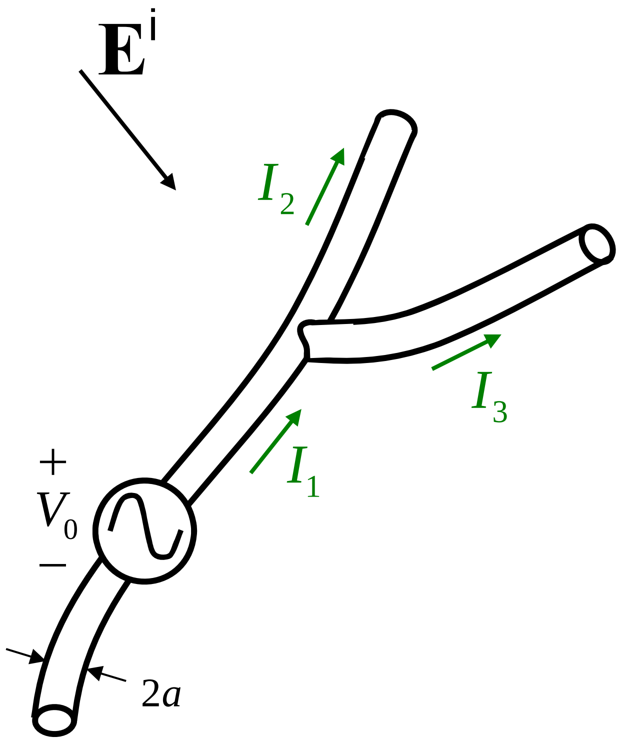
Thin Wire Modeling

Donald R. Wilton

Nathan Champagne

Thin Wire Assumptions



- Current has only an axial component
- Current is azimuthally invariant

$$\Rightarrow \quad a \leq 0.01\lambda$$

- Kirchhoff's law applies at junctions: $I_1 = I_2 + I_3$
- Current vanishes at wire ends

$$I(\mathbf{r}) \equiv \int_0^{2\pi} \mathbf{J}(\mathbf{r}+a\hat{\boldsymbol{\rho}})\cdot\hat{\ell}\, a\, d\phi \;\approx\; 2\pi a\, J_\ell(\mathbf{r}),$$

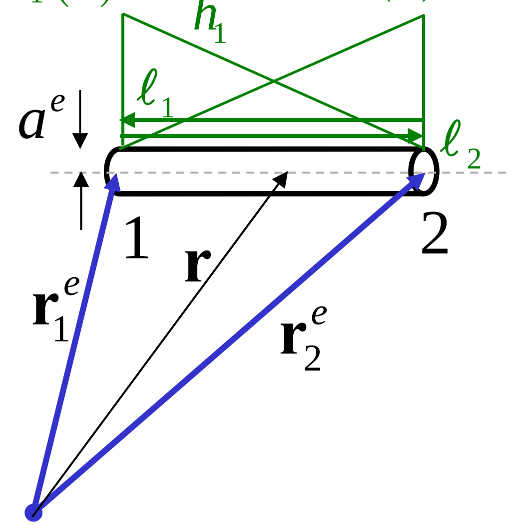
$\mathbf{r}$ on wire axis

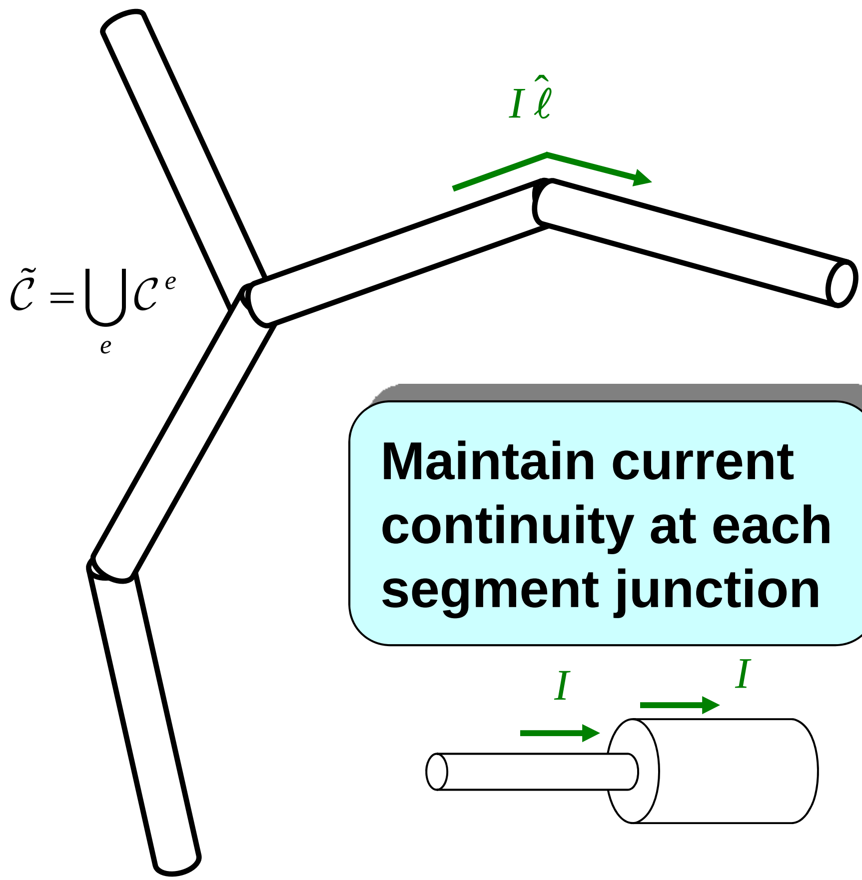
A Wire Is Modeled as Collection of Straight, Thin Tubular Conductors

$$I\,\hat{\ell} = \sum_{n=1}^{N} I_n \Lambda_n(\mathbf{r}), \ \ \mathbf{r} \in \tilde{\mathcal{C}} \qquad \text{(global)}$$

$$= I_1^e\, \Lambda_1^e(\mathbf{r}) + I_2^e \Lambda_2^e(\mathbf{r}), \ \ \mathbf{r} \in \mathcal{C}^e \quad \text{(local)}$$

$$\Lambda_1^e(\mathbf{r}) = \frac{\xi_1 \ell_1}{h_1} \qquad \Lambda_2^e(\mathbf{r}) = \frac{\xi_2 \ell_2}{h_2}$$





Maintain current continuity at each segment junction

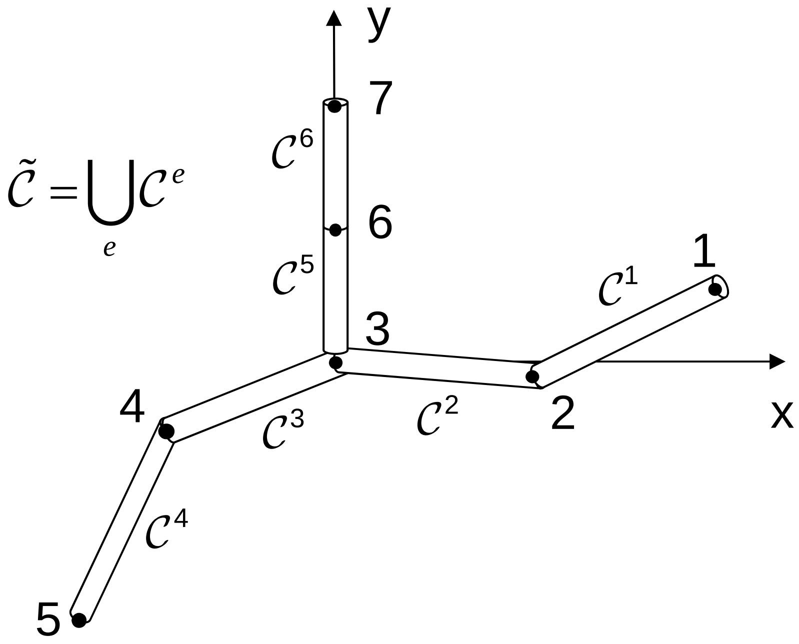
Wire axis parameterization:

$$\mathbf{r} = \xi_1 \mathbf{r}_1^e + \xi_2 \mathbf{r}_2^e,$$

$$\xi_1 + \xi_2 = 1$$

Discretization and Geometry Data Structure

Discretized wire structure

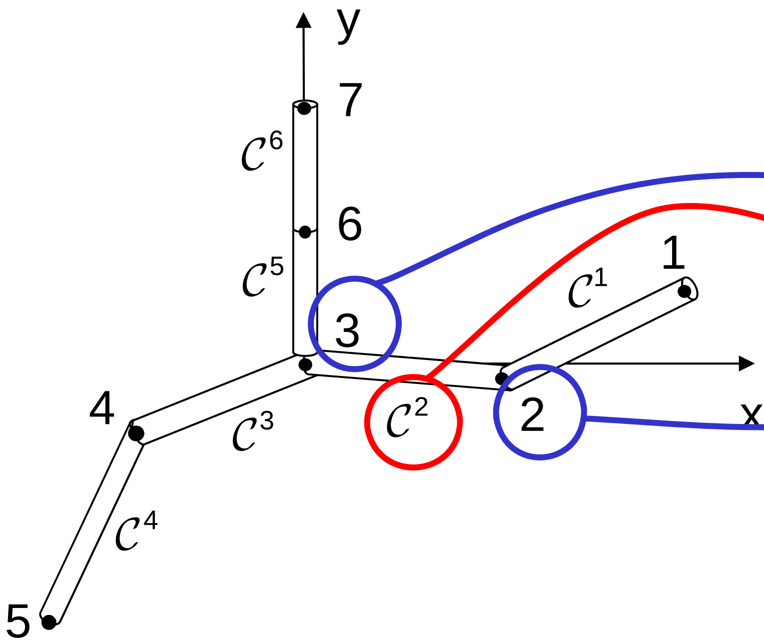


Data structure for element nodes

Global Node Number	Coordinates		
	x	y	z
1	1.0000	0.1500	0.1000
2	0.5000	- 0.0500	0.0500
3	0.0000	0.0000	0.0000
4	-0.5000	-0.1500	0.0000
5	-0.8000	-0.8000	0.0000
6	0.0000	0.4000	0.0000
7	0.0000	0.8000	0.0000

Element Connectivity Data Structure

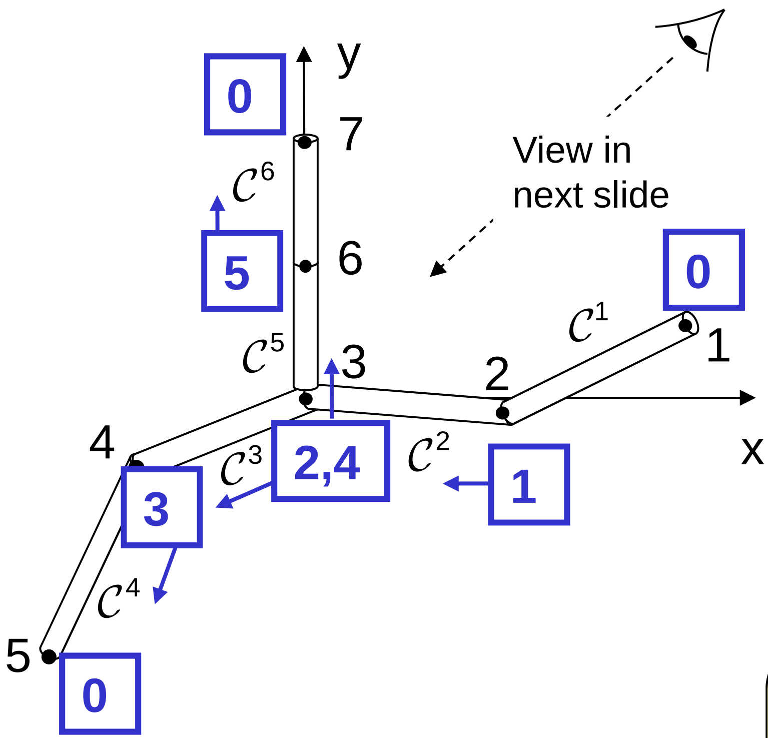
Element to node mapping



Element Number	Global Node Numbers	
	Local Node 1	Local Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	3	6
6	6	7

- In addition to the connection data, we associate a radius a^e with each segment

Element DoF Data

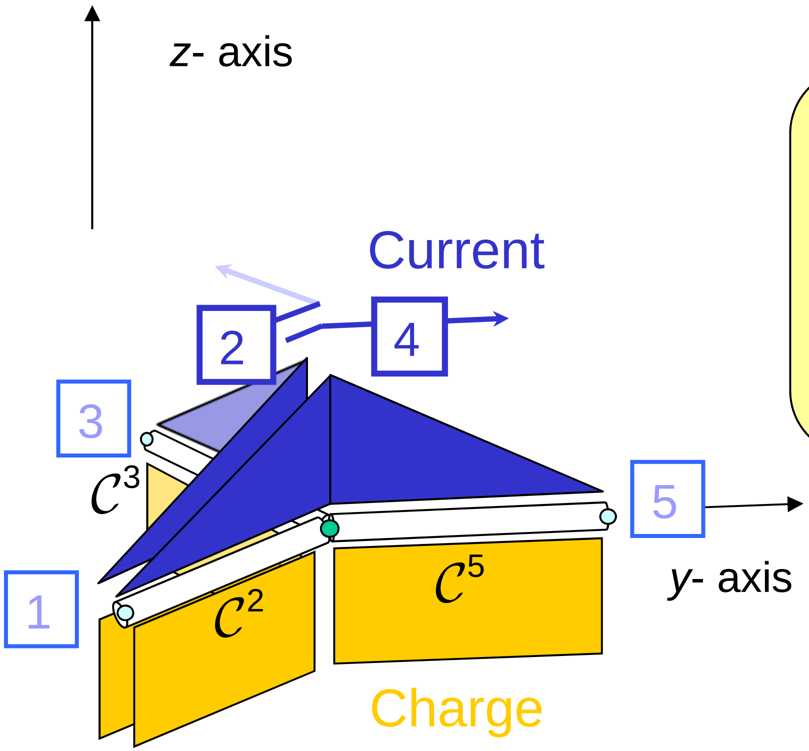


Local Element-to-Global DoF mapping

e	Local Indices, Element *e*			
	1		2	
	# DoF's	DoF Index	# DoF's	DoF Index
1	0	0	1	+1
2	1	-1	2	+2,+4
3	1	-2	1	+3
4	1	-3	0	0
5	1	-4	1	+5
6	1	-5	0	0

$$\sigma_i^e = \begin{cases} +1 \text{ if sign of global DoF corresponding to} \\ \quad i \text{ th DoF of element } e \text{ is positive,} \\ \\ -1 \text{ if sign of global DoF corresponding to} \\ \quad i \text{ th DoF of element } e \text{ is negative} \end{cases}$$

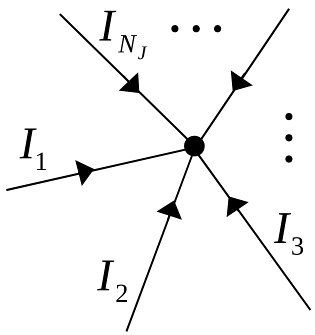
KCL Easily Enforced Using PWL Bases



Current out of $\mathcal{C}^2$ at junction $= \; I_2 + I_4$

$$I_1 + I_2 + I_3 + \cdots + I_{N_J} = 0$$

- N_J -1 independent currents at a junction of N_J line segments
- Select independent bases in N_J -1 arms of junction and overlap them onto remaining arm



EFIE (Pocklington) Formulation for Total Wire Current

Applying boundary condition

$$-\mathbf{E}^s \cdot \hat{\ell} = \mathbf{E}^i \cdot \hat{\ell}$$

on the wire surface leads to the EFIE with

moment equation $\left[Z_{mn}\right][I_n] = [V_m]$ where $[V_m] = [< \Lambda_m ; \mathbf{E}^\mathrm{i} >]$,

$$\left[Z_{mn}\right] = j\omega\mu\left[< \Lambda_m ; K, \Lambda_n >\right] + \frac{1}{j\omega\varepsilon}\left[< \nabla\cdot\Lambda_m , K, \nabla\cdot\Lambda_n >\right]$$

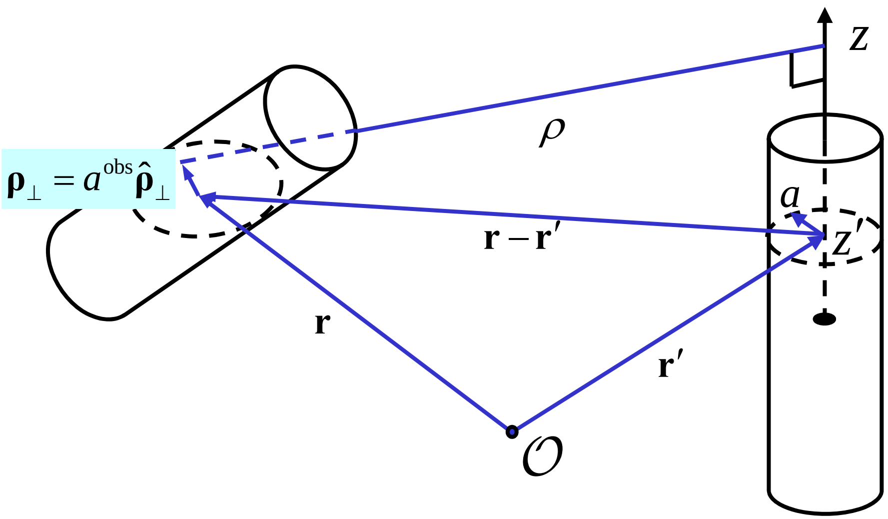
with element matrix

Obs. pt. integral is usually reduced to a one-point angular integration

$$\left[Z_{ij}^{ef}\right] = j\omega\mu\left[< \Lambda_i^e ; K, \Lambda_j^f >\right] + \frac{1}{j\omega\varepsilon}\left[< \nabla\cdot\Lambda_i^e , K, \nabla\cdot\Lambda_j^f >\right]$$

and element excitation vector $[< \Lambda_i^e ; \mathbf{E}^\mathrm{i} >]$.

Test Segment Geometry Details



- We choose the test segment observation point as a point a test segment radius a^{obs} away from the test segment axis and perpendicular to the plane defined by the source wire axis $\hat{\ell}'$ ($\hat{\mathbf{z}}$ in the local coordinate shown), and the vector $\mathbf{r}-\mathbf{r}'$. Hence,

$$\mathbf{r}^{\mathrm{obs}} = \mathbf{r} \pm \boldsymbol{\rho}_\perp, \quad \boldsymbol{\rho}_\perp = a^{\mathrm{obs}}\hat{\boldsymbol{\rho}}_\perp$$

where $\hat{\boldsymbol{\rho}}_\perp = \dfrac{(\mathbf{r}-\mathbf{r}')\times\hat{\ell}'}{\left|(\mathbf{r}-\mathbf{r}')\times\hat{\ell}'\right|}$

Element Matrix Derivation Details

Note for $\hat{\boldsymbol{\rho}} \approx \hat{\boldsymbol{\rho}}_\perp$ a unit vector perpendicular to both the source wire axis and the component of $\mathbf{r}-\mathbf{r}'$ transverse to the source wire axis,

$$Z_{ij}^{ef} = j\omega\mu \int_{C^e}\int_0^{2\pi}\left(\int_{C^f}\int_0^{2\pi}\frac{\boldsymbol{\Lambda}_i^e(\mathbf{r})\cdot\boldsymbol{\Lambda}_j^f(\mathbf{r}')}{(2\pi a^e)(2\pi a^f)}G(\mathbf{r}+a^e\hat{\boldsymbol{\rho}},\mathbf{r}'+a^f\hat{\boldsymbol{\rho}}')a^f\,d\phi'\,d\ell'\right)a^e\,d\phi\,d\ell$$

$$+\frac{1}{j\omega\varepsilon}\int_{C^e}\int_0^{2\pi}\left(\int_{C^f}\int_0^{2\pi}\frac{\nabla\cdot\boldsymbol{\Lambda}_i^e(\mathbf{r})\;\nabla\cdot\boldsymbol{\Lambda}_j^f(\mathbf{r}')}{(2\pi a^e)(2\pi a^f)}G(\mathbf{r}+a^e\hat{\boldsymbol{\rho}},\mathbf{r}'+a^f\hat{\boldsymbol{\rho}}')a^f\,d\phi'\,d\ell'\right)a^e\,d\phi\,d\ell$$

$$\approx j\omega\mu\int_{C^e}\int_{C^f}\left(\frac{1}{2\pi}\int_0^{2\pi}G(\mathbf{r}+a^e\hat{\boldsymbol{\rho}}_\perp,\mathbf{r}'+a^f\hat{\boldsymbol{\rho}}')\,d\phi'\right)\boldsymbol{\Lambda}_i^e(\mathbf{r})\cdot\boldsymbol{\Lambda}_j^f(\mathbf{r}')\,d\ell'\,d\ell$$

$$+\frac{1}{j\omega\varepsilon}\int_{C^e}\int_{C^f}\left(\frac{1}{2\pi}\int_0^{2\pi}G(\mathbf{r}+a^e\hat{\boldsymbol{\rho}}_\perp,\mathbf{r}'+a^f\hat{\boldsymbol{\rho}}')\,d\phi'\right)\nabla\cdot\boldsymbol{\Lambda}_i^e(\mathbf{r})\,\nabla\cdot\boldsymbol{\Lambda}_j^f(\mathbf{r}')\,d\ell'\,d\ell$$

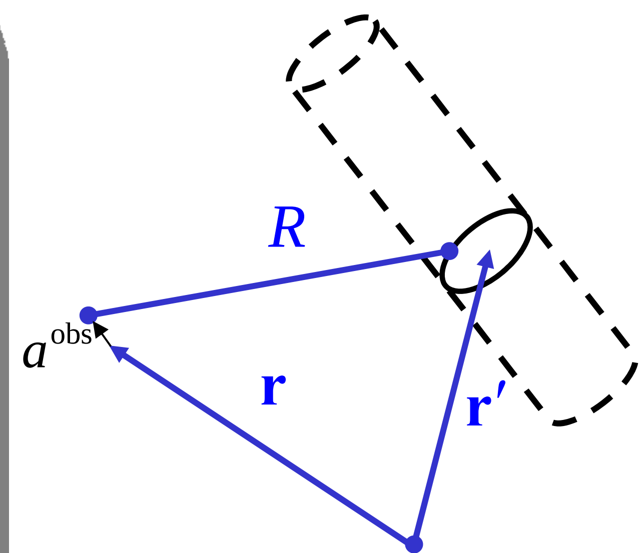
$$= j\omega\mu<\boldsymbol{\Lambda}_i^e;K,\boldsymbol{\Lambda}_j^f>+\frac{1}{j\omega\varepsilon}<\nabla\cdot\boldsymbol{\Lambda}_i^e,K,\nabla\cdot\boldsymbol{\Lambda}_j^f>$$

Kernel for Thin Wire Integral Equation

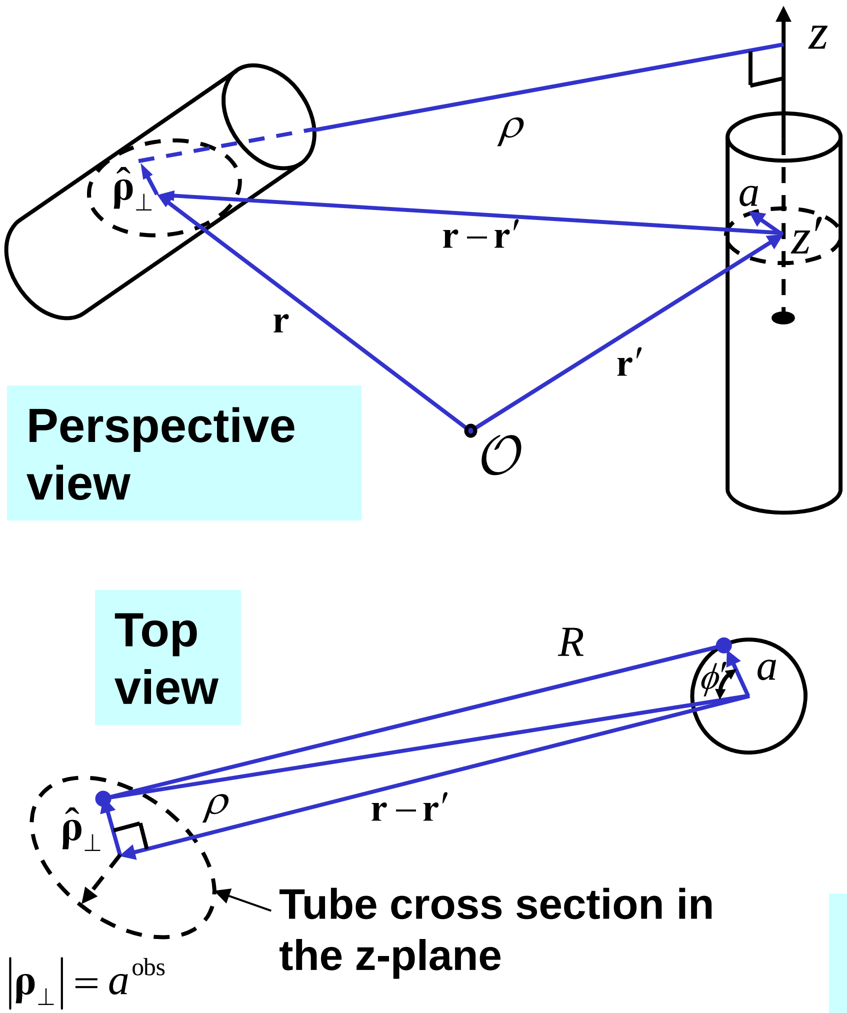
Kernel is the potential at $\mathbf{r}$ produced by a ring source of radius a centered along the wire axis.

$$K(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi \cancel{a}} \int_{-\pi}^{\pi} \frac{e^{-jkR}}{4\pi R} \cancel{a}\, d\phi' = \frac{1}{\pi} \int_{0}^{\pi} \frac{e^{-jkR}}{4\pi R} d\phi',$$

Geometry is simply described if a local coordinate system is introduced with tube along the z - axis.



Geometry Relating Source and Observation Wire Segments



- $\mathbf{r}$ is the vector to the observation segment axis
- $\mathbf{r}'$ is the vector to the source segment axis

Define in local source segment coords.:

$$z - z' = \hat{\ell}\cdot(\mathbf{r}-\mathbf{r}')$$

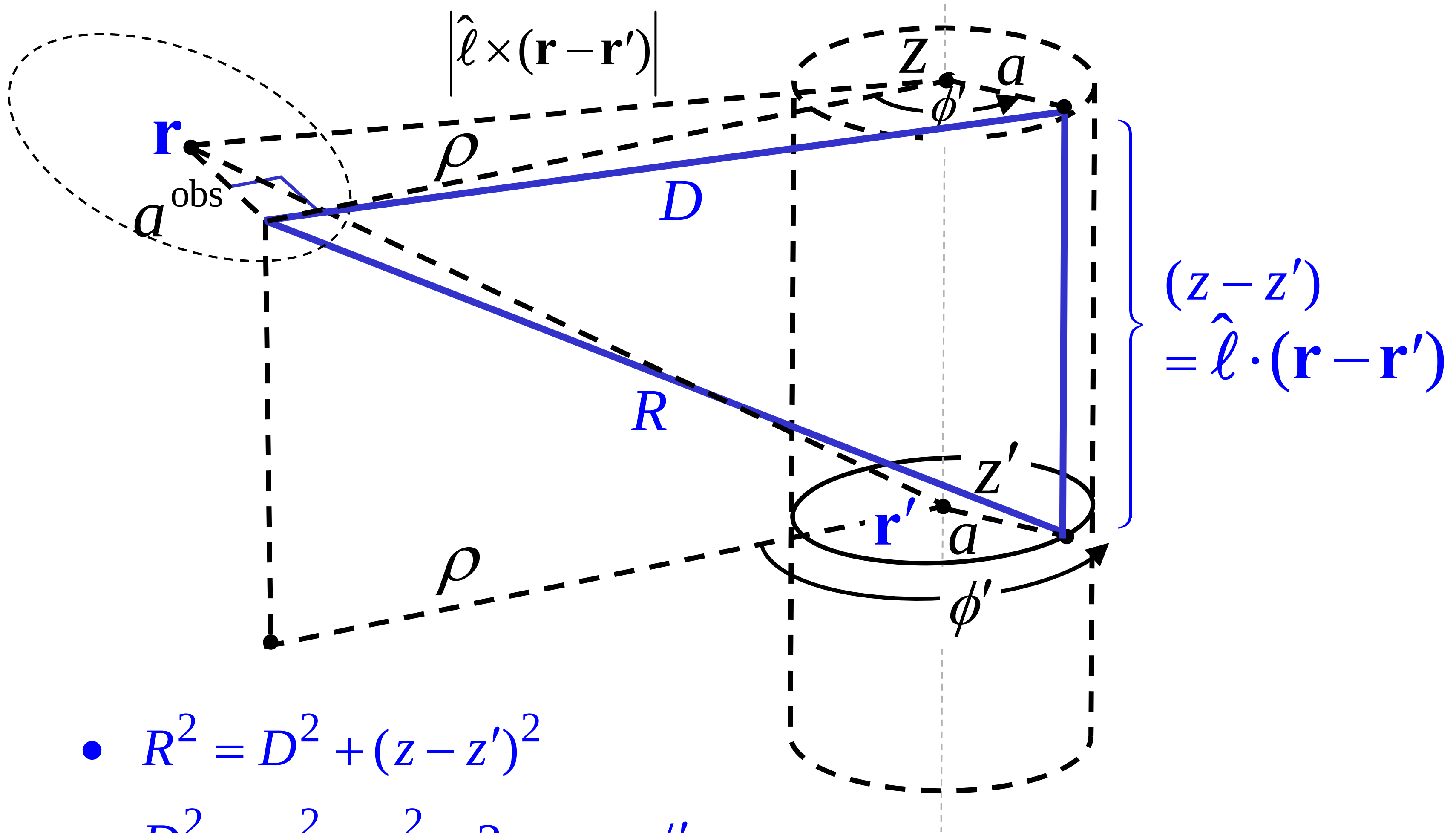
$$\rho^2 = (a^{\text{obs}})^2 + \left|\hat{\ell}\times(\mathbf{r}-\mathbf{r}')\right|^2$$

$$R^2 = (z-z')^2 + \underbrace{\rho^2 + a^2 - 2\rho a\cos\phi'}_{\text{Law of cosines}}$$

$$= \left|\mathbf{r}-\mathbf{r}'\right|^2 + (a^{\text{obs}})^2 + a^2 - 2\rho a\cos\phi'$$

- **Note offset observation point is not necessarily on wire surface unless wires are parallel, but yields correct self terms !**

Tube Segment Geometry



- $R^2 = D^2 + (z-z')^2$
- $D^2 = \rho^2 + a^2 - 2\rho a \cos\phi'$
- $\rho^2 = \left(a^{\mathrm{obs}}\right)^2 + \left|\hat{\ell}\times(\mathbf{r}-\mathbf{r}')\right|^2$

Tube Segment Geometry, cont'd

$$R = \sqrt{(z-z')^2 + \rho^2 + a^2 - 2\rho a \cos\phi'}$$
$$= \sqrt{(z-z')^2 + (\rho + a)^2 - 4\rho a \sin^2\alpha}$$
$$= R_{\max}\sqrt{1-\beta^2 \sin^2\alpha}, \qquad \left(\alpha = \frac{\pi - \phi'}{2}\right)$$

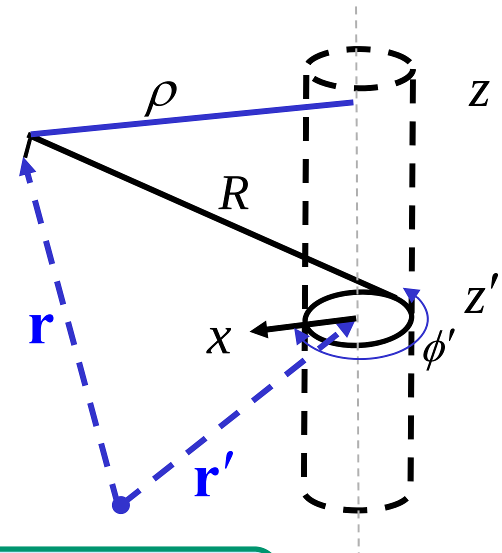
where

$$R_{\max} = \sqrt{(z-z')^2 + (\rho + a)^2}, \;\; \beta^2 = \frac{4\rho a}{R_{\max}^2}$$

hence the kernel becomes

$$K(z,z') = \frac{2}{\pi R_{\max}} \int_0^{\pi/2} \frac{e^{-jk R_{\max}\sqrt{1-\beta^2 \sin^2\alpha}}}{4\pi\sqrt{1-\beta^2 \sin^2\alpha}} d\alpha$$

$$\cos\phi' = \cos(\pi - 2\alpha)$$
$$= -\cos 2\alpha$$
$$= 2\sin^2\alpha - 1$$



Scalar potential of a ring source

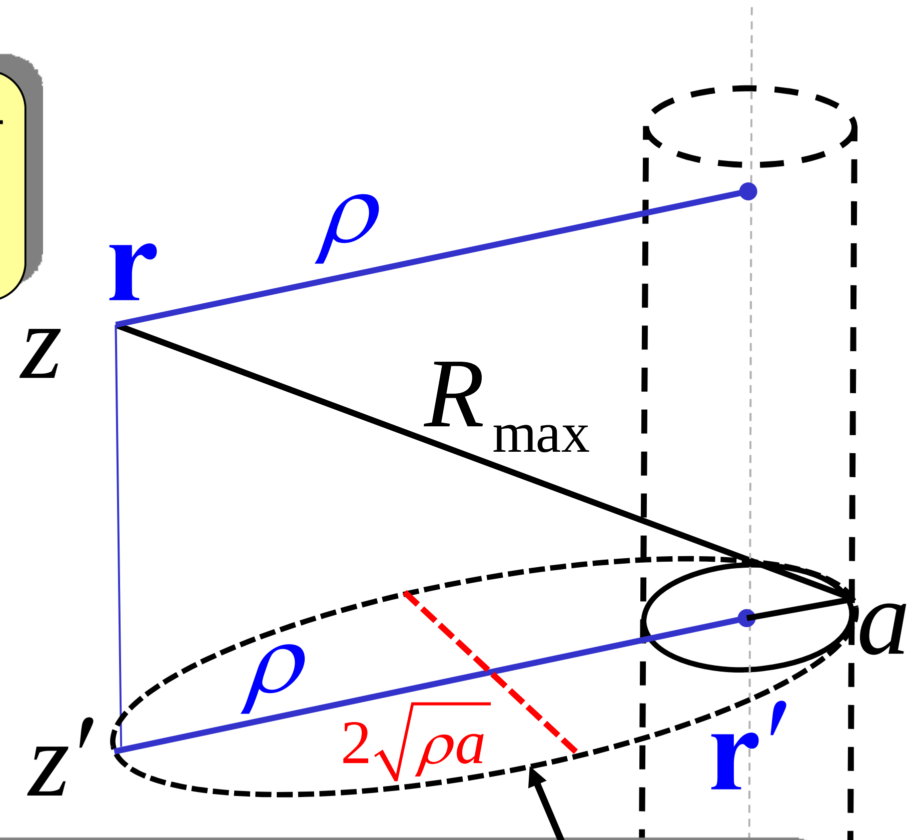
Tube SegmentGeometry, cont'd

$$R_{\max} = \sqrt{(z - z')^2 + (\rho + a)^2}$$

$$\beta^2 = \frac{4\rho a}{R_{\max}^2}$$

Geometric Mean of ρ and a

$$\beta = \frac{2\overbrace{\sqrt{\rho a}}}{R_{\max}} \xrightarrow[\rho \to a,\, z' \to z]{} 1$$



Ellipse with focus at wire center,

$a =$ distance from focus to nearest pt. on ellipse.

$\rho =$ distance from focus to furthest pt. on ellipse

$\rho + a \;\; =$ major axis of ellipse

$2\sqrt{\rho a} =$ minor axis of ellipse

Transform Kernel to Cancel Singularity

Let $u = F(\alpha,\beta) = \int_0^{\alpha} \frac{d\alpha}{\sqrt{1-\beta^2 \sin^2 \alpha}}$,

incomplete elliptic integral of the first kind

$\Rightarrow du = \frac{\partial}{\partial \alpha} F(\alpha,\beta)\, d\alpha = \frac{d\alpha}{\sqrt{1-\beta^2 \sin^2 \alpha}}$

Thus the kernel becomes

$$\boxed{\begin{aligned} K(z,z') &= \frac{2}{4\pi^2 R_{\max}} \int_0^{K(\beta)} e^{-jk R_{\max} \sqrt{1-\beta^2 \operatorname{sn}^2 u}}\, du \\ &= \frac{2}{4\pi^2 R_{\max}} \int_0^{K(\beta)} e^{-jkR_{\max} \operatorname{dn} u}\, du , \end{aligned}}$$

$F(\frac{\pi}{2},\beta) = K(\beta)$ (complete elliptic integral of the first kind)

$$\left.\begin{aligned} \operatorname{sn} u &= \sin \alpha, \\ \operatorname{dn} u &= \sqrt{1-\beta^2 \operatorname{sn}^2 u} \end{aligned}\right\}$$ (Jacobi elliptic functions)

Kernel Evaluation

- Hence

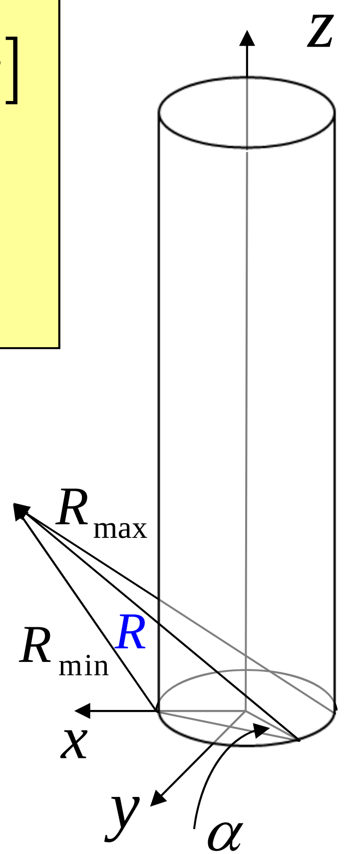
$$K(z,z') = \frac{2}{4\pi^2 R_{\max}} \int_0^{K(\beta)} e^{-jk\overbrace{R_{\max}\mathrm{dn}\,u}^{R}}\,du \qquad \text{[Let } u = K(\beta)\eta\text{]}$$

$$\approx \frac{2K(\beta)}{4\pi^2 R_{\max}} \sum_{k'} w_{k'} e^{-jkR_{\max}\mathrm{dn}\left(\eta^{(k')}K(\beta)\right)}$$

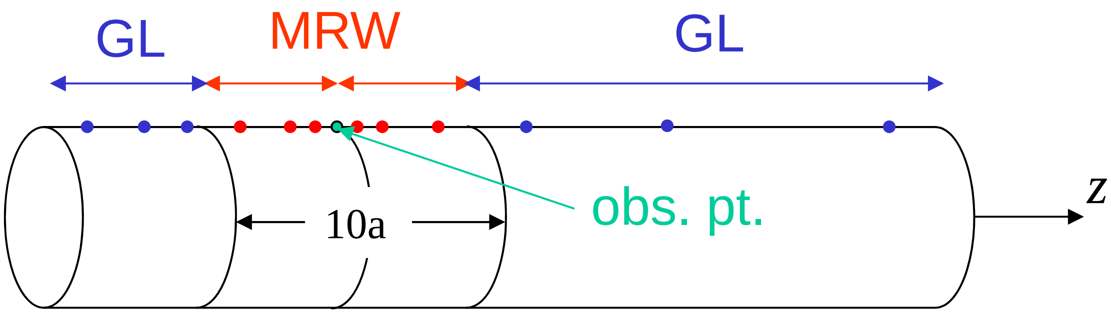
- $w_k, \eta^{(k)}$ are Gauss-Legendre weights and samples
- Integrand above is *almost constant* for thin wires, and

$$K(\beta) \xrightarrow[z\to z',\,\rho\to a,\,\beta\to 1]{} P(z-z')\ln|z-z'| + Q(z-z'),$$

$$\beta \xrightarrow[R\to\infty]{} 0, \qquad \frac{1}{R_{\max}} \xrightarrow[R\to\infty]{} \frac{1}{R}$$



Self-Term Integration Along Wire Axis



- Ma-Rokhlin-Wandzura (MRW) quadrature integrates $P_n(z)\ln|z| + Q_n(z)$ exactly

  J. Ma, V. Rokhlin, and S. Wandzura, *SIAM J. Numer. Anal.* 33, 1996, pp. 971–996.

- Gauss-Legendre (GL) quadrature *after* substituting

$$du = \frac{dz'}{R} \Rightarrow \; u = -\sinh^{-1}\frac{z - z'}{R},$$

to smooth the integrand

- Non-self segment contributions are often simply modeled as line sources

Tables of Sample Points and Weights for MRW Quadrature

Table 4 Sample points and weighting coefficients for K-point quadratures of form $\int_0^1 f(\xi_1)\, d\xi_1 \approx \sum_{k=1}^{K} w_k f(\xi_1^{(k)})$ where $f(\xi_1)$ has a logarithmic singularity at $\xi_1 = 0$. *

Sample Points, $\xi_1^{(k)}$	Weights, w_k
K=1:	
0.367879441171442	1.000000000000000
K=2:	
0.882968651376531 $\times 10^{-1}$	0.298499893705525
0.675186490909887	0.701500106294475
K=3:	
0.288116625309523 $\times 10^{-1}$	0.103330707964930
0.304063729612140	0.454636525970100
0.811669225344079	0.442032766064970
K=5:	
0.565222820508010 $\times 10^{-2}$	0.210469457918546 $\times 10^{-1}$
0.734303717426523 $\times 10^{-1}$	0.130705540744447
0.284957404462558	0.289702301671314
0.619482264084778	0.350220370120399
0.915758083004698	0.208324841671986

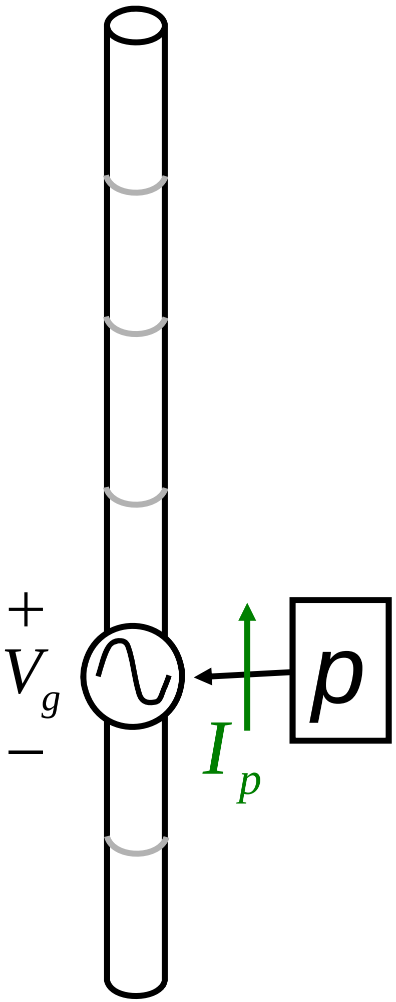
***Ma, Rokhlin,Wandzura *SIAM J. Numer. Anal.* 33, 1996**

Thin Wire Kernel Evaluation

- Wilton, D.R. and N.J. Champagne, "Evaluation and integration of the thin wire kernel," *IEEE Trans. Antennas and Propagat.,* **54**, 4, pp. 1200 – 1206, April 2006.

- Champagne, N. J., D. R. Wilton, J. D. Rockway, "The Analysis of Thin Wires Using Higher Order Elements and Basis Functions," *IEEE Trans. Antennas and Propagat.,* **54**, 12, pp. 3815 – 3821, Dec. 2006.

Modeling Voltage Sources



- Current maintains a positive voltage V_g if references are as shown at DoF p.
- Change sign of V_g if its reference is opposite the current reference

p **th row**

$$[Z_{mn}][I_n] = \begin{bmatrix} 0 \\ \vdots \\ V_g \\ \vdots \\ 0 \end{bmatrix} = \left[\delta_{mp} V_g\right]$$

Kronecker delta:

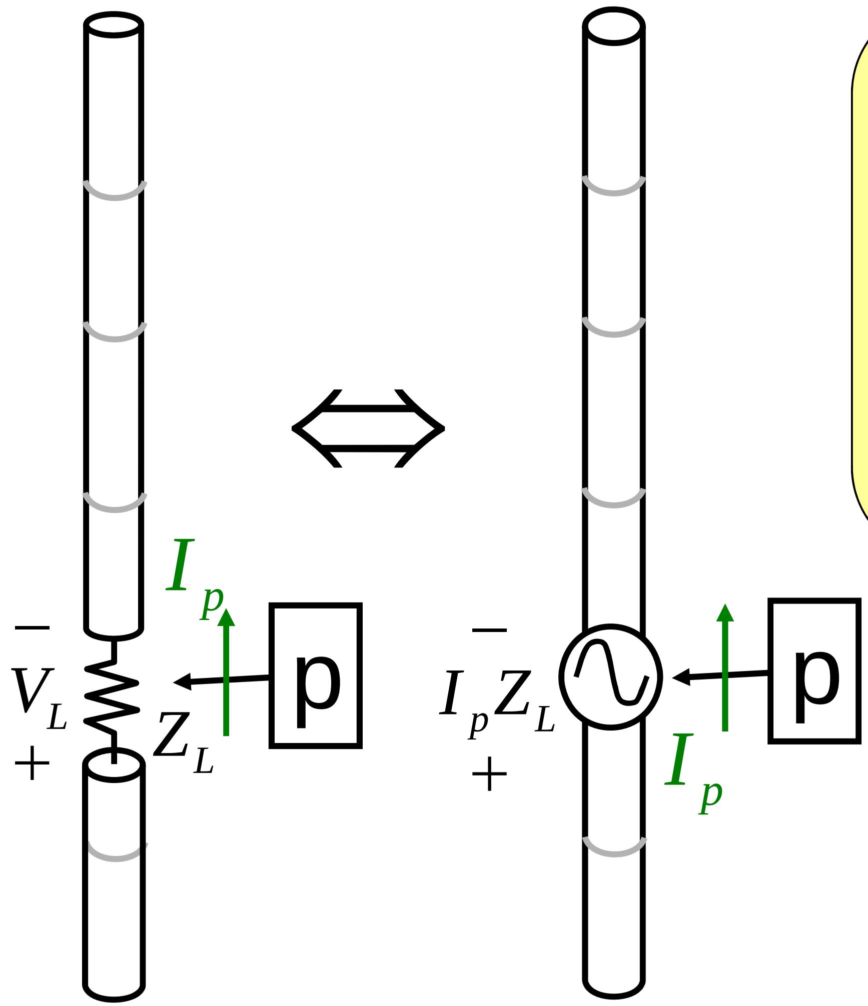
$$\delta_{mp} = \begin{cases} 1, & m = p \\ 0, & m \neq p \end{cases}$$

Experience with Voltage Source Models and Input Impedance Evaluation

- Input *conductance* converges as number of subdomains increases; *susceptance* does not due to knife-edge capacitance

- Reasonable susceptance can often be obtained using finite subdomains, but better results are obtained with distributed source or other improved feed models

Modeling Lumped Loads



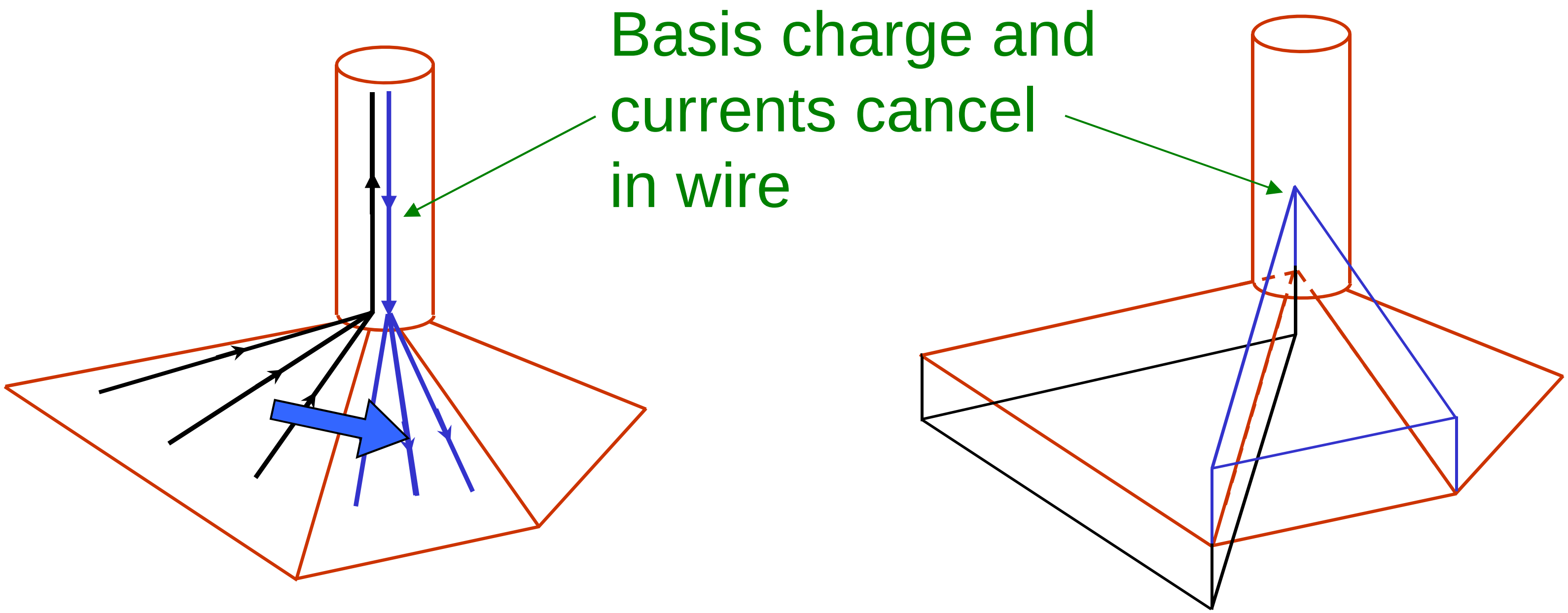
$$[Z_{mn}][I_n] = \begin{bmatrix} \vdots \\ -I_p Z_L \\ \vdots \end{bmatrix}$$

$$\Rightarrow \left[Z_{mn} + \delta_{mp}\delta_{np} Z_L\right][I_n] = [\vdots]$$

Kronecker delta:

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Ill-Conditioning at Junctions: Junction Basis Pairs $\Leftrightarrow$ Surface Bases



Junction and surface bases have same dipole moments $\Rightarrow$ nearly equal vector potentials

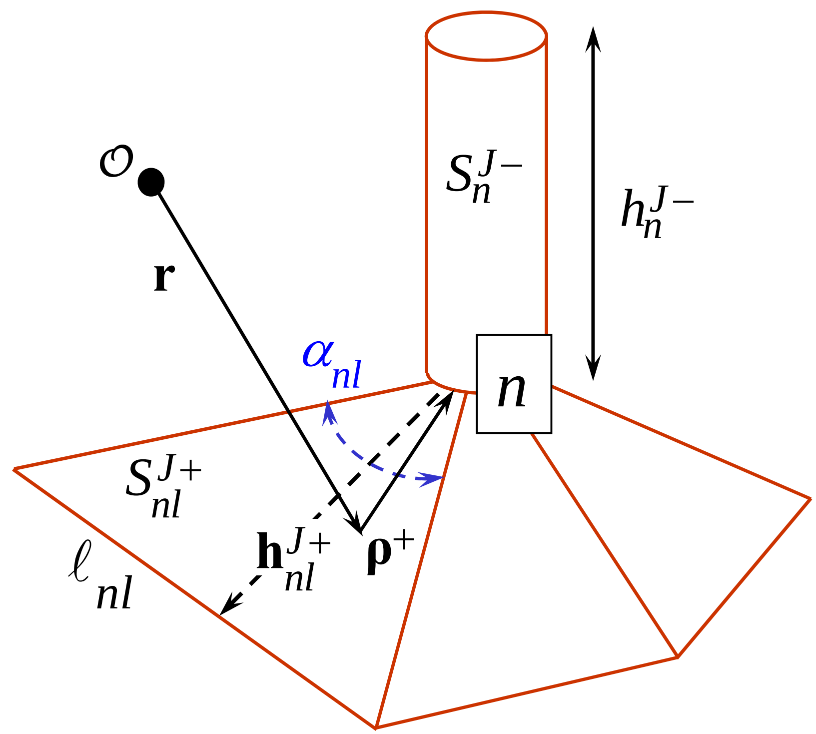
Junction and surface bases have same charge density $\Rightarrow$ equal scalar potentials

Junction Basis Definition

$$\mathbf{\Lambda}_n^J(\mathbf{r}) = \begin{cases} K_{nl}\left[1 - \dfrac{\left(h_{nl}^{J+}\right)^2}{\left(\boldsymbol{\rho}^+ \cdot \hat{\mathbf{h}}_{nl}^{J+}\right)^2}\right]\mathbf{\Lambda}_{nl}^B(\mathbf{r}), & \mathbf{r} \text{ on } S_{nl}^{J+} \\ \mathbf{\Lambda}_n^W(\mathbf{r}), & \mathbf{r} \text{ on } S_n^{J-} \\ 0, & \text{otherwise} \end{cases}$$

$$K_{nl} = \frac{\alpha_{nl}}{\ell_{nl} \sum_{l=1}^{N_{Jn}} \alpha_{nl}} = \frac{\alpha_{nl}}{\ell_{nl}\,\alpha_n^t}$$

$$\nabla_S \cdot \mathbf{\Lambda}_n^J(\mathbf{r}) = \begin{cases} \dfrac{2K_{nl}}{h_{nl}^{J+}}, & \mathbf{r} \text{ on } S_{nl}^{J+} \\ -\dfrac{1}{h_n^{J-}}, & \mathbf{r} \text{ on } S_{nl}^{J-} \\ 0, & \text{otherwise} \end{cases}$$



Local Junction Basis Definition

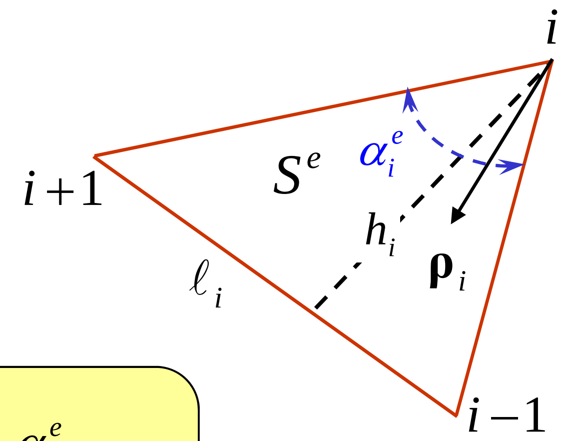
$$\Lambda_i^{e,J}(\mathbf{r}) = K_i^e \hat{\boldsymbol{\rho}}_i \left[\frac{\rho_i}{h_i} - \underbrace{\frac{h_i^{\cancel{2}}}{\rho_i^{\cancel{2}} \cos^2\phi}}_{\text{divergenceless!}} \left(\cancel{\frac{\rho_i}{h_i}} \right) \right], \quad \nabla \cdot \Lambda_i^{e,J}(\mathbf{r}) = \frac{2K_i^e}{h_i}$$

$$K_i^e = \frac{\alpha_i^e}{\ell_i \sum\limits_{\substack{e \text{ attached} \\ \text{to junction}}} \alpha_i^e}$$

Since $\hat{\boldsymbol{\rho}}_i \cdot \hat{\mathbf{h}}_i = \cos\phi$, and $\rho_i \big|_{\xi_i = 0} = \dfrac{h_i}{\cos\phi}$, . . . ,

flux density out of edge i is

$$\Lambda_i^{e,J}(\mathbf{r}) \cdot \hat{\mathbf{h}}_i \Big|_{\xi_i = 0} = K_i^e \cos\phi \left[\frac{\cancel{h_i}}{\cancel{h_i} \cos\phi} - \frac{\cancel{h_i} \cancel{\cos\phi}}{\cancel{h_i} \cos^{\cancel{2}}\phi} \right] = 0$$



Net flux out of vertex i :

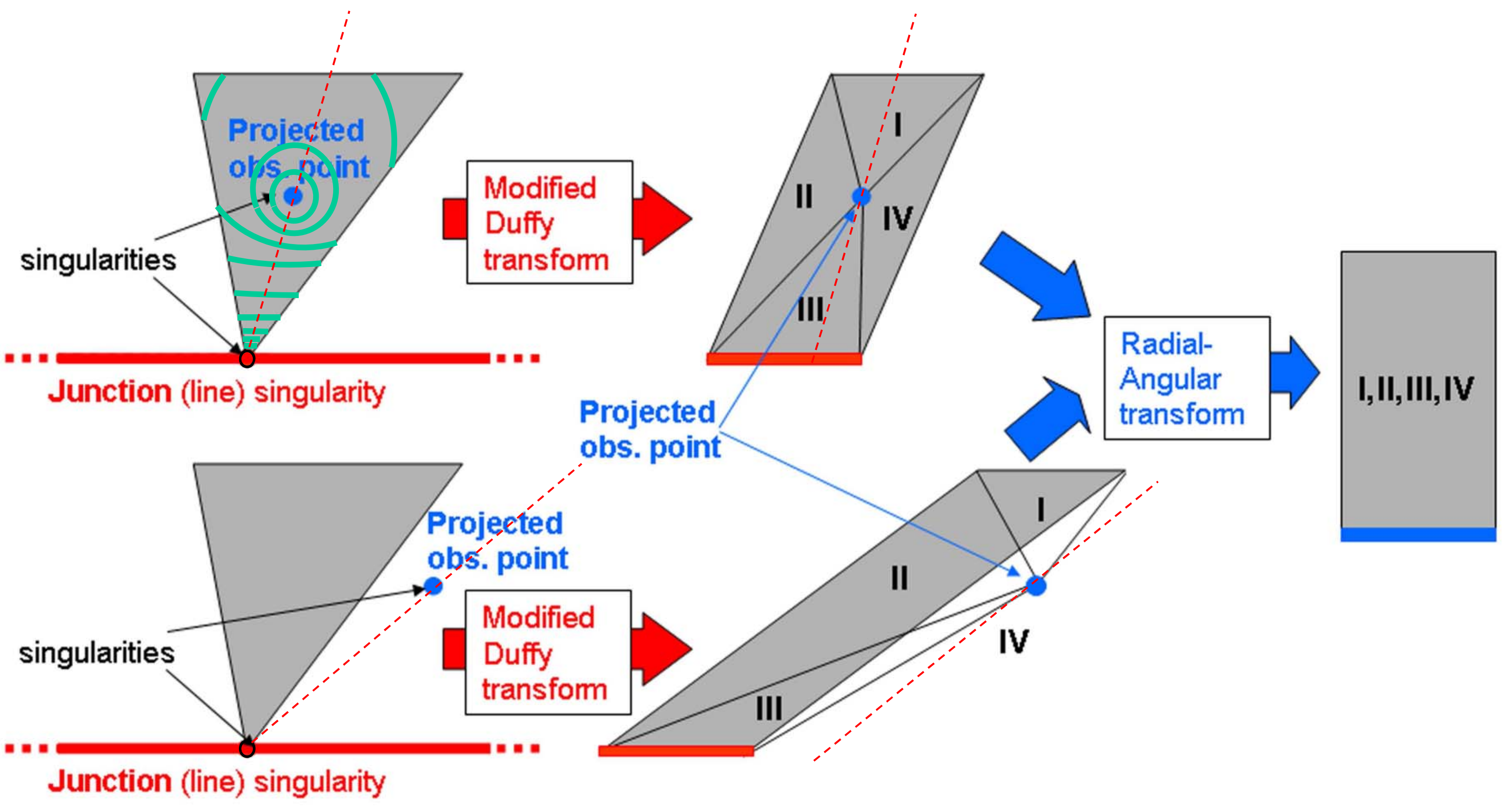
$$\underbrace{K_i^e \ell_i \frac{\rho_i \hat{\boldsymbol{\rho}}_i}{h_i} \cdot \hat{\mathbf{h}}_i \Big|_{\xi_i = 0}}_{\text{Net flux of } \Lambda_i^e \text{ out of edge } i\text{ !}} = K_i^e \ell_i \cos\phi \frac{h_i}{h_i \cos\phi} = K_i^e \ell_i = \frac{\alpha_i^e}{\sum\limits_{\substack{e \text{ attached} \\ \text{to junction}}} \alpha_i^e}$$

Summary of Junction Basis Properties

- Contains a surface basis term $\mathbf{\Lambda}_n$ plus a *divergenceless* term proportional to $\dfrac{\hat{\boldsymbol{\rho}}}{\rho \cos^2 \alpha}$ in attached triangle.
- Has a constant divergence arising from the $\mathbf{\Lambda}_n$ term.
- The normal component of basis vanishes at all edges; hence basis is divergence-conforming.
- Normalized such that 1) each triangle carries current proportional to its angular extent and 2) net current flux into wire is unity.
- Any angular variation of current at base of junction is modeled by the ordinary surface bases there.

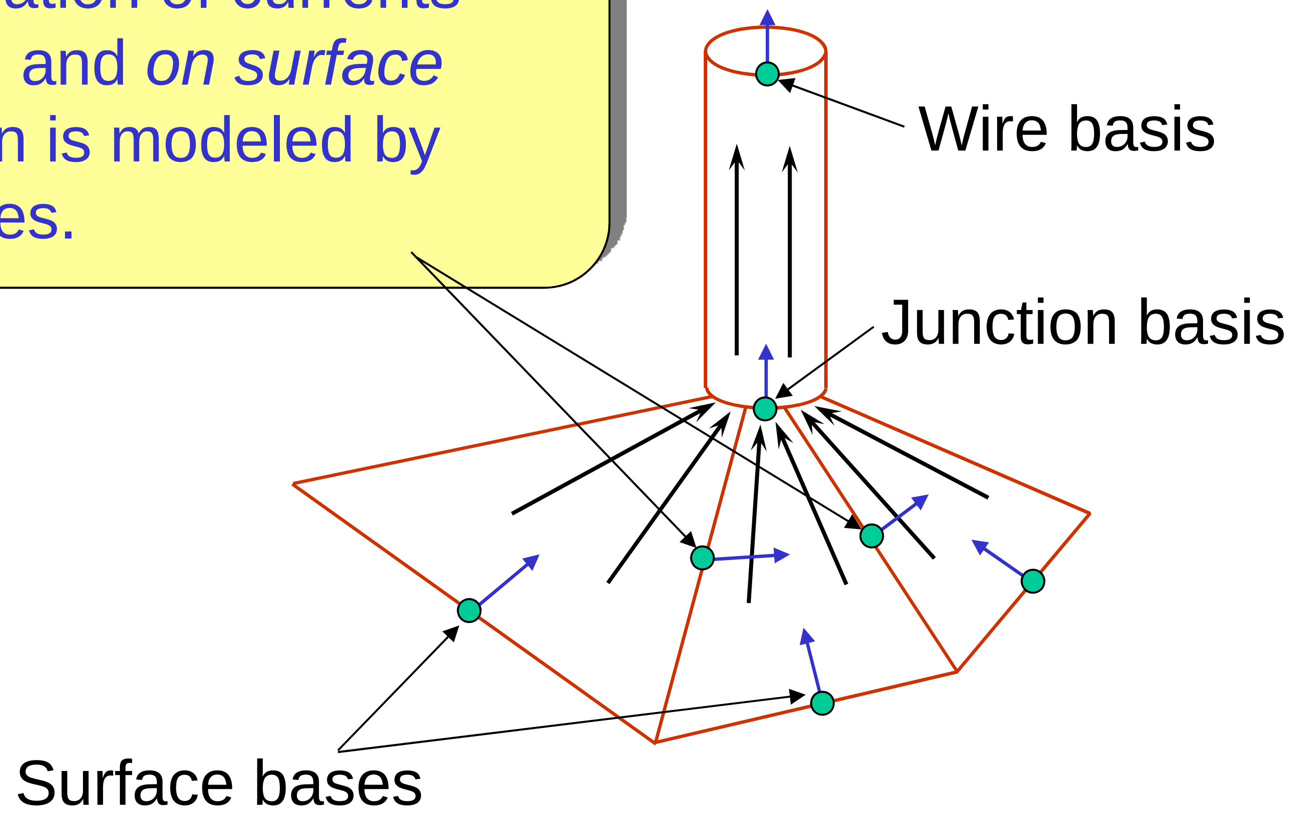
Handling Singular Integrals for Junctions

- Vipiana, F. and D.R. Wilton, "Optimized Numerical Evaluation of Singular and Near-Singular Potential Integrals Involving Junction Basis Functions," *IEEE Trans. Antennas and Propagat.,* ,Vol. 59, 1, Jan. 2011, pp. 162 – 171.



Distribution of Degrees of Freedom Near a Junction

Angular variation of currents *into junction* and *on surface* near junction is modeled by surface bases.



Matrix Equation

- The matrix equation that results from using junctions is given by

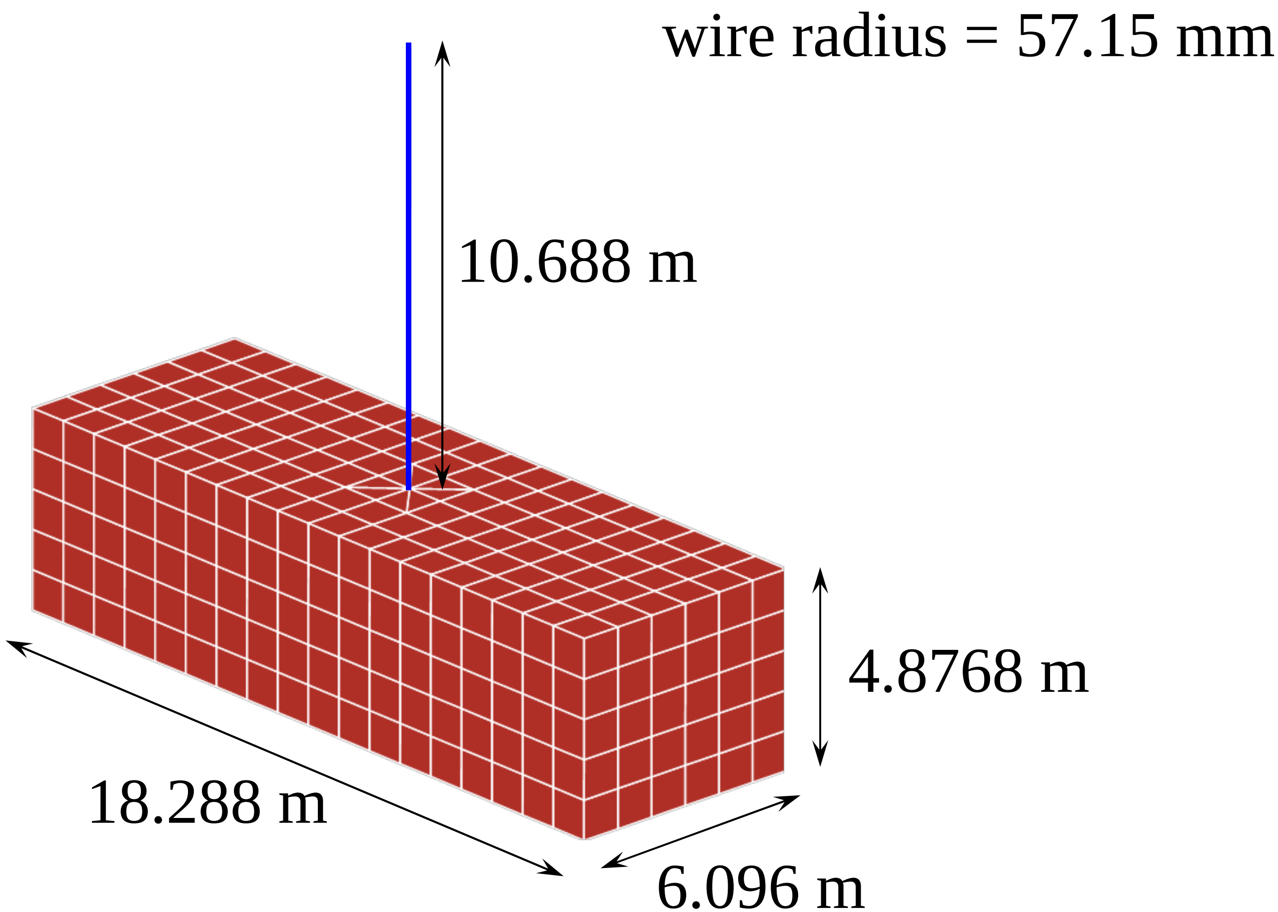
$$\begin{bmatrix} \left[Z^{PP}\right] & \left[Z^{PW}\right] & \left[Z^{PJ}\right] \\ \left[Z^{WP}\right] & \left[Z^{WW}\right] & \left[Z^{WJ}\right] \\ \left[Z^{JP}\right] & \left[Z^{JW}\right] & \left[Z^{JJ}\right] \end{bmatrix} \begin{bmatrix} \left[I^{P}\right] \\ \left[I^{W}\right] \\ \left[I^{J}\right] \end{bmatrix} = \begin{bmatrix} \left[V^{P}\right] \\ \left[V^{W}\right] \\ \left[V^{J}\right] \end{bmatrix},$$

where P = patch**,** W = wire, and J = junction.

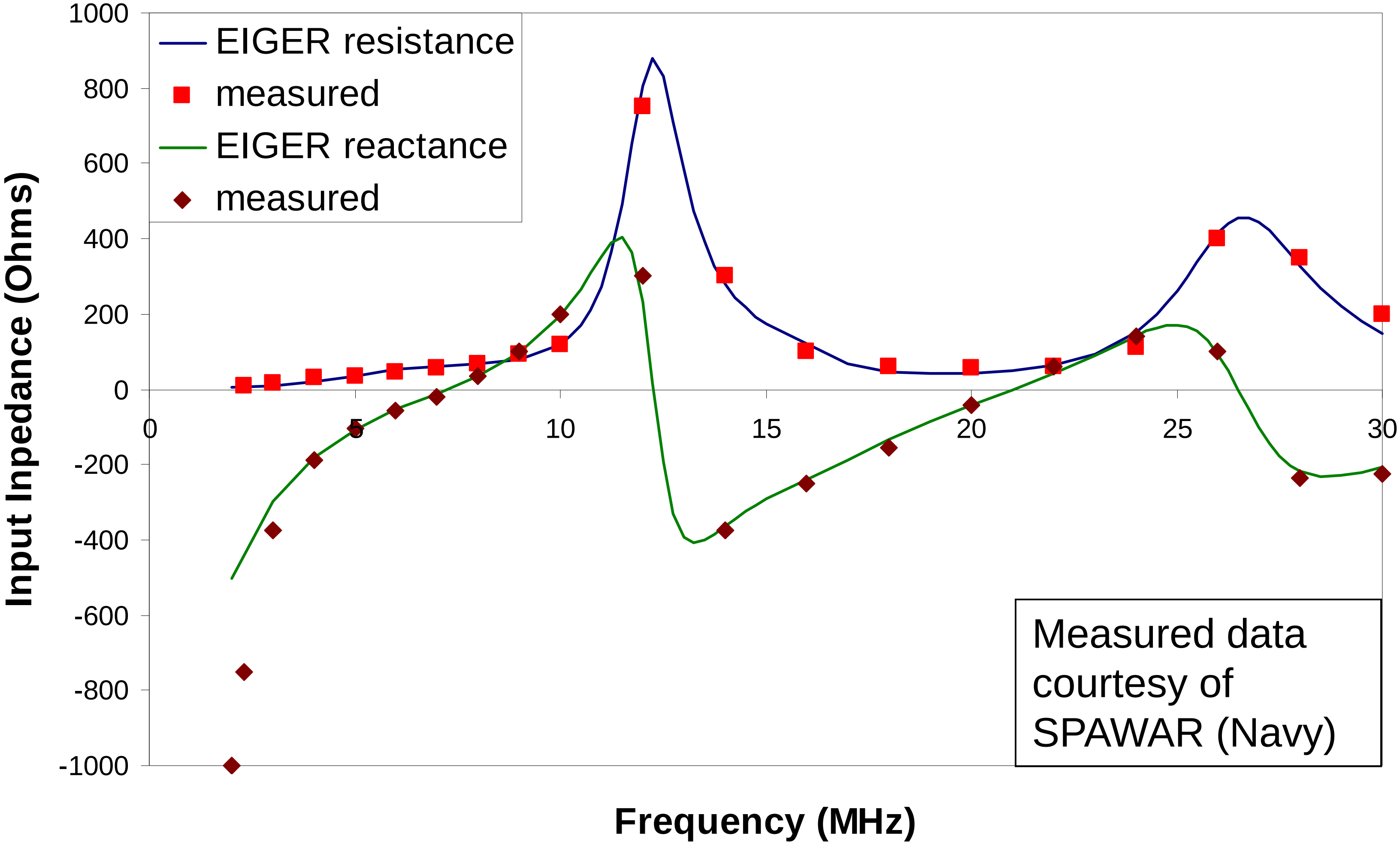
- The general interaction submatrix may be written

$$\left[Z^{\alpha\beta}\right],\ \alpha,\beta = \text{P,W,J}.$$

Topside Calibration Model

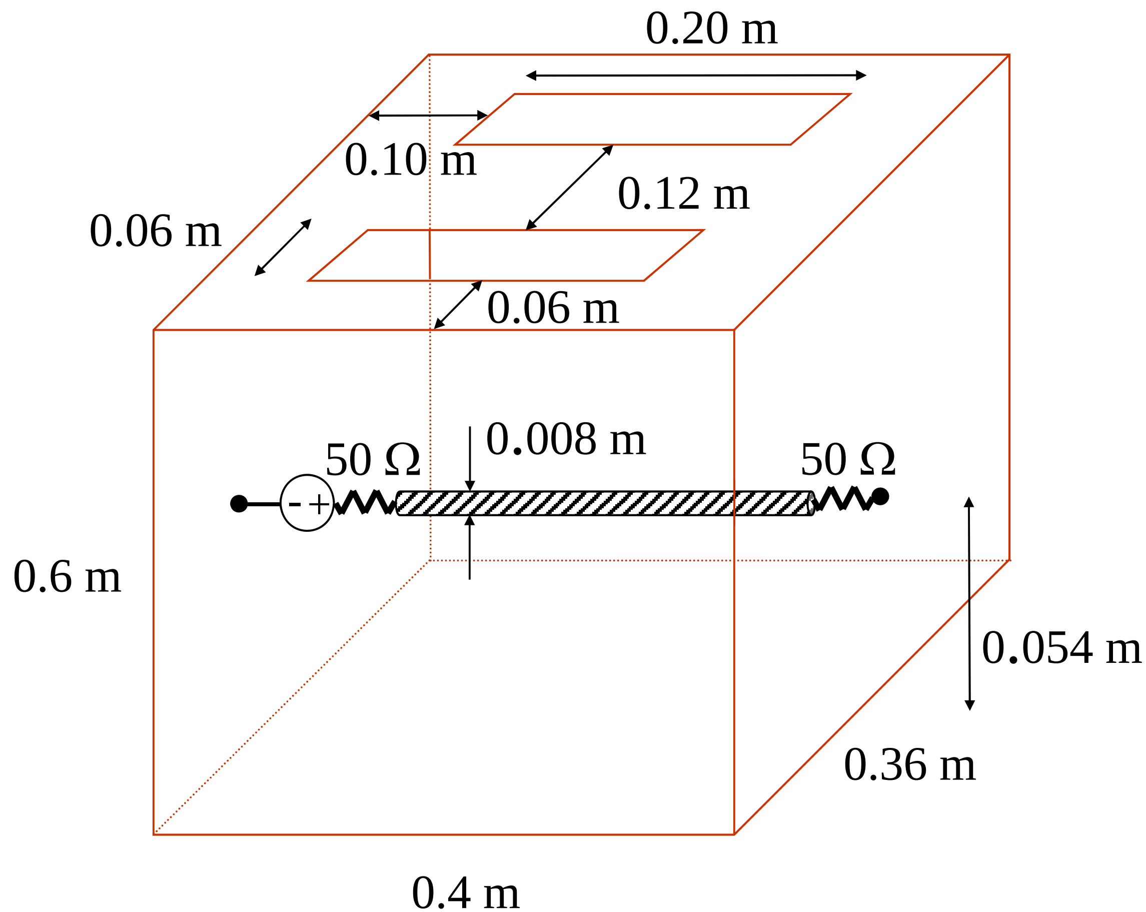


Topside Calibration Model Results

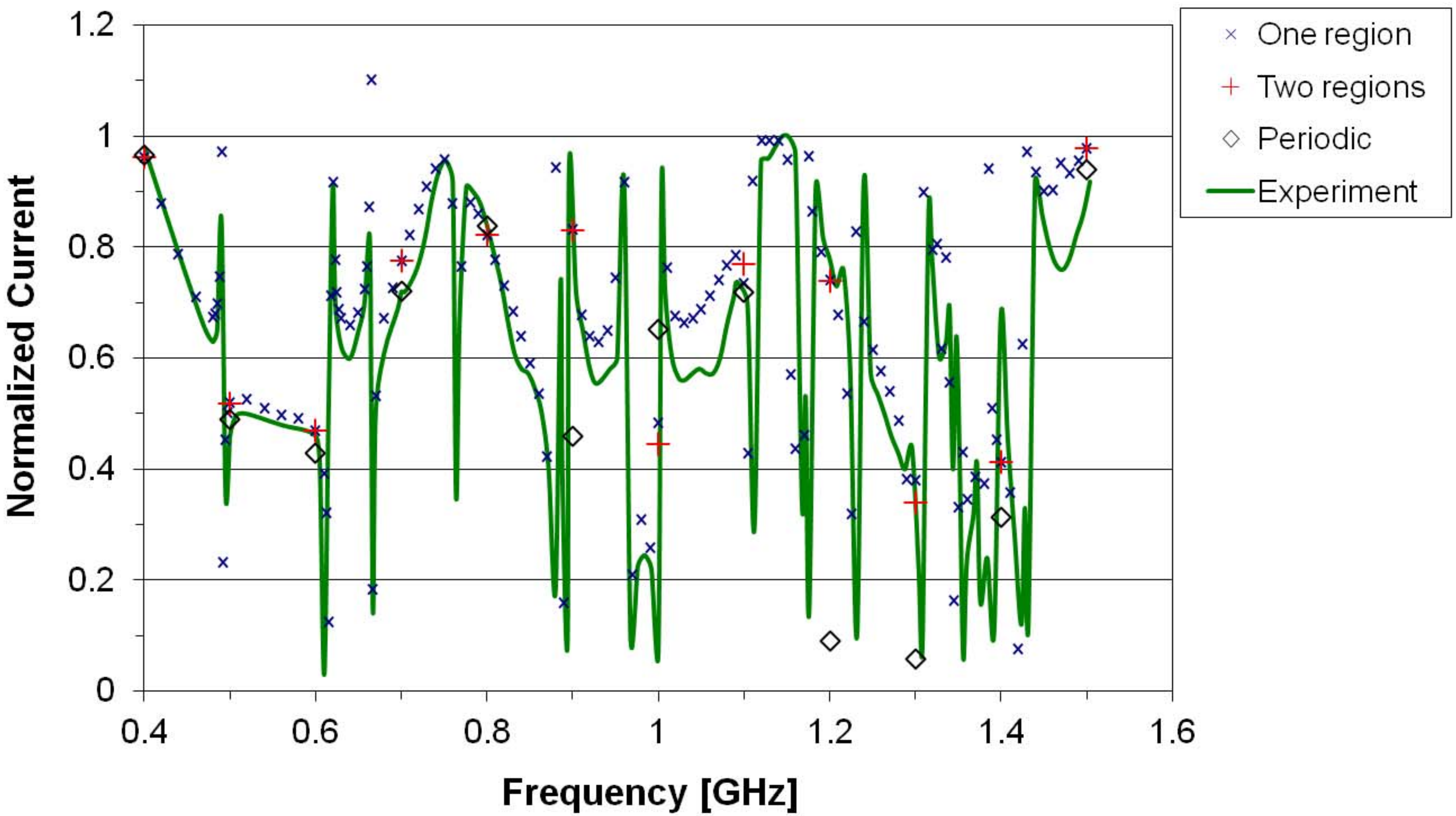


Coupling to Wires in a Cavity

- **Duffy et al., 1994**



Current at Load Opposite Source



The End