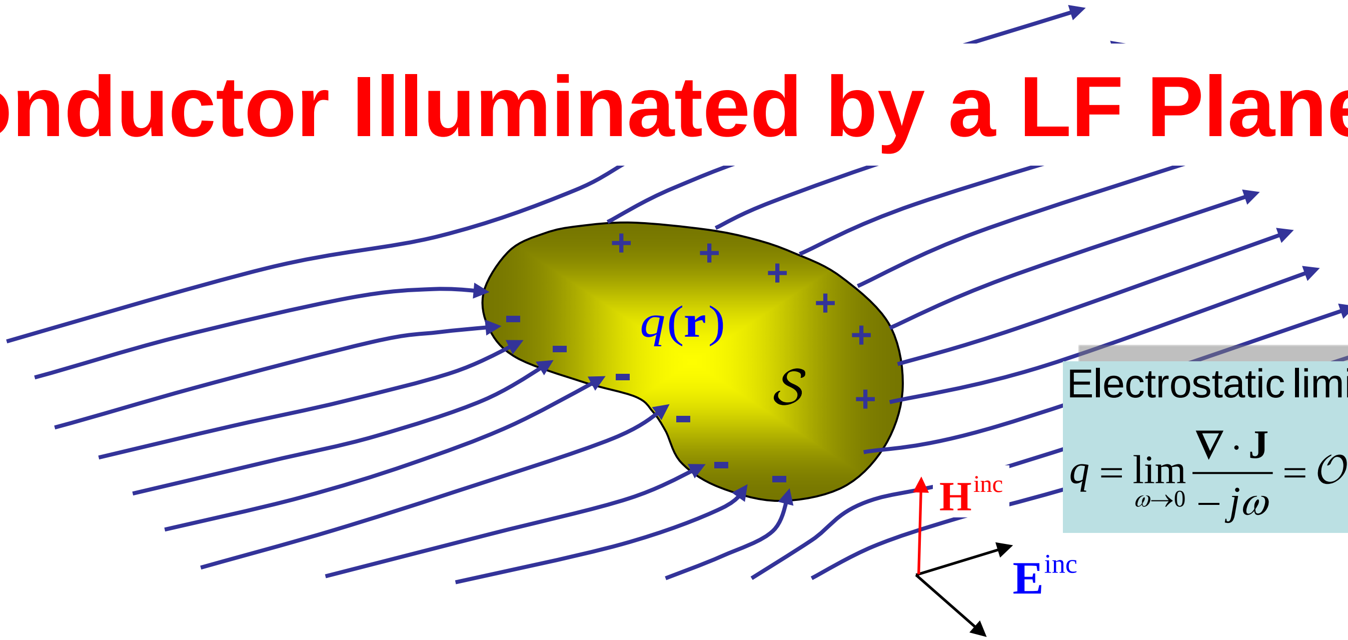
Low Frequency Breakdown

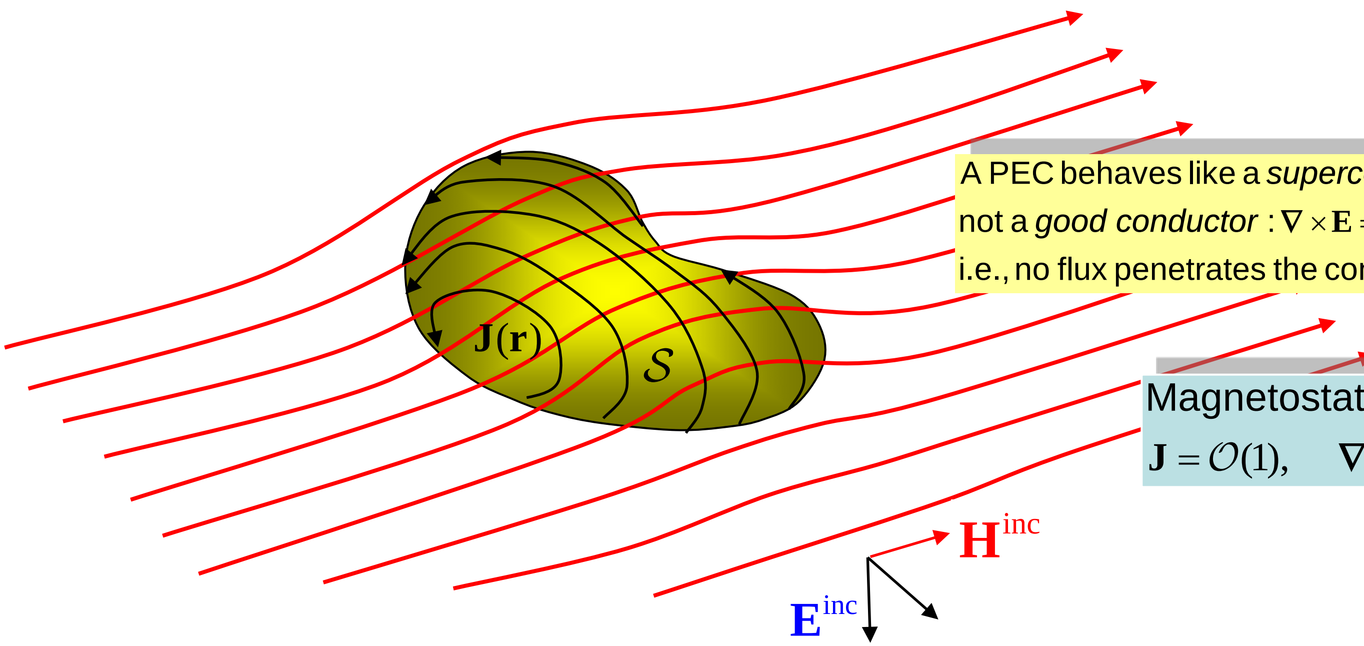
Donald R. Wilton
Dept. of Electrical and Computer Engineeering
University of Houston
Houston, TX 77096 USA
wilton@uh.edu

Conductor Illuminated by a LF Plane Wave



Electrostatic limit :

$$q = \lim_{\omega \to 0} \frac{\nabla \cdot \mathbf{J}}{-j\omega} = \mathcal{O}(1) \;\; \Rightarrow \nabla \cdot \mathbf{J} = \mathcal{O}(\omega)$$



A PEC behaves like a *superconductor* at DC, not a *good conductor* : $\nabla \times \mathbf{E} = 0 \Rightarrow d\mathbf{B}/dt = 0$, i.e., no flux penetrates the conductor!

Magnetostatic problem :

$$\mathbf{J} = \mathcal{O}(1), \qquad \nabla \cdot \mathbf{J} = 0$$

EFIE and Vector Helmholtz Equations at Low Frequencies

- **EFIE (strong form)**:

$$\mathcal{L}\mathbf{J} \equiv \left[j\omega\mu \int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}' - \frac{\nabla}{j\omega\varepsilon} \int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{J}(\mathbf{r}')d\mathcal{S}' \right]_{\tan} = \mathbf{E}_{\tan}^{\text{inc}}, \ \mathbf{r}\in\mathcal{S}$$

$$\xrightarrow{\omega\to 0} \mathcal{L}_{\text{LF}}\mathbf{J} \equiv \left[-\frac{\nabla}{j\omega\varepsilon} \int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{J}(\mathbf{r}')d\mathcal{S}' \right]_{\tan} = \mathbf{E}_{\tan}^{\text{inc}}$$

$\Rightarrow$ Any divergenceless current $(\nabla\cdot\mathbf{J}_h(\mathbf{r}')=0)$ distribution on $\mathcal{S}$ is a homogeneous solution, $\mathcal{L}_{\text{LF}}\mathbf{J}_h = 0$, and implies *low frequency solutions are non-unique*.

- **Helmholtz Eq. (strong form):**

$\mathcal{L}\mathbf{E} \equiv \nabla\times\mu_r^{-1}\nabla\times\mathbf{E} - \omega^2\mu_0\varepsilon_0\varepsilon_r\mathbf{E} = -j\omega\mu_0\mathbf{J}, \ \mathbf{r}\in \ \ \mathcal{D}=\mathcal{S}$ (2-D) or $\mathcal{V}$ (3-D)

$$\xrightarrow{\omega\to 0} \mathcal{L}_{\text{LF}}\mathbf{E} \equiv \nabla\times\mu_r^{-1}\nabla\times\mathbf{E} = -j\omega\mu_0\mathbf{J}$$

$\Rightarrow$ Any curl-free field $(\nabla\times\mathbf{E}_h = 0)$ in $\mathcal{D}$ is a homogeneous solution, $\mathcal{L}_{\text{LF}}\mathbf{E}_h = 0$, and implies *low frequency solutions are non-unique*.

Helmholtz Decomposition of EFIE Current

- **EFIE current splitting**:

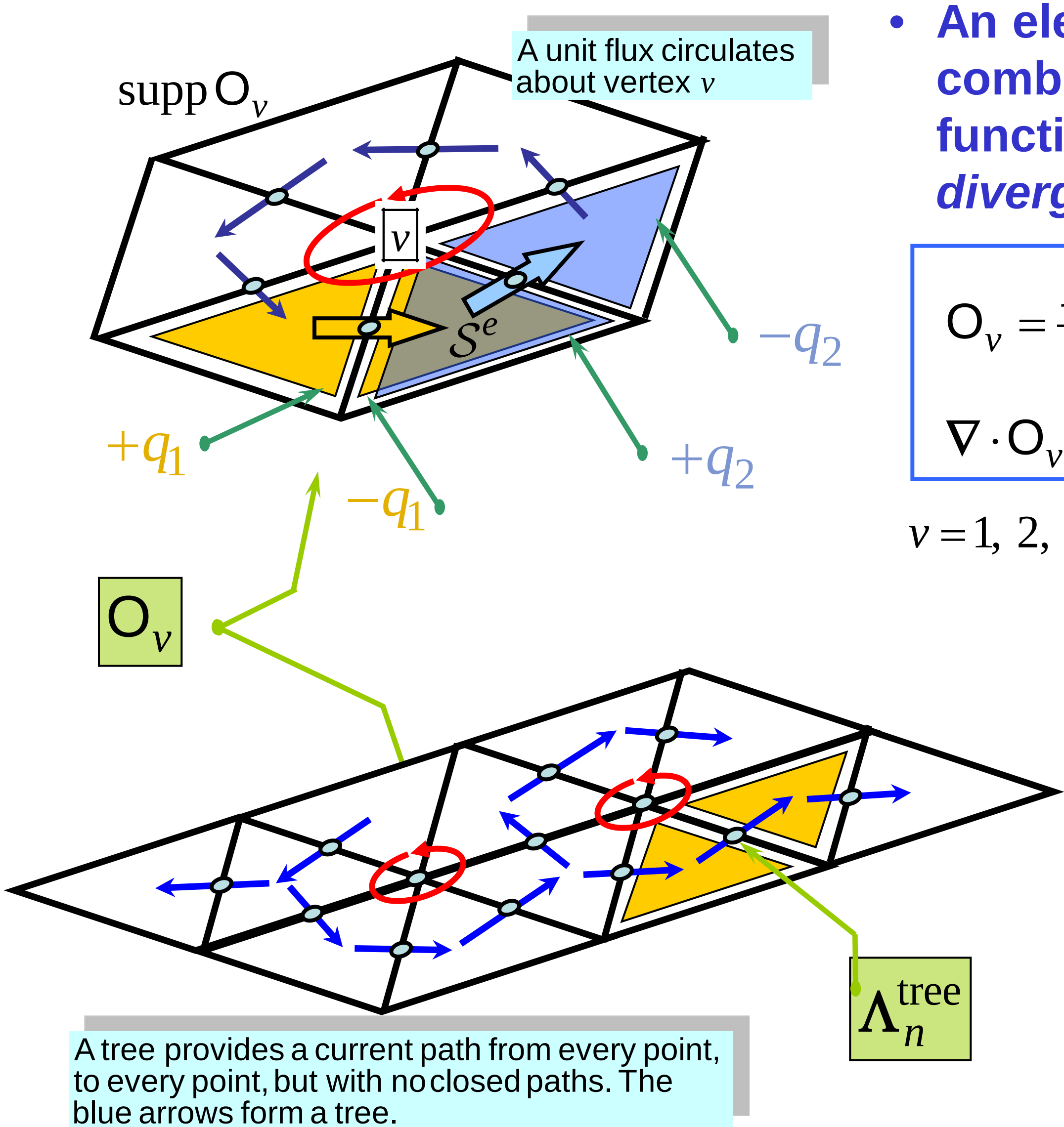
$$\mathbf{J} = \overbrace{\mathbf{J}^{\circ}}^{\text{divergenceless, magnetostatic}} + \overbrace{\mathbf{J}^{\star}}^{\text{non-divergenceless, electrostatic}}$$

- **Low frequency behavior :**

$$\mathbf{J} \xrightarrow{\omega\to 0} \mathbf{J}^{\circ} \qquad \Rightarrow \boxed{\mathbf{J}^{\circ} = \mathcal{O}(1) \quad \text{(real)}}$$

$$q = \frac{\nabla\cdot\mathbf{J}^{\star}}{-j\omega} = \mathcal{O}(1) \ \text{(real)} \ \Rightarrow \boxed{\mathbf{J}^{\star} = \mathcal{O}(\omega) \quad \text{(imaginary)}}$$

Loop-Tree Basis Decomposition



- **An elemental "loop" is a linear combination of patch basis functions that produces a *divergence-free* basis function**

$$\mathrm{O}_v = \frac{\Lambda_{i+1}^e}{\ell_{i+1}} - \frac{\Lambda_{i-1}^e}{\ell_{i-1}}, \quad \mathbf{r} \in \mathcal{S}^e \subset \mathrm{supp}\,\mathrm{O}_v$$

$$\nabla \cdot \mathrm{O}_v = 0 \quad \Rightarrow \quad q_1 = q_2$$

$v = 1, 2, \ldots,$ # interior vertices = V - B

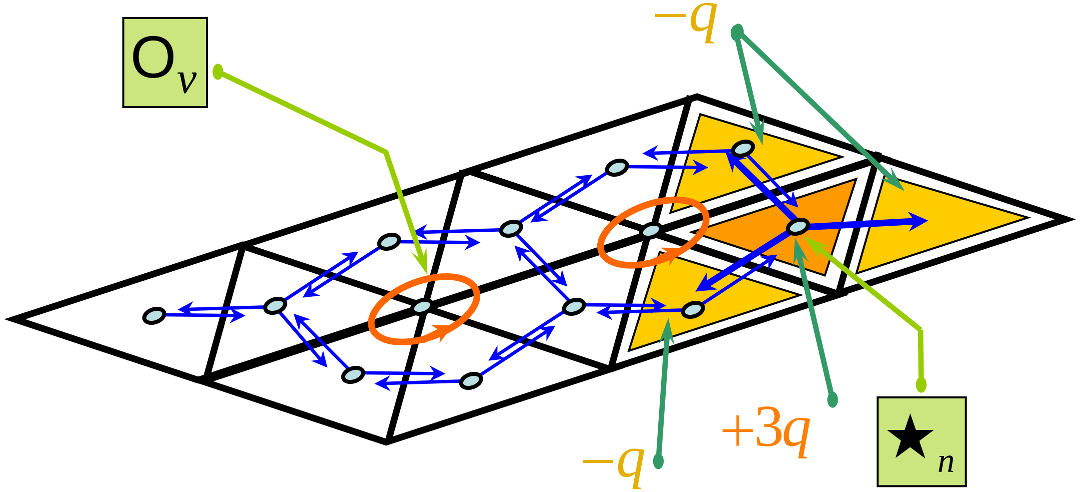
At low frequencies:

- O_v **forms a magnetostatic source (current loop)**
- Λ_n^{tree} **forms an electrostatic source (charge dipole)**

$n = 1, 2, \ldots,$ # triangles $-1 = \mathrm{F} - 1,$

$\mathrm{V} - \mathrm{B} + \mathrm{F} - 1 = \mathrm{E} - \mathrm{B} = N$

Loop-Star Basis Decomposition



At low frequencies:

- O_v becomes a magnetostatic source (current loop)
- $\bigstar_n$ becomes an electrostatic source (charge multipole)

- O_v is a vertex-based source
- $\bigstar_n$ is a face-based source

$\Rightarrow$

In principle, the loop-star decomposition eliminates the tedious procedure of specifying a "tree" on the triangular patch surface.

In practice, it usually does not yield as well-conditioned MoM matrix as the loop-tree decomposition.

EFIE and Vector Helmholtz Equations at Low Frequencies

EFIE Unknowns:

$$\mathbf{J} \approx \sum_{n=1}^{N} I_n \mathbf{\Lambda}_n,\ q_S = -\frac{\nabla \cdot \mathbf{J}}{j\omega} \approx -\frac{\sum_{n=1}^{N} I_n \nabla \cdot \mathbf{\Lambda}_n}{j\omega}$$

$$\left[Z_{mn} \right] = j\omega\mu \left[< \mathbf{\Lambda}_m ; G, \mathbf{\Lambda}_n > \right] + \frac{1}{j\omega\varepsilon} \left[< \nabla \cdot \mathbf{\Lambda}_m , G, \nabla \cdot \mathbf{\Lambda}_n > \right]$$

$$\xrightarrow{\omega \to 0} \frac{1}{j\omega\varepsilon} \left[< \nabla \cdot \mathbf{\Lambda}_m , G, \nabla \cdot \mathbf{\Lambda}_n > \right]$$

$\Rightarrow$ Homogeneous solutions exist: $\left[Z_{mn} \right] \left[I_n \right] = 0$

$$\mathrm{O}_v = \sum_{n=1}^{N} \sigma_{vn}^{\mathrm{o}}\ \mathbf{\Lambda}_n,\quad \nabla \cdot \mathrm{O}_v = 0$$

$$\bigstar_f = \sum_{n=1}^{N} \sigma_{fn}^{\bigstar}\ \mathbf{\Lambda}_n,\quad \nabla \cdot \bigstar_f \neq 0$$

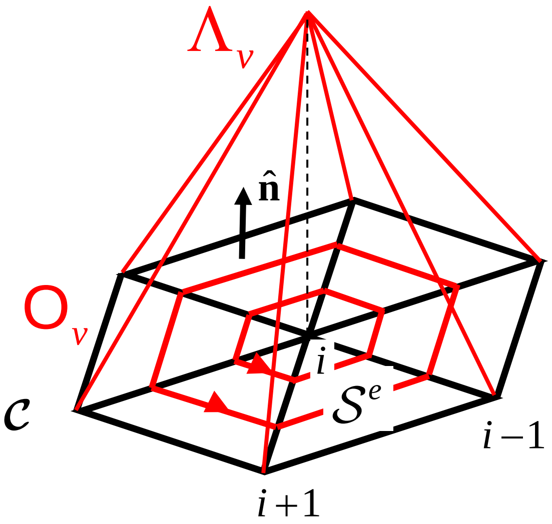
Loop Basis Representation

- We can write a loop basis O_v about interior vertex v containing triangle $\mathcal{S}^e$ in its support, $\operatorname{supp}\mathrm{O}_v$, and with the i-th local vertex of $\mathcal{S}^e$ corresponding to vertex v, in various ways:

$$\mathrm{O}_v = \frac{\boldsymbol{\Lambda}_{i+1}^e}{\ell_{i+1}} - \frac{\boldsymbol{\Lambda}_{i-1}^e}{\ell_{i-1}} = \frac{\boldsymbol{\ell}_i}{2A^e} = \frac{\hat{\boldsymbol{\ell}}_i}{h_i} = \frac{\hat{\mathbf{n}}\times\hat{\mathbf{h}}_i}{h_i}, \quad \mathbf{r}\in\mathcal{S}^e\subset\operatorname{supp}\mathrm{O}_v,$$

but perhaps most useful and illuminating is

$$\boxed{\mathrm{O}_v = \nabla\xi_i\times\hat{\mathbf{n}}, \quad \mathbf{r}\in\mathcal{S}^e\subset\operatorname{supp}\mathrm{O}_v} \Rightarrow \boxed{\mathrm{O}_v = \nabla\Lambda_v\times\hat{\mathbf{n}}, \quad \mathbf{r}\in\operatorname{supp}\mathrm{O}_v}$$



- For an arbitrary, continuous vector $\mathbf{A}$ on $\mathcal{S}^e$, we have

$$\int_{\mathcal{S}^e}(\nabla\xi_i\times\hat{\mathbf{n}})\cdot\mathbf{A}\,d\mathcal{S} = -\int_{\mathcal{S}^e}(\nabla\xi_i\times\mathbf{A})\cdot\hat{\mathbf{n}}\,d\mathcal{S} = \int_{\mathcal{S}^e}(\xi_i\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A})\,d\mathcal{S} - \oint_{\partial\mathcal{S}^e}\xi_i\mathbf{A}\cdot d\boldsymbol{\mathcal{C}}$$

(Van Bladel, A3.57)

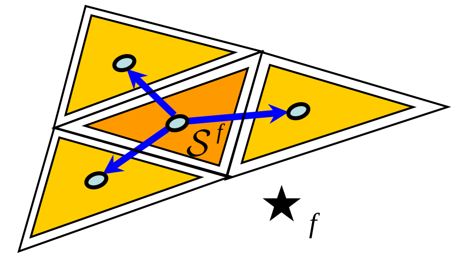
where the contour integral vanishes when contributions from all adjacent triangles with a common vertex are added, so that

$$\boxed{<\mathrm{O}_v;\mathbf{A}> = \int_{\mathcal{S}}\mathrm{O}_v\cdot\mathbf{A}\,d\mathcal{S} = \int_{\mathcal{S}}(\Lambda_v\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A})\,d\mathcal{S} = <\Lambda_v\hat{\mathbf{n}};\nabla\times\mathbf{A}>},$$

where $\Lambda_v = \xi_i, \mathbf{r}\in\mathcal{S}^e\subset\operatorname{supp}\mathrm{O}_v$ is the scalar rooftop (pyramidal) function with peak at node v. Hence, *testing a continuous vector with a loop function is equivalent to averaging the rooftop-weighted normal component of the vector's curl over the loop's support.*

Tree, Star Basis Representations

- *Tree bases* are usual basis set but with any tree links forming closed loops removed from the set : $\left\{\mathbf{\Lambda}_n^{\text{tree}}\right\} \subset \left\{\mathbf{\Lambda}_n\right\}$

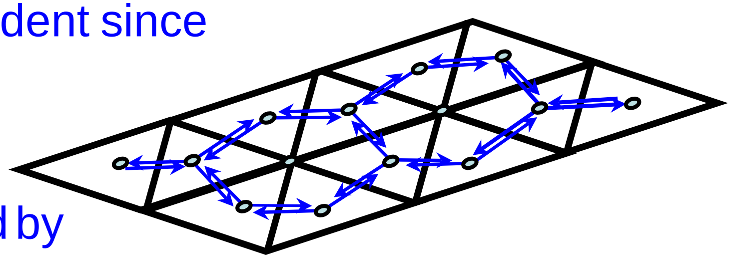


- *Star bases* are not uniquely defined; two possible definitions are

$$\bigstar_f = \sum_n \sigma_{fn}^{\bigstar} \mathbf{\Lambda}_n, \quad \sigma_{fn}^{\bigstar} = \pm 1 \ \ \textit{or} \ \ \sigma_{fn}^{\bigstar} = \pm \frac{1}{\ell_n}$$

where the sum is over edge DoFs for edges of face f ($\mathcal{S}^f$) and the signs are chosen such that current flows out of triangular face f and into adjacent faces.

- We note that only $\mathrm{F}-1$ of the star bases are independent since



$$\sum_{n=1}^{\mathrm{F}} \bigstar_f = 0.$$

- The divergence of star bases may be simply defined by

$$\nabla \cdot \bigstar_f = \sum_n \sigma_{fn}^{\bigstar} \nabla \cdot \mathbf{\Lambda}_n$$

- The star and loop bases form a quasi-Helmholtz decomposition of $\mathbf{J}$:

$$\boxed{\mathbf{J}(\mathbf{r}') \approx \overbrace{\sum_{v=1}^{V-B} I_v^{\mathrm{O}} \mathrm{O}_v}^{\text{divergenceless}} + j\omega \overbrace{\sum_{n=1}^{F-1} P_n^{\bigstar} \bigstar_n}^{\text{non-divergenceless}}}$$

Loop- and Star-Tested EFIE

- Testing EFIE with a loop basis O_v :

$$\cancel{j\omega\mu} < \mathrm{O}_v; G(\mathbf{r},\mathbf{r}'), \mathbf{J}(\mathbf{r}') > + \frac{1}{j\omega\varepsilon} < \cancel{\nabla\cdot\mathrm{O}_v}, G(\mathbf{r},\mathbf{r}'), \nabla\cdot\mathbf{J} >$$

$$= < \mathrm{O}_v; \mathbf{E}^{\mathrm{inc}} > = < \Lambda_v \hat{\mathbf{n}}; \nabla\times\mathbf{E}^{\mathrm{inc}} > = - \cancel{j\omega\mu} < \Lambda_v \hat{\mathbf{n}}; \mathbf{H}^{\mathrm{inc}} >$$

$$\Rightarrow \boxed{< \mathrm{O}_v; G(\mathbf{r},\mathbf{r}'), \mathbf{J}(\mathbf{r}') > = - < \Lambda_v \hat{\mathbf{n}}; \mathbf{H}^{\mathrm{inc}} >, \; v = 1, 2, \ldots, V - B,}$$

(weak form of magnetostatic integral eq., $\hat{\mathbf{n}}\cdot\mathbf{H}^{\mathrm{sc}}[\mathbf{J}^{\mathrm{O}}] = \hat{\mathbf{n}}\cdot\frac{1}{\mu}\nabla\times\mathbf{A}[\mathbf{J}^{\mathrm{O}}] = -\hat{\mathbf{n}}\cdot\mathbf{H}^{\mathrm{inc}}$)

- Now expand the surface current in terms of loops and star (or tree) bases:

$$\mathbf{J}(\mathbf{r}') \approx \sum_{v=1}^{V-B} I_v^{\mathrm{O}} \mathrm{O}_v + j\omega \sum_{n=1}^{F-1} P_n^{\star} \star_n \qquad \left(\mathbf{J}(\mathbf{r}') \approx \sum_{v=1}^{V-B} I_v^{\mathrm{O}} \mathrm{O}_v + j\omega \sum_{n=1}^{F-1} P_n^{\mathrm{tree}} \boldsymbol{\Lambda}_n^{\mathrm{tree}} \right)$$

Substitute into the EFIE and above eq. and test with star (or tree) bases, yielding

$$\boxed{\begin{bmatrix} [<\mathrm{O}_u; G, \mathrm{O}_v>] & j\omega[<\mathrm{O}_u; G, \star_n>] \\ j\omega\mu[<\star_m; G, \mathrm{O}_v>] & \left[\frac{1}{\varepsilon}<\nabla\cdot\star_m, G \nabla\cdot\star_n> - \omega^2\mu <\star_m; G, \star_n>\right] \end{bmatrix} \begin{bmatrix} [I_v^{\mathrm{O}}] \\ [P_n^{\star}] \end{bmatrix} = \begin{bmatrix} [-<\Lambda_v \hat{\mathbf{n}}; \mathbf{H}^{\mathrm{inc}}>] \\ [<\star_m; \mathbf{E}^{\mathrm{inc}}>] \end{bmatrix}}$$

$$\xrightarrow{\omega\to 0} \begin{bmatrix} \left[<\mathrm{O}_u; \frac{1}{4\pi R}, \mathrm{O}_v>\right] & 0 \\ 0 & \left[\frac{1}{\varepsilon}<\nabla\cdot\star_m, \frac{1}{4\pi R}, \nabla\cdot\star_n>\right] \end{bmatrix} \begin{bmatrix} [I_v^{\mathrm{O}}] \\ [P_n^{\star}] \end{bmatrix} = \begin{bmatrix} [-<\Lambda_v \hat{\mathbf{n}}; \mathbf{H}^{\mathrm{inc}}>] \\ [<\star_m; \mathbf{E}^{\mathrm{inc}}>] \end{bmatrix} \; (\text{or } \star \to \boldsymbol{\Lambda}^{\mathrm{tree}})$$

Summary of EFIE Low Frequency Treatment

- Split surface current $\mathbf{J}$ into a divergenceless and non-divergenceless part using loop and star (or tree) bases, respectively.

- Equate the EFIE's surface curl and quasi-divergence parts by testing with loop and star (or tree) bases, respectively.

- The separated parts require frequency scaling the electrostatic limit exists.

- The electrostatic limit approximates the integral equation $-\nabla\Phi[q]=\mathbf{E}^{\mathrm{inc}}$ with constraint $\int_{\mathcal{S}} q\,d\mathcal{S}=0$, by the matrix equation $\left[<\nabla\cdot\bigstar_m,\dfrac{1}{4\pi\varepsilon R},\nabla\cdot\bigstar_n>\right]\left[P_n^{\bigstar}\right]=<\bigstar_m;\mathbf{E}^{\mathrm{inc}}>$, where $\Phi[q]$ is the electrostatic scalar potential in terms of surface charge q, expanded as a superposition of charge dipoles to satisfy the constraint. Testing the equation with stars ensures no closed paths of the (conservative) electrostatic field are formed.

- The magneostatic limit approximates the integral equation $-\left(1/\mu\right)\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A}[\mathbf{J}]=\hat{\mathbf{n}}\cdot\mathbf{H}^{\mathrm{inc}}$ with constraint $\nabla\cdot\mathbf{J}=0$, by the matrix equation $\left[-<\mathrm{O}_u;\dfrac{1}{4\pi R},\mathrm{O}_v>\right]\left[I_v^{\mathrm{O}}\right]=\left[<\Lambda_v\,\hat{\mathbf{n}};\mathbf{H}^{\mathrm{inc}}>\right]$, where $\mathbf{A}[\mathbf{J}]$ is the magnetostatic vector potential as a function of the associated surface current $\mathbf{J}$ expanded in divergence-less loop bases to satisfy the constraint.

Helmholtz Decomposition of Electric Field

- **Vector wave equation**:

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{E} - \omega^2 \mu_0 \varepsilon_0 \varepsilon_r \mathbf{E} = -j\omega\mu_0 \mathbf{J}, \qquad \mathbf{r} \in \mathcal{D} = \mathcal{S} \text{ (2-D) or } \mathcal{V} \text{ (3-D)}$$

- **E-field splitting**:

$$\mathbf{E} = \overbrace{\mathbf{E}^{\circ}}^{\text{non-curl free}} + \overbrace{\mathbf{E}^{\star}}^{\text{curl-free}}$$

- **Low frequency behavior :**

$$\mathbf{J} \xrightarrow{\omega \to 0} \mathbf{J}^{\circ} \qquad \Rightarrow \boxed{\mathbf{J}^{\circ} = \mathcal{O}(1) \quad \text{(real)}}$$

$$q = \frac{\nabla \cdot \mathbf{J}^{\star}}{-j\omega} = \mathcal{O}(1) \quad \text{(real)} \quad \Rightarrow \boxed{\mathbf{J}^{\star} = \mathcal{O}(\omega) \quad \text{(imaginary)}}$$

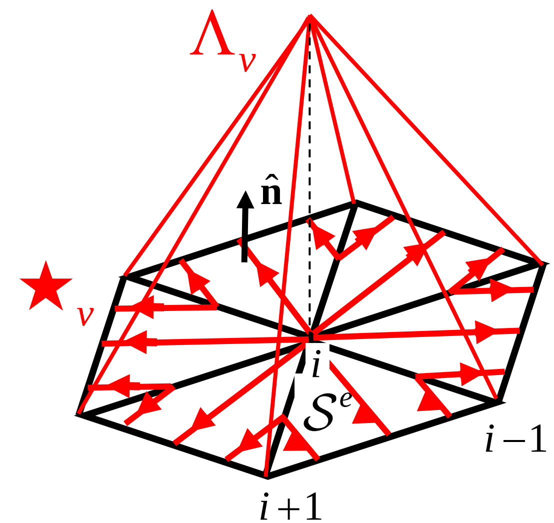
Dual Star Basis Representation

- We can write a *dual* star basis $\bigstar_v$ about an interior vertex v containing $\mathcal{D}^e$ in its support, $\text{supp}\,\bigstar_v$, and with the i-th local vertex of $\mathcal{D}^e$ corresponding to vertex v, as

$$\boxed{\bigstar_v = O_v \times \hat{\mathbf{n}} = \frac{\mathbf{\Omega}^e_{i-1}}{\ell_{i-1}} - \frac{\mathbf{\Omega}^e_{i+1}}{\ell_{i+1}} = \frac{\hat{\mathbf{h}}_i}{h_i}, \qquad \mathbf{r} \in \mathcal{D}^e \subset \text{supp}\,\bigstar_v}$$

  but perhaps most useful is

$$\boxed{\bigstar_v = -\nabla\xi, \qquad \mathbf{r} \in \mathcal{D}^e \subset \text{supp}\,\bigstar_v}$$



- Note $\nabla \times \bigstar_v = -\nabla \times \nabla\xi = 0$, i.e. $\bigstar_v$ is curl-free.

- For an arbitrary, continuous vector $\mathbf{A}$ on $\mathcal{D}^e$, we have

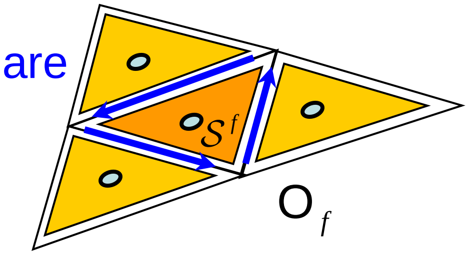
$$-\int_{\mathcal{D}^e} \nabla\xi \cdot \mathbf{A}\, d\mathcal{D} = \int_{\mathcal{D}^e} \left(\xi_i \nabla \cdot \mathbf{A}\right) d\mathcal{D} - \oint_{\partial\mathcal{D}^e} \xi_i \mathbf{A} \cdot \hat{\mathbf{n}}\, d\mathcal{B}$$

  where the contour integral vanishes when contributions from all adjacent elements with a common vertex are added, so that

$$\boxed{<\bigstar_v; \mathbf{A}> \; = \int_{\mathcal{D}} \bigstar_v \cdot \mathbf{A}\, d\mathcal{D} = \int_{\mathcal{D}} \left(\Lambda_v \nabla \cdot \mathbf{A}\right) d\mathcal{D} \; = \; <\Lambda_v, \nabla \cdot \mathbf{A}>},$$

  where $\Lambda_v = \xi_i, \mathbf{r} \in \mathcal{D}^e \subset \text{supp}\,\bigstar_v$ is the scalar rooftop function with peak at node v.
  Hence, *testing a continuous vector with a star function is equivalent to averaging its rooftop-weighted divergence over the star's support.*

Dual Loop Basis Representations



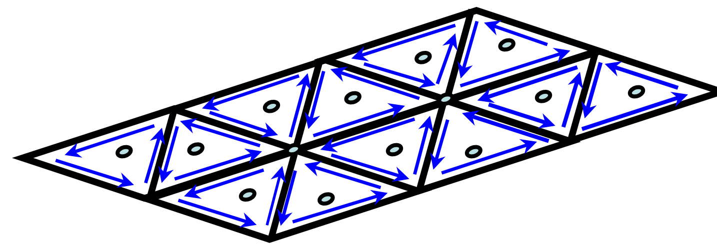
- Dual *loop bases* are not uniquely defined; two possible definitions are

$$O_f = \sum_n \sigma^{\bigstar}_{fn} \boldsymbol{\Omega}_n, \quad \sigma^{\bigstar}_{fn} = \pm 1 \quad \textit{or} \quad \sigma^{\bigstar}_{fn} = \pm \frac{1}{\ell_n}$$

where the sum is over edge DoFs for edges of face f ($\mathcal{S}^f$) and the signs are chosen so flux is parallel to edges of f and defined on adjacent elements.

- We note that only $\mathrm{F}-1$ of the star bases are independent since

$$\sum_{n=1}^{\mathrm{F}} O_n = 0.$$



- The curl of loop bases may be simply defined by

$$\nabla \times O_f = \sum_n \sigma^{\bigstar}_{fn} \nabla \times \boldsymbol{\Omega}_n$$

- The star and loop bases form a quasi-Helmholtz decomposition of $\mathbf{E}$:

$$\boxed{\mathbf{E}(\mathbf{r}') \approx \overbrace{\sum_{n=1}^{F-1} P_n^{O} O_n}^{\text{non-curl free}} + \overbrace{\frac{1}{j\omega} \sum_{v=1}^{V-B} V_v^{\bigstar} \bigstar_v}^{\text{curl-free}}}$$

Loop- and Star-Tested Helmholtz Eq.

- Testing Helholtz Eq. with a star basis $\bigstar_v$:

$$<\bigstar_v, \nabla\times\mu_r^{-1}\nabla\times\mathbf{E}> - \omega^2\mu_0\varepsilon_0\varepsilon_r <\bigstar_v, \mathbf{E}> = -j\omega\mu_0 <\bigstar_v, \mathbf{J}>, \qquad \mathbf{r}\in\mathcal{D}=\mathcal{S}\text{ (2-D) or }\mathcal{V}\text{ (3-D)}$$

$$\Rightarrow \cancel{<\Lambda_v, \nabla\cdot\nabla\times\mu_r^{-1}\nabla\times\mathbf{E}>} \cancel{-\omega^2}\mu_0\,\varepsilon_0\varepsilon_r <\Lambda_v, \nabla\cdot(\mathbf{E})> = -j\omega\mu_0 <\Lambda_v, \nabla\cdot\mathbf{J}> = \cancel{-\omega^2}\mu_0 <\Lambda_v, \frac{\nabla\cdot\mathbf{J}}{-j\omega}>$$

$$\Rightarrow \boxed{\varepsilon_0\varepsilon_r <\Lambda_v, \nabla\cdot\mathbf{E}> \; = \; <\Lambda_v, q>}$$

(weak form of electrostatic Poisson's eq.)

- Now expand the field in terms of loop and star bases:

$$\mathbf{E}(\mathbf{r}') \approx \overbrace{\sum_{n=1}^{F-1} V_n^{\bigstar}\bigstar_n}^{\text{curl-free}} + \overbrace{\frac{1}{j\omega}\sum_{v=1}^{V-B} P_v^{\text{O}}\text{O}_v}^{\text{non-curl free}}$$

$$\mathbf{J}(\mathbf{r}') \approx \sum_{v=1}^{V-B} I_v^{\text{O}}\text{O}_v + j\omega\sum_{n=1}^{F-1} P_n^{\text{star}}\bigstar_n$$

Substitute into the Helmholtz and Poisson's eqs. and test with loop bases, yielding

$$\boxed{\begin{bmatrix} [<\text{O}_u;G,\text{O}_v>] & j\omega[<\text{O}_u;G,\bigstar_n>] \\ j\omega\mu[<\bigstar_m;G,\text{O}_v>] & \left[\frac{1}{\varepsilon}<\nabla\cdot\bigstar_m, G\nabla\cdot\bigstar_n> - \omega^2\mu<\bigstar_m;G,\bigstar_n>\right] \end{bmatrix} \begin{bmatrix} [I_v^{\text{O}}] \\ [P_n^{\bigstar}] \end{bmatrix} = \begin{bmatrix} [-<\Lambda_v\,\hat{\mathbf{n}};\mathbf{H}^{\text{inc}}>] \\ [<\bigstar_m;\mathbf{E}^{\text{inc}}>] \end{bmatrix}}$$

$$\xrightarrow{\omega\to 0} \begin{bmatrix} \left[<\text{O}_u;\frac{1}{4\pi R},\text{O}_v>\right] & 0 \\ 0 & \left[\frac{1}{\varepsilon}<\nabla\cdot\bigstar_m,\frac{1}{4\pi R},\nabla\cdot\bigstar_n>\right] \end{bmatrix} \begin{bmatrix} [I_v^{\text{O}}] \\ [P_n^{\text{star}}] \end{bmatrix} = \begin{bmatrix} [-<\Lambda_v\,\hat{\mathbf{n}};\mathbf{H}^{\text{inc}}>] \\ [<\bigstar_m;\mathbf{E}^{\text{inc}}>] \end{bmatrix} \text{ (or } \bigstar\to\mathbf{\Lambda}^{\text{tree}}\text{)}$$

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