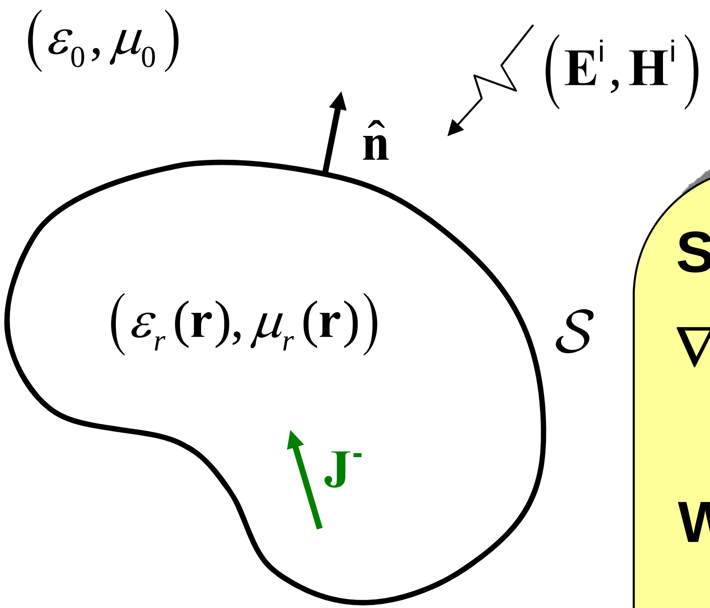
Coupled 3D Finite and Boundary Element Formulation

Donald R. Wilton

Scattering notes, pp. 39-41 3D-FEM

pp. 41-43, hybrid FEM/BEM

Strong and Weak Forms of the 3-D Helmholtz Equation



Strong form :

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = -j\omega\mu_0 \mathbf{J}^{-}, \quad \mathbf{r} \in \mathcal{V}$$

Weak form :

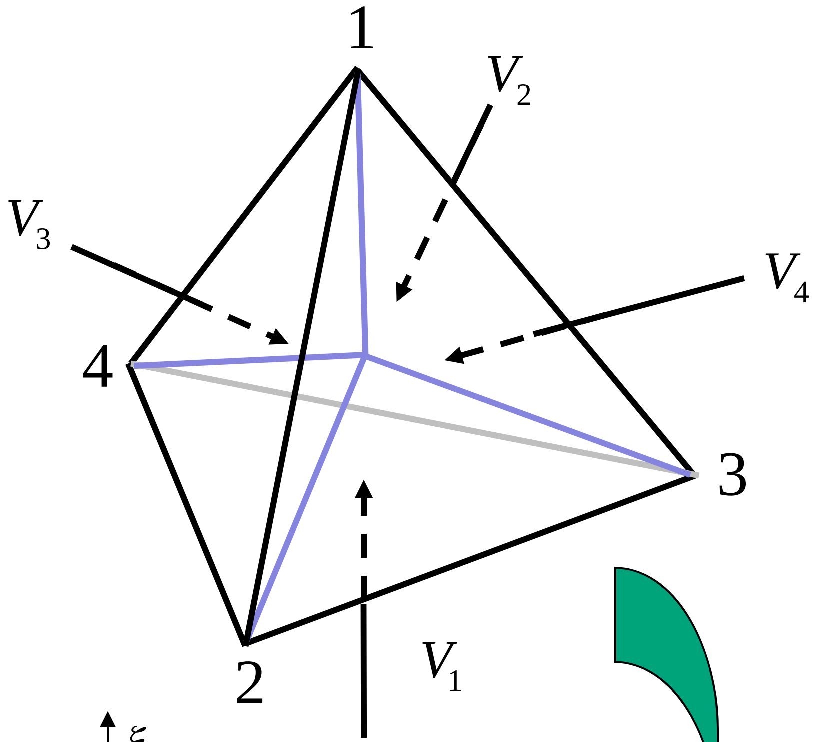
$$\frac{1}{j\omega\mu_0} < \nabla \times \boldsymbol{\Omega}_m, \mu_r^{-1} \nabla \times \mathbf{E} > + j\omega\varepsilon_0 < \boldsymbol{\Omega}_m, \varepsilon_r \mathbf{E} >$$

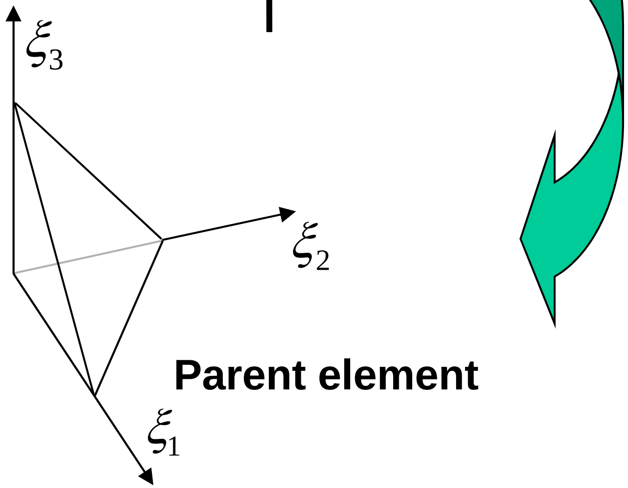
$$- \oint_S \boldsymbol{\Omega}_m \cdot \hat{\mathbf{n}} \times \mathbf{H}\, dS = - < \boldsymbol{\Omega}_m, \mathbf{J}^{-} >, \quad \mathbf{r} \in \mathcal{V}$$

Vanishes if either

$\boldsymbol{\Omega}_m$ **or** $\hat{\mathbf{n}} \times \mathbf{H}$ **vanishes on** $\mathcal{S}$

Volume Coordinates for Tetrahedral Meshes





Volume Coordinates :

$$\xi_i = \frac{V_i}{V^e}, \quad i = 1, 2, 3, 4$$

$$\Rightarrow \quad \xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$$

Geometry Parametrization :

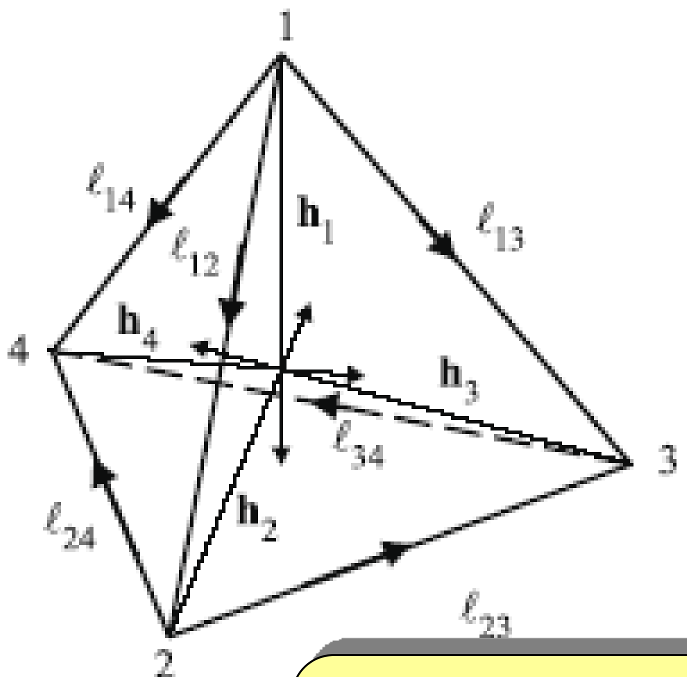
$$\mathbf{r} = \mathbf{r}_1^e \xi_1 + \mathbf{r}_2^e \xi_2 + \mathbf{r}_3^e \xi_3 + \mathbf{r}_4^e \xi_4,$$

$\mathbf{r}_i^e$ = **vector from global origin to** ***i*** **th vertex of element** ***e***

Traversing the path from vertices 1-2-3 should determine outward normal $\hat{\mathbf{n}}$ according to the right hand rule.

Geometrical Parameters Associated with a Tetrahedron

Our convention: Choose vertices 1,2,3, such that traversing them in order produces an outward normal by right hand rule



Edge vectors	$\boldsymbol{\ell}_{ij} = \boldsymbol{r}_j^e - \boldsymbol{r}_i^e;\ \ \ell_{ij} = \lvert\boldsymbol{\ell}_{ij}\rvert;$ $\hat{\boldsymbol{\ell}}_{ij} = \frac{\boldsymbol{\ell}_{ij}}{\ell_{ij}}\ ,\ i \neq j \in \{1,2,3,4\}$
Volume	$V^e = \frac{\lvert\boldsymbol{\ell}_{14}\cdot(\boldsymbol{\ell}_{24}\times\boldsymbol{\ell}_{34})\rvert}{6}$ $= \frac{A_i h_i}{3}\ ,$ A_i = area of face i, h_i = height of vertex i
Coordinate gradients, $\nabla\xi_i = -\frac{\hat{\boldsymbol{h}}_i}{h_i}$	$\nabla\xi_1 = \frac{\boldsymbol{\ell}_{24}\times\boldsymbol{\ell}_{34}}{6V^e}\ ,$ $\nabla\xi_2 = \frac{\boldsymbol{\ell}_{34}\times\boldsymbol{\ell}_{14}}{6V^e}\ ,$ $\nabla\xi_3 = \frac{\boldsymbol{\ell}_{14}\times\boldsymbol{\ell}_{24}}{6V^e}\ ,$ $\nabla\xi_4 = -\nabla\xi_1 - \nabla\xi_2 - \nabla\xi_3$

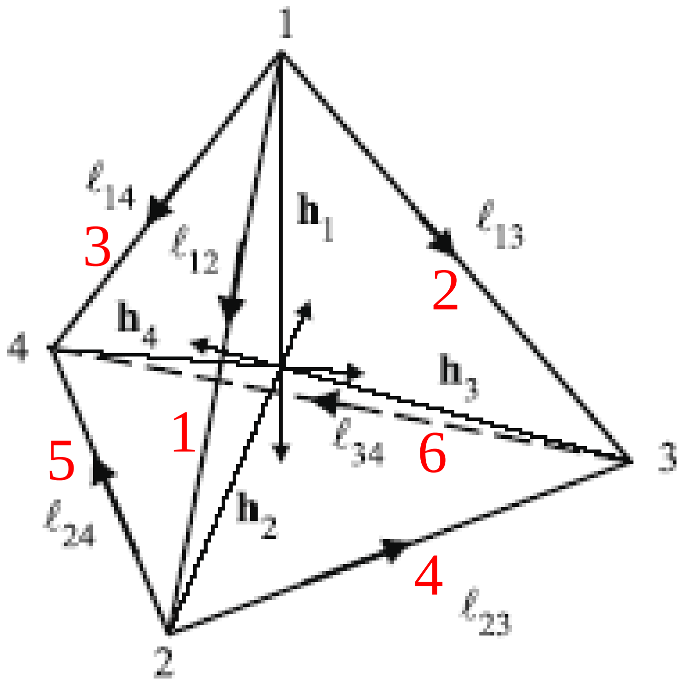
- **Gradient *direction* is from face i to vertex i $= -\hat{\mathbf{h}}_i$**
- **Gradient *magnitude* = change in ξ_i from face i to vertex i $\div$ distance ($= 1/h_i \Rightarrow \lvert\nabla\xi_i\rvert = 1/h_i$)**

  $\Rightarrow \nabla\xi_i = -\hat{\mathbf{h}}_i / h_i$

Local Edge Numbering and Reference Directions

- **Note** $\ell_{ii} = \mathbf{0}.$
- **Also** $\ell_{ij} = -\ell_{ji}$**, so for independent edge vectors, use only** $\ell_{ij},\ j > i.$

Local edge number	Edge reference direction
1	ℓ_{12}
2	ℓ_{13}
3	ℓ_{14}
4	ℓ_{23}
5	ℓ_{24}
6	ℓ_{34}



Parameterization of Integrals and Numerical Integration over Tetrahedrons

$$\int_{V^e} f(\mathbf{r})\,d\mathcal{V} = 6V^e \int_0^1 \int_0^{1-\xi_k} \int_0^{1-\xi_j-\xi_k} f(\mathbf{r})\,d\xi_i\,d\xi_j\,d\xi_k,\ i \neq j \neq k$$

$$\approx \underbrace{6V^e}_{\mathcal{J}^e} \sum_{k=1}^{K} w_k\, f(\mathbf{r}_1^e \xi_1^{(k)} + \mathbf{r}_2^e \xi_2^{(k)} + \mathbf{r}_3^e \xi_3^{(k)} + \mathbf{r}_4^e \xi_4^{(k)})$$

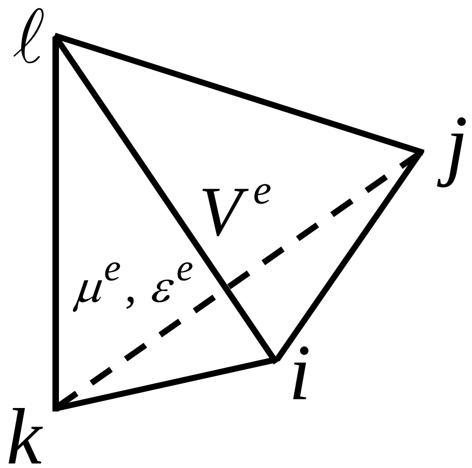
Table 13 Sample points and weighting coefficients for K-point quadrature on tetrahedrons.

Sample Points, $\left(\xi_1^{(k)}, \xi_2^{(k)}, \xi_3^{(k)}\right)$ $(\xi_4^{(k)} = 1 - \xi_1^{(k)} - \xi_2^{(k)} - \xi_3^{(k)})$	**Weights,** w_k
K=1, error $\mathcal{O}(\xi_i^2)$**:**	
(0.25000000,0.25000000,0.25000000)	0.16666667
K=4, error $\mathcal{O}(\xi_i^3)$**:**	
(0.58541020, 0.13819660, 0.13819660)	0.041666667
(0.13819660, 0.58541020, 0.13819660)	0.041666667
(0.13819660, 0.13819660, 0.58541020)	0.041666667
(0.13819660, 0.13819660, 0.13819660)	0.041666667

...or use the exact result

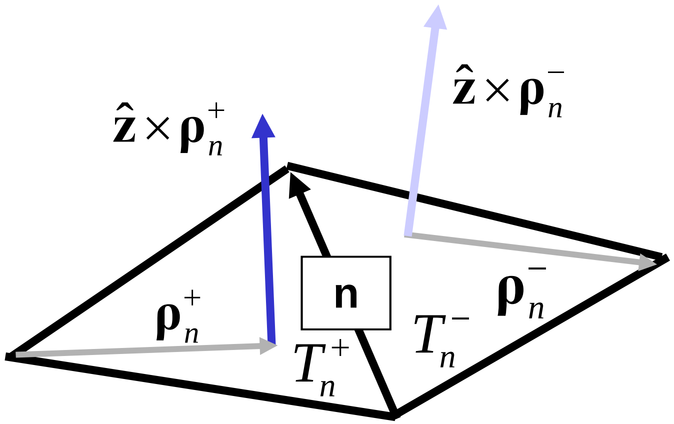
$$\int_{V^e} \xi_1^\alpha \xi_2^\beta \xi_3^\gamma \xi_4^\delta\, d\mathcal{V} = \frac{6V^e \alpha!\beta!\gamma!\delta!}{(\alpha+\beta+\gamma+\delta+3)!}$$

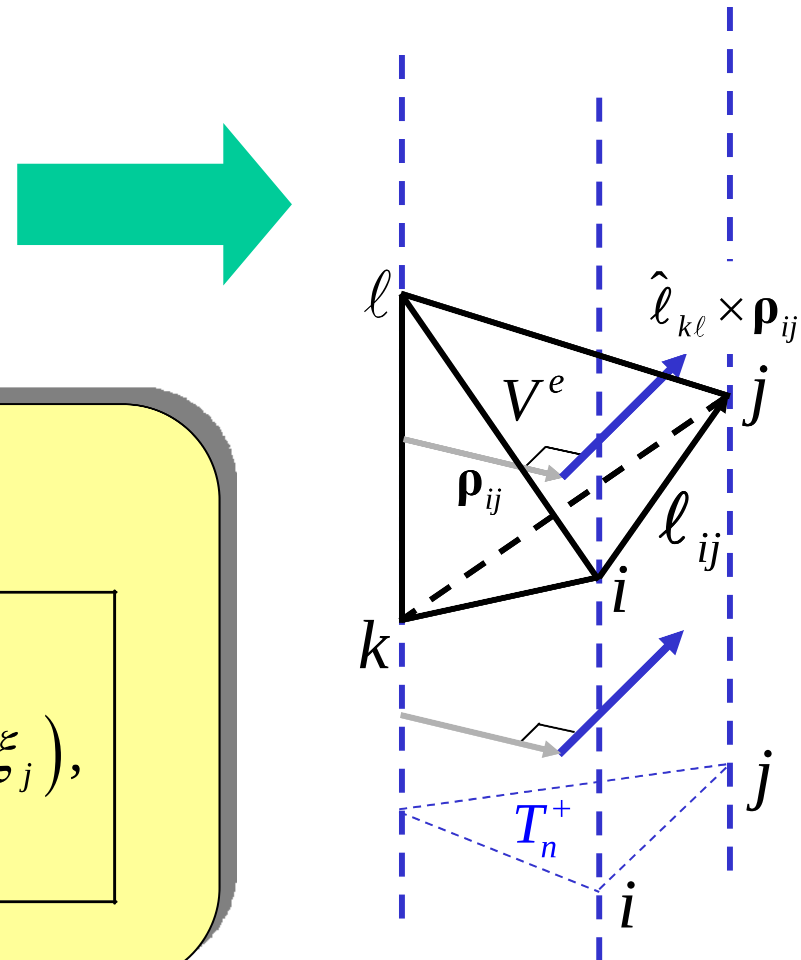
Properties of Curl-Conforming Bases Needed on Tetrahedral Meshes



- **Assume medium parameters constant within a tetrahedron**
- **Make tangential electric field continuous across medium boundaries by defining DOFs at edges**
- **Make interpolatory by allowing *only* the basis associated with the DOF at an edge to have a unit tangential component there**

Planar Triangle Curl-Conforming Bases May Be Extended to Tetrahedrons by Viewing Them as Embedded in Infinite Cylinders





$$\boldsymbol{\Omega}_{ij}^e(\mathbf{r}) \propto \hat{\ell}_{k\ell} \times \boldsymbol{\rho}_{ij} \quad \Rightarrow$$

$$\boxed{\begin{aligned} \boldsymbol{\Omega}_{ij}^e(\mathbf{r}) &= \ell_{ij}\left(\xi_i \nabla \xi_j - \xi_j \nabla \xi_i\right), \\ \nabla \times \boldsymbol{\Omega}_{ij}^e(\mathbf{r}) &= 2\ell_{ij}\left(\nabla \xi_i \times \nabla \xi_j\right), \\ & j > i, \quad \mathbf{r} \in V^e \end{aligned}}$$

Demonstration of Interpolatory Properties of Curl-Conforming Bases

$$\mathbf{\Omega}_{ij}^{e}(\mathbf{r}) = \ell_{ij}\left(\xi_i \nabla \xi_j - \xi_j \nabla \xi_i\right)$$

- $\nabla\xi_i$ **is** $\perp$ **to face** i **and hence to edges** jk, $k\ell$, ℓj

$$\Rightarrow \mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_i=0}\cdot\hat{\ell}_{jk} = \mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_i=0}\cdot\hat{\ell}_{k\ell} = \mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_i=0}\cdot\hat{\ell}_{\ell j} = 0$$

- $\nabla\xi_j$ **is** $\perp$ **to face** j **and hence to edges** $i\ell$, ℓk, ki

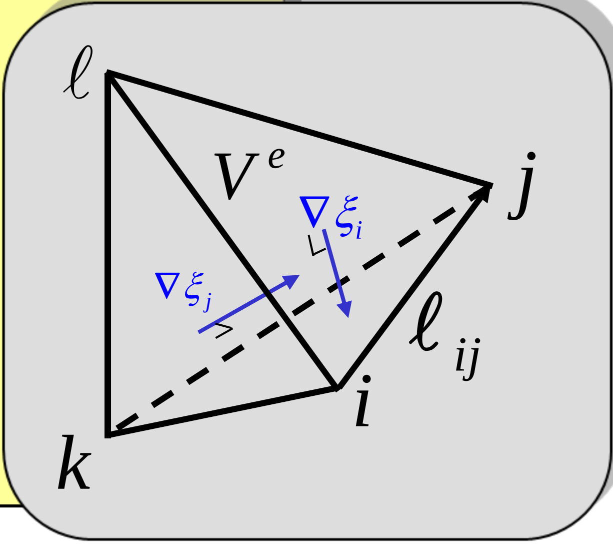
$$\Rightarrow \mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_j=0}\cdot\hat{\ell}_{il} = \mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_j=0}\cdot\hat{\ell}_{\ell k} = \mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_j=0}\cdot\hat{\ell}_{ki} = 0$$

- **Finally,**

$$\mathbf{\Omega}_{ij}^{e}(\mathbf{r})\Big|_{\xi_k=\xi_\ell=0}\cdot\hat{\ell}_{ij} = \ell_{ij}\hat{\ell}_{ij}\cdot\left(\xi_i\nabla\xi_j - \xi_j\nabla\xi_i\right)$$

$$= -\ell_{ij}\cdot\left(\xi_i\frac{\hat{\mathbf{h}}_j}{h_j} - \xi_j\frac{\hat{\mathbf{h}}_i}{h_i}\right)\Bigg|_{\xi_k=\xi_\ell=0} = \xi_i\frac{\cancel{h_j}}{\cancel{h_j}} + \xi_j\frac{\cancel{h_i}}{\cancel{h_i}}$$

$$= \xi_i + \xi_j = 1$$



Discretized Equations Obtained by Substituting Representations for Electric and Surface Magnetic Fields

$$\mathbf{E} \approx = \sum_{n=1}^{N_V} \underbrace{V_n^V}_{\text{volume DoFs}} \boldsymbol{\Omega}_n(\mathbf{r}) + \sum_{n=1}^{N_S} \underbrace{V_n^S}_{\text{surface DoFs}} \boldsymbol{\Omega}_{N_V+n}(\mathbf{r})$$

$$\mathbf{J} \equiv \hat{\mathbf{n}} \times \mathbf{H} \approx \sum_{n=1}^{N_S} I_n^S \boldsymbol{\Omega}_n^S(\mathbf{r})$$

$$\Rightarrow$$

$$\left[Y_{mn}^{VV}\right]\left[V_n^V\right] + \left[Y_{mn}^{VS}\right]\left[V_n^S\right] = \left[I_m^{-V}\right]$$

$$\left[Y_{mn}^{SV}\right]\left[V_n^V\right] + \left[Y_{mn}^{SS}\right]\left[V_n^S\right] + \left[\gamma_{mn}^{SS}\right]\left[I_n^S\right] = \left[I_m^{-S}\right]$$

- 2 equation blocks,
  3 vector unknown blocks
- If S is not PEC or PMC, need another block of equations to determine I_n^S and supply *radiation conditions*

$\mathcal{S}$ is PEC

$\mathcal{S}$ is PMC

PEC: $\hat{\mathbf{n}} \times \mathbf{E}\big|_S = 0 \Rightarrow \left[V_n^S\right] = 0$

PMC: $\hat{\mathbf{n}} \times \mathbf{H}\big|_S = 0 \Rightarrow \left[I_n^S\right] = 0$

Matrix Block Definitions

$$Y_{mn}^{VV} = \frac{1}{j\omega\mu_0} < \nabla\times\boldsymbol{\Omega}_m, \mu_r^{-1}\nabla\times\boldsymbol{\Omega}_n > + \; j\omega\varepsilon_0 < \boldsymbol{\Omega}_m, \varepsilon_r\boldsymbol{\Omega}_n >$$

$$Y_{mn}^{VS} = \frac{1}{j\omega\mu_0} < \nabla\times\boldsymbol{\Omega}_m, \mu_r^{-1}\nabla\times\boldsymbol{\Omega}_{N_V+n} > + \; j\omega\varepsilon_0 < \boldsymbol{\Omega}_m, \varepsilon_r\boldsymbol{\Omega}_{N_V+n} >$$

$$Y_{mn}^{SV} = \frac{1}{j\omega\mu_0} < \nabla\times\boldsymbol{\Omega}_{N_V+m}, \mu_r^{-1}\nabla\times\boldsymbol{\Omega}_n > + \; j\omega\varepsilon_0 < \boldsymbol{\Omega}_{N_V+m}, \varepsilon_r\boldsymbol{\Omega}_n >$$

$$Y_{mn}^{SS} = \frac{1}{j\omega\mu_0} < \nabla\times\boldsymbol{\Omega}_{N_V+m}, \mu_r^{-1}\nabla\times\boldsymbol{\Omega}_{N_V+n} > + \; j\omega\varepsilon_0 < \boldsymbol{\Omega}_{N_V+m}, \varepsilon_r\boldsymbol{\Omega}_{N_V+n} >$$

Volume integrals!

$$\gamma_{mn}^{SS} = - < \boldsymbol{\Omega}_{N_V+m}, \boldsymbol{\Omega}_n^S > \;\; = - < \boldsymbol{\Omega}_m^S, \boldsymbol{\Omega}_n^S > = - < \boldsymbol{\Lambda}_m^S, \boldsymbol{\Lambda}_n^S >$$

Surface integral!

Index ranges:

$${}_p^V \Rightarrow p = 1, 2, \ldots, N_V$$

$${}_p^S \Rightarrow p = 1, 2, \ldots, N_S$$

$$\boldsymbol{\Lambda}_m^S = \boldsymbol{\Omega}_m^S \times \hat{\mathbf{n}}$$

$$-\hat{\mathbf{n}}\times\hat{\mathbf{n}}\times\boldsymbol{\Omega}_{N_V+m}\Big|_{\mathbf{r}\in\mathcal{S}} \overset{!}{=} \boldsymbol{\Omega}_m^S$$

$$I_m^{-V} = - < \boldsymbol{\Omega}_m, \mathbf{J}^- >$$

$$I_m^{-S} = - < \boldsymbol{\Omega}_{N_V+m}, \mathbf{J}^- >$$

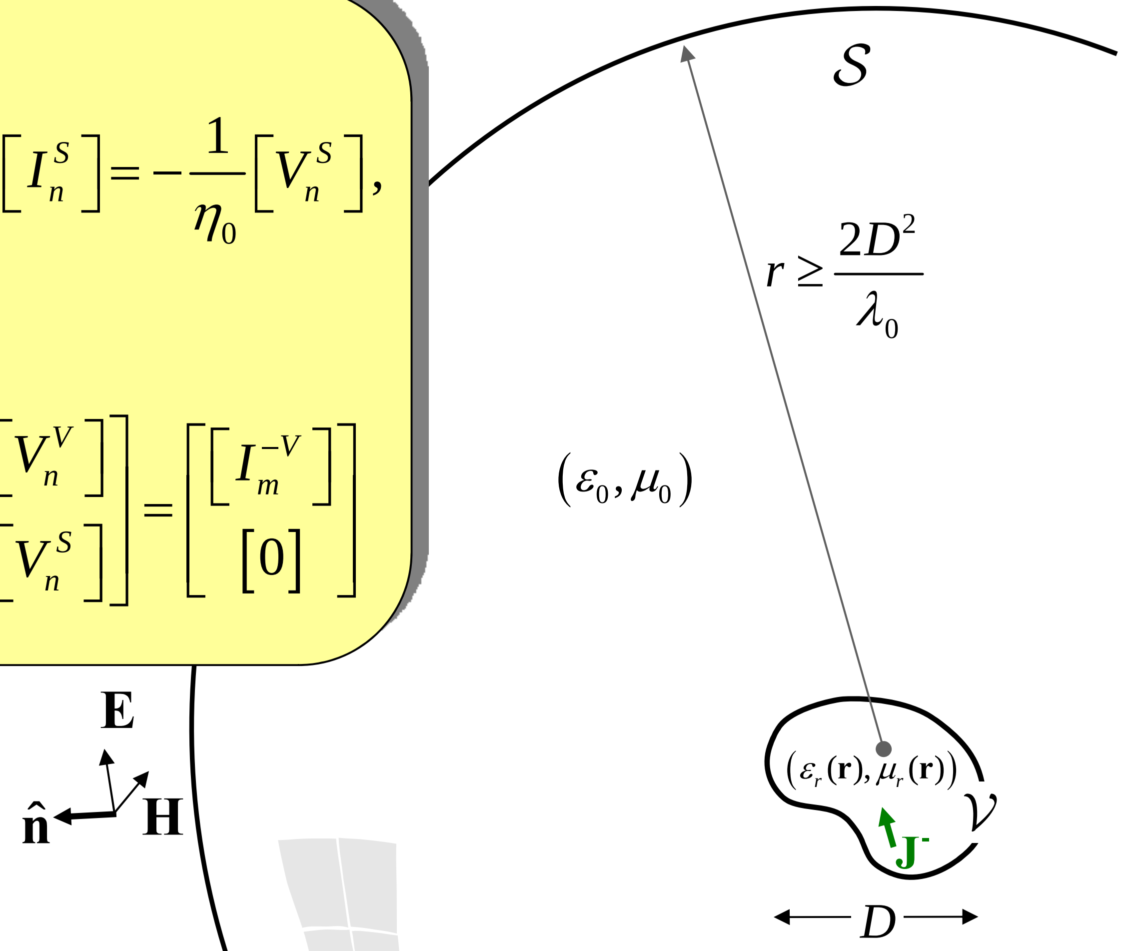
In Far Field, We Also Know the *Ratio* of Tangential Surface Electric and Magnetic Fields …

In the far field,

$$\hat{\mathbf{r}} \times \mathbf{H} = \hat{\mathbf{n}} \times \mathbf{H} = -\frac{\mathbf{E}_{\tan}}{\eta_0} \Rightarrow \left[I_n^S \right] = -\frac{1}{\eta_0} \left[V_n^S \right],$$

$$\mathbf{J}^- \Big|_S = 0 \;\; \Rightarrow \left[I_m^{-S} \right] = 0 :$$

$$\Rightarrow \begin{bmatrix} \left[Y_{mn}^{VV} \right] & \left[Y_{mn}^{VS} \right] \\ \left[Y_{mn}^{SV} \right] & \left[Y_{mn}^{SS} - \dfrac{\gamma_{mn}^{SS}}{\eta_0} \right] \end{bmatrix} \begin{bmatrix} \left[V_n^V \right] \\ \left[V_n^S \right] \end{bmatrix} = \begin{bmatrix} \left[I_m^{-V} \right] \\ [0] \end{bmatrix}$$



Note we *also* know their ratio for *surface impedances*:

$$\hat{\mathbf{n}} \times \mathbf{H} = -\frac{\mathbf{E}_{\tan}}{Z_s} \Rightarrow \left[I_n^S \right] = -\frac{1}{Z_s} \left[V_n^S \right]$$

Mesh fills scatterer and surrounding region *out to the far field!*

…Or Null Field Condition Provides Add'l Eq. at Boundary

Expressing the magnetic null field condition (see PMCHWT formulation) using equivalent currents and potentials provides the missing equation:

$$-\hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{sc}}[\mathbf{J}, \mathbf{M}] = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{inc}}, \quad \mathbf{r} \uparrow S$$

$$\Rightarrow \ \frac{\mathbf{J}}{2} - \frac{1}{\mu_0} \hat{\mathbf{n}} \times \int_S \nabla \times \mathcal{G}^A(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, dS' + j\omega\varepsilon_0\, \hat{\mathbf{n}} \times \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{M}(\mathbf{r}')\, dS'$$

$$-\frac{1}{j\omega\mu_0} \hat{\mathbf{n}} \times \nabla \int_S G(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{M}(\mathbf{r}')\, dS' = \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{inc}}$$

Relate equivalent currents to fields of the Helmholtz eq.:

$$\mathbf{J} \equiv \hat{\mathbf{n}} \times \mathbf{H} = \sum_{n=1}^{N_S} I_n^S\, \mathbf{\Omega}_n^S(\mathbf{r}), \quad \mathbf{M} \equiv \mathbf{E} \times \hat{\mathbf{n}} = \sum_{n=1}^{N_S} V_n^S\, \mathbf{\Lambda}_n^S(\mathbf{r})$$

Substituting into above and testing with Ω_n^S yields final matrix equation:

$$\Rightarrow \left[\widehat{\beta}_{mn}^{SS}\right]\left[I_n^S\right] + \left[\frac{Z_{mn}^{SS}}{\eta_0^2}\right]\left[V_n^S\right] = \left[I_m^{\mathrm{inc}}\right]$$

Homogeneous background medium is assumed to be free space

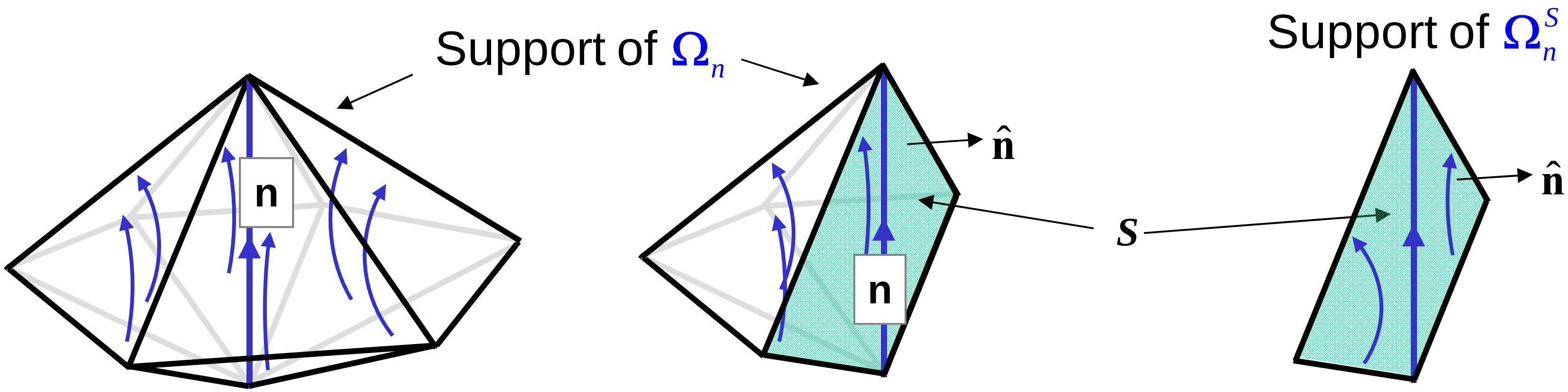
Integral Equation Matrix Block Definitions

$$\frac{Z_{mn}^{SS}}{\eta_0^2} = j\omega\varepsilon_0 < \boldsymbol{\Lambda}_m^S; G, \boldsymbol{\Lambda}_n^S > + \frac{1}{j\omega\mu_0} < \nabla\cdot\boldsymbol{\Lambda}_m^S, G, \nabla\cdot\boldsymbol{\Lambda}_n^S >, \ m,n = 1,\ldots,N_S$$

$$\begin{aligned}\widehat{\beta}_{mn}^{SS} &= \frac{1}{2} < \boldsymbol{\Omega}_m^S, \boldsymbol{\Omega}_n^S > - \frac{1}{\mu_0} < \boldsymbol{\Omega}_m^S; \hat{\mathbf{n}}\times\nabla\times\mathcal{G}^A; \boldsymbol{\Omega}_n^S > \\ &= -\frac{1}{2}\gamma_{mn}^{SS} - \frac{1}{\mu_0} < \boldsymbol{\Omega}_m^S; \hat{\mathbf{n}}\times\nabla\times\mathcal{G}^A; \boldsymbol{\Omega}_n^S >, \quad m,n = 1,\ldots,N_S,\end{aligned}$$

$$I_m^{\text{inc}} = < \boldsymbol{\Lambda}_m^S, \mathbf{H}^{\text{inc}} >$$

Pictorial Representation of Surface Bases, Interior and Surface Volume Bases



$\boldsymbol{\Omega}_n, \quad n \le N_V$

$\boldsymbol{\Omega}_n, \quad N_V + 1 \le n \le N_V + N_S$

$\boldsymbol{\Omega}_n^S, \quad n \le N_S$

$$\boldsymbol{\Lambda}_m^S = \boldsymbol{\Omega}_m^S \times \hat{\mathbf{n}}$$

$$-\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \boldsymbol{\Omega}_{N_V+m}\Big|_{\mathbf{r}\in\mathcal{S}} = \boldsymbol{\Omega}_m^S$$

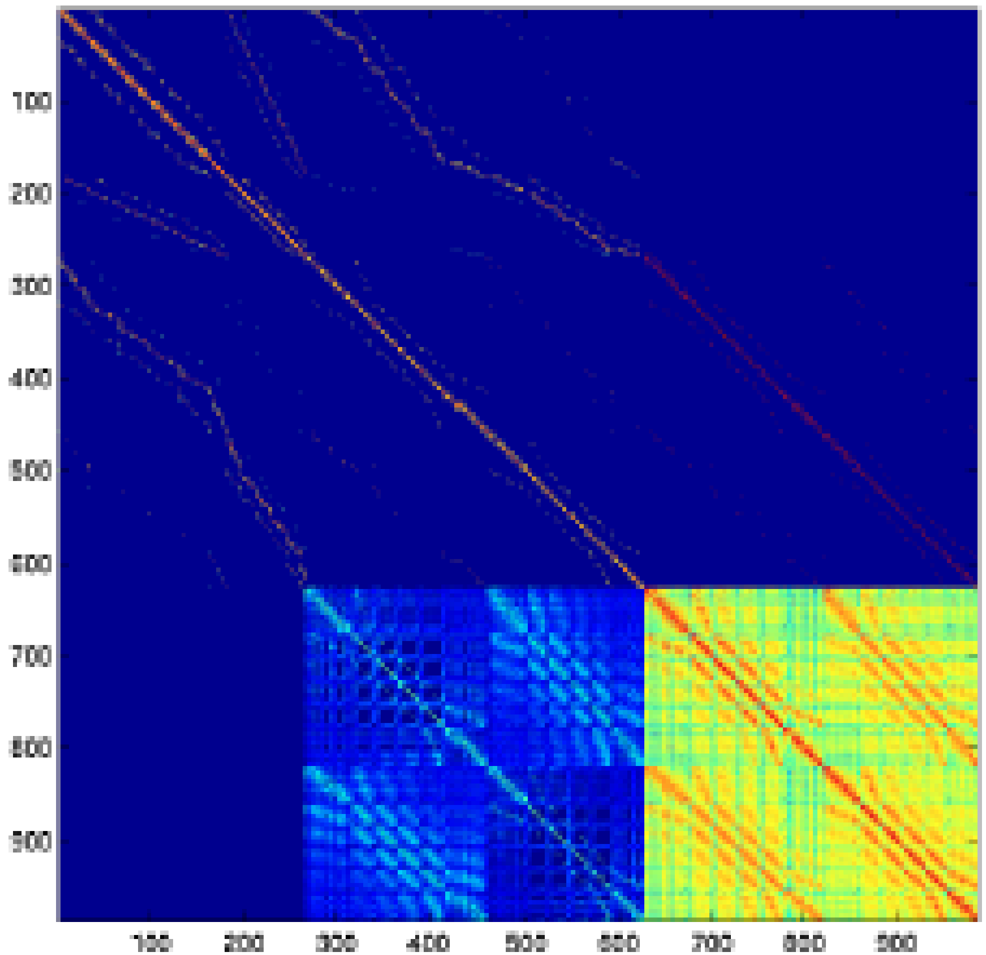
$$\mathbf{E} \approx \; = \sum_{n=1}^{N_V} \underbrace{V_n^V}_{\text{volume DoFs}} \boldsymbol{\Omega}_n(\mathbf{r}) + \sum_{n=1}^{N_S} \underbrace{V_n^S}_{\text{surface DoFs}} \boldsymbol{\Omega}_{N_V+n}(\mathbf{r})$$

$$\mathbf{J} \equiv \hat{\mathbf{n}} \times \mathbf{H} \approx \sum_{n=1}^{N_S} I_n^S \boldsymbol{\Omega}_n^S(\mathbf{r})$$

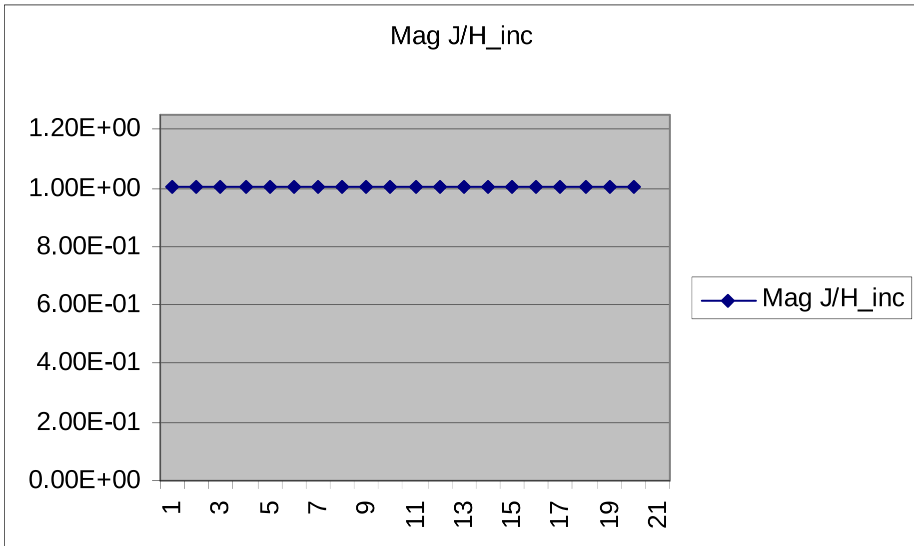
Complete System of Coupled FEM and Integral (Hybrid) Equations

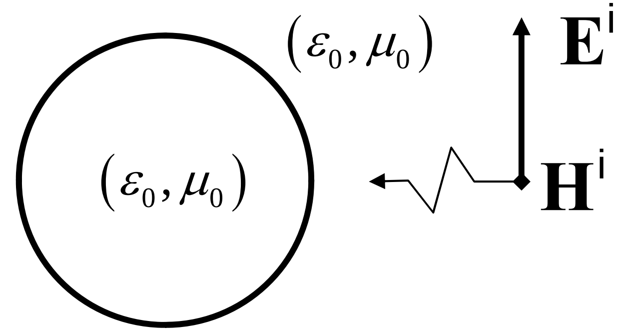
$$\begin{bmatrix} \left[Y_{mn}^{VV} \right] & \left[Y_{mn}^{VS} \right] & [0] \\ \left[Y_{mn}^{SV} \right] & \left[Y_{mn}^{SS} \right] & \left[\gamma_{mn}^{SS} \right] \\ [0] & \left[Z_{mn}^{SS} / \eta_0^2 \right] & \left[\widehat{\beta}_{mn}^{SS} \right] \end{bmatrix} \begin{bmatrix} \left[V_n^V \right] \\ \left[V_n^S \right] \\ \left[I_n^S \right] \end{bmatrix} = \begin{bmatrix} \left[I_m^{-,V} \right] \\ \left[I_m^{-,S} \right] \\ \left[I_m^{\mathrm{inc}} \right] \end{bmatrix}$$

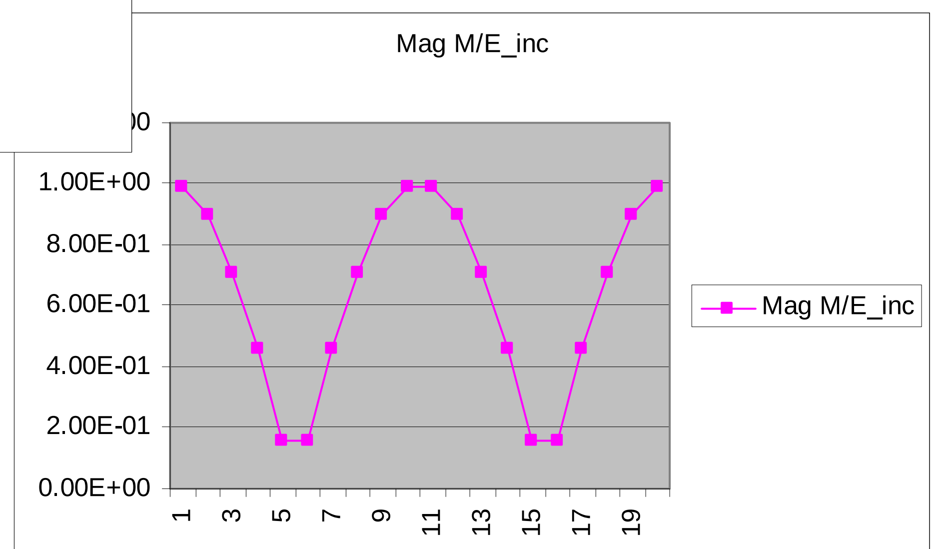
Typical Matrix Structure for Hybrid Problems



Check: Reduce Medium to Free Space







The End