ECE 6350

Solution For Surface Currents Induced on PEC Infinite Cylinder with TE Excitation

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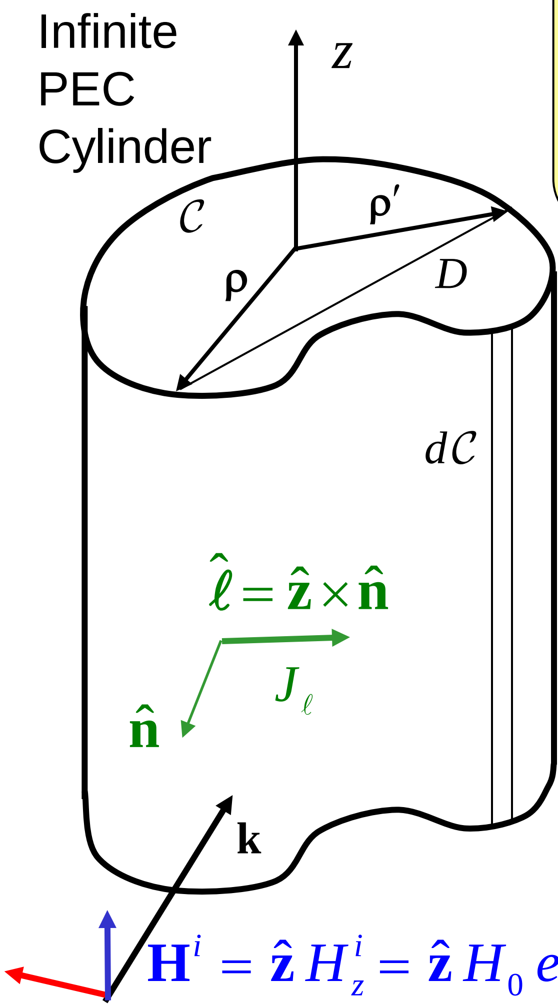
Ref: Scattering Notes, pp. 23 - 27

Additional Features of TE Problem vis-à-vis the TM Problem

- Problem is vectorial in nature
- Scalar potential is non-vanishing
- Current representation must be differentiable
- Current splitting at a junction
- It is not generally possible to directly associate element and unknown (DoF) indices

Normally Incident, TE Polarized Plane Wave Illumination of PEC Cylinder

- If illumination has no z **variation**, there is no z variation of surface currents, scattered fields, potentials, etc.
- If illumination is also transversely polarized (TEz), there is only a transverse component of surface current.



$\Rightarrow$ • $\mathbf{J}(\mathbf{r}) = J_\ell(\boldsymbol{\rho})\,\hat{\boldsymbol{\ell}}, \quad \boldsymbol{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$

$\Rightarrow$ • $\nabla \cdot \mathbf{J}(\mathbf{r}) = \dfrac{dJ_\ell}{d\ell} \neq 0, \ \Rightarrow \Phi \neq 0$

$\Rightarrow$ •
$$\begin{aligned}
\mathbf{E}^s(\mathbf{r}) &= -j\omega\mathbf{A} - \nabla\Phi \\
&= -j\omega\mu \int_{\mathcal{C}} G(\boldsymbol{\rho},\boldsymbol{\rho}') J_\ell(\boldsymbol{\rho}')\hat{\boldsymbol{\ell}}'\, d\mathcal{C}' \\
&\quad + \frac{1}{j\omega\varepsilon}\nabla \int_{\mathcal{C}} G(\boldsymbol{\rho},\boldsymbol{\rho}') \frac{dJ_\ell(\boldsymbol{\rho}')}{d\ell'}\, d\mathcal{C}' \\
&= \hat{\boldsymbol{\ell}} E_\ell^s + \hat{\mathbf{n}} E_n^s
\end{aligned}$$

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') \equiv \frac{H_0^{(2)}(kD)}{4j}, \quad D \equiv |\boldsymbol{\rho} - \boldsymbol{\rho}'|$$

$\mathbf{H}^i = \hat{\mathbf{z}} H_z^i = \hat{\mathbf{z}} H_0 e^{-j\mathbf{k}\cdot\mathbf{r}},$

$\mathbf{E}^i = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$
$= \eta \mathbf{H}_0 \times \hat{\mathbf{k}}\, e^{-j\mathbf{k}\cdot\mathbf{r}}$

$\mathbf{k} \equiv k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} = -k\cos\phi^i\hat{\mathbf{x}} - k\sin\phi^i\hat{\mathbf{y}}$

Mixed Potential Representation of Scattered Electric Field

$$\mathbf{E}^s = -j\omega\mathbf{A} - \nabla\Phi$$

where the magnetic vector potential is

$$\mathbf{A} = \mu \int_{\mathcal{C}} G(\boldsymbol{\rho}, \boldsymbol{\rho}') J_\ell(\boldsymbol{\rho}') \hat{\boldsymbol{\ell}}' \, d\mathcal{C}'$$

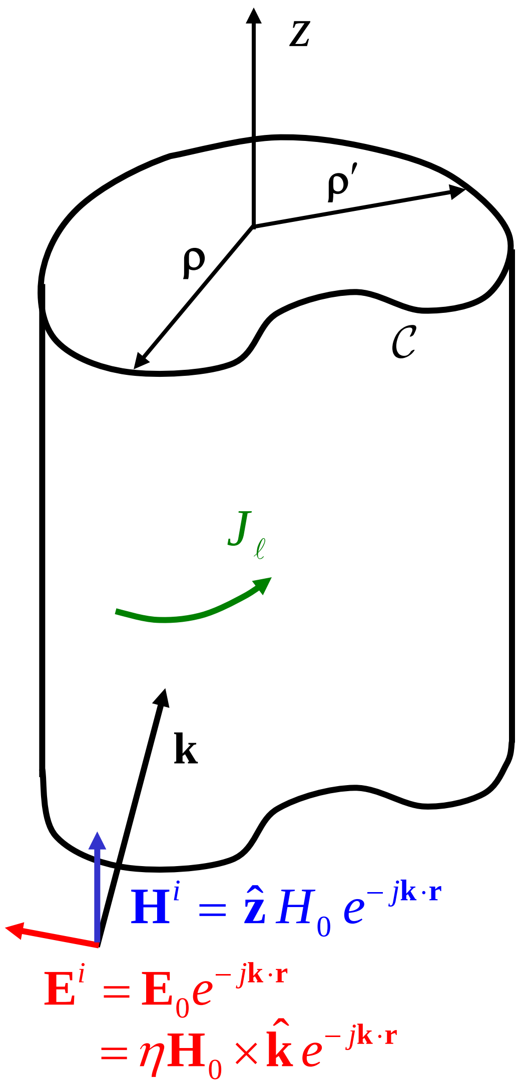
and the electric scalar potential is

$$\Phi = \frac{1}{\varepsilon} \int_{\mathcal{C}} G(\boldsymbol{\rho}, \boldsymbol{\rho}') q(\boldsymbol{\rho}') d\mathcal{C}' = -\frac{1}{j\omega\varepsilon} \int_{\mathcal{C}} G(\boldsymbol{\rho}, \boldsymbol{\rho}') \frac{dJ_\ell(\boldsymbol{\rho}')}{d\ell'} d\mathcal{C}'$$

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') \equiv \frac{H_0^{(2)}\left(kD\right)}{4j},$$

$$D \equiv \left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|$$

Electric Field Integral Equation (EFIE)



$$\mathbf{E}_{\tan} = \left(\hat{\ell}\cdot\mathbf{E}\right)\hat{\ell} = \mathbf{E}^{\mathrm{i}}_{\tan} + \mathbf{E}^{\mathrm{s}}_{\tan} = \mathbf{0}, \quad \boldsymbol{\rho}\in\mathcal{C}$$

$$\Rightarrow \; j\omega\mu\int_{\mathcal{C}} G(\boldsymbol{\rho},\boldsymbol{\rho}')\,\hat{\ell}\cdot\hat{\ell}' J_\ell(\boldsymbol{\rho}')\,d\mathcal{C}'$$

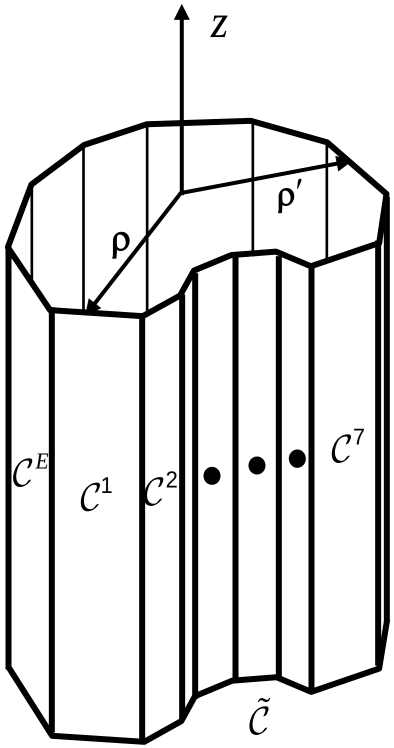
$$-\frac{1}{j\omega\varepsilon}\frac{d}{d\ell}\int_{\mathcal{C}} G(\boldsymbol{\rho},\boldsymbol{\rho}')\frac{dJ_\ell(\boldsymbol{\rho}')}{d\ell'}\,d\mathcal{C}'$$

$$= \hat{\ell}\cdot\mathbf{E}^{\mathrm{i}}, \quad \boldsymbol{\rho}\in\mathcal{C}$$

Integro-differential equation!

- This *strong* form of EFIE holds at every point of $\mathcal{C}$
- Must solve for vector-valued current $\mathbf{J}$ at each point $\boldsymbol{\rho}$ of $\mathcal{C}$
- $\nabla\cdot\mathbf{J} = dJ_\ell / d\ell$ must exist

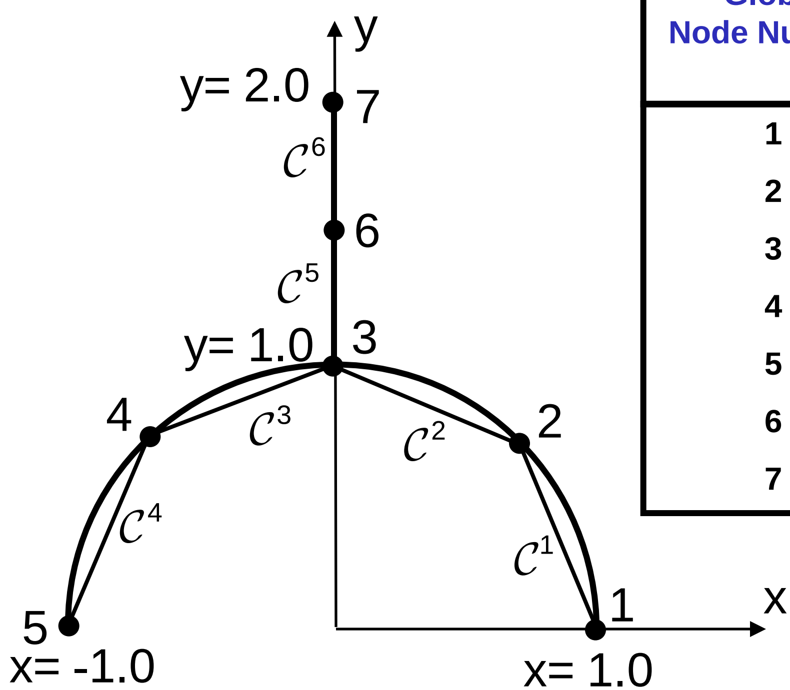
Discretization and Geometry Data Structure



Piecewise linear discretization of geometry

$$\mathcal{C} \approx \tilde{\mathcal{C}} = \bigcup_{e=1}^{N_e} \mathcal{C}^e$$

Example: Cross section of hemicylinder with fin

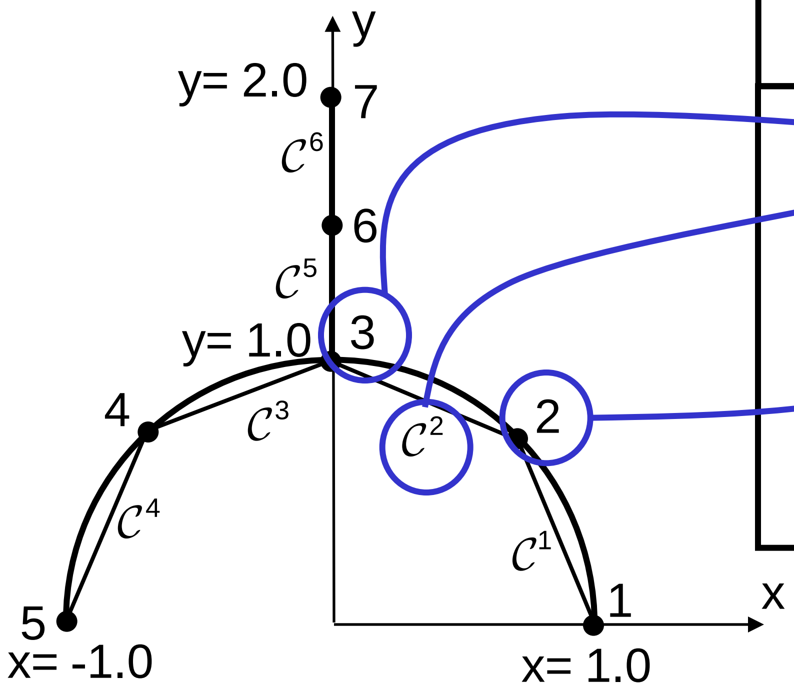


Data structure for element nodes

Global Node Number	Coordinates (z=0) x	Coordinates (z=0) y
1	1.0000	0.0000
2	0.7071	0.7071
3	0.0000	1.0000
4	-0.7071	0.7071
5	-1.0000	0.0000
6	0.0000	1.5000
7	0.0000	2.0000

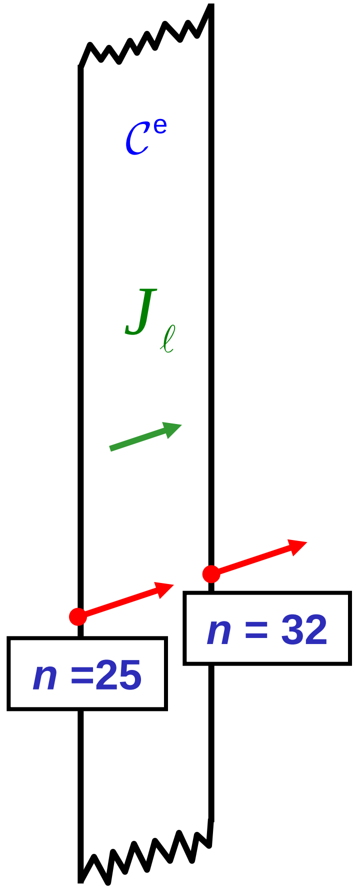
Element Connectivity Data Structure

Element to node mapping



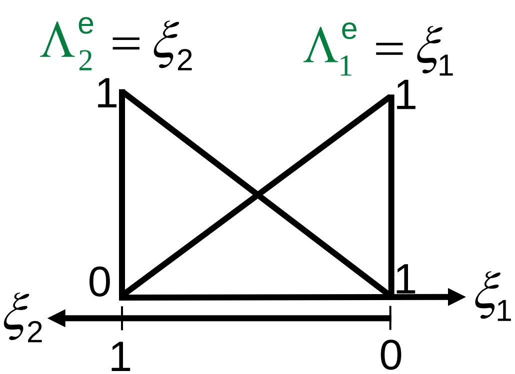
Element Number	Global Node Numbers	
	Local Node 1	Local Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	3	6
6	6	7

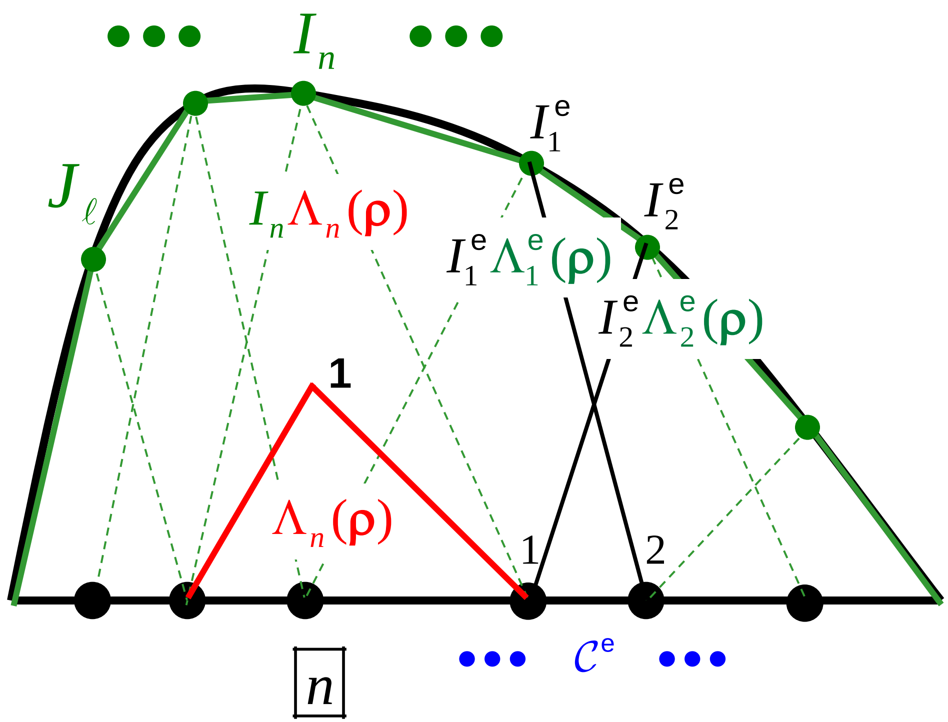
Approximation of Transverse Current Component

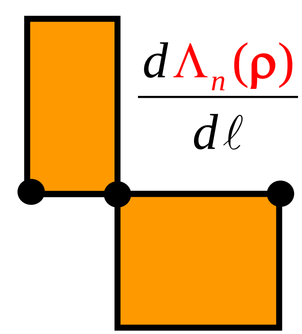


$$J_\ell(\rho) \approx \sum_{n=1}^{N} I_n \Lambda_n(\rho)$$ (current is PWL)

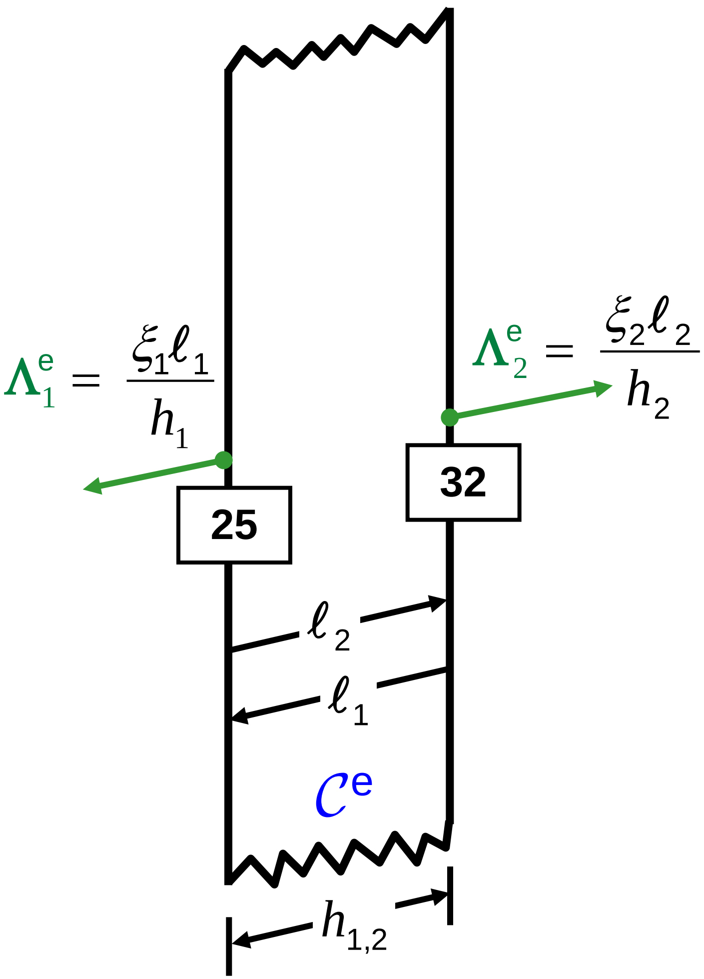
$$\nabla \cdot \left(\hat{\ell}\, J_\ell\right) = \frac{d J_\ell}{d\ell} \approx \sum_{n=1}^{N} I_n \frac{d}{d\ell}\Lambda_n(\rho)$$ (charge is PWC)







"Vectorization" and Summary of Surface Current Representation

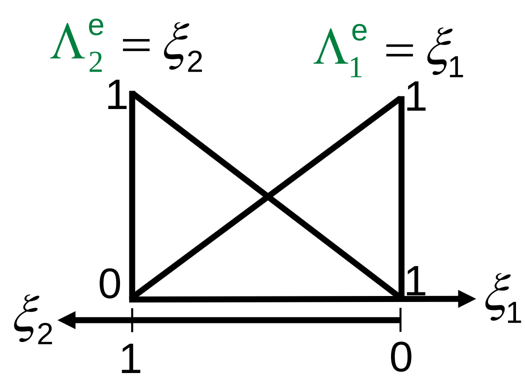


$$\mathbf{J}(\boldsymbol{\rho}) \approx \sum_{\boxed{n}=1}^{N} I_{\boxed{n}} \boldsymbol{\Lambda}_{\boxed{n}}(\boldsymbol{\rho}), \qquad (\boldsymbol{\Lambda}_n \equiv \Lambda_n \hat{\boldsymbol{\ell}}, \ \boldsymbol{\rho} \in \tilde{\mathcal{C}})$$

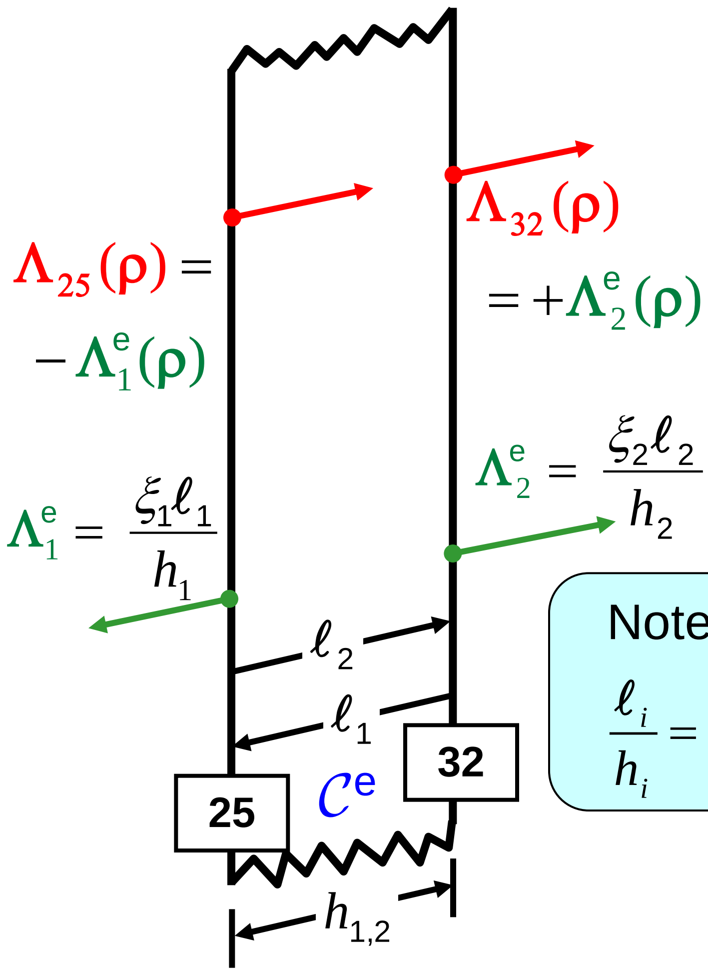
(***global representation***, boxed n is global DoF index)

$$= \sum_{i=1}^{2} I_i^{\mathrm{e}} \sigma_i^{\mathrm{e}} \boldsymbol{\Lambda}_i^{\mathrm{e}}(\boldsymbol{\rho}), \quad \boldsymbol{\rho} \in \mathcal{C}^{\mathrm{e}}$$

(***local representation***, $\sigma_i^{\mathrm{e}} = \pm 1$, i is local basis index in element e)



Note There Exists Both Local and a Global Current Representations on Each Element

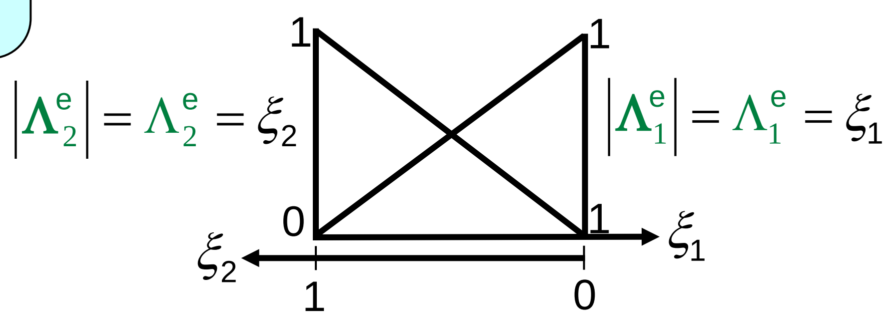


$$\mathbf{J}(\rho) \approx \begin{cases} \underbrace{I_{25}\Lambda_{25}(\rho) + I_{32}\Lambda_{32}(\rho)}_{\textit{global representation}} \\ \underbrace{\sum_{i=1}^{2} I_i^e \sigma_i^e \Lambda_i^e(\rho)}_{\textit{local representation}}, \quad \sigma_i^e = \pm 1 \end{cases}, \quad \rho \in \mathcal{C}^e$$

- Λ_{25}, Λ_{32} reference directions specified by user
- Λ_1^e, Λ_2^e reference directions *out* of element

Note:

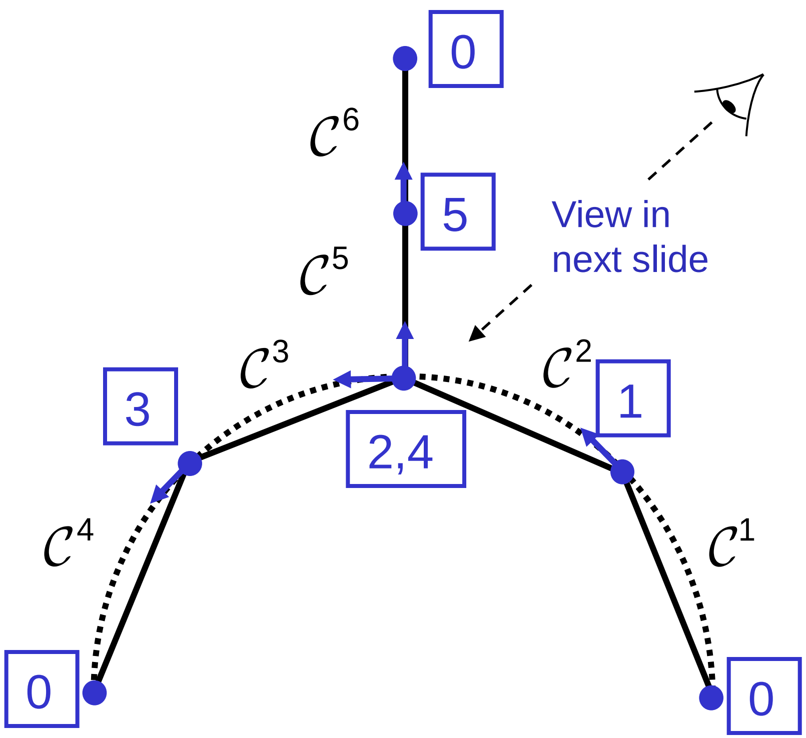
$$\frac{\ell_i}{h_i} = \hat{\ell}_i$$



Element DoF Data

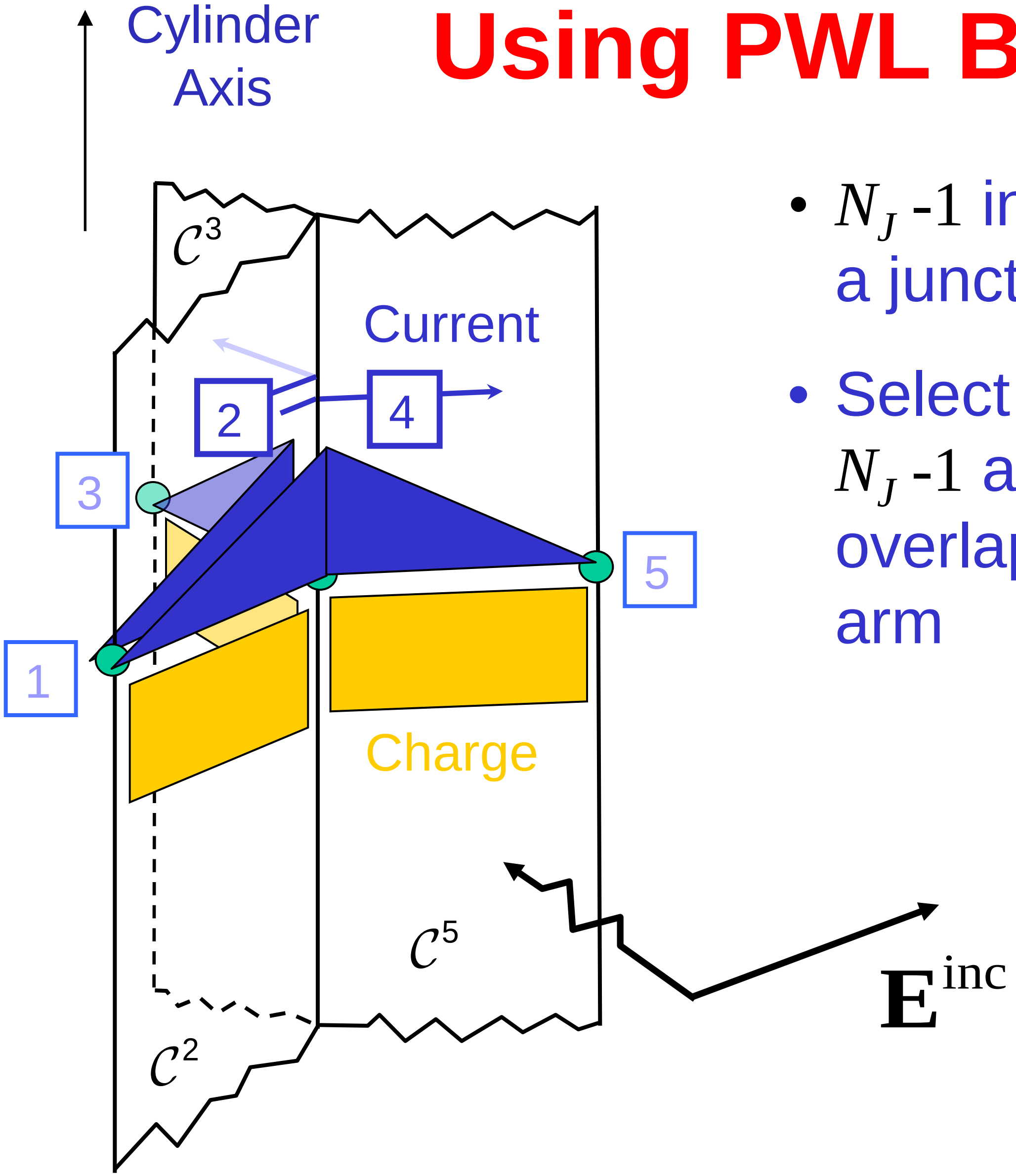
Local Element-to-Global DoF mapping

e	Local Indices, Element e: 1 — # DoF's	1 — DoF Index	2 — # DoF's	2 — DoF Index
1	0	0	1	+1
2	1	-1	2	+2,+4
3	1	-2	1	+3
4	1	-3	0	0
5	1	-4	1	+5
6	1	-5	0	0



$$\sigma_i^e = \begin{cases} +1 \text{ if sign of global DoF corresponding to } i \text{ th DoF of element } e \text{ is positive,} \\ -1 \text{ if sign of global DoF corresponding to } i \text{ th DoF of element } e \text{ is negative} \end{cases}$$

KCL Easily Enforced Using PWL Bases



- N_J -1 independent currents at a junction of N_J line segments
- Select independent bases on N_J -1 arms of junction and overlap them onto remaining arm

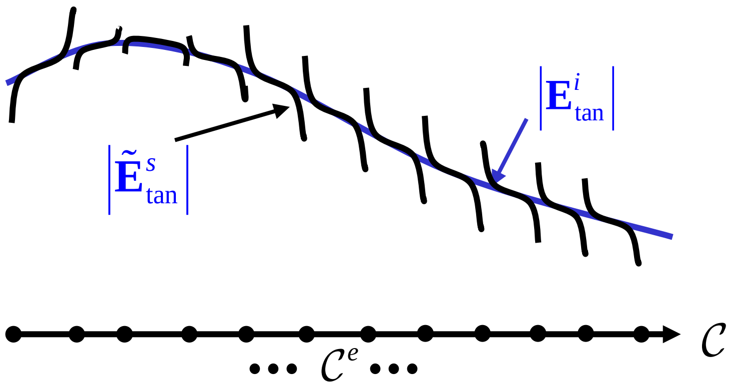
Current out of $\mathcal{C}^2$ at junction $= \; I_2 + I_4$

Formation of Moment Equations

Since $\mathbf{J}(\boldsymbol{\rho}) \approx \sum_{n=1}^{N} I_n \boldsymbol{\Lambda}_n(\boldsymbol{\rho}), \quad \mathcal{C} \approx \tilde{\mathcal{C}}$

the EFIE becomes

$$\overbrace{\sum_{n=1}^{N} I_n \left[j\omega\mu \int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}')\boldsymbol{\Lambda}_n(\boldsymbol{\rho}')\,d\mathcal{C}' - \frac{1}{j\omega\varepsilon}\nabla\int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}')\nabla'\cdot\boldsymbol{\Lambda}_n(\boldsymbol{\rho}')\,d\mathcal{C}' \right]_{\tan}}^{\tilde{\mathbf{E}}^{s}_{\tan}} \approx \mathbf{E}^{i}_{\tan}, \quad \boldsymbol{\rho} \in \tilde{\mathcal{C}}$$



Enforce "equality" on subdomains:

- Point match:

$$-\int_{\mathcal{C}} \mathbf{E}^{s}(\boldsymbol{\rho})\cdot\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_m)\,d\mathcal{C} = \int_{\mathcal{C}} \mathbf{E}^{i}(\boldsymbol{\rho})\cdot\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_m)\,d\mathcal{C}$$

- Equate average fields:

$$-\int_{\mathcal{C}} \mathbf{E}^{s}(\boldsymbol{\rho})\cdot\pi(\boldsymbol{\rho}-\boldsymbol{\rho}_m)\,d\mathcal{C} = \int_{\mathcal{C}} \mathbf{E}^{i}(\boldsymbol{\rho})\cdot\pi(\boldsymbol{\rho}-\boldsymbol{\rho}_m)\,d\mathcal{C}$$

- Weighted average with "testing function":

$$-\int_{\mathcal{C}} \mathbf{E}^{s}(\boldsymbol{\rho})\cdot\mathbf{T}_m(\boldsymbol{\rho})\,d\mathcal{C} = \int_{\mathcal{C}} \mathbf{E}^{i}(\boldsymbol{\rho})\cdot\mathbf{T}_m(\boldsymbol{\rho})\,d\mathcal{C}$$

($\mathbf{T}_m(\boldsymbol{\rho}) = \boldsymbol{\Lambda}_m(\boldsymbol{\rho})$, Galerkin's method)

Differentiable Testing Function Permits Transfer of Derivative on Scalar Potential to Testing Function

Generalized Divergence Theorem:

$$\nabla \cdot \left(\mathbf{\Lambda}_m \Phi \right) = \mathbf{\Lambda}_m \cdot \nabla \Phi + \Phi \nabla \cdot \mathbf{\Lambda}_m$$

$$\Rightarrow \int_{\partial \mathcal{D}} \Phi\, \mathbf{\Lambda}_m \cdot \hat{\mathbf{u}}\, d\mathcal{B}$$

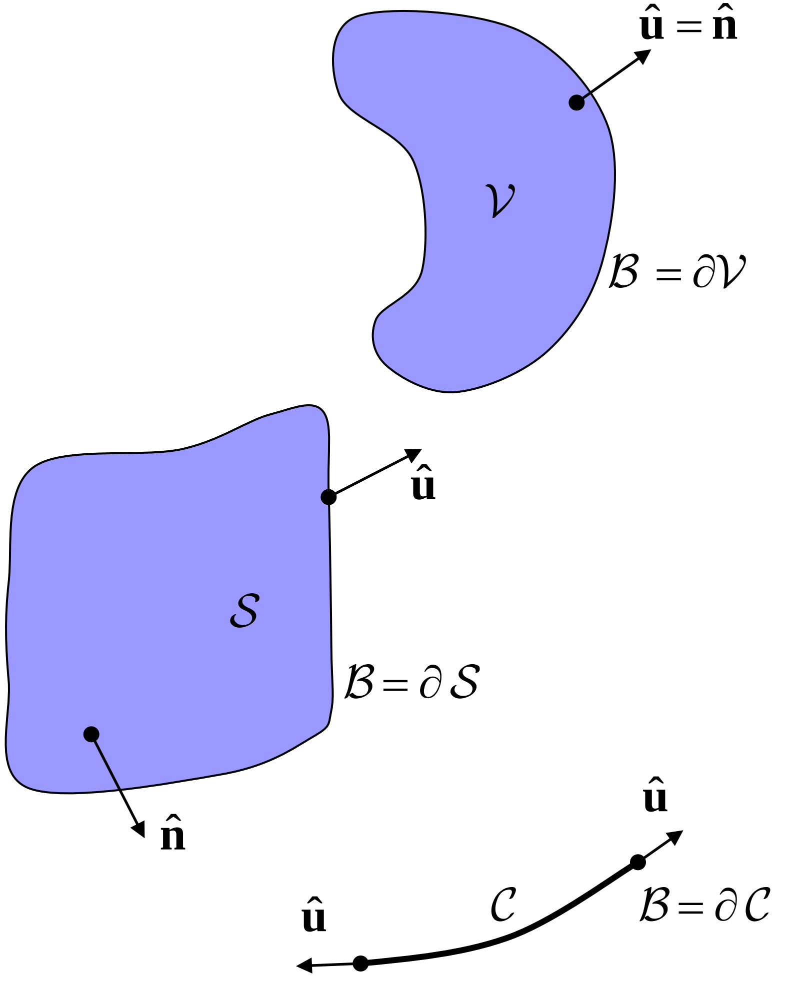
$$= \int_{\mathcal{D}} \mathbf{\Lambda}_m \cdot \nabla \Phi \, d\mathcal{D} + \int_{\mathcal{D}} \Phi \nabla \cdot \mathbf{\Lambda}_m \, d\mathcal{D}$$

$$\mathcal{D} = \mathcal{C}, \mathcal{S}, \mathcal{V}, \quad \mathcal{B} = \partial \mathcal{D}$$

$$\hat{\mathbf{u}} \in \mathcal{D}, \quad \hat{\mathbf{u}} \perp \mathcal{B}$$

Note if net flux out of $\mathcal{B} = \partial \mathcal{D}$ vanishes,

$$\int_{\mathcal{D}} \mathbf{\Lambda}_m \cdot \nabla \Phi \, d\mathcal{D} = -\int_{\mathcal{D}} \Phi \nabla \cdot \mathbf{\Lambda}_m \, d\mathcal{D}$$



Testing the Equations

To apply Galerkin's method, multiply both sides by "testing function" $\boldsymbol{\Lambda}_m(\boldsymbol{\rho})$ and integrate over the conductor; the scalar potential term can be integrated by parts via the generalized divergence theorem:

$$\int_{\tilde{\mathcal{C}}} \boldsymbol{\Lambda}_m(\boldsymbol{\rho}) \cdot \left(\nabla \int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}') \nabla' \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}')\, d\mathcal{C}' \right) d\mathcal{C} = -\int_{\tilde{\mathcal{C}}} \nabla \cdot \boldsymbol{\Lambda}_m(\boldsymbol{\rho}) \left(\int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}') \nabla' \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}')\, d\mathcal{C}' \right) d\mathcal{C}$$

$$+ \underbrace{\hat{\ell} \cdot \boldsymbol{\Lambda}_m(\boldsymbol{\rho})}_{=\,\mathbf{0} \text{ on } \partial\tilde{\mathcal{C}}} \left(\int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}') \nabla' \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}')\, d\mathcal{C}' \right) \Bigg|_{\partial\tilde{\mathcal{C}}}$$

The result is the linear system

$$\sum_{n=1}^{N} I_n \left[j\omega\mu \int_{\tilde{\mathcal{C}}} \int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}') \boldsymbol{\Lambda}_m(\boldsymbol{\rho}) \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}') d\mathcal{C}' d\mathcal{C} \right.$$

$$\left. + \frac{1}{j\omega\varepsilon} \int_{\tilde{\mathcal{C}}} \int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho},\boldsymbol{\rho}') \nabla \cdot \boldsymbol{\Lambda}_m(\boldsymbol{\rho}) \nabla' \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}')\, d\mathcal{C}' d\mathcal{C} \right] = \langle \boldsymbol{\Lambda}_m ; \mathbf{E}^{\mathrm{i}} \rangle,$$

$$m = 1, 2, \ldots, N$$

Matrix Form of Moment Equations

In matrix format

$$\sum_{n=1}^{N} Z_{mn} I_n = V_m, \quad m = 1, 2, \ldots, N \quad \text{or} \quad [Z_{mn}][I_n] = [V_m]$$

Global impedance matrix :

$$[Z_{mn}] = j\omega [L_{mn}] + \frac{1}{j\omega}[S_{mn}]$$

Global *inductance matrix :*

$$L_{mn} = \mu \int_{\tilde{\mathcal{C}}} \int_{\tilde{\mathcal{C}}} G(\boldsymbol{\rho}, \boldsymbol{\rho}') \boldsymbol{\Lambda}_m(\boldsymbol{\rho}) \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}') d\mathcal{C}' d\mathcal{C} \quad \equiv \quad \boxed{\mu < \boldsymbol{\Lambda}_m; G, \boldsymbol{\Lambda}_n >}$$

Global *elastance matrix :*

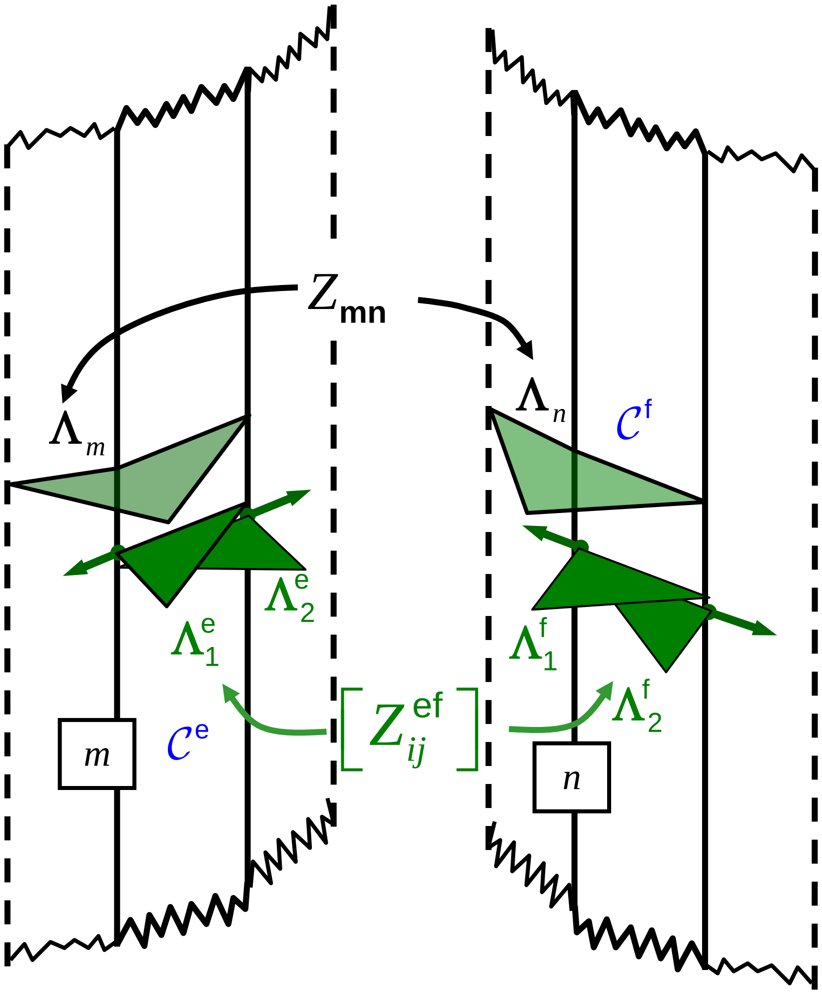
$$S_{mn} = \frac{1}{\varepsilon} \int_{\tilde{\mathcal{C}}} \int_{\tilde{\mathcal{C}}} \nabla \cdot \boldsymbol{\Lambda}_m(\boldsymbol{\rho})\, G(\boldsymbol{\rho}, \boldsymbol{\rho}') \nabla' \cdot \boldsymbol{\Lambda}_n(\boldsymbol{\rho}') d\mathcal{C}' d\mathcal{C} \quad \equiv \quad \boxed{\frac{1}{\varepsilon} < \nabla \cdot \boldsymbol{\Lambda}_m, G, \nabla \cdot \boldsymbol{\Lambda}_n >},$$

Global *excitation voltage vector :*

$$V_m = < \boldsymbol{\Lambda}_m; \mathbf{E}^{\mathrm{i}} >$$

Unfortunately, it is both inefficient and impractical to form the impedance matrix this way!

Forming the Moment Matrix



Since $\int_{\tilde{\mathcal{C}}} d\mathcal{C} = \sum_{\mathrm{e}=1}^{\mathrm{E}} \int_{\mathcal{C}^{\mathrm{e}}} d\mathcal{C}$, instead of directly calculating

$$Z_{mn} = j\omega\mu < \Lambda_m ; G, \Lambda_n > + \frac{1}{j\omega\varepsilon} < \nabla\cdot \Lambda_m , G, \nabla \cdot \Lambda_n > \;\ldots$$

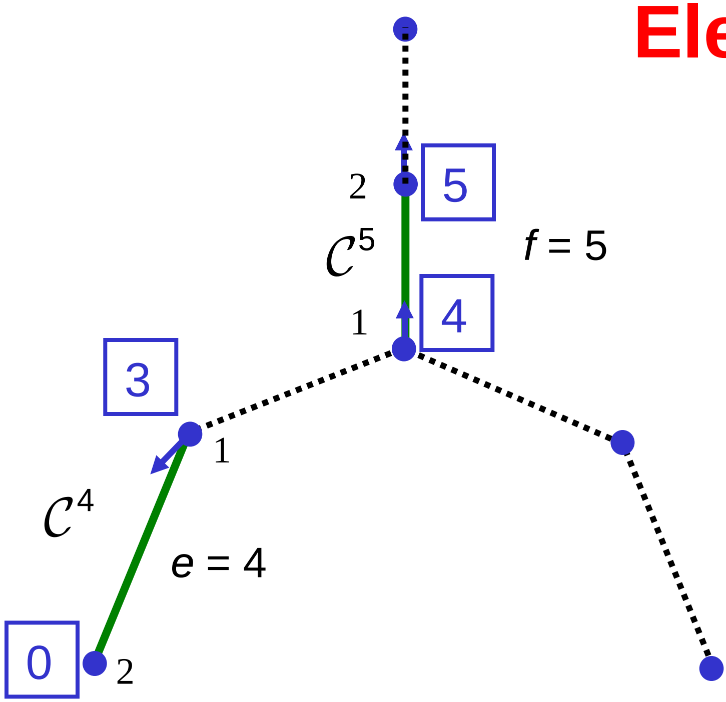
accumulate *partial* contributions from element pairs,

$$\left[Z_{ij}^{\mathrm{ef}}\right] = j\omega\mu\left[< \Lambda_i^{\mathrm{e}} ; G, \Lambda_j^{\mathrm{f}} > \right] + \frac{1}{j\omega\varepsilon}\left[< \nabla\cdot \Lambda_i^{\mathrm{e}} , G, \nabla \cdot \Lambda_j^{\mathrm{f}} > \right],$$

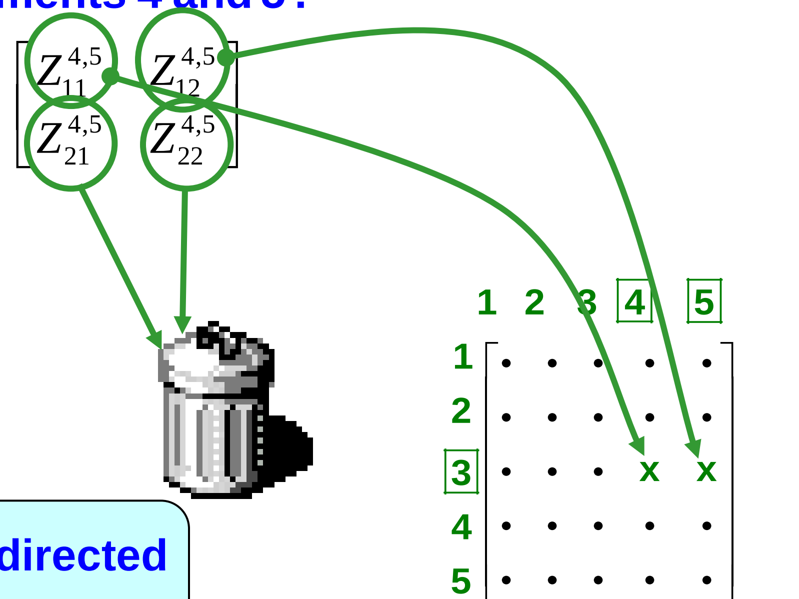
$$i, j = 1, 2$$

"Think globally; act locally"

System Matrix Assembly from Element Matrices



Element matrix for interactions between elements 4 and 5 :



Global System Matrix

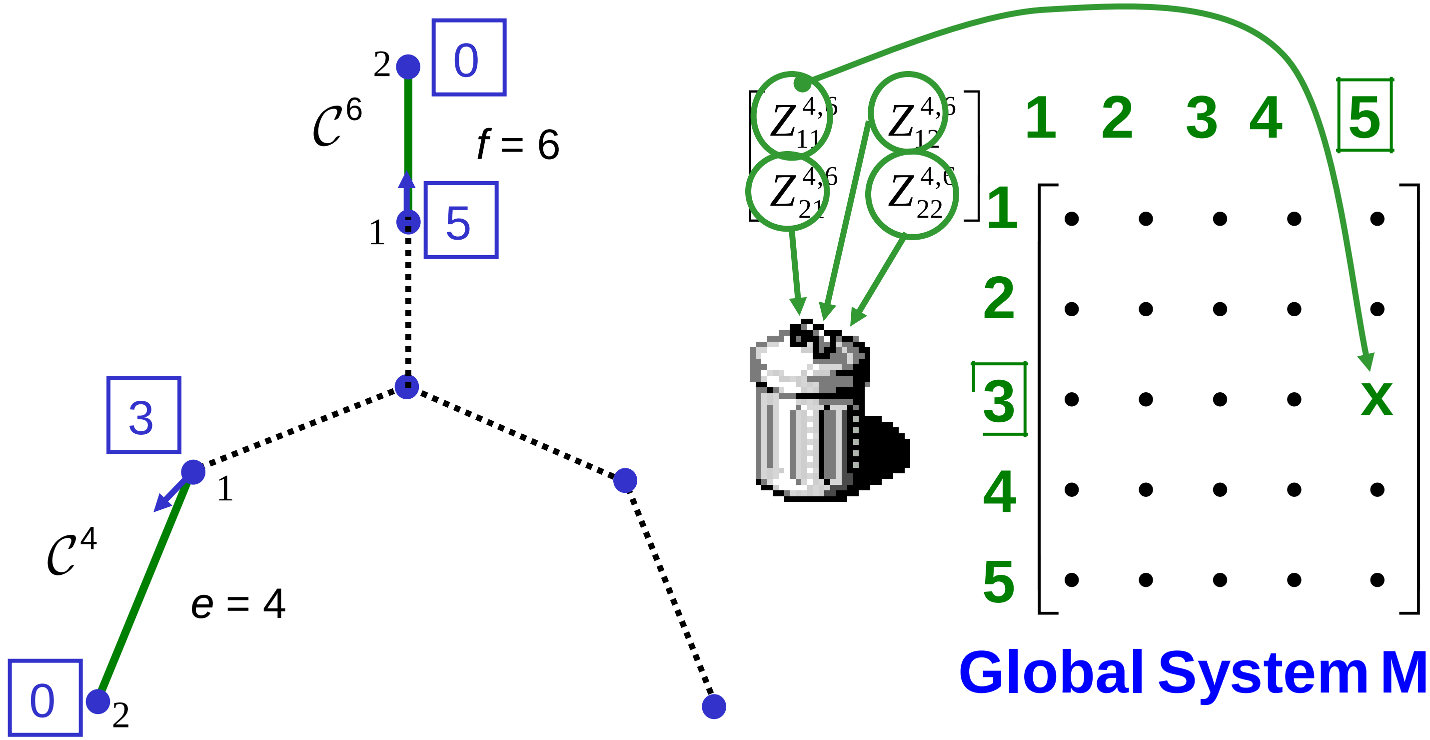
$$\sigma_i^e = \begin{cases} 1, & \text{local and global DOFs similarly directed} \\ -1, & \text{otherwise} \end{cases}$$

Matrix Assembly Rule:

Element $\sigma_i^e \sigma_j^f Z_{ij}^{ef}$ of the element matrix is added to row m and column n of the system matrix if m is the global DoF corresponding to the i th local DOF of element e and n is the global DoF corresponding to the j th local DOF of element f.

System Matrix Assembly from Element Matrices, Cont'd

Element matrix for interactions between elements 4 and 6 :



Global System Matrix

$$\sigma_i^e = \begin{cases} 1, & \textbf{local and global DOFs similarly directed} \\ -1, & \textbf{otherwise} \end{cases}$$

Element Matrix Calculation—Non-Self Terms

For $e \neq f$, *simultaneously* compute local inductance and elastance matrices as

$$L_{ij}^{ef} = \mu < \Lambda_i^e ; G, \Lambda_j^f > \approx \ell^e \ell^f \mu \sum_{k=1}^{K} \sum_{k'=1}^{K'} w_k w_{k'} G\left(\boldsymbol{\rho}^{(k)}, \boldsymbol{\rho}^{(k')}\right) \Lambda_i^e(\boldsymbol{\rho}^{(k)}) \cdot \Lambda_j^f(\boldsymbol{\rho}^{(k')})$$

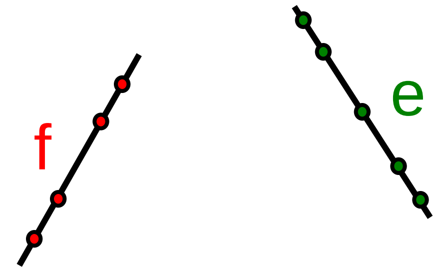
$$S_{ij}^{ef} = \frac{1}{\varepsilon} < \nabla \cdot \Lambda_i^e, G, \nabla \cdot \Lambda_j^f > \approx \frac{\ell^e \ell^f}{\varepsilon} \sum_{k=1}^{K} \sum_{k'=1}^{K'} w_k w_{k'} \nabla \cdot \Lambda_i^e(\boldsymbol{\rho}^{(k)}) G\left(\boldsymbol{\rho}^{(k)}, \boldsymbol{\rho}^{(k')}\right) \nabla' \cdot \Lambda_j^f(\boldsymbol{\rho}^{(k')})$$

where

$$\boldsymbol{\rho}^{(k)} = \xi_1^{(k)} \boldsymbol{\rho}_1^e + \xi_2^{(k)} \boldsymbol{\rho}_2^e, \quad \boldsymbol{\rho}^{(k')} = \xi_1^{(k')} \boldsymbol{\rho}_1^f + \xi_2^{(k')} \boldsymbol{\rho}_2^f,$$

$$\Lambda_i^e(\boldsymbol{\rho}^{(k)}) = \xi_i^{(k)} \hat{\ell}_i^e, \quad i = 1, 2$$

$$\tilde{\nabla} \cdot \Lambda_i^e(\boldsymbol{\rho}^{(k)}) = \frac{1}{h_i}, \quad i = 1, 2$$



and $\left(\xi_1^{(k)}, w_k\right)$ are Gauss Legendre quadrature samples and weights.

Element Matrix Calculation—Self Terms

- For $e = f$, kernel is logarithmically singular at $\boldsymbol{\rho} = \boldsymbol{\rho}'$, so use special (MRW) quadrature rule designed for log singularity :

$$L_{ij}^{ee} = \mu < \Lambda_i^e; G, \Lambda_j^e >$$

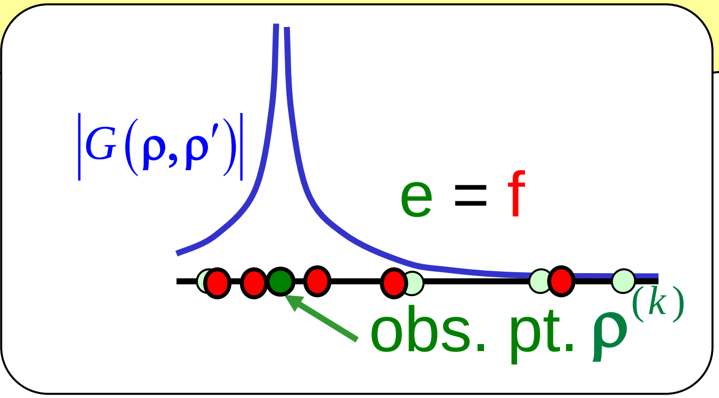
$$\approx \ell^e \ell^e \mu \sum_{k=1}^{K} \sum_{k'=1}^{K'} w_k w_{k'} G\left(\boldsymbol{\rho}^{(k)}, \boldsymbol{\rho}^{(k')}\right) \Lambda_i^e(\boldsymbol{\rho}^{(k)}) \cdot \Lambda_j^e(\boldsymbol{\rho}^{(k')})$$

$$S_{ij}^{ee} = \frac{1}{\varepsilon} < \nabla \cdot \Lambda_i^e, G, \nabla \cdot \Lambda_j^e >$$

$$\approx \frac{\ell^e \ell^e}{\varepsilon} \sum_{k=1}^{K} \sum_{k'=1}^{K'} w_k w_{k'} \nabla \cdot \Lambda_i^e(\boldsymbol{\rho}^{(k)}) G\left(\boldsymbol{\rho}^{(k)}, \boldsymbol{\rho}^{(k')}\right) \nabla' \cdot \Lambda_j^e(\boldsymbol{\rho}^{(k')})$$

Note : When $e = f$,

$\left(w_{k'}, \xi_i^{k'}\right)$ depend on $\boldsymbol{\rho}^{(k)}$



Element Matrix Calculation is at the Heart of Computational Electromagnetics

$$L_{ij}^{ef} = \mu < \Lambda_i^e ; G, \Lambda_j^f >$$

$$\approx \ell^e \ell^f \mu \sum_{k=1}^{K} \sum_{k'=1}^{K'} w_k w_{k'} G\left(\rho^{(k)}, \rho^{(k')}\right) \Lambda_i^e(\rho^{(k)}) \cdot \Lambda_j^f(\rho^{(k')})$$

$$S_{ij}^{ef} = \frac{1}{\varepsilon} < \nabla \cdot \Lambda_i^e, G, \nabla \cdot \Lambda_j^f >$$

$$\approx \frac{\ell^e \ell^f}{\varepsilon} \sum_{k=1}^{K} \sum_{k'=1}^{K'} w_k w_{k'} \nabla \cdot \Lambda_i^e(\rho^{(k)}) G\left(\rho^{(k)}, \rho^{(k')}\right) \nabla' \cdot \Lambda_j^f(\rho^{(k')})$$

Element Matrix Contributions are Assembled into the Global Matrix

$$\left[Z_{ij}^{\mathrm{ef}}\right]_{2\times 2} = j\omega \underbrace{\mu\left[<\Lambda_i^{\mathrm{e}};G,\Lambda_j^{\mathrm{f}}>\right]_{2\times 2}}_{\left[L_{ij}^{\mathrm{ef}}\right]} + \frac{1}{j\omega}\frac{1}{\varepsilon}\underbrace{\left[<\nabla\cdot\Lambda_i^{\mathrm{e}},G,\nabla\cdot\Lambda_j^{\mathrm{f}}>\right]_{2\times 2}}_{\left[S_{ij}^{\mathrm{ef}}\right]}$$

Matrix Assembly Rule:

Element $\sigma_i^e \sigma_j^f Z_{ij}^{ef}$ of the element matrix is added to row *m* and column *n* of the system matrix if *m* is the global DoF corresponding to the *i* th local DOF of element *e* and *n* is the global DoF corresponding to the *j* th local DOF of element *f*.

$$\left[Z_{mn}\right]_{N\times N} = j\omega \underbrace{\mu\left[<\Lambda_m;G,\Lambda_n>\right]_{N\times N}}_{\left[L_{mn}\right]} + \frac{1}{j\omega}\frac{1}{\varepsilon}\underbrace{\left[<\nabla\cdot\Lambda_m,G,\nabla\cdot\Lambda_n>\right]_{N\times N}}_{\left[S_{mn}\right]}$$

Computation of Voltage Excitation Vector

RHS voltage vector :

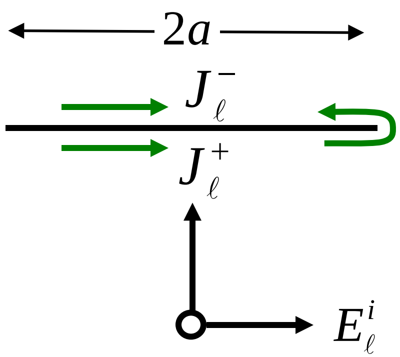
$$V_m \; = \; <\boldsymbol{\Lambda}_m;\mathbf{E}^{\mathrm{i}}> \; \equiv \int_{\mathcal{C}} \boldsymbol{\Lambda}_m(\boldsymbol{\rho})\cdot\mathbf{E}^{\mathrm{i}}(\boldsymbol{\rho})\,d\mathcal{C}.$$

Local voltage element vector :

$$V_i^e \; = \; <\boldsymbol{\Lambda}_i^e;\mathbf{E}^{\mathrm{i}}> \; \approx \; \ell^e \sum_{k=1}^{K} w_k\, \boldsymbol{\Lambda}_i^e(\boldsymbol{\rho}^{(k)})\cdot\mathbf{E}^{\mathrm{i}}(\boldsymbol{\rho}^{(k)})$$

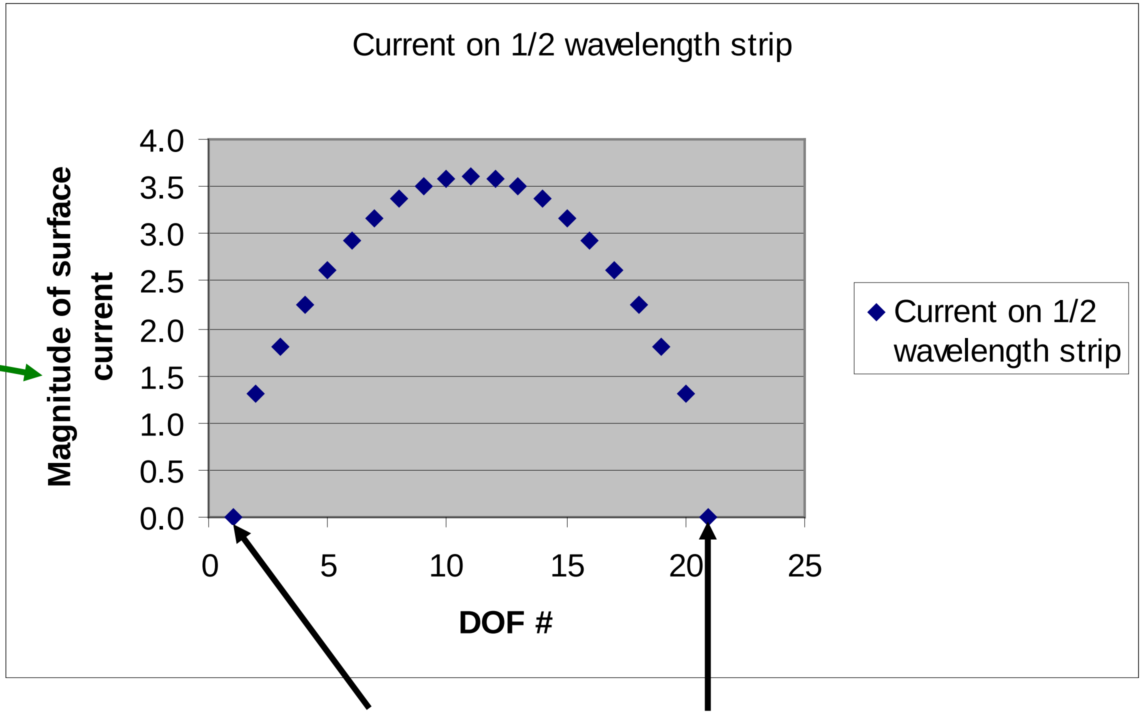
Accumulate *signed* contributions $\sigma_i^e V_i^e$ to the global RHS voltage vector by the assembly rule.

TE Scattering by Conducting Strip



$$J_\ell \equiv J_\ell^+ + J_\ell^-$$

$= 0$ **on strip ends**



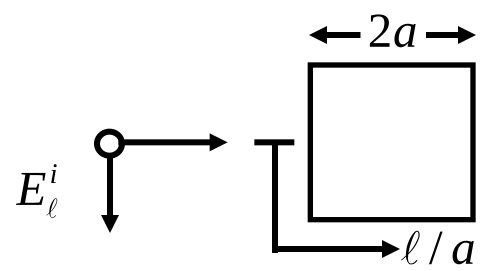
Zero currents at strip ends added to output data file for plotting!

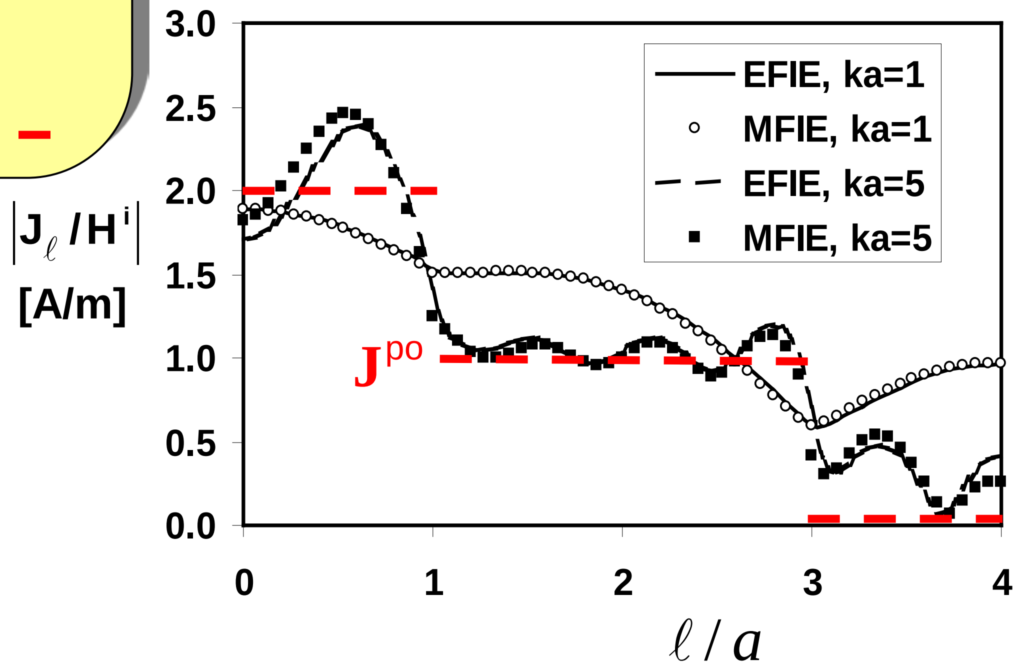
TE Scattering by a Square Cylinder

- Current normal to edges is continuous, but with infinite slope

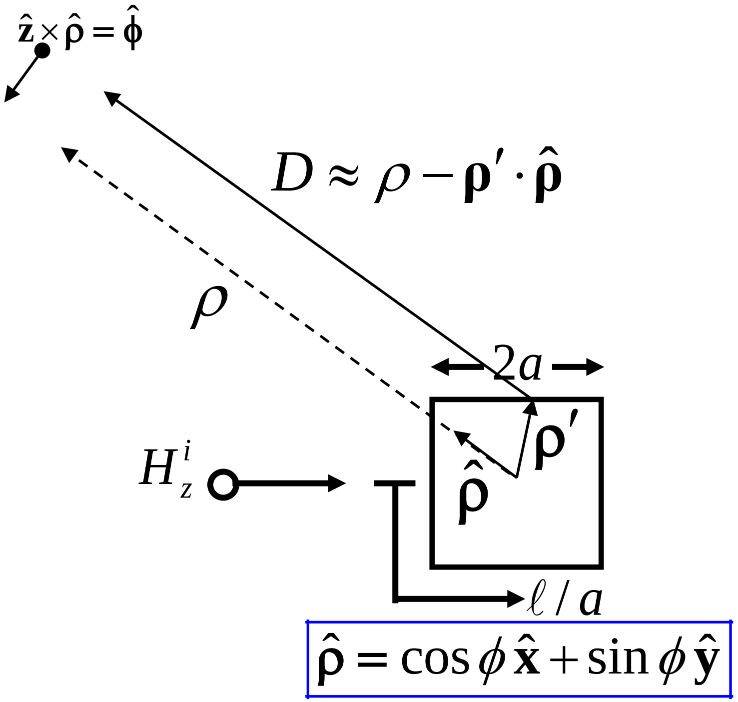
- At high frequencies, surface current approaches physical optics result, $\mathbf{J}^{po}$ – – – –

$$\mathbf{J} \xrightarrow{\omega \to \infty} \mathbf{J}^{po} = \hat{\mathbf{n}} \times \mathbf{H}^{inc}$$





TE Far Scattered Field for a Square Cylinder



Using

$$H_0^{(2)}(kD) \xrightarrow[\rho\to\infty]{} \sqrt{\frac{2}{\pi k\rho}}\, e^{-j(k\rho - \frac{\pi}{4})} e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'},$$

$$\hat{\boldsymbol{\rho}} \equiv \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}},$$

the far electric field is given by

$$E_\phi^s = -j\omega\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}\cdot\mathbf{A} \xrightarrow[\rho\to\infty]{} \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j(k\rho + \frac{\pi}{4})} \hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}\cdot\int_{\tilde{C}} \mathbf{J}(\boldsymbol{\rho}') e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'} d\mathcal{C}'$$

$$= \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j(k\rho + \frac{\pi}{4})} \hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}\cdot\left[\int_{\tilde{C}} \boldsymbol{\Lambda}_n(\boldsymbol{\rho}') e^{jk\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'} d\mathcal{C}'\right]^t [I_n]$$

$$= \frac{-j\omega\mu}{\sqrt{8\pi k\rho}} e^{-j(k\rho + \frac{\pi}{4})} \underbrace{\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}}}_{\hat{\boldsymbol{\phi}}}\cdot\left[\tilde{\boldsymbol{\Lambda}}_n(k\hat{\boldsymbol{\rho}})\right]^t [I_n]$$

The End