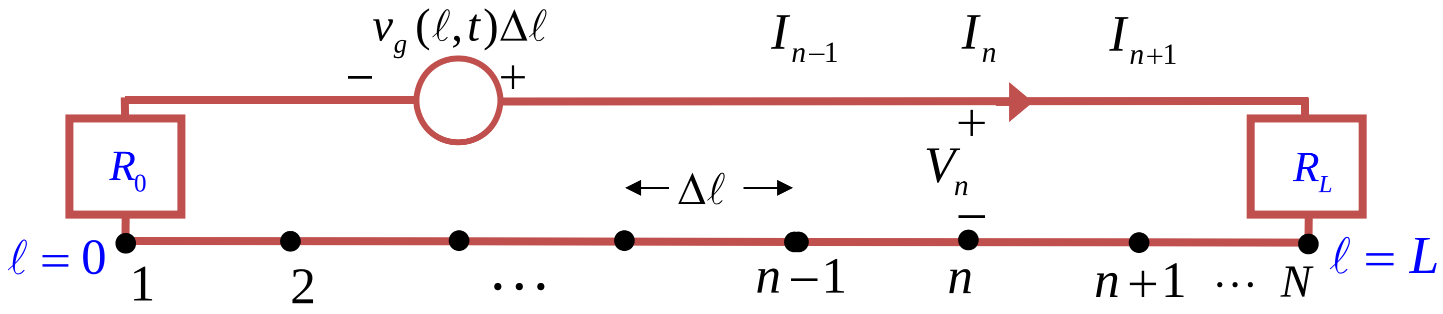
ECE 6350

Introduction to Finite Difference Time Domain (FDTD) Solution of Transmission Lines and Maxwell's Equations

D. R. Wilton

University of Houston

Time-domain Transmission line Solution



Time domain

TX - Line Eqs :

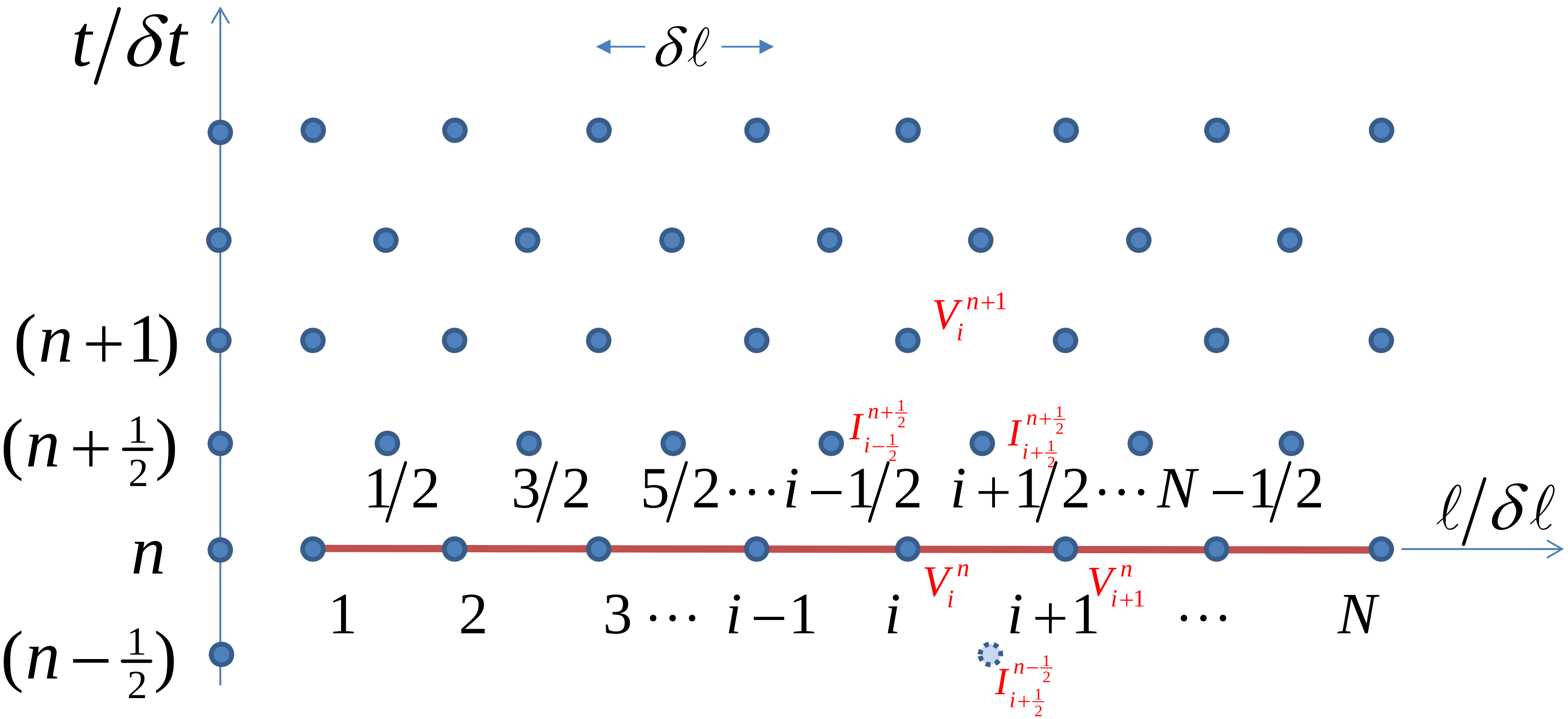
$$-\frac{\partial V(\ell,t)}{\partial \ell} = L\,\frac{\partial I(\ell,t)}{\partial t}$$

$$-\frac{\partial I(\ell,t)}{\partial \ell} = C\,\frac{\partial V(\ell,t)}{\partial t}$$

Time-domain Transmission line Solution

TX-Line Eqs. discretized in both space and time:

$$-\frac{V_{i+1}^{n}-V_{i}^{n}}{\delta\ell}=L\,\frac{I_{i+\frac{1}{2}}^{n+\frac{1}{2}}-I_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{\delta t},\qquad -\frac{I_{i+\frac{1}{2}}^{n+\frac{1}{2}}-I_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\delta\ell}=C\,\frac{V_{i}^{n+1}-V_{i}^{n}}{\delta t}$$



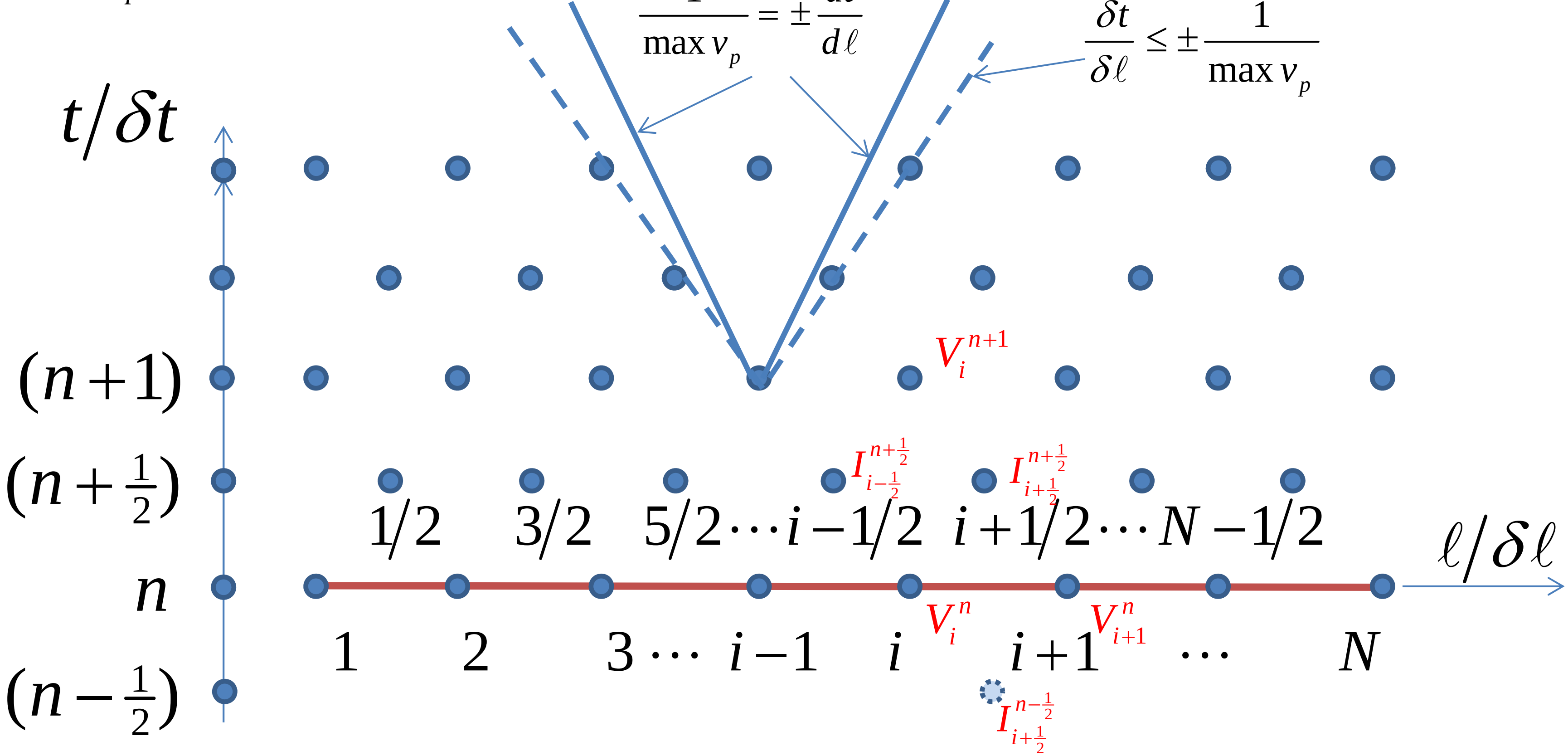
Courant Condition

TX-Line Eqs. discretized in both space and time:

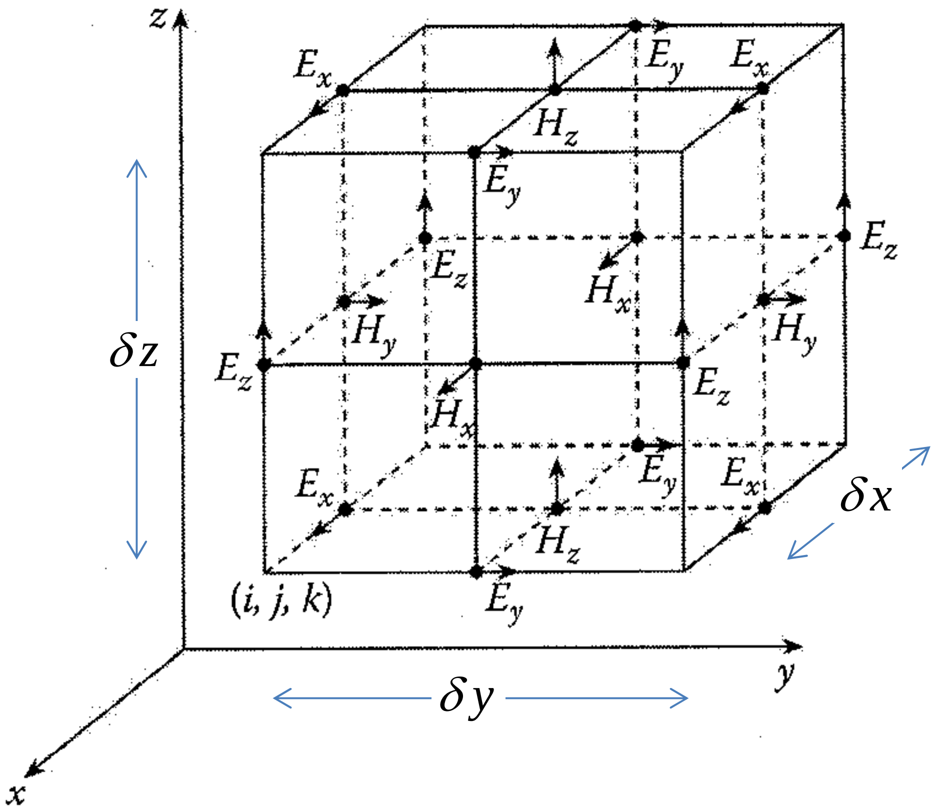
$$-\frac{V_{i+1}^{n}-V_{i}^{n}}{\delta\ell}=L\,\frac{I_{i+\frac{1}{2}}^{n+\frac{1}{2}}-I_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{\delta t},\qquad -\frac{I_{i+\frac{1}{2}}^{n+\frac{1}{2}}-I_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\delta\ell}=C\,\frac{V_{i}^{n+1}-V_{i}^{n}}{\delta t}$$

Courant-Friedrichs-Lewy (CFL) condition on time step:

$$\max v_p \times \delta t \le \delta\ell$$



Yee Lattice for the FDTD Method



Notation:

For $f(x, y, z, t)$**,**

$$f(i\delta x, j\delta y, k\delta z, n\delta t) \equiv f^n(i, j, k)$$

E.g., for $E_y(x, y, z, t)$**,**

$$E_y\left(i\delta x, (j+\tfrac{1}{2})\delta y, (k+1)\delta z, (n+\tfrac{1}{2})\delta t\right)$$

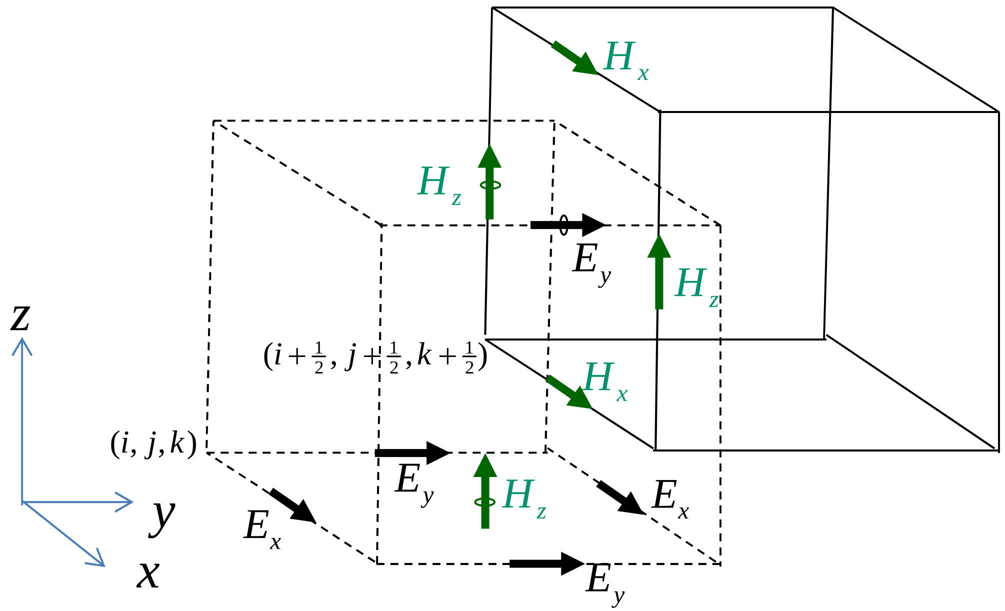
$$\equiv E_y^{n+\frac{1}{2}}(i, j+\tfrac{1}{2}, k+1)$$

Path Integration Method

Integral form of Maxwell's eqs.:

$$\oint_C \mathbf{E}\cdot d\ell = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}, \quad \mathbf{B}=\mu\mathbf{H}$$

$$\oint_C \mathbf{H}\cdot d\ell = \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}, \quad \mathbf{D}=\varepsilon\mathbf{E}$$



$$C_{i+\frac{1}{2},j+\frac{1}{2},k}: \oint_{C_{i+\frac{1}{2},j+\frac{1}{2},k}} \mathbf{E}\cdot d\ell \approx -\left[E_x^n(i+\tfrac{1}{2}, j+1,k)-E_x^n(i+\tfrac{1}{2}, j,k)\right]\delta x+\left[E_y^n(i+1, j+\tfrac{1}{2},k)-E_y^n(i, j+\tfrac{1}{2},k)\right]\delta y$$

$$= -\frac{d}{dt}\int_{S_{i+\frac{1}{2},j+\frac{1}{2},k}} \mathbf{B}\cdot d\mathbf{S} \approx -\frac{\mu_{i+\frac{1}{2},j+\frac{1}{2},k}}{\delta t}\left[H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2},k)-H_z^{n-\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2},k)\right]\delta x\delta y$$

$$C_{i,j,k+\frac{1}{2}}: \oint_{C_{i,j,k+\frac{1}{2}}} \mathbf{H}\cdot d\ell \approx -\left[H_x^{n+\frac{1}{2}}(i, j+\tfrac{1}{2},k+\tfrac{1}{2})+H_x^{n+\frac{1}{2}}(i, j-\tfrac{1}{2},k+\tfrac{1}{2})\right]\delta x+\left[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j,k+\tfrac{1}{2})-H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2}, j,k+\tfrac{1}{2})\right]\delta y$$

$$= \frac{d}{dt}\int_{S_{i,j,k+\frac{1}{2}}} \mathbf{D}\cdot d\mathbf{S} \approx \frac{\varepsilon_{i,j,k+\frac{1}{2}}}{\delta t}\left[E_z^{n+1}(i, j,k+\tfrac{1}{2})-E_z^{n}(i, j,k+\tfrac{1}{2})\right]\delta x\delta y$$

Path Integration Method, Con'd

Similarly for all field components, and expressing in terms of "updated" fields, we obtain the FDTD update equations:

$$H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) = H_x^{n-\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) + \frac{\delta t}{\mu_{i,j+\frac{1}{2},k+\frac{1}{2}}\delta y \delta z}\left\{\left[E_y^n(i,j+\tfrac{1}{2},k+1)-E_y^n(i,j+\tfrac{1}{2},j,k)\right]\delta y-\left[E_z^n(i,j+1,k+\tfrac{1}{2})-E_z^n(i,j,k+\tfrac{1}{2})\right]\delta z\right\}$$

$$H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k+\tfrac{1}{2}) = H_y^{n-\frac{1}{2}}(i+\tfrac{1}{2},j,k+\tfrac{1}{2}) + \frac{\delta t}{\mu_{i+\frac{1}{2},j,k+\frac{1}{2}}\delta x \delta z}\left\{\left[E_z^n(i+1,j,k+\tfrac{1}{2})-E_z^n(i,j,k+\tfrac{1}{2})\right]\delta z-\left[E_x^n(i+\tfrac{1}{2},j,k+1)-E_x^n(i+\tfrac{1}{2},j,k)\right]\delta x\right\}$$

$$H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k) = H_z^{n-\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k) + \frac{\delta t}{\mu_{i+\frac{1}{2},j+\frac{1}{2},k}\delta x \delta y}\left\{\left[E_x^n(i+\tfrac{1}{2},j+1,k)-E_x^n(i+\tfrac{1}{2},j,k)\right]\delta x-\left[E_y^n(i+1,j+\tfrac{1}{2},k)-E_y^n(i,j+\tfrac{1}{2},k)\right]\delta y\right\}$$

Path Integration Method, Con'd

$$E_x^{n+1}(i+\tfrac{1}{2},j,k) = E_x^n(i+\tfrac{1}{2},j,k)$$
$$+\frac{\delta t}{\varepsilon_{i+\frac{1}{2},j,k}\delta y\delta z}\left\{\left[H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k)-H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2},j-\tfrac{1}{2},k)\right]\delta z-\left[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k+\tfrac{1}{2})-H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k-\tfrac{1}{2})\right]\delta y\right\}$$

$$E_y^{n+1}(i,j+\tfrac{1}{2},k) = E_y^n(i,j+\tfrac{1}{2},k)$$
$$+\frac{\delta t}{\varepsilon_{i,j+\frac{1}{2},k}\delta x\delta z}\left\{\left[H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2})-H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k-\tfrac{1}{2})\right]\delta x-\left[H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k)-H_z^{n+\frac{1}{2}}(i-\tfrac{1}{2},j+\tfrac{1}{2},k)\right]\delta z\right\}$$

$$E_z^{n+1}(i,j,k+\tfrac{1}{2}) = E_z^n(i,j,k+\tfrac{1}{2})$$
$$+\frac{\delta t}{\varepsilon_{i,j,k+\frac{1}{2}}\delta x\delta y}\left\{\left[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k+\tfrac{1}{2})-H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2},j,k+\tfrac{1}{2})\right]\delta y-\left[H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2})-H_x^{n+\frac{1}{2}}(i,j-\tfrac{1}{2},k+\tfrac{1}{2})\right]\delta x\right\}$$

Courant-Friedrichs-Lewy (CFL) Stability Condition

The method is stable only if the time step satisfies the CFL stability condition

$$\max v_p \ \ \delta t \leq \left[\frac{1}{\delta x^2} + \frac{1}{\delta y^2} + \frac{1}{\delta z^2} \right]^{-1/2}$$

where $\max v_p$ is the maximum phase velocity in the excitation spectrum.

http://en.wikipedia.org/wiki/Courant%E2%80%93Friedrichs%E2%80%93Lewy_condition
http://www.stanford.edu/class/cme324/classics/courant-friedrichs-lewy.pdf