EFIE in 3-D: Rectangular and Triangular Surface Patch Modeling

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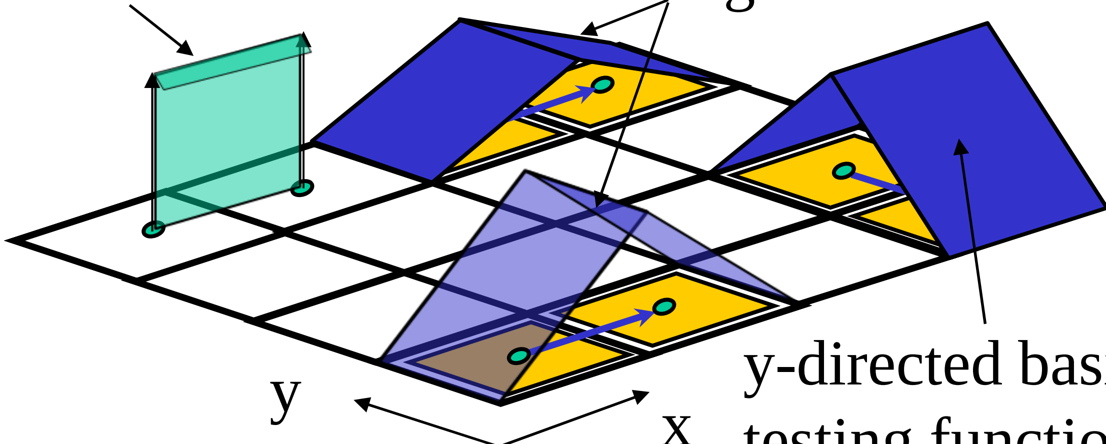
New Features of 3D Electric Field Integral Equation (EFIE)

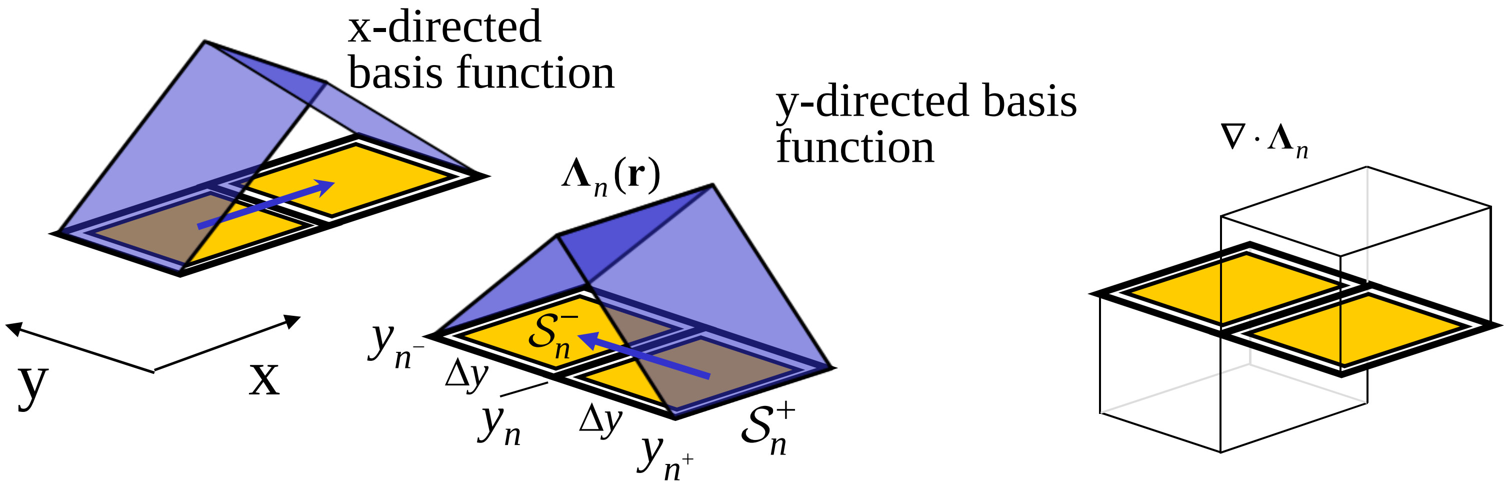
- Dynamic, 3D Green's function
- Vector basis and testing functions defined on triangles

On Rectangular Elements, Rooftop Bases Provide Good Compromise Between Simplicity and Effectiveness



- Normal current components are continuous, even at bends (*div-conforming*)
- Tangential current components are discontinuous
- Piecewise constant charge representation
- Current vanishes at plate edges
- Charge, current qualitatively satisfy edge and corner conditions
- Alternative “razor blade” testing stays away from edges

Rooftop Bases Model Surface Charge Density as Piecewise Constant



For y-directed bases, for instance:

$$\mathbf{\Lambda}_n(\mathbf{r}) = \hat{\mathbf{y}}\,\Lambda_n(y) = \begin{cases} \hat{\mathbf{y}}\,\dfrac{\left|y - y_{n^\pm}\right|}{\Delta y}, & \mathbf{r} \in \mathcal{S}_n^\pm \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

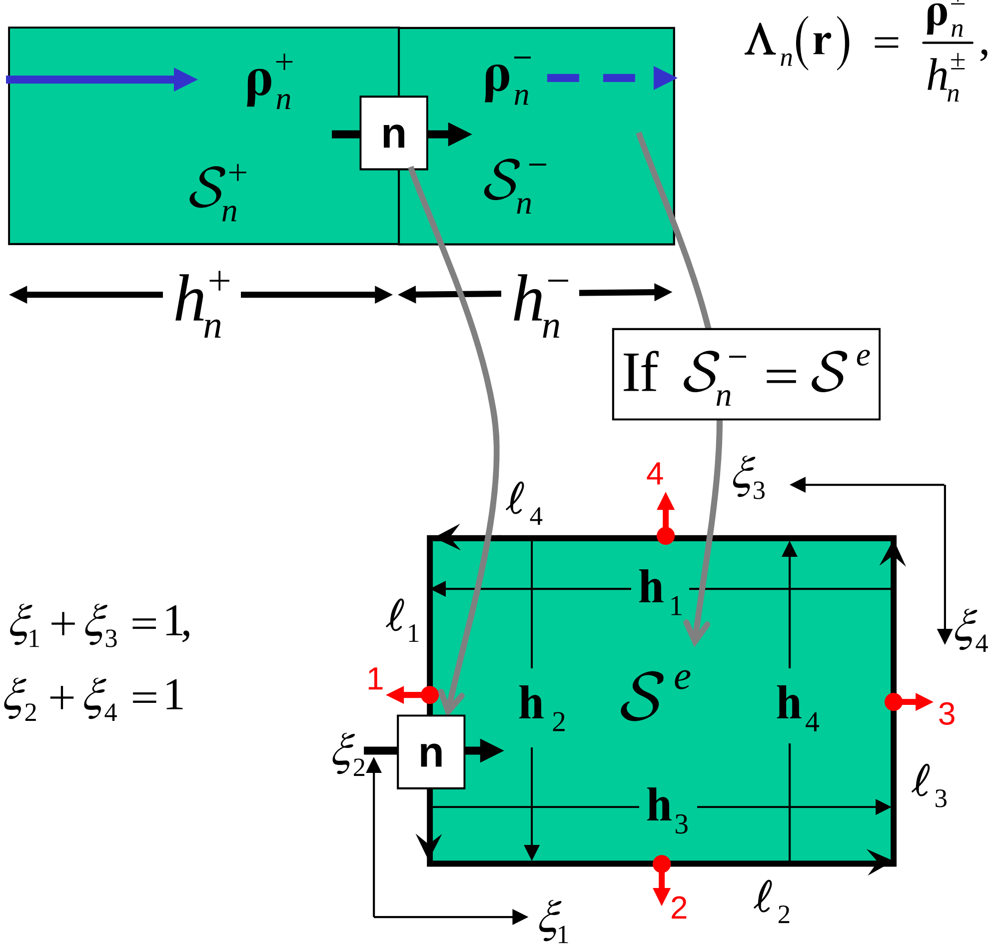
$$\nabla \cdot \mathbf{\Lambda}_n = \hat{\mathbf{y}} \cdot \nabla \Lambda_n = \frac{d\Lambda_n}{dy} = \begin{cases} \dfrac{\pm 1}{\Delta y}, & \mathbf{r} \in \mathcal{S}_n^\pm \\ 0, & \text{otherwise} \end{cases}$$

Interpolation properties of y-directed bases:

$$\hat{\mathbf{y}} \cdot \mathbf{\Lambda}_n(\mathbf{r})\Big|_{\substack{x = x_m \\ y = y_m}} = \delta_{mn}$$

$$\hat{\mathbf{x}} \cdot \mathbf{\Lambda}_n(\mathbf{r})\Big|_{\substack{x = x_n \\ y = y_n}} = 0$$

Global and Local Bases on Rectangular Elements



$$\xi_1 + \xi_3 = 1,$$
$$\xi_2 + \xi_4 = 1$$

Global definition:

$$\boldsymbol{\Lambda}_n(\mathbf{r}) = \frac{\boldsymbol{\rho}_n^\pm}{h_n^\pm}, \ \mathbf{r} \in \mathcal{S}_n^\pm$$

Local definition, $\mathbf{r} \in \mathcal{S}^e$:

$$\boldsymbol{\Lambda}_1^e(\mathbf{r}) = \frac{\xi_3 \, \boldsymbol{\ell}_4}{h_1} \equiv -\boldsymbol{\Lambda}_n(\mathbf{r}),$$

$$\boldsymbol{\Lambda}_2^e(\mathbf{r}) = \frac{\xi_4 \, \boldsymbol{\ell}_1}{h_2} \equiv \pm\boldsymbol{\Lambda}_?(\mathbf{r}), \text{etc.}$$

$$\boldsymbol{\Lambda}_3^e(\mathbf{r}) = \frac{\xi_1 \, \boldsymbol{\ell}_2}{h_3},$$

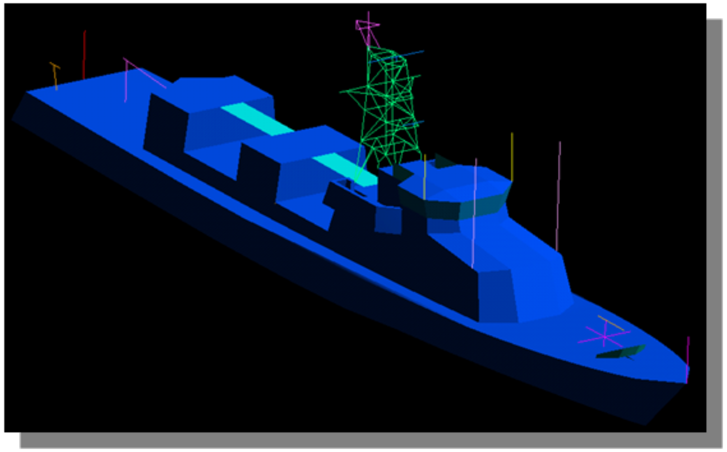
$$\boldsymbol{\Lambda}_4^e(\mathbf{r}) = \frac{\xi_2 \, \boldsymbol{\ell}_3}{h_4}$$

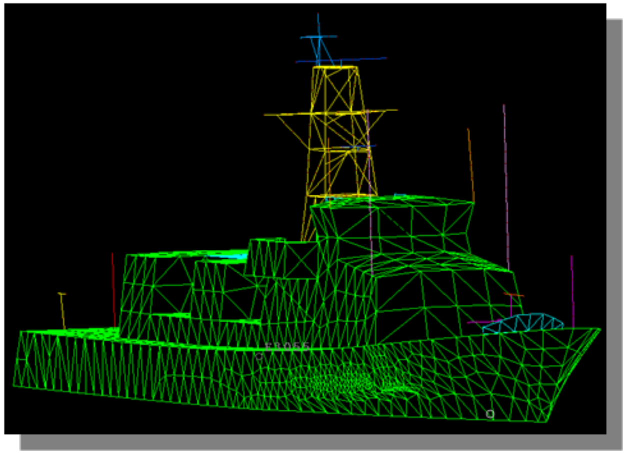
$$\Rightarrow \boldsymbol{\Lambda}_i^e(\mathbf{r}) = \frac{\xi_{i+2} \, \boldsymbol{\ell}_{i-1}}{h_i}$$

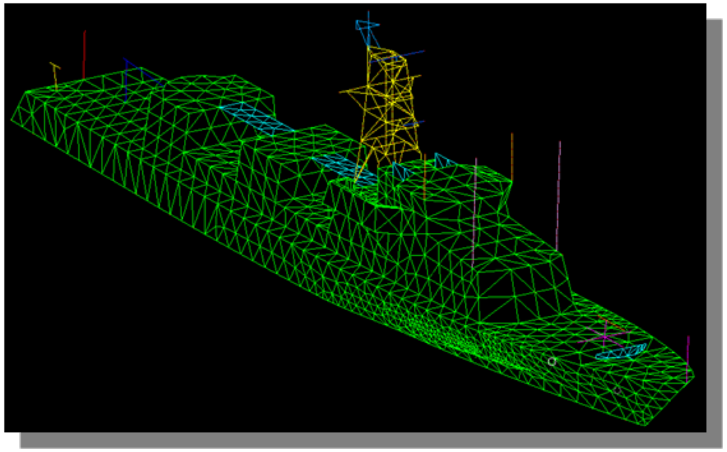
(Index arithmetic performed modulo 4)

But Modern Problems Require the Flexibility of Triangular Surface Patch Modeling

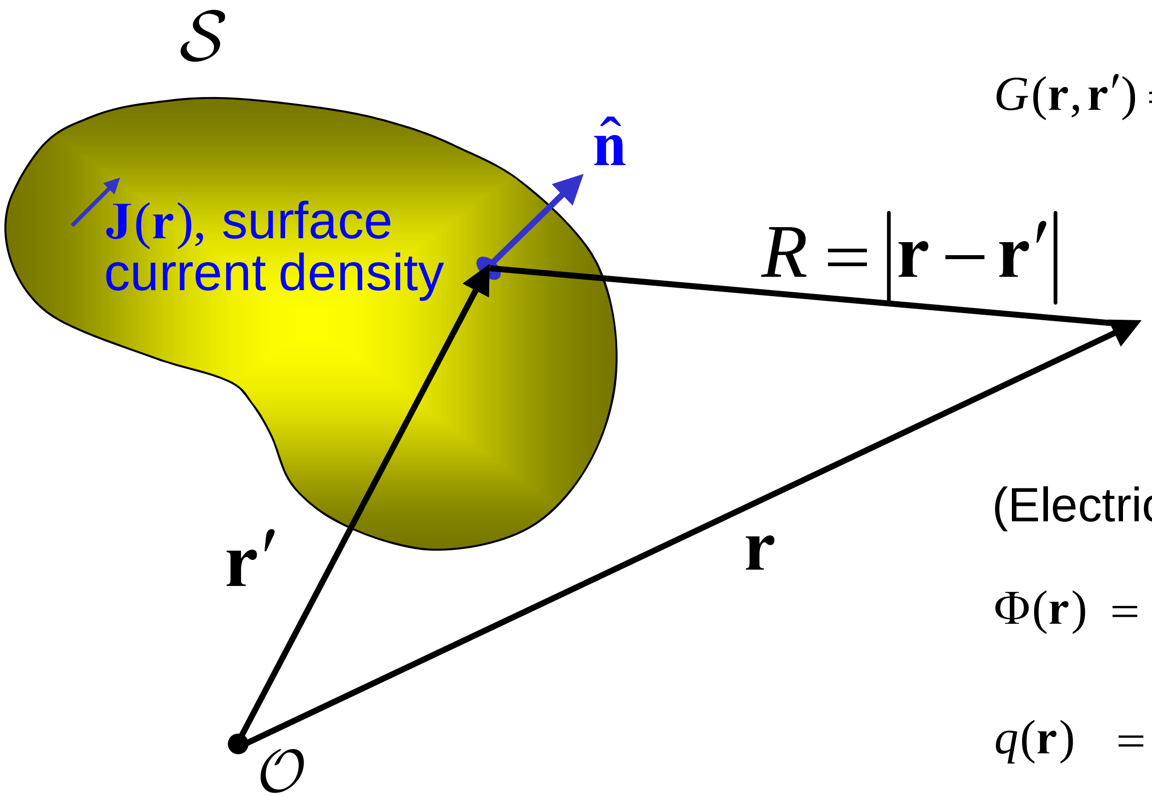
Cyclone Class Patrol Craft, PC-1







Definitions of Geometrical and Electrical Quantities for Current on a *Surface*



(Magnetic) Vector Potential:

$$\mathbf{A}(\mathbf{r}) = \mu \int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\,\mathbf{J}(\mathbf{r}')\,d\mathcal{S}',$$

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-jkR}}{4\pi R} \quad \text{(Green's function)}$$

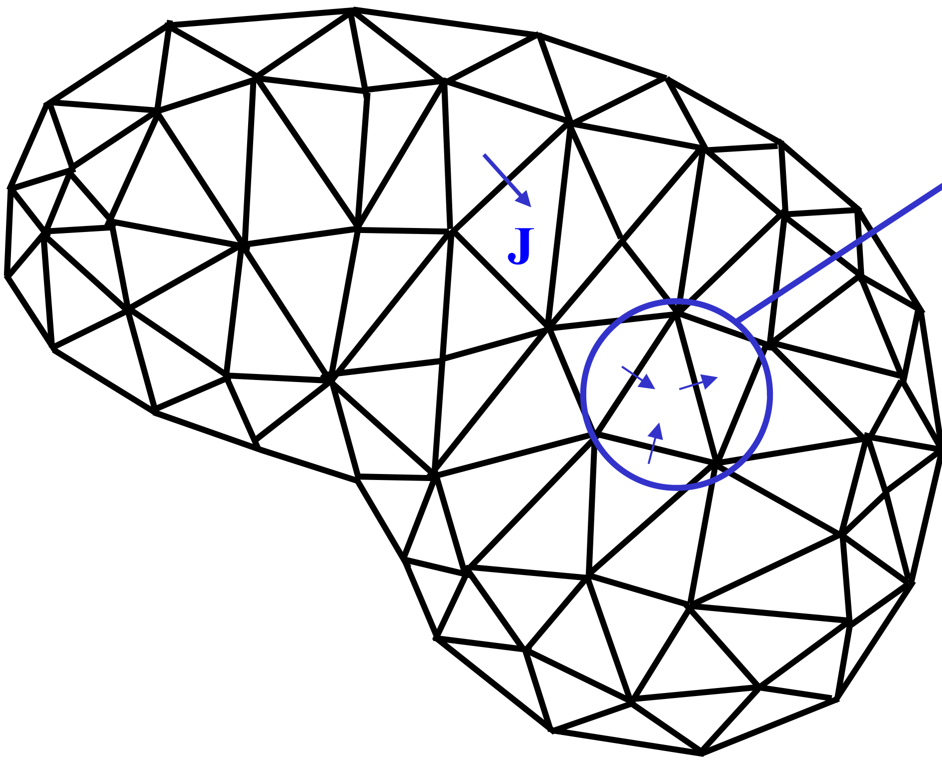
(Electric) Scalar Potential:

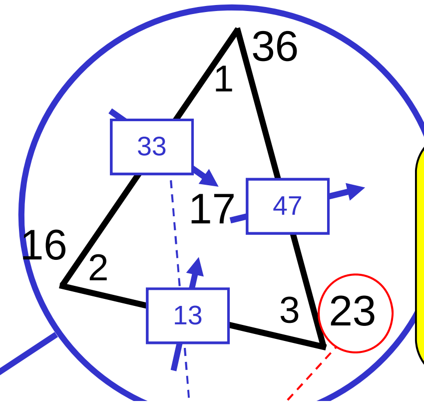
$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon} \int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\; q(\mathbf{r}')\,d\mathcal{S}',$$

$$q(\mathbf{r}) = \frac{1}{-j\omega} \nabla \cdot \mathbf{J}(\mathbf{r}) \quad \text{(continutity eq.)}$$

Surface Discretization

$$\mathcal{S} \approx \tilde{\mathcal{S}} = \bigcup_{e=1}^{E(=N)} \mathcal{S}^e$$





Note:

$$\underbrace{\mathbf{r}_{23}}_{\text{global}} = \underbrace{\mathbf{r}_3^{17}}_{\text{local}}$$

- **A *Global Node list* defines vertex locations**

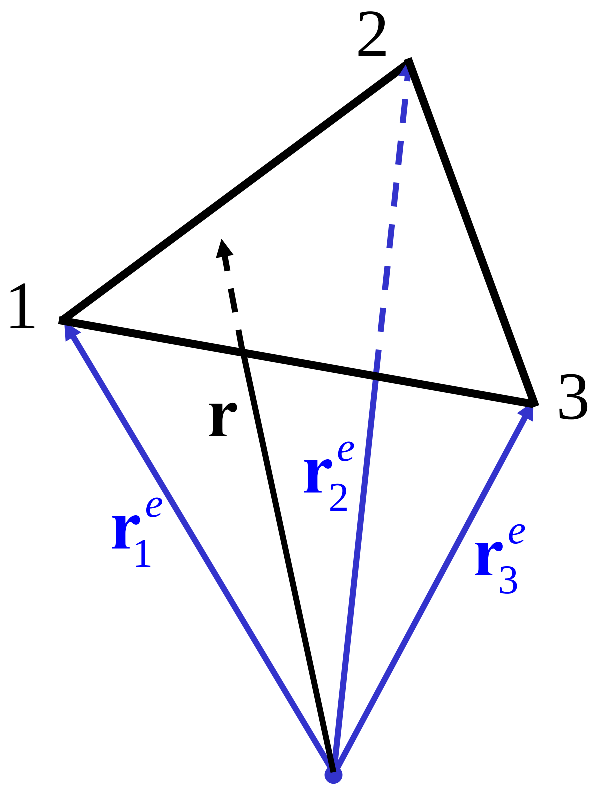
Node #	x	y	z
… 23 …	… x_{23} …	… y_{23} …	… z_{23} …

- **An *element list* contains both global node and DoF numbers**

Element e	Global node number/ DoF number		
	1	2	3
⋮ 17 ⋮	⋮ 36 / −13 ⋮	⋮ 16 / 47 ⋮	⋮ 23 / $\boxed{-33}$ ⋮

DoF's are current density components normal to triangle edges

Triangular Surface Patches



$$\mathbf{r} = \xi_1 \mathbf{r}_1^e + \xi_2 \mathbf{r}_2^e + \xi_3 \mathbf{r}_3^e$$

- **Geometry modeling is a straight-forward extension to 2D modeling**
- **EFIE has the form**

$$-\mathbf{E}_{\tan}^{\mathrm{s}}(\mathbf{J}) = \mathbf{E}_{\tan}^{\mathrm{i}}$$

- **Expand the current in div-conforming bases $\Lambda_n(\mathbf{r})$,**

$$\mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} I_n \Lambda_n(\mathbf{r})$$

and use them for testing functions.

3D EFIE Formulation

$$-\mathbf{E}^{\mathrm{s}}_{\tan} = j\omega\mathbf{A}_{\tan} + \nabla_{\tan}\Phi = \mathbf{E}^{\mathrm{i}}_{\tan}, \quad \mathbf{r}\in\mathcal{S}$$

$$\Rightarrow \left[j\omega\mu\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}' - \frac{1}{j\omega\varepsilon}\nabla\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{J}(\mathbf{r}')d\mathcal{S}' \right]_{\tan} = \mathbf{E}^{\mathrm{i}}_{\tan},$$

Test with $\Lambda_m(\mathbf{r})$, to obtain the weak form

$$j\omega<\Lambda_m;\mathbf{A}> + <\Lambda_m;\nabla\Phi> = <\Lambda_m;\mathbf{E}^{\mathrm{i}}>$$

where

$$\mathbf{A} = \mu\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathcal{S}', \quad \Phi = \frac{1}{-j\omega\varepsilon}\int_{\mathcal{S}} G(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{J}(\mathbf{r}')d\mathcal{S}',$$

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{-jkR}}{4\pi R}, \qquad <\mathbf{A};\mathbf{B}> = \int_{\mathcal{S}} \mathbf{A}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r})d\mathcal{S}$$

or

$$j\omega\mu<\Lambda_m;G,\mathbf{J}> + \frac{1}{-j\omega\varepsilon}<\Lambda_m;\nabla G,\nabla\cdot\mathbf{J}> = <\Lambda_m;\mathbf{E}^{\mathrm{i}}>$$

Integration by Parts

$$j\omega < \Lambda_m; \mathbf{A} > + < \Lambda_m; \nabla\Phi > \; = < \Lambda_m; \mathbf{E}^{\mathrm{i}} >$$

Using $\nabla\cdot(\Lambda_m\Phi) = \Phi\nabla\cdot\Lambda_m + \Lambda_m\cdot\nabla\Phi \;\Rightarrow$

$$\int_{\mathcal{S}} \nabla\cdot(\Lambda_m\Phi)\, d\mathcal{S} \overset{\text{divergence theorem}}{=} \int_{\partial\mathcal{S}} \Phi\Lambda_m\cdot\hat{\mathbf{u}}\, d\mathcal{C} = \underbrace{\int_{\mathcal{S}} \Phi\nabla\cdot\Lambda_m\, d\mathcal{S}}_{<\nabla\cdot\Lambda_m,\Phi>} + \underbrace{\int_{\mathcal{S}} \Lambda_m\cdot\nabla\Phi\, d\mathcal{S}}_{<\Lambda_m;\nabla\Phi>}$$

$$\Rightarrow \; < \Lambda_m; \nabla\Phi > \; = \; - < \nabla\cdot\Lambda_m, \Phi > \; + \int_{\partial\mathcal{S}} \Phi \; \underbrace{\Lambda_m\cdot\hat{\mathbf{u}}}_{\substack{=0 \text{ on } \partial\mathcal{S}, \\ \text{cancels on } \partial\mathcal{S}^e}} \; d\mathcal{S}$$

Hence the weak form becomes

$$\boxed{j\omega < \Lambda_m; \mathbf{A} > - < \nabla\cdot\Lambda_m, \Phi > \; = < \Lambda_m; \mathbf{E}^{\mathrm{i}} >}$$

or

$$\boxed{j\omega\mu < \Lambda_m; G, \mathbf{J} > + \frac{1}{j\omega\varepsilon} < \nabla\cdot\Lambda_m; G, \nabla\cdot\mathbf{J} > \; = < \Lambda_m; \mathbf{E}^{\mathrm{i}} >}$$

EFIE MoM Formulation

Setting $\mathbf{J}(\mathbf{r}') = \sum_n I_n \Lambda_n(\mathbf{r}')$, **and substituting yields**

$$\left[Z_{mn}\right]\left[I_n\right] = \left[V_m\right]$$

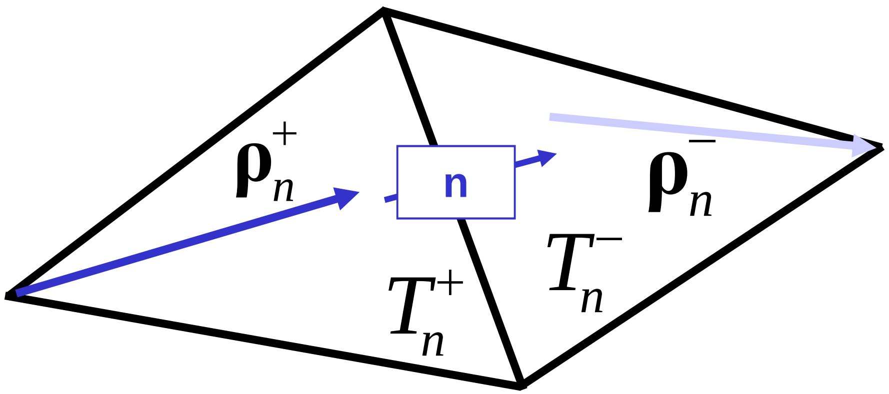
where $\left[Z_{mn}\right] = j\omega\left[L_{mn}\right] + \frac{1}{j\omega}\left[S_{mn}\right]$ **and**

$$L_{mn} = \mu \int_{\mathcal{S}} \int_{\mathcal{S}} G\left(\mathbf{r},\mathbf{r}'\right) \Lambda_m(\mathbf{r}) \cdot \Lambda_n(\mathbf{r}') d\mathcal{S}' d\mathcal{S} \quad \equiv \quad \mu < \Lambda_m; G, \Lambda_n >$$

$$S_{mn} = \frac{1}{\varepsilon} \int_{\mathcal{S}} \int_{\mathcal{S}} \nabla \cdot \Lambda_m(\mathbf{r})\, G\left(\mathbf{r},\mathbf{r}'\right) \nabla' \cdot \Lambda_n(\mathbf{r}') d\mathcal{S}' d\mathcal{S} \quad \equiv \quad \frac{1}{\varepsilon} < \nabla \cdot \Lambda_m, G, \nabla \cdot \Lambda_n >,$$

$$V_m \;=\; < \Lambda_m; \mathbf{E}^{\mathrm{i}} >, \quad G\left(\mathbf{r},\mathbf{r}'\right) \;=\; \frac{e^{-jkR}}{4\pi R}, \quad R \;=\; \left|\mathbf{r} - \mathbf{r}'\right|$$

Basis Functions for Surface Currents on Triangular Elements (Global Representation)

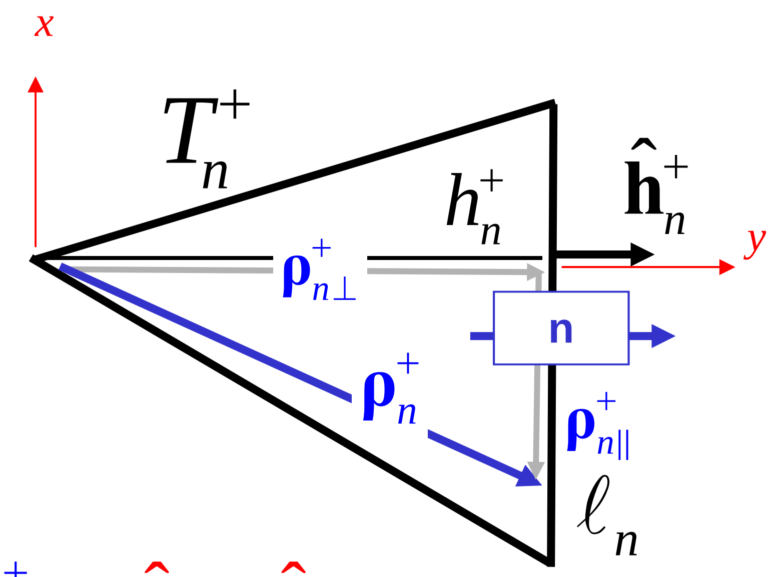


Global basis definition:

$$\boldsymbol{\Lambda}_n(\mathbf{r}) = \begin{cases} \dfrac{\boldsymbol{\rho}_n^\pm}{h_n^\pm}, & \mathbf{r} \in T_n^\pm \\ \mathbf{0}, & \mathbf{r} \notin T_n^\pm \end{cases}$$

- Rao, S.S.M., D.R. Wilton, and A.W. Glisson, "Electromagnetic Scattering by Surfaces of Arbitrary Shape," *IEEE Trans. Antennas and Propagation*, AP-30, No. 3, pp. 409-4l8, May l982.

Interpolation and Divergence Properties



$$\boldsymbol{\rho}_n^+ = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

Interpolation property:

$$\hat{\mathbf{h}}_n^+ \cdot \boldsymbol{\Lambda}_n \Big|_{\mathbf{r}\,\in\,\textbf{edge}\,n} = \frac{\hat{\mathbf{h}}_n^+ \cdot \boldsymbol{\rho}_n^+}{h_n^+}$$

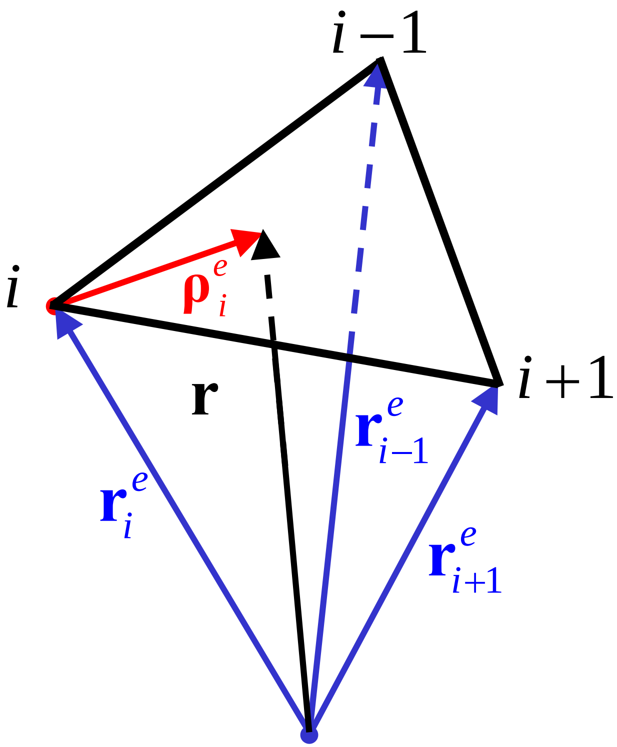
$$= \frac{\hat{\mathbf{h}}_n^+ \cdot \boldsymbol{\rho}_{n\perp}^+}{h_n^+} = \frac{\cancel{h_n^+}}{\cancel{h_n^+}} = 1,$$

$$\hat{\mathbf{h}}_m^\pm \cdot \boldsymbol{\Lambda}_n \Big|_{\mathbf{r}\,\in\,\textbf{edge}\,m} = 0,\; m \neq n$$

Divergence property:

$$\nabla \cdot \boldsymbol{\Lambda}_n = \frac{\nabla \cdot \boldsymbol{\rho}_n^\pm}{h_n^\pm} = \frac{\pm\nabla \cdot \left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right)}{h_n^\pm} = \pm\frac{2}{h_n^\pm}, \quad \mathbf{r} \in T_n^\pm$$

Local Representation of Basis Functions for for Triangular Elements



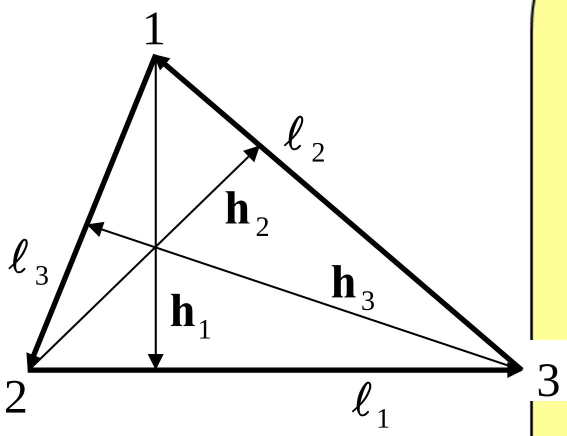
$$\mathbf{r} = \xi_i \mathbf{r}_i^e + \xi_{i+1}\mathbf{r}_{i+1}^e + \xi_{i-1}\mathbf{r}_{i-1}^e$$

Local basis function:

$$\begin{aligned}
\Lambda_i^e(\mathbf{r}) &= \frac{\boldsymbol{\rho}_i^e}{h_i} = \frac{\mathbf{r}-\mathbf{r}_i^e}{h_i} \\
&= \frac{\xi_i \mathbf{r}_i^e + \xi_{i+1}\mathbf{r}_{i+1}^e + \xi_{i-1}\mathbf{r}_{i-1}^e - \mathbf{r}_i^e}{h_i} \\
&= \frac{(\cancel{1} - \xi_{i+1} - \xi_{i-1})\mathbf{r}_i^e + \xi_{i+1}\mathbf{r}_{i+1}^e + \xi_{i-1}\mathbf{r}_{i-1}^e - \cancel{\mathbf{r}_i^e}}{h_i} \\
&= \frac{\xi_{i+1}\left(\mathbf{r}_{i+1}^e - \mathbf{r}_i^e\right) - \xi_{i-1}\left(\mathbf{r}_i^e - \mathbf{r}_{i-1}^e\right)}{h_i}
\end{aligned}$$

$$\Rightarrow \Lambda_i^e(\mathbf{r}) = \frac{\xi_{i+1}\boldsymbol{\ell}_{i-1} - \xi_{i-1}\boldsymbol{\ell}_{i+1}}{h_i}, \quad \mathbf{r} \in \mathcal{S}^e,\ i = 1,2,3$$

Recall Local Geometry Definitions



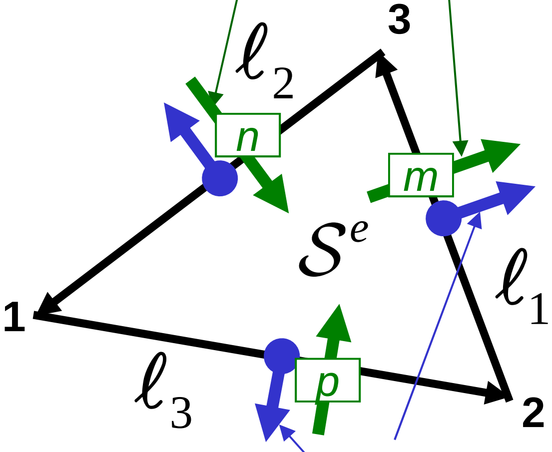
$$\hat{\mathbf{n}} = \frac{\ell_{i+1} \times \ell_{i-1}}{2A^e},$$

$i = 1, 2$ or 3

Table 8 Geometrical quantities defined on triangular elements.

Edge vectors	$\boldsymbol{\ell}_i = \boldsymbol{\rho}^e_{i-1} - \boldsymbol{\rho}^e_{i+1}; \ \ell_i = \lvert\boldsymbol{\ell}_i\rvert;$ $\hat{\boldsymbol{\ell}}_i = \frac{\boldsymbol{\ell}_i}{\ell_i}$, $i = 1, 2, 3$
Area	$A^e = \frac{\lvert\boldsymbol{\ell}_{i-1} \times \boldsymbol{\ell}_{i+1}\rvert}{2}$, $i = 1, 2,$ or 3
Height vectors	$h_i = \frac{2A^e}{\ell_i}; \ \hat{\boldsymbol{h}}_i = -\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\ell}}_i$; $\boldsymbol{h}_i = h_i \hat{\boldsymbol{h}}_i, \ i = 1, 2, 3$
Coordinate gradients	$\boldsymbol{\nabla}\xi_i = -\frac{\hat{\boldsymbol{h}}_i}{h_i}$, $i = 1, 2, 3$

Local Basis Functions on Triangular Elements



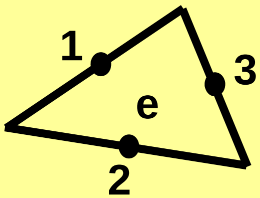
Local basis functions:

$$\Lambda_i^e(\mathbf{r}) = \frac{\xi_{i+1}\ell_{i-1} - \xi_{i-1}\ell_{i+1}}{h_i}, \ \mathbf{r} \in \mathcal{S}^e$$

$$\nabla \cdot \Lambda_i^e(\mathbf{r}) = \frac{2}{h_i}, \ \mathbf{r} \in \mathcal{S}^e$$

$$\sigma_i^e = \begin{cases} 1, \text{ Global reference direction} \\ \quad \text{for } i\text{th DoF is out of element e} \\ -1, \text{ Global reference direction} \\ \quad \text{for } i\text{th DoF is into element e} \end{cases}$$

Element Matrix for 3D EFIE



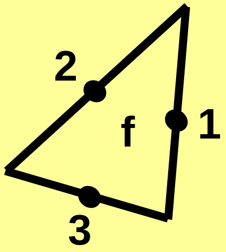
Element matrix :

$$\left[Z_{ij}^{ef}\right] = j\omega\left[L_{ij}^{ef}\right] + \frac{1}{j\omega}\left[S_{ij}^{ef}\right],$$

$$L_{ij}^{ef} = \mu < \Lambda^{e}_{i}; G, \Lambda^{f}_{j} > ,$$

$$S_{ij}^{ef} = \frac{1}{\varepsilon} < \nabla\cdot\Lambda^{e}_{i}, G, \nabla\cdot\Lambda^{f}_{j} >,$$

$$i, j = 1, 2, 3$$



Element excitation vector :

$$V_i^e = < \Lambda^{e}_{i}; \mathbf{E}^{\mathrm{i}} >, \quad i = 1, 2, 3$$

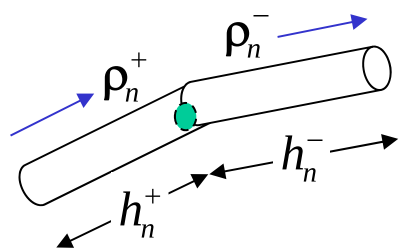
Matrix assembly :

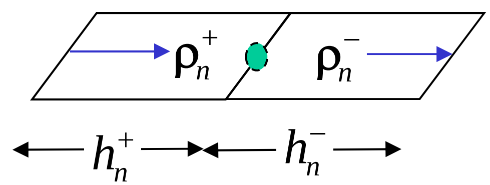
$$\sigma_i^e \sigma_j^f Z_{ij}^{ef} \rightarrow \left[Z_{mn}\right]$$

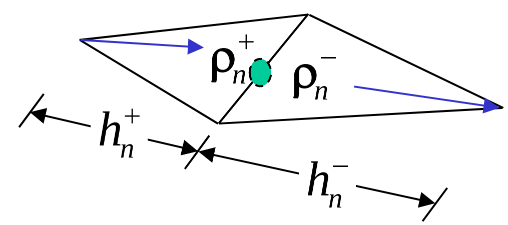
Excitation vector assembly :

$$\sigma_i^e V_i^e \rightarrow \left[V_m\right]$$

Note that Global Forms of Line Segment, Rectangular and Triangular, Tetrahedral, etc. Bases Can All Be Similarly Expressed …

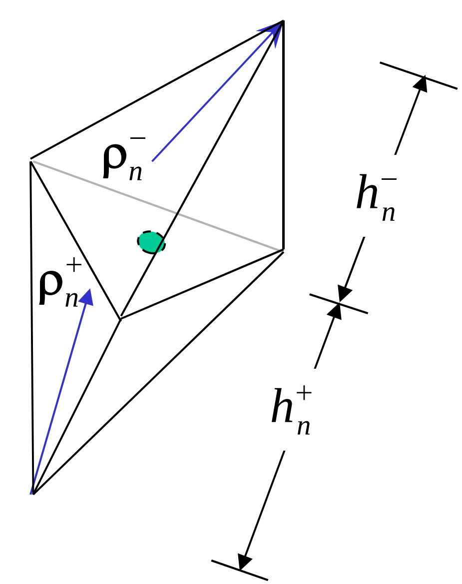






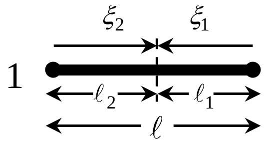
$$\boldsymbol{\Lambda}_n(\mathbf{r}) = \begin{cases} \dfrac{\boldsymbol{\rho}_n^+}{h_n^+}, & \mathbf{r} \in \mathcal{D}_n^+ \\[2ex] \dfrac{\boldsymbol{\rho}_n^-}{h_n^-}, & \mathbf{r} \in \mathcal{D}_n^- \\[2ex] \mathbf{0}, & \text{elsewhere} \end{cases} \qquad \nabla \cdot \boldsymbol{\Lambda}_n(\mathbf{r}) = \begin{cases} \dfrac{\dim \boldsymbol{\rho}_n^+}{h_n^+}, & \mathbf{r} \in \mathcal{D}_n^+ \\[2ex] -\dfrac{\dim \boldsymbol{\rho}_n^-}{h_n^-}, & \mathbf{r} \in \mathcal{D}_n^- \\[2ex] 0, & \text{elsewhere} \end{cases}$$

$$\dim \boldsymbol{\rho}_n^\pm = \begin{cases} 1, & \boldsymbol{\rho}_n^\pm \text{ varies in 1D} \\ 2, & \boldsymbol{\rho}_n^\pm \text{ varies in 2D} \\ 3, & \boldsymbol{\rho}_n^\pm \text{ varies in 3D} \end{cases}$$



…But Local Basis Definitions Require Local Coordinates

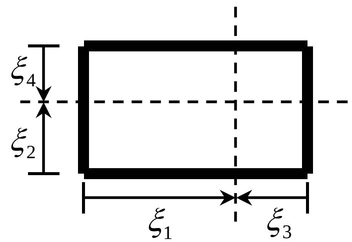
D=1



$$\xi_1 + \xi_2 = \frac{\ell_1}{\ell} + \frac{\ell_2}{\ell} = 1$$

$\Rightarrow$

D=2

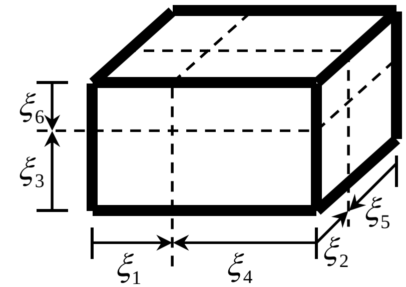


$$\xi_1 + \xi_3 = 1$$

$$\xi_2 + \xi_4 = 1$$

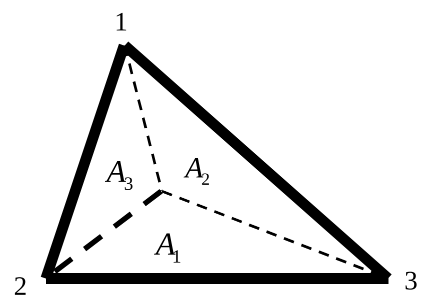
$\Rightarrow$

D=3



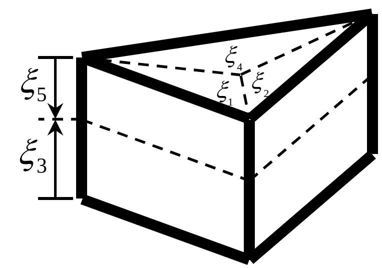
$$\xi_1 + \xi_4 = \xi_2 + \xi_5 = \xi_3 + \xi_6 = 1$$

- *The i-th boundary vertex, edge, or face is the zero coordinate surface for ξ_i; at the opposite boundary, $\xi_i = 1$*

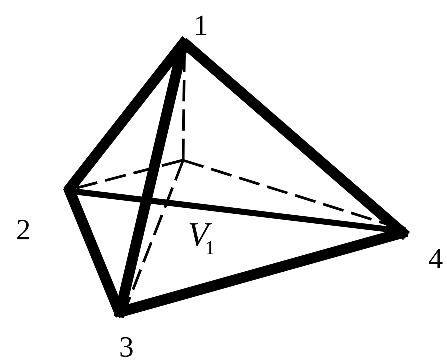


$$\xi_1 + \xi_2 + \xi_3 = \frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = 1$$

$\Rightarrow$



$$\xi_1 + \xi_2 + \xi_4 = \xi_3 + \xi_5 = 1$$



$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = \frac{V_1}{V} + \frac{V_2}{V} + \frac{V_3}{V} + \frac{V_4}{V} = 1$$

- $\xi_1, \ldots, \xi_D$, *D=1,2,3, are considered the independent coordinates for D-dimensional elements; $\nabla\xi_1, \nabla\xi_2$ and $\nabla\xi_3$ (or $\hat{\mathbf{n}}$ in 2-D) form a right-handed system.*

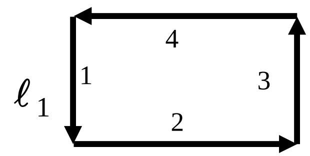
Summary of Local Vector Basis Definitions

D=1

$$\Lambda_i^e = \frac{\xi_i \ell_i}{h_i}$$

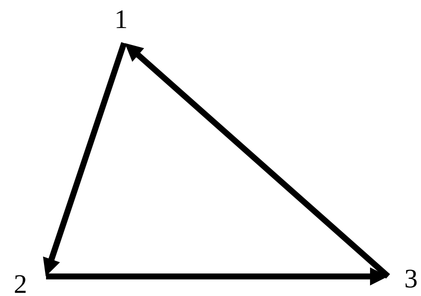
Line Segment

D=2



$$\Lambda_i^e(\mathbf{r}) = \frac{\xi_{i+2}\,\ell_{i-1}}{h_i}$$

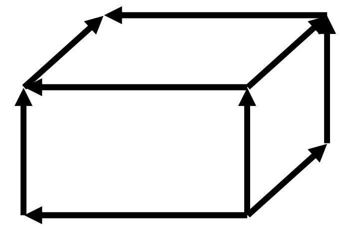
Rectangle



$$\Lambda_i^e(\mathbf{r}) = \frac{\xi_{i+1}\,\ell_{i-1} - \xi_{i-1}\,\ell_{i+1}}{h_i}$$

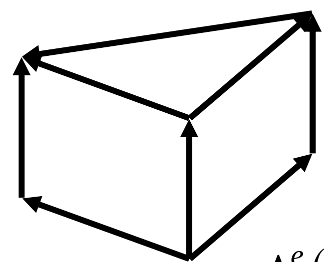
Triangle

D=3



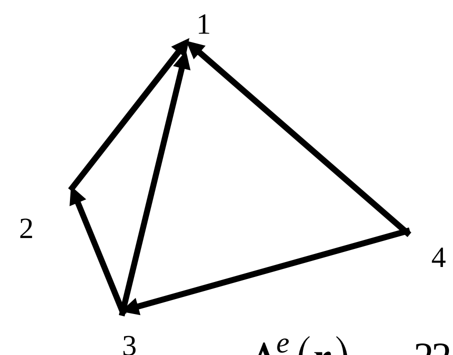
$$\Lambda_i^e(\mathbf{r}) = ??$$

Brick



Wedge

$$\Lambda_i^e(\mathbf{r}) = ??$$



$$\Lambda_i^e(\mathbf{r}) = ??$$

Tetrahedron

In 2D, require:

- $\nabla \xi_i \equiv -\dfrac{\hat{\mathbf{h}}_i}{h_i},$
- $\hat{\ell}_i = \hat{\mathbf{n}} \times \hat{\mathbf{h}}_i$

$$\nabla \cdot \Lambda_i^e(\mathbf{r}) = \begin{cases} \dfrac{\dim \Lambda_i^e}{h_i}, & \mathbf{r} \in \mathcal{D}^e \\ 0, & \textbf{elsewhere} \end{cases}$$

Numerical Integration to Form Element Matrices

- **Typical element matrix has the form**

$$< \Lambda_i^e ; G, \Lambda_j^f >$$

- **For $e \neq f$ use the result**

$$\int_{A^e} f(\mathbf{r})\, d\mathcal{S}$$
$$= 2A^e \int_0^1 \int_0^{1-\xi_2} f(\xi_1 \mathbf{r}_1^e + \xi_2 \mathbf{r}_2^e + \xi_3 \mathbf{r}_3^e)\, d\xi_1 d\xi_2$$
$$\underbrace{\approx \mathcal{J}^e \sum_{k=1}^{K} w_k f(\xi_1^{(k)} \mathbf{r}_1^e + \xi_2^{(k)} \mathbf{r}_2^e + \xi_3^{(k)} \mathbf{r}_3^e)}_{\text{Numerical integration}}, \quad \mathcal{J}^e = 2A^e$$

- **For $e = f$ use a singularity subtraction or cancellation scheme to handle the $1/R$ singularity**

Singularity Subtraction vs. Singularity Cancellation

Singularity subtraction:

$$\int_S f(\mathbf{r}')\frac{e^{-jkR}}{4\pi R}\,d\mathcal{S}' = \underbrace{\int_S \left(\frac{f(\mathbf{r}')e^{-jkR}}{4\pi R} - \sum_{n\ge0,m\ge0}^{N,M}\frac{(-jk)^m}{m!}\frac{\overbrace{P^n(\mathbf{r}')}^{\text{Degree } N \text{ polynom. approx. of } f(\mathbf{r}')}\;\overbrace{R^m}^{\text{Degree } M \text{ power series approx. of } e^{-jkR}}}{4\pi R}\right)d\mathcal{S}'}_{\text{Integrate numerically}} + \underbrace{\sum_{n\ge0,m\ge0}^{N,M}\frac{(-jk)^m}{m!}\int_S \frac{P^n(\mathbf{r}')R^m}{4\pi R}\,d\mathcal{S}'}_{\text{Integrate analytically}}$$

Singularity subtraction has been used very successfully, but has drawbacks:

- Accuracy of numerical integral limited by non-analytic form of difference integrand
  (i.e., $R=\sqrt{x^2+y^2+z^2}$ is not "smooth" or "polynomial-like" at $(x,y,z)=(0,0,0)$)
- Method is sometimes unsuitable for *nearly*-singular integrands
- Occasionally a singular form *cannot* be analytically integrated
- Analytical integrals are complicated, difficult, and must be performed for *every separate combination* of basis, element, and Green's function.

  --Hence the approach is poorly-suited to object-oriented programming !

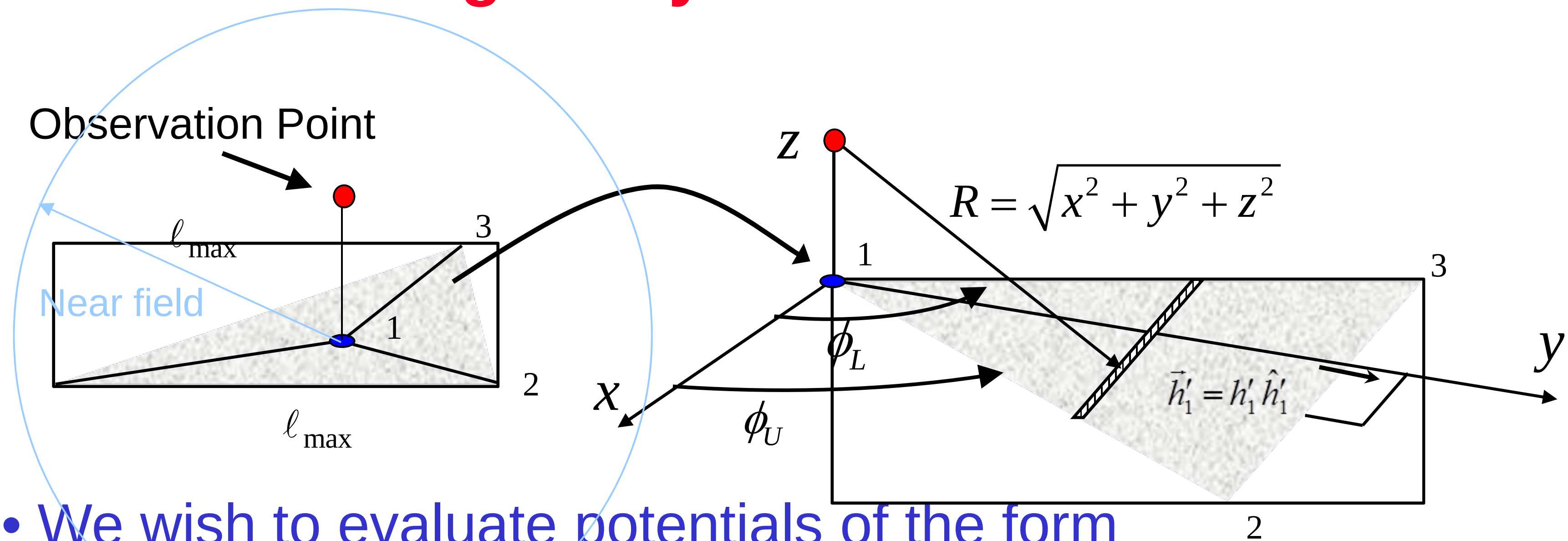
- But Green's functions of the form $\underbrace{G_0}_{\text{Singular}} + \underbrace{\Delta G}_{\text{Near singular image term}}$ may require separate handling of each term

Singularity Subtraction Methods Appear in the Following References

- S. Järvenpää, M. Taskinen, and P. Ylä-Oijala, “Singularity Subtraction Technique for High-Order Polynomial Vector Basis Functions on Planar Triangles,” *IEEE Trans. Antennas and Propagat*.,**54**, 1, pp. 42—49, Jan. 2006.

- Wilton, D.R., S.M. Rao, A.W. Glisson, D.H. Schaubert, O.M. Al-Bundak, and C.M. Butler, "Potential Integrals for Uniform and Linear Source Distributions on Polygonal and Polyhedral Domains," *IEEE Trans. Antennas and Propagat.*, **32**, 3, pp. 276—28l, March l984.

Singularity Cancellation



- We wish to evaluate potentials of the form

$$\mathbf{I} = \int_{\mathcal{D}} \boldsymbol{\Lambda}(\mathbf{r}') \frac{e^{-jkR}}{4\pi R} d\mathcal{D}$$

- Subtriangle integral has the general form

$$\int_0^h \int_{y\cot\phi_L}^{y\cot\phi_U} h(x,y)\,dxdy = \int_{v_L}^{v_U} \int_{u_L}^{u_U} \underbrace{h\left[x(u,v), y(u,v)\right] \mathcal{J}(u,v)}_{\mathcal{J} \text{ cancels singularity of } h} dudv$$

Various Transforms for 1/R Singularities

$$\int_0^h \int_{y\cot\phi_L}^{y\cot\phi_U} h(x,y)\,dxdy = \int_{v_L}^{v_U}\int_{u_L}^{u_U} h\left[x(u,v), y(u,v)\right]\mathcal{J}(u,v)\,dudv$$

Reverse the order of integration and normalize the interval on the inner integral

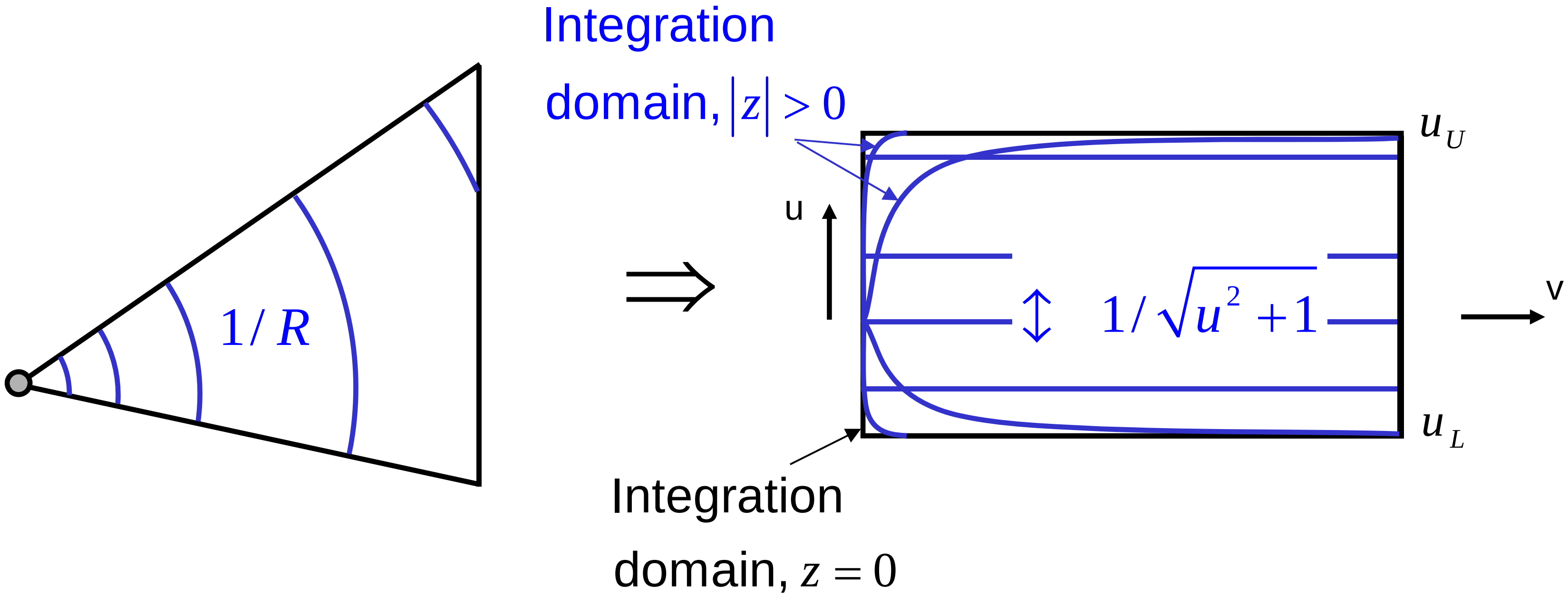
$$= \int_{u_L}^{u_U}(v_U - v_L)\int_0^1 h\left[x(u,v(\eta)), y(u,v(\eta))\right]\mathcal{J}(u,v(\eta))\,d\eta du, \quad v = v_L(1-\eta) + v_U\eta$$

	TRANSFORMATION	$\mathcal{J}(u,v)$	INTEGRATION LIMITS
Extended Duffy	$u = \dfrac{x}{\sqrt{y^2+z^2}}$ $v = y$	$\sqrt{y^2+z^2}$	$u_{L,U} = \dfrac{y\cot\phi_{L,U}}{\sqrt{y^2+z^2}}$ $v_{L,U} = 0,\ h$
Arcsinh	$u = \sinh^{-1}\dfrac{x}{\sqrt{y^2+z^2}}$ $v = y$	R	$u_{L,U} = \sinh^{-1}\left(\dfrac{y\cot\phi_{L,U}}{\sqrt{y^2+z^2}}\right)$ $v_{L,U} = 0,\ h$
Radial (Extended Polar)	$u = \tan^{-1}\dfrac{y}{x} = \phi$ $v = R$	R	$u_{L,U} = \phi_{L,U}$ $v_{L,U} = \lvert z\rvert,\ \sqrt{z^2 + (h/\sin u)^2}$
Radial-Angular	$u = \ln\tan\dfrac{\phi}{2} = -\sinh^{-1}\dfrac{x}{y}$ $v = R$	$\dfrac{R}{\cosh u}$	$u_{L,U} = \ln\tan\dfrac{\phi_{L,U}}{2}$ $v_{L,U} = \lvert z\rvert,\ \sqrt{z^2 + (h\cosh u)^2}$

$$\mathcal{J}(u,v) \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \left|\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv}\right|$$

- For more possible transforms, see M. M. Botha, “A family of augmented Duffy Transformations for near- singularity cancellation quadrature,” *IEEE Trans. Antennas Propagat.*, 2013.

Extended Duffy

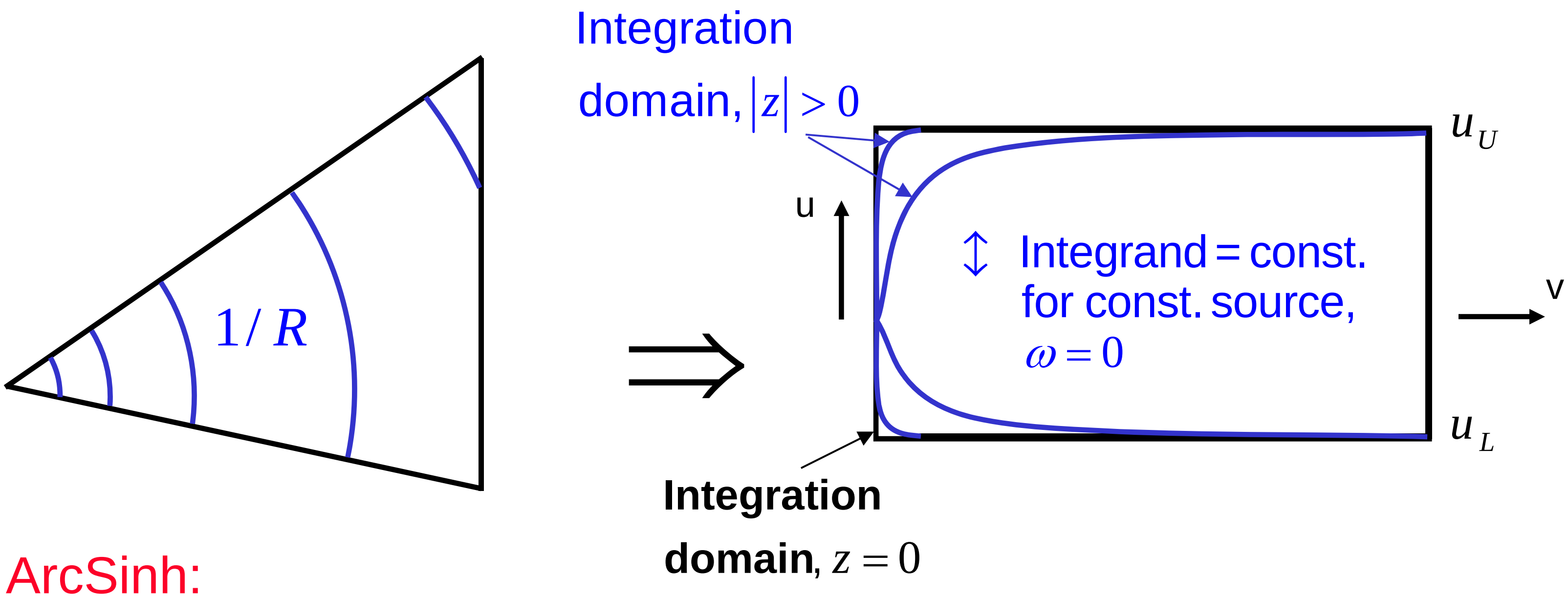


Extended Duffy:

- Results in $1/\sqrt{u^2+1}$ variation of integrand for constant source density and static case $(\omega = 0)$ 

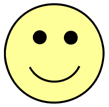
- Integration domain sensitive to z variation of obs. pt. 

ArcSinh



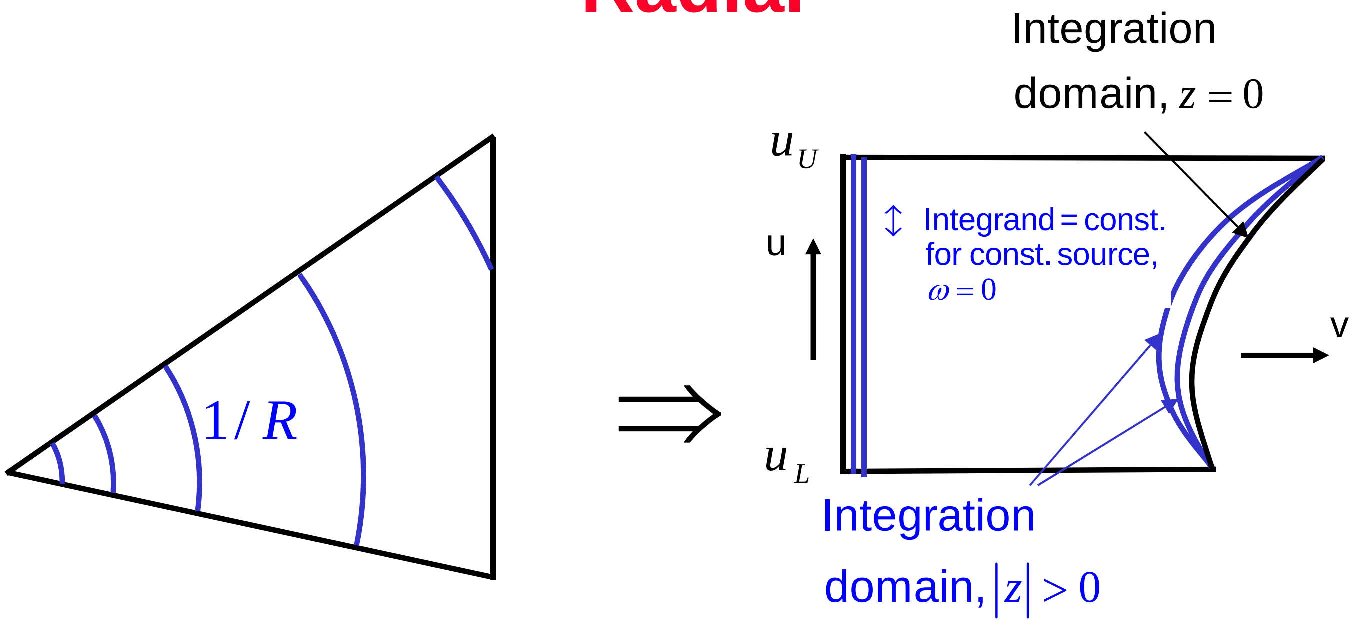
ArcSinh:

- Integrates *static* kernel with constant source *exactly* (one sample pt) for $z = 0$ (more sample pts. needed to handle basis and exponential phase variations)



- But integration domain sensitive to z variation of obs. pt. for small z

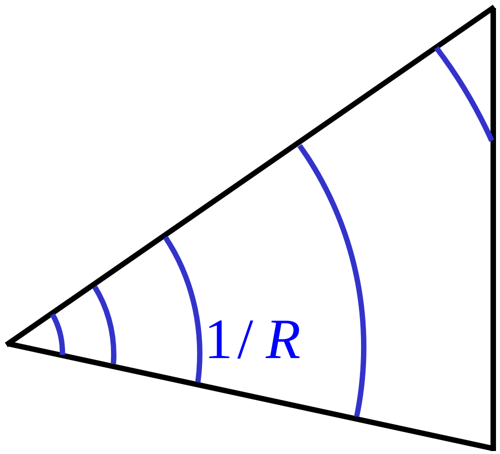
Radial

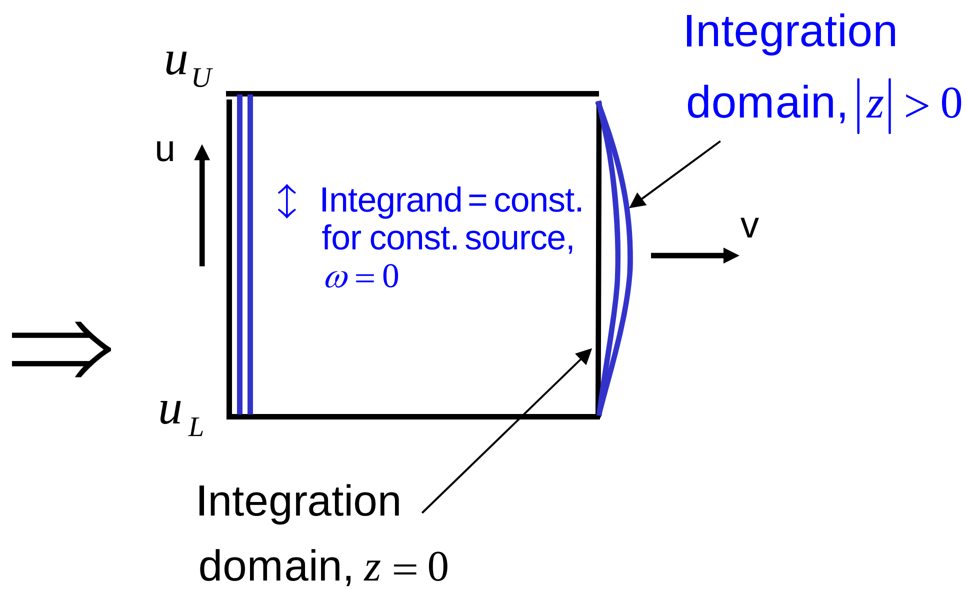


Radial:

- Exactly cancels static kernel singularity, but integration domain is not rectangular, even when $z=0$ 
- Integration domain insensitive to *z* variation of obs. pt. 

Radial-Angular





<u>Radial-Angular:</u>

- Integrates *static* kernel with constant bases exactly (rectangular integration domain needs one sample pt. only; more sample pts. needed to handle variation of bases and exponential phase factor)

- Integration domain insensitive to z variation of obs. pt.

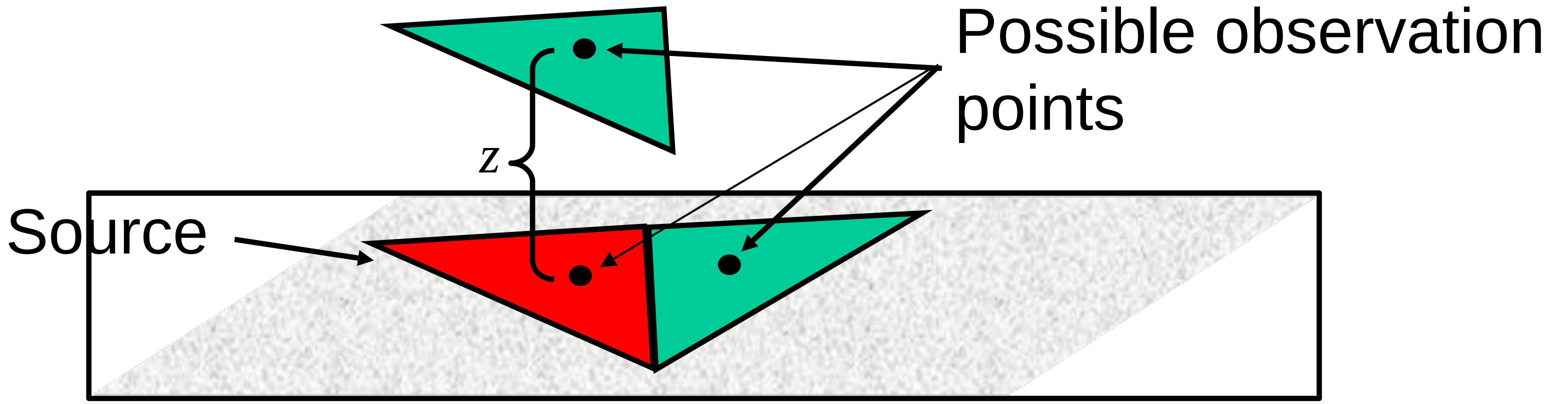
- Above features suggest this as the method of choice

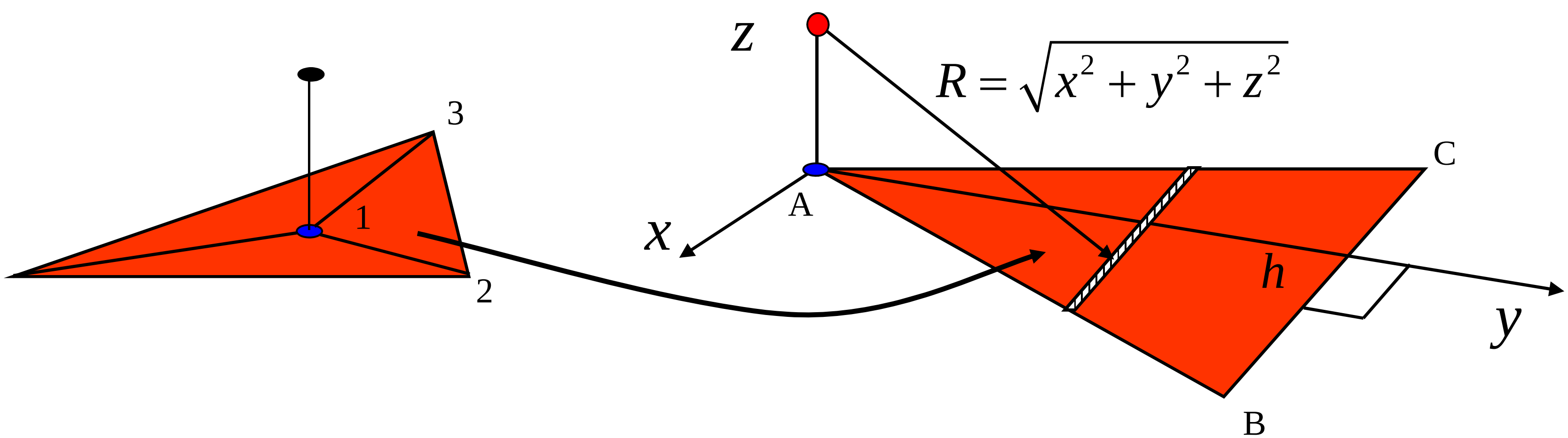
These and Several Additional Transformations Have Been Analyzed in Detail

- Khayat, M. A., D. R. Wilton, and P. W. Fink, "An Improved Transformation and Optimized Sampling Scheme for the Numerical Evaluation of Singular and Near-Singular Potentials," IEEE Antennas and Wireless Propagation Letters, Vol. 7, pp. 377 – 380, July 2008.

- M. M. Botha, "A family of augmented Duffy Transformations for near-singularity cancellation quadrature," *IEEE Trans. Antennas Propagat*., 2013. (Tests and compares several schemes; concludes that the radial-angular scheme appears to be the most effective of those tested.)

Singularity Cancellation Approach for Self *and* Near Terms



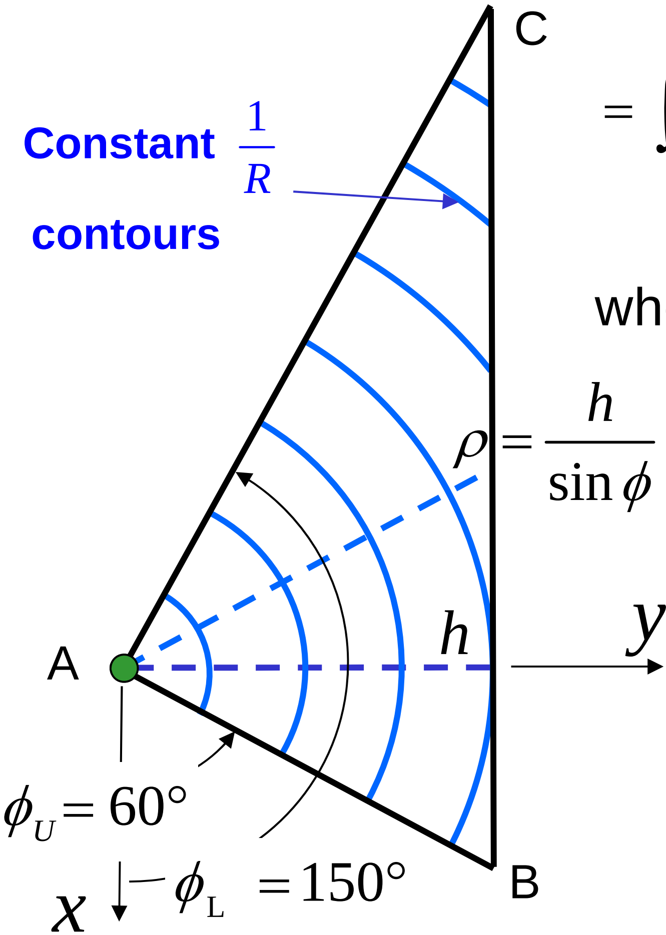


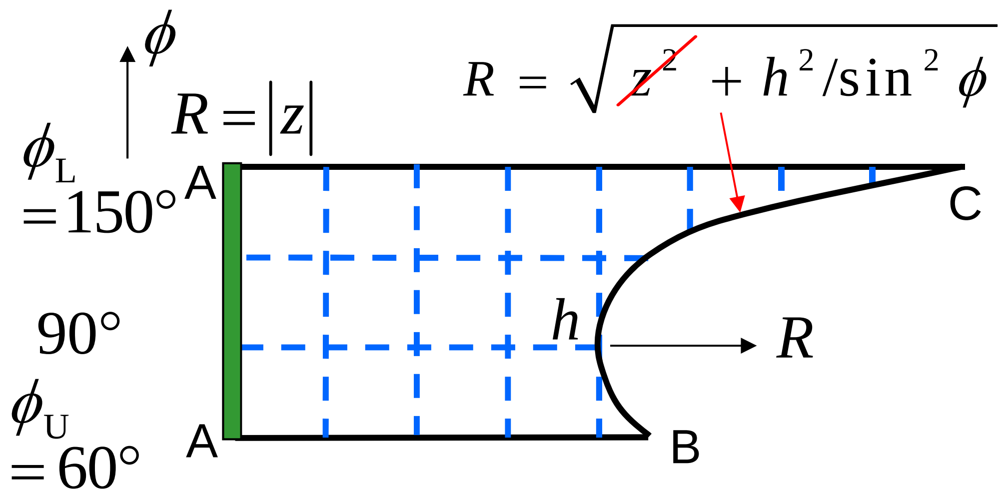
Radial Transformation Removes Singularity, But Leaves a Non-Rectangular Domain

$$\int_0^h \int_{y\cot\phi_L}^{y\cot\phi_U} \Lambda_j^e(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\,dxdy = \int_{\phi_U}^{\phi_L}\int_{|z|}^{\sqrt{z^2+h^2/\sin^2\phi}} \Lambda_j^e(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\overbrace{R\,dR}^{\rho\,d\rho}\,d\phi$$

$$= \int_{\phi_U}^{\phi_L}\left[\left(\underbrace{\sqrt{z^2+h^2/\sin^2\phi}-|z|}_{\xrightarrow[z\to 0]{} h/\sin\phi}\right)\int_0^1 \Lambda_j^e(\mathbf{r}')G(\mathbf{r},\mathbf{r}')R\,d\eta\right]d\phi\,,$$

where $R^2 = z^2+\rho^2$, $R=(1-\eta)|z|+\eta\sqrt{z^2+h^2/\sin^2\phi}$





A Second (Angular)Transformation on ϕ Regularizes the Domain

$$\int_{\phi_{\mathrm{U}}}^{\phi_{\mathrm{L}}}\left[\left(\sqrt{z^2+h^2/\sin^2\phi}-|z|\right)\int_0^1 \Lambda_j^e(\mathbf{r}')G(\mathbf{r},\mathbf{r}')R\,d\eta\right]d\phi$$

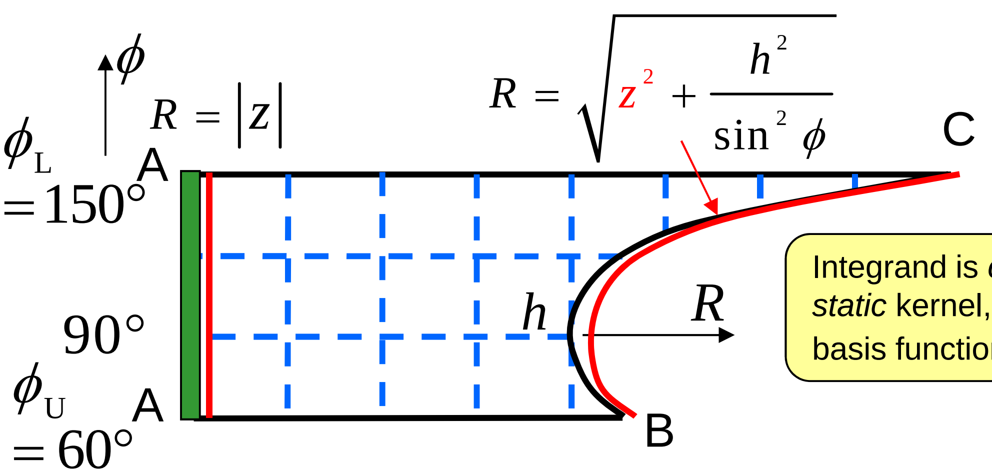
$$=\int_{u_{\mathrm{U}}}^{u_{\mathrm{L}}}\left[\left(\frac{\sqrt{z^2+h^2\cosh^2 u}-|z|}{\cosh u}\right)\int_0^1 \Lambda_j^e(\mathbf{r}')G(\mathbf{r},\mathbf{r}')R\,d\eta\right]du$$

$$\approx 2A^e\sum_i\sum_j \underbrace{w_i w_j\left(\frac{(u_{\mathrm{U}}-u_{\mathrm{L}})\left(\sqrt{z^2+h^2\cosh^2 u^{(i)}}-|z|\right)}{2A^e\cosh u^{(i)}}\right)R^{(i,j)}}_{W_k}\Lambda_j^e\left(\mathbf{r}^{(i,j)}\right)G\left(\mathbf{r},\mathbf{r}^{(i,j)}\right)$$

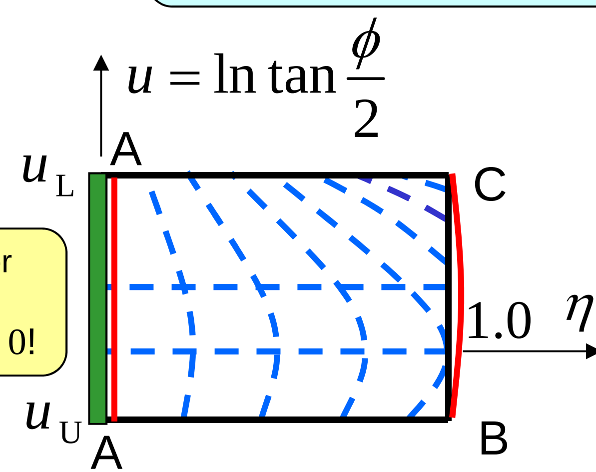
Let $du=\dfrac{d\phi}{\sin\phi}$

$$\Rightarrow\begin{cases} u=\ln\tan\dfrac{\phi}{2}\,, \\ (\sin\phi)^{-1}=\cosh u\,, \\ R=\sqrt{z^2+h^2\cosh^2 u} \end{cases}$$

Domain is insensitive to z for small z/h !

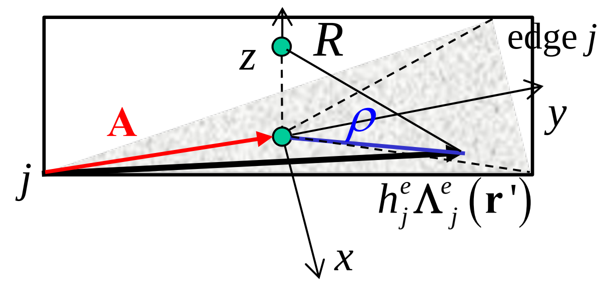


Integrand is *constant* for *static* kernel, *constant* basis function, and $z=0$!



The Radial Transformation Introduces Branch Points into the Basis Functions

Transformation results in a "branch point" singularity in the basis functions:

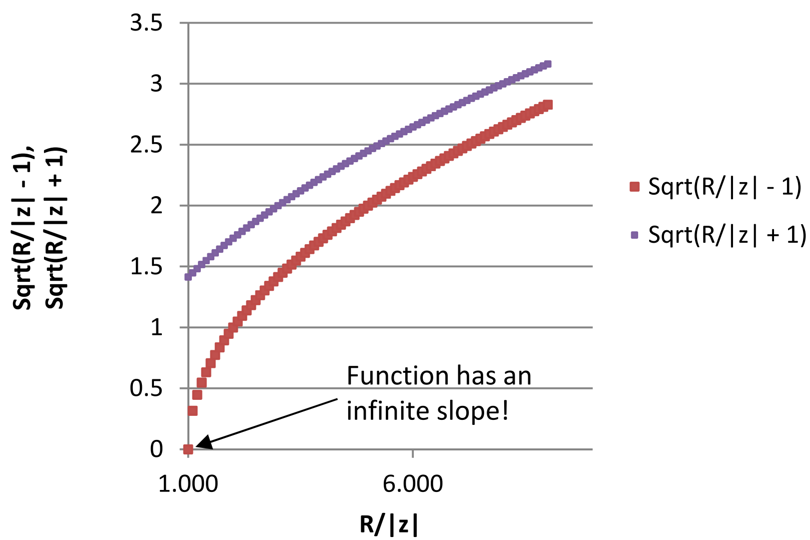


$$h_j^e \Lambda_j^e(\mathbf{r}') = \mathbf{A} + \hat{\mathbf{x}} \underbrace{\rho \cos\phi}_{x} + \hat{\mathbf{y}} \underbrace{\rho \sin\phi}_{y}$$

$$= \mathbf{A} + \hat{\mathbf{x}}\sqrt{R^2 - z^2}\cos\phi + \hat{\mathbf{y}}\sqrt{R^2 - z^2}\sin\phi,$$

but $\sqrt{R^2 - z^2} = z^2\sqrt{R/|z|+1}\,\underbrace{\sqrt{R/|z|-1}}_{\substack{\text{non-smooth} \\ \text{as } R\to|z|}}$.

The non-polynomial-like behavior as $R \to |z|$ implies that Gauss-Legendre quadrature will be ineffective.



A Special Quadrature Scheme or an Additional Transformation Handles Branch Points in the Basis Functions

- Because of the branch point, the basis functions have the form

$$h_j^e \Lambda_j^e(\mathbf{r}') = \mathbf{A} + \hat{\mathbf{x}}\sqrt{R^2 - z^2}\cos\phi + \hat{\mathbf{y}}\sqrt{R^2 - z^2}\sin\phi = \mathbf{A} + \underbrace{\sqrt{R - |z|}}_{\substack{\text{branch pt.} \\ \text{at } R=|z|}} \underbrace{\left(\hat{\mathbf{x}}\sqrt{R + |z|}\cos\phi + \hat{\mathbf{y}}\sqrt{R + |z|}\sin\phi\right)}_{\substack{\text{smooth function} \\ \text{of } R}}$$

- Hence the radial integrals are of the form

$$\int_{|z|}^{\sqrt{z^2 + h^2/\sin^2\phi}} \left(f(R) + \sqrt{R - |z|}\, g(R) \right) dR \text{ where } f(R) \text{ and } g(R) \text{ smooth (polynomial-like) functions;}$$

the integration interval normalization, $R = (1-\eta)|z| + \eta\sqrt{z^2 + h^2/\sin^2\phi}$, thus yields

$$\boxed{\int_0^1 \left(F(\eta) + \sqrt{\eta}\, G(\eta) \right) d\eta}, \text{ where } F(\eta) = \left(\sqrt{z^2 + h^2/\sin^2\phi} - |z| \right) f\left((1-\eta)|z| + \eta\sqrt{z^2 + h^2/\sin^2\phi} \right),$$

and $G(\eta) = \left(\sqrt{z^2 + h^2/\sin^2\phi} - |z| \right)^{\frac{3}{2}} g\left((1-\eta)|z| + \eta\sqrt{z^2 + h^2/\sin^2\phi} \right)$ are polynomial-like in η.

Hence we develop rules for exactly integrating integrals of the form $\boxed{\int_0^1 \left(P_n(\eta) + \sqrt{\eta}\, Q_n(\eta) \right) d\eta}$,

where $P_n(\eta)$, $Q_n(\eta)$ are polynomials of degree n.

- Or one can make the substitution $\eta^2 = R - |z|$ and use Gauss-Legendre quadrature.

Special Gauss Quadrature Scheme to Handle the Square Root-Type Branch Point in the Basis Functions

Weights and sample points for integrating the function set

$$\left\{\eta^n,\ \eta^n\sqrt{\eta}\right\},\ n = 0,1,2,\ldots,N$$

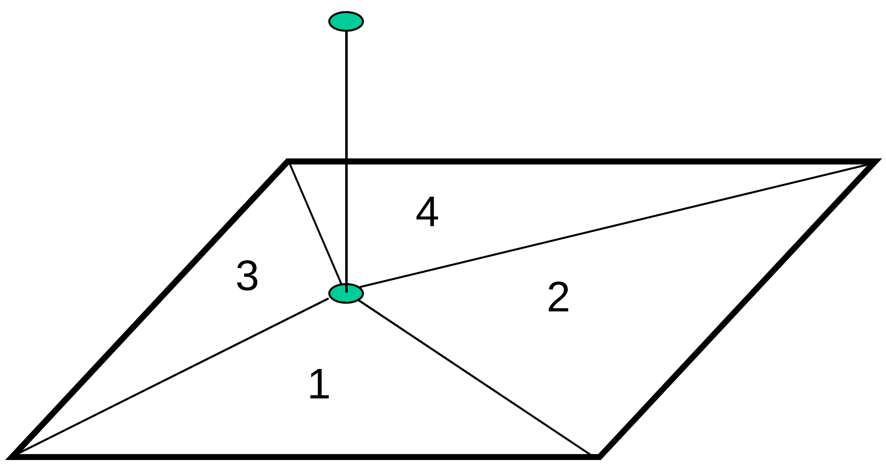
$$\boxed{\int_0^1 f(\eta)d\eta \approx \sum_{k=1}^{N} w_k f(\eta_k)}$$

N	Nodes η_i	Weights w_i
2	0.12606123086601956	0.3639172365120473
	0.7139387691339825	0.6360827634879527
3	0.045088504179695364	0.13965395980291434
	0.34872938419346483	0.45848221271917206
	0.8306719075452189	0.4018638274779136
4	0.019532819681463730	0.06236194190019799
	0.17339692801497078	0.25969509521658130
	0.522956026924229700	0.40692913602039693
	0.88905249698491430	0.27101382686282377

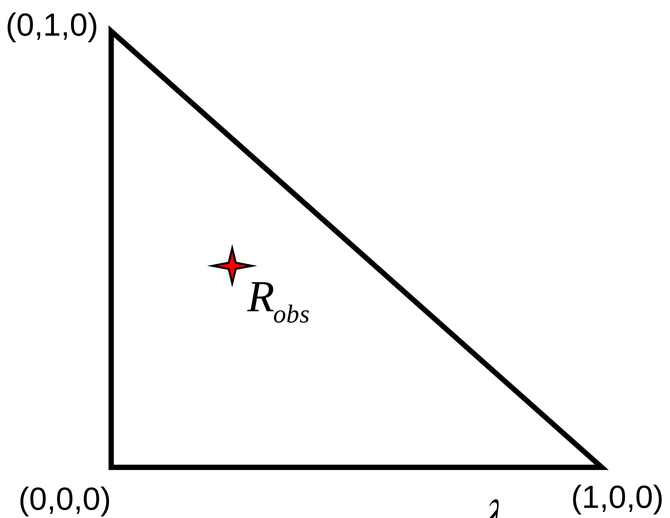
- The weights w_k and sample points η_k integrate $f(\eta)$ *exactly* if it has the form $f(\eta) = P_n(\eta) + \sqrt{\eta}Q_n(\eta)$ where P_n and Q_n are polynomials of degree $N-1$; they may be used to *approximately* integrate $f(\eta)$ if it can be approximated by the same form.

The Same Subtriangle Approach Can Also Be Used to Handle Singular and Near-Singular Integrals on Rectangular Domains

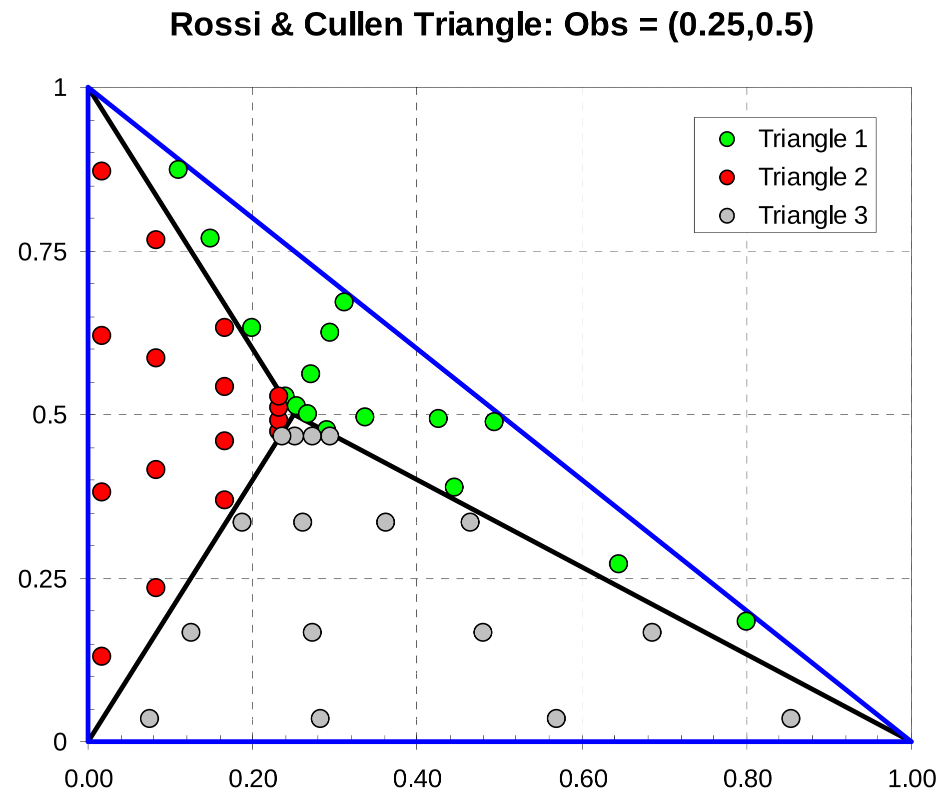
Only the Subtriangle-to-Rectangle Mapping Eqs. Change



Distribution of Sampled Points in Example Triangle

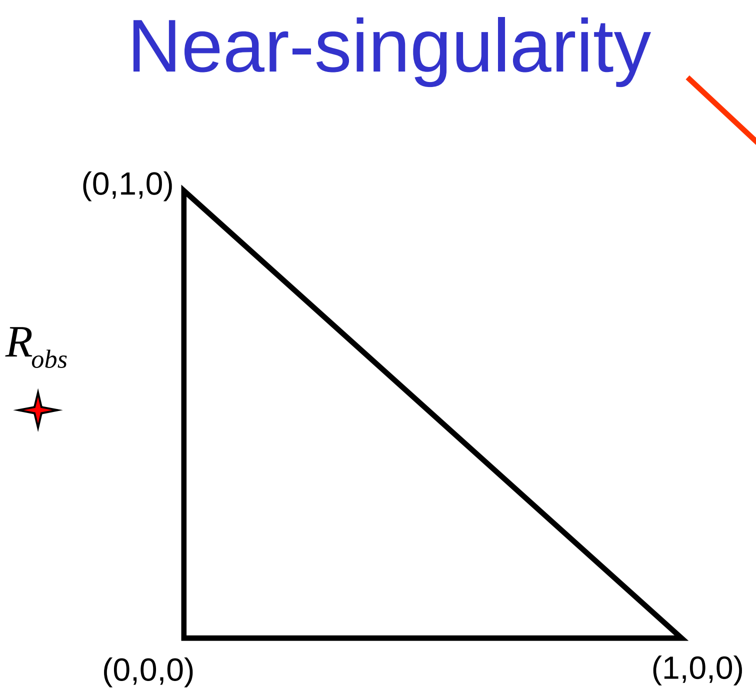


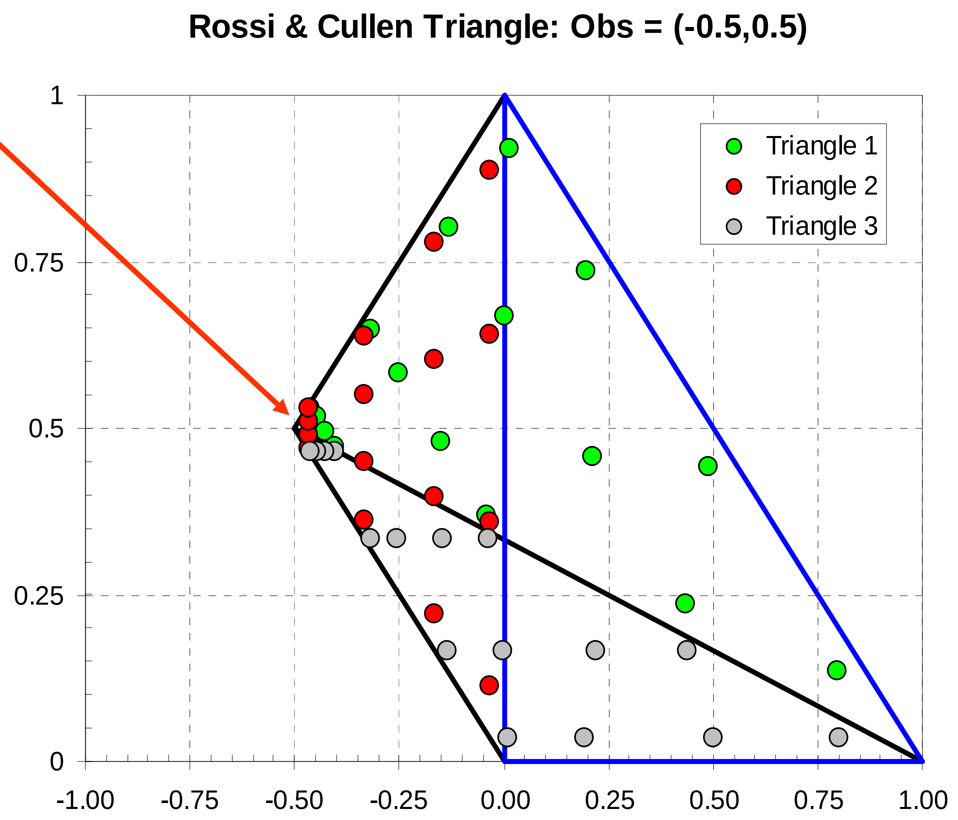
edge length $= \frac{\lambda}{10}$



* L. Rossi and P.J. Cullen, *IEEE Trans. AP*-47, pp. 398-402, April 1999

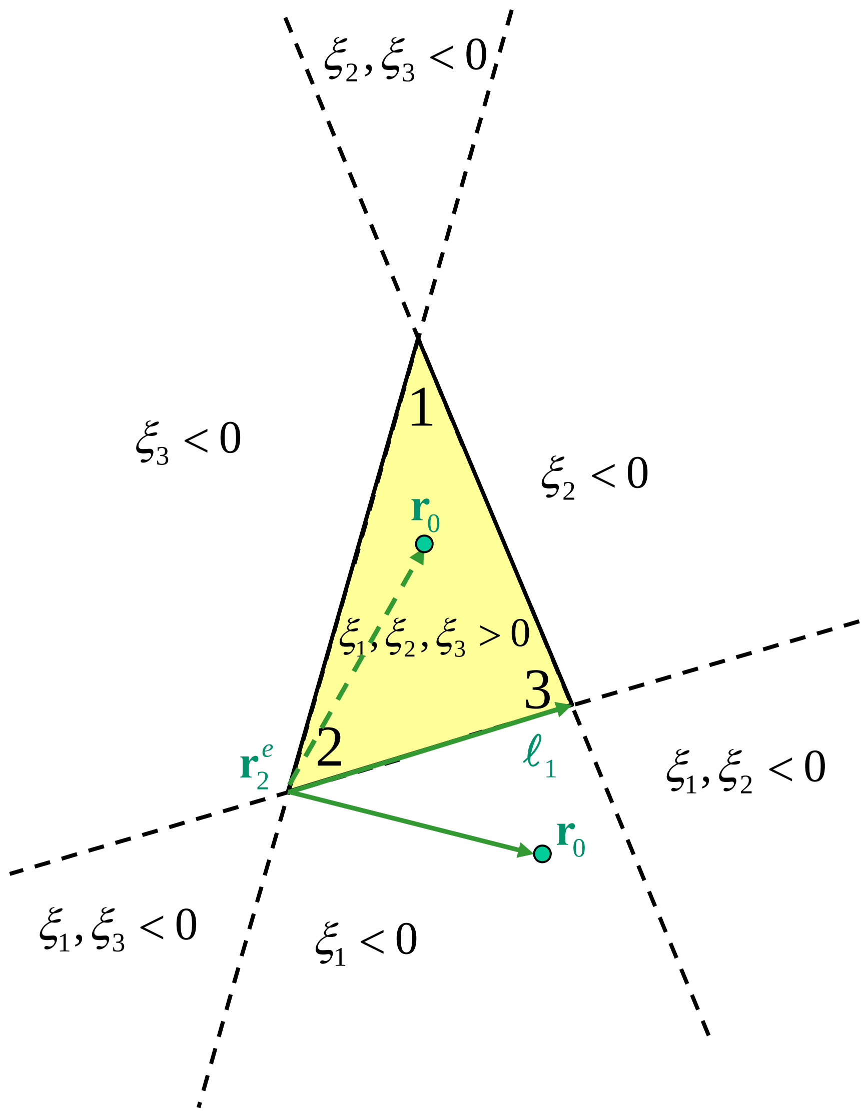
Calculation for Near-Singularities with Projected Obs. Pt. Outside Triangle





- **Note that the contributions from the integration domains of subtriangles 1 and 3 that lie outside the original triangle are completely canceled by the (negative) contribution of subtriangle 2**
- **Note also we've introduced a ficticious singularity at the obs. pt. from each of the three subtriangles, but the singularity *cancels* when contributions are summed**

If the Projected Obs. Pt. Falls Outside a Triangle, at Least One of Its Area Coordinates is Negative



Projected Obs. Pt.:

$$\mathbf{r}_0 = \mathbf{r} - \hat{\mathbf{n}}\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}_j^e), \quad j = 1, 2, \text{ or } 3$$

Area Coords. of Projected Obs. Pt.:

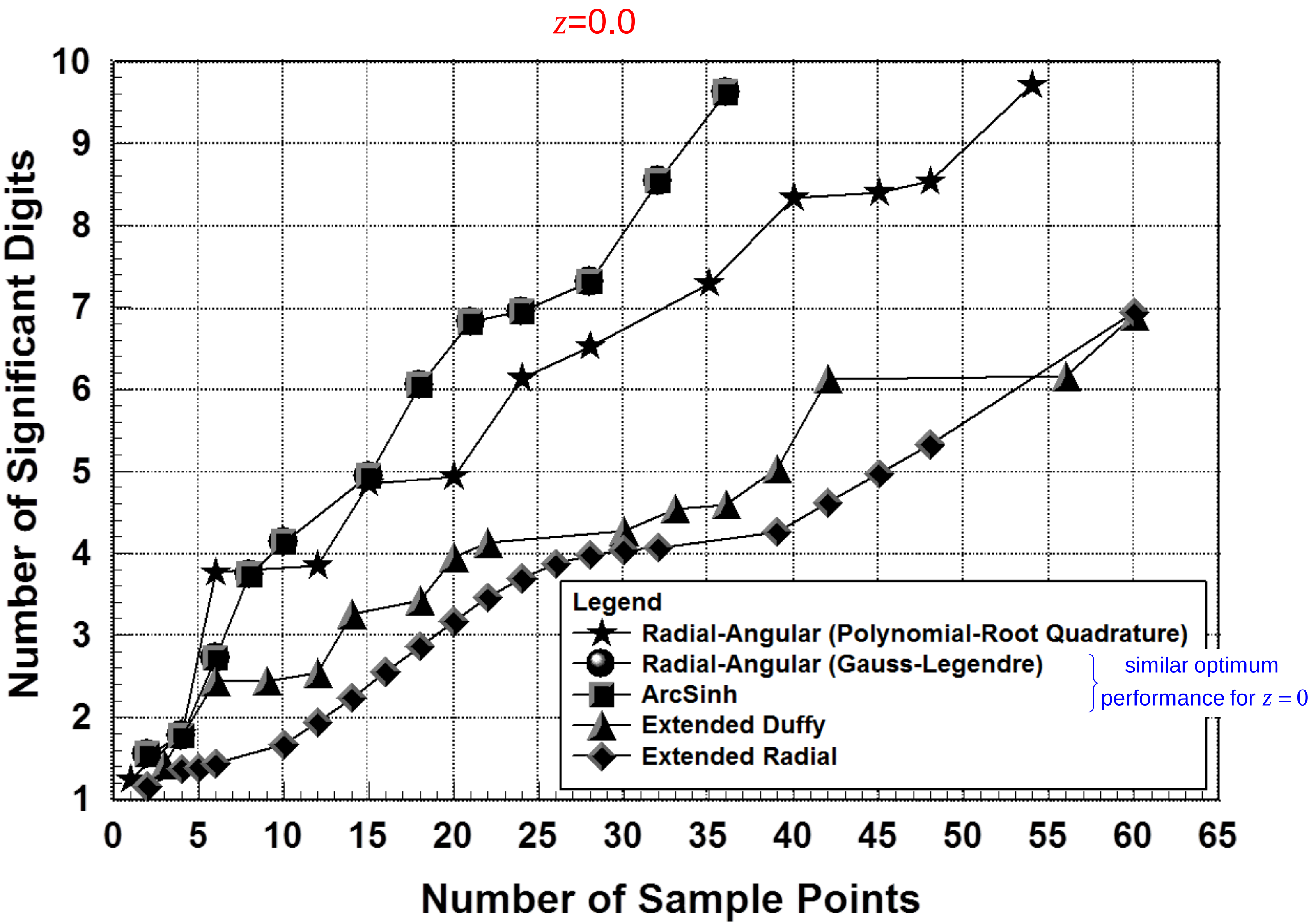
$$\xi_1 = \hat{\mathbf{n}} \cdot \frac{\boldsymbol{\ell}_1 \times (\mathbf{r}_0 - \mathbf{r}_2^e)}{2A^e}$$

$$\xi_2 = \hat{\mathbf{n}} \cdot \frac{\boldsymbol{\ell}_2 \times (\mathbf{r}_0 - \mathbf{r}_3^e)}{2A^e}$$

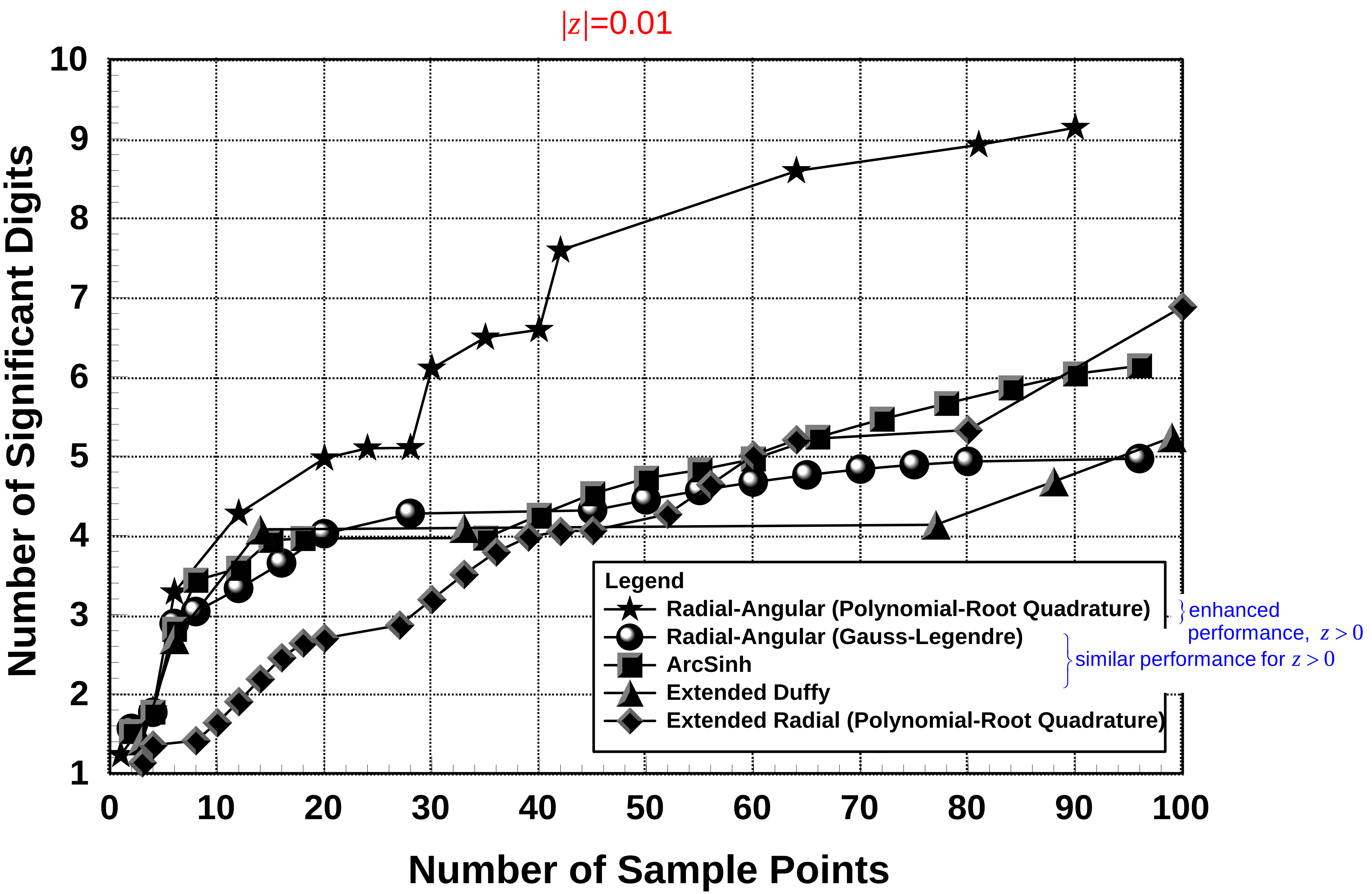
$$\xi_3 = 1 - \xi_1 - \xi_2$$

If area coordinate ξ_i is negative, then the contribution to the integral from subtriangle i must also be negative.

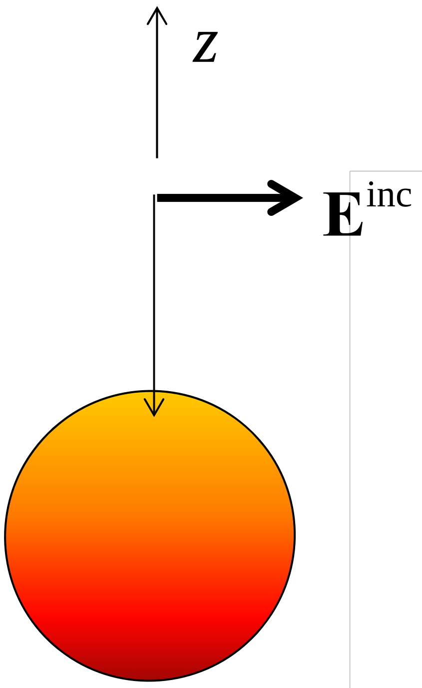
Scheme is Efficient and Essentially Arbitrary Accuracy Can Be Obtained…

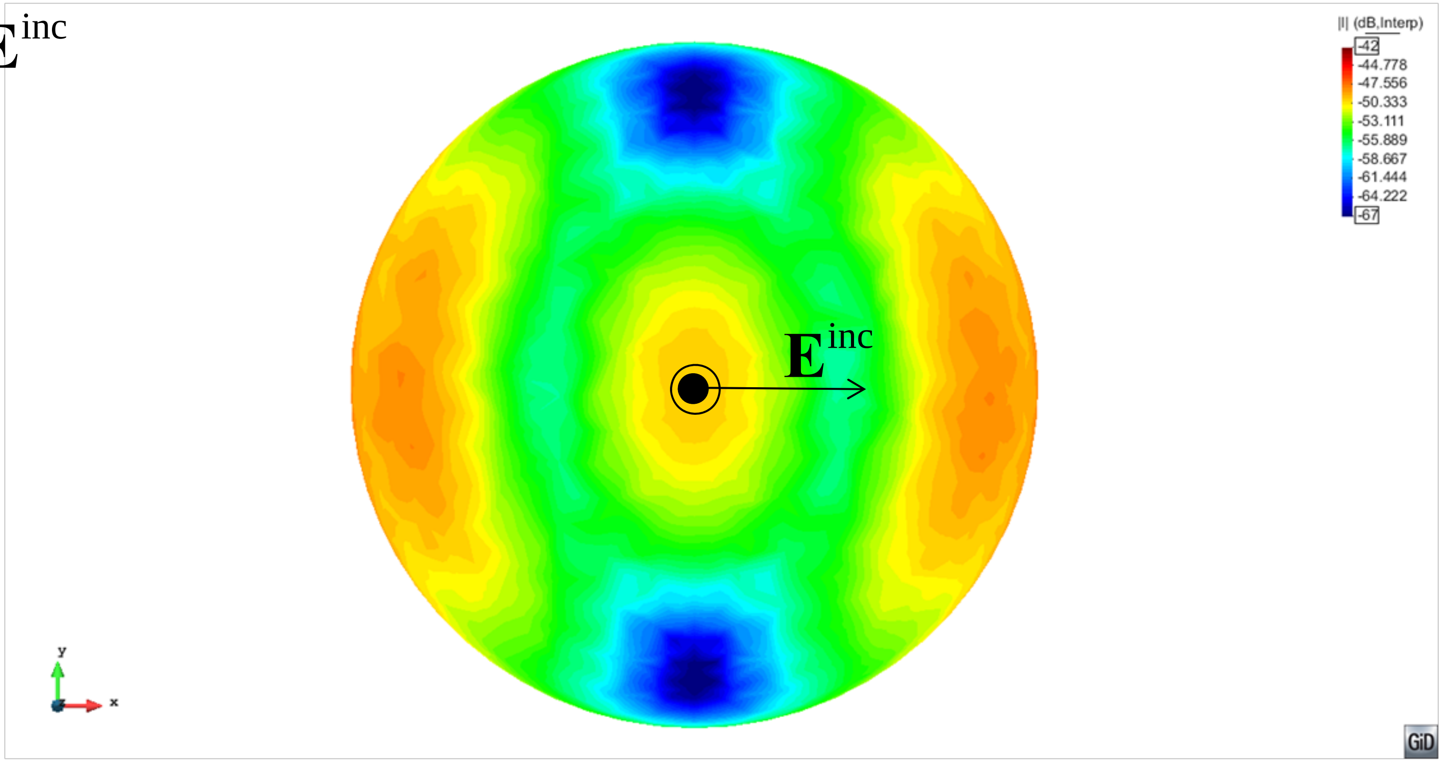


… Including the Nearly Singular Case

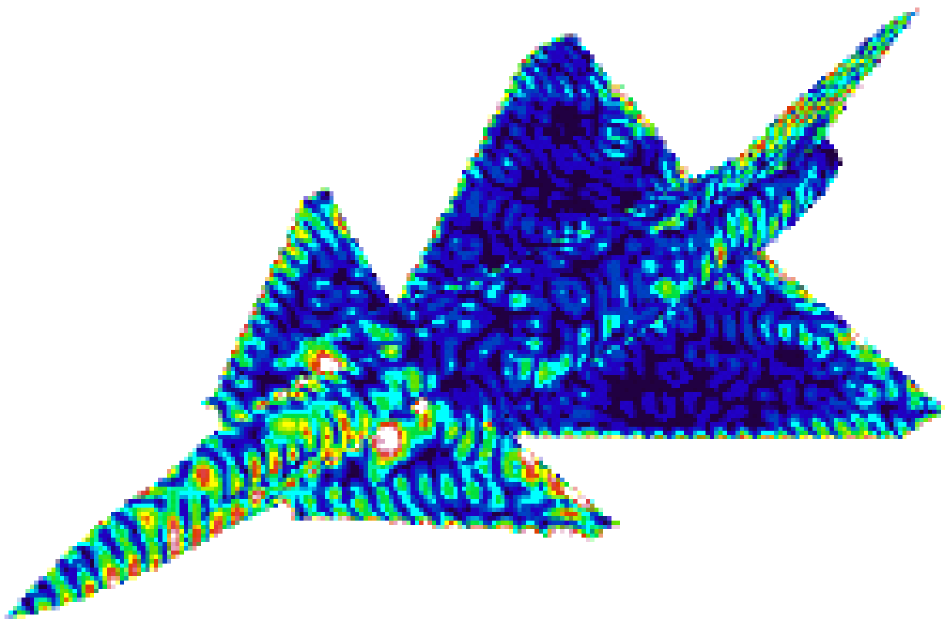


Example: Current Induced on Sphere by a Plane Wave Incident along the Negative z-Axis





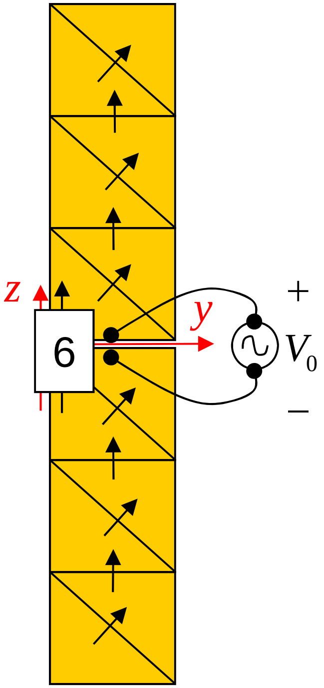
Example: Current Induced by Plane Wave Incident on VFY-218



500 MHz
PC computed with Mercury MOM
•157,000 unknowns

Courtesy of John Schaeffer

Voltage Sources



- $\mathbf{E}^{\mathrm{i}}_{\tan} = 0 \quad \Rightarrow \quad \mathbf{E}^{\mathrm{s}}_{\tan} = 0$ except at voltage source
- $\mathbf{J}$ must produce a potential difference between triangles at source terminals :

$$\Phi = V_0 \underbrace{u(z)}_{\text{unit step function}} \quad \text{on} \quad \mathcal{S}$$

$$\Rightarrow \mathbf{E}^{\mathrm{s}}_{\tan} = -\nabla_{\tan}\Phi = -\hat{\mathbf{z}} V_0\, \delta(z) \text{ on } \mathcal{S}$$

$$j\omega \mathbf{A}_{\tan}(\mathbf{J}) + \nabla_{\tan}\Phi(\mathbf{J}) = -\mathbf{E}^{\mathrm{s}}_{\tan}$$

$$\Rightarrow -<\boldsymbol{\Lambda}_m ; \mathbf{E}^{\mathrm{s}}> = V_0 \int_{\mathcal{S}} \underbrace{\boldsymbol{\Lambda}_m \cdot \hat{\mathbf{z}}}_{=\begin{cases} 1 \text{ at } z=0, m=6 \\ 0, \text{otherwise} \end{cases}} \delta(z)\, dz\, dy$$

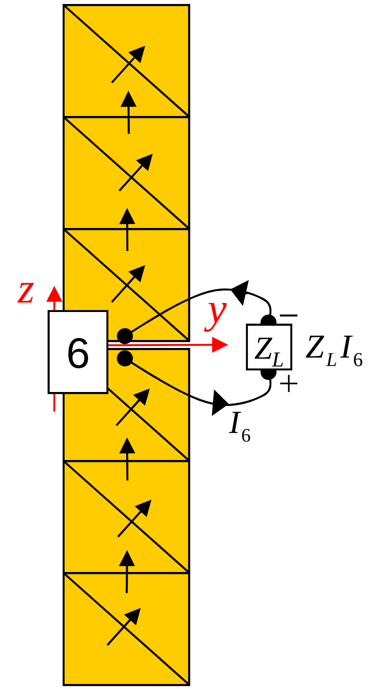
$$= \begin{cases} V_0 \ell_6, & m = 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow <\boldsymbol{\Lambda}_m , \mathbf{E}^i> = \begin{bmatrix} 0 \\ \vdots \\ V_0 \ell_6 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{row } 6 \\ \\ \\ \end{matrix}$$

Impedance Loading

- Load is equivalent to a voltage source $V_0 = -Z_L I_6$
- Replace voltage vector by

$$< \mathbf{\Lambda}_m, \mathbf{E}^i > \; = \begin{bmatrix} 0 \\ \vdots \\ -Z_L I_6 \ell_6 \\ \vdots \\ 0 \end{bmatrix} = -I_6 \left[Z_L \ell_6 \delta_{m,6} \right]$$



- $\Rightarrow \left[Z_{mn} \right] \left[I_n \right] = I_6 \left[Z_L \ell_6 \delta_{m,6} \right] +$ voltage /and or $\mathbf{E}^i$ terms
- Transfer load terms to other side of matrix :

$$\left[Z_{mn} + \underbrace{Z_L \ell_6 \delta_{m,6}}_{\text{add load to matrix diagonal}} \right] \left[I_n \right] = \text{voltage /and or } \mathbf{E}^i \text{ terms}$$

The End