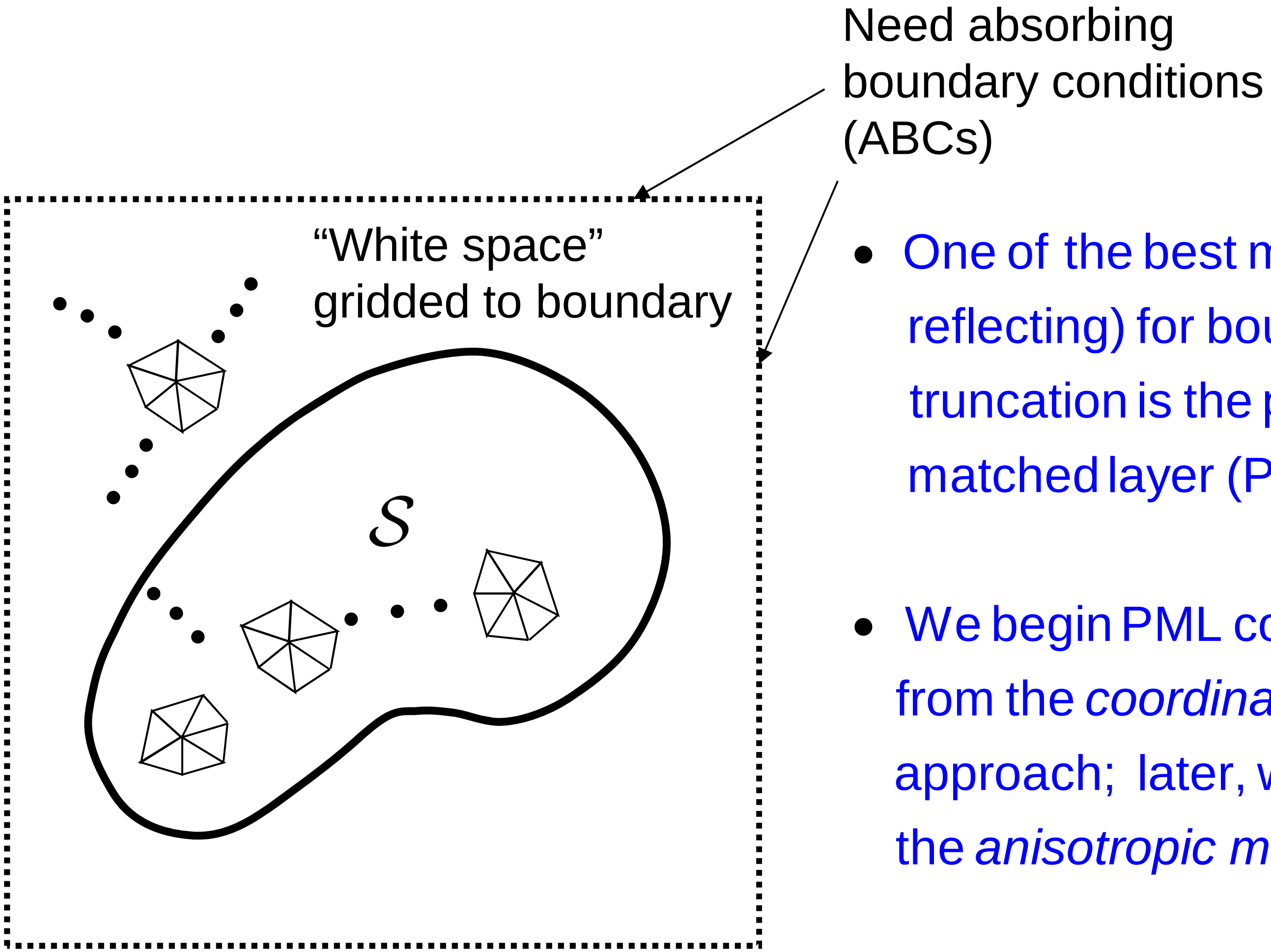
FEM Domain Boundary Truncation

Donald R. Wilton

Ref: *Finite Element Analysis of Antennas and Arrays*, IEEE Press, J.-M. Jin, D.J. Riley, 2008

For Unbounded FEM Problems Boundary Truncation is an Important Consideration



- One of the best methods (least reflecting) for boundary truncation is the perfectly matched layer (PML) approach.

- We begin PML consideration from the *coordinate stretching* approach; later, we switch to the *anisotropic medium* approach.

Coordinate Stretching Approach

- Consider modified source - free Maxwell's equations:

$$\nabla_s \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla_s \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\nabla_s \cdot (\varepsilon\mathbf{E}) = 0$$

$$\nabla_s \cdot (\mu\mathbf{H}) = 0$$

where

$$\nabla_s \equiv \hat{\mathbf{x}}\frac{1}{s_x}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{1}{s_y}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{1}{s_z}\frac{\partial}{\partial z}$$

is the standard ∇ operator operating on stretched coordinates with (dimensionless) stretch factors s_x, s_y, s_z. In general,

$$s_x = s_x(x)$$

$$s_y = s_y(y) \qquad \text{(may also be } \textit{complex} \text{ and / or } \textit{constant}\text{)}$$

$$s_z = s_z(z)$$

Plane Wave Propagation in Stretched System

- Assume plane waves can propagate in the stretched medium,

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}, \quad \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$\mathbf{H} = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}},$ substitute into Maxwell's equations to find

$$\Rightarrow \left(\frac{k_x}{s_x}\right)^2 + \left(\frac{k_y}{s_y}\right)^2 + \left(\frac{k_z}{s_z}\right)^2 = \omega^2\mu\varepsilon \equiv k^2$$

so that

$$k_x = k\, s_x \sin\theta \cos\phi$$

$$k_y = k\, s_y \sin\theta \sin\phi$$

$$k_z = k\, s_z \cos\theta$$

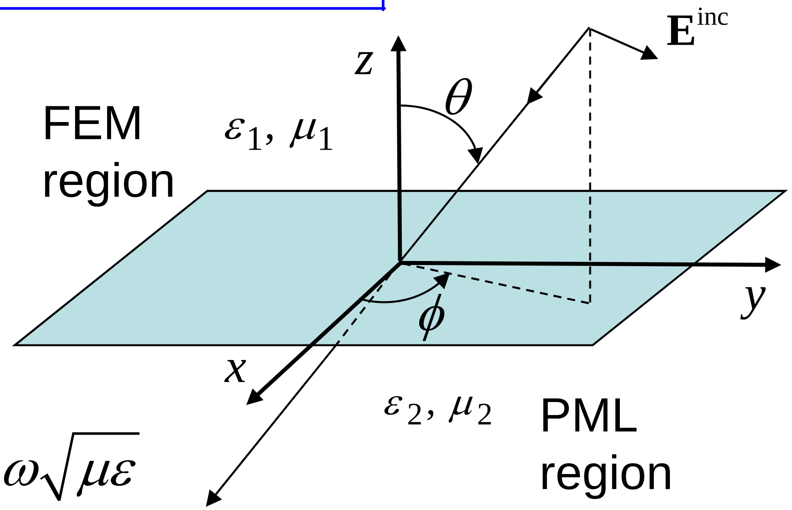
Note that if $\operatorname{Im} s_x < 0$, the plane wave *attenuates* in the $+x$ direction, etc.

Plane Wave Reflection at a Material Interface

- Assume plane waves in both materials at a planar interface and apply boundary conditions, $\mathbf{E}^{\text{inc}}_{\text{tan}} + \mathbf{E}^{\text{r}}_{\text{tan}} = \mathbf{E}^{\text{t}}_{\text{tan}}, \quad \mathbf{H}^{\text{inc}}_{\text{tan}} + \mathbf{H}^{\text{r}}_{\text{tan}} = \mathbf{H}^{\text{t}}_{\text{tan}}$, and the phase matching condition, $\boxed{k_{1\alpha}s_{1\alpha} = k_{2\alpha}s_{2\alpha}, \quad \alpha = x, y}$, to find reflection coefficients

$$R_{TE} = \frac{k_{1z}s_{2z}\mu_2 - k_{2z}s_{1z}\mu_1}{k_{1z}s_{2z}\mu_2 + k_{2z}s_{1z}\mu_1},$$

$$R_{TM} = \frac{k_{1z}s_{2z}\varepsilon_2 - k_{2z}s_{1z}\varepsilon_1}{k_{1z}s_{2z}\varepsilon_2 + k_{2z}s_{1z}\varepsilon_1}.$$



- If $\varepsilon_1 = \varepsilon_2 \equiv \varepsilon, \ \mu_1 = \mu_2 \equiv \mu, \ \Rightarrow \ k_1 = k_2 \equiv k = \omega\sqrt{\mu\varepsilon}$
- Also, if $s_{1x} = s_{2x} \equiv s_x, \quad s_{1y} = s_{2y} \equiv s_y$, then for any s_{1z}, s_{2z} choice,

$$\Rightarrow \boxed{\theta_1 = \theta_2 \equiv \theta, \quad \phi_1 = \phi_2 \equiv \phi} \Rightarrow \boxed{\frac{k_{x1}}{s_{x1}} = \frac{k_{x2}}{s_{x2}} \equiv \frac{k_x}{s_x}, \quad \frac{k_{y1}}{s_{y1}} = \frac{k_{y2}}{s_{y2}} \equiv \frac{k_y}{s_x}}$$

$$\Rightarrow \left(\frac{k_x}{s_x}\right)^2 + \left(\frac{k_y}{s_y}\right)^2 + \left(\frac{k_{1,2z}}{s_{1,2z}}\right)^2 = k^2 \Rightarrow \frac{k_{1z}}{s_{1z}} = \frac{k_{2z}}{s_{2z}} \Rightarrow \boxed{R_{TE} = R_{TM} = 0} \text{ (PML)}$$

PML as an Anisotropic Absorber

- The stretched coordinate approach is difficult to implement in FEM, but *anisotropic* materials are relatively easy to implement; recall they just appear as *dyadic* operators in the FEM element matrices:

$$< \nabla \times \boldsymbol{\Omega}_i^e \ ; \ \boldsymbol{\mu}_r^{-1} \cdot \nabla \times \boldsymbol{\Omega}_j^e >, \quad < \boldsymbol{\Omega}_i^e ; \boldsymbol{\varepsilon}_r \cdot \boldsymbol{\Omega}_j^e >$$

- Let $\mathbf{E}^a, \mathbf{H}^a$ be anisotropic field variables, $\mathbf{E}^c, \mathbf{H}^c$ be stretched coordinate field variables related as

$$\begin{bmatrix} \mathbf{E}^a \\ \mathbf{H}^a \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}^c \\ \mathbf{H}^c \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \mathbf{E}^c \\ \mathbf{H}^c \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & \frac{1}{s_z} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}^a \\ \mathbf{H}^a \end{bmatrix}.$$

We find

$$\nabla_s \times \begin{bmatrix} \mathbf{E}^c \\ \mathbf{H}^c \end{bmatrix} = \begin{bmatrix} \frac{1}{s_y s_z} & 0 & 0 \\ 0 & \frac{1}{s_z s_x} & 0 \\ 0 & 0 & \frac{1}{s_x s_y} \end{bmatrix} \cdot \nabla \times \begin{bmatrix} \mathbf{E}^a \\ \mathbf{H}^a \end{bmatrix}.$$

Maxwell's Eqs. in an Anisotropic PML

- Substituting into $\nabla_s \times \mathbf{E}^c = -j\omega\mu\mathbf{H}^c$ etc., we find

$$\nabla \times \mathbf{E}^a = -j\omega\mu\Lambda \cdot \mathbf{H}^a$$

$$\nabla \times \mathbf{H}^a = j\omega\varepsilon\Lambda \cdot \mathbf{E}^a$$

$$\nabla\left(\varepsilon\Lambda \cdot \mathbf{E}^a\right) = 0$$

$$\nabla\left(\mu\Lambda \cdot \mathbf{H}^a\right) = 0$$

an anisotropic material for which

$$\boldsymbol{\varepsilon} = \varepsilon\Lambda,\ \ \boldsymbol{\mu} = \mu\Lambda,\ \ \boldsymbol{\Lambda} = \begin{bmatrix} \frac{1}{s_y s_z} & 0 & 0 \\ 0 & \frac{1}{s_z s_x} & 0 \\ 0 & 0 & \frac{1}{s_x s_y} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & \frac{1}{s_z} \end{bmatrix} = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_z s_x}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

- Within the PML, $\mathbf{E}^a$ and $\mathbf{E}^c$ are different, but outside the PML region they are identical (Λ, etc. $= \mathbf{I} =$ identity matrix), so we must still have

$$R_{TE} = R_{TM} = 0$$

- The result is independent of frequency, polarization, and incidence angle!

Choosing the Stretching Parameter

- Note that if

  $$s_{2z} = s' - js'',$$

  $$\Rightarrow \quad k_{2z} = k_2 s_{2z} \cos\theta \; = k_2 \left(s' - js''\right)\cos\theta$$

  and $\left|R(\theta)\right| = e^{-2k_2 \cos\theta \int_0^L s''(z)dz}$

- If we also choose

  $$s_{2z} = 1 - j\frac{\sigma}{\omega\varepsilon}$$

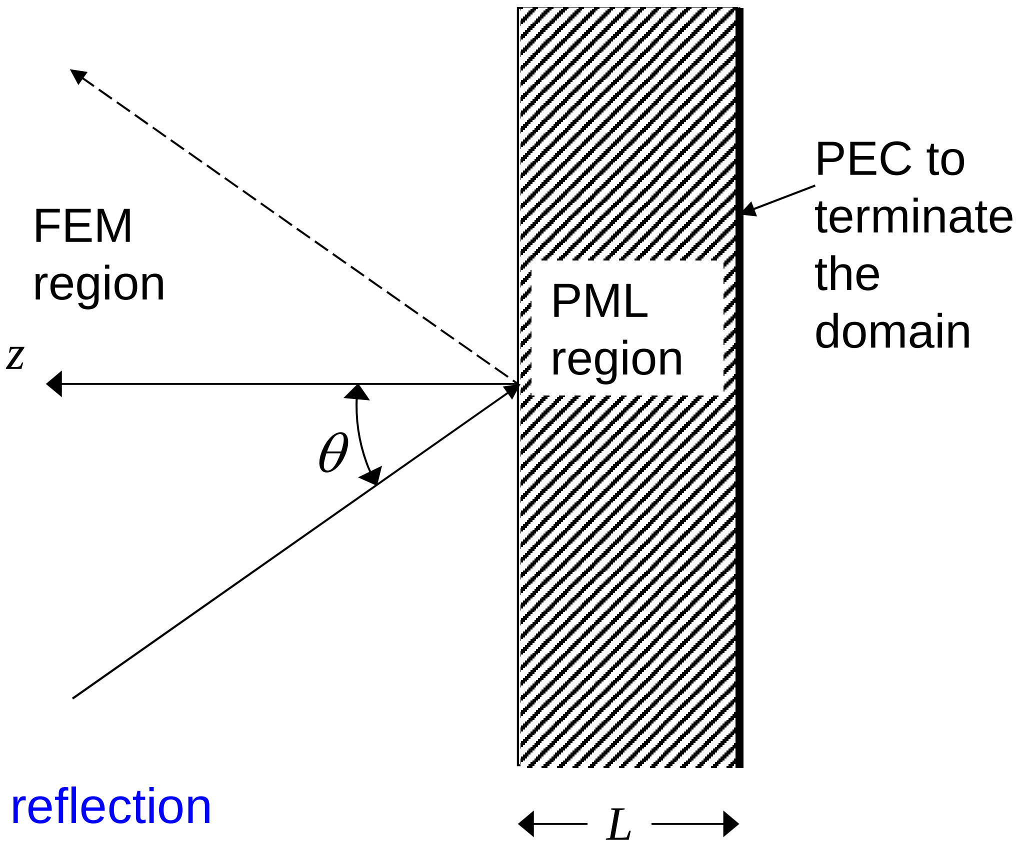
  then

  $$k_{2z}s_{2z} = k - jk\frac{\sigma}{\omega\varepsilon} = k - j\eta\sigma$$

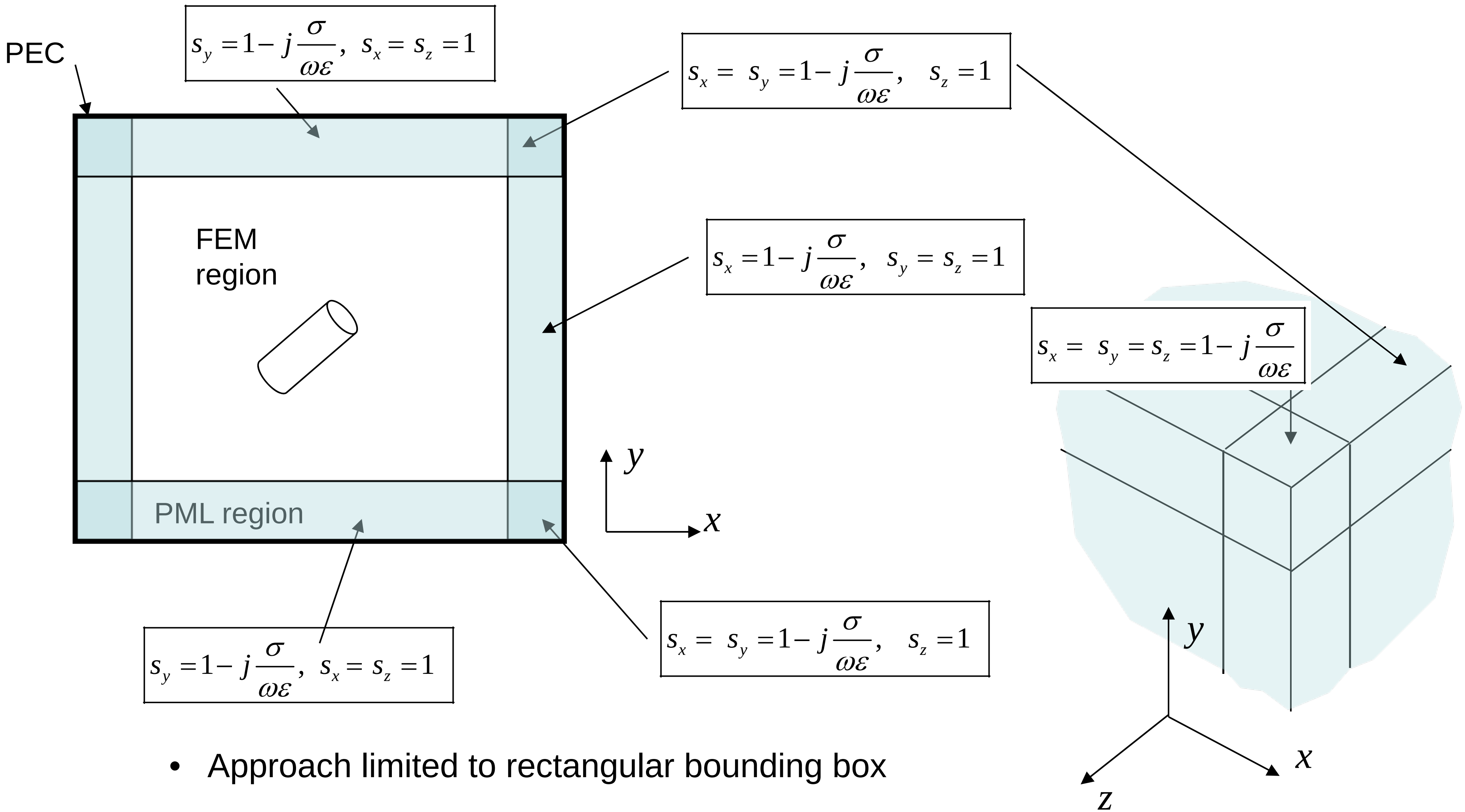
  and the decay factor is frequency independent :

  $$\left|R(\theta)\right| = e^{-2\eta\sigma L \cos\theta}$$

- $\sigma, \; L$ determine the amount of reflection



PML Construction for Domain Truncation



- Approach limited to rectangular bounding box
- 27 PML regions