Combined Field Integral Equation (CFIE)

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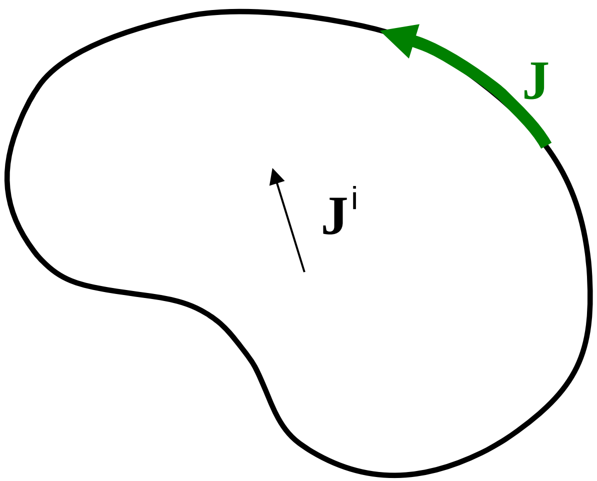
Interior Resonances

- **For closed bodies, the EFIE cannot distinguish whether the excitation sources J^i of E^i are interior or exterior to the PEC**

- **At cavity resonant frequencies, *source-free* solutions of the EFIE exist (if an interior source of the same frequency, exists, the resulting fields will generally be infinite).**

- **The surface currents corresponding to source-free solutions of the EFIE are simply the cavity wall surface currents of the associated resonant cavity mode.**

Interior Resonance Properties of EFIE



$@\ \omega=\omega_p,\ \ p=1,2,\cdots$

- At interior resonance freqs., there exist homogeneous solutions $\mathbf{J}_h$ to the EFIE :

$$\left[j\omega\mu \int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}_h(\mathbf{r}')d\mathcal{S}' - \frac{\nabla}{j\omega\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{J}_h(\mathbf{r}')d\mathcal{S}' \right]_{\tan} = \mathbf{0},$$

- In matrix form, this means

$$\left[Z_{mn}\right]\left[I_n\right]=0 \qquad \Rightarrow \qquad \det\left[Z_{mn}\right]=0$$

- Unless the Green's function is replaced by a non-radiating form, e.g., $\dfrac{e^{-jkR}}{4\pi R}\rightarrow\dfrac{\cos kR}{4\pi R}$, the determinant doesn't *completely* vanish at real resonant frequencies because discretization errors "leak" radiation.
- The problem becomes ill-conditioned, however, and solutions can be contaminated by homogeneous solutions from nearby complex frequencies.

The MFIE at Interior Resonances

- Though the physical explanation differs from the EFIE, the MFIE also has homogeneous solutions at interior resonant frequencies

- The MFIE homogeneous form is

$$\frac{\mathbf{J}_h(\mathbf{r})}{2} - \hat{\mathbf{n}} \times \int_{\mathcal{S}} \nabla \times \mathcal{G}^A(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_h(\mathbf{r}') d\mathcal{S}' = \mathbf{0}$$

- The matrix MFIE homogeneous form is

$$[\beta_{mn}][I_n] = 0 \qquad \Rightarrow \qquad \det[\beta_{mn}] = 0$$

Linear Operator Problems and Uniqueness

- $\mathcal{L}$ is a ***linear operator*** if

$$\mathcal{L}\left(au_1 + bu_2\right) = a\mathcal{L}u_1 + b\mathcal{L}u_2$$

for any functions u_1, u_2 **, any constants** a, b**. (Both the EFIE and MFIE are linear operator eqs. with** $u \equiv \mathbf{J}$**.)**

- **The operator** $\mathcal{L}$ **has a non-trivial *homogeneous solution* if there exists a function** $u_h \neq 0$ **such that** $\mathcal{L}u_h = 0$ **(**u_h **is not unique since** Cu_h **is also a homogeneous solution:** $\mathcal{L}\left(Cu_h\right) = C\mathcal{L}u_h = 0$**)**
- **If** $\mathcal{L}$ **has a non-trivial homogeneous solution, the operator equation** $\mathcal{L}u = f$ **has no unique solution, since for every** u **a solution,** $u + Cu_h, C \neq 0,$ **is also a solution:**

$$\mathcal{L}\left(u + Cu_h\right) = \mathcal{L}u + C\,\cancel{\mathcal{L}u_h} = \mathcal{L}u = f$$

- **Different solutions** $\mathcal{L}u_1 = f,\ \mathcal{L}u_2 = f, u_1 \neq u_2,$ **may differ only by a homogeneous solution:**

$$f - f \equiv 0 = \mathcal{L}u_1 - \mathcal{L}u_2 = \mathcal{L}\left(u_1 - u_2\right) \Rightarrow\ u_1 - u_2 = u_h$$

Uniqueness is proved by assuming $u_1 - u_2 \equiv u_h \neq 0$ **and proving a contradiction!**

Linear Operator Problems and Uniqueness

- $\mathcal{L}u = f$ **has a unique solution if and only if the only solution to the homogeneous equation $\mathcal{L}u_h = 0$ is the trivial solution, $u_h = 0$**

Uniqueness is proved by assuming $u_1 - u_2 \equiv u_h \neq 0$ and proving a contradiction!

The Combined Field Integral Equation (CFIE)

- **Remarkably, linearly combining the EFIE and MFIE eliminates difficulties with interior resonances!**
- **Write the EFIE in the abbreviated form**

$$-\mathbf{E}_{\text{tan}}(\mathbf{J}) = \mathbf{E}^{\text{i}}_{\text{tan}} \quad \overset{\text{discretize}}{\Rightarrow} \quad [Z_{mn}][I_n] = [V_m]$$

and the MFIE as

$$-\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{J}) = \hat{\mathbf{n}} \times \mathbf{H}^{\text{i}}, \quad \overset{\text{discretize}}{\Rightarrow} \quad [\beta_{mn}][I_n] = [I^{\text{i}}_m]$$

with $\mathbf{r} \uparrow \mathcal{S}$ **understood, and combine them as**

$$-\frac{\mathbf{E}_{\text{tan}}(\mathbf{J})}{\eta} - \alpha \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{J}) = \frac{\mathbf{E}^{\text{i}}_{\text{tan}}}{\eta} + \alpha \hat{\mathbf{n}} \times \mathbf{H}^{\text{i}}$$

- **In discrete form, this is**

$$\left[\frac{Z_{mn}}{\eta} + \alpha \beta_{mn}\right][I_n] = \left[\frac{V_m}{\eta} + \alpha I^{\text{i}}_m\right] \quad \textbf{(CFIE)}$$

Uniqueness of the CFIE

- To prove CFIE uniqueness, assume $\exists\, \mathbf{J}_h \neq \mathbf{0}$ satisfying

$$-\frac{\mathbf{E}_{\tan}(\mathbf{J}_h)}{\eta} - \alpha \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{J}_h) = \mathbf{0},\ \mathbf{r} \in \mathcal{S}$$

- Multiply eq. by its conjugate and integrate over $\mathcal{S}$:

$$\int_{\mathcal{S}} \left[\left| \frac{\mathbf{E}_{\tan}(\mathbf{J}_h)}{\eta} \right|^2 + |\alpha|^2 \left| \mathbf{H}_{\tan}(\mathbf{J}_h) \right|^2 \right] d\mathcal{S}$$

$$+ \frac{2\alpha}{\eta} \overbrace{\mathrm{Re} \int_{\mathcal{S}} \left[\mathbf{E}(\mathbf{J}_h) \times \mathbf{H}^*(\mathbf{J}_h) \cdot (-\hat{\mathbf{n}}) \right] d\mathcal{S}}^{\text{Power radiated into } \mathcal{S},\ \geq 0} = 0$$

where $\dfrac{\alpha}{\eta}$ is chosen positive and real.

Uniqueness theorem:
If 1) no sources exterior to $\mathcal{S}^+$,
2) $\mathbf{E}_{\tan} = \mathbf{0}$ or $\mathbf{H}_{\tan} = \mathbf{0}$ on $\mathcal{S}^+$,
$\Longrightarrow \mathbf{E} = \mathbf{H} = \mathbf{0}$ exterior to $\mathcal{S}^+$

- $\Rightarrow \mathbf{H}_{\tan} = \mathbf{0}$ on $\mathcal{S}^-$, $\mathbf{E}_{\tan} = \mathbf{0}$ on $\mathcal{S}^\pm$, $\overset{\text{unique. thm.}}{\Longrightarrow}$ $\mathbf{H}_{\tan} = \mathbf{0}$ on $\mathcal{S}^+$

$$\Rightarrow \mathbf{J}_h = \hat{\mathbf{n}} \times \mathbf{H}\big|_{\mathcal{S}^+} - \hat{\mathbf{n}} \times \mathbf{H}\big|_{\mathcal{S}^-} = \mathbf{0} \quad \text{(contradiction!)}$$

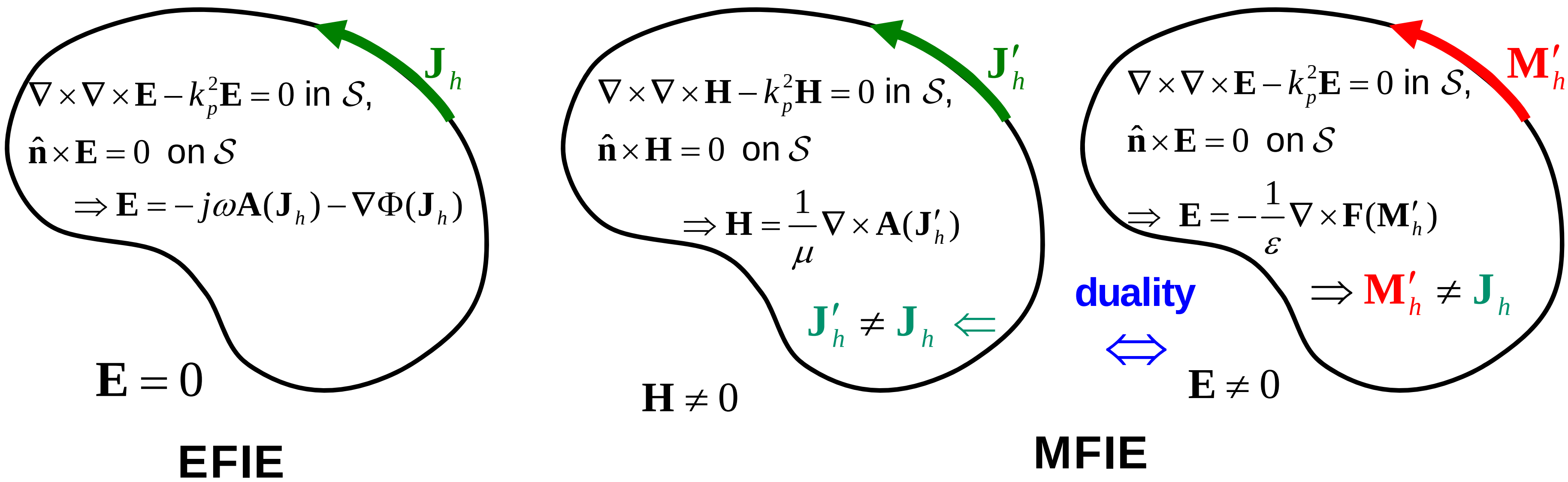
Uniqueness of the CFIE, cont'd

- **If** $\mathcal{L}\,\mathbf{J}_h = 0$ (**EFIE**) **and** $\mathcal{K}\,\mathbf{J}_h = 0$ (**MFIE**) **at** $\omega = \omega_p$ **then why doesn't the linear combination also have a homogeneous solution:**

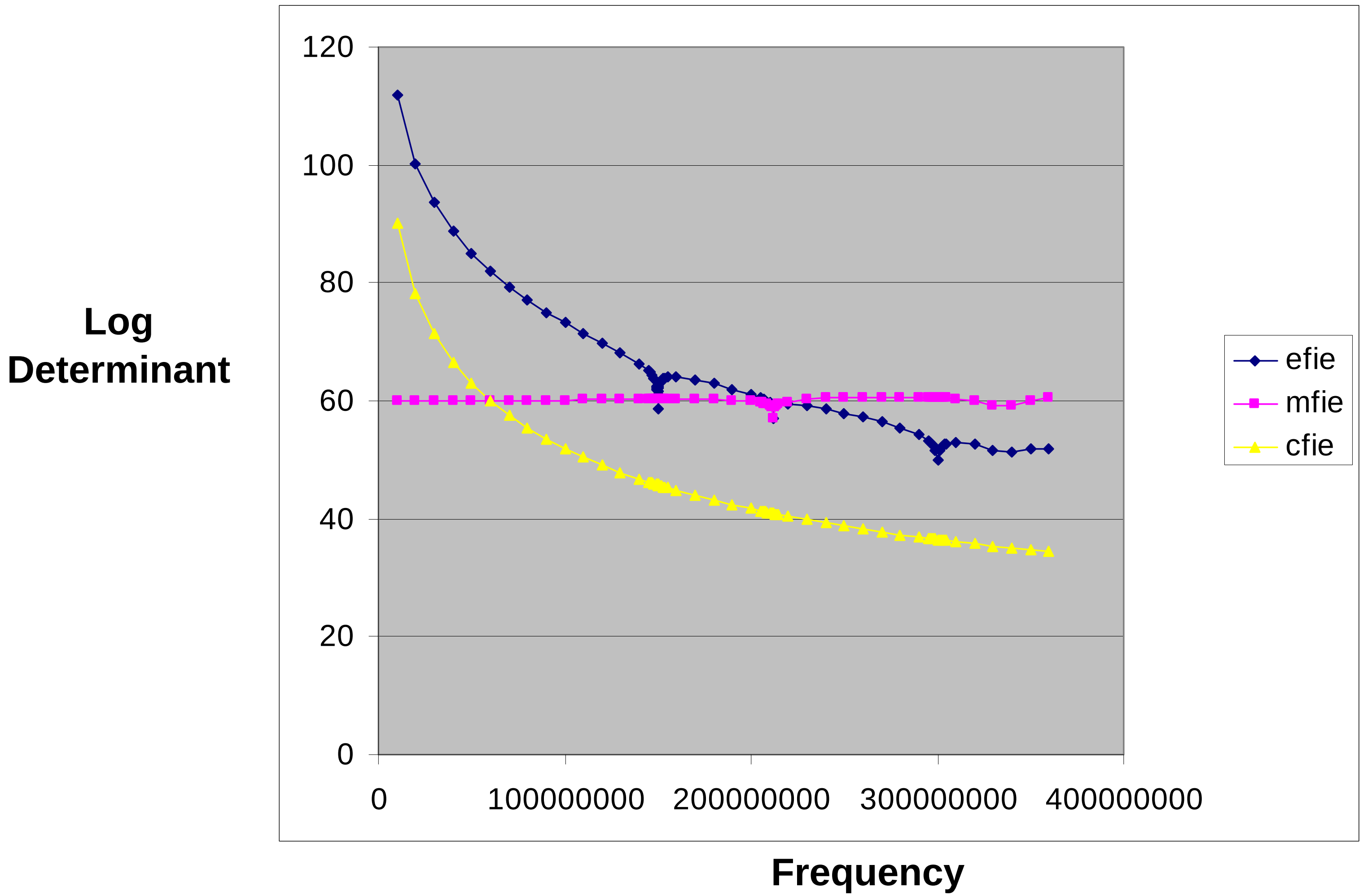
$$\mathcal{L}\,\mathbf{J}_h + \alpha\mathcal{K}\,\mathbf{J}_h = \left(\mathcal{L} + \alpha\mathcal{K}\right)\mathbf{J}_h = 0 \text{ (CFIE)}$$

- **Ans: The EFIE and MFIE homogeneous solutions are different!**

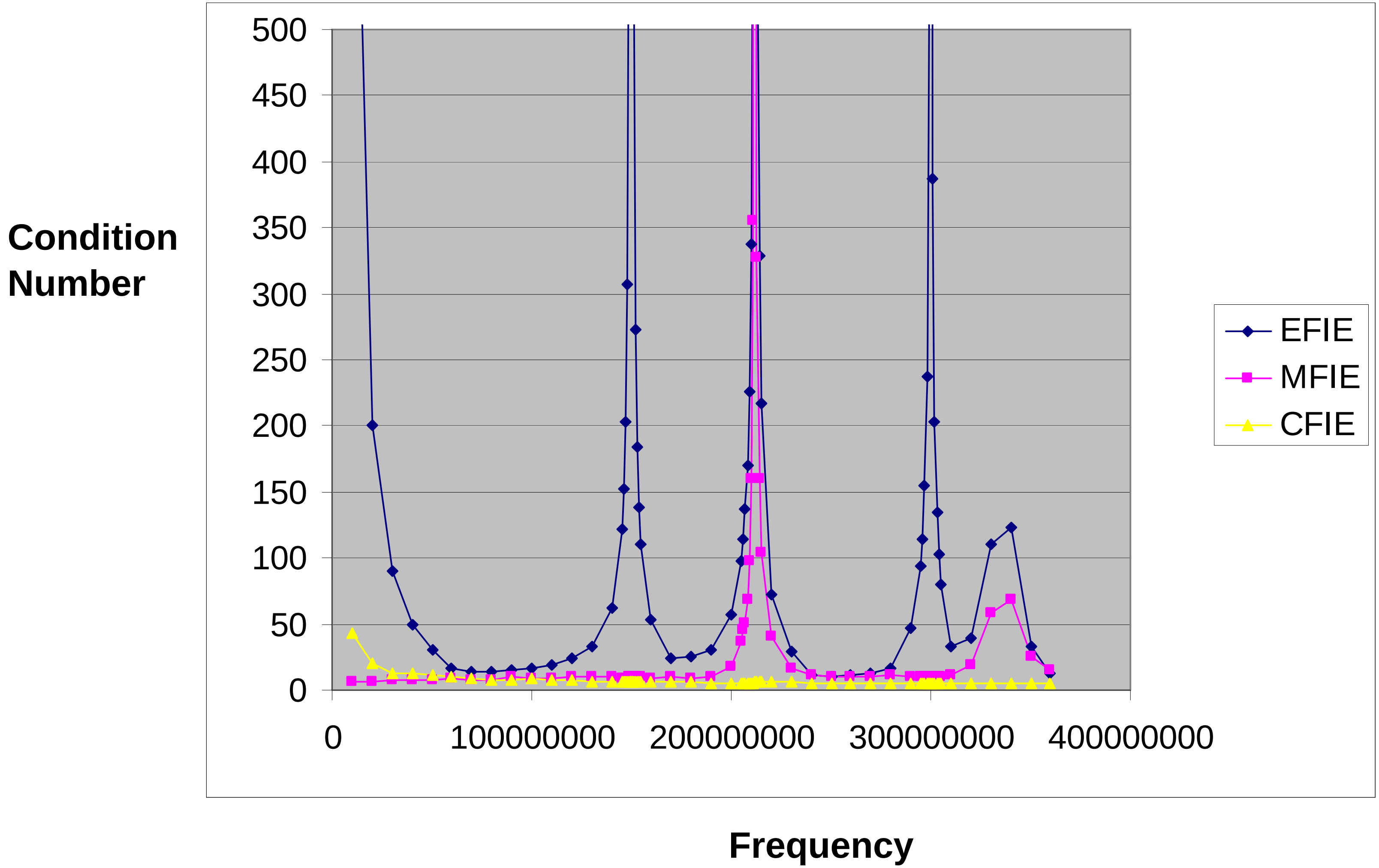
$$\mathcal{L}\,\mathbf{J}_h = 0 \text{ and } \mathcal{K}\,\mathbf{J}'_h = 0 \not\Rightarrow \mathcal{L}\,\mathbf{J}_h + \alpha\mathcal{K}\,\mathbf{J}_h = 0 \text{ if } \mathbf{J}'_h \neq \mathbf{J}_h$$



Log_{10} of Determinant vs. Frequency, TE Circular Cylinder



Approx. Condition Number vs. Frequency, TE Circular Cylinder



Condition Number

If $Ax = b$

then $\operatorname{cond} A \equiv \dfrac{\text{Largest eigenvalue of } A^H A}{\text{Smallest eigenvalue of } A^H A} \geq 1, \quad A^H \equiv \left(A^*\right)^t$

and

$$\frac{\|\delta x\|}{\|x\|} \approx \operatorname{cond} A \left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right]$$

and where

$$\left(A + \delta A\right)(x + \delta x) = b + \delta b.$$

Condition number is the single most important figure of merit in solving linear systems!!

Roughly, $\operatorname{cond} A$ measures how much relative errors in A and b magnify the relative error of the solution.

Alternatively, $\log_{10} \operatorname{cond} A$ estimates how many (decimal) digits are lost in solving $Ax = b$. I.e, it estimates the worst - case loss of precision.

End