ECE 6350

3D Electrostatic Potential Integral Equation

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New Features of Static 3D Potential Integral Equation

- 3D geometry and Green's function
- Triangular elements
  - Data structure
  - Local coordinate system
    (area coordinates—both for IE and FEM)
  - Linear interpolation on triangles
  - Numerical integration on triangles
- Handling 1/R singularities in 3D

Equations of Electrostatics in Homogeneous Media

- $\nabla \times \mathbf{E} = \mathbf{0} \quad \Rightarrow \quad \mathbf{E} = -\nabla\Phi$
- $\nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E} = q \qquad [\mathrm{C/m^3}]$

$$\Rightarrow \nabla^2 \Phi = -\frac{q}{\varepsilon}$$

where

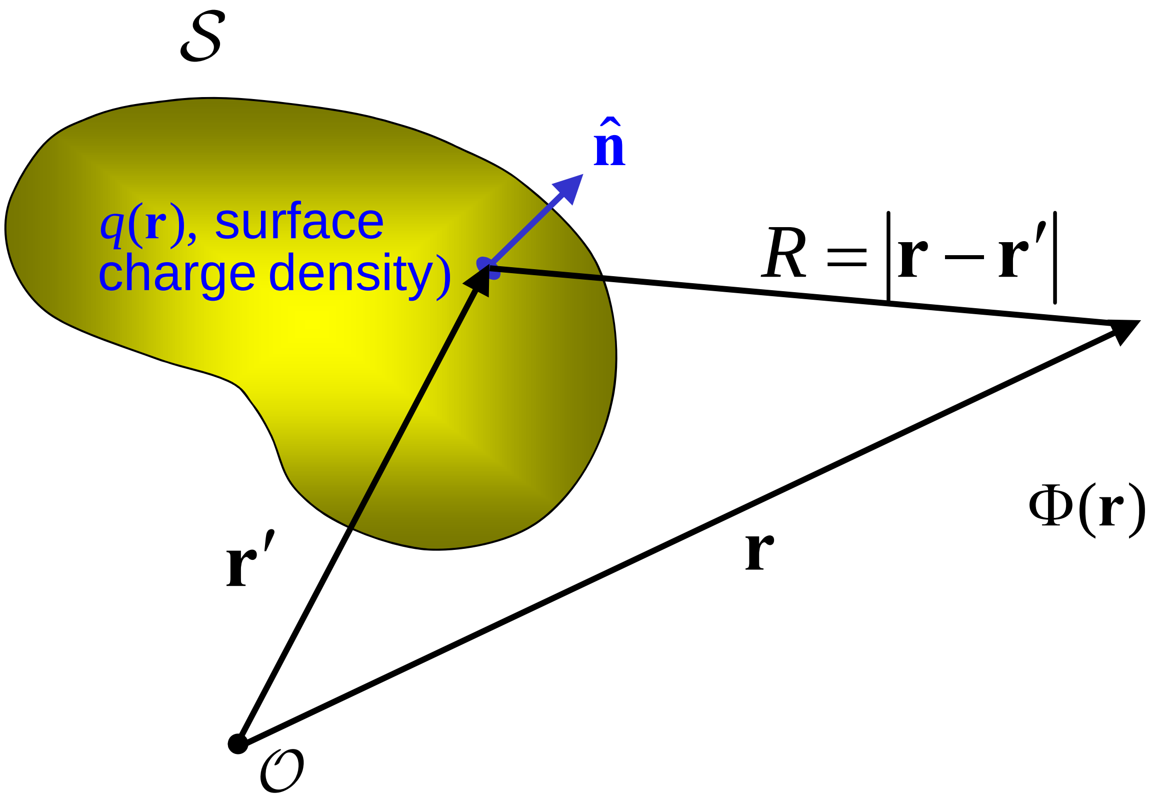
$$\Phi = \frac{1}{\varepsilon} \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}')\, q(\mathbf{r}')\, d\mathcal{V}'$$

and $G(\mathbf{r}, \mathbf{r}') = \dfrac{1}{4\pi R}$, $R = |\mathbf{r} - \mathbf{r}'|$

Note:

$\nabla^2 G = -\delta(\mathbf{r} - \mathbf{r}')$

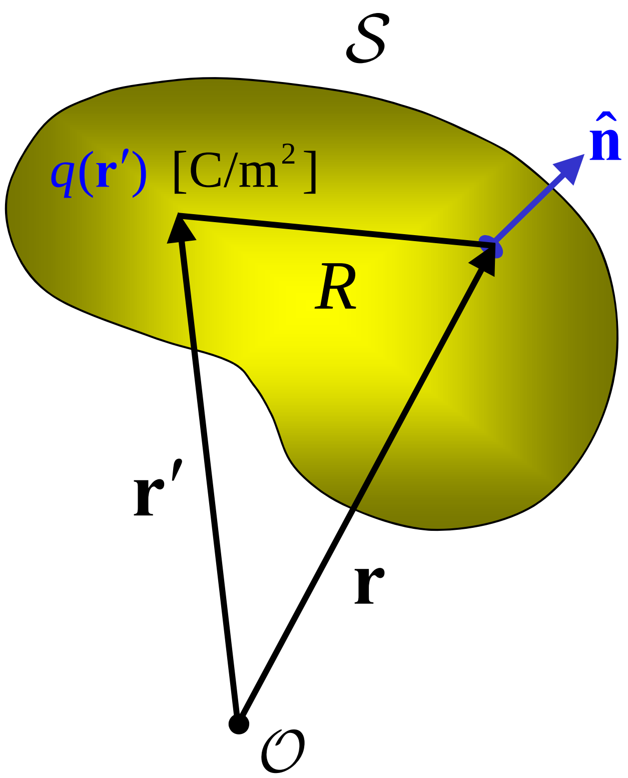
Definitions of Geometrical and Electrical Quantities for Charges on a *Surface*



$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\; q(\mathbf{r}')\, d\mathcal{S}'$$

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi R}$$

Conductor Charged to a Given, Constant Potential Φ_0



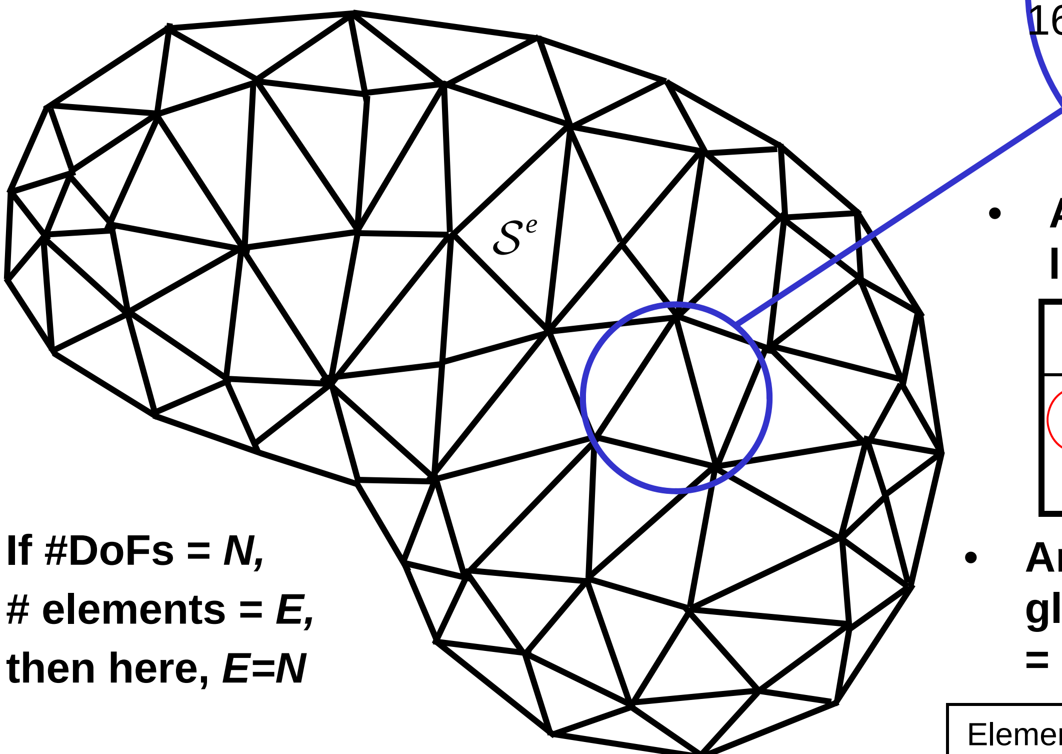
Apply the boundary condition,

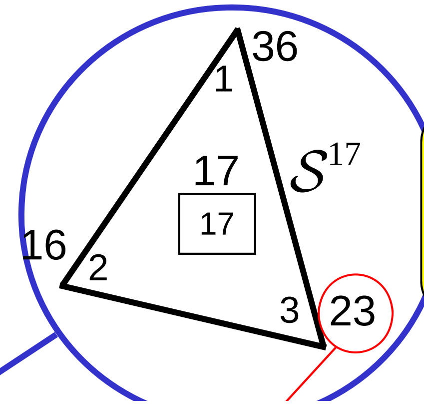
$$\frac{1}{\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}') \underbrace{q(\mathbf{r}')}_{\text{unknown}} \, d\mathcal{S}' = \underbrace{\Phi_0}_{\text{known}} \;, \quad \mathbf{r} \in \mathcal{S}$$

Reminder : Here $q(\mathbf{r})$ denotes a *surface* charge density!

Surface Discretization

$$\mathcal{S} \approx \tilde{\mathcal{S}} = \bigcup_{e=1}^{N} \mathcal{S}^e$$





Note that

$$\underbrace{\mathbf{r}_{23}}_{\text{global}} = \underbrace{\mathbf{r}_3^{17}}_{\text{local}}$$

If #DoFs = *N*,
elements = *E*,
then here, *E*=*N*

- **A *Global Node list* defines vertex locations**

Node #	x	y	z
… 23 …	… x_{23} …	… y_{23} …	… z_{23} …

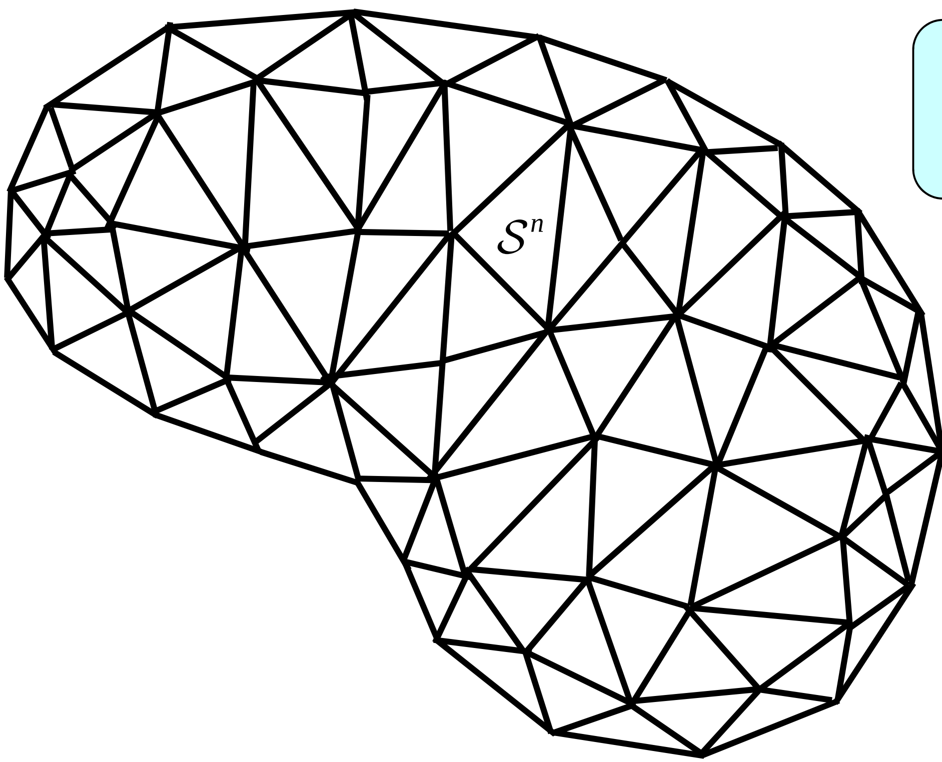
- **An *element list* contains the global node numbers; here DoF# = element #**

Element e	DoF	1	2	3
… 17 …	17	… 36 …	… 16 …	… 23 …

Constant surface charge density assumed in each triangle => Piecewise constant representation

Piecewise Constant Surface Charge Approximation

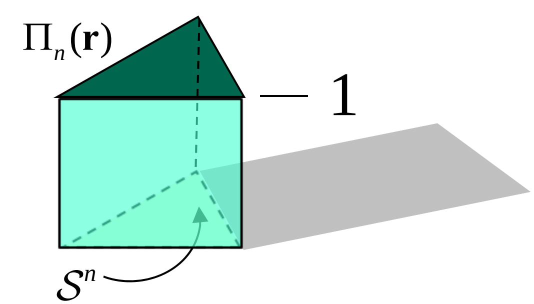
$\boxed{e \to n,}$ $\quad \mathcal{S} \approx \tilde{\mathcal{S}} = \bigcup_{n=1}^{N} \mathcal{S}^n$



Constant surface charge density assumed in each triangle => Piecewise constant representation

$$q(\mathbf{r}') \approx \sum_{n=1}^{N} Q_n \Pi_n(\mathbf{r}')$$

$$\Pi_n(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \mathcal{S}^n \\ 0, & \text{otherwise} \end{cases}$$



Substitute Charge Approximation and Enforce Equality at Subdomain Centroids

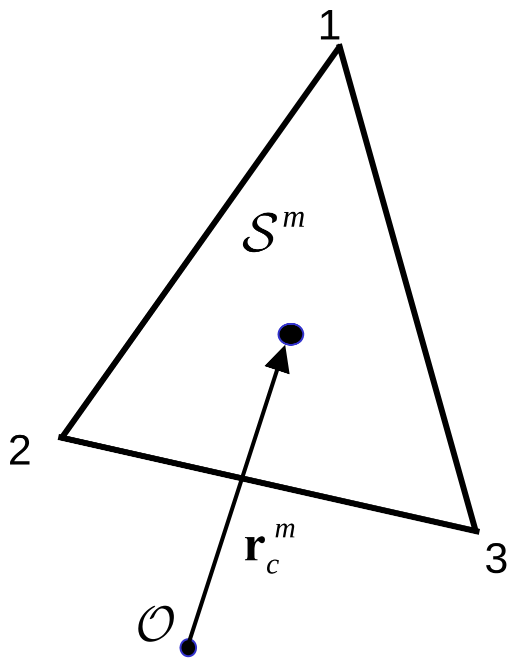
$$q(\mathbf{r}') \approx \sum_{n=1}^{N} Q_n \Pi_n(\mathbf{r}') \quad [\mathrm{C/m^2}],$$

$$\int_{\mathcal{S}} \frac{q(\mathbf{r}')}{4\pi\varepsilon\left|\mathbf{r}-\mathbf{r}'\right|} d\mathcal{S}' = \Phi_0 \quad [\mathrm{V}],\ \mathbf{r}\in\mathcal{S}$$

$$\Rightarrow \sum_{n=1}^{N} Q_n \int_{\tilde{\mathcal{S}}} \frac{\Pi_n(\mathbf{r}')}{4\pi\varepsilon\left|\mathbf{r}-\mathbf{r}'\right|} d\mathcal{S}' \approx \Phi_0,$$

Enforce equality at centroid of $\mathcal{S}^m$:

let $\mathbf{r} = \mathbf{r}_c^m \equiv \dfrac{\mathbf{r}_1^m + \mathbf{r}_2^m + \mathbf{r}_3^m}{3}$, $m = 1, 2, \ldots, N$



Matrix Equation for Approximate Surface Charge Distribution

$$\sum_{n=1}^{N} Q_n \int_{\tilde{S}} \frac{\Pi_n(\mathbf{r}')}{4\pi\varepsilon\left|\mathbf{r}_c^m - \mathbf{r}'\right|} d\mathcal{S}' = \Phi_0, \quad m = 1, 2, \cdots, N$$

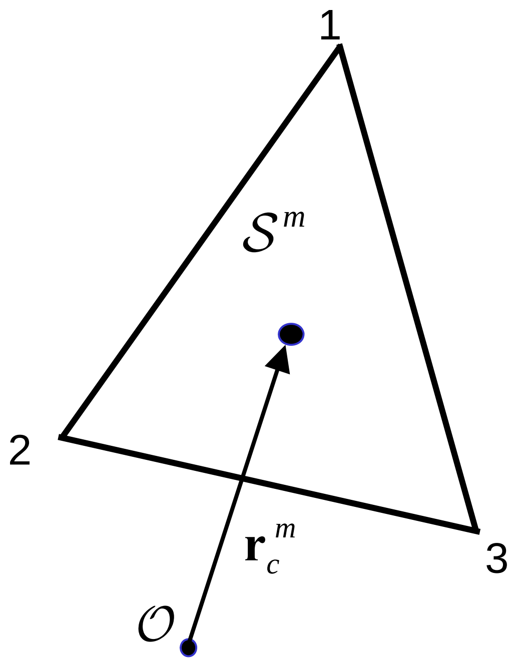
or

$$[S_{mn}][Q_n] = [V_m]$$

where

$$S_{mn} = \int_{\tilde{S}} \frac{\Pi_n(\mathbf{r}')}{4\pi\varepsilon\left|\mathbf{r}_c^m - \mathbf{r}'\right|} d\mathcal{S}' = \int_{\mathcal{S}^n} \frac{d\mathcal{S}'}{4\pi\varepsilon\left|\mathbf{r}_c^m - \mathbf{r}'\right|}$$

$$\mathbf{r}_c^m = \frac{\mathbf{r}_1^m + \mathbf{r}_2^m + \mathbf{r}_3^m}{3}, \quad V_m = \Phi_0$$

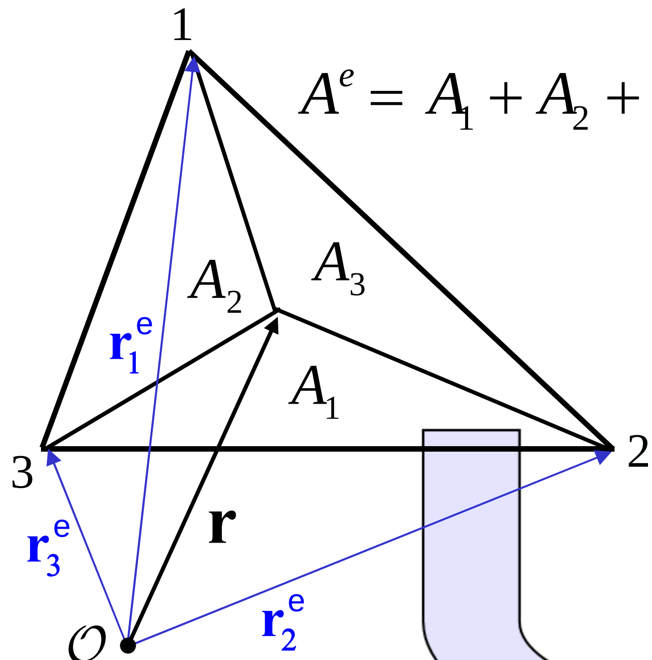


Alternative interpretation as delta f'n testing:

$$S_{mn} = \frac{1}{\varepsilon} \int_{\tilde{S}} \int_{\tilde{S}} \delta\left(\mathbf{r} - \mathbf{r}_c^m\right) G\left(\mathbf{r} - \mathbf{r}'\right) \Pi_n\left(\mathbf{r}'\right) d\mathcal{S}' d\mathcal{S}$$

$$\equiv \frac{1}{\varepsilon} < \delta\left(\mathbf{r} - \mathbf{r}_c^m\right), G, \Pi_n >$$

Area Coordinates Are Used to Represent Bases and Parameterize Element Geometry



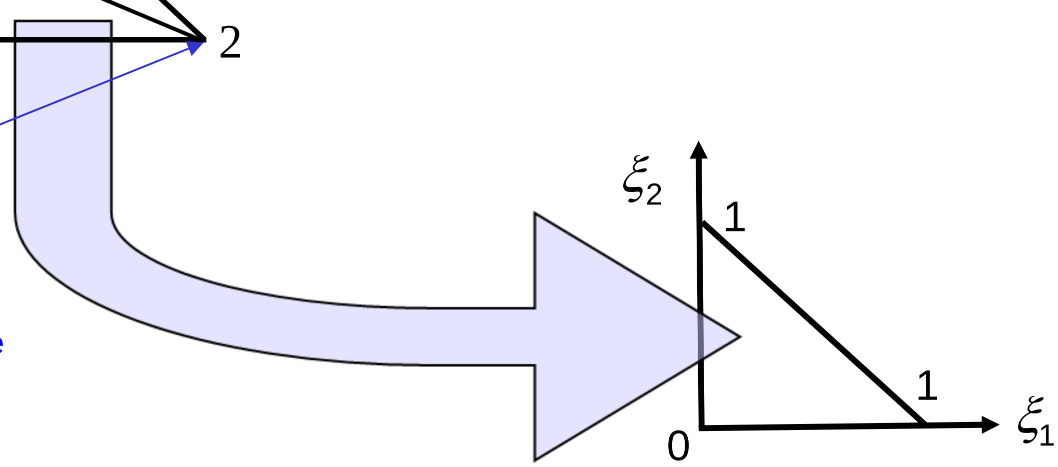
$$A^e = A_1 + A_2 + A_3$$

$$\xi_i = \frac{A_i}{A^e}, \quad i = 1, 2, 3$$

$$\Rightarrow \quad \xi_1 + \xi_2 + \xi_3 = 1$$

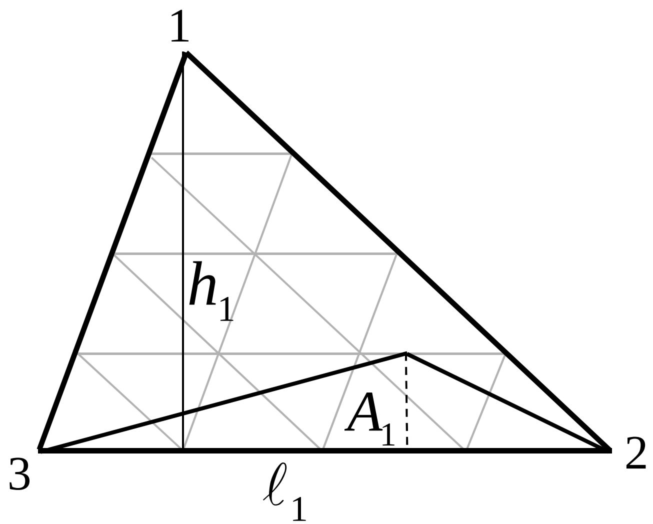
$$\mathbf{r} = \xi_1 \mathbf{r}_1^e + \xi_2 \mathbf{r}_2^e + \xi_3 \mathbf{r}_3^e$$

(proved later!)

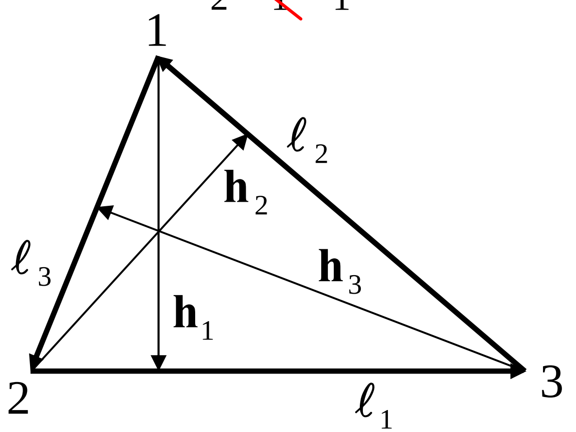


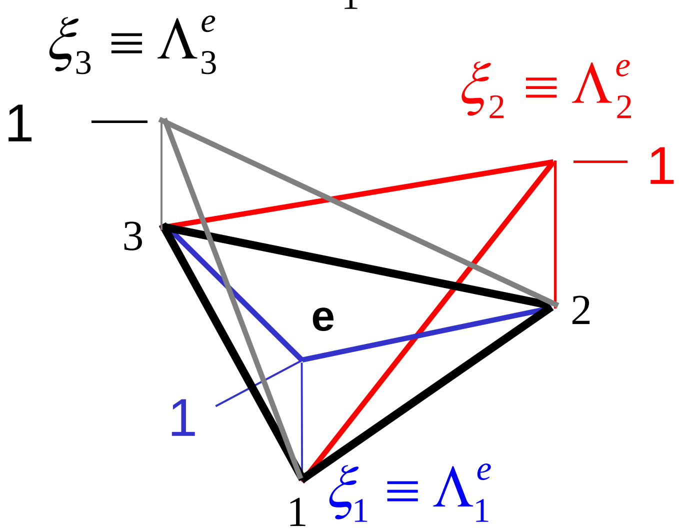
All elements mapped to "parent element"

An Area Coordinate Is Also the Fractional Distance from an Edge to the Opposite Vertex



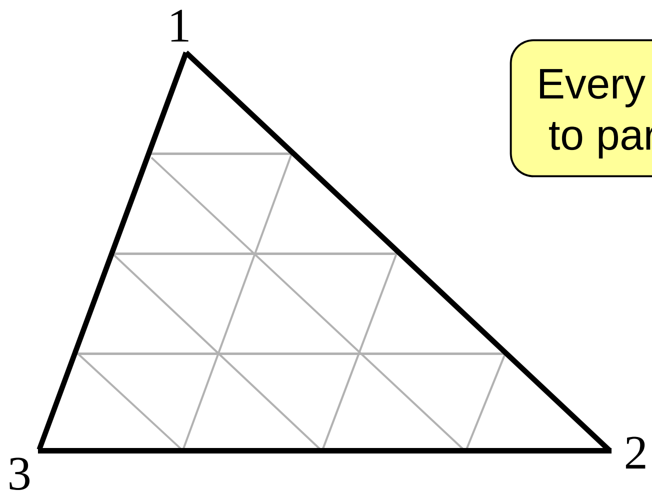
$$\xi_1 = \frac{\cancel{\tfrac{1}{2}\ell_1} \times \text{ height of } A_1}{\cancel{\tfrac{1}{2}\ell_1}\, h_1} = \frac{\text{height of } A_1}{h_1}$$



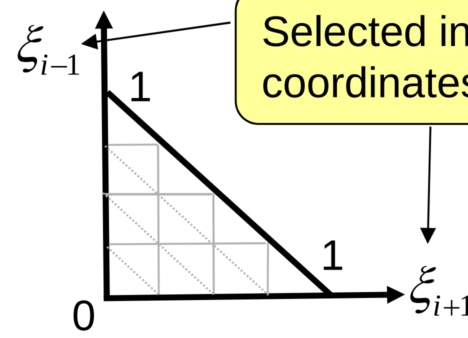


It is convenient to define local edge vectors ℓ_i associated with the edge opposite each vertex and local height vectors $\mathbf{h}_i$ associated with each vertex.

Coordinate Mapping and Modulo 3 Indexing

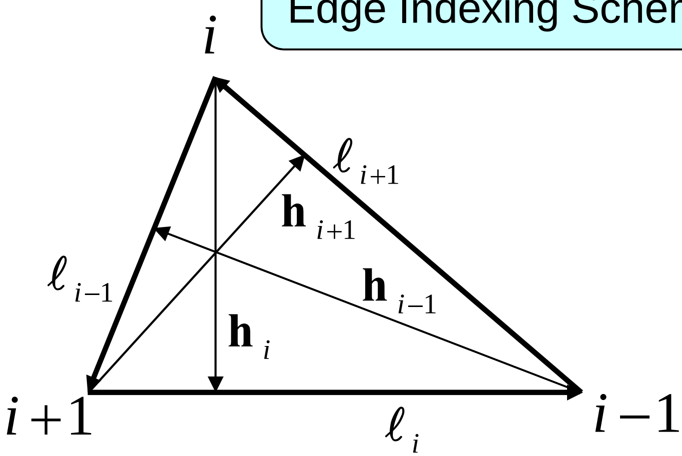


Every triangle is mapped to parent triangle



Selected independent coordinates

Modulo Vertex & Edge Indexing Scheme



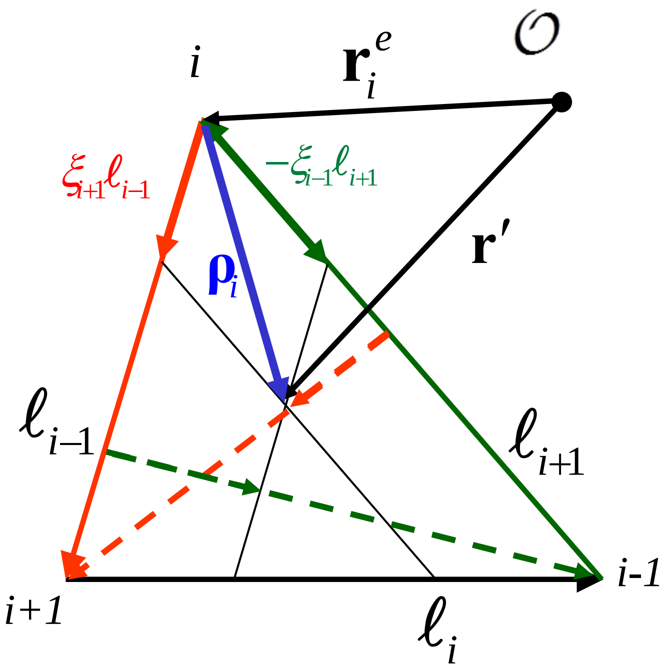
i	i+1 (= i-2)	i-1 (= i+2)
1	2	3
2	3	1
3	1	2

Index arithmetic performed modulo 3 :

$$i \pm j \equiv \left(i \pm j - 1 \right)_{\mathrm{mod}3} + 1$$

Parameterization of a Triangular Patch

Parameterization proof:



$$\mathbf{r}' = \mathbf{r}_i^e + \underbrace{\xi_{i+1}\ell_{i-1} - \xi_{i-1}\ell_{i+1}}_{\rho_i}$$

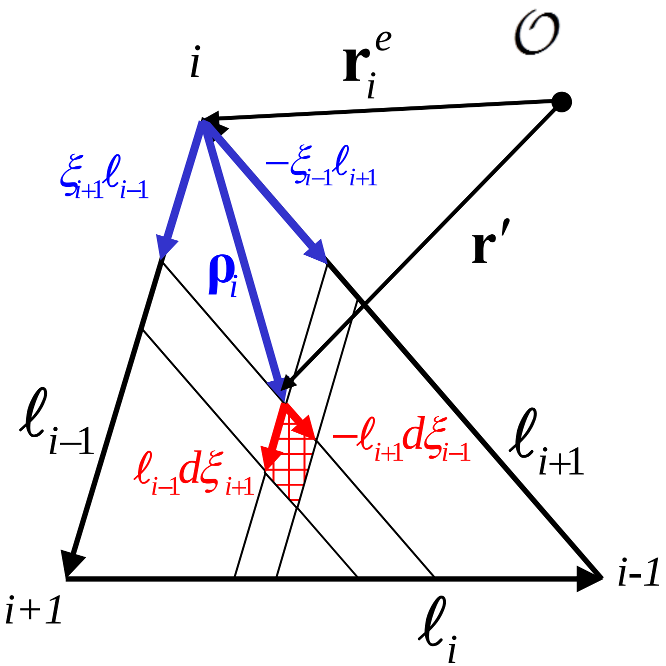
$$= \mathbf{r}_i^e + \xi_{i+1}\left(\mathbf{r}_{i+1}^e - \mathbf{r}_i^e\right) - \xi_{i-1}\left(\mathbf{r}_i^e - \mathbf{r}_{i-1}^e\right)$$

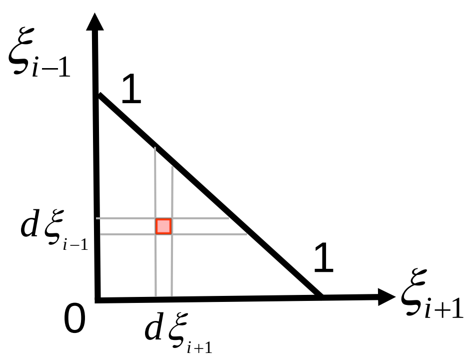
$$= \mathbf{r}_i^e\left(1 - \xi_{i+1} - \xi_{i-1}\right) + \mathbf{r}_{i+1}^e\xi_{i+1} + \mathbf{r}_{i-1}^e\xi_{i-1}$$

$$\Rightarrow \boxed{\mathbf{r}' = \mathbf{r}_i^e\xi_i + \mathbf{r}_{i+1}^e\xi_{i+1} + \mathbf{r}_{i-1}^e\xi_{i-1}}$$

$$\Rightarrow \boxed{\mathbf{r}' = \sum_{i=1}^{3}\mathbf{r}_i^e\xi_i}$$

Parameterization of Integrals





If ξ_{i+1} and ξ_{i-1} are independent variables, what is the surface area $d\mathcal{S}$ swept out when (ξ_{i+1}, ξ_{i-1}) changes to $(\xi_{i+1} + d\xi_{i+1}, \xi_{i-1} + d\xi_{i-1})$?

Ans: $d\mathcal{S} = \left|\ell_{i-1} \times \ell_{i+1}\right| d\xi_{i+1} d\xi_{i-1}$

Hence integrals are evaluated as

$$\int_{A^e} f(\mathbf{r})\, d\mathcal{S}$$

$$= \left|\ell_{i-1} \times \ell_{i+1}\right| \int_0^1 \int_0^{1-\xi_{i-1}} f\left(\mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2 + \mathbf{r}_3^e \xi_3\right) d\xi_{i+1} d\xi_{i-1}$$

$$= \underbrace{2A^e}_{\mathcal{J}} \int_0^1 \int_0^{1-\xi_{i-1}} f\left(\mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2 + \mathbf{r}_3^e \xi_3\right) d\xi_{i+1} d\xi_{i-1}$$

For $m \neq n$, Integrate over Triangles Using Gaussian Area Coordinate Rules

$$\int_{A^e} f(\mathbf{r})\, d\mathcal{S}$$
$$= \; 2A^e \int_0^1 \int_0^{1-\xi_2} f(\xi_1 \mathbf{r}_1^e + \xi_2 \mathbf{r}_2^e + \xi_3 \mathbf{r}_3^e) d\xi_1 d\xi_2$$
$$\underbrace{\approx 2A^e \sum_{k=1}^{K} w_k\, f(\xi_1^{(k)} \mathbf{r}_1^e + \xi_2^{(k)} \mathbf{r}_2^e + \xi_3^{(k)} \mathbf{r}_3^e)}_{\text{Numerical integration}}$$

Table 9 Sample points and weighting coefficients for K-point quadrature on triangles.

Sample Points, $\left(\xi_1^{(k)}, \xi_2^{(k)}\right)$ $\left(\xi_3^{(k)} = 1 - \xi_1^{(k)} - \xi_2^{(k)}\right)$	**Weights,** w_k
K=1, error $\mathcal{O}(\xi_i^2)$**:**	
(0.33333333333333, 0.33333333333333)	0.50000000000000
K=3, error $\mathcal{O}(\xi_i^3)$**:**	
(0.66666666666667, 0.16666666666667)	0.16666666666667
(0.16666666666667, 0.66666666666667)	0.16666666666667
(0.16666666666667, 0.16666666666667)	0.16666666666667
K=7, error $\mathcal{O}(\xi_i^6)$**:**	
(0.33333333333333, 0.33333333333333)	0.11250000000000
(0.79742698535309, 0.10128650732346)	0.06296959027241
(0.10128650732346, 0.79742698535309)	0.06296959027241
(0.10128650732346, 0.10128650732346)	0.06296959027241
(0.47014206410512, 0.47014206410512)	0.06619707639425
(0.47014206410512, 0.05971587178977)	0.06619707639425
(0.05971587178977, 0.47014206410512)	0.06619707639425

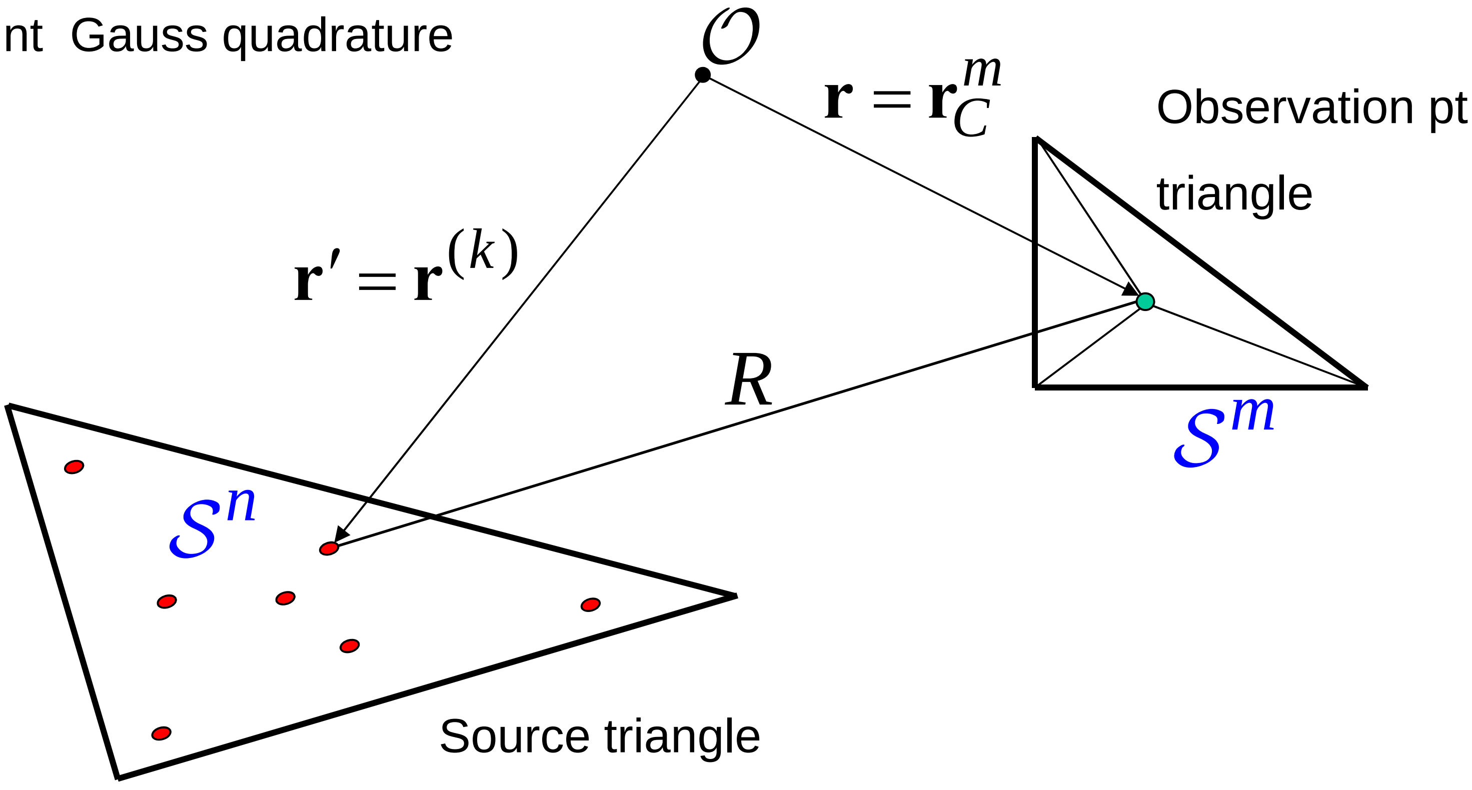
Hence,

$$S_{mn} = \int_{\mathcal{S}^n} \frac{d\mathcal{S}'}{4\pi\varepsilon \left|\mathbf{r}_c^m - \mathbf{r}'\right|}$$
$$\approx \frac{2A^n}{\varepsilon} \sum_{k=1}^{K} w_k\, G\left(\frac{\mathbf{r}_1^m + \mathbf{r}_2^m + \mathbf{r}_3^m}{3},\; \xi_1^{(k)} \mathbf{r}_1^n + \xi_2^{(k)} \mathbf{r}_2^n + \xi_3^{(k)} \mathbf{r}_3^n \right)$$

Integration for Non-Self Terms, $m \neq n$

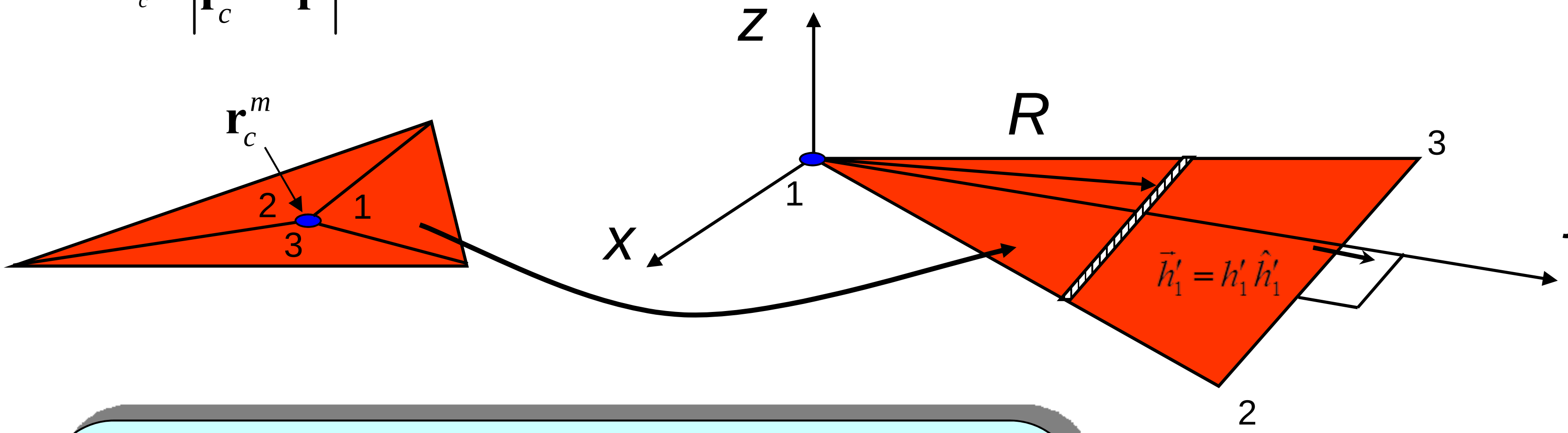
Non-singular case,

7 point Gauss quadrature



For $m = n$, Use a Singularity Cancellation Approach

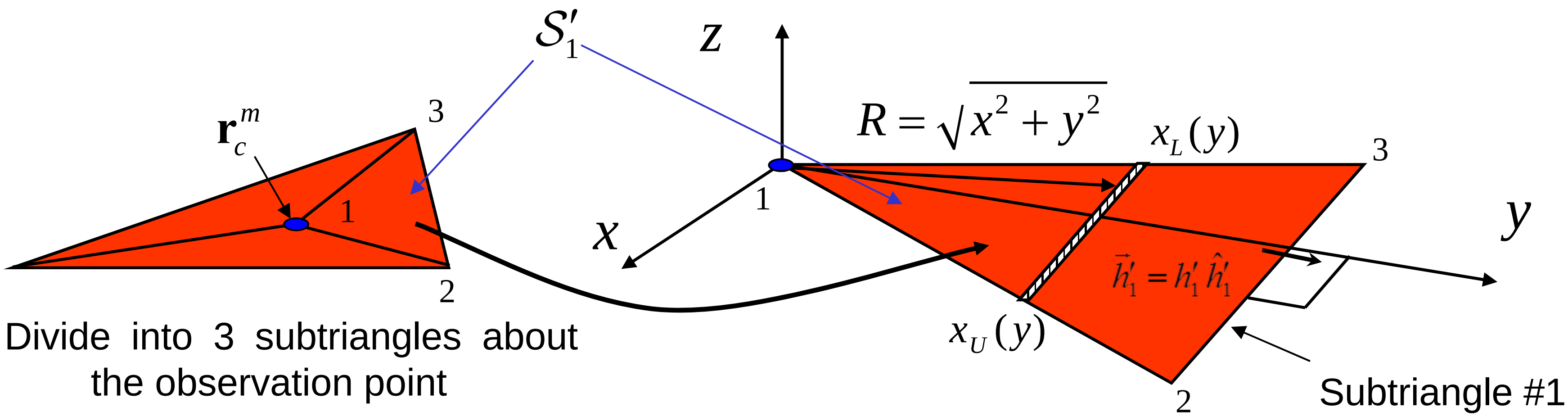
$$\lim_{\mathbf{r}' \to \mathbf{r}_c^m} \frac{1}{\left|\mathbf{r}_c^m - \mathbf{r}'\right|} = \infty$$



- Split observation triangle into three subtriangles about the observation point

- Each subtriangle, which has a singularity at one of its vertices, is treated separately using a local *x*-*y* coordinate system

Transformation to Remove Singularity

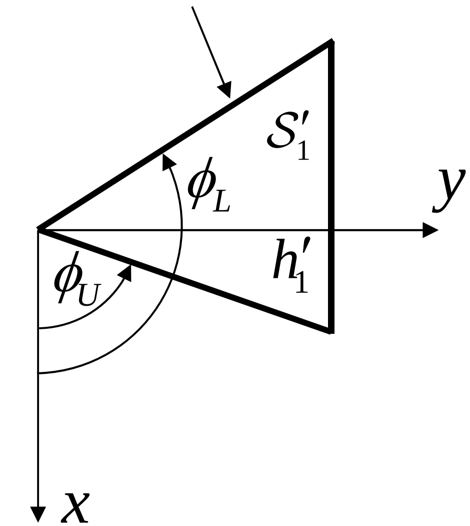


Divide into 3 subtriangles about the observation point

$$\int_{\mathcal{S}_1'} \frac{1}{4\pi R} d\mathcal{S}' = \int_0^{h_1'} \int_{x_L(y)}^{x_U(y)} \frac{1}{4\pi R} dx\, dy$$

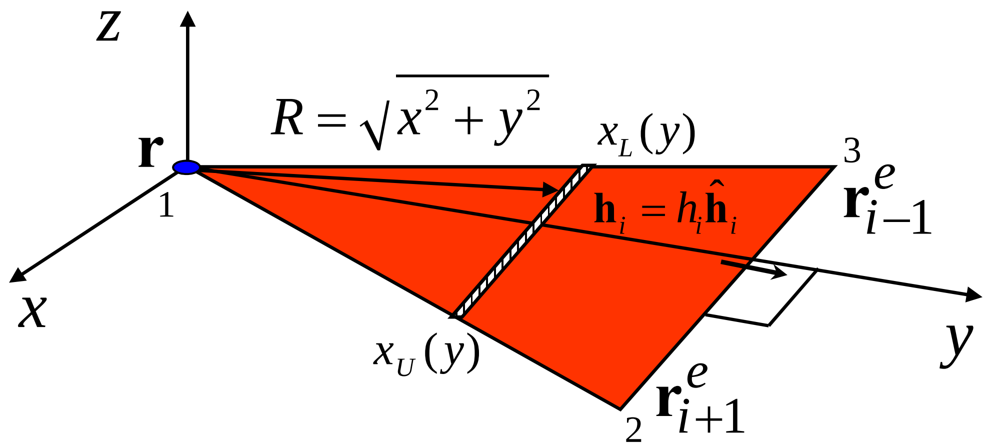
Let $\boxed{du = \frac{dx}{R} \;\Rightarrow\; u = \sinh^{-1}\left(\frac{x}{y}\right) \left(= \ln \frac{x + \sqrt{x^2 + y^2}}{y} \right)}$,

$$\Rightarrow x = y \sinh u, \quad R = \sqrt{x^2 + y^2} = y\sqrt{1 + \sinh^2 u} = y \cosh u$$



$$\sinh u_L = \frac{x_L(y)}{y} = \cot \phi_L\,; \quad \sinh u_U = \frac{x_U(y)}{y} = \cot \phi_U$$

Evaluation of Integral

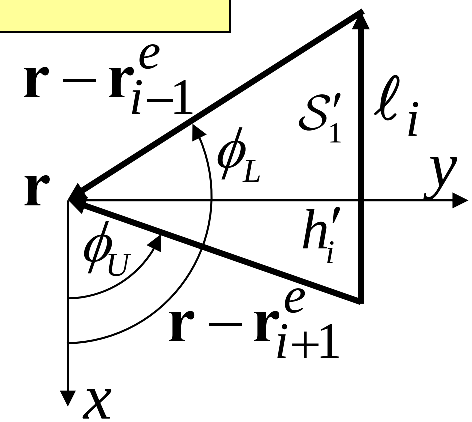


$$\int_{\mathcal{S}'_i}\frac{1}{4\pi R}\,d\mathcal{S}' = \int_0^{h'_i}\int_{x_L(y)}^{x_U(y)}\frac{1}{4\pi R}\,dx\,dy = \frac{1}{4\pi}\int_0^{h'_i}\int_{u_L=\sinh^{-1}\cot\phi_L}^{u_U=\sinh^{-1}\cot\phi_U} du\,dy$$

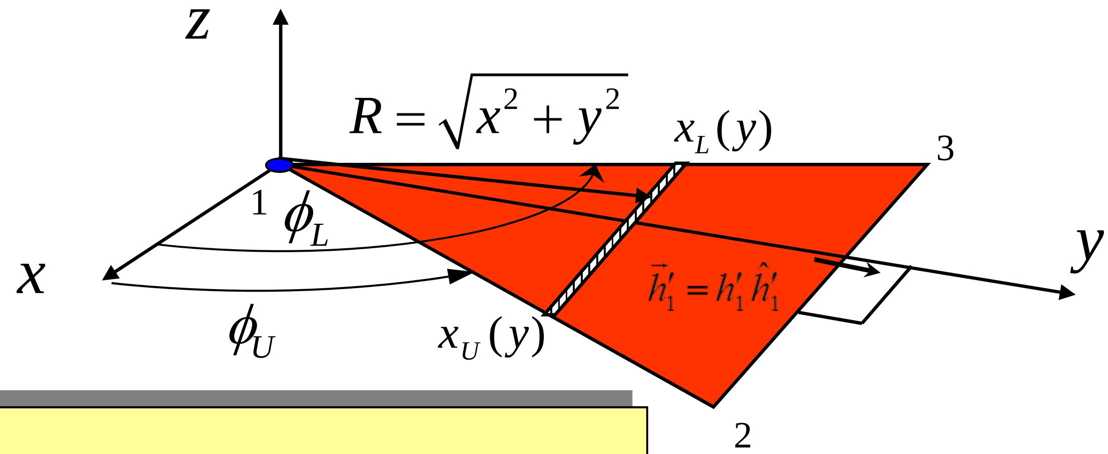
$$= \frac{h'_i}{4\pi}\left(\sinh^{-1}\cot\phi_U - \sinh^{-1}\cot\phi_L\right)$$

Repeat and sum over all three subtriangles

$$\cot\phi_U = \frac{\ell_i\cdot\left(\mathbf{r}-\mathbf{r}^e_{i+1}\right)}{\left|\ell_i\times\left(\mathbf{r}-\mathbf{r}^e_{i+1}\right)\right|},\qquad \cot\phi_L = \frac{\ell_i\cdot\left(\mathbf{r}-\mathbf{r}^e_{i-1}\right)}{\left|\ell_i\times\left(\mathbf{r}-\mathbf{r}^e_{i-1}\right)\right|}$$



Determining a Quadrature Rule



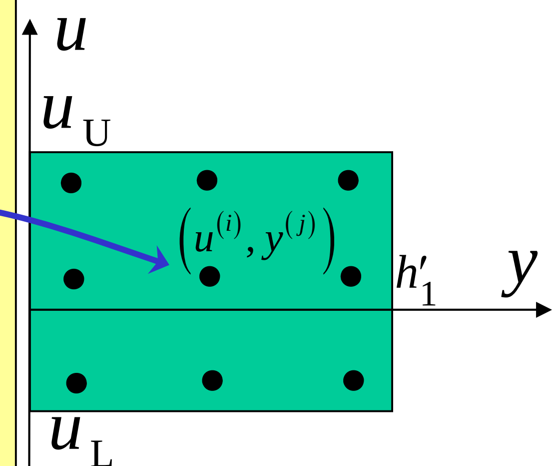
$$\int_{\mathcal{S}_1'} \frac{\overbrace{f(\mathbf{r})}^{\text{generalized to allow for phase factor, bases, etc.}}}{4\pi R}\, d\mathcal{S}' = \frac{1}{4\pi}\int_0^{h_1'}\int_{u_L}^{u_U} \overbrace{f(\mathbf{r}(u,y))}^{\text{smoothed integrand}}\, du dy$$

$$= \frac{h_1'(u_U - u_L)}{4\pi}\sum_i \sum_j w_i w_j f\left(\mathbf{r}\left(u^{(i)}, y^{(j)}\right)\right)$$

where $u^{(i)} = u_U \xi_1^{(i)} + u_L \xi_2^{(i)}, \quad y^{(j)} = h_1' \xi_1^{(j)},$

$\left(w_k, \xi_1^{(k)}\right)$ are Gauss - Legendre weights & samples,

$$u_{U,L} = \sinh^{-1}\frac{x_{U,L}(y)}{y} = \sinh^{-1}\left(\cot\phi_{U,L}\right)$$



Note only *one* sample pt. needed to integrate *exactly* if *f*(**r**)=1!

We Can Hide Transformation Details by Mapping Weights & Sample Points Back to Parent Triangle

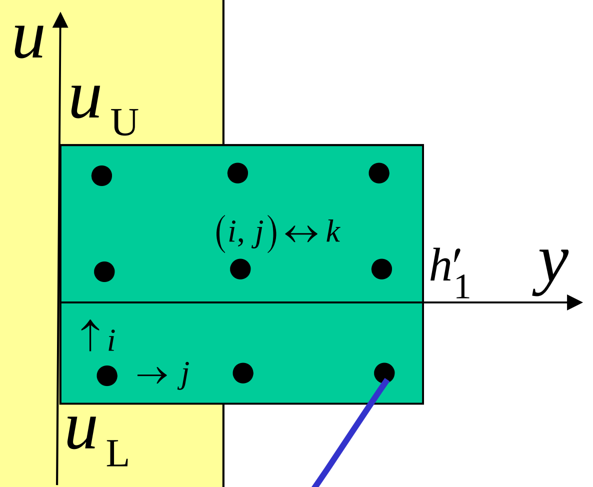
- Map the index pair (i,j) to a single index k: $k \leftrightarrow (i,j)$
- Then *force* the integral into the standard parent triangle form,

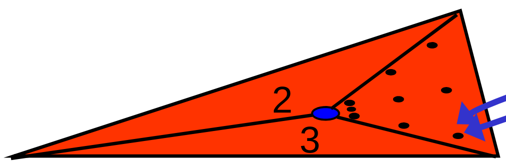
$$\int_{\mathcal{S}_1'} \frac{f(\mathbf{r})}{4\pi R}\, d\mathcal{S}' \approx 2A^e \sum_{k=(i,j)} W_k \underbrace{\frac{f\left(\mathbf{r}^{(k)}\right)}{4\pi R^{(k)}}}_{\text{Sampled values of integrand}}$$

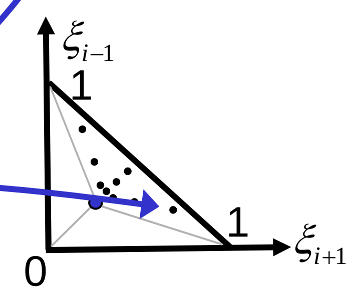
- Since $\displaystyle\int_{\mathcal{S}_1'} \frac{f(\mathbf{r})}{4\pi R}\, d\mathcal{S}' \approx \frac{h_1'\left(u_U - u_L\right)}{4\pi} \sum_i \sum_j w_i w_j f\left(\mathbf{r}^{(i,j)}\right)$

$$\Rightarrow W_k = \frac{w_i w_j\, h_1'\left(u_U - u_L\right) R^{(k)}}{2A^e}$$

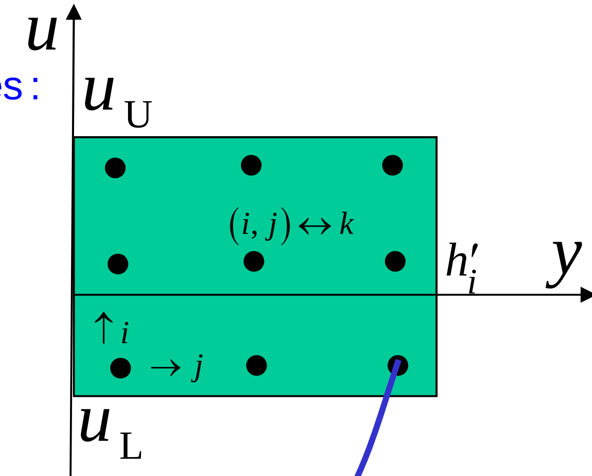
(repeat for each subtriangle)







Mapping (u, y) Sample Points & Weights Back to ξ_i Coordinates

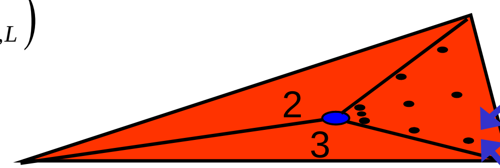


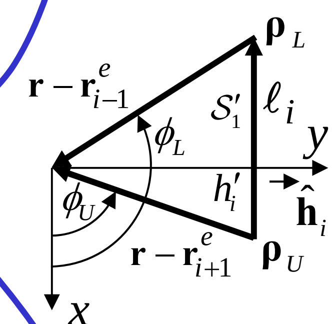
- Find corner of i-th subtriangle in local $\boldsymbol{\rho}_{\mathrm{loc}} = (x_{\mathrm{loc}}, y_{\mathrm{loc}})$ coordinates:

$$\boldsymbol{\rho}_U \equiv \left(x_U(h_i'),\, h_i'\right),\ \boldsymbol{\rho}_L \equiv \left(x_L(h_i'),\, h_i'\right),$$

where $h_i' = \hat{\mathbf{h}}_i \cdot (\mathbf{r}^e_{i\pm1} - \mathbf{r}),\ x_L(h_i') = -\hat{\ell}_i \cdot (\mathbf{r}^e_{i-1} - \mathbf{r})$

$x_U(h_i') = -\hat{\ell}_i \cdot (\mathbf{r}^e_{i+1} - \mathbf{r})$ (Note $\hat{\mathbf{x}}_{\mathrm{loc}} = -\hat{\ell}_i, \hat{\mathbf{y}}_{\mathrm{loc}} = \hat{\mathbf{h}}_i$!)





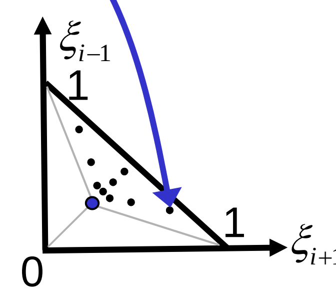
- Determine angular limits: $\cot\phi_{L,U} = \dfrac{x_{L,U}(h_i')}{h'}$
- Determine u-parameter limits: $u_{U,L} = \sinh^{-1}\left(\cot\phi_{U,L}\right)$
- Determine transverse and radial sample points:

$$u^{(i)} = u_U \xi_1^{(i)} + u_L \xi_2^{(i)},\ i = 1, 2, \ldots, K_{\mathrm{transverse}}$$

$$y^{(j)} = h_i' \, \xi_1^{(j)},\ j = 1, 2, \ldots, K_{\mathrm{radial}}$$

- Map (u, y) sample points back to $(x_{\mathrm{loc}}, y_{\mathrm{loc}})$, then global (x, y, z) coordinates:

$$\boldsymbol{\rho}^{(k)} \equiv (x^{(k)}, y^{(k)}) \overset{k \leftrightarrow (i,j)}{=} \left(y^{(j)} \sinh u^{(i)},\ y^{(j)}\right) \Rightarrow \mathbf{r}^{(k)} = \mathbf{r} + y^{(k)} \hat{\mathbf{h}}_i - x^{(k)} \hat{\ell}_i$$



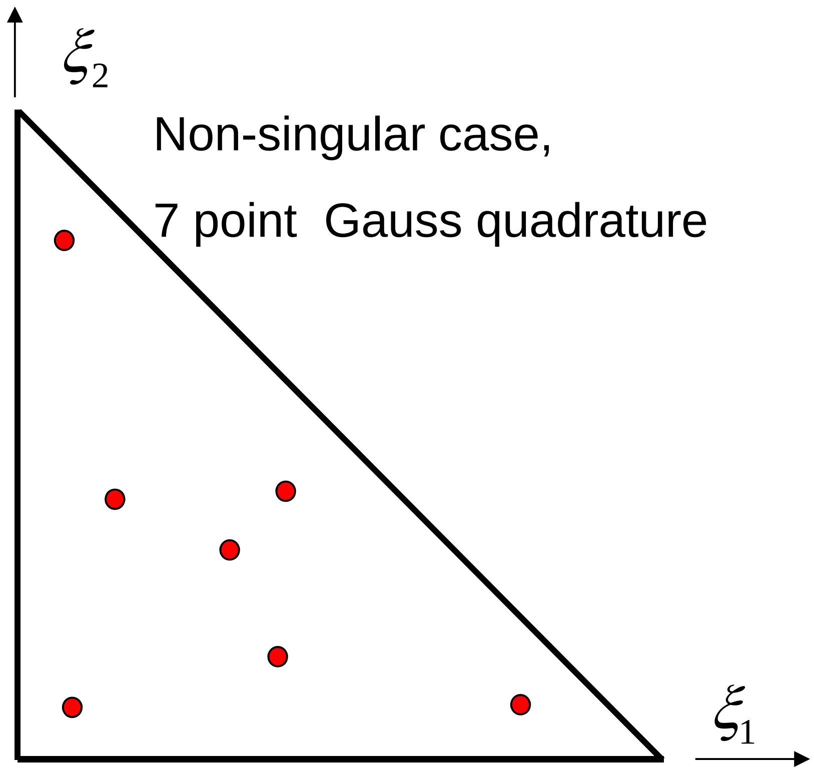
- Map $\mathbf{r}^{(k)}$ coordinates to area coordinates:

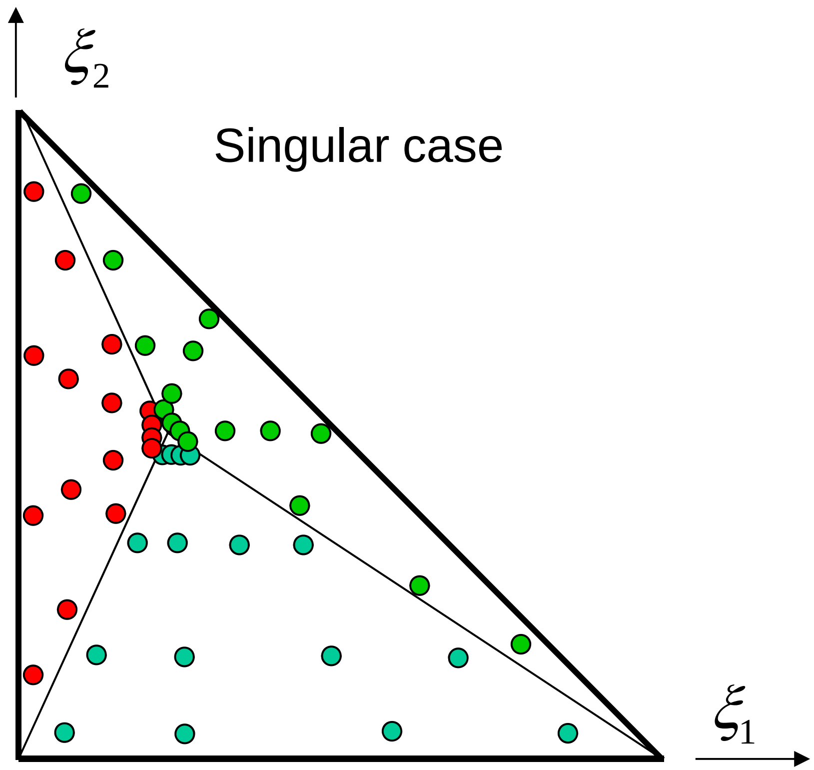
$$A_i^{(k)} = \left|\ell_i \times \left(\mathbf{r}^{(k)} - \mathbf{r}^e_{i+1}\right)\right| \big/ 2,\quad A_{i-1}^{(k)} = \left|\ell_{i-1} \times \left(\mathbf{r}^{(k)} - \mathbf{r}^e_{i+1}\right)\right| \big/ 2,$$

$$(\xi_i^{(k)}, \xi_{i-1}^{(k)}, \xi_{i+1}^{(k)}) = \left(A_i^{(k)}/A^e,\ A_{i-1}^{(k)}/A^e,\ 1 - \xi_i^{(k)} - \xi_{i-1}^{(k)}\right)$$

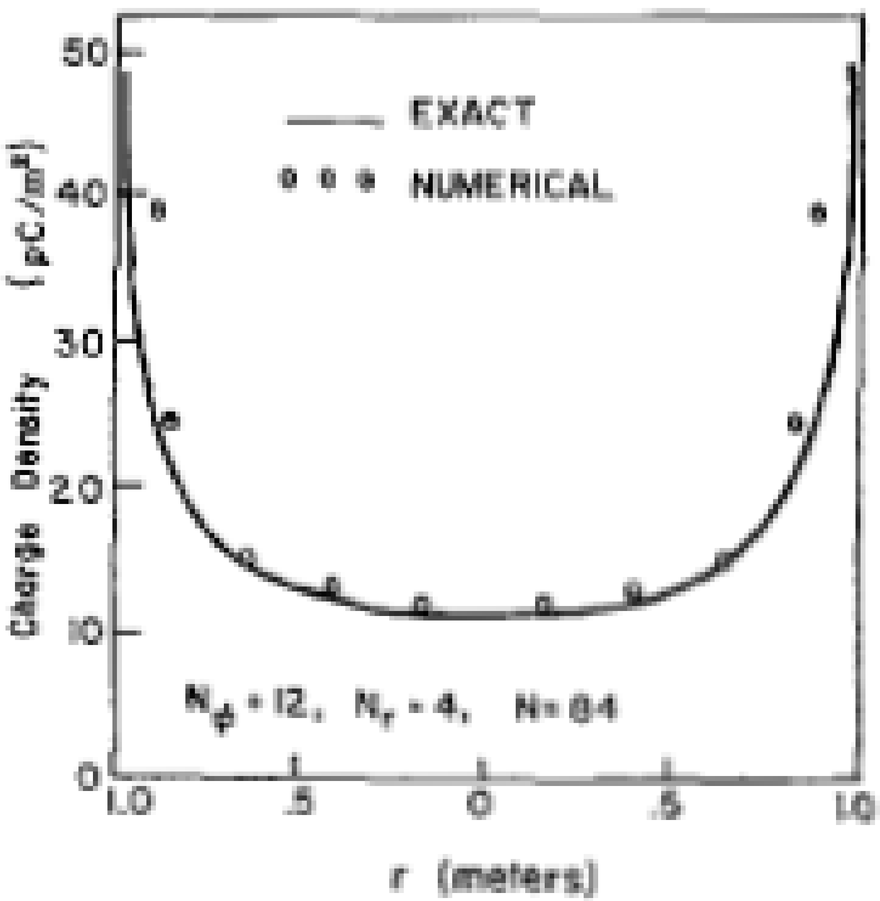
$$W_k = \frac{w_i w_j \, h_1' \left(u_U - u_L\right) y^{(j)} \cosh u^{(i)}}{2A^e}$$

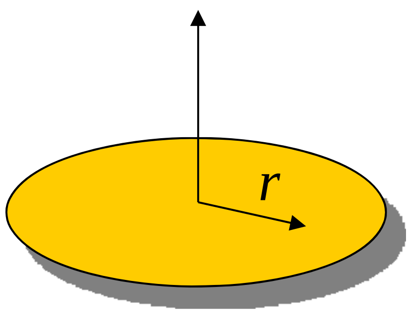
Typical Sample Point Schemes





Charge Distribution on a Conducting Circular Disk





Capacitance of a Conducting Sphere

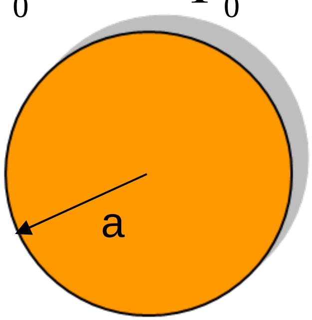
TABLE I
NORMALIZED CAPACITANCE OF A SPHERE (IN PICOFARADS/METER)

N_ϕ	N_θ	N	C/a
6	3	24	94.03
6	4	36	98.35
6	5	48	100.39
6	6	60	101.51
6	8	84	102.64
8	3	32	96.81
8	4	48	101.20
8	5	64	103.28
8	6	80	104.43
exact			111.26

$$C = 4\pi\varepsilon_0 a \;\; [\mathrm{F}]$$

$$C = \frac{Q}{\Phi_0} \approx \frac{\int_{\mathcal{S}} \sum_{n=1}^{N} Q_n \Pi_n(\mathbf{r})\, d\mathcal{S}}{\Phi_0}$$

$$\approx \frac{\sum_{n=1}^{N} Q_n \mathcal{S}^n}{\Phi_0} = \frac{1}{\Phi_0}[Q_n]^t\left[\mathcal{S}^n\right]$$



Charge Distribution on a Bent Conducting Circular Disk

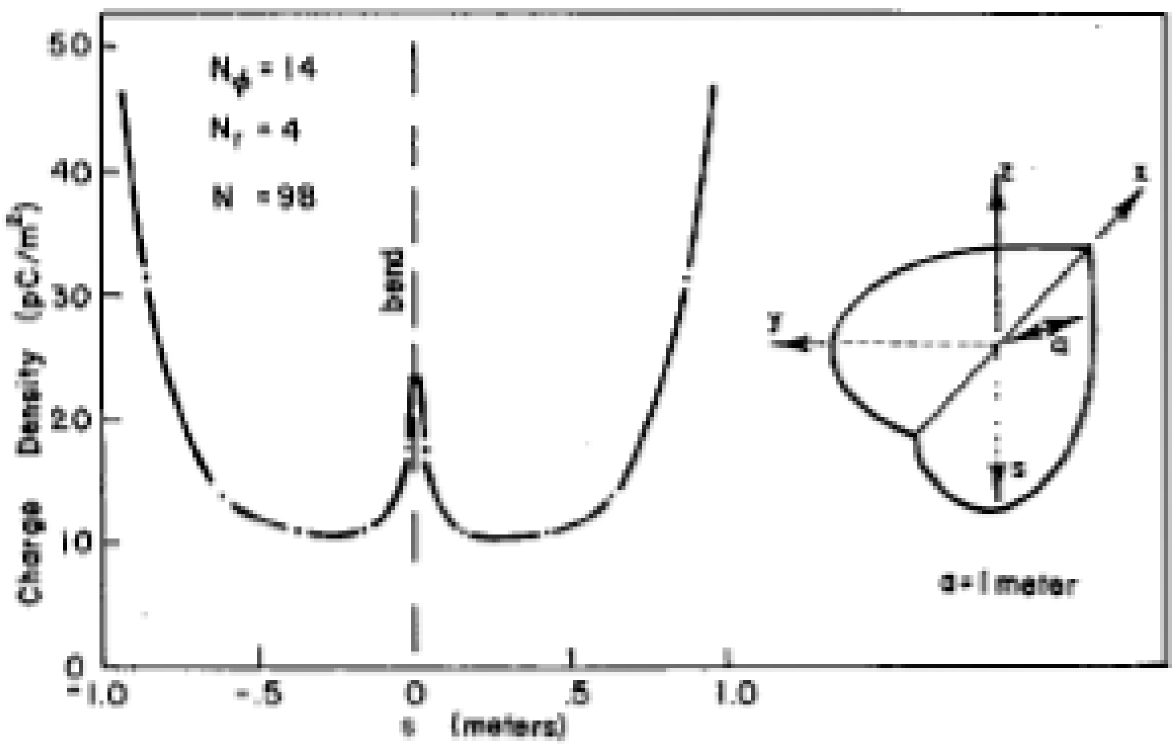
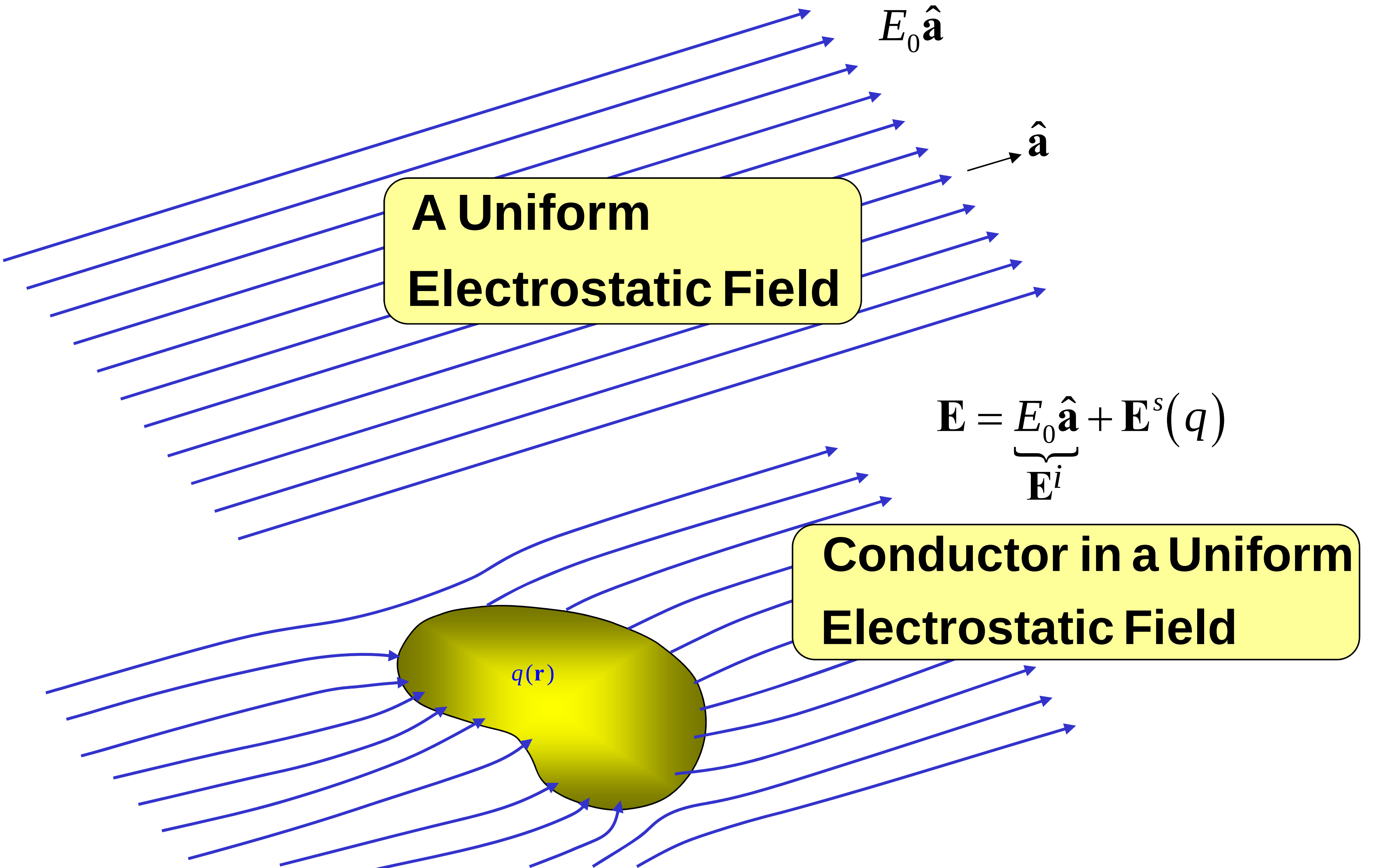


Fig. 3. Calculated charge density distribution on unit disk bent 90° along diameter. Distribution is plotted along symmetry plane perpendicular to bend.

Conductor in a Uniform Static Electric Field



Modifications for a Conductor in a Uniform Impressed Field

- To produce a constant electric field in the direction of $\hat{\mathbf{a}}$, choose

$$\Phi^i = -E_0\,\hat{\mathbf{a}}\cdot\mathbf{r} = -E_0\left(\hat{a}_x x + \hat{a}_y y + \hat{a}_z z\right)$$

since

$$\mathbf{E}^i = -\nabla\Phi^i = E_0\left(\hat{a}_x\hat{\mathbf{x}} + \hat{a}_y\hat{\mathbf{y}} + \hat{a}_z\hat{\mathbf{z}}\right) = E_0\hat{\mathbf{a}}$$

Assumed given!

- $\left(\mathbf{E}^i + \mathbf{E}^s\right)_{\text{tan}} = 0$ **on** $\mathcal{S}$

$\Rightarrow -\nabla_{\text{tan}}\left(\Phi^i + \Phi^s\right) = 0$ **on** $\mathcal{S}$

$\Rightarrow \Phi^i + \Phi^s = \Phi_0$ **on** $\mathcal{S}$

$$\Rightarrow \frac{1}{\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\,q(\mathbf{r}')\,d\mathcal{S}' = \underbrace{-\Phi^i(\mathbf{r})}_{=E_0\,\hat{\mathbf{a}}\cdot\mathbf{r}} + \Phi_0,\ \mathbf{r}\ \textbf{on}\ \mathcal{S}$$

Constraint: $\int_{\mathcal{S}} q(\mathbf{r}')\,d\mathcal{S}' = Q_0$

Assumed given!

Problem Discretization

Potential integral equation:

$$\Rightarrow \quad \frac{1}{\varepsilon}\int_{\mathcal{S}} G(\mathbf{r}_c^m,\mathbf{r}')\, q(\mathbf{r}')\, d\mathcal{S}' \approx \sum_{n=1}^{N} Q_n \int_{\tilde{\mathcal{S}}} \frac{\Pi_n(\mathbf{r}')}{4\pi\varepsilon\left|\mathbf{r}_c^m-\mathbf{r}'\right|}\, d\mathcal{S}' = -\Phi^i(\mathbf{r}_c^m)+\Phi_0,$$

Charge constraint:

$$m = 1, 2, \cdots, N$$

$$\Rightarrow \quad \int_{\mathcal{S}} q(\mathbf{r}')\, d\mathcal{S}' \approx \sum_{n=1}^{N} Q_n \int_{\tilde{\mathcal{S}}} \Pi_n(\mathbf{r}')\, d\mathcal{S}' = \sum_{n=1}^{N} Q_n \mathcal{S}^n = \left[\mathcal{S}^n\right]^t \left[Q_n\right] = Q_0$$

$$\Rightarrow \quad \boxed{\begin{bmatrix} [S_{mn}] & [-1] \\ [\mathcal{S}^n]^t & 0 \end{bmatrix} \begin{bmatrix} [Q_n] \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} [V_m] \\ Q_0 \end{bmatrix}}$$

$\mathcal{S}^n \equiv$ area of $\mathcal{S}^n$

where

$$S_{mn} = \int_{\tilde{\mathcal{S}}} \frac{\Pi_n(\mathbf{r}')}{4\pi\varepsilon\left|\mathbf{r}_c^m-\mathbf{r}'\right|}\, d\mathcal{S}' = \int_{\mathcal{S}^n} \frac{d\mathcal{S}'}{4\pi\varepsilon\left|\mathbf{r}_c^m-\mathbf{r}'\right|}$$

$$V_m = -\Phi^i(\mathbf{r}_c^m) = E_0\hat{\mathbf{a}}\cdot\mathbf{r}_c^m$$

The End