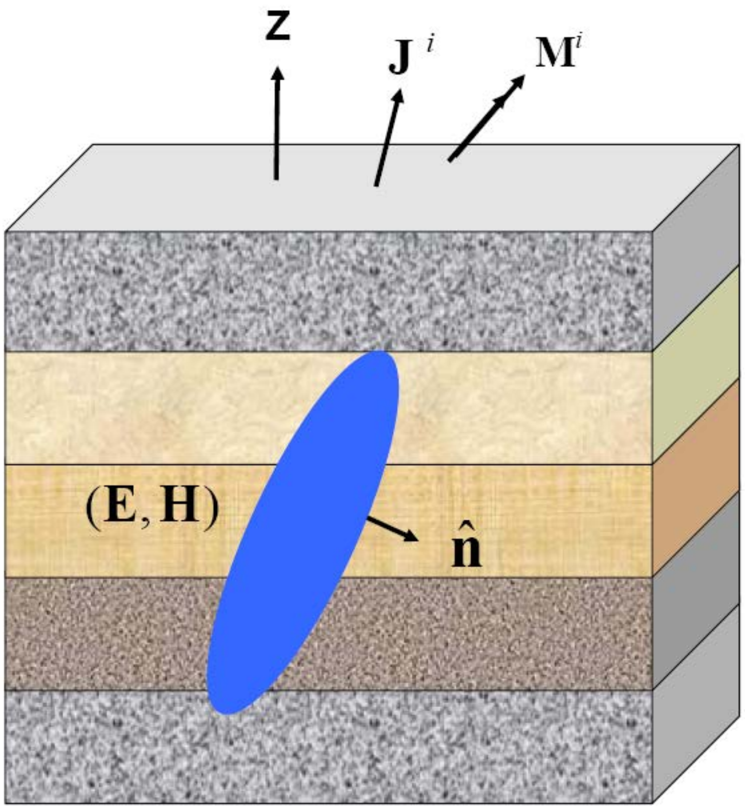
$$S_n^{(k+1)} = \frac{S_n^{(k)} + \eta_n^{(k)} S_{n+1}^{(k)}}{1 + \eta_n^{(k)}}, \quad n, k \geq 0, \quad \text{where} \quad \eta_n^{(k)} = e^{q|z-z'|} \left( \frac{k_{t,n+1}}{k_{t,n}} \right)^{\alpha + 2k}$$


K. A. Michalski, IEEE Trans. Antennas and Propagat., 46, Oct. 1998.

# Example: A Practical Problem Involving a Multilayered Medium



# Exterior Equivalence




$$\mathbf{E}[\mathbf{J},\mathbf{M}] = -j\omega\mathbf{A} - \nabla\Phi - \varepsilon^{-1}\nabla\times\mathbf{F}$$

$$\mathbf{H}[\mathbf{J},\mathbf{M}] = -j\omega\mathbf{F} - \nabla\Psi + \mu^{-1}\nabla\times\mathbf{A}$$

$$\mathbf{A} = \int_S \mathcal{G}^A(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dS'$$

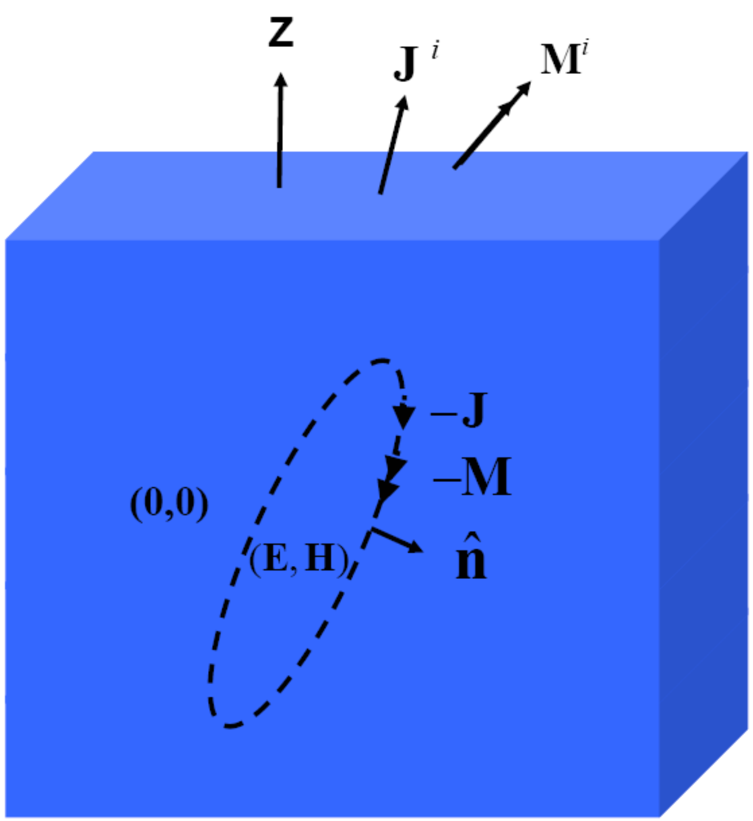
$$\mathbf{F} = \int_S \mathcal{G}^F(\mathbf{r},\mathbf{r}')\cdot\mathbf{M}(\mathbf{r}')\,dS'$$

$$\Phi = \int_S K^\Phi(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{J}(\mathbf{r}')\,dS' + \int_S P_z(\mathbf{r},\mathbf{r}')\,J_z(\mathbf{r}')\,dS'$$

$$\Psi = \int_S K^\Psi(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{M}(\mathbf{r}')\,dS' + \int_S Q_z(\mathbf{r},\mathbf{r}')\,M_z(\mathbf{r}')\,dS'$$

Impressed fields due to $\mathbf{M}^i,\mathbf{J}^i$ : $\mathbf{E}^i,\mathbf{H}^i$

# Interior Equivalence




$$\mathbf{E}[-\mathbf{J},-\mathbf{M}] = j\omega\mathbf{A} + \nabla\Phi + \varepsilon^{-1}\nabla\times\mathbf{F}$$

$$\mathbf{H}[-\mathbf{J},-\mathbf{M}] = j\omega\mathbf{F} + \nabla\Psi - \mu^{-1}\nabla\times\mathbf{A}$$

$$\mathbf{A} = \mu\int_S G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\,dS'$$

$$\mathbf{F} = \mu\int_S G(\mathbf{r},\mathbf{r}')\mathbf{M}(\mathbf{r}')\,dS'$$

$$\Phi = -\frac{1}{j\omega\varepsilon}\int_S G(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{J}(\mathbf{r}')\,dS'$$

$$\Psi = -\frac{1}{j\omega\mu}\int_S G(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{M}(\mathbf{r}')\,dS'$$

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-jkR}}{4\pi R}$$

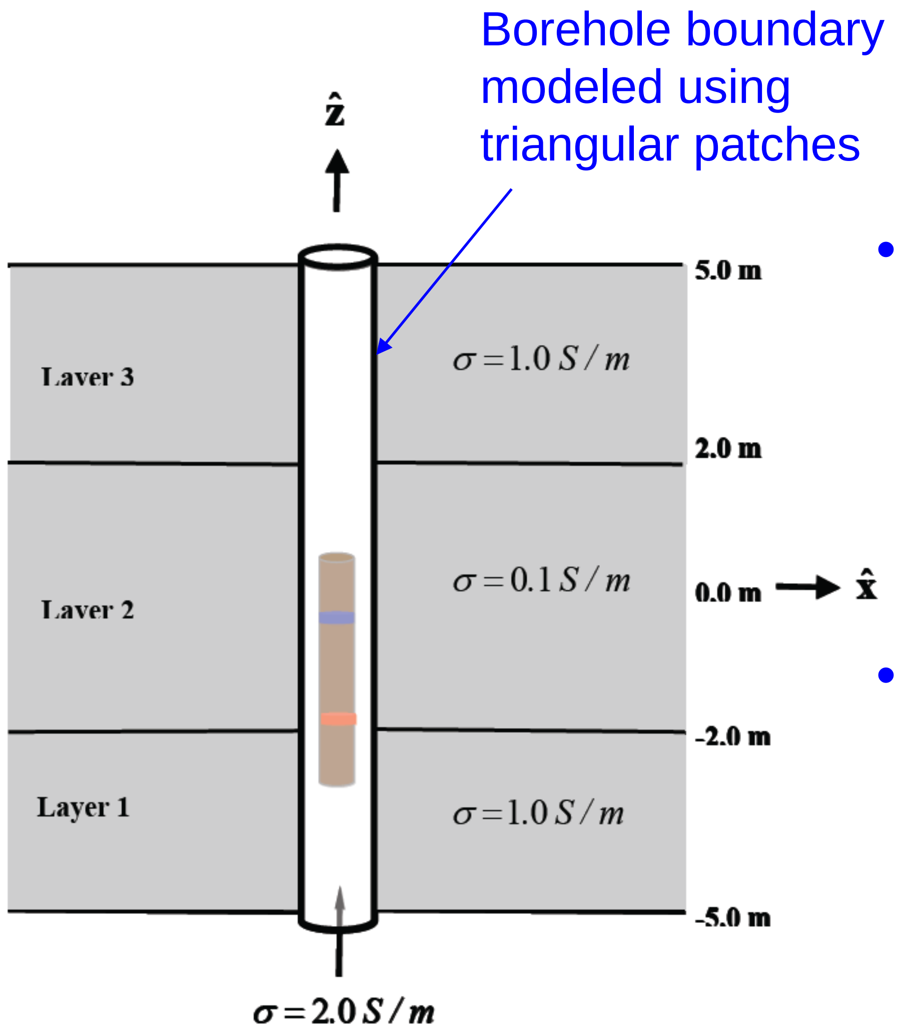
# Coupled Integral Equations

- Equating tangential components of exterior and interior fields at interface results in a pair of coupled integral equations (PMCHWT):

$$\hat{\mathbf{n}}\times\left\{\mathbf{E}\left[-\mathbf{J},-\mathbf{M}\right]-\mathbf{E}\left[\mathbf{J},\mathbf{M}\right]\right\}=\hat{\mathbf{n}}\times\mathbf{E}^{i}$$

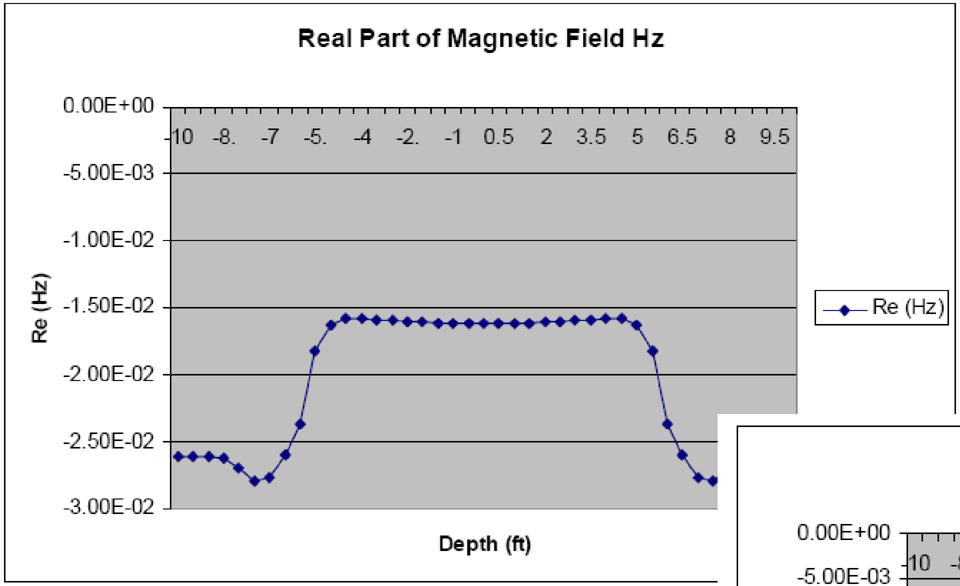
$$\hat{\mathbf{n}}\times\left\{\mathbf{H}\left[-\mathbf{J},-\mathbf{M}\right]-\mathbf{H}\left[\mathbf{J},\mathbf{M}\right]\right\}=\hat{\mathbf{n}}\times\mathbf{H}^{i}$$

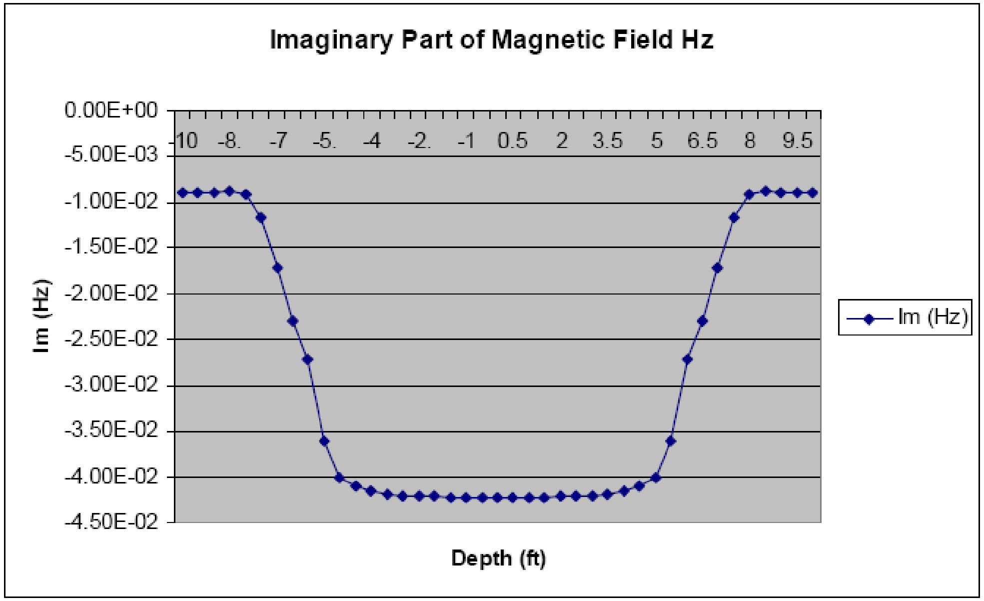
# Example




- Point magnetic current source representing a coil on the borehole axis
- Find the magnetic field on the axis at a fixed distance away as source and observation points with fixed separation slide down the borehole

# Calculated Magnetic Field



# Conclusions and Summary

- Express point current sources as superpositions over current sheet sources parallel to layer boundaries.
- Decompose transverse components of current in each constituent sheet into components parallel and perpendicular to the transverse wavevector.
- Fields produced by the sheet current components are $TM_z$ and $TE_z$ fields, respectively, propagating independently through layered media, and are conveniently analyzed by transmission line theory.

# Conclusions and Summary, cont'd

- Michalski's mixed potential representation expresses the relationship between vector sources and vector fields in a form convenient for numerical processing. The representation is in terms of scalar transmission line voltage and current Green's functions.

- The resulting Sommerfeld-type integrals are evaluated as follows:
  - The integration is accelerated by removing asymptotic direct and quasi-static contributions from the integrand. Their contributions are evaluated in closed form.
  - The integral on $k_t$ over the visible spectrum is evaluated on a deformed path in the complex plane.
  - The method of averages is used to accelerate the "tail" integral.

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