Finite Element Solution of Helmholtz Equation for Inhomogeneously Filled Cylindrical Waveguide

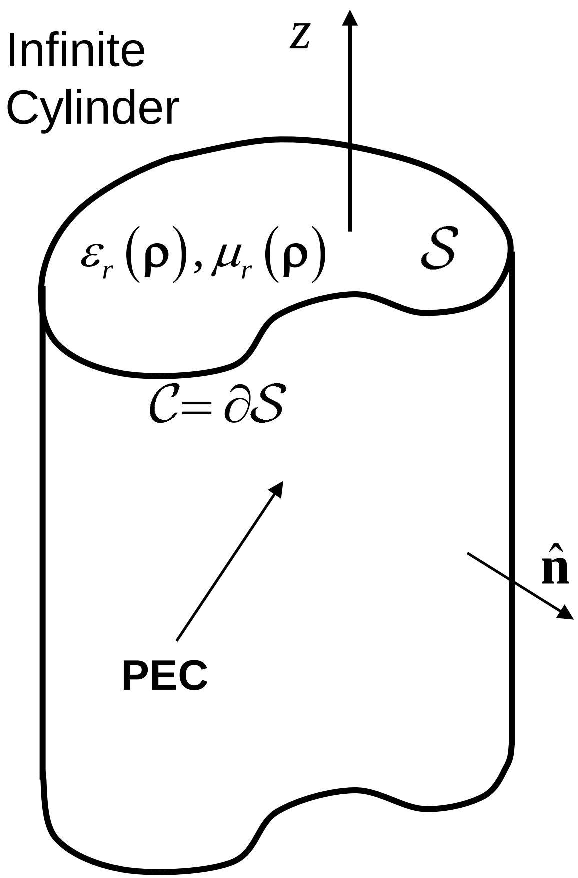
D. R. Wilton and T. Kennedy

University of Houston

2.5D --- Main Extensions from the 2D, TM Case

- **Both axial (**E_z**) and transverse (**$\mathbf{E}_t$ **) component exist and they are coupled**
- **We can model** E_z **as before, but now with** $\exp(-jk_z z)$ **phase variation**
- **Also must model** $\mathbf{E}_t$**, which also has** $\exp(-jk_z z)$ **phase variation**

Helmholtz Equation for Inhomogeneously Filled Cylindrical Waveguide



Obtain the Helmholtz wave equation by eliminating the magnetic field between Maxwell's curl equations :

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mu_r(\boldsymbol{\rho})\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon_0\varepsilon_r(\boldsymbol{\rho})\mathbf{E} + \mathbf{J}$$

$$\Rightarrow \nabla \times \left(\mu_r^{-1}\nabla \times \mathbf{E}\right) - k_0^2\varepsilon_r\mathbf{E} = -j\omega\mu_0\mathbf{J}$$

Strong form, (unreduced) wave eq.

Assume fields and current vary as $e^{-jk_z z}$:

$$\mathbf{J}(\mathbf{r}) = \tilde{\mathbf{J}}(\boldsymbol{\rho})\,[e^{-jk_z z}]$$

$$\mathbf{E}(\mathbf{r}) = \tilde{\mathbf{E}}(\boldsymbol{\rho})\,[e^{-jk_z z}]$$

$$\mathbf{H}(\mathbf{r}) = \tilde{\mathbf{H}}(\boldsymbol{\rho})\,[e^{-jk_z z}]$$

e.g. $\dfrac{\partial}{\partial z}\mathbf{J}(\mathbf{r}) = -jk_z\,\tilde{\mathbf{J}}(\boldsymbol{\rho})\,[e^{-jk_z z}]$

$\Rightarrow \quad \dfrac{\partial}{\partial z} \Leftrightarrow -jk_z$

Reduced Helmholtz Equation

no z dependence

Strong **form of "reduced" Helmholtz equation:**

$$\tilde{\nabla}\times\left(\mu_r^{-1}\tilde{\nabla}\times\tilde{\mathbf{E}}\right)-k_0^2\varepsilon_r\tilde{\mathbf{E}}=-j\omega\mu_0\tilde{\mathbf{J}},\quad \boldsymbol{\rho}\in\mathcal{S}$$

or

$$\boxed{\tilde{\nabla}\times\left(\left(j\omega\mu\right)^{-1}\tilde{\nabla}\times\tilde{\mathbf{E}}\right)+j\omega\varepsilon\tilde{\mathbf{E}}=-\tilde{\mathbf{J}},\ \boldsymbol{\rho}\in\mathcal{S}}$$

$$\nabla_t\equiv\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}$$

$$\tilde{\nabla}\equiv\nabla_t-jk_z\hat{\mathbf{z}}$$

$$\tilde{\nabla}^*\equiv\nabla_t+jk_z\hat{\mathbf{z}}$$

for *both* real and complex k_z

- **Test *un*reduced equation with $\boldsymbol{\Omega}_m(\boldsymbol{\rho})\left[e^{+jk_zz}\right]$, obtaining**

$$<\boldsymbol{\Omega}_m;\nabla\times\left(\mu_r^{-1}\nabla\times\mathbf{E}\right)>-k_0^2<\boldsymbol{\Omega}_m;\varepsilon_r\mathbf{E}>=-j\omega\mu_0<\boldsymbol{\Omega}_m;\mathbf{J}>,$$

$\boldsymbol{\rho}\in\mathcal{S}, z\in(-\infty,\infty)$, **and where** $<\mathbf{A};\mathbf{B}>\equiv\int_{\mathcal{S}}\mathbf{A}\cdot\mathbf{B}\,d\mathcal{S}.$

- **Reduce differentiability requirement on $\mathbf{E}$ using**

$\nabla\cdot\left(\mathbf{A}\times\mathbf{B}\right)=\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$ **and divergence theorem, yielding**

the *weak* form of the Helmholtz equation:

$$\boxed{\begin{aligned}\frac{1}{j\omega\mu_0}<\tilde{\nabla}^*\times\boldsymbol{\Omega}_m;\mu_r^{-1}\tilde{\nabla}\times\tilde{\mathbf{E}}>+j\omega\varepsilon_0<\boldsymbol{\Omega}_m;\varepsilon_r\tilde{\mathbf{E}}>\\=-<\boldsymbol{\Omega}_m;\tilde{\mathbf{J}}>-\oint_{\mathcal{C}}\boldsymbol{\Omega}_m\cdot(\tilde{\mathbf{H}}\times\hat{\mathbf{n}})\,d\mathcal{C}\end{aligned}}$$

$$\frac{1}{j\omega\mu_0}\int_{\mathcal{S}}\nabla\cdot\left[\boldsymbol{\Omega}_m\times\left(\mu_r^{-1}\nabla\times\mathbf{E}\right)\right]d\mathcal{S}$$

$$=\frac{1}{j\omega\mu_0}\int_{\mathcal{C}}\left[\boldsymbol{\Omega}_m\times\left(\mu_r^{-1}\nabla\times\mathbf{E}\right)\right]\cdot\hat{\mathbf{n}}\,d\mathcal{C}$$

$$=-\oint_{\mathcal{C}}\boldsymbol{\Omega}_m\cdot(\mathbf{H}\times\hat{\mathbf{n}})\,d\mathcal{C}$$

System Matrix

The boundary integral vanishes if $\boldsymbol{\Omega}_m(\boldsymbol{\rho})\left[e^{-jk_z z}\right]$ are also interpolatory basis functions for $\tilde{\mathbf{E}}$, (or tangential $\tilde{\mathrm{E}}$)

$$\Rightarrow \boxed{\tilde{\mathbf{E}} = \sum_{n=1}^{N} V_n \boldsymbol{\Omega}_n(\boldsymbol{\rho}),}$$

since $\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = \sum_{n=1}^{N} V_n \left[\hat{\mathbf{n}} \times \boldsymbol{\Omega}_n(\boldsymbol{\rho})\right] = \mathbf{0}$ on the boundary $\Rightarrow \hat{\mathbf{n}} \times \boldsymbol{\Omega}_n = \mathbf{0}$ on $\mathcal{C}$.

Substituting $\tilde{\mathrm{E}}$ into the weak form yields $\boxed{\left[Y_{mn}\right]\left[V_n\right] = \left[I_m\right]}$ where

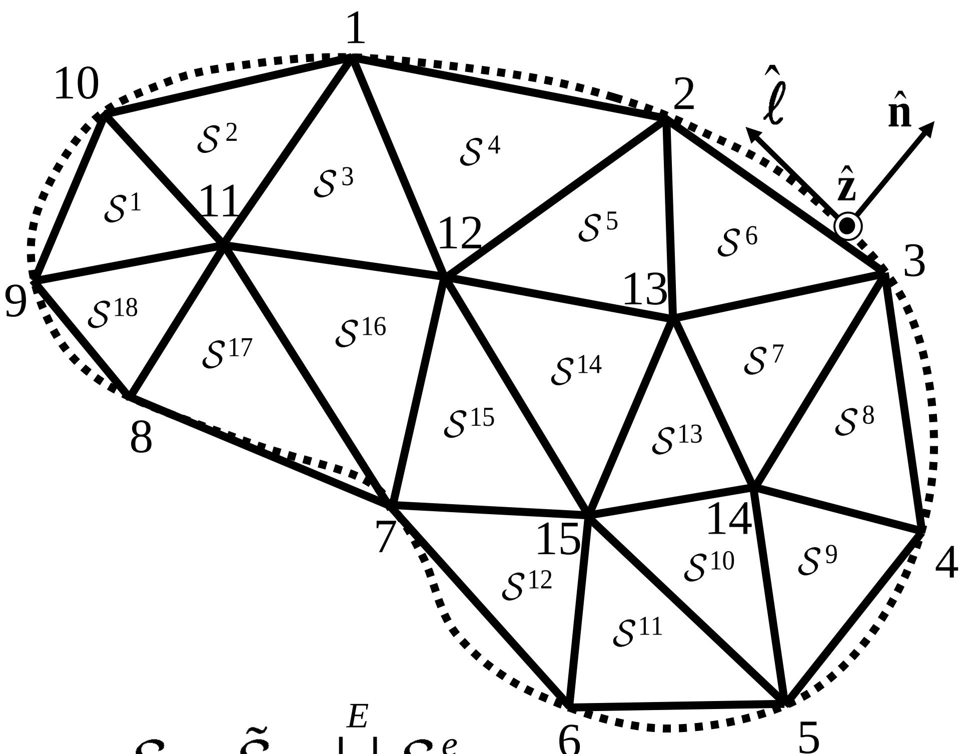
$$\left[Y_{mn}\right] = \frac{1}{j\omega}\left[\Gamma_{mn}\right] + j\omega\left[C_{mn}\right],$$ **(admittance or system matrix)**

$$\left[\Gamma_{mn}\right] = \frac{1}{\mu_0}\left[< \tilde{\nabla}^* \times \boldsymbol{\Omega}_m ; \mu_r^{-1} \tilde{\nabla} \times \boldsymbol{\Omega}_n >\right],$$ **(reciprocal inductance matrix)**

$$\left[C_{mn}\right] = \varepsilon_0\left[< \boldsymbol{\Omega}_m ; \varepsilon_r \boldsymbol{\Omega}_n >\right],$$ **(capacitance matrix)**

$$\left[I_m\right] = \left[- < \boldsymbol{\Omega}_m ; \tilde{\mathbf{J}} >\right]$$ **(excitation vector)**

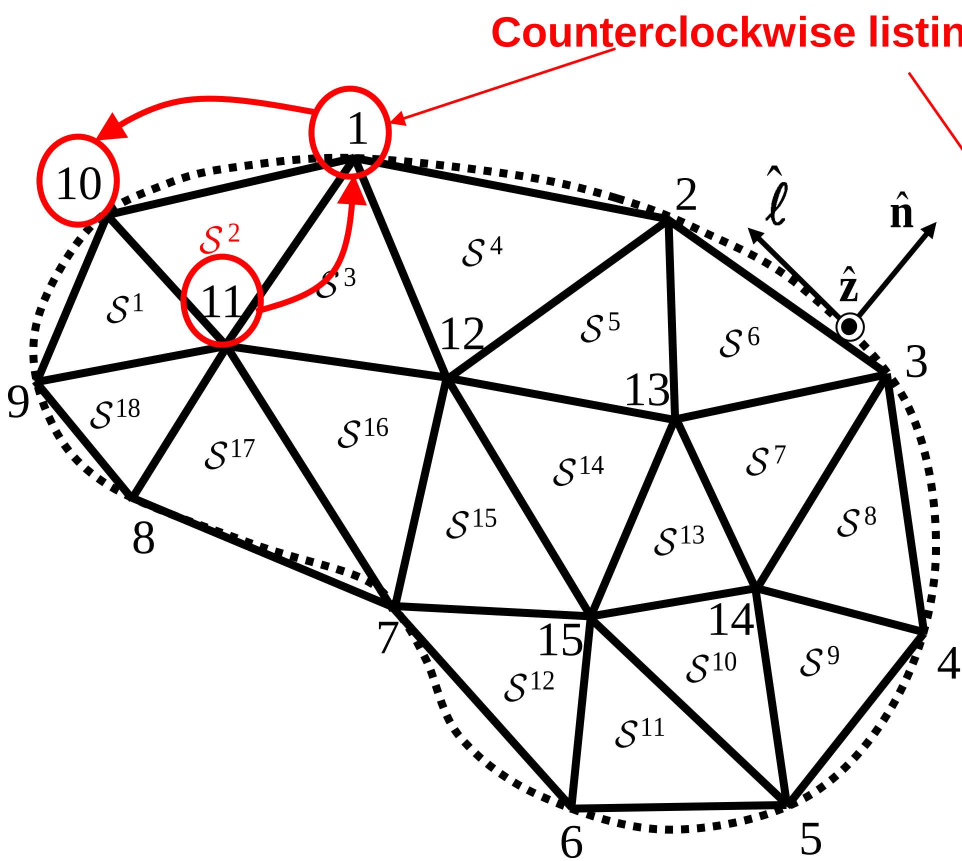
Discretize the Guide Cross Section --- Nodal Data



$$\mathcal{S} \approx \tilde{\mathcal{S}} = \bigcup_{e=1}^{E} \mathcal{S}^e$$

Global Node Index v	Coordinates	
	x_v	y_v
1	-0.500	1.100
2	1.100	0.700
$\vdots$	$\vdots$	$\vdots$
12	0.000	0.000
$\vdots$	$\vdots$	$\vdots$
15	0.700	-1.100

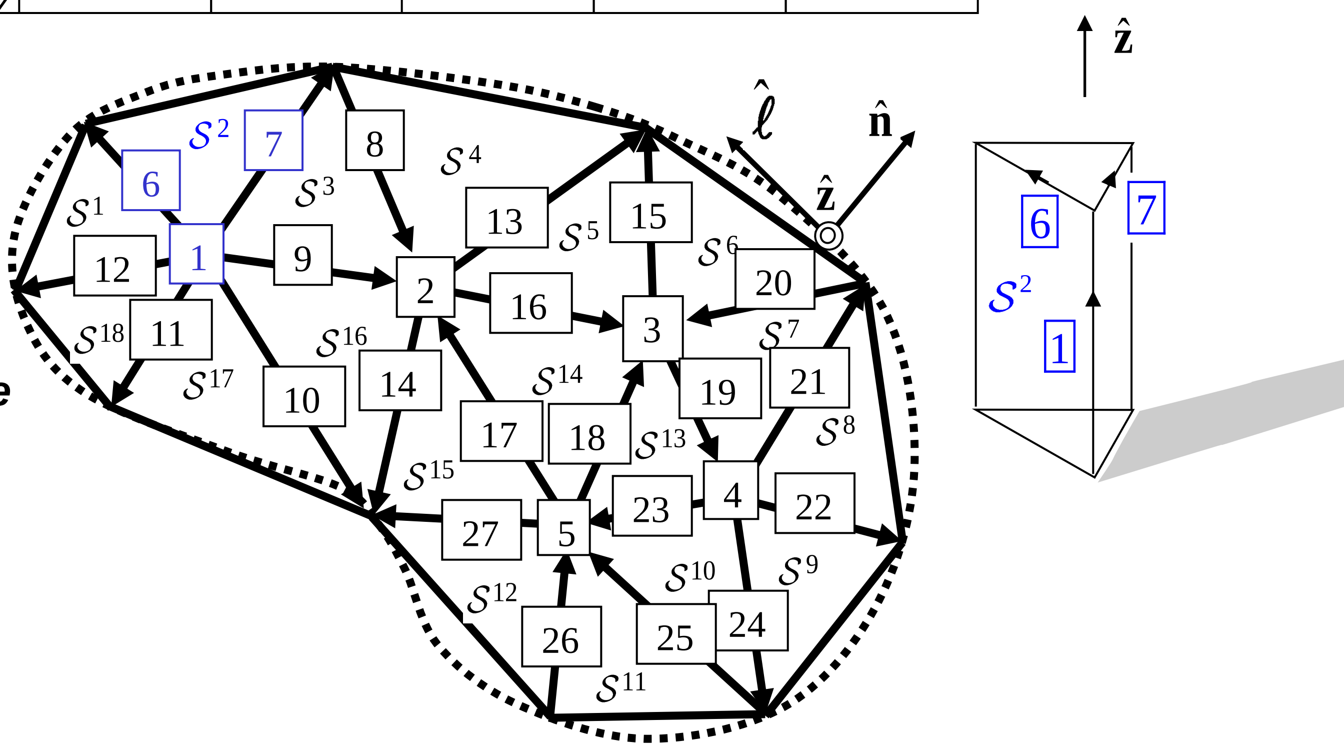
Element Connection Data



Local Node	1	2	3
e	Global Node No.	Global Node No.	Global Node No.
1	9	11	10
2	11	1	10
⋮	⋮	⋮	⋮
14	15	13	12
⋮	⋮	⋮	⋮
18	8	11	9

Element DoF Data

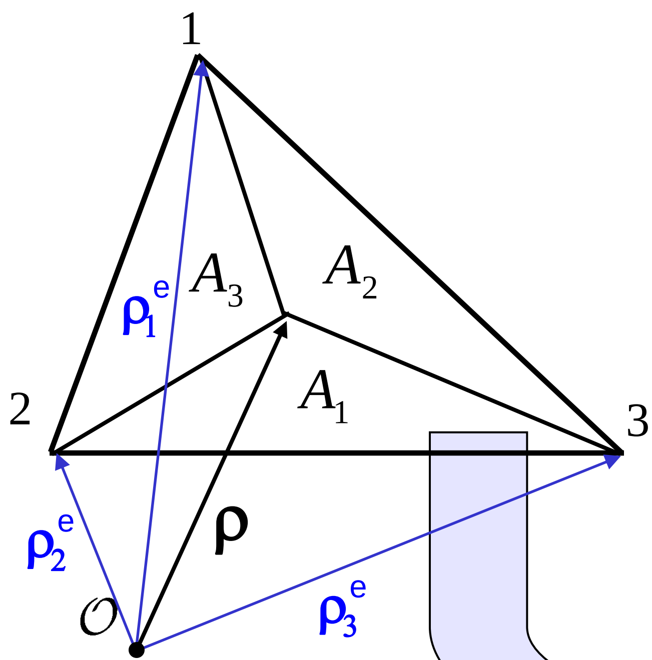
Local DoF #	← Transverse DoFs → 1	2	3	← Axial DoFs → 4	5	6
e	Global DoF#	Global DoF#	Global DoF#	Global DoF#	Global DoF#	Global DoF#
1	6	0	-12	0	1	0
2	0	-6	7	1	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮



Ref. direction is *opposite* counterclockwise direction

$$\Rightarrow \sigma_2^e = -1$$

Area Coordinates Used to Represent Bases, Parameterize Element Geometry

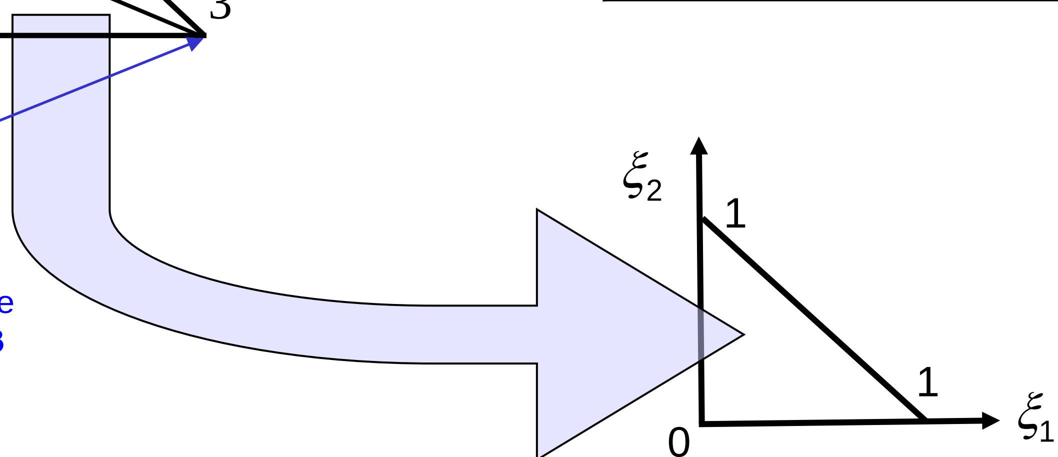


$$\xi_i = \frac{A_i}{A^e}, \quad i = 1, 2, 3$$

$$\Rightarrow \quad \xi_1 + \xi_2 + \xi_3 = 1$$

$$\Rightarrow \boxed{\Lambda_i^e = \xi_i, \quad i = 1, 2, 3}$$

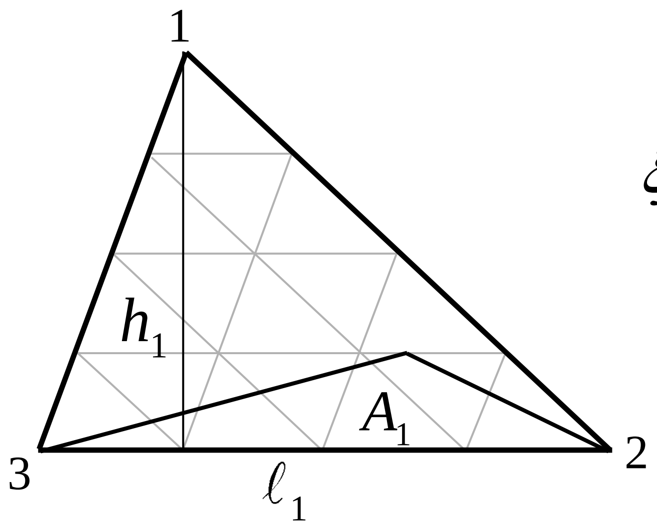
$$\boldsymbol{\rho} = \xi_1 \boldsymbol{\rho}_1^e + \xi_2 \boldsymbol{\rho}_2^e + \xi_3 \boldsymbol{\rho}_3^e$$



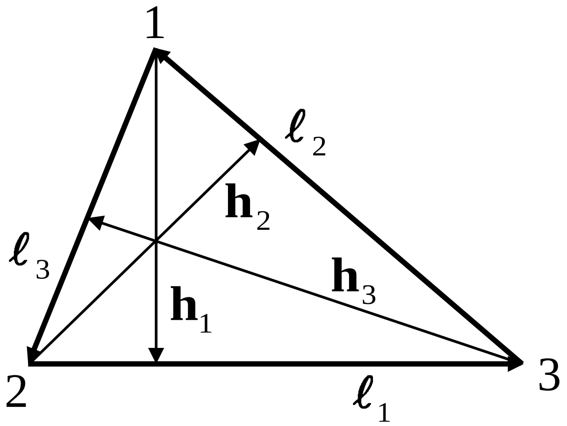
$$\Rightarrow \boxed{\Lambda_i^e = \xi_i = \frac{\hat{\mathbf{z}} \cdot \boldsymbol{\ell}_i \times (\boldsymbol{\rho} - \boldsymbol{\rho}_{i+1}^e)}{\hat{\mathbf{z}} \cdot \boldsymbol{\ell}_{i+1} \times \boldsymbol{\ell}_{i-1}}, \quad i = 1, 2, 3}$$

All elements mapped to "parent element"

An Area Coordinate Is Also the Fractional Distance from an Edge to the Opposite Vertex

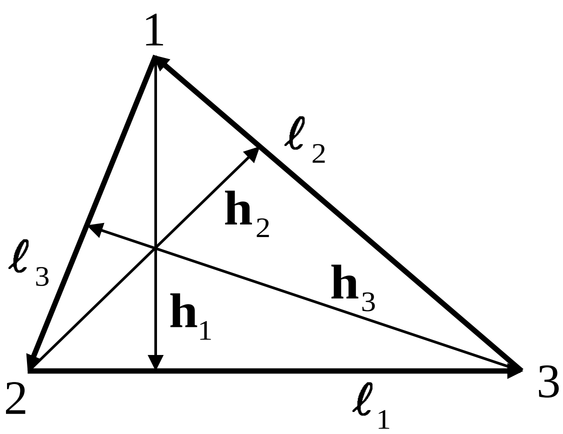


$$\xi_1 = \frac{\frac{1}{2}\ell_1 \times (\text{height of } A_1)}{\frac{1}{2}\ell_1 h_1} = \frac{\text{height of } A_1}{h_1}$$



It is convenient to define edge vectors associated with each edge and height vectors associated with each vertex.

Recall Local Geometry Definitions

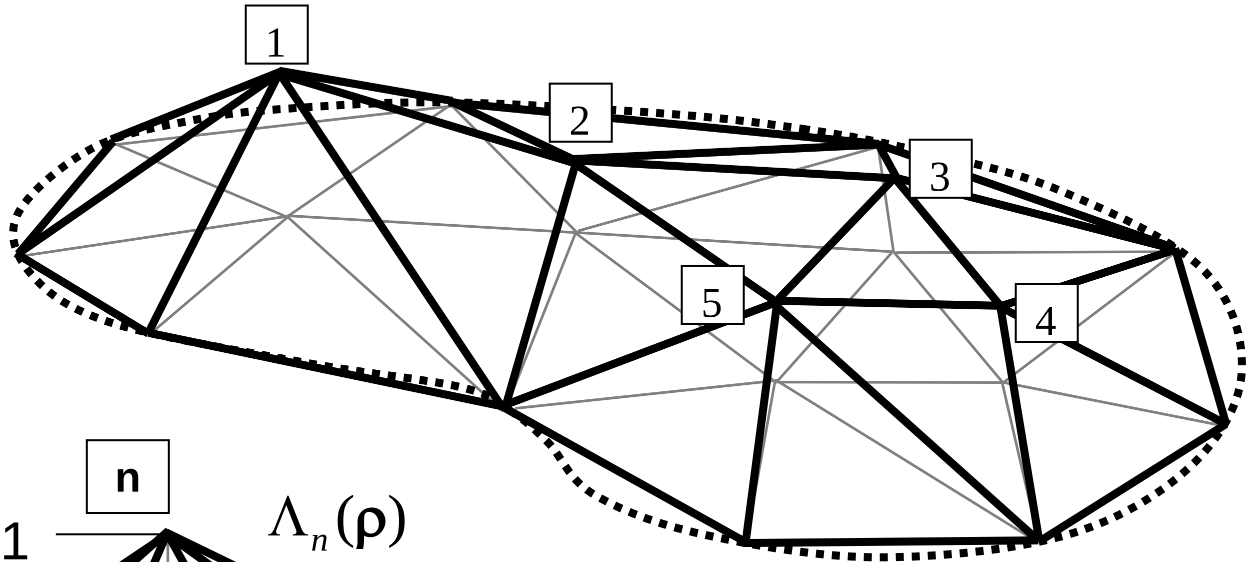


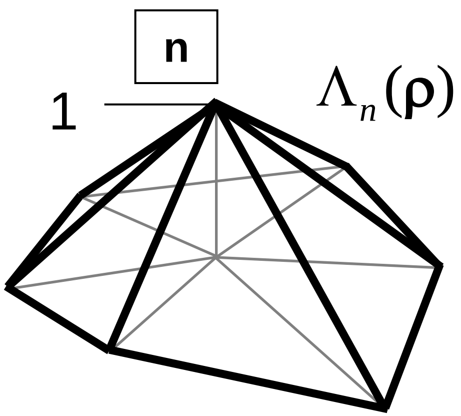
$$\hat{\mathbf{n}} = \frac{\boldsymbol{\ell}_{i+1} \times \boldsymbol{\ell}_{i-1}}{2A^e}$$

Table 8 Geometrical quantities defined on triangular elements.

Edge vectors	$\boldsymbol{\ell}_i = \boldsymbol{\rho}^e_{i-1} - \boldsymbol{\rho}^e_{i+1}$; $\ell_i = \lvert\boldsymbol{\ell}_i\rvert$; $\hat{\boldsymbol{\ell}}_i = \frac{\boldsymbol{\ell}_i}{\ell_i}$, $i = 1, 2, 3$
Area	$A^e = \frac{\lvert\boldsymbol{\ell}_{i-1} \times \boldsymbol{\ell}_{i+1}\rvert}{2}$, $i = 1, 2,$ or 3
Height vectors	$h_i = \frac{2A^e}{\ell_i}$; $\hat{\boldsymbol{h}}_i = -\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\ell}}_i$; $\boldsymbol{h}_i = h_i \hat{\boldsymbol{h}}_i$, $i = 1, 2, 3$
Coordinate gradients	$\boldsymbol{\nabla}\xi_i = -\frac{\hat{\boldsymbol{h}}_i}{h_i}$, $i = 1, 2, 3$

Piecewise Linear Model of Axial Electric Field, $\tilde{E}_z$





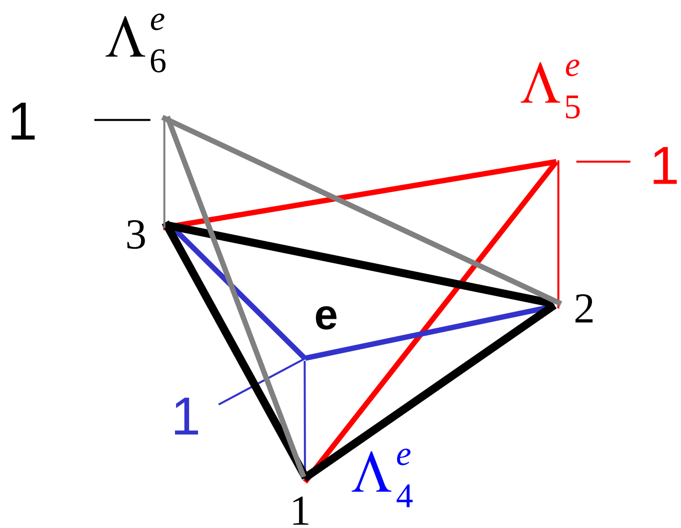
Global basis function associated with DoF *n*

$$\tilde{E}_z(\boldsymbol{\rho}) \approx \sum_{n=1}^{5} V_n \Lambda_n(\boldsymbol{\rho}), \ \ \boldsymbol{\rho} \in \tilde{\mathcal{S}}$$

or $$\hat{\mathbf{z}}\tilde{E}_z(\boldsymbol{\rho}) \approx \sum_{n=1}^{5} V_n \boldsymbol{\Omega}_n(\boldsymbol{\rho}), \ \ \boldsymbol{\Omega}_n \equiv \hat{\mathbf{z}}\Lambda_n$$

Global Representation

Local Representations of $\tilde{E}_z$



$$\Lambda_4^e = \xi_1$$

$$\Lambda_5^e = \xi_2$$

$$\Lambda_6^e = \xi_3$$

$$\tilde{E}_z(\boldsymbol{\rho}) \approx \sum_{i=4}^{6} V_i^e \Lambda_i^e(\boldsymbol{\rho}), \ \ \boldsymbol{\rho} \in \mathcal{S}^e$$

Local Scalar Representation

$$\hat{\mathbf{z}}\, \tilde{E}_z(\boldsymbol{\rho}) \approx \sum_{i=4}^{6} V_i^e \underbrace{\left[\hat{\mathbf{z}}\, \Lambda_i^e(\boldsymbol{\rho})\right]}_{\text{vector basis, } \boldsymbol{\Omega}_i^e(\boldsymbol{\rho})},$$

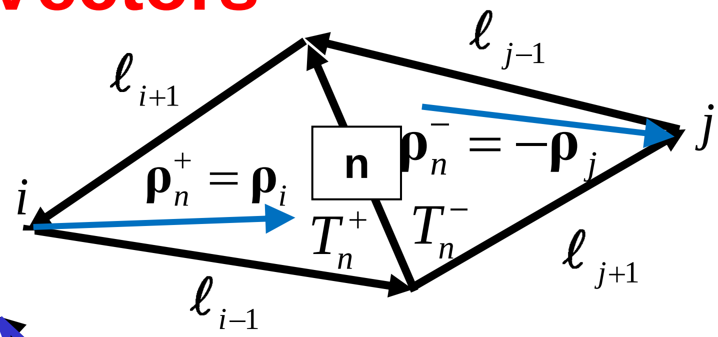
and $\tilde{\nabla} \times \left[\hat{\mathbf{z}}\, \Lambda_i^e(\boldsymbol{\rho})\right] = \nabla \xi_{i-3} \times \hat{\mathbf{z}}, \ \boldsymbol{\rho} \in \mathcal{S}^e,$

Local Vector Representation

$$\Rightarrow \boxed{\begin{array}{l} \Lambda_i^e = \xi_{i-3}, \\ \boldsymbol{\Omega}_i^e = \hat{\mathbf{z}}\, \xi_{i-3}, i = 4, 5, 6 \end{array}}$$

Local bases and triangle parameterization can be easily expressed in area coordinates

Properties of $\boldsymbol{\rho}_n^{\pm}$ Vectors



- Representation in area coords :

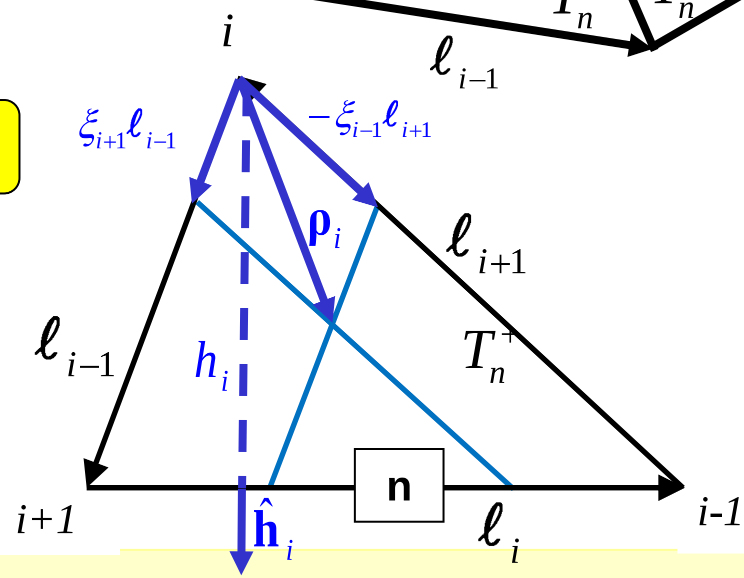
$$\boldsymbol{\rho}_n^{+} = \boldsymbol{\rho}_i = \xi_{i+1}\boldsymbol{\ell}_{i-1} - \xi_{i-1}\boldsymbol{\ell}_{i+1},$$

$$\boldsymbol{\rho}_n^{-} = -\boldsymbol{\rho}_j = \xi_{j-1}\boldsymbol{\ell}_{j+1} - \xi_{i+1}\boldsymbol{\ell}_{i-1}$$

- Unit normal component at edges :

$$\hat{\mathbf{h}}_i \cdot \frac{\boldsymbol{\rho}_n^{+}\big|_{\xi_i=0}}{h_i} = \frac{\boldsymbol{\rho}_i \cdot \hat{\mathbf{h}}_i\big|_{\xi_i=0}}{h_i} = \frac{\cancel{h_i}}{\cancel{h_i}} = 1,$$

$$-\hat{\mathbf{h}}_j \cdot \frac{\boldsymbol{\rho}_n^{-}\big|_{\xi_j=0}}{h_j} = \frac{\boldsymbol{\rho}_j \cdot \hat{\mathbf{h}}_j\big|_{\xi_j=0}}{h_j} = \frac{\cancel{h_j}}{\cancel{h_j}} = 1$$

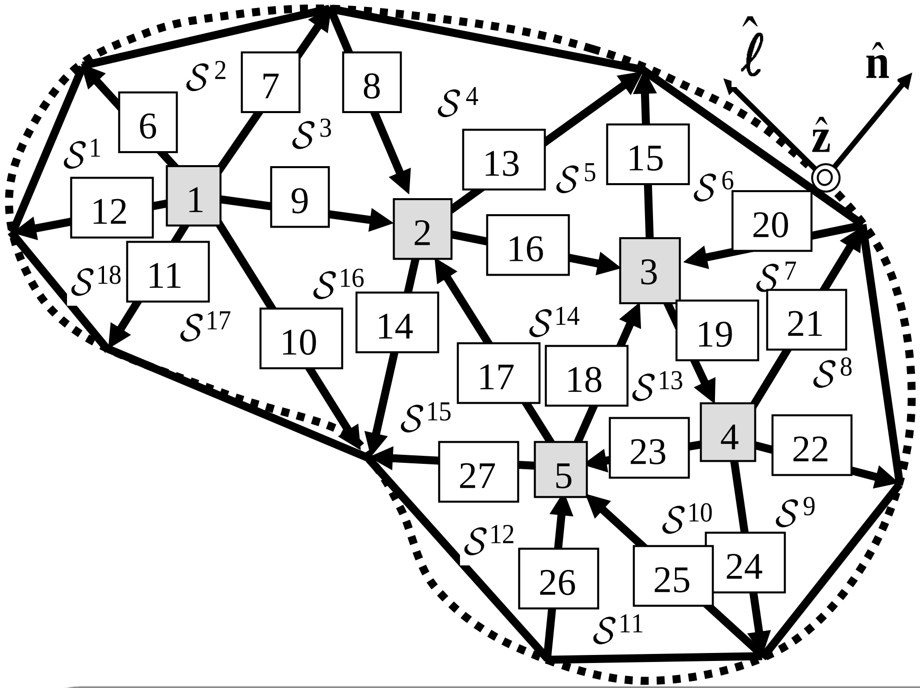


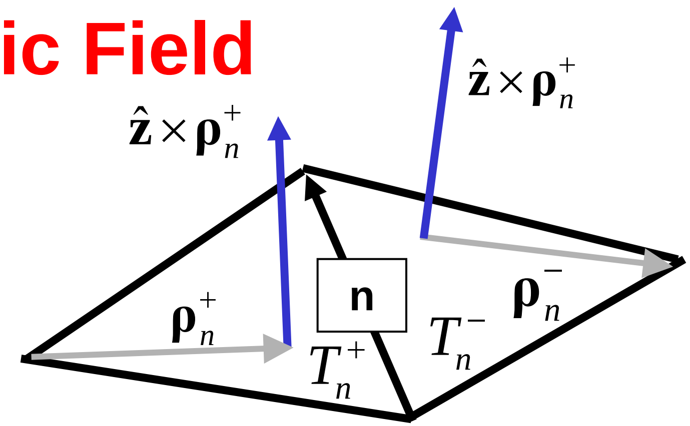
- Since $\boldsymbol{\rho}_i \cdot \hat{\mathbf{h}}_{i\pm1} = \boldsymbol{\rho}_j \cdot \hat{\mathbf{h}}_{j\pm1} = 0$, $\dfrac{\boldsymbol{\rho}_n^{\pm}}{\mathbf{h}_n^{\pm}}$ interpolates normal components!

I.e., given normal components, $\mathbf{A}\cdot\hat{\mathbf{h}}_i$, $i = 1,2,3$, of a vector $\mathbf{A}$ at triangle edges, a possible approximation for $\mathbf{A}$ is $\boxed{\mathbf{A} \approx \sum_{i=1}^{3}\left(\mathbf{A}\cdot\hat{\mathbf{h}}_i\right)\dfrac{\boldsymbol{\rho}_i}{h_i}}$. Similarly,

$$\mathbf{A} \approx \sum_{i=1}^{3}\left(\mathbf{A}\cdot\hat{\boldsymbol{\ell}}_i\right)\frac{\hat{\mathbf{n}}\times\boldsymbol{\rho}_i}{h_i}$$ interpolates *tangential* components!

Representation of Transverse Vector Electric Field





Global basis representation:

$$\boldsymbol{\Omega}_n(\boldsymbol{\rho}) = \begin{cases} \dfrac{\hat{\mathbf{z}}\times\boldsymbol{\rho}_n^\pm}{h_n^\pm}, \; \boldsymbol{\rho}\in T_n^\pm \\ \mathbf{0}, \text{ otherwise} \end{cases}$$

$$=\hat{\mathbf{z}}\times\boldsymbol{\Lambda}_n(\boldsymbol{\rho})$$

$$\tilde{\nabla}\times\boldsymbol{\Omega}_n(\boldsymbol{\rho}) = \frac{\pm 2}{h_n^\pm}\hat{\mathbf{z}} - jk_z\hat{\mathbf{z}}\times\boldsymbol{\Omega}_n, \; \boldsymbol{\rho}\in T_n^\pm$$

- DoFs defined at edge centers
- DoF is the component of the transverse electric field parallel to edge
- Component is positive if directed the same as the reference (counterclockwise) direction there, negative otherwise

$$\nabla\times\left[\boldsymbol{\Omega}_n(\boldsymbol{\rho})e^{-jk_z z}\right] = \left[\nabla\times\boldsymbol{\Omega}_n(\boldsymbol{\rho})\right]e^{-jk_z z} - jk_z\hat{\mathbf{z}}\times\boldsymbol{\Omega}_n\, e^{-jk_z z}$$

$$\nabla\times\left[\mathbf{A}\times\mathbf{B}\right] = \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

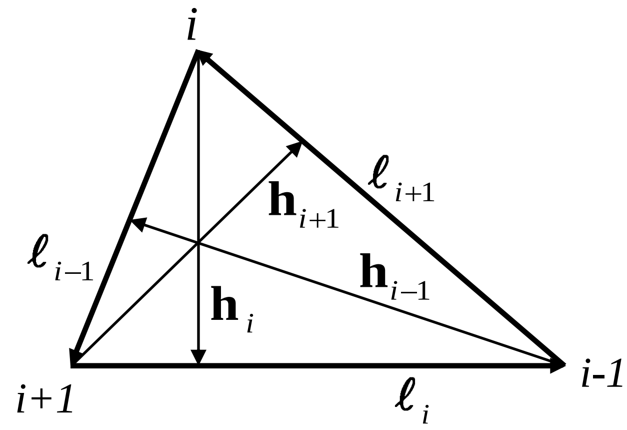
Local Representation of Transverse Bases

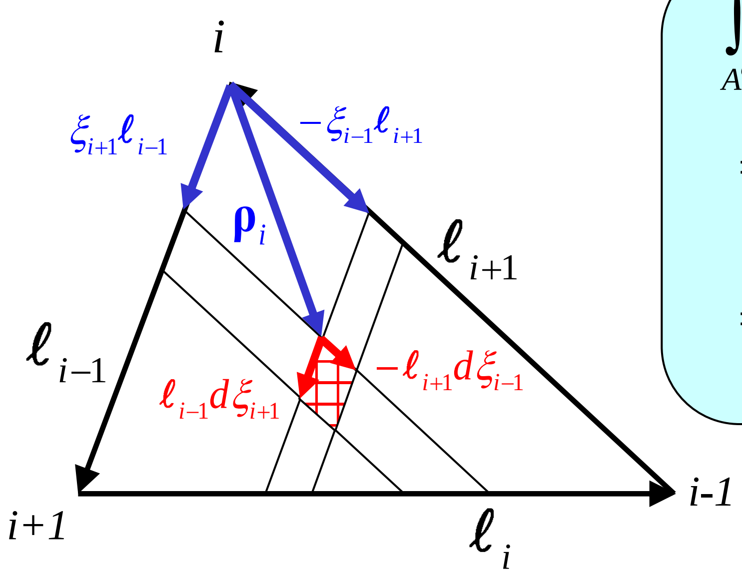
$$\boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \frac{\hat{\mathbf{z}}\times\boldsymbol{\rho}_i}{h_i} = \ell_i \frac{\hat{\mathbf{z}}\times\left(\xi_{i+1}\boldsymbol{\ell}_{i-1} - \xi_{i-1}\boldsymbol{\ell}_{i+1}\right)}{2A^e}$$

$$= \ell_i\left(\frac{-\xi_{i+1}\cancel{\ell_{i-1}}\,\hat{\mathbf{h}}_{i-1}}{\cancel{\ell_{i-1}}\,h_{i-1}} + \frac{\xi_{i-1}\cancel{\ell_{i+1}}\,\hat{\mathbf{h}}_{i+1}}{\cancel{\ell_{i+1}}\,h_{i+1}}\right)$$

$$\Rightarrow \boxed{\boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \ell_i\left(\xi_{i+1}\nabla\xi_{i-1} - \xi_{i-1}\nabla\xi_{i+1}\right)}$$

$$\boxed{\tilde{\nabla}\times\boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \frac{2}{h_i}\hat{\mathbf{z}} - jk_z\hat{\mathbf{z}}\times\boldsymbol{\Omega}_i^e}$$





$$\int_{A^e} f(\boldsymbol{\rho})\,d\mathcal{S}$$

$$= \left|\boldsymbol{\ell}_{i-1}\times\boldsymbol{\ell}_{i+1}\right| \int_0^1 \int_0^{1-\xi_{i-1}} f\left(\boldsymbol{\rho}_1^e\xi_1 + \boldsymbol{\rho}_2^e\xi_2 + \boldsymbol{\rho}_3^e\xi_3\right) d\xi_{i+1}\,d\xi_{i-1}$$

$$= \underbrace{2A^e}_{\mathcal{J}} \int_0^1 \int_0^{1-\xi_{i-1}} f\left(\boldsymbol{\rho}_1^e\xi_1 + \boldsymbol{\rho}_2^e\xi_2 + \boldsymbol{\rho}_3^e\xi_3\right) d\xi_{i+1}\,d\xi_{i-1}$$

Summary of Vectorized Bases and Field Representation

Global representation, $\tilde{\mathbf{E}} \approx \sum_{n=1}^{N} V_n \boldsymbol{\Omega}_n(\boldsymbol{\rho}),\ \boldsymbol{\rho} \in \mathcal{S}$,

$$\boldsymbol{\Omega}_n(\boldsymbol{\rho}) \equiv \begin{cases} \hat{\mathbf{z}} \times \boldsymbol{\Lambda}_n(\boldsymbol{\rho}) & \textbf{(edge-based DoFs)} \\ \hat{\mathbf{z}}\, \Lambda_n(\boldsymbol{\rho}) & \textbf{(vertex-based DoFs)} \end{cases}$$

Local representation, $\tilde{\mathbf{E}} \approx \sum_{i=1}^{6} V_i^e \boldsymbol{\Omega}_i^e(\boldsymbol{\rho}),\ \boldsymbol{\rho} \in \mathcal{S}^e$ **:**

$$\left.\begin{aligned} \boldsymbol{\Omega}_1^e(\boldsymbol{\rho}) &= \ell_1\left(\xi_2 \nabla \xi_3 - \xi_3 \nabla \xi_2\right) \\ \boldsymbol{\Omega}_2^e(\boldsymbol{\rho}) &= \ell_2\left(\xi_3 \nabla \xi_1 - \xi_1 \nabla \xi_3\right) \\ \boldsymbol{\Omega}_3^e(\boldsymbol{\rho}) &= \ell_3\left(\xi_1 \nabla \xi_2 - \xi_2 \nabla \xi_1\right) \end{aligned}\right\} \textbf{edge-based DoFs:}\quad \boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \ell_i\left(\xi_{i+1} \nabla \xi_{i-1} - \xi_{i-1} \nabla \xi_{i+1}\right)$$

$$\left.\begin{aligned} \boldsymbol{\Omega}_4^e(\boldsymbol{\rho}) &= \hat{\mathbf{z}}\xi_1 \\ \boldsymbol{\Omega}_5^e(\boldsymbol{\rho}) &= \hat{\mathbf{z}}\xi_2 \\ \boldsymbol{\Omega}_6^e(\boldsymbol{\rho}) &= \hat{\mathbf{z}}\xi_3 \end{aligned}\right\} \textbf{vertex-based DoFs:}\quad \boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \hat{\mathbf{z}}\xi_{i-3}$$

$$\tilde{\nabla} \times \boldsymbol{\Omega}_i^e(\boldsymbol{\rho}) = \frac{2}{h_i}\hat{\mathbf{z}} - jk_z \hat{\mathbf{z}} \times \boldsymbol{\Omega}_i^e$$

$$\tilde{\nabla} \times \boldsymbol{\Omega}_{i+3}^e = \nabla \xi_i \times \hat{\mathbf{z}}, \qquad i = 1, 2, 3$$

Element Matrix and Excitation Vector

Local admittance matrices and current column vectors corresponding to $\left[Y_{mn}\right]\left[V_n\right]=\frac{1}{j\omega}\left[\Gamma_{mn}\right]\left[V_n\right]+j\omega\left[C_{mn}\right]\left[V_n\right]=\left[I_m\right]$**:**

$\left[Y_{ij}^e\right]=\frac{1}{j\omega}\left[\Gamma_{ij}^e\right]+j\omega\left[C_{ij}^e\right],$ (**admittance element matrix)**

$\left[\Gamma_{ij}^e\right]=\frac{1}{\mu_0}\left[<\tilde{\nabla}^*\times\boldsymbol{\Omega}_i^e;\mu_r^{-1}\tilde{\nabla}\times\boldsymbol{\Omega}_j^e>\right],$ (**reciprocal inductance element matrix)**

$\left[C_{ij}^e\right]=\varepsilon_0\left[<\boldsymbol{\Omega}_i^e;\varepsilon_r\boldsymbol{\Omega}_j^e>\right],$ (**capacitance element matrix)**

$\left[I_i^e\right]=\left[-<\boldsymbol{\Omega}_i^e;\tilde{\mathbf{J}}>\right]$ (**excitation current element vector)**

Add $\sigma_i^e\sigma_j^e Y_{ij}^e$ **to system matrix**

using matrix assembly rule !

Integration over Triangles Using Area Coordinates

$$\int_{A^e} f(\boldsymbol{\rho})\, d\mathcal{S}$$

$$= 2A^e \int_0^1 \int_0^{1-\xi_2} f(\xi_1\boldsymbol{\rho}_1^e + \xi_2\boldsymbol{\rho}_2^e + \xi_3\boldsymbol{\rho}_3^e)\, d\xi_1\, d\xi_2$$

$$\underbrace{\approx 2A^e \sum_{k=1}^{K} w_k\, f(\xi_1^{(k)}\boldsymbol{\rho}_1^e + \xi_2^{(k)}\boldsymbol{\rho}_2^e + \xi_3^{(k)}\boldsymbol{\rho}_3^e)}_{\text{Numerical integration}}$$

Or evaluate analytically using

$$\int_0^1 \int_0^{1-\xi_2} \xi_1^{\alpha} \xi_2^{\beta} \xi_3^{\gamma}\, d\xi_1\, d\xi_2 = \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!}$$

Table 9 Sample points and weighting coefficients for K-point quadrature on triangles.

Sample Points, $\left(\xi_1^{(k)}, \xi_2^{(k)}\right)$ $(\xi_3^{(k)} = 1 - \xi_1^{(k)} - \xi_2^{(k)})$	**Weights,** w_k
K=1, error $\mathcal{O}(\xi_i^2)$:	
(0.33333333333333, 0.33333333333333)	0.50000000000000
K=3, error $\mathcal{O}(\xi_i^3)$:	
(0.66666666666667, 0.16666666666667)	0.16666666666667
(0.16666666666667, 0.66666666666667)	0.16666666666667
(0.16666666666667, 0.16666666666667)	0.16666666666667
K=7, error $\mathcal{O}(\xi_i^6)$:	
(0.33333333333333, 0.33333333333333)	0.11250000000000
(0.79742698535309, 0.10128650732346)	0.06296959027241
(0.10128650732346, 0.79742698535309)	0.06296959027241
(0.10128650732346, 0.10128650732346)	0.06296959027241
(0.47014206410512, 0.47014206410512)	0.06619707639425
(0.47014206410512, 0.05971587178977)	0.06619707639425
(0.05971587178977, 0.47014206410512)	0.06619707639425

Source-Free Problems—Waveguide Cutoff Frequencies and Dispersion Data

- $\mathbf{J}=\mathbf{0} \Rightarrow \left[I_m\right]=0 \Rightarrow \left[V_n\right]=0$ ***except*** **for eigenfrequencies:**

$$\left[\Gamma_{mn}\right]\left[V_n^p\right] = \omega_p^2\left[C_{mn}\right]\left[V_n^p\right], \quad p=1,2,\ldots,N$$

where

$$\left[\Gamma_{mn}\right]=\frac{1}{\mu_0}\left[<\tilde{\nabla}^* \times \mathbf{\Omega}_m ; \mu_r^{-1} \tilde{\nabla} \times \mathbf{\Omega}_n >\right]$$

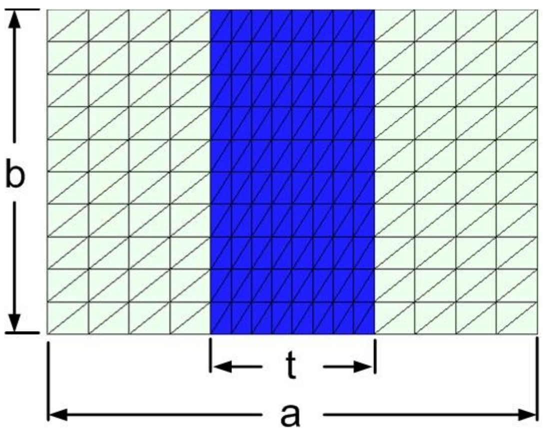
$$\left[C_{mn}\right]=\varepsilon_0\left[<\mathbf{\Omega}_m ; \varepsilon_r \mathbf{\Omega}_n >\right]$$

Generalized **eigenvalue problem of the form**
[*A*] [*x*^{p}] = λ_p[*B*][*x*^{p}]

A quadratic function of k_z !

- **Setting** $k_z=0$ **yields cutoff frequencies**
- **For** $k_z \neq 0$, **obtain dispersion information in the form** $\omega_p\left(k_z\right)$ **for mode** p
- **Field distribution for mode** p **is** $\mathbf{E}_p=\sum_n V_n^p \mathbf{\Omega}_n e^{-jk_z z}$

Example: Slab-loaded Rectangular Waveguide

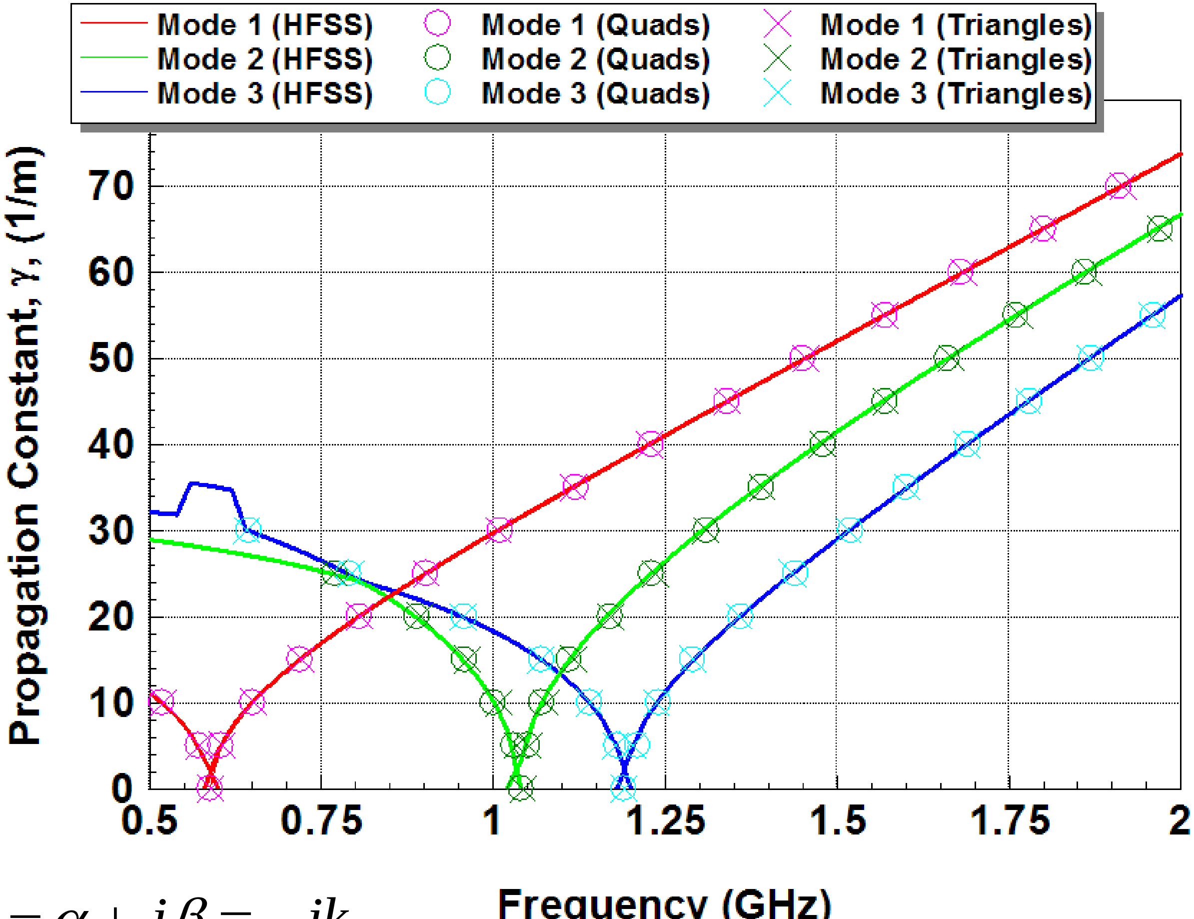


a = 15 cm

b = 10 cm

t = 5 cm

ε_r = 4.0



$$\gamma = \alpha + j\beta \equiv -jk_z$$

The End