ECE 6350

Grad-, Div-, and Curl-Conforming Bases on 2- and 3-D Simplexes

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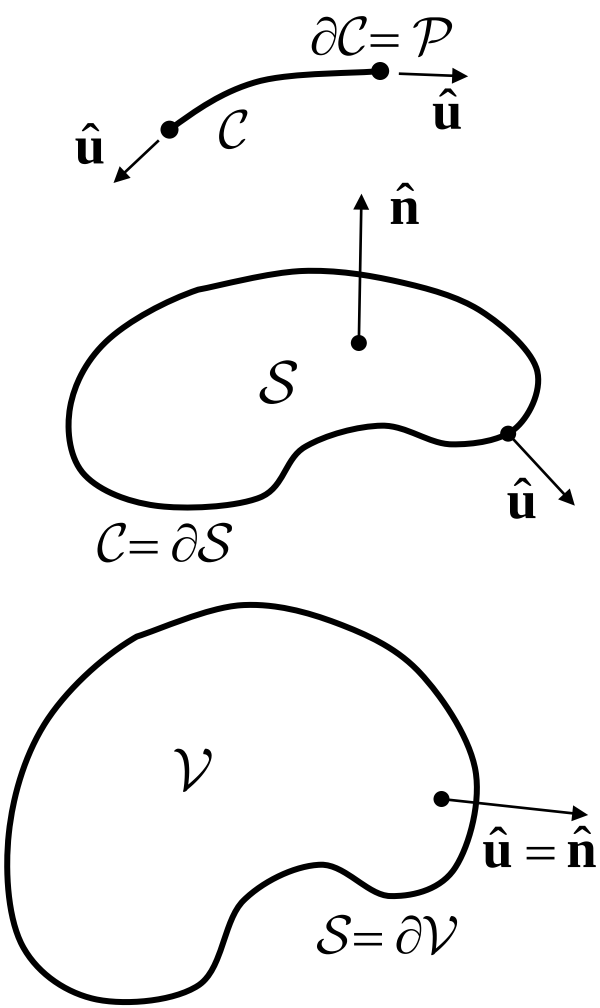
Dimension-Independent Divergence Definition and Theorem



Definitions:

- $\mathbf{J}, \mathbf{D}, \mathbf{B}$ are flux vectors
- Domain $\mathcal{D} = \mathcal{P}, \mathcal{C}, \mathcal{S}, \mathcal{V}$ (point, curve, surface, volume)
- Boundary of $\mathcal{D} = \partial\mathcal{D} \equiv \mathcal{B} = \begin{cases} \mathcal{P} \text{ if } \mathcal{D} = \mathcal{C} \text{ (open)}, \\ \mathcal{C} \text{ if } \mathcal{D} = \mathcal{S} \text{ (open)}, \\ \mathcal{S} \text{ if } \mathcal{D} = \mathcal{V} \end{cases}$
- "Measure" of $\mathcal{D} \equiv \text{meas}\,\mathcal{D} = \begin{cases} \text{length of } \mathcal{C} \\ \text{area of } \mathcal{S} \\ \text{volume of } \mathcal{V} \end{cases}$
- $\hat{\mathbf{u}}$ is normal to $\mathcal{B}$ and "tangent" to $\mathcal{D}$
- Flux of a vector $\mathbf{F} \equiv \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{u}}\, d\mathcal{B}$
- Divergence of a vector $\mathbf{F} \equiv \nabla \cdot \mathbf{F} \equiv \lim_{\text{meas}\,\mathcal{D} \to 0} \dfrac{1}{\text{meas}\,\mathcal{D}} \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{u}}\, d\mathcal{B}$
- Divergence Thm: $\int_{\mathcal{D}} \nabla \cdot \mathbf{F}\, d\mathcal{D} = \oint_{\partial\mathcal{D}} \mathbf{F} \cdot \hat{\mathbf{u}}\, d\mathcal{B}$

Modeling Flux Quantities

In the EFIE :

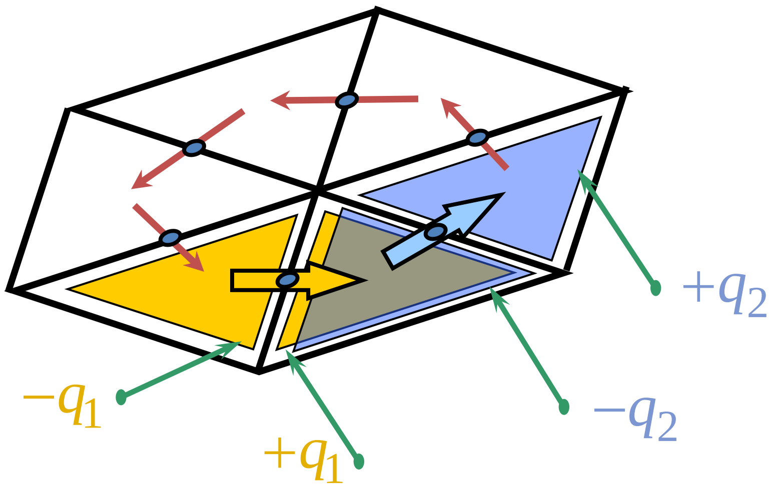
- We selected $\boldsymbol{\Lambda}_i^e$ such that $\nabla \cdot \boldsymbol{\Lambda}_i^e$ was constant, $\boldsymbol{\Lambda}_i^e$ has constant (unit) normal component on element subboundary

- Hence $\nabla \cdot \boldsymbol{\Lambda}_i^e = \dfrac{1}{\text{meas}\,\mathcal{D}} \displaystyle\int_{\mathcal{D}} \nabla \cdot \boldsymbol{\Lambda}_i^e \, d\mathcal{D} = \dfrac{1}{\text{meas}\,\mathcal{D}} \oint_{\partial\mathcal{D}} \boldsymbol{\Lambda}_i^e \cdot \hat{\mathbf{u}} \, d\mathcal{B}$

$\Rightarrow$ Note the *consistent* modeling of divergence on both point and discrete (element) scales.

This appears to be a strongly desirable condition for a good numerical scheme

Modeling Flux Quantities, cont'd



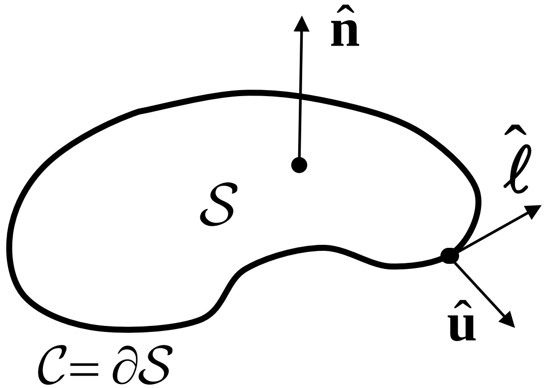
- We should alos be able to create a *divergence-free basis function* as a linear combination of ordinary div-conforming basis functions

$$\mathbf{\Lambda}_{loop} = \sum_i \alpha_i \mathbf{\Lambda}_i \;\; \Rightarrow \;\; \nabla \cdot \mathbf{\Lambda}_{loop} = 0$$

$$(\text{choose } \alpha_i \text{ such that } q_{i+1} = q_i)$$

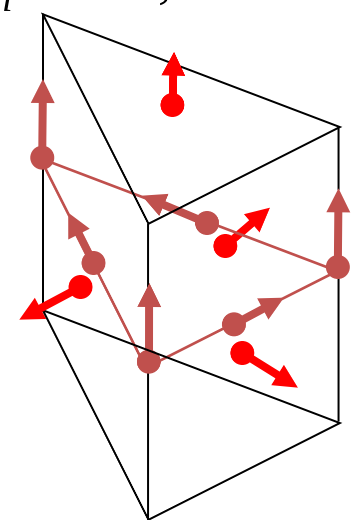
Modeling Line Integral Quantities

- $\mathbf{E}, \mathbf{H}, \nabla\Phi$ are all line integral quantities $\Rightarrow$
  circulation of $\mathbf{F} = \oint_{\mathcal{C}} \mathbf{F} \cdot \hat{\ell}\, d\mathcal{C}$



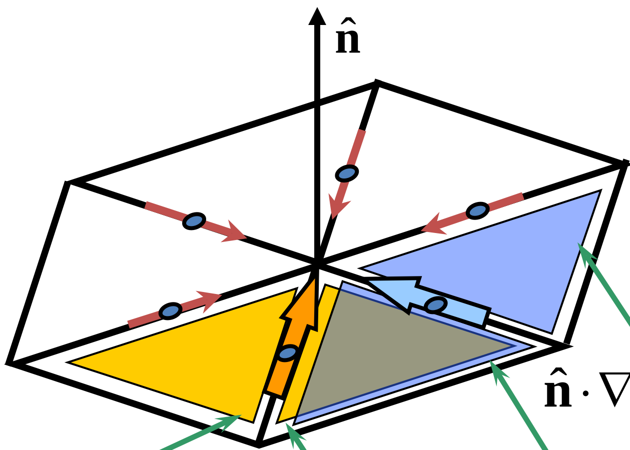
- Associated bases $\mathbf{\Omega}_i^e$ should be *curl - conforming*
- Select $\mathbf{\Omega}_i^e$ such that $\nabla \times \mathbf{\Omega}_i^e$ is constant, $\mathbf{\Omega}_i^e$ has constant (unit) tangential component on element subboundary

- Hence $\nabla \times \mathbf{\Omega}_i^e = \dfrac{1}{\text{meas}\,\mathcal{S}} \displaystyle\int_{\mathcal{S}} \nabla \times \mathbf{\Omega}_i^e \cdot \hat{\mathbf{n}}\, d\mathcal{S} \overset{\text{Stokes's Theorem}}{=} \dfrac{1}{\text{meas}\,\mathcal{S}} \displaystyle\oint_{\partial\mathcal{S}} \mathbf{\Omega}_i^e \cdot \hat{\ell}\, d\mathcal{C}, \ \hat{\ell} = \hat{\mathbf{n}} \times \hat{\mathbf{u}}$



$\Rightarrow$ Hence the curl is modeled *consistently* on both point and discrete (element) scales. This is a strongly desirable condition for schemes involving the curl operator

Modeling of Line Integral Quantities, cont'd



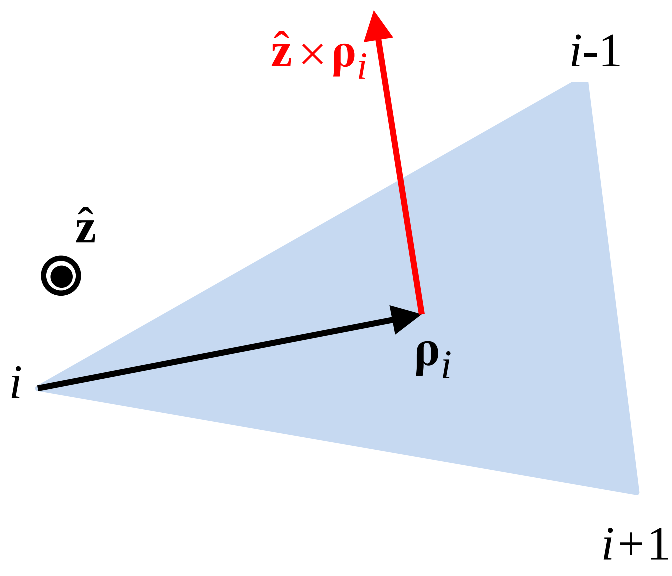
- We should also be able to create a curl-free basis function as a linear combination of ordinary curl-conforming basis functions

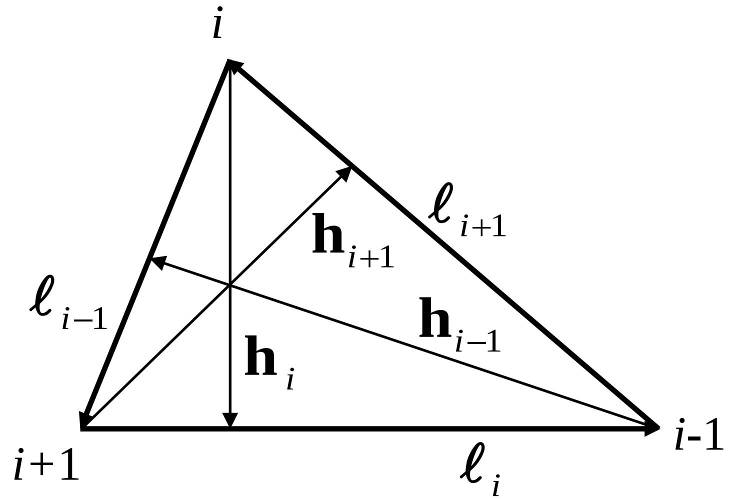
$$\mathbf{\Omega}_{vertex} = \sum_i \alpha_i \mathbf{\Omega}_i \Rightarrow \nabla \times \mathbf{\Omega}_{vertex} = 0$$

(choose α_i such that $\alpha_i \, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{\Omega}_i = \alpha_{i+1} \hat{\mathbf{n}} \cdot \nabla \times \mathbf{\Omega}_{i+1}$)

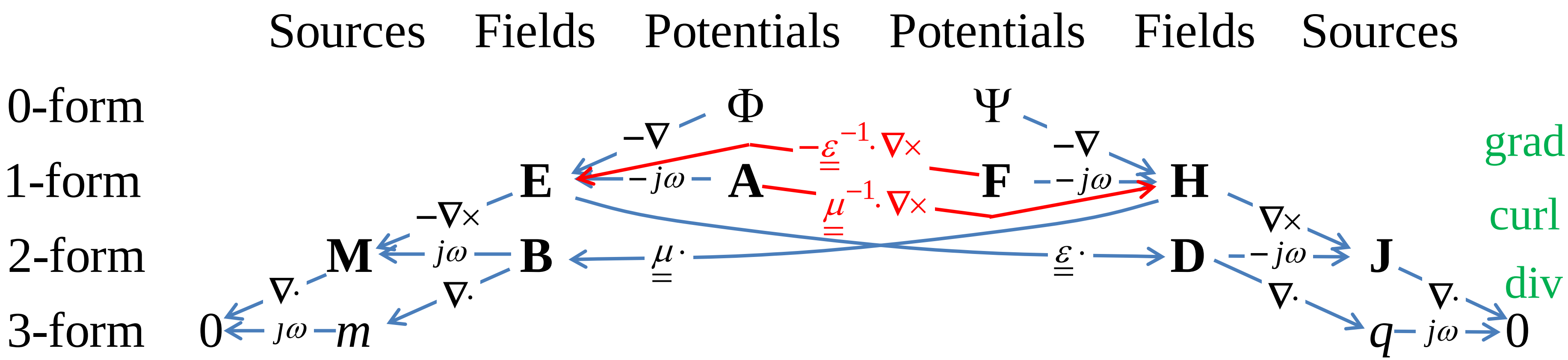
Properties of $\boldsymbol{\rho}_i$, $\hat{\mathbf{z}}\times\boldsymbol{\rho}_i$ Vectors

	local coordinates	area coordinates	normal comp., edge i	tangential comp., edge i	div ($\nabla\cdot$)	curl ($\nabla\times$)
$\boldsymbol{\rho}_i$	$x\hat{\mathbf{x}}+y\hat{\mathbf{y}}$	$\xi_{i+1}\ell_{i-1}-\xi_{i-1}\ell_{i+1}$	h_i	$---$	2	$0+\delta$'s@ boundary
$\hat{\mathbf{z}}\times\boldsymbol{\rho}_i$	$x\hat{\mathbf{y}}-y\hat{\mathbf{x}}$	$2A^e\left(\xi_{i+1}\nabla\xi_{i-1}-\ \xi_{i-1}\nabla\xi_{i+1}\right)$	$---$	h_i	$0+\delta$'s@ boundary	$2\hat{\mathbf{z}}$





Character of Potential, Field, and Source Quantities



0- forms: scalar point functions: Φ, Ψ

1- forms: vector line-integral functions: $\int \mathbf{E}\cdot d\mathbf{r},\ \int \mathbf{H}\cdot d\mathbf{r}$

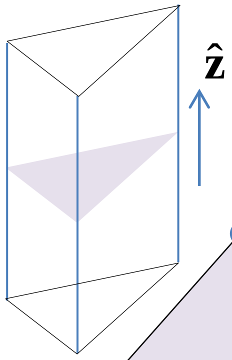
2- forms: vector flux integral functions: $\int \mathbf{B}\cdot d\mathbf{S},\ \int \mathbf{D}\cdot d\mathbf{S},\ \int \mathbf{J}\cdot d\mathbf{S},\ \int \mathbf{M}\cdot d\mathbf{S}$

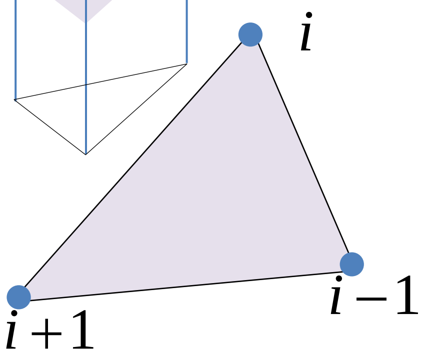
3- forms: scalar density functions: $\int q dV,\ \int m dV$

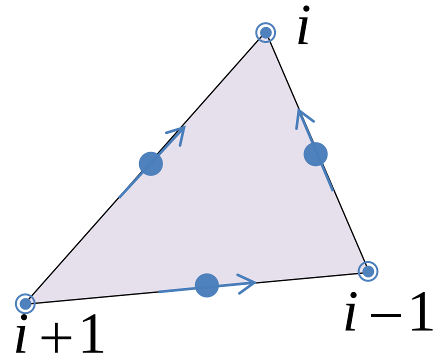
p-forms represent quantities integrated over *p*-dimensions

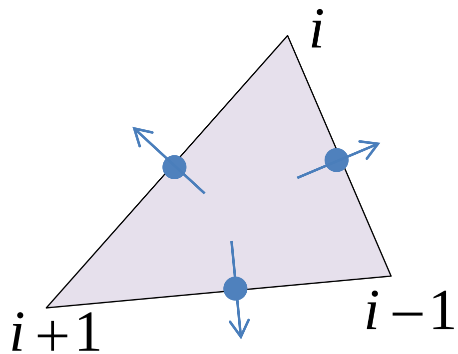
Warnick, K.F., Russer, P., “Two, Three and Four-Dimensional Electromagnetics Using Differential Forms,“ http://journals.tubitak.gov.tr/elektrik/issues/elk-06-14-1/elk-14-1-14-0509-13.pdf

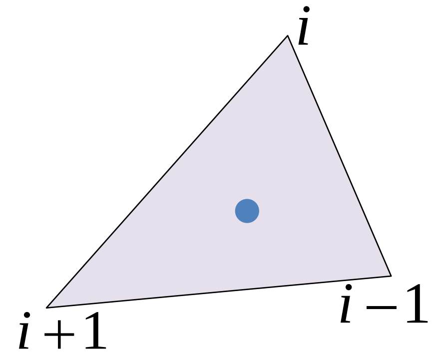
Interpolation of 2-D 0-, 1-, 2-, and 3-forms











- 0– form
- grad-conforming
- $\Lambda_i^e = \xi_i , i = 1,2,3$
- $\Lambda_i^e \Big|_{\xi_j=1} = \delta_{ij}$
- $\nabla \Lambda_i^e = \dfrac{-\hat{\mathbf{h}}_i}{h_i}$

Example:

$$\Phi = \sum_{i=1}^{3} V_i^e \ \Lambda_i^e , \quad \mathbf{r} \in S^e$$

- 1– form
- curl-conforming
- $\boldsymbol{\Omega}_i^e = \ell_i (\xi_{i+1} \nabla \xi_{i-1} - \xi_{i-1} \nabla \xi_{i+1}),$
  $\boldsymbol{\Omega}_{i+3}^e = \xi_i \hat{\mathbf{z}}, \ i = 1,2,3;$
- $\boldsymbol{\Omega}_i^e \cdot \hat{\boldsymbol{\ell}}_j \Big|_{\xi_j=0} = \delta_{ij},$
  $\boldsymbol{\Omega}_i^e \cdot \hat{\mathbf{z}} \Big|_{\xi_j=1} = \delta_{ij}$
- $\nabla \times \boldsymbol{\Omega}_i^e = \dfrac{2\hat{\mathbf{z}}}{h_i} ; \nabla \times \boldsymbol{\Omega}_{i+3}^e = \dfrac{\ell_i}{2A^e}$

Example:

$$\mathbf{E} = \sum_{i=1}^{6} V_i^e \, \boldsymbol{\Omega}_i^e , \quad \mathbf{r} \in S^e$$

- 2– form
- div-conforming
- $\boldsymbol{\Lambda}_i^e = \dfrac{\xi_{i+1} \boldsymbol{\ell}_{i-1} - \xi_{i-1} \boldsymbol{\ell}_{i+1}}{h_i},$
  $i = 1,2,3$
- $\boldsymbol{\Lambda}_i^e \cdot \hat{\mathbf{h}}_j \Big|_{\xi_j=0} = \delta_{ij}$
- $\nabla \cdot \boldsymbol{\Lambda}_i^e = \dfrac{2}{h_i}$

Example:

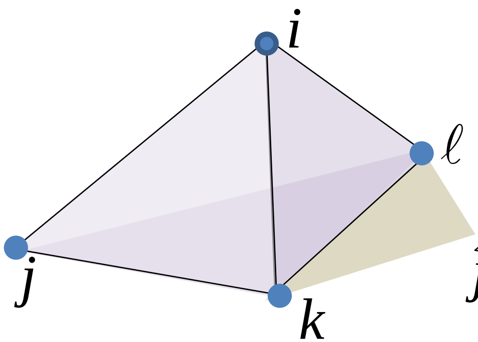
$$\mathbf{J} = \sum_{i=1}^{3} J_i^e \boldsymbol{\Lambda}_i^e , \mathbf{r} \in S^e$$

- 3– form
- density function
- $\Pi^e = 1, \mathbf{r} \in S^e$

Example:

$$q = Q^e \, \Pi^e , \ \mathbf{r} \in S^e$$

Interpolation of 3-D 0-, 1-, 2-, and 3-forms

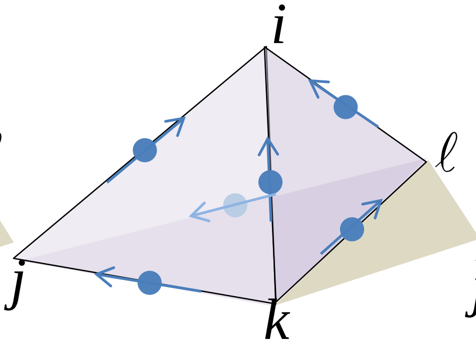


- $0-$ form
- grad-conforming
- $\Lambda_i^e = \xi_i,\ i = 1,\dots,4$
- $\Lambda_i^e\Big|_{\xi_j=1} = \delta_{ij}$
- $\nabla\Lambda_i^e = \dfrac{-\hat{\mathbf{h}}_i}{h_i}$

Example:

$$\Phi = \sum_{i=1}^{4} V_i^e\ \Lambda_i^e,$$

$$\mathbf{r}\in V^e$$



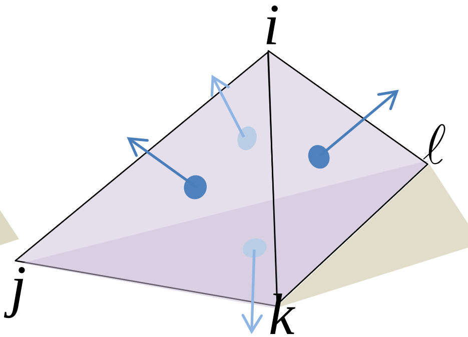
- $1-$ form
- curl-conforming
- $\boldsymbol{\Omega}_{ij}^e = \ell_{ij}(\xi_j\nabla\xi_i - \xi_i\nabla\xi_j),$

  $i,j\in\{1,\dots,4\},\ i<j$

  $\boldsymbol{\ell}_{ij} = \mathbf{r}_i^e - \mathbf{r}_j^e,\ \mathbf{r}\in V^e$
- $\boldsymbol{\Omega}_{ij}^e\cdot\hat{\boldsymbol{\ell}}_{\alpha\beta}\Big|_{\xi_k=\xi_\ell=0} = \delta_{i\alpha,j\beta}$
- $\nabla\times\boldsymbol{\Omega}_{ij}^e = \dfrac{\ell_{ij}\boldsymbol{\ell}_{k\ell}}{3V^e}$

Example:

$$\mathbf{E} = \sum_{i<j} V_{ij}^e\ \boldsymbol{\Omega}_{ij}^e,\ \mathbf{r}\in V^e,$$



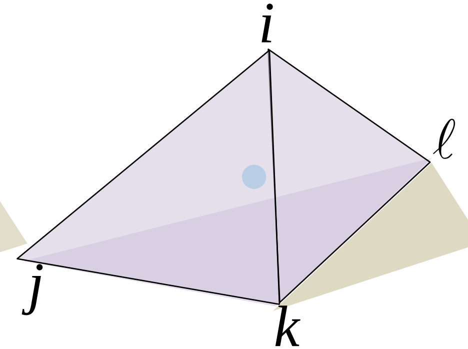
- $2-$ form
- div-conforming
- $\boldsymbol{\Lambda}_i^e = \displaystyle\sum_{\alpha\in\{j,k,\ell\}} \frac{\xi_\alpha\boldsymbol{\ell}_{\alpha i}}{h_i},$

  $i\in\{1,\dots,4\}$

  $\boldsymbol{\ell}_{\alpha i} = \mathbf{r}_\alpha^e - \mathbf{r}_i^e,\ \mathbf{r}\in V^e,$
- $\boldsymbol{\Lambda}_i^e\cdot\hat{\mathbf{h}}_j\Big|_{\xi_j=0} = \delta_{ij}$
- $\nabla\cdot\boldsymbol{\Lambda}_i^e = \dfrac{3}{h_i}$

Example:

$$\mathbf{J} = \sum_{i=1}^{4} J_i^e\boldsymbol{\Lambda}_i^e,\ \ \mathbf{r}\in V^e$$



- $3-$ form
- $\Pi^e = 1,\quad \mathbf{r}\in V^e$

Example:

$$q = Q^e\,\Pi^e,\ \mathbf{r}\in V^e$$