Method of Moments and Finite Element Methods

Donald R. Wilton
Dept. of Electrical and Computer Engineeering
University of Houston
Houston, TX 77096 USA
wilton@uh.edu

Linear Operator Equations

- A *linear operator equation* is represented symbolically as

$$\mathcal{L}u = f \tag{1}$$

where $\mathcal{L}$ is a *linear operator,*

$u = u(\mathbf{r})$ = unknown field or current, $\mathbf{r} \in \mathcal{D}$

$f = f(\mathbf{r})$ = known source, incident field or other forcing function, $\mathbf{r} \in \mathcal{D}$

- Operator $\mathcal{L}$ is linear if it satisfies

$$\mathcal{L}(au_1 + bu_2) = a\mathcal{L}u_1 + b\mathcal{L}u_2$$

- $\mathcal{L}$ is generally a differential, integral, or integro - differential operator on $\mathcal{D}$ relating sources f to fields u in $\mathcal{D}$ or on its boundary, $\partial\mathcal{D}$. *Matrices* are also linear operators.

Linear Operator Examples

- $\mathcal{L}V = \dfrac{d^2V}{dx^2} + k_0^2 V$ + BC's
- $\mathcal{L}q_S = \displaystyle\int_{\mathcal{S}} \frac{q_S(\mathbf{r}')}{4\pi\varepsilon|\mathbf{r}-\mathbf{r}'|} d\mathcal{S}'$
- $\mathcal{L}\mathbf{J} = j\omega\mu \displaystyle\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}' - \frac{\nabla}{j\omega\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{J}(\mathbf{r}')d\mathcal{S}'$ + BC's
- $\mathcal{L}\mathbf{J} = \dfrac{\mathbf{J}(\mathbf{r})}{2} - \hat{\mathbf{n}} \times \displaystyle\int_{\mathcal{S}} \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{J}(\mathbf{r}') d\mathcal{S}'$
- $\mathcal{L}\Phi = \nabla^2\Phi + k_0^2\varepsilon_r\Phi$ + BC's
- $\mathcal{L}\mathbf{E} = \nabla\times\mu_r^{-1}\nabla\times\mathbf{E} - k_0^2\varepsilon_r\mathbf{E}$ + BC's
- $\mathcal{L}[x_m] = [L_{mn}][x_m]$

In statics:

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi R}, \qquad R = |\mathbf{r}-\mathbf{r}'|, \text{ (3D)}$$

$$= -\frac{1}{2\pi}\ln D, \; D = |\boldsymbol{\rho}-\boldsymbol{\rho}'|, \text{(2D)}$$

In dynamics:

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-jkR}}{4\pi R}, \qquad \text{(3D)}$$

$$= -\frac{H_0^{(2)}(kD)}{4j}, \qquad \text{(2D)}$$

Inner Products

- Numerical solution methods are *projection* or *moment methods* involving *inner products*.
- Inner (dot) product between pairs of N - component vectors, $u = (u_1, u_2, \dots, u_N)$, $v = (v_1, v_2, \dots, v_N)$:

$$u \cdot v = < u, v > = \sum_{n=1}^{N} u_n v_n = [u_n]^t \, [v_n] \tag{3}$$

- Inner product between two scalar functions u and v:

$$< u, v > = \int_D uv \, d\mathcal{D}, \ \mathcal{D} = \mathcal{C}, \mathcal{S}, \text{or } \mathcal{V}, \tag{4}$$

Projection of "u on v" or "v on u."

$\mathcal{D} =$ line or curve $\mathcal{C}$ (one - dimension), surface $\mathcal{S}$ (two dimensions), or volume $\mathcal{V}$ (three - dimensions) (Note : Often v appears conjugated in inner product definitions!)

- (Bi-) linearity of inner product :

$$< au_1 + bu_2, v > = a < u_1, v > + b < u_2, v >,$$
$$< u, cv_1 + dv_2 > = c < u, v_1 > + d < u, v_2 > \tag{5}$$

Inner Products: Examples

- u and v vector functions, $u = \mathbf{u}$ and $v = \mathbf{v}$:

$$< u\,; v > = \int_{\mathcal{D}} u \cdot v \; d\mathcal{D}\,. \tag{6}$$

- Scalars u and v convolved with scalar Green's function $G(\mathbf{r}, \mathbf{r}')$ ("kernel" of integral equation):

$$< u, G, v > \;\equiv\; < u, < G, v >> \int_{\mathcal{D}} \int_{\mathcal{D}} u(\mathbf{r}) G(\mathbf{r}, \mathbf{r}')\, v(\mathbf{r}')\; d\mathcal{D}' d\mathcal{D} \tag{7}$$

- Vectors u and v convolved with scalar Green's function $G(\mathbf{r}, \mathbf{r}')$:

$$< u\,; G, v > \; = \int_{\mathcal{D}} \int_{\mathcal{D}} u(\mathbf{r}) \cdot v(\mathbf{r}') \;\; G(\mathbf{r}, \mathbf{r}') \;\; d\mathcal{D}' d\mathcal{D} \tag{8}$$

- Vectors u and v convolved with dyadic Green's function $\mathcal{G}(\mathbf{r}, \mathbf{r}')$:

$$< u; \mathcal{G}; v > \; = \int_{\mathcal{D}} \int_{\mathcal{D}} u(\mathbf{r}) \cdot \mathcal{G}(\mathbf{r}, \mathbf{r}') \cdot v(\mathbf{r}') \;\; d\mathcal{D}' d\mathcal{D} \tag{9}$$

- In general, $< u, v > \; = \; < v, u >$

Weak and Strong Forms of Operator Equations

- *Strong form* (equality holds at every point in $\mathcal{D}$):

$$\mathcal{L}u = f \qquad (10)$$

- *Weak form* (equality holds in a weighted average sense):

$$< w, Lu > = < w, f > \qquad (11)$$

where $\{w\}$ is a set of *weighting functions*.

Notes:

– For differential operators, integration by parts is often used to transfer differentiability requirements from u to w.

– If continuous, solutions u of (10) and (11) are identical

– If u is infinite dimensional, then so the set of functions w

- In numerical solutions, w's are chosen from a *finite* set of weighting or testing functions $\{w_m^{(N)}\}$, $m = 1, 2, \ldots, N$.

Bases and Unknown Representations

- Approximate u as

$$u \approx u = \sum_{n=1}^{N} U_n u_n \;=\; [U_n]^{\mathrm{t}}\,[u_n] \qquad (12)$$

where coefficients U_n are unknown and $u_n,\ n = 1, \ldots, N$ are known *basis functions*.

- u_n must be "independent" and capable of approximating u.

- Independence of bases is measured by their projections on one another,

$$< u_m, u_n > \;\equiv\; \text{Gram Matrix} \qquad (13)$$

Independence of Basis Functions

- Ideal are *orthonormal* bases u_n, for which

$$< u_m, u_n > = \int_{\mathcal{D}} u_m u_n \, d\mathcal{D} = \delta_{mn} = \begin{cases} 1, \; m = n, \\ 0, \; m \neq n, \end{cases} \qquad (14)$$

(δ_{mn} is the "Kronecker delta") but are difficult to discover for arbitrary $\mathcal{D}$

- Instead, first approximate $\mathcal{D}$ by subdividing into *subdomains* or *elements* (e.g., line segments, triangles, rectangles, tetrahedrons) $\mathcal{D}^e$, $e = 1, 2, \ldots, E.$ Then $\mathcal{D} \approx \tilde{\mathcal{D}} = \bigcup_{e=1}^{E} \mathcal{D}^e.$

- Then *interpolatory polynomial* bases are usually used. They satisfy the property $u_m(\mathbf{r}_j) = \delta_{mj}$ where $\mathbf{r}_j$, $j = 1, 2, \ldots, N$ are interpolation points on $\tilde{\mathcal{D}}$. In addition, they also satisfy the following "approximation" to (14):

$$\sum_{j=1}^{N} u_m(\mathbf{r}_j) u_n(\mathbf{r}_j) = \delta_{mn} \qquad (15)$$

Method of Moments

- Substituting representation for u into operator equation and testing with $\{w = w_m, m = 1, 2, \ldots, N\}$ yields

$$\sum_{n=1}^{N} < w_m, \mathcal{L}u_n > U_n = < w_m, f >, \quad m = 1, 2, \ldots, N \tag{16}$$

or in matrix form,

$$[L_{mn}][U_n] = [F_m], \tag{17}$$

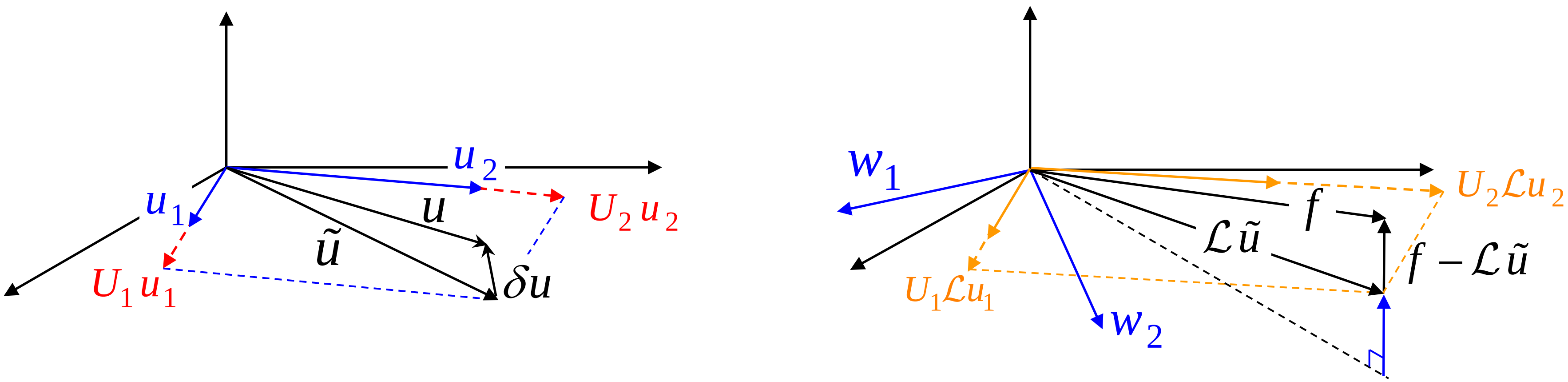
where $L_{mn} =< w_m, \mathcal{L}u_n >$ and $F_m =< w_m, f >$.

- Solving the linear system yields unknowns $[U_n]$ that provide an approximation to u in $\mathcal{D}$. The result can also be written as

$$u \approx u = [U_n]^{\mathrm{t}} [u_n] = [u_n]^{\mathrm{t}} [U_n], \tag{18}$$

where $[u_n]^{\mathrm{t}}$ denotes transpose of $[u_n]$.

Abstract Vector Space Interpretation of the Method of Moments



- The unknown is approximated in the "subspace of basis vectors u_n" as $u \approx \tilde{u} = \sum_n U_n u_n$
- Both $\mathcal{L}\tilde{u} = \sum_n U_n \mathcal{L} u_n$ and f are *projected* onto the "subspace of testing vectors w_m"; equating the projections determines $\{U_n\}$.
- The projection both minimizes the residual error $f - \sum_n U_n \mathcal{L} u_n$ and makes it orthogonal to the testing vector subspace.

Linear Functionals

- A *linear functional* $I[u]$ is a scalar physical parameter or figure of merit that depends linearly on u (e.g., $I[au] = aI[u]$).

  Examples:
  - Capacitance where u is surface charge
  - Input admittance where u is a surface current
  - Vector component of far field where u is a surface current
  - Value of $u(\mathbf{r})$ at point $\mathbf{r}$ (may be unbounded at edge or corner!)

- **Riesz representation theorem:** For any *bounded* linear functional, a function g exists such that $I[u]$ can be represented as an inner product,

$$I[u] = -< u, g > \tag{19}$$

- For $u \approx \tilde{u}$,

$$I[u] \approx I[\tilde{u}] = -< \tilde{u}, g > = -\sum_{n=1}^{N} U_n < u_n, g > = -[U_n]^t \, [< u_n, g >] \tag{20}$$

- Note: Sampled values of the unknown involve unbounded functionals:

$$J(\mathbf{r}') = \int_{\mathcal{S}} J(\mathbf{r}) \; \delta(\mathbf{r} - \mathbf{r}') \; d\mathcal{S} \quad \Rightarrow \quad g(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}')$$ is *unbounded*

Summary of Method of Moments

- Subdivide $\mathcal{D}$ into E subdomains or elements $\mathcal{D}^e$; approximate the solution domain as $\mathcal{D} \approx \tilde{\mathcal{D}} = \bigcup_{e=1}^{E} \mathcal{D}^e$.

- Choose (usually interpolating) basis functions $\{u_n\}$ and approximate u as

$$u \approx \tilde{u} = \sum_n U_n u_n = [U_n]^{\mathrm{t}} [u_n].$$

- Choose weighting (testing) functions $\{w_m\}$. (*Galerkin's method* : $\{w_m\} \equiv \{u_m\}$)

- Substitute $\tilde{u}$ into operator equation and test with w_m. (For differential or integro-differential operators, integrate by parts to reduce differentiability requirements on u_n and incorporate boundary conditions.)

Summary of Method of Moments, Cont'd

- Solve the resulting linear matrix system

$$[L_{mn}][U_n]=[F_m]$$

where

$$L_{mn} = < w_m, \mathcal{L}u_n >,$$
$$F_m = < w_m, f >$$

for unknown coefficients U_n. A direct or iterative solution procedure may be used.

- Compute desired figure-of-merit (functional) $I[u]$ as

$$I[u] \approx I[\tilde{u}] = -< \tilde{u}, g > = -\sum_{n=1}^{N} U_n < u_n, g > = -[U_n]^{\mathrm{t}} [< u_n, g >]. \qquad (21)$$

The Variational Approach

- Variational and MoM approaches appear to be quite different, but really are equivalent, as we'll show.

- As a first step, we define an adjoint operator $\mathcal{L}^\dagger$ such that

$$< w, \mathcal{L}u > = < \mathcal{L}^\dagger w, u > \qquad (22)$$

for arbitrary u and w.

- Adjoints exist and are unique; to find:
  - Differential operators: Successively integrate by parts
  - Integral operators: Interchange source and observation points in the kernel
  - Matrix operator: Simply transpose the original matrix

The Adjoint Problem

- The variational approach to solving $\mathcal{L}u = f$ begins by considering the linear functional

$$I[u] = - < u, g >$$

- Next define the *adjoint* problem,

$$\mathcal{L}^{\dagger} w = g. \qquad (23)$$

where g plays role of source or forcing function, w is solution of adjoint problem.

- Physical significance of w may not always be clear, but note it does provide an alternative means to compute the functional:

$$\begin{aligned} I[u] &= - < u, g > = - < u, \mathcal{L}^{\dagger} w > \\ &= - < \mathcal{L}u, w > = - < f, w > \end{aligned} \qquad (24)$$

In electromagnetics, this dual representation is usually a consequence of *reciprocity*, which also often implies that $\mathcal{L} = \mathcal{L}^{\dagger}$ ($\mathcal{L}$ is *self - adjoint*)

Adjoint Operator Examples

- $\mathcal{L}^{\dagger} V = \dfrac{d^2 V}{dx^2} + k_0^2 V \quad + \quad \text{BC}^{\dagger}\text{s}$
- $\mathcal{L}^{\dagger} q_S = \displaystyle\int_{\mathcal{S}} \frac{q_S(\mathbf{r}')}{4\pi\varepsilon \left|\mathbf{r} - \mathbf{r}'\right|} d\mathcal{S}'$
- $\mathcal{L}^{\dagger} \mathbf{J} = j\omega\mu \displaystyle\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}' - \frac{\nabla}{j\omega\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla' \cdot \mathbf{J}(\mathbf{r}')d\mathcal{S}' \quad + \quad \text{BC}^{\dagger}\text{s}$
- $\mathcal{L}^{\dagger} \mathbf{J} = \dfrac{\mathbf{J}(\mathbf{r})}{2} - \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \nabla \times \text{PV}\displaystyle\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}' \times \mathbf{J}(\mathbf{r}')\, d\mathcal{S}') \quad + \quad \text{BC}^{\dagger} \text{ (see Appendix)}$
- $\mathcal{L}^{\dagger} \Phi = \nabla^2 \Phi + k_0^2 \varepsilon_r \Phi \quad + \quad \text{BC}^{\dagger}\text{s}$
- $\mathcal{L}^{\dagger} \mathbf{E} = \nabla \times \mu_r^{-1} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} \quad + \quad \text{BC}^{\dagger}\text{s}$
- $\mathcal{L}^{\dagger} [x_m] = [L_{mn}]^{\mathrm{t}} [x_m]$

Most of the above operators are "self - adjoint!"

Bi-Variational Functional

- Define the *bivariational* functional **

$$I[\tilde{u},\ \tilde{w}] = <\mathcal{L}\tilde{u}, \tilde{w}> - <\tilde{u}, g> - <f, \tilde{w}> \qquad (25)$$

Note that $I[u, w] = \cancel{<\mathcal{L}u, w>} - <u, g> - \cancel{<f, w>} = I[u]$.

We regard $\tilde{u}$ and $\tilde{w}$ as approximate or *trial* solutions to the original and adjoint problems, respectively.

- Define *solution errors* in the original and adjoint problems as

$$\delta u = \tilde{u} - u, \quad \delta w = \tilde{w} - w. \qquad (26)$$

Then we can easily show that

$$I[\tilde{u},\ \tilde{w}] = - <u, g> + <\mathcal{L}\delta u, \delta w> \qquad (27)$$

or $\delta I[u, w] = <\mathcal{L}\delta u, \delta w>$ with *second order error* in δu and δw.
(Functional is said to be *stationary* or to have only a *second order variation* about the functions u and w.)

** Other, less general functionals may actually restrict the form of the resulting linear system, e.g., to Galerkin's method!

Rayleigh-Ritz Procedure

- Approximate u and w in terms of basis sets $\{u_n\}$ and $\{w_m\}$ as

$$\tilde{u} = \sum_n U_n u_n \tag{28}$$

$$\tilde{w} = \sum_m W_m w_m \,. \tag{29}$$

- Substitute above expansions into the bi - variational functional,

$$I[\tilde{u}, \tilde{w}] = \sum_m \sum_n W_m U_n < \mathcal{L} u_n, w_m > - \sum_n U_n < u_n, g > - \sum_m W_m < f, w_m > \tag{30}$$

and set $\partial I[\tilde{u}, \tilde{w}] / \partial W_p = \partial I[\tilde{u}, \tilde{w}] / \partial U_p = 0$ (*stationarity* condition).

- Replace dummy index p by m in first set, p by n in the second.

The surprising result is that …

. . . One Obtains Independent Moment Equations for Both the Original and the Adjoint Problems!

- (Moment equations for original problem, which are independent of g)

$$\sum_n < w_m, \mathcal{L}u_n > U_n = < w_m, f >, \quad m = 1, 2, \ldots, N \Rightarrow \left([L_{mn}][U_n] = [F_m] \right) \qquad (31)$$

- (Moment equations for original problem, which are independent of f)

$$\sum_m < \mathcal{L}u_n, w_m > W_m = < u_n, g >, \quad n = 1, 2, \ldots, N. \qquad (32)$$

(recall $< \mathcal{L}u_n, w_m > \; = \; < u_n, \mathcal{L}^\dagger w_m >$) $\Rightarrow \left([L_{mn}]^t [W_n] = [G_m] \right)$ (33)

- Note also the independence of equation sets (31) and (32), and the *reversed roles of basis and testing functions* in the adjoint problem.

Idea : Why not insert the resulting $\tilde{u}, \tilde{w}$ into the *variational* form possibly yielding more accurate results than substituting into the *non - variational* form of the functional?

Evaluation of Functional

Write bivariational functional as

$$I(\tilde{u}, \tilde{w}) = <\mathcal{L}\tilde{u}\,, \tilde{w}> - < f\,, \tilde{w}> \; - <\tilde{u}, g>$$
$$= <\mathcal{L}\tilde{u} - f\,, \tilde{w}> \; - <\tilde{u}, g>. \qquad (34)$$

The first term on the right hand side vanishes :

$$<\mathcal{L}\tilde{u} - f\,, \tilde{w}> = \sum_m W_m \left[\sum_n <\mathcal{L}u_n, w_m > U_n \; - < f\,, w_m > \right] = \; [W_m]^t \left([L_{mn}][U_n] - [F_m] \right)$$

$= 0.$ (by (31)) (35)

Hence

$$I(\tilde{u}, \tilde{w}) = \; - <\tilde{u}, g> \; = -\sum_n U_n < u_n, g > \qquad (36)$$

i.e.,

$$I(\tilde{u}) = I(\tilde{u}, \tilde{w}), \qquad (37)$$

so we obtain the same result using either the (bi-)variational or non - variational forms of the functional!

Equivalence of MoM and Variational Approach

- $\tilde{u}$ can be determined from the MoM equations independent of w and g. The solution is same as that obtained by the (bi-)variational approach.

- $I[\ \tilde{u}\ ] = I[\tilde{u}, \tilde{w}]$ independent of g.

- The variational approach (and adjoint problem) is useful in proving stationarity, but seems otherwise largely superfluous in arriving at a numerical formulation. The moment method yields the same solution but is generally simpler to apply. The variational problem tells us...
  - Moment method solutions are *automatically stationary*.
  - Error in $I[\tilde{u}, \tilde{w}]$ ($\delta I[u, w] = \ < \mathcal{L}\delta u, \delta w >$) is proportional to that in both u, w; so to reduce error, *choose* w_m *to well.approximate* w, the solution of the adjoint problem.

Appendix: Derivation of MFIE Adjoint Operator

- $$\mathcal{L}\mathbf{J} = \frac{\mathbf{J}(\mathbf{r})}{2} - \hat{\mathbf{n}} \times \mathrm{PV}\int_{\mathcal{S}} \nabla G(\mathbf{r},\mathbf{r}') \times \mathbf{J}(\mathbf{r}')\, d\mathcal{S}' = \left(\hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{inc}}\right)$$

We first write the MFIE in the non - standard form

$$-\hat{\mathbf{n}} \times \mathcal{L}\mathbf{J} = \underbrace{-\hat{\mathbf{n}} \times \frac{\mathbf{J}(\mathbf{r})}{2} + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \nabla \times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\,d\mathcal{S}')}_{\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\mathbf{H}^{S^-}\right) = -\mathbf{H}^{S^-}_{\tan} = -\frac{1}{\mu}\lim_{\mathbf{r}\uparrow\mathcal{S}} \nabla\times\mathbf{A}} = -\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{inc}}\right)$$

Then for a tangential surface testing vector $\mathbf{M}(\mathbf{r})$, we have that

$$< \mathbf{M}, -\hat{\mathbf{n}} \times \frac{\mathbf{J}(\mathbf{r})}{2} + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \nabla \times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\,d\mathcal{S}') >$$

which is of the form $< \mathbf{M}, -\lim_{\mathbf{r}\uparrow\mathcal{S}} \mathbf{H}[\mathbf{J}] >$ and which can be interpreted as a reaction intgral.

Hence, by the reaction theorem, we have

$$< \mathbf{M}, -\lim_{\mathbf{r}\uparrow\mathcal{S}} \mathbf{H}[\mathbf{J}] > \; = \; < \mathbf{J}, \lim_{\mathbf{r}\downarrow\mathcal{S}} \mathbf{E}[\mathbf{M}] > \; = \; < \mathbf{J}, \underbrace{\hat{\mathbf{n}} \times \frac{\mathbf{M}(\mathbf{r})}{2} + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \nabla \times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{M}(\mathbf{r}')\,d\mathcal{S}')}_{-\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\mathbf{E}^{S^+}\right) = \mathbf{E}^{S^+}_{\tan} = -\frac{1}{\varepsilon}\lim_{\mathbf{r}\downarrow\mathcal{S}} \nabla\times\mathbf{F}} >$$

Appendix: Derivation of MFIE Adjoint Operator, cont'd

Thus,

$$\boxed{<\mathbf{M}, -\hat{\mathbf{n}}\times\frac{\mathbf{J}(\mathbf{r})}{2}+\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\nabla\times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}')> \;=\; <\mathbf{J}, \hat{\mathbf{n}}\times\frac{\mathbf{M}(\mathbf{r})}{2}+\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\nabla\times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{M}(\mathbf{r}')d\mathcal{S}')>}$$

and even though $\mathbf{M}\cdot\left(-\hat{\mathbf{n}}\times\frac{\mathbf{J}(\mathbf{r})}{2}\right)=\mathbf{J}\cdot\left(\hat{\mathbf{n}}\times\frac{\mathbf{M}(\mathbf{r})}{2}\right)$, the operator $-\hat{\mathbf{n}}\times\mathcal{L}$ is non-self-adjoint.

For the original operator, set $\mathbf{M}=-\hat{\mathbf{n}}\times\mathbf{w}$ and the above becomes

$$\boxed{<\mathbf{w}, \frac{\mathbf{J}(\mathbf{r})}{2}-(\hat{\mathbf{n}}\times\nabla\times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}')> \;=\; <\mathbf{J}, \frac{\mathbf{w}(\mathbf{r})}{2}-\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\nabla\times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}'\times\mathbf{w}(\mathbf{r})\,d\mathcal{S}')>}$$

and the adjoint operator is

$$\boxed{\mathcal{L}^{\dagger}\mathbf{w}=\frac{\mathbf{w}(\mathbf{r})}{2}-\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\nabla\times \mathrm{PV}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\hat{\mathbf{n}}'\times\mathbf{w}(\mathbf{r})\,d\mathcal{S}')}$$