ECE 6350

Solution of Transmission Line Currents–Introduction to FEM

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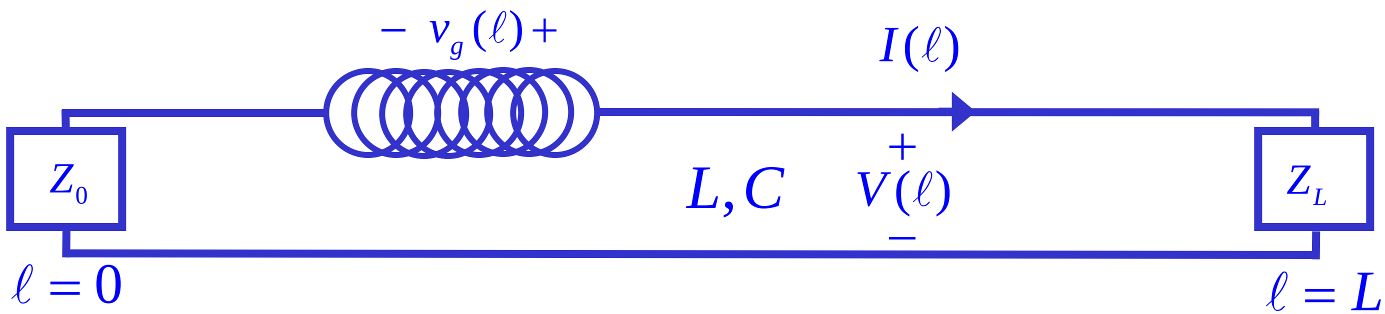
Transmission Line with Per Unit Length Voltage Sources



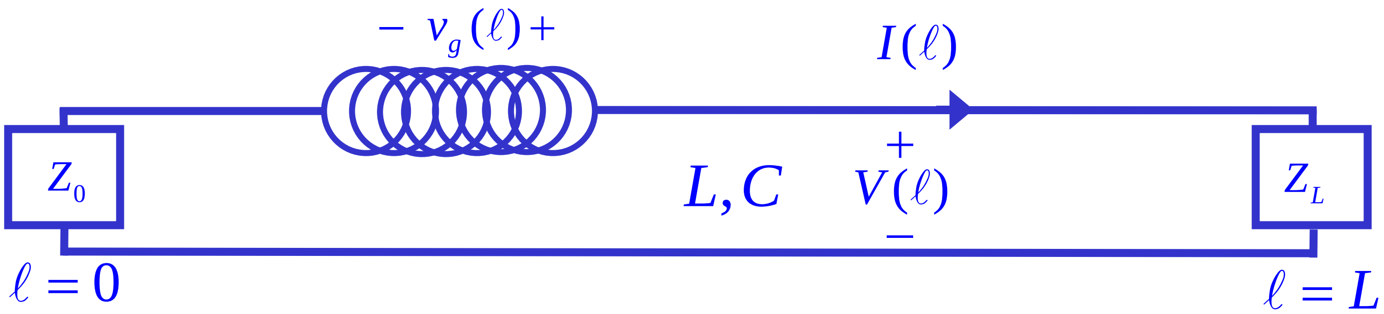
$$\left.\begin{aligned} -\frac{dV}{d\ell} &= j\omega L\, I - v_g \\ -\frac{dI}{d\ell} &= j\omega C\, V \end{aligned}\right\} \quad \overset{\text{eliminate } V}{\Longrightarrow} \quad \boxed{-\frac{1}{j\omega C}\frac{d^2 I}{d\ell^2} + j\omega L\, I \;=\; v_g}$$

or more often written as

$$\boxed{\frac{d^2 I}{d\ell^2} + k^2 I \;=\; -\, j\omega C v_g,}$$

where $\quad k^2 \equiv \omega^2 LC$

Problem Statement

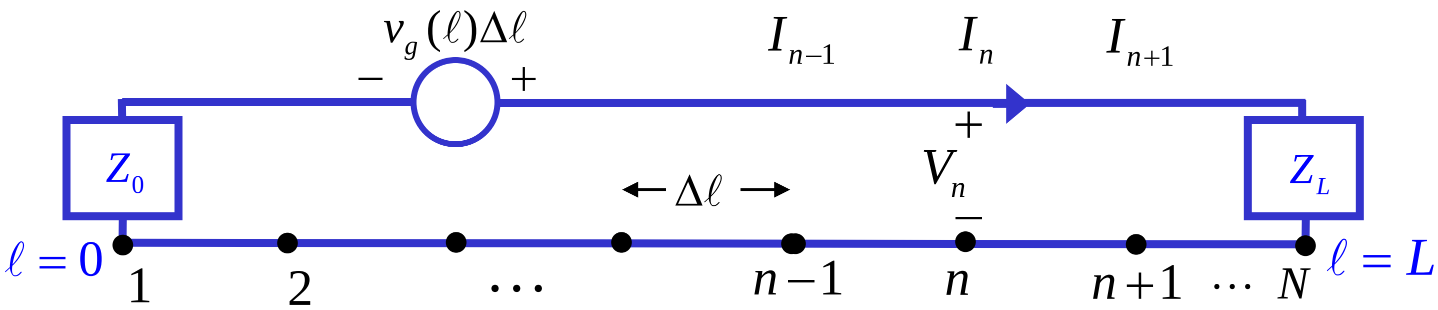


$$-\frac{1}{j\omega C}\frac{d^2 I}{d\ell^2} + j\omega L\, I = v_g$$

Given , $v_g(\ell)$, find $I(\ell)$ subject to boundary conditions

$$\frac{V(0)}{I(0)} = -Z_0, \qquad \frac{V(L)}{I(L)} = Z_L$$

Traditional Finite Difference Approach

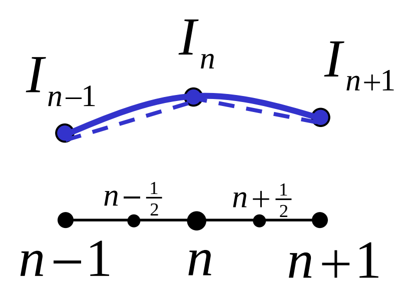


$$\boxed{-\frac{1}{j\omega C}\frac{d^2 I}{d\ell^2} + j\omega L\, I = v_g}$$

$$\boxed{\Delta\ell = \frac{L}{N-1}}$$

Approximate derivatives by finite differences :

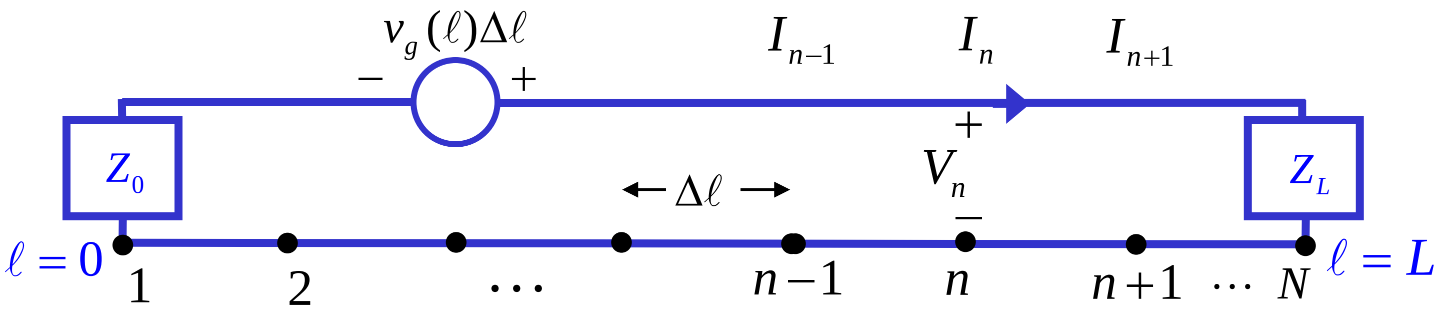
$$\left.\frac{dI}{d\ell}\right|_{n+\frac{1}{2}} \approx \frac{I_{n+1} - I_n}{\Delta\ell}, \qquad \left.\frac{dI}{d\ell}\right|_{n-\frac{1}{2}} \approx \frac{I_n - I_{n-1}}{\Delta\ell}$$



$$\left.\frac{d^2 I}{d\ell^2}\right|_n \approx \frac{\left.\frac{dI}{d\ell}\right|_{n+\frac{1}{2}} - \left.\frac{dI}{d\ell}\right|_{n-\frac{1}{2}}}{\Delta\ell} \approx \frac{I_{n+1} - 2I_n + I_{n-1}}{\Delta\ell^2}$$

$$n = 2, \ldots, N-1$$

Substitute Finite Diff. Approx. Into Wave Equation



$$\ell_n \equiv \frac{(n-1)L}{N-1}, \quad n = 1,\ldots,N$$

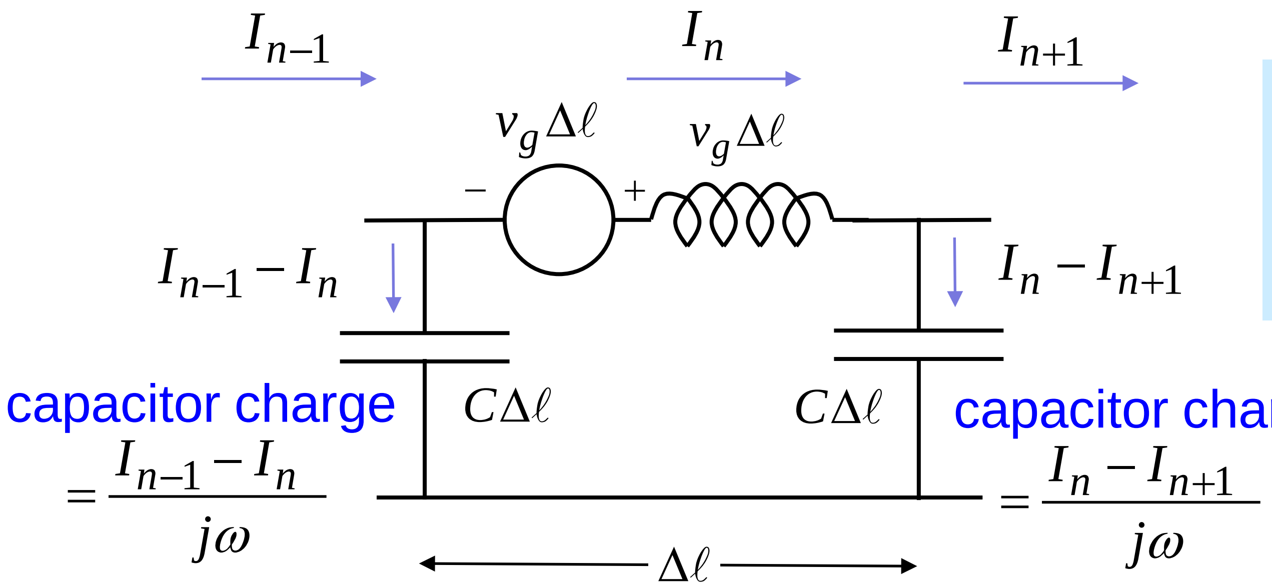
$$-\frac{1}{j\omega C}\frac{d^2 I}{d\ell^2} + j\omega L\, I = v_g$$

$$\Rightarrow \boxed{-\frac{I_{n+1} - 2I_n + I_{n-1}}{j\omega C\Delta\ell} + j\omega L\Delta\ell\, I_n \approx \Delta\ell v_g(\ell_n)}$$

or, more traditionally,

$$\Rightarrow \boxed{I_{n+1} - \left(2 - k^2\Delta\ell^2\right)I_n + I_{n-1} = -j\omega C\Delta\ell^2 v_g(\ell_n), \;\; n = 2,3,\cdots,N-1}$$

Circuit Interpretation of Finite Difference Eq.



From $i=\dfrac{dq}{dt}\Rightarrow\quad I=j\omega Q$

$\Rightarrow Q=\dfrac{I}{j\omega}$

- From KVL :

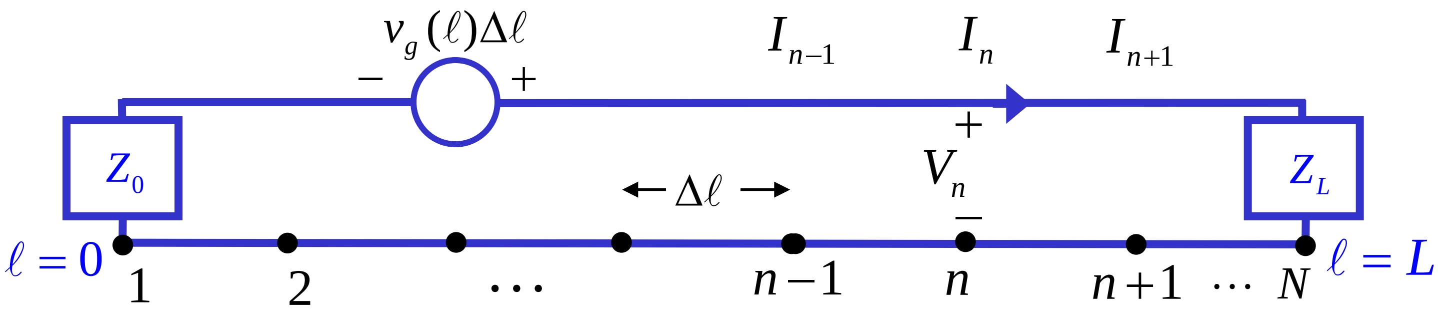
$$v_g\Delta\ell = j\omega L\Delta\ell I_n + \frac{I_n-I_{n+1}}{j\omega C\Delta\ell} - \frac{I_{n-1}-I_n}{j\omega C\Delta\ell}$$

$$\Rightarrow \boxed{-\frac{I_{n+1}-2I_n+I_{n-1}}{j\omega C\Delta\ell} + j\omega L\Delta\ell\, I_n \approx \Delta\ell v_g(\ell_n)}$$

For a *lumped* voltage source V_0 at node n, replace $\Delta\ell v_g(\ell_n)$ by V_0.

- Simulation could be performed using a circuit simulator, such as SPICE.

Discretize the Boundary Conditions



Boundary conditions:

$$\frac{V(0)}{I(0)} = -Z_0 = -\frac{1}{j\omega C I(0)}\frac{dI(0)}{d\ell} \approx \frac{I_1 - I_2}{j\omega C\Delta\ell I_1}, \quad \Rightarrow \boxed{\left(1 + j\omega C\Delta\ell Z_0\right) I_1 - I_2 = 0}$$

$$\frac{V(L)}{I(L)} = Z_L = -\frac{1}{j\omega C I(L)}\frac{dI(L)}{d\ell} \approx \frac{I_{N-1} - I_N}{j\omega C\Delta\ell I_N} \quad \Rightarrow \boxed{-I_{N-1} + \left(1 + j\omega C\Delta\ell Z_0\right) I_N = 0}$$

Write the Resulting Linear System in Matrix Form

$$[Z_{mn}][I_n] = [V_m]$$ **where ,**

$$[I_n] = \begin{bmatrix} I_1 \\ \vdots \\ I_{n-1} \\ I_n \\ I_{n+1} \\ \vdots \\ I_N \end{bmatrix}, \qquad [V_m] = \begin{bmatrix} 0 \\ -j\omega C\Delta\ell^2 v_g(\ell_2) \\ \vdots \\ -j\omega C\Delta\ell^2 v_g(\ell_m) \\ \vdots \\ -j\omega C\Delta\ell^2 v_g(\ell_{N-1}) \\ 0 \end{bmatrix},$$

$$[Z_{mn}] = \begin{bmatrix}
-(1+j\omega C\Delta\ell Z_0) & 1 & 0 & & & \cdots & & & 0 \\
1 & -2+k^2\Delta\ell^2 & 1 & \ddots & & & & & \\
0 & \ddots & \ddots & \ddots & & & & & \\
& \ddots & 1 & -2+k^2\Delta\ell^2 & 1 & & & & \vdots \\
& & & 1 & -2+k^2\Delta\ell^2 & 1 & & & \\
\vdots & & & & 1 & -2+k^2\Delta\ell^2 & 1 & & \\
& & & & & \ddots & \ddots & \ddots & 0 \\
& & & & & \ddots & 1 & -2+k^2\Delta\ell^2 & 1 \\
0 & & & \cdots & & & 0 & 1 & -(1+j\omega C\Delta\ell Z_L)
\end{bmatrix}$$

Some Observations On the Matrix System

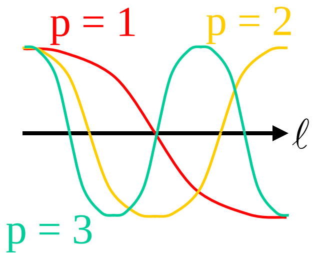
- Each unknown current I_n couples only to its nearest neighbor. As a result the matrix is *sparse* --- in fact, *tri-diagonal*

- The system is *resonant* at frequencies f_p such that $\det[Z_{mn}] = 0$ i.e., this determinantal equation determines frequencies f_p such that there exists a non-vanishing set of currents [I_n] satisfying the matrix equation---*even* with no excitation (i.e., $[V_n] = 0$).

- E.g. if $Z_0 = Z_L = 0$ (short-circuited line) then we should expect that

$$I_n^p \approx A\cos\frac{p\pi\ell_n}{L}, \quad L = p\frac{\lambda_0}{2}, \quad f_p = p\frac{c}{\lambda_0} = p\frac{c}{2L},$$

$$1 \le n \le N, \quad p = 1, 2, \ldots$$



Observations On the Matrix System Form

- The finite difference approach here may be viewed as a special case of more general approaches that go by the names *method of moments* or *finite element method*

- These methods are just more general means for turning an equation like the wave equation into a corresponding *discrete* form of the equation

- All discrete, linear forms may be solved as a system of linear equations

Steps in the Moment or Finite Element Methods

- Approximate the unknown as a linear combination of (known) basis functions with unknown coefficients

$$I(\ell) \approx \sum_{n=1}^{N} \underbrace{I_n}_{\text{unknown cofficients}} \underbrace{b_n(\ell)}_{\text{known basis functions}}$$

- Substitute the unknown representation into the system equation

- Enforce the original equality as a weighted average

- Solve the resulting system of linear equations

Express $I(\ell)$ in Series Form

$$I(\ell) \approx \sum_{n=1}^{N} \underbrace{I_n}_{\text{unknown cofficients}} \underbrace{b_n(\ell)}_{\text{known basis functions}}$$

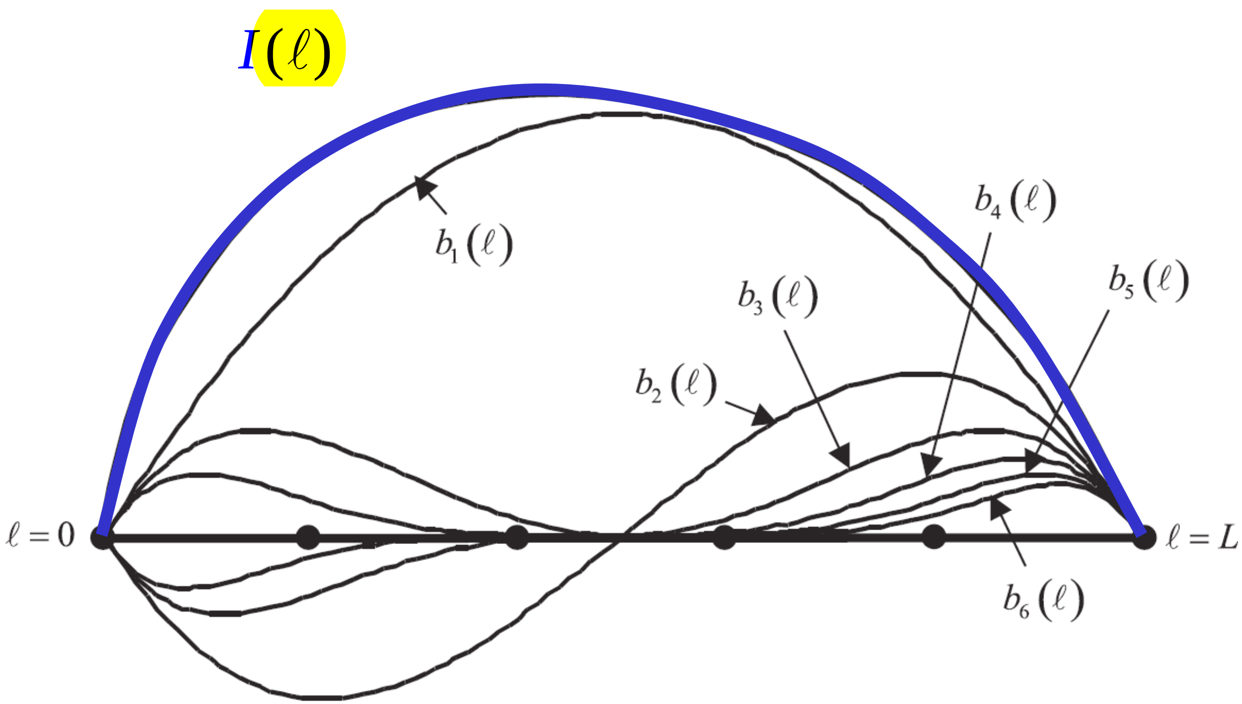
- The unknown coefficients are also known as *degrees of freedom* (DoF)

- Possible *basis functions* include power series, polynomials, Fourier sine or cosine series terms, interpolation functions, etc.

- For simplicity, we'll assume an open circuited line, $I(0) = I(L) = 0$ and consider several bases

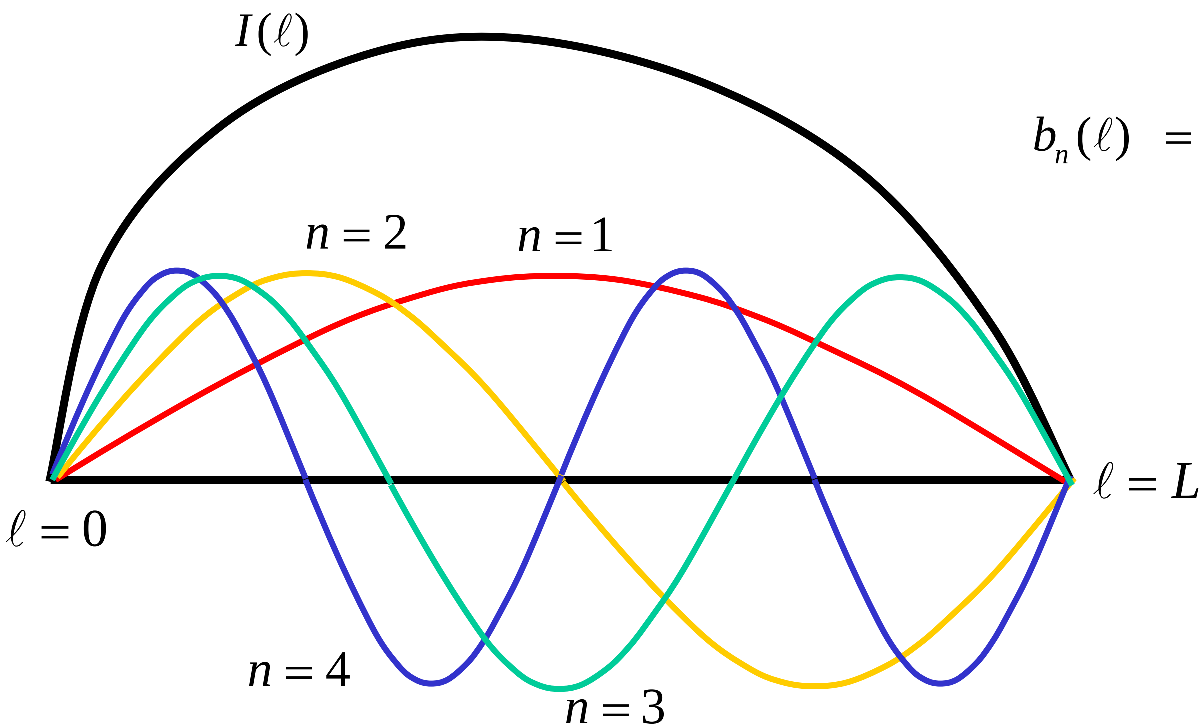
Power Series as Bases

$$b_n(\ell) = \underbrace{\left(\ell - \frac{L}{2}\right)^{n-1}}_{\text{powers of } \ell \text{ centered about } \ell = \frac{L}{2}} \times \underbrace{\ell(L-\ell)}_{\substack{\text{satisfies BCs} \\ I(0)=I(L)=0}}, \quad n = 1, 2, \ldots, N$$



- Power series bases become less and less independent for large n
  $\Rightarrow$ we can't easily determine coefficients of b_n and b_{n+1} when we can't *numerically* distinguish between the bases.
- Smoothness is built into representation, but sometimes we *need* to at least allow slope discontinuities

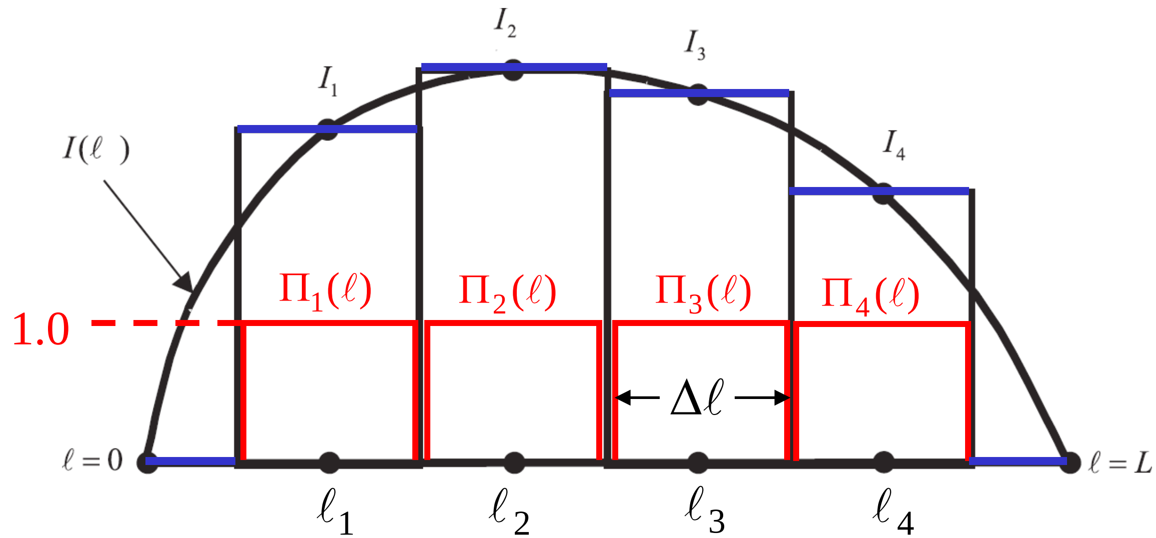
Fourier Series as Bases



$$b_n(\ell) = \sin\frac{n\pi\ell}{L}, \quad n = 1, 2, \ldots, N$$

- Fourier series bases, since they are orthogonal, have maximal independence, but are *slow to converge* if the solution is *discontinuous* or has *slope discontinuities*. (Theorem: If the αth derivative of $I(\ell)$ is piecewise continuous and all lower order derivatives are continuous, then the Fourier coefficients I_n decay as $\mathcal{O}(1/n^{\alpha+1})$.)

Piecewise Constant (PWC) Bases Are Independent and Allow Slope Discontinuities



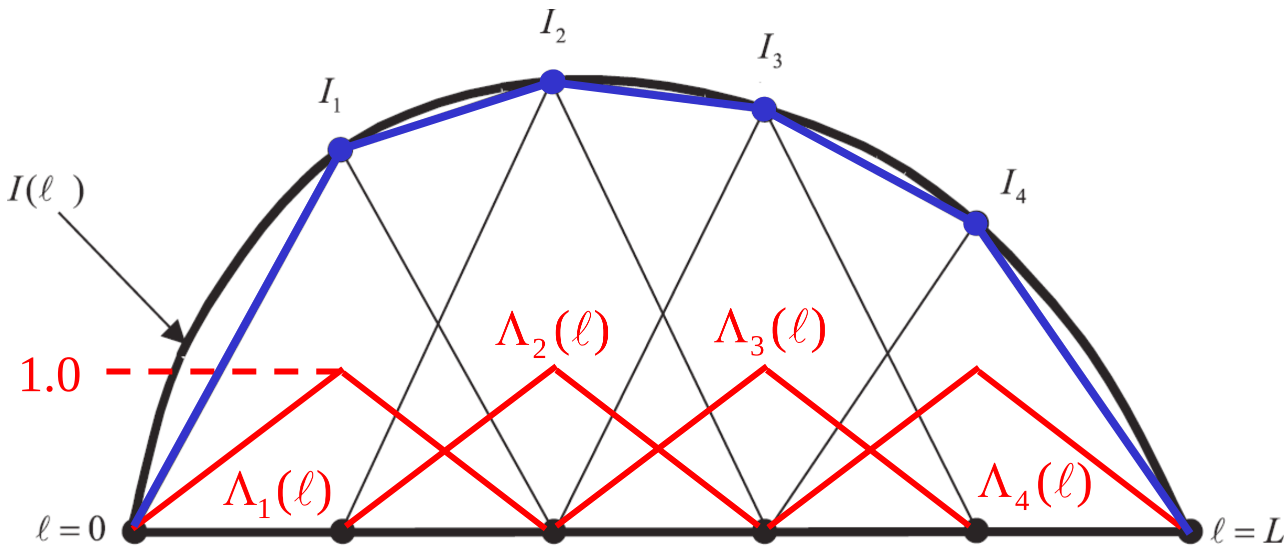
$$I(\ell) \approx \sum_{n=1}^{N} I_n b_n(\ell)$$

$$b_n(\ell) = \Pi_n(\ell) \equiv \begin{cases} 1, & \ell \in \left(\ell_n - \Delta\ell/2,\ \ell_n + \Delta\ell/2\right) \\ 0, & \text{otherwise} \end{cases}$$

- But first and second derivatives do not "exist" :

$$\frac{d\Pi_n(\ell)}{d\ell} = \delta\left[\ell - \left(\ell_n - \Delta\ell/2\right)\right] - \delta\left[\ell - \left(\ell_n + \Delta\ell/2\right)\right], \text{etc.}$$

Piecewise Linear Bases (PWL) Are Relatively Independent and Allow Slope Discontinuities



$$I(\ell) \approx \sum_{n=1}^{N} I_n b_n(\ell)$$

$$b_n(\ell) = \boxed{\Lambda_n(\ell)} \equiv \begin{cases} \dfrac{\ell - \ell_{n-1}}{\Delta\ell}, & \ell \in \left(\ell_{n-1}, \ell_n\right) \\ \dfrac{\ell_{n+1} - \ell}{\Delta\ell}, & \ell \in \left(\ell_n, \ell_{n+1}\right) \\ 0, & \text{otherwise} \end{cases}$$

- Here, *second* derivative does not exist, but we'll use a trick to circumvent the problem!

- This method is essentially *linear interpolation* of the unknown

Moment / Finite Element Approach Using PWL Bases

- Substitute the current representation $I(\ell) \approx \sum_{n=1}^{N} I_n \Lambda_n(\ell)$ into the wave equation $\frac{d^2 I}{d\ell^2} + k^2 I = -j\omega C v_g$

$$\Rightarrow \sum_{n=1}^{N} I_n \left(\frac{d^2 \Lambda_n}{d\ell^2} + k^2 \Lambda_n \right) \approx -j\omega C v_g$$

- Enforce the approximate equality to be exact in a *weighted average* sense, where $\Lambda_m(\ell)$ is the *weighting function* :

$$\sum_{n=1}^{N} I_n \int_0^L \Lambda_m(\ell) \left(\frac{d^2 \Lambda_n(\ell)}{d\ell^2} + k^2 \Lambda_n(\ell) \right) d\ell = -j\omega C \int_0^L \Lambda_m(\ell) v_g \, d\ell, \quad m = 1, 2, \ldots, N$$

- Integrate the second derivative term by parts to reduce the differentiability requirement on Λ_n

Integration by Parts Details

- Integration by parts :

Why?

$$\int_0^L \Lambda_m(\ell)\frac{d^2\Lambda_n(\ell)}{d\ell^2}d\ell = \cancel{\Lambda_m(\ell)\frac{d\Lambda_n(\ell)}{d\ell}\Bigg|_{\ell=0}^{L}} - \int_0^L \frac{d\Lambda_m(\ell)}{d\ell}\frac{d\Lambda_n(\ell)}{d\ell}d\ell$$

or in *scalar* or *inner product* notation,

$$\boxed{\left\langle \Lambda_m, \frac{d^2\Lambda_n}{d\ell^2}\right\rangle = -\left\langle \frac{d\Lambda_m}{d\ell}, \frac{d\Lambda_n}{d\ell}\right\rangle}$$

where

$$< A, B > \equiv \int_0^L A(\ell)\, B(\ell)\, d\ell$$

Product of functions followed by integration over continuous variable ℓ

which generalizes the ordinary dot or scalar product,

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i$$

$$\mathbf{A} = A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}} + A_3\hat{\mathbf{z}}\ ,\ \ \mathbf{B} = B_1\hat{\mathbf{x}} + B_2\hat{\mathbf{y}} + B_3\hat{\mathbf{z}}$$

Product of components followed by summation over discrete index *i*

System Equation

- Substituting into the above result, we obtain the linear system

$$\sum_{n=1}^{N} I_n \left(\int_0^L -\frac{d\Lambda_m(\ell)}{d\ell}\frac{d\Lambda_n(\ell)}{d\ell} + k^2 \Lambda_m(\ell)\Lambda_n(\ell)\, d\ell \right) = -j\omega C \int_0^L \Lambda_m(\ell) v_g\, d\ell, \quad m = 1, 2, \ldots, N$$

or dividing by $-j\omega C$, and using scalar product and matrix notation,

$$\boxed{[Z_{mn}][I_n] = [V_m]}$$

System equation

where

$$\boxed{\begin{aligned} [Z_{mn}] &= \frac{1}{j\omega C}\left\langle \frac{d\Lambda_m}{d\ell}, \frac{d\Lambda_n}{d\ell} \right\rangle + j\omega L < \Lambda_m, \Lambda_n >, \\ [V_m] &= < \Lambda_m, v_g > \end{aligned}}$$

Note:

$$\frac{k^2}{-j\omega C} = j\frac{\omega^{\cancel{2}} L \cancel{C}}{\cancel{\omega} \cancel{C}} = j\omega L$$

- The resulting sparse matrix may be solved for the column vector $[I_n]$; the current is then given by $I(\ell) \approx \sum_{n=1}^{N} I_n \Lambda_n(\ell)$

Linear System in Matrix Form

To compare to FD scheme, write as

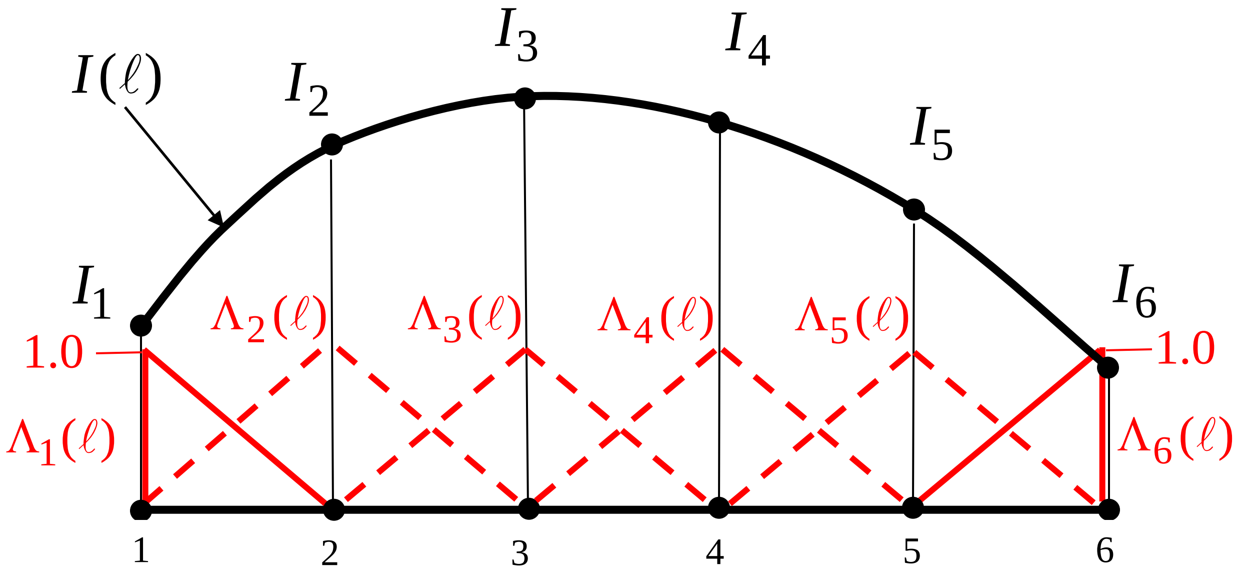
$$-\cancel{j\omega C}\left[Z_{mn}\right]\left[I_n\right] = -\cancel{j\omega C}\left[V_m\right] \text{ ,} \quad \left[I_n\right] = \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ \vdots \\ I_N \end{bmatrix} \text{ ,} \qquad \left[V_m\right] = \begin{bmatrix} <\Lambda_1, v_g> \\ \vdots \\ <\Lambda_m, v_g> \\ \vdots \\ <\Lambda_N, v_g> \end{bmatrix} \text{ ,}$$

where

$$-j\omega C\left[Z_{mn}\right] = \begin{bmatrix} -2+\frac{2}{3}k^2\Delta\ell^2 & 1+\frac{1}{6}k^2\Delta\ell^2 & 0 & \cdots & 0 \\ 1+\frac{1}{6}k^2\Delta\ell^2 & \ddots & \ddots & \ddots & \vdots \\ 0 & 1+\frac{1}{6}k^2\Delta\ell^2 & -2+\frac{2}{3}k^2\Delta\ell^2 & 1+\frac{1}{6}k^2\Delta\ell^2 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1+\frac{1}{6}k^2\Delta\ell^2 \\ 0 & \cdots & 0 & 1+\frac{1}{6}k^2\Delta\ell^2 & -2+\frac{2}{3}k^2\Delta\ell^2 \end{bmatrix}$$

which has a very similar structure to the finite difference formulation!

Modifications to Incorporate Impedance Boundary Conditions



- With loads at ends, current no longer vanishes there so add DoF's and associated (half-) bases at each end.
- $I(\ell) = \sum_{n=1}^{N} I_n \Lambda_n(\ell)$; note $I(0) = I_1,\ I(L) = I_N$.

BC's:
$$V(0)/I(0) = -Z_0 = -I'(0)/\left(j\omega C I(0)\right),$$
$$V(L)/I(L) = Z_L = -I'(L)/\left(j\omega C I(L)\right),$$

Modifications to Incorporate Impedance Boundary Conditions (cont'd)

- Return to the wave equation and boundary conditions :

BC's :
$$\frac{V(0)}{I(0)} = -Z_0 = -\frac{\boxed{I'(0)}}{j\omega C I(0)},$$
$$\frac{V(L)}{I(L)} = Z_L = -\frac{\boxed{I'(L)}}{j\omega C I(L)},$$

Wave equation :
$$\frac{d^2 I}{d\ell^2} + k^2 I = -j\omega C v_g$$

- Test wave equation with $\Lambda_m(\ell)$ *first*. Then integrate by parts, noting

$$\Lambda_m(0) = \Lambda_m(L) = 0, \quad m \neq 0, N; \;\; \Lambda_N(0) = \Lambda_1(L) = 0, \; \Lambda_1(0) = \Lambda_N(L) = 1:$$

$$\int_0^L \Lambda_m \left(\frac{d^2 I}{d\ell^2} + k^2 I \right) d\ell = -j\omega C \int_0^L \Lambda_m v_g \, d\ell, \quad m = 1, 2, \ldots, N$$

$$\Rightarrow \boxed{\overbrace{\Lambda_m(L) I'(L)}^{-j\omega C Z_L I_N \delta_{mN}} \; \overbrace{-\Lambda_m(0) I'(0)}^{-j\omega C Z_0 I_1 \delta_{m1}} - \int_0^L \frac{d\Lambda_m}{d\ell} \frac{dI}{d\ell} d\ell + k^2 \int_0^L \Lambda_m I \, d\ell = -j\omega C \int_0^L \Lambda_m v_g \, d\ell, \quad m = 1, 2, \ldots, N}$$

Kronecker delta :
$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Modifications to Incorporate Impedance Boundary Conditions (cont'd)

- Substitute the BC's to eliminate the derivatives $I'(0),\ I'(L)$, and note that $I(0) \equiv I_1,\ I(L) \equiv I_N$:

$$\Rightarrow -j\omega C\left[Z_0 I_1 \delta_{m1} + Z_L I_N \delta_{mN}\right] + \int_0^L \left(-\frac{d\Lambda_m}{d\ell}\frac{dI}{d\ell} + k^2 \Lambda_m I \right) d\ell$$

$$= -j\omega C \int_0^L \Lambda_m v_g \, d\ell, \qquad m = 1, 2, \ldots, N$$

- Now substitute the current expansion, $\sum_{n=1}^{N} I_n \Lambda_n(\ell)$:

$$-j\omega C\left[Z_0 I_1 \delta_{m1} + Z_L I_N \delta_{mN}\right] + \sum_{n=1}^{N} I_n \int_0^L \left(-\frac{d\Lambda_m}{d\ell}\frac{d\Lambda_n}{d\ell} + k^2 \Lambda_m \Lambda_n \right) d\ell$$

$$= -j\omega C \int_0^L \Lambda_m v_g \, d\ell, \qquad m = 1, 2, \ldots, N \quad \text{or}$$

$$\left[Z_0 I_1 \delta_{m1} + Z_L I_N \delta_{mN}\right] + \sum_{n=1}^{N} I_n \int_0^L \left(\frac{1}{j\omega C}\frac{d\Lambda_m}{d\ell}\frac{d\Lambda_n}{d\ell} + j\omega L \, \Lambda_m \Lambda_n \right) d\ell = \int_0^L \Lambda_m v_g \, d\ell \,,$$

$$m = 1, 2, \ldots, N$$

Matrix Equation Incorporating Impedance Boundary Conditions

$$\underbrace{\left(Z_0 I_1 \delta_{m1} + Z_L I_N \delta_{mN}\right)}_{\sum_{n=1}^{N}\left(Z_0 I_n \delta_{n1}\delta_{m1} + Z_L I_n \delta_{nN}\delta_{mN}\right)} + \sum_{n=1}^{N} I_n \int_0^L \left(\frac{1}{j\omega C} \frac{d\Lambda_m}{d\ell} \frac{d\Lambda_n}{d\ell} + j\omega L \Lambda_m \Lambda_n \right) d\ell$$

$$= \int_0^L \Lambda_m v_g \, d\ell, \quad m = 1, 2, \ldots, N$$

$$\Rightarrow \quad [Z_{mn}][I_n] = [V_m]$$ (Linear system of equations)

Impedance Matrix : $[Z_{mn}] = \left[Z_{mn}^L\right] + \frac{1}{j\omega}[S_{mn}] + j\omega[L_{mn}].$

Current Column Vector

$$[I_n] = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix},$$

Voltage Column Vector

$$[V_m] = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} \int_0^L \Lambda_1 v_g \, d\ell \\ \int_0^L \Lambda_2 v_g \, d\ell \\ \vdots \\ \int_0^L \Lambda_N v_g \, d\ell \end{bmatrix},$$

Load matrix

$$\left[Z_{mn}^L\right] = \begin{bmatrix} Z_0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & Z_L \end{bmatrix}$$

Matrix Equation Incorporating Impedance Boundary Conditions (cont'd)

Impedance Matrix :

$$\left[Z_{mn}\right]\left[I_n\right]=\left[V_m\right] \text{ where } \left[Z_{mn}\right]=\left[Z_{mn}^L\right]+\frac{1}{j\omega}\left[S_{mn}\right]+j\omega\left[L_{mn}\right]$$

Elastance Matrix :

$$\frac{1}{j\omega}\left[S_{mn}\right],\quad \left[S_{mn}\right]=\frac{1}{C}\left[\int_0^L \frac{d\Lambda_m}{d\ell}\frac{d\Lambda_n}{d\ell}\,d\ell\right]=\frac{1}{C\Delta\ell}\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

Inductance Matrix :

$$j\omega\left[L_{mn}\right],\quad \left[L_{mn}\right]=L\left[\int_0^L \Lambda_m\Lambda_n\,d\ell\right]=\frac{L\Delta\ell}{6}\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & 1 & 4 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

Summary of Generalized Moment / Finite Element Approach

- To clarify and generalize, use the operator notation,

$$\underbrace{\mathcal{L}}_{\substack{\text{linear operator,} \\ \text{e.g., } \frac{d^2}{d\ell^2}+k^2}} \quad \underbrace{u}_{\substack{\text{unknown,} \\ \text{e.g., } I}} = \underbrace{f}_{\substack{\text{forcing function,} \\ \text{e.g., } -j\omega C v_g}}$$

where the *linear operator* satisfies

$$\mathcal{L}(au_1 + bu_2) = a\mathcal{L}u_1 + b\mathcal{L}u_2$$
$$= af_1 + bf_2$$

with a, b constant

- Approximate $u(\mathbf{r})$ via a set of bases $b_n(\mathbf{r})$,

$$u(\mathbf{r}) \approx \sum_{n=1}^{N} U_n b_n(\mathbf{r}) = [U_n]^t [b_n(\mathbf{r})]$$

- Substitute into the operator equation, yielding by linearity

$$\mathcal{L}\left(\sum_{n=1}^{N} U_n b_n\right) = \sum_{n=1}^{N} U_n \mathcal{L} b_n \approx f$$

Typical Linear Operators

Matrix: $[L_{mn}][U_n] = [V_m]$

Differential: $d^2u/dx^2 + k^2 u = f$

Partial Differential: $\nabla^2 u = -f$,

$\nabla^2 u + k^2 u = -f$

Integral: $\int_0^L k(x,x')u(x')dx' = f(x)$

Integrodifferential:

$$\left(\frac{d^2}{dx^2}+k^2\right)\int_0^L k(x,x')u(x')dx' = f(x),$$

$$j\omega \mathbf{A}_{\tan} + \nabla_{\tan}\Phi = \mathbf{E}_{\tan},$$

where $\mathbf{A} = \mu \int_S \frac{e^{-jkR}}{4\pi R}\mathbf{J}(\mathbf{r}')dS'$,

$$\Phi = \frac{-1}{j\omega\varepsilon}\int_S \frac{e^{-jkR}}{4\pi R}\nabla'\cdot \mathbf{J}(\mathbf{r}')dS'$$

Moment / Finite Element Approach, cont'd

- Enforce equality in a weighted avg. sense using weight $w_m(\mathbf{r})$;

  i.e., multiply both sides of equation by w_m and integrate :

$$\sum_{n=1}^{N} U_n < w_m \,, \mathcal{L} b_n > \; = \; < w_m \,, f >, \qquad m = 1, 2, \ldots, N$$

- If the operator involves differentials, integrate by parts, if possible, and incorporate boundary conditions; the result can be written in matrix form as

$$[L_{mn}][U_n] = [F_m]$$

  where $L_{mn} = < w_m \,, \mathcal{L} b_n >, \quad F_m = < w_m \,, f >$

- Solve the system for the unknown column vector $[U_n] = [L_{mn}]^{-1}[F_m]$;

  then $u(\mathbf{r}) \approx \sum_{n=1}^{N} U_n b_n(\mathbf{r}) \; = \; [b_n(\mathbf{r})]^{t} [U_n]$

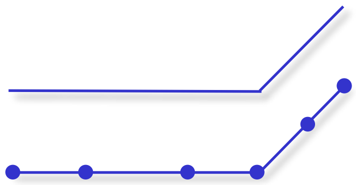
Comments on Moment / Finite Element Approach

- Notes :
  - The case $w_m = b_m$ is known as *Galerkin's method*
  - Often the bases must be chosen such that BC's are satisfied
  - $\mathcal{L}$ can be *any* linear operator, so the method is extremely general
  - Multiplication by w_m followed by integration is called *testing*
  - The weight functions w_m are also called *testing functions*
  - The order of 1) testing the operator equation and 2) expanding the unknown may be reversed
  - The tested operator equation, $< w_m, \mathcal{L}u > \; = \; < w_m, f >$, is known as the *weak form* of the operator equation; $\mathcal{L}u = f$ is the *strong form*

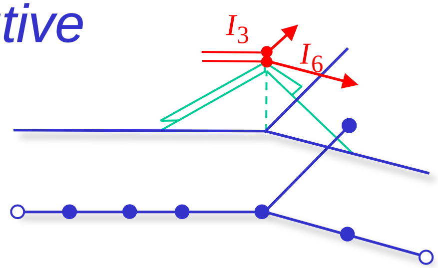
Generalized Transmission Line Problem

Now let's generalize the transmission line problem in several ways:

- The transmission line need *not be straight*
- We subdivide the line into segments but the *segment lengths* need not be e*qual*

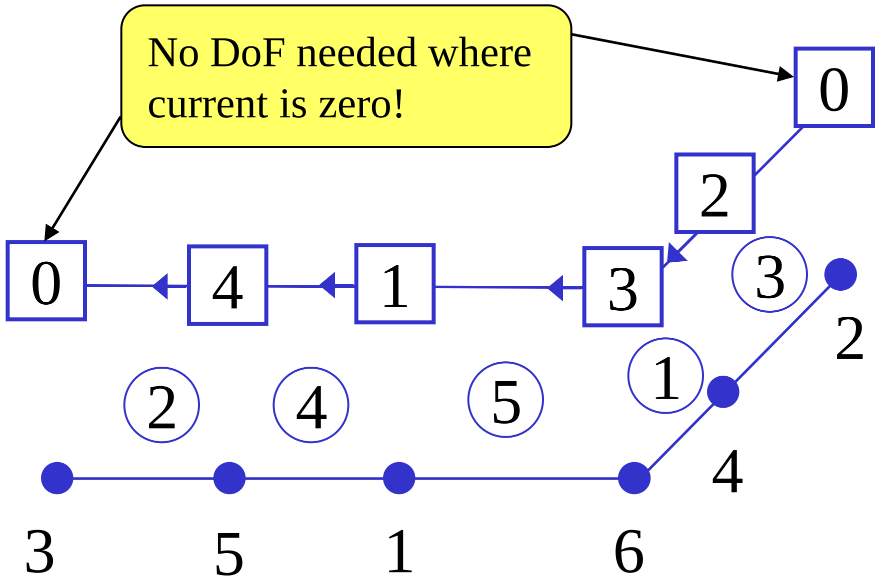


- We number the segments and their endpoints (nodes), but the *segment and node numbers need not be consecutive*



- We number the unknowns (degrees of freedom) and provide current reference directions at each node, but since currents may or may not exist at the end nodes, and more than one current may appear at a junction or load, the *node and DoF numbers need not correspond*
- For now, we'll keep the line ends open-circuited, but later modify to allow arbitrary loads

Node, Element, and DoF Specification

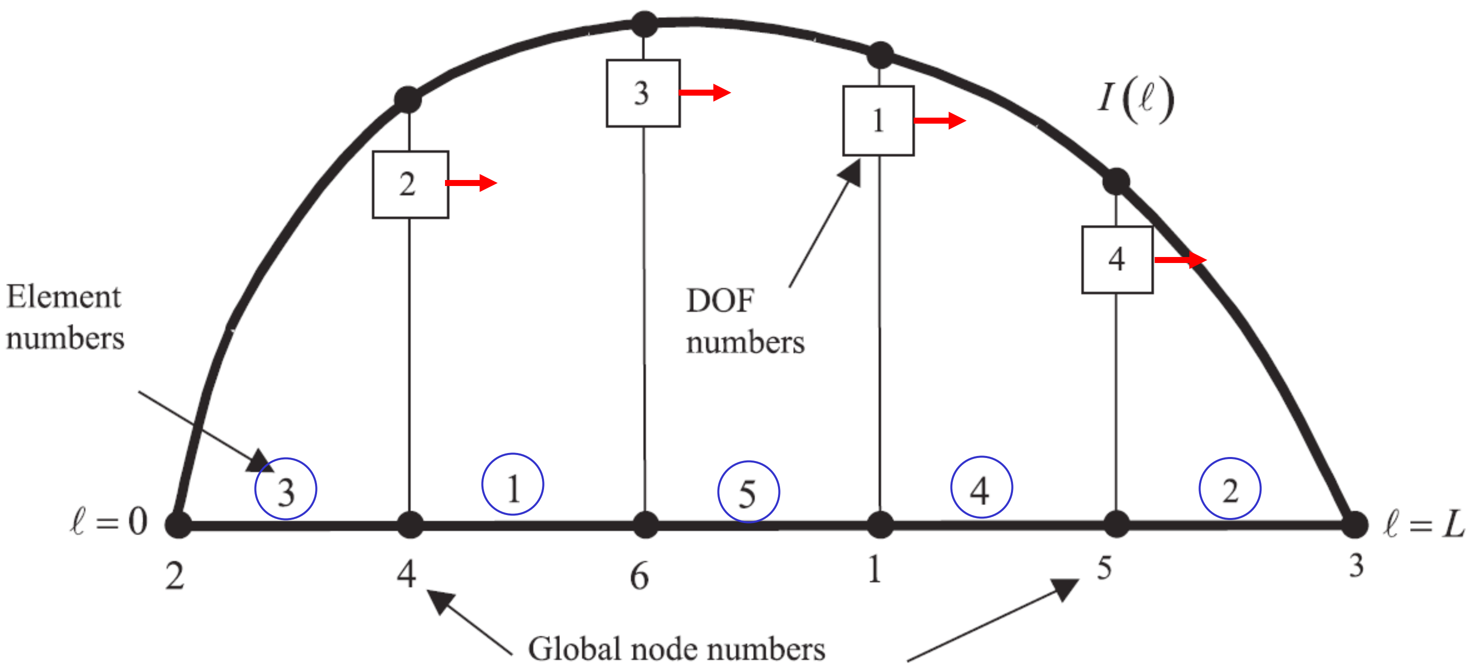


5 - Node number

3 - Element (segment) number

4 - DoF (unknown) number *with reference direction*

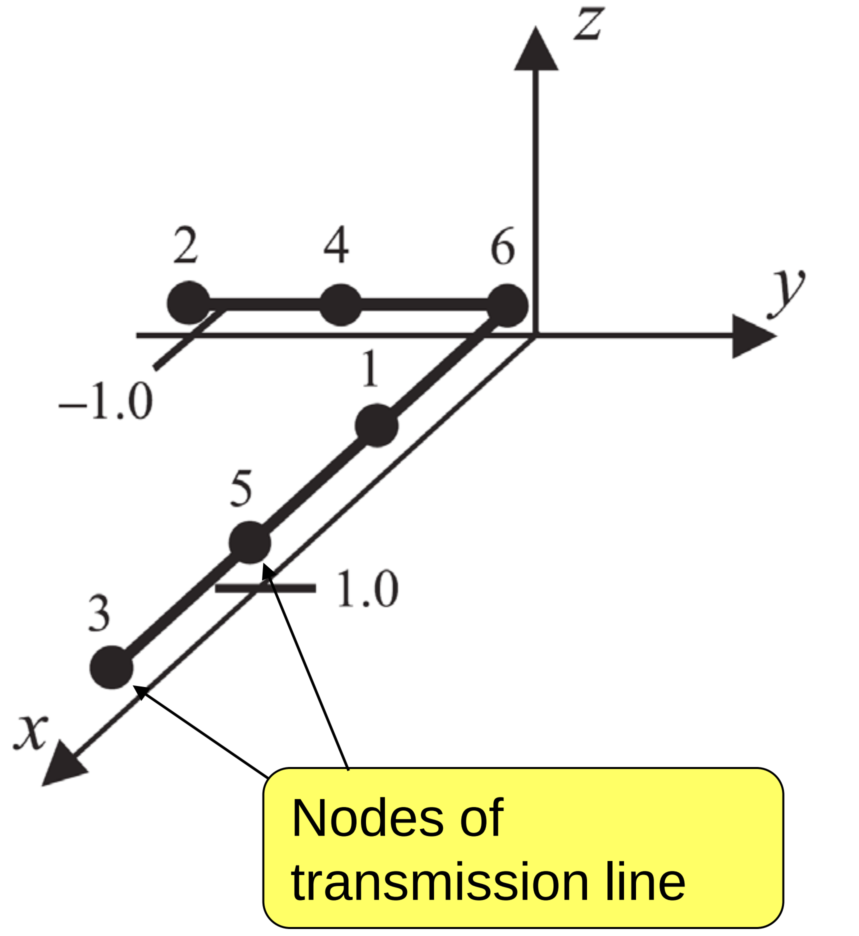
Geometry and Current Representations



The Global Node List Defines the Geometry

Geometry (Nodal) Specification

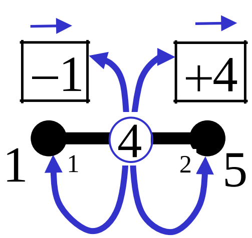
Global Node Number	Coordinates x	y	z
1	0.5	0.0	0.0
2	0.0	–1.0	0.0
3	1.5	0.0	0.0
4	0.0	–0.5	0.0
5	1.0	0.0	0.0
6	0.0	0.0	0.0

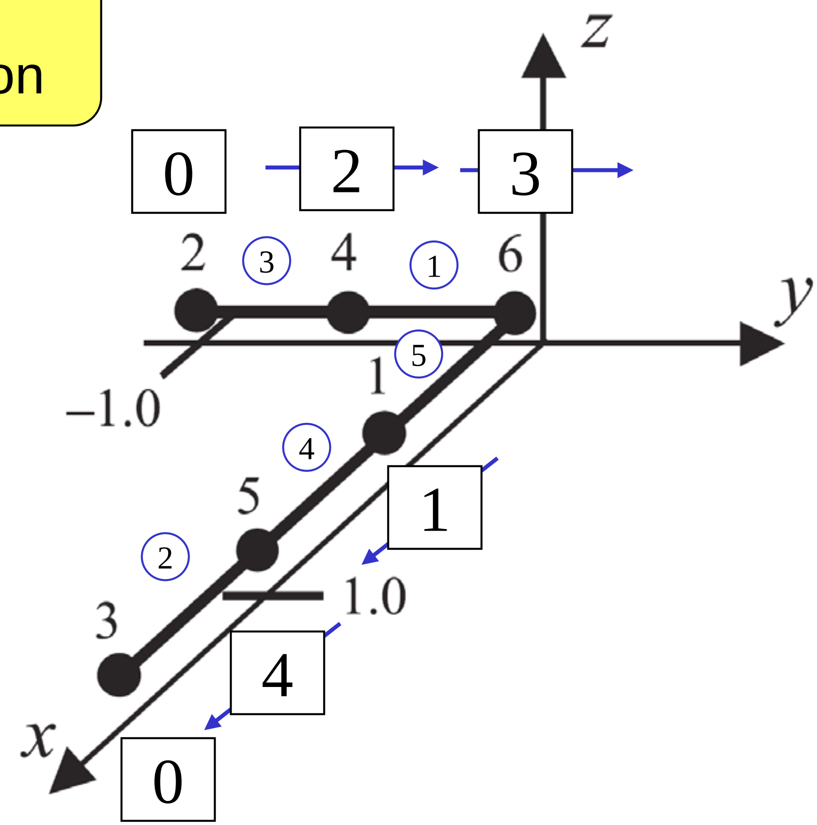


The Element List Defines Connectivity, DoFs, and Their Reference Directions

Degree of Freedom & Current Reference Direction Specification

e	Local Nodes, Element e: 1 — Global Node No.	1 — No. DoF's	1 — DoF index	2 — Global Node No.	2 — No. DoF's	2 — DoF index
1	4	1	$\boxed{-2}$	6	1	$\boxed{+3}$
2	5	1	$\boxed{-4}$	3	0	$\boxed{0}$
3	2	0	$\boxed{0}$	4	1	$\boxed{+2}$
4	1	1	$\boxed{-1}$	5	1	$\boxed{+4}$
5	6	1	$\boxed{-3}$	1	1	$\boxed{+1}$

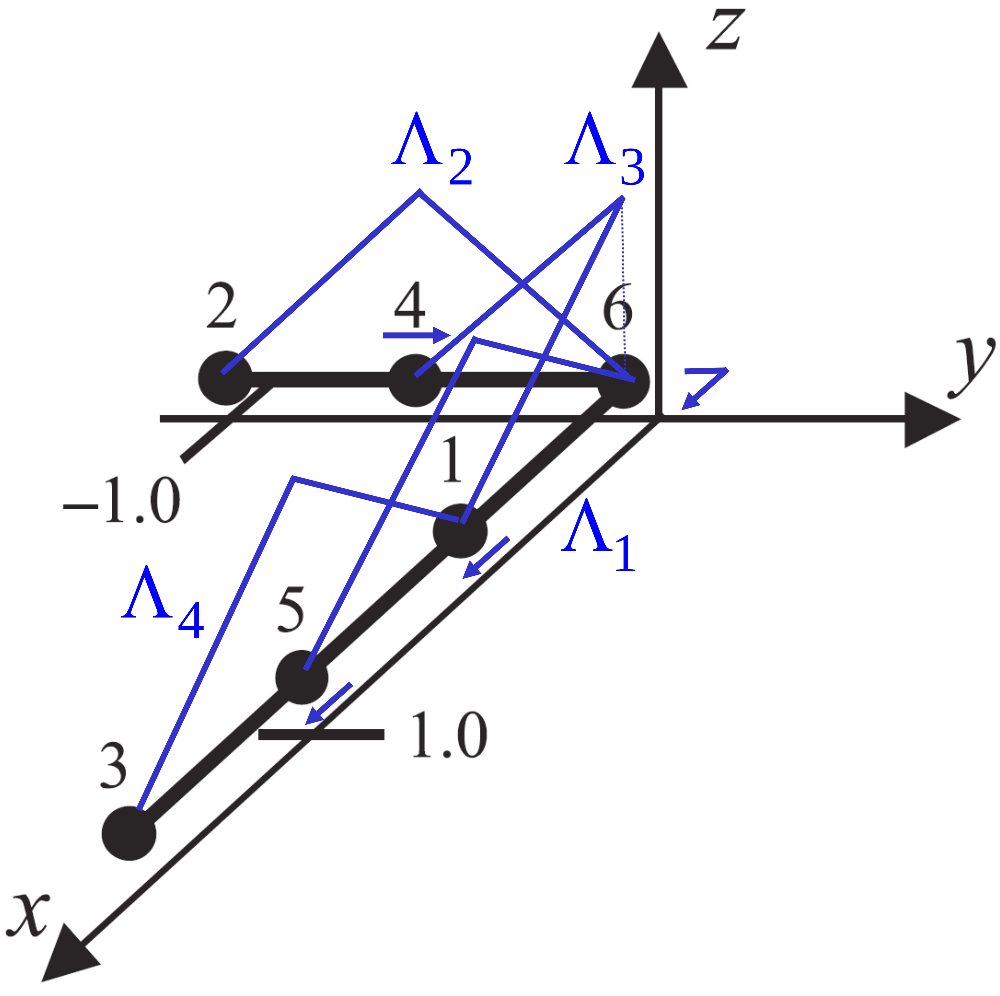




\+ sign points *out* of segment

\- sign points *into* segment

Linear Basis and Testing Functions



Tested Transmission Line Equation

Testing the wave equation $\dfrac{-1}{j\omega C}\dfrac{d^2 I}{d\ell^2} + j\omega L I = v_g\,,\ \ \ell \in (0, L)$

with Λ_m yields

$$\frac{-1}{j\omega C}\left\langle \Lambda_m, \frac{d^2 I}{d\ell^2} \right\rangle + j\omega L < \Lambda_m, I > \; = \; < \Lambda_m, v_g > \,, \quad m = 1, 2, \ldots, N$$

Integrating by parts,

$$\left\langle \Lambda_m, \frac{d^2 I}{d\ell^2} \right\rangle = \int_0^L \Lambda_m(\ell) \frac{d^2 I}{d\ell^2} d\ell = \left[\Lambda_m(\ell) \overbrace{\frac{dI}{d\ell}}^{-j\omega C V(\ell)} \right]_{\ell=0}^{L} - \int_0^L \frac{d\Lambda_m}{d\ell}\frac{dI}{d\ell} d\ell$$

$$\Rightarrow \boxed{\begin{aligned} &\Lambda_m(0)\underbrace{Z_0 I(0)}_{-V(0)} + \Lambda_m(L)\underbrace{Z_L I(L)}_{V(L)} + \frac{1}{j\omega C} < \frac{d\Lambda_m}{d\ell}, \frac{dI}{d\ell} > + j\omega L < \Lambda_m, I > \\ &\qquad = \; < \Lambda_m, v_g >, \quad m = 1, 2, \ldots, N \end{aligned}}$$

The Weak Form and Boundary Conditions

Weak form of the wave equation:

$$\Lambda_m(0)\underbrace{Z_0 I(0)}_{-V(0)} + \Lambda_m(L)\underbrace{Z_L I(L)}_{V(L)} + \frac{1}{j\omega C} < \frac{d\Lambda_m}{d\ell}, \frac{dI}{d\ell} > + j\omega L < \Lambda_m, I >$$
$$= \; < \Lambda_m, v_g >, \;\; m = 1, 2, \ldots, N$$

Note the boundary terms *disappear* under the following conditions:

- If the line is shorted at both ends, $I'(0) \propto V(0) = -Z_0 I(0) = 0$ and $I'(L) \propto V(L) = Z_L I(L) = 0$ so that the boundary terms vanish. This condition places no requirements on the testing functions or current approximations. Hence these *Neumann boundary* conditions ($I'(0) = I'(L) = 0$) on the unknown $I(\ell)$ are *natural boundary conditions.*

The Weak Form and Boundary Conditions, Cont'd

Weak form :

$$\Lambda_m(0)\underbrace{Z_0 I(0)}_{-V(0)}+\Lambda_m(L)\underbrace{Z_L I(L)}_{V(L)}+\frac{1}{j\omega C}<\frac{d\Lambda_m}{d\ell},\frac{dI}{d\ell}>+j\omega L<\Lambda_m,I>$$

$$=\;<\Lambda_m,v_g>,\;\; m=1,2,\ldots,N$$

- If the line is open-circuited at both ends, $I(0)=I(L)=0$, but $V(0)=-Z_0 I(0)\neq 0,\; V(L)=Z_L I(L)\neq 0$. Since we know the current vanishes at the ends, however, we need not test the equation at the line ends; indeed, we choose our testing functions such that $\Lambda_m(0)=\Lambda_m(L)=0$ so that the boundary terms vanish. Since we have to explicitly choose testing functions to enforce *Dirichlet boundary* conditions ($I(0)=I(L)=0$), the Dirichlet condition is an *essential boundary condition.*

Finite Element Equations

- With open-circuit boundary conditions, the weak form simplifies to

$$\frac{1}{j\omega C} < \frac{d\Lambda_m}{d\ell}, \frac{dI}{d\ell} > + j\omega L < \Lambda_m, I > \ = \ < \Lambda_m, v_g >, \ \ m = 1, 2, \ldots, N$$

- Expand the current in terms of bases,

$$I(\ell) \approx \sum_{n=1}^{N} I_n \Lambda_n(\ell)$$

$I(0) = I(L) = 0$ is an essential boundary condition since $\Lambda_m(0) = \Lambda_m(L) = 0$ for all m

and substitute into the weak form:

$$\sum_{n=1}^{N} I_n \left(\frac{1}{j\omega C} < \frac{d\Lambda_m}{d\ell}, \frac{d\Lambda_n}{d\ell} > + j\omega L < \Lambda_m, \Lambda_n > \right) \ = \ < \Lambda_m, v_g >, \ \ m = 1, 2, \ldots, N$$

or, expressing in matrix form,

$$[Z_{mn}][I_n] = [V_m]$$

where ...

Matrix Form of Finite Element Equations

$$\left[Z_{mn}\right]\left[I_n\right]=\left[V_m\right]$$ (system matrix)

where

$$\left[I_n\right]$$ (current vector)

$$\left[Z_{mn}\right]=\frac{1}{j\omega}\left[S_{mn}\right]+j\omega\left[L_{mn}\right]$$ (impedance matrix)

$$\left[S_{mn}\right]=\frac{1}{C}\left[<\frac{d\Lambda_m}{d\ell},\frac{d\Lambda_n}{d\ell}>\right]$$ (elastance matrix)

$$\left[L_{mn}\right]=L\left[<\Lambda_m,\Lambda_n>\right]$$ (inductance matrix)

$$\left[V_m\right]=\left[<\Lambda_m,v_g>\right]$$ (voltage vector)

What's Left?

It remains only to

- Evaluate the system matrix elements

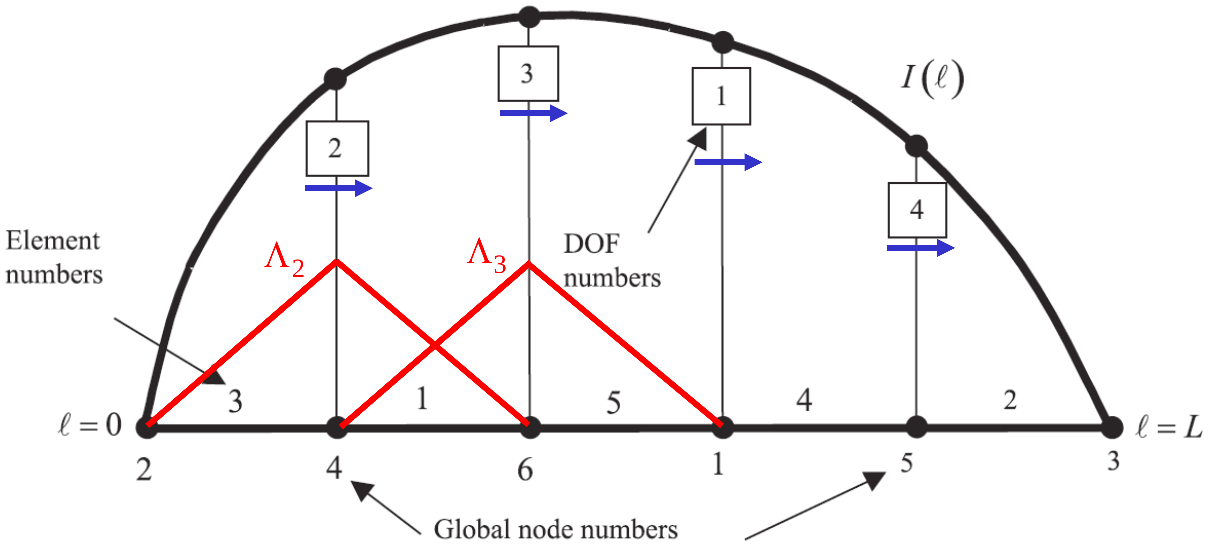
  - -This is our next task, but is not as straightforward as it might seem!

- Include boundary terms for loads

- Solve the resulting linear system of equations

  - -for efficiency, we should take into account its sparsity

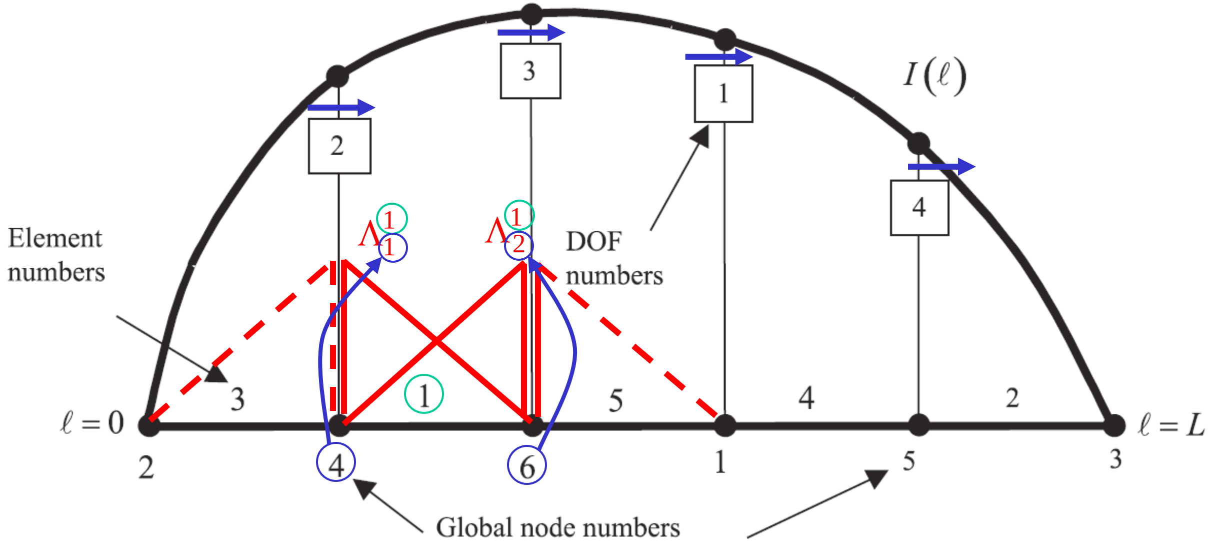
Direct Evaluation of System Matrix Elements is Both Difficult and Inefficient



- To evaluate e.g., Z_{23}, we'd need to find which elements are associated with DoFs 2 and 3 , but this information is unavailable; either a search or an auxiliary set of tables mapping DoFs to elements is required.

- Repeated integrations over the same element (e.g. #1) are needed, for example, to find partial contributions to $Z_{23}, Z_{32}, Z_{22}, Z_{33}$

Define Local Basis and Testing Functions and Evaluate a Matrix of Element Interactions



e	Local Nodes, Element e			
	1 Global Node No.		2 Global Node No.	
	No. DoF's	DoF index	No. DoF's	DoF index
1	4		6	
	1	-2	1	$+3$

- A solution to both difficulties is to evaluate a matrix of all interactions for a given element, add those partial contributions to the system matrix, then repeat the procedure for every element.
- The *element matrix* is defined as

$$\left[Z_{ij}^e\right] = \frac{1}{j\omega}\left[S_{ij}^e\right] + j\omega\left[L_{ij}^e\right]$$ (element impedance matrix)

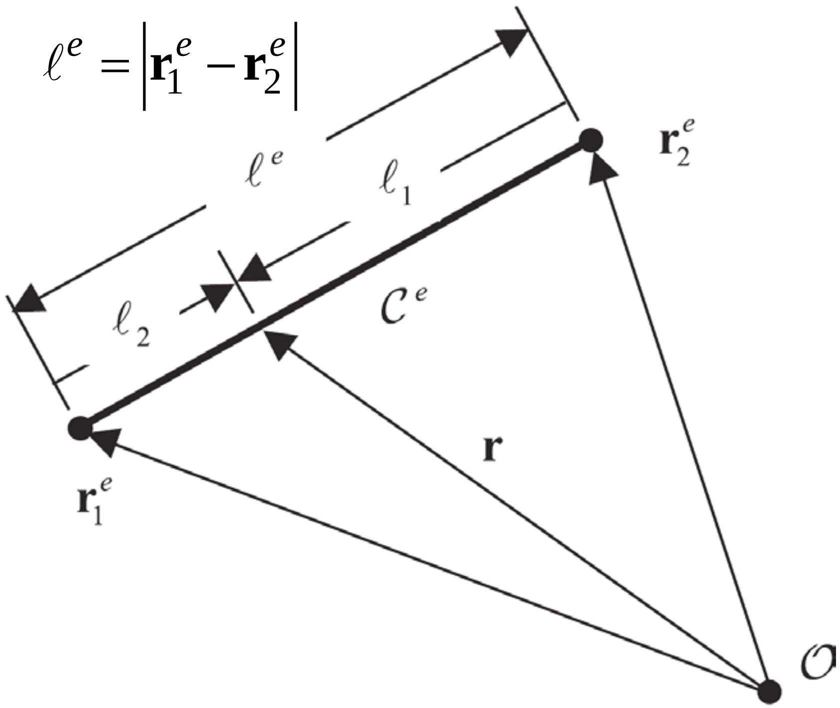
$$\left[S_{ij}^e\right] = \frac{1}{C} < \frac{d\Lambda_i^e}{d\ell}, \frac{d\Lambda_j^e}{d\ell} >$$ (element elastance matrix)

$$\left[L_{ij}^e\right] = L < \Lambda_i^e, \Lambda_j^e >$$ (element inductance matrix)

Z_{ij}^e represents the interaction between the ith testing and jth basis functions of element e; $i, j = 1, 2$
$e = 1, 2, \ldots, E$ = #elements

Element Parameterization

$$\ell^e = \left| \mathbf{r}_1^e - \mathbf{r}_2^e \right|$$



$$\mathbf{r} = \mathbf{r}_2^e + \frac{\mathbf{r}_1^e - \mathbf{r}_2^e}{\ell^e}\ell_1$$

$$= \mathbf{r}_1^e \frac{\ell_1}{\ell^e} + \mathbf{r}_2^e \frac{\ell_2}{\ell^e}$$

$$= \mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2$$

where

$$\xi_1 + \xi_2 = 1$$

since $\ell_1 + \ell_2 = \ell^e$

e	Local Nodes, Element e			
	1		2	
	Global Node No.		Global Node No.	
	No. DoF's	DoF index	No. DoF's	DoF index
1	4		6	
	1	-2	1	$+3$

element #, e

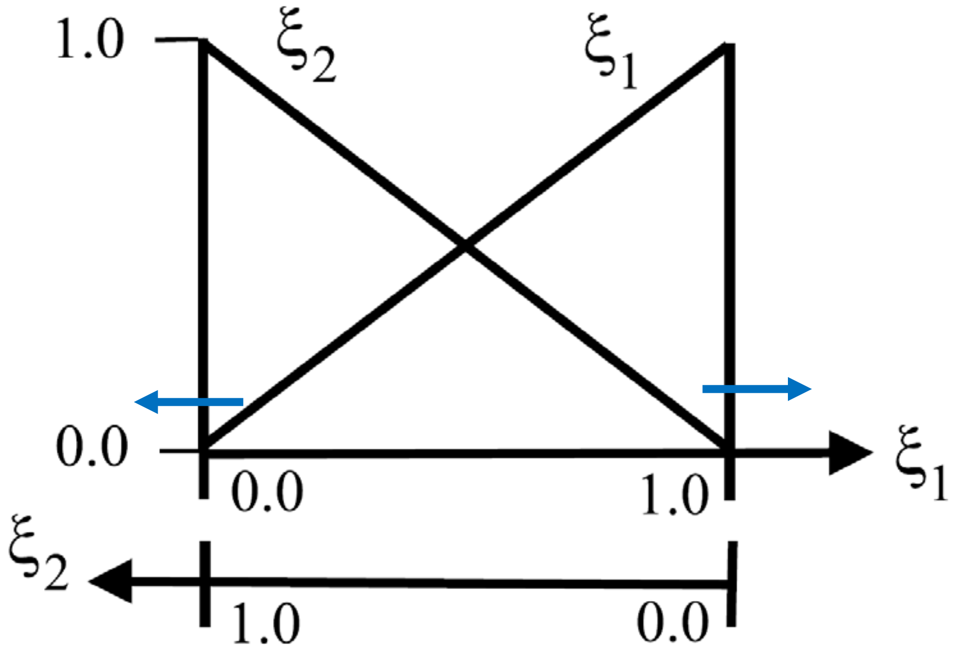
e.g. $\Rightarrow$ $\mathbf{r}_1^1 = \mathbf{r}_4$

$\mathbf{r}_2^1 = \mathbf{r}_6$

global node #'s

local node #'s

Normalized Element Coordinates, Local Bases



$\Lambda_i^e = \xi_i, \; i = 1, 2,$

local reference direction assumed *out* of element

Element coordinates are not independent since $\xi_1 + \xi_2 = 1\,!$

All elements are mapped to this "parent element"!

Integration in Normalized Coordinates

$$\xi_1 + \xi_2 = 1$$

$$d\ell = |d\mathbf{r}| = \left|\mathbf{r}_1^e d\xi_1 + \mathbf{r}_2^e d\xi_2\right| = \left|\mathbf{r}_1^e + \mathbf{r}_2^e \frac{d\xi_2}{d\xi_1}\right| d\xi_1 = \left|\mathbf{r}_1^e - \mathbf{r}_2^e\right| d\xi_1 = \ell^e d\xi_1$$

$$\Rightarrow \int_0^{\ell^e} f(\mathbf{r})\, d\ell_{1,2} = \ell^e \int_0^1 f(\mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2)\, d\xi_1$$

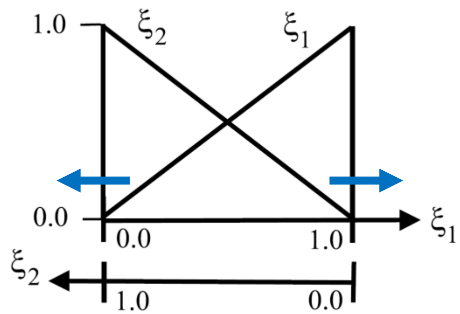
$$= \ell^e \int_0^1 f(\mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2)\, d\xi_2$$

$$\approx \ell^e \sum_{k=1}^{K} w_k f\left(\mathbf{r}_1^e \xi_1^{(k)} + \mathbf{r}_2^e \xi_2^{(k)}\right)$$ (if numerically integrated)

$d\xi_1 = -d\xi_2$, but limits are also reversed!

where we must observe the dependency $\xi_1 + \xi_2 = 1$ or $\xi_1^{(k)} + \xi_2^{(k)} = 1$!

Element Matrix Evaluation



- Observing that in normalized coordinates

$$\Lambda_i^e = \xi_i, \;\; i = 1,2; \qquad \frac{d\Lambda_i^e}{d\ell} = \frac{1}{\ell^e}$$

we evaluate the *element matrix* as

$$\left[Z_{ij}^e\right] = \frac{1}{j\omega}\left[S_{ij}^e\right] + j\omega\left[L_{ij}^e\right]$$

element impedance matrix

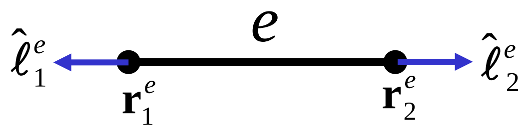
where the *element elastance matrix* is

$$\left[S_{ij}^e\right] = \frac{1}{C}\left[< \frac{d\Lambda_i^e}{d\ell}, \frac{d\Lambda_j^e}{d\ell} > \right] = \frac{\ell^e}{C}\left[\left(2\delta_{ij} - 1\right)^2 \int_0^1 \frac{1}{(\ell^e)^2} d\xi_{1,2} \right] = \frac{1}{C\ell^e}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

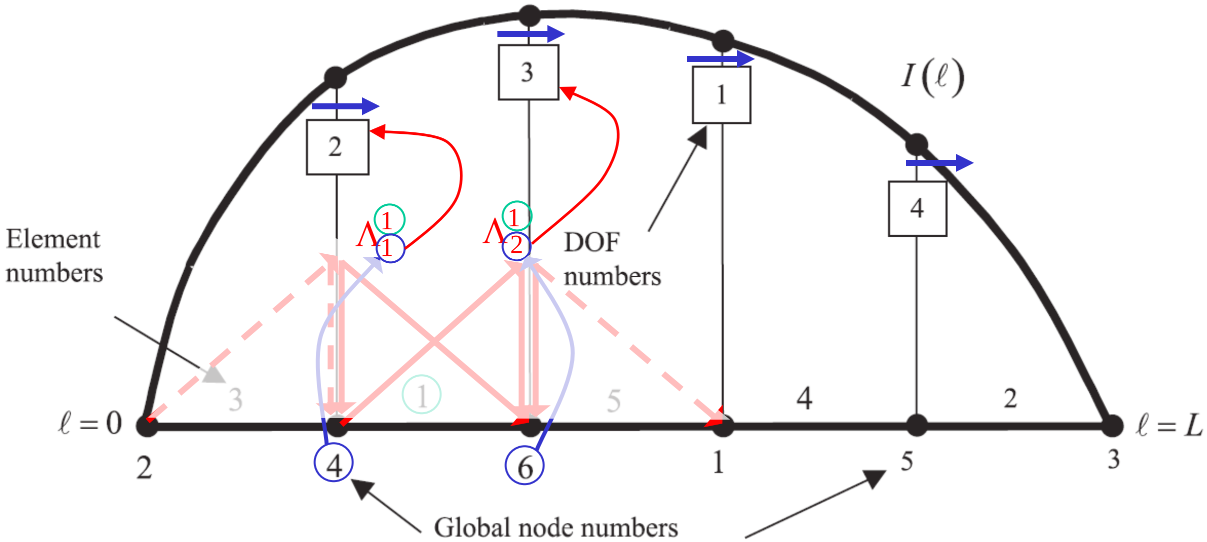
and the *element inductance matrix* is

$$\left[L_{ij}^e\right] = L\left[< \Lambda_i^e, \Lambda_j^e > \right] = L\ell^e\left[\left(2\delta_{ij} - 1\right) \int_0^1 \xi_i \xi_j \, d\xi_{1,2} \right] = \frac{L\ell^e}{6}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and where $2\delta_{ij} - 1 = \begin{cases} 1, \; i = j \\ -1, \; i \neq j \end{cases}$ since ref. directions are opposite for $\Lambda_i^e, \Lambda_j^e, \; i \neq j$



Associating Local and Global Degrees of Freedom



e	Local Nodes, Element e			
	1		2	
	Global Node No.		Global Node No.	
	No. DoF's	DoF index	No. DoF's	DoF index
1	4		6	
	1	−2	1	+3

$\sigma_1^1 = -1$ $\qquad$ $\sigma_2^1 = +1$

$$\begin{bmatrix} Z_{11}^1 & Z_{12}^1 \\ Z_{21}^1 & Z_{22}^1 \end{bmatrix} = \frac{1}{j\omega C \ell^1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{j\omega L \ell^1}{6}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\sigma_i^e = \begin{cases} 1, & i \text{ th node reference direction } \textit{out of} \text{ element } e, \\ -1, & i \text{ th node reference direction } \textit{into} \text{ element } e \end{cases}$$

Associating Local and Global Bases

On element 1:

element #, e

$$\Lambda_2 = \sigma_1^1 \Lambda_1^1$$

global DoF #'s

$$\Lambda_3 = \sigma_2^1 \Lambda_2^1$$

local DoF #'s

e	Local Nodes, Element e			
	1		2	
	Global Node No.		Global Node No.	
	No. DoF's	DoF index	No. DoF's	DoF index
1	4		6	
	1	−2	1	+3
2	5		3	
	1	−4	0	0
3	2		4	
	0	0	1	+2

On element 3:

element #, e

$$\Lambda_2 = \sigma_2^3 \Lambda_2^3$$

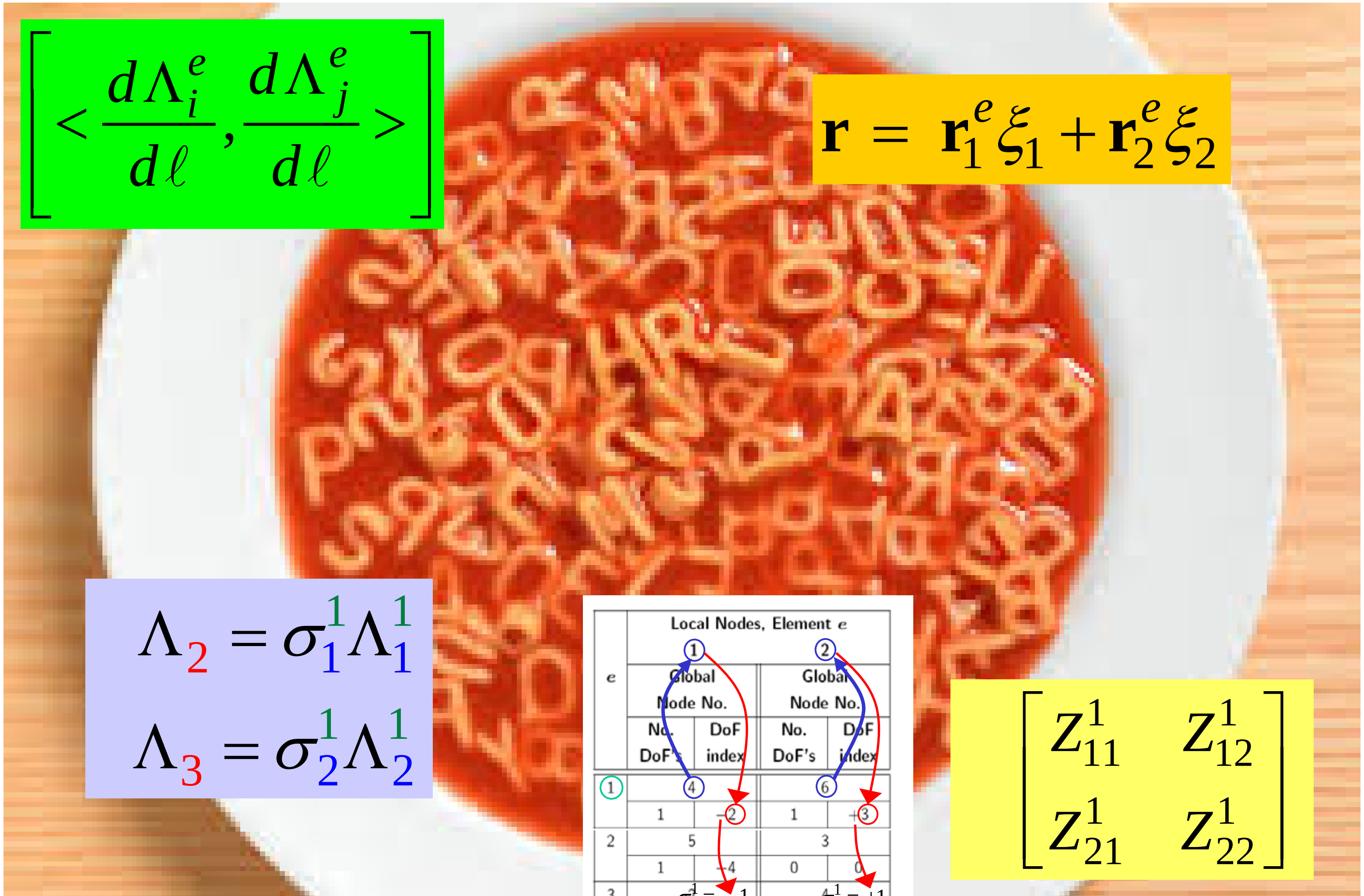
global DoF #

local DoF #'s

$$\Rightarrow \quad \Lambda_2(\mathbf{r}) = \begin{cases} \sigma_1^1 \Lambda_1^1(\mathbf{r}), & \mathbf{r} \in \text{element } \#1 \\ \sigma_2^3 \Lambda_2^3(\mathbf{r}), & \mathbf{r} \in \text{element } \#3 \end{cases}$$

In practice, we never construct global bases directly, but only *assemble their contributions* from the elements forming their support !

Our Formulation So Far Reminds Us of "Index Soup!"



$$\left[< \frac{d\Lambda_i^e}{d\ell}, \frac{d\Lambda_j^e}{d\ell} > \right]$$

$$\mathbf{r} = \mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2$$

$$\Lambda_2 = \sigma_1^1 \Lambda_1^1$$

$$\Lambda_3 = \sigma_2^1 \Lambda_2^1$$

$$\begin{bmatrix} Z_{11}^1 & Z_{12}^1 \\ Z_{21}^1 & Z_{22}^1 \end{bmatrix}$$

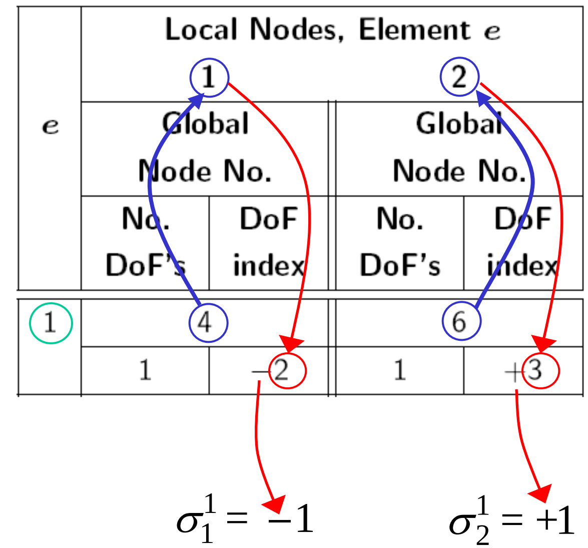
e	Local Nodes, Element e: 1 — Global Node No.		2 — Global Node No.	
	No. DoF's	DoF index	No. DoF's	DoF index
1	4		6	
	1	−2	1	+3
2	5		3	
	1	−4	0	0
3	2		4	
	0	0	1	+2
4	1		5	
	1	−1	1	+4
5	6		1	
	1	−3	1	+1

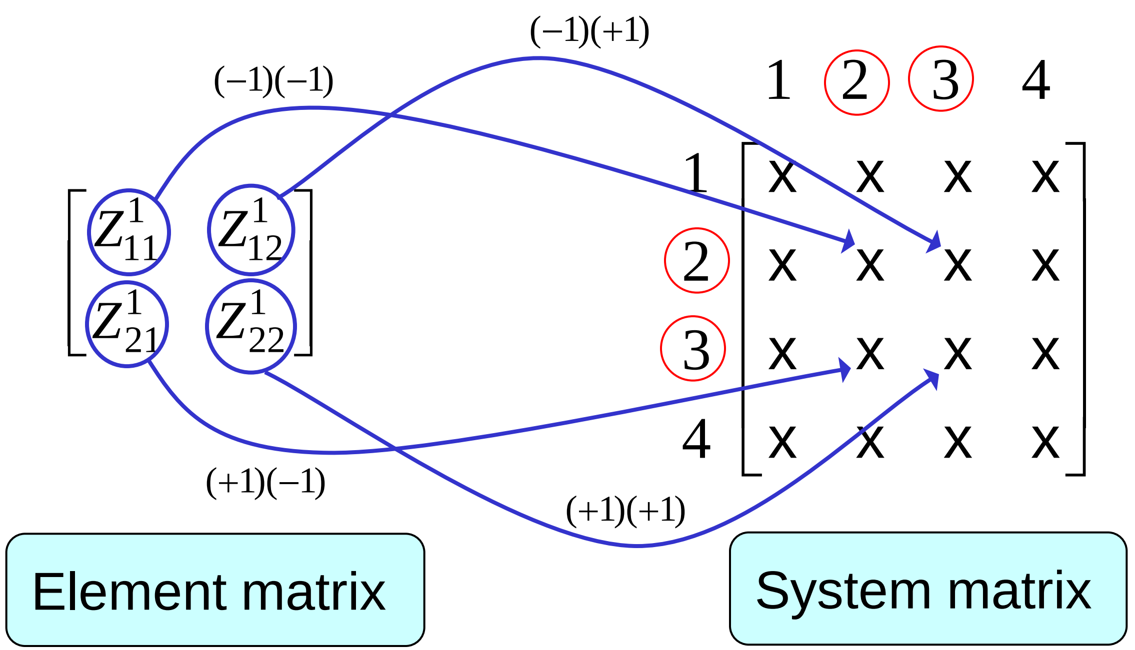
$\sigma_1^1 = -1$ $\sigma_2^1 = +1$

Element Matrix Assembly

Matrix Assembly Rule :

$\sigma_i^e \sigma_j^e Z_{ij}^e$ is added to Z_{mn} where m, n are the nodal degree of freedom indices associated with local nodes i and j, respectively, of element e.

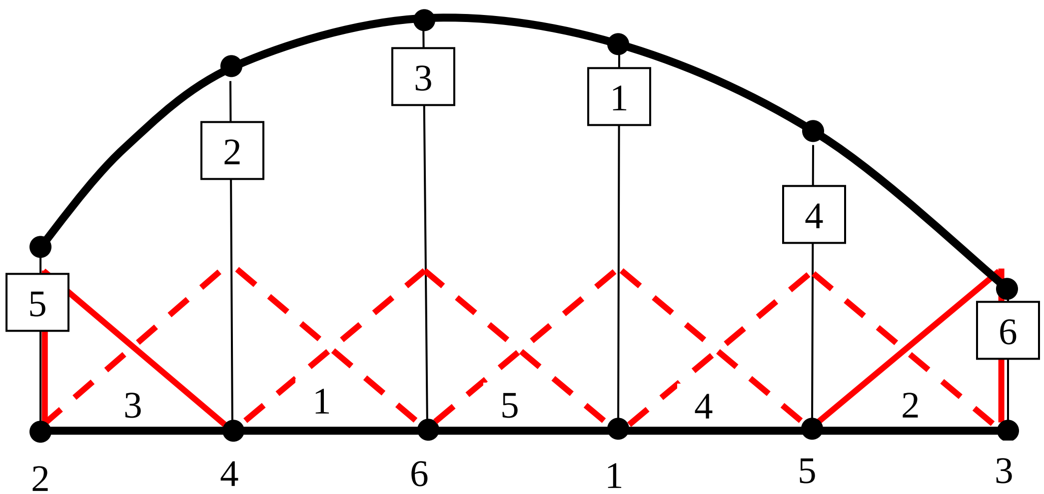




Modifications for a Loaded Line

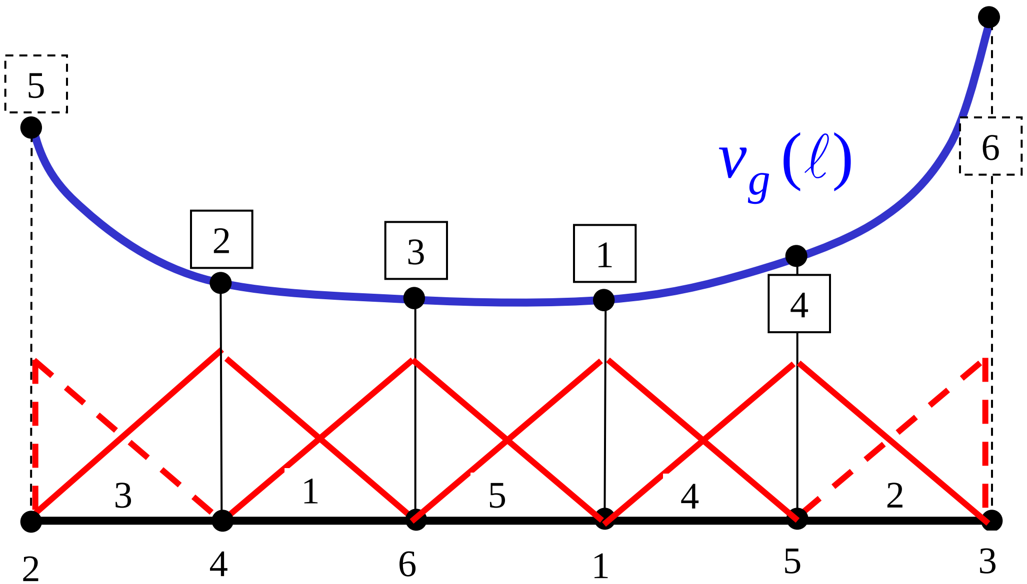
$$\underbrace{\Lambda_m(0)Z_0 I(0) + \Lambda_m(L)Z_L I(L)}_{\text{New terms} = 0 \text{ except for } m=5,6} + \frac{1}{j\omega C} < \frac{d\Lambda_m}{d\ell}, \frac{dI}{d\ell} > + j\omega L < \Lambda_m, I > \; = \; < \Lambda_m, v_g >$$

$m = 1, 2, \ldots, N$ (= 6)



- New degrees of freedom, 5 and 6, and associated half-triangles added at line ends
- Note $\Lambda_5(0)\,Z_0\,I(0) = 1 \cdot Z_0 \cdot I_5$ and $\Lambda_6(L)Z_L I(L) = 1 \cdot Z_L \cdot I_6$
- $\therefore$ add new terms Z_0 , Z_L to system matrix diagonal, rows 5 and 6, rsp.

Filling the RHS System (Forcing) Vector



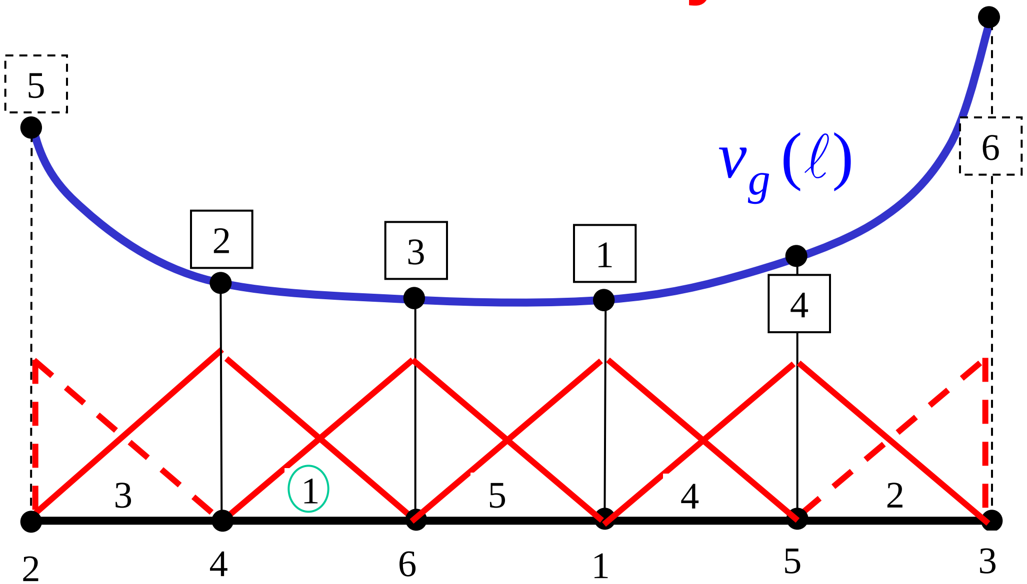
$$[V_m] = [\langle \Lambda_m, v_g \rangle],$$
$$m = 1, 2, \ldots, N$$

Three possible approaches to evaluating $\langle \Lambda_m, v_g \rangle = \int_0^L \Lambda_m(\ell) v_g(\ell)\, d\ell$:

- Integrate analytically (over each of two elements spanned by Λ_m)
- Numerically integrate (... "..." ...)
- Interpolate $v_g \approx \sum_{p=1}^{N} v_g(\ell_p) \Lambda_p(\ell)$, then

  evaluate $\langle \Lambda_m, v_g \rangle \approx \sum_{p=1}^{N} v_g(\ell_p) \langle \Lambda_m, \Lambda_p(\ell) \rangle$ (... "..." ...)

Define and Use an *Element Vector* to Fill the System Vector



e	Local Nodes, Element e			
	1		2	
	Global Node No.		Global Node No.	
	No. DoF's	DoF index	No. DoF's	DoF index
1	4		6	
	1	−2	1	+3

$\sigma_1^1 = -1$ $\sigma_2^1 = +1$

$$\left[V_i^e\right] \equiv \left[<\Lambda_i^e, v_g>\right] = \left[\ell^e \int_0^1 \xi_i v_g(\mathbf{r})\,d\xi\right], \quad i = 1, 2$$

e.g. for $e = 1$,

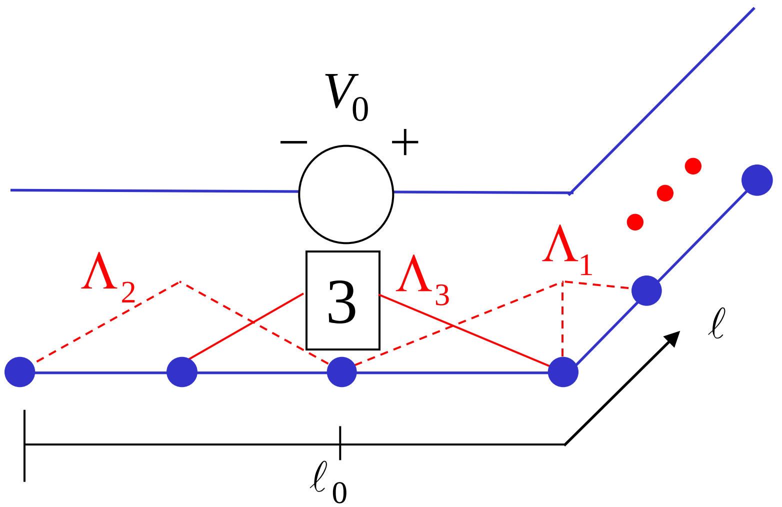
$$\begin{bmatrix} V_1^1 \\ V_2^1 \end{bmatrix} \xrightarrow{(-1),\ (+1)} \begin{array}{c} \vdots \\ 2 \\ 3 \\ \vdots \end{array} \begin{bmatrix} \mathrm{x} \\ \mathrm{x} \\ \mathrm{x} \\ \mathrm{x} \end{bmatrix} = \left[V_m\right]$$

Element vector

System vector

Filling the System Vector for a Discrete Source

$$v_g(\mathbf{r}) = V_0\ \delta(\ell - \ell_0)$$

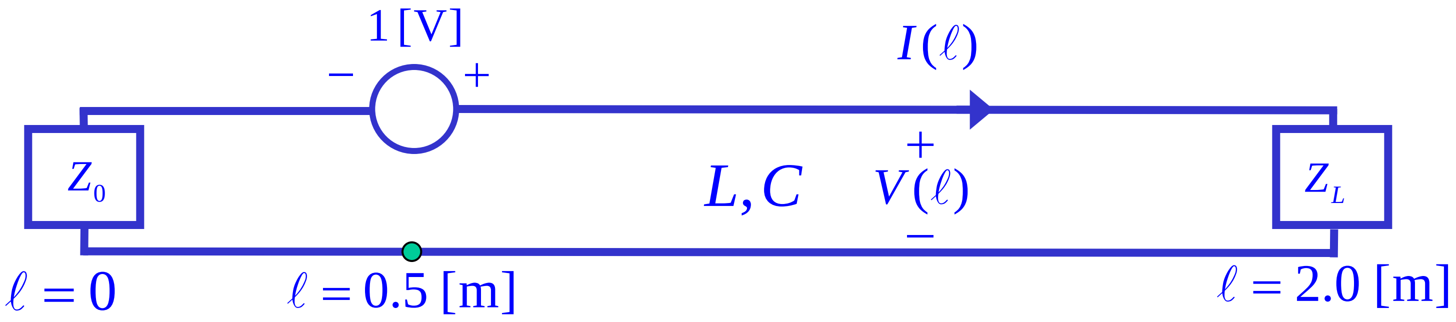


$$V_m = \int_0^L \Lambda_m(\ell) v_g(\mathbf{r})\, d\ell = V_0 \int_0^L \Lambda_m(\ell)\delta(\ell - \ell_0)\, d\ell = \begin{cases} V_0, & m = 3 \\ 0, & \text{otherwise} \end{cases}$$

This is the only case in which we do not first fill an element vector!

$$\begin{array}{c} 1 \\ \vdots \\ 3 \\ \vdots \end{array} \begin{bmatrix} 0 \\ \vdots \\ V_0 \\ 0 \end{bmatrix} = [V_m]$$

Numerical Results



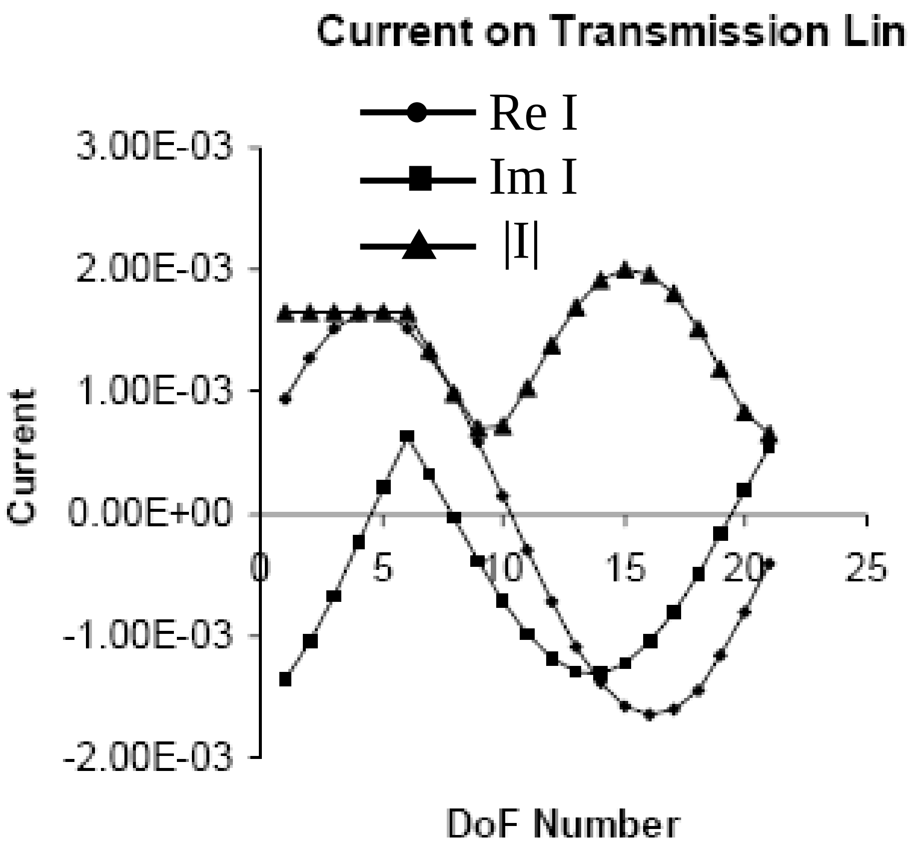
- Line length $= 2\ [\mathrm{m}]$
- $L = \mu_0\ [\mathrm{H/m}]$
- $C = \varepsilon_0\ [\mathrm{F/m}]$

$\Rightarrow$ characteristic impedance $= \sqrt{\dfrac{L}{C}} \approx 377\,\Omega$

- $Z_0 = \sqrt{\dfrac{L}{C}},\quad Z_L = 3\sqrt{\dfrac{L}{C}}$ $\Rightarrow$ SWR $\Rightarrow 3.0$

- $f = 130\ [\mathrm{MHz}]$
- Unit voltage source $0.5\ [\mathrm{m}]$ from matched end of line
- 21 DoF's (20 elements) numbered starting from the matched end of line

Numerical Results, cont'd



- Line length $= 2$ [m]
- $L = \mu_0$ [H/m]
- $C = \varepsilon_0$ [F/m]

  $\Rightarrow$ characteristic impedance,

  $\sqrt{L/C} \approx 377\,\Omega$
- $Z_0 = \sqrt{\frac{L}{C}}, \quad Z_L = 3\sqrt{\frac{L}{C}}$ $\quad \Rightarrow$ SWR $= 3.0$
- $f = 130$ [MHz]
- Unit voltage source 0.5 [m] from matched end of line
- DoF's numbered starting from matched end of line

Determining Line Resonances

Recall the system matrix equation

$$\boxed{[Z_{mn}][I_n]=[V_m]}$$ (system matrix)

where

$$\boxed{[Z_{mn}]=\frac{1}{j\omega}[S_{mn}]+j\omega[L_{mn}]}$$ (impedance matrix)

$$\boxed{[S_{mn}]=\frac{1}{C}\left[<\frac{d\Lambda_m}{d\ell},\frac{d\Lambda_n}{d\ell}>\right]}$$ (elastance matrix)

$$\boxed{[L_{mn}]=L[<\Lambda_m,\Lambda_n>]}$$ (inductance matrix)

Resonances are *source free solutions*, so set $[V_m]=[0]$:

$$\Rightarrow \frac{1}{j\omega}[S_{mn}][I_n]+j\omega[L_{mn}][I_n]=[0]$$

or $$\boxed{[S_{mn}][I_n^p]=\omega_p{}^2[L_{mn}][I_n^p]}$$ *(generalized eigen-value problem)*

$$p=1,2,\ldots$$

Note : 1×1 matrix case reduces to $\omega=\sqrt{\frac{S}{L}}=\frac{1}{\sqrt{LC}}$!