Modeling Homogeneous Penetrable Materials --- *PMCHWT Formulation

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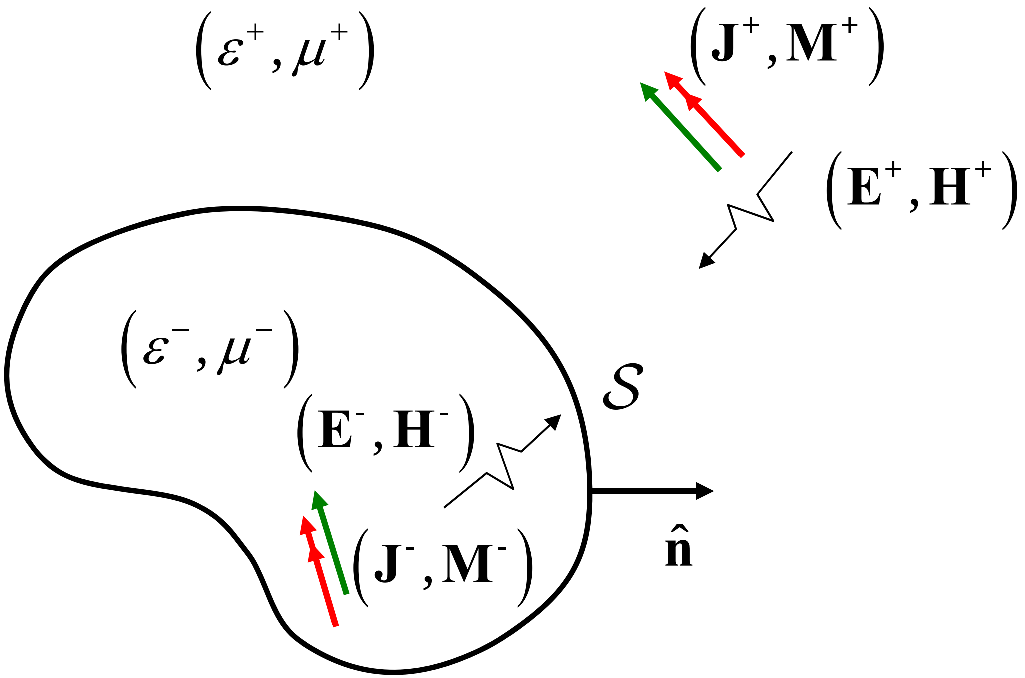
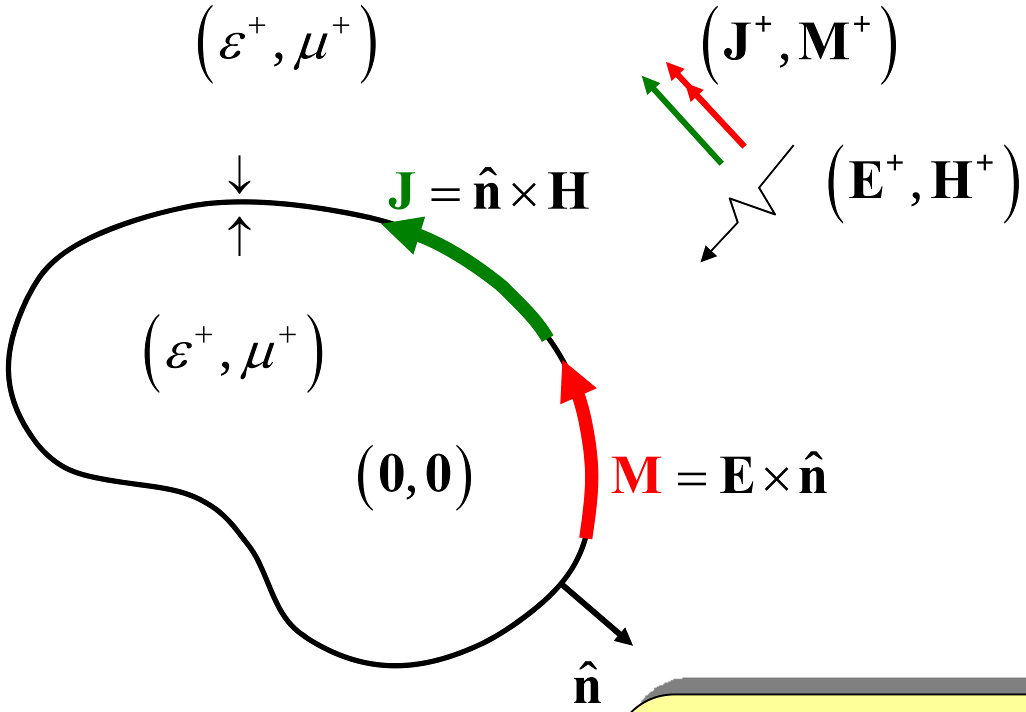
Formulation of Problems Involving Piecewise Homogeneous Media



- $(\mathbf{E}^{\pm}, \mathbf{H}^{\pm})$ are *incident* fields i.e., they are radiated by $(\mathbf{J}^{\pm}, \mathbf{M}^{\pm})$ in a *homogeneous* medium with parameters $(\varepsilon^{\pm}, \mu^{\pm})$

Exterior Equivalence, Interior Null Field Conditions



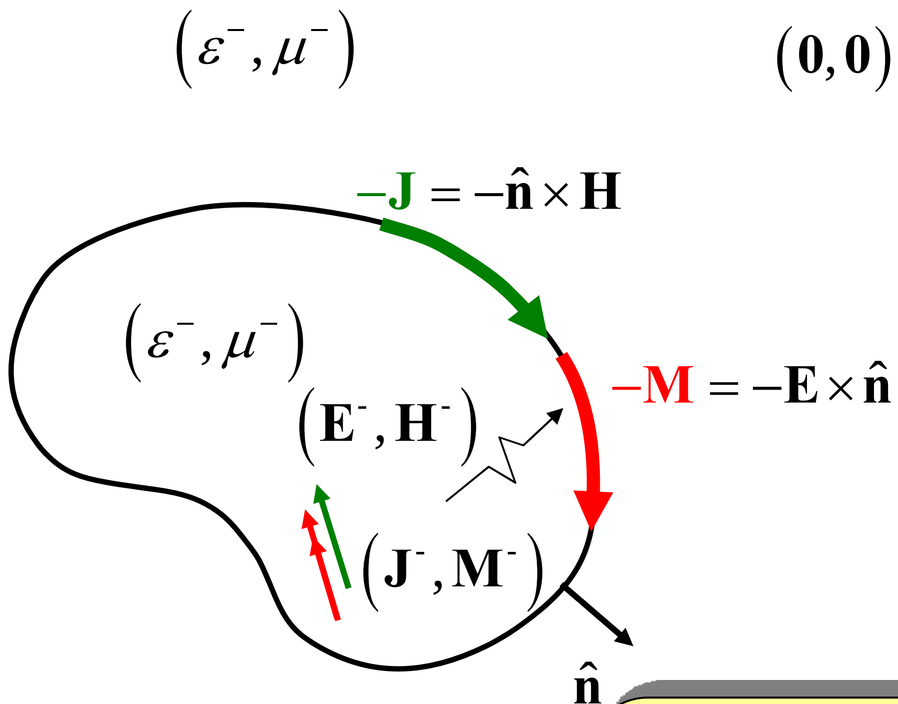
$$k^+ \equiv \omega\sqrt{\mu^+\varepsilon^+},$$
$$\eta^+ \equiv \sqrt{\frac{\mu^+}{\varepsilon^+}}$$

Tested null field conditions:

1) $<\Lambda_m;\mathbf{E}(\mathbf{J},\mathbf{M})>+<\Lambda_m;\mathbf{E}^+> \;=\; 0, \qquad \mathbf{r}\in\lim_{\mathbf{r}\uparrow\mathcal{S}}\mathcal{S}$

2) $<\Lambda_m;\mathbf{H}(\mathbf{J},\mathbf{M})>+<\Lambda_m;\mathbf{H}^+> \;=\; 0, \qquad \mathbf{r}\in\lim_{\mathbf{r}\uparrow\mathcal{S}}\mathcal{S}$

Interior Equivalence, Exterior Null Field Conditions



$$k^- \equiv \omega\sqrt{\mu^- \varepsilon^-},$$
$$\eta^- \equiv \sqrt{\frac{\mu^-}{\varepsilon^-}}$$

Tested null field conditions:

3) $< \Lambda_m; \mathbf{E}(-\mathbf{J}, -\mathbf{M}) > + < \Lambda_m; \mathbf{E}^- > \; = \; 0, \qquad \mathbf{r} \in \lim_{\mathbf{r} \downarrow \mathcal{S}} \mathcal{S}$

4) $< \Lambda_m; \mathbf{H}(-\mathbf{J}, -\mathbf{M}) > + < \Lambda_m; \mathbf{H}^- > \; = \; 0, \qquad \mathbf{r} \in \lim_{\mathbf{r} \downarrow \mathcal{S}} \mathcal{S}$

The PMCHWT Equations

Tested null field conditions:

1) $<\Lambda_m;\mathbf{E}(\mathbf{J},\mathbf{M})>+<\Lambda_m;\mathbf{E}^+> \;=\; 0, \quad \mathbf{r}\in\lim_{\mathbf{r}\uparrow\mathcal{S}}\mathcal{S}$

2) $<\Lambda_m;\mathbf{H}(\mathbf{J},\mathbf{M})>+<\Lambda_m;\mathbf{H}^+> \;=\; 0, \quad \mathbf{r}\in\lim_{\mathbf{r}\uparrow\mathcal{S}}\mathcal{S}$

3) $<\Lambda_m;\mathbf{E}(-\mathbf{J},-\mathbf{M})>+<\Lambda_m;\mathbf{E}^-> \;=\; 0, \quad \mathbf{r}\in\lim_{\mathbf{r}\downarrow\mathcal{S}}\mathcal{S}$

4) $<\Lambda_m;\mathbf{H}(-\mathbf{J},-\mathbf{M})>+<\Lambda_m;\mathbf{H}^-> \;=\; 0, \quad \mathbf{r}\in\lim_{\mathbf{r}\downarrow\mathcal{S}}\mathcal{S}$

Any linear combination of 1) and 2) and of 3) and 4) constitutes a valid coupled pair of integral equations for unknowns J and M ….though their solution may not be unique at all frequencies

Most common formulation is PMCHWT, obtained by equating 1) to 3) and 2) to 4); ***it is equivalent to enforcing continuity of tangential*** $\mathbf{E}$ ***and*** $\mathbf{H}$ ***at*** $\mathcal{S}$:

$$\boxed{\begin{aligned}\lim_{\mathbf{r}\uparrow\mathcal{S}}<\Lambda_m;\mathbf{E}(\mathbf{J},\mathbf{M})>+<\Lambda_m;\mathbf{E}^+> &= \lim_{\mathbf{r}\downarrow\mathcal{S}}<\Lambda_m;\mathbf{E}(-\mathbf{J},-\mathbf{M})>+<\Lambda_m;\mathbf{E}^-> \\ \lim_{\mathbf{r}\uparrow\mathcal{S}}<\Lambda_m;\mathbf{H}(\mathbf{J},\mathbf{M})>+<\Lambda_m;\mathbf{H}^+> &= \lim_{\mathbf{r}\downarrow\mathcal{S}}<\Lambda_m;\mathbf{H}(-\mathbf{J},-\mathbf{M})>+<\Lambda_m;\mathbf{H}^->\end{aligned}}$$

PMCHWT is both unique and well-conditioned

Field and Current Representations

Represent fields via their potentials :

$$\mathbf{E}(\pm\mathbf{J},\pm\mathbf{M}) = \mp j\omega\mathbf{A}^{\pm}(\mathbf{J}) \mp \nabla\Phi^{\pm}(\mathbf{J}) \mp \frac{1}{\varepsilon^{\pm}}\nabla\times\mathbf{F}^{\pm}(\mathbf{M})$$

$$= \mp j\omega\mu^{\pm}\int_{\mathcal{S}} G^{\pm}(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\,d\mathcal{S}' \pm \frac{\nabla}{j\omega\varepsilon^{\pm}}\int_{\mathcal{S}} G^{\pm}(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{J}(\mathbf{r}')\,d\mathcal{S}'$$

$$\mp \lim_{\mathbf{r}\uparrow\downarrow\mathcal{S}} \nabla\times\int_{\mathcal{S}} G^{\pm}(\mathbf{r},\mathbf{r}')\mathbf{M}(\mathbf{r}')\,d\mathcal{S}'$$

$$G^{\pm}(\mathbf{r},\mathbf{r}') = \frac{e^{-jk^{\pm}R}}{4\pi R}$$

$$\mathbf{H}(\pm\mathbf{J},\pm\mathbf{M}) = \mp j\omega\mathbf{F}^{\pm}(\mathbf{M}) \mp \nabla\Psi^{\pm}(\mathbf{M}) \pm \frac{1}{\mu^{\pm}}\nabla\times\mathbf{A}^{\pm}(\mathbf{J})$$

$$= \mp j\omega\varepsilon^{\pm}\int_{\mathcal{S}} G^{\pm}(\mathbf{r},\mathbf{r}')\mathbf{M}(\mathbf{r}')\,d\mathcal{S}' \pm \frac{\nabla}{j\omega\mu^{\pm}}\int_{\mathcal{S}} G^{\pm}(\mathbf{r},\mathbf{r}')\nabla'\cdot\mathbf{M}(\mathbf{r}')\,d\mathcal{S}'$$

$$\pm \lim_{\mathbf{r}\uparrow\downarrow\mathcal{S}} \nabla\times\int_{\mathcal{S}} G^{\pm}(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\,d\mathcal{S}'$$

Expansion of Equivalent Currents

Represent currents using div - conforming bases :

$$\mathbf{J}(\mathbf{r}) \approx \sum_{n=1}^{N} I_n \Lambda_n(\mathbf{r})$$

$$\mathbf{M}(\mathbf{r}) \approx \sum_{n=1}^{N} V_n \Lambda_n(\mathbf{r})$$

Substitute these into the tested (weak form) of the PMCHWT equations and rearrange.

Discretized Form of PMCHWT Equations

$$\begin{bmatrix} \left[Z_{mn}^{+} + Z_{mn}^{-}\right] & \left[-\beta_{mn}^{+} - \beta_{mn}^{-}\right] \\ \left[\beta_{mn}^{+} + \beta_{mn}^{-}\right] & \left[\ Y_{mn}^{+} + Y_{mn}^{-}\ \right] \end{bmatrix} \begin{bmatrix} I_n \\ V_n \end{bmatrix} = \begin{bmatrix} V_m^{\mathrm{i}} \\ I_m^{\mathrm{i}} \end{bmatrix}$$

where

$$Z_{mn}^{\pm} = \eta_{mn}^{\pm 2} Y_{mn}^{\pm} = j\omega L_{mn}^{\pm} + \frac{1}{j\omega} S_{mn}^{\pm} = j\eta_{mn}^{\pm} \left[k^{\pm} \iint_{\mathcal{S}\ \mathcal{S}} G^{\pm} \left(\Lambda_m \cdot \Lambda_n - \frac{1}{k^{\pm 2}} \nabla \cdot \Lambda_m \nabla \cdot \Lambda_n \right) d\mathcal{S}' d\mathcal{S} \ \right]$$

$$L_{mn}^{\pm} = \mu^{\pm} < \Lambda_m ; G^{\pm}, \Lambda_n >, \ S_{mn}^{\pm} = \frac{1}{\varepsilon^{\pm}} < \nabla \cdot \Lambda_m, G^{\pm}, \nabla \cdot \Lambda_n >$$

$$\beta_{mn}^{\pm} = - < \Lambda_m ; \nabla G^{\pm} \times, \Lambda_n > = - \iint_{\mathcal{S}\ \mathcal{S}} \Lambda_m \cdot \nabla G^{\pm} \times \Lambda_n \, d\mathcal{S}' d\mathcal{S} \ , \ G^{\pm} = \frac{e^{-jk^{\pm}R}}{4\pi R},$$

$$V_m^{\mathrm{i}} = < \Lambda_m ; \mathbf{E}^{+} - \mathbf{E}^{-} >, \quad I_m^{\mathrm{i}} = \ < \Lambda_m ; \mathbf{H}^{+} - \mathbf{H}^{-} >,$$

Note that $\pm\frac{\mathbf{J}}{2}, \pm\frac{\mathbf{M}}{2}$ terms *cancel* both in formulation and in self term element matrices (i.e., $\frac{1}{2} < \Lambda_m ; \Lambda_n >$ terms).

Row Scaling

Consider

$$[a_{mn}][x_n] = [b_m] \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

To scale row p **by** C_R**,**

- **Multiply all elements of** p**th row of** $[a_{mn}]$ **by** C_R
- **Multiply** p**th row of** $[b_m]$ **by** C_R

$$\begin{array}{r} \\ \\ \text{row p ->} \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ C_R\, a_{pq} & \cdots & C_R\, a_{pN} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ C_R\, b_p \\ \vdots \\ b_N \end{bmatrix}$$

Column Scaling

Consider

$$\left[a_{mn}\right]\left[x_n\right]=\left[b_m\right] \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

To scale column q by C_C ,

- Multiply all elements of qth column of $\left[a_{mn}\right]$ by C_C
- Divide qth row of $\left[x_n\right]$ by C_C

$$\begin{array}{c} \text{column q} \\ \begin{bmatrix} a_{11} & \cdots & C_C a_{1q} & \cdots & a_{1N} \\ \vdots & & \vdots & & \vdots \\ a_{N1} & \cdots & C_C a_{Nq} & \cdots & a_{NN} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_q / C_C \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

- Note that column scaling scales the solution vector, $\left[x_n\right]$, which must be "unscaled" after the system is solved!

Normalization and Symmetrization of PMCHWT Equations

Normalize and symmetrize the system matrix by

- **multiplying first column block by $1/\eta_0$, renormalizing the current vector I_n**
- **multiplying the second row block by $-j\eta_0$**
- **multiplying the second column block by j:**

$$\begin{bmatrix} \left[\dfrac{Z_{mn}^{+} + Z_{mn}^{-}}{\eta_0} \right] & -j\left[\beta_{mn}^{+} + \beta_{mn}^{-} \right] \\ -j\dfrac{\cancel{\eta_0}}{\cancel{\eta_0}}\left[\beta_{mn}^{+} + \beta_{mn}^{-} \right] & \left[\eta_0 \left(Y_{mn}^{+} + Y_{mn}^{-} \right) \right] \end{bmatrix} \begin{bmatrix} \eta_0 I_n \\ -jV_n \end{bmatrix} = \begin{bmatrix} V_m^{\mathrm{i}} \\ -j\eta_0 I_m^{\mathrm{i}} \end{bmatrix}$$

Note that, unfortunately,

$$\eta_0\left(Y_{mn}^{+} + Y_{mn}^{-} \right) = \eta_0 \left(\frac{Z_{mn}^{+}}{\eta^{+2}} + \frac{Z_{mn}^{-}}{\eta^{-2}} \right) \neq \frac{Z_{mn}^{+} + Z_{mn}^{-}}{\eta_0}$$

Far Field Computation

$$\mathbf{E} \xrightarrow[r\to\infty]{} -j\omega(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}})\cdot\mathbf{A}+j\omega\eta(\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}})\cdot\mathbf{F} \quad (\textbf{Note } \nabla\Phi \xrightarrow[r\to\infty]{} -j\omega\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{A})$$

$$\mathbf{H} \xrightarrow[r\to\infty]{} -j\omega(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}})\cdot\mathbf{F}+\frac{j\omega}{\eta}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}-\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}})\cdot\mathbf{A} \quad (\textbf{Note } \nabla\Psi \xrightarrow[r\to\infty]{} -j\omega\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{F})$$

where

$$\mathbf{A}=\frac{\mu^+e^{-jk^+r}}{4\pi r}\int_{\mathcal{S}}\mathbf{J}(\mathbf{r}')e^{jk^+\hat{\mathbf{r}}\cdot\mathbf{r}'}d\mathcal{S}' \approx \frac{\mu^+e^{-jk^+r}}{4\pi r}[\tilde{\Lambda}_n]^t[I_n]$$

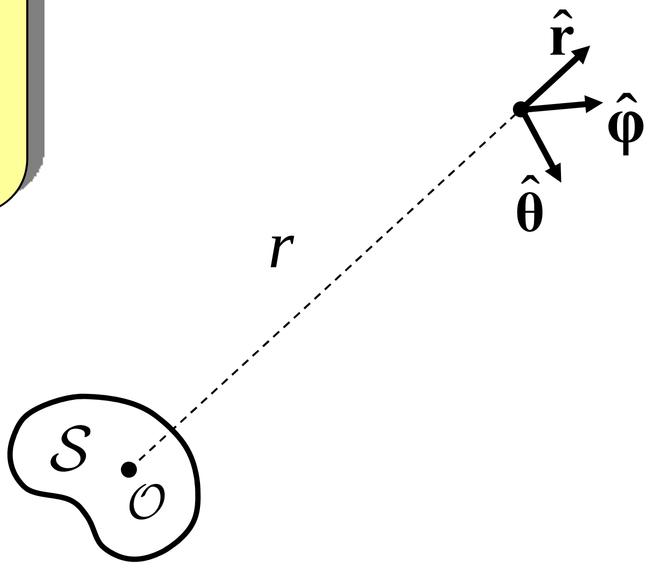
$$\mathbf{F}=\frac{\varepsilon^+e^{-jk^+r}}{4\pi r}\int_{\mathcal{S}}\mathbf{M}(\mathbf{r}')e^{jk^+\hat{\mathbf{r}}\cdot\mathbf{r}'}d\mathcal{S}' \approx \frac{\varepsilon^+e^{-jk^+r}}{4\pi r}[\tilde{\Lambda}_n]^t[V_n]$$

$$\hat{\mathbf{r}}=\hat{\mathbf{x}}\cos\phi\sin\theta+\hat{\mathbf{y}}\sin\phi\sin\theta+\hat{\mathbf{z}}\cos\theta$$

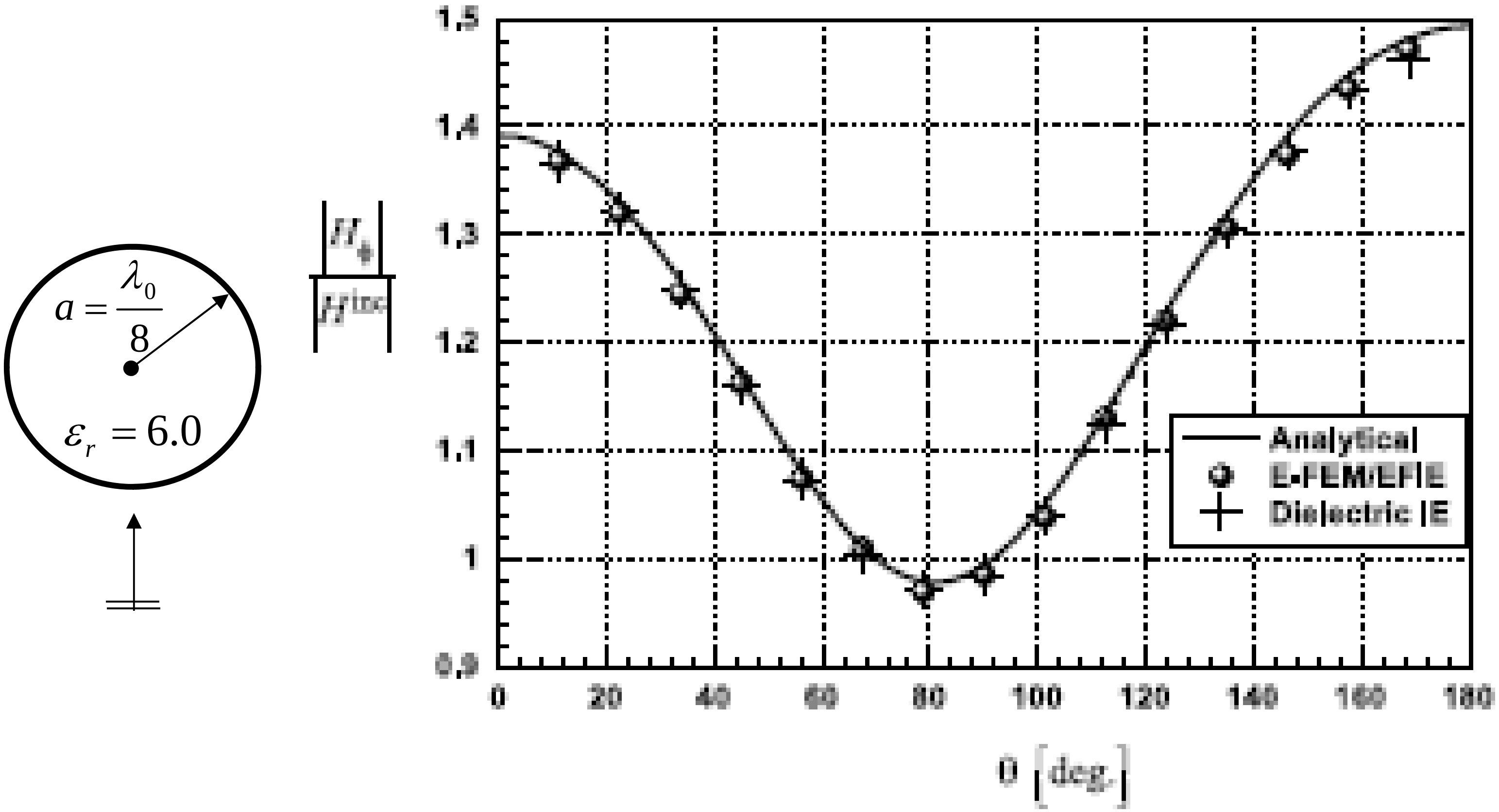
$$\hat{\boldsymbol{\theta}}=\hat{\mathbf{x}}\cos\phi\cos\theta+\hat{\mathbf{y}}\sin\phi\cos\theta-\hat{\mathbf{z}}\sin\theta$$

$$\hat{\boldsymbol{\varphi}}=-\hat{\mathbf{x}}\sin\phi+\hat{\mathbf{y}}\cos\phi$$

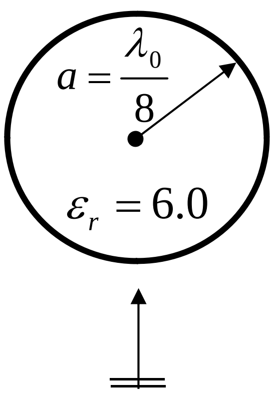
$$[\tilde{\Lambda}_n]\equiv\left[\int_{\mathcal{S}}\Lambda_n(\mathbf{r}')e^{jk^+\hat{\mathbf{r}}\cdot\mathbf{r}'}d\mathcal{S}'\right]$$

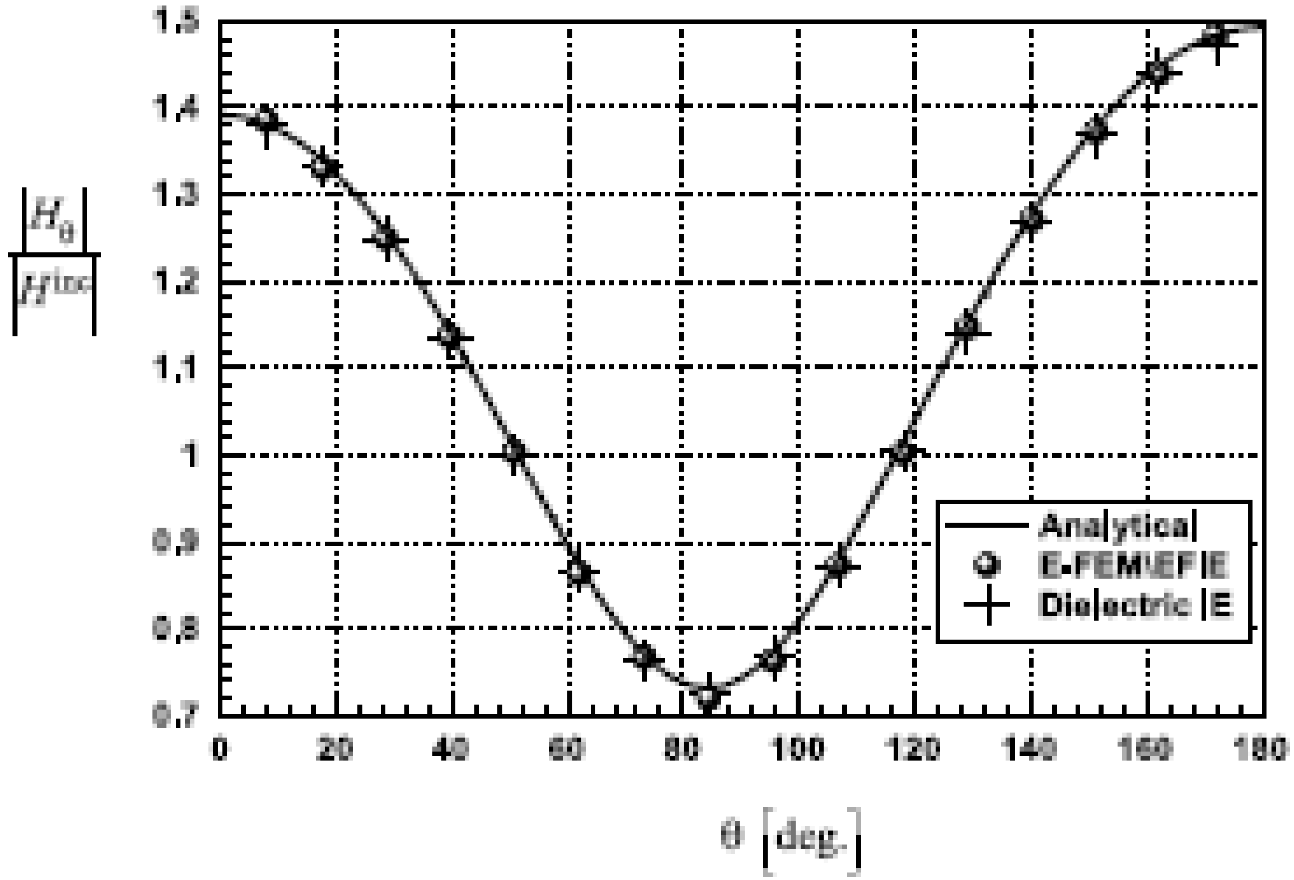


Surface Magnetic Field, $\mathbf{H}_{\phi}$
Dielectric Sphere



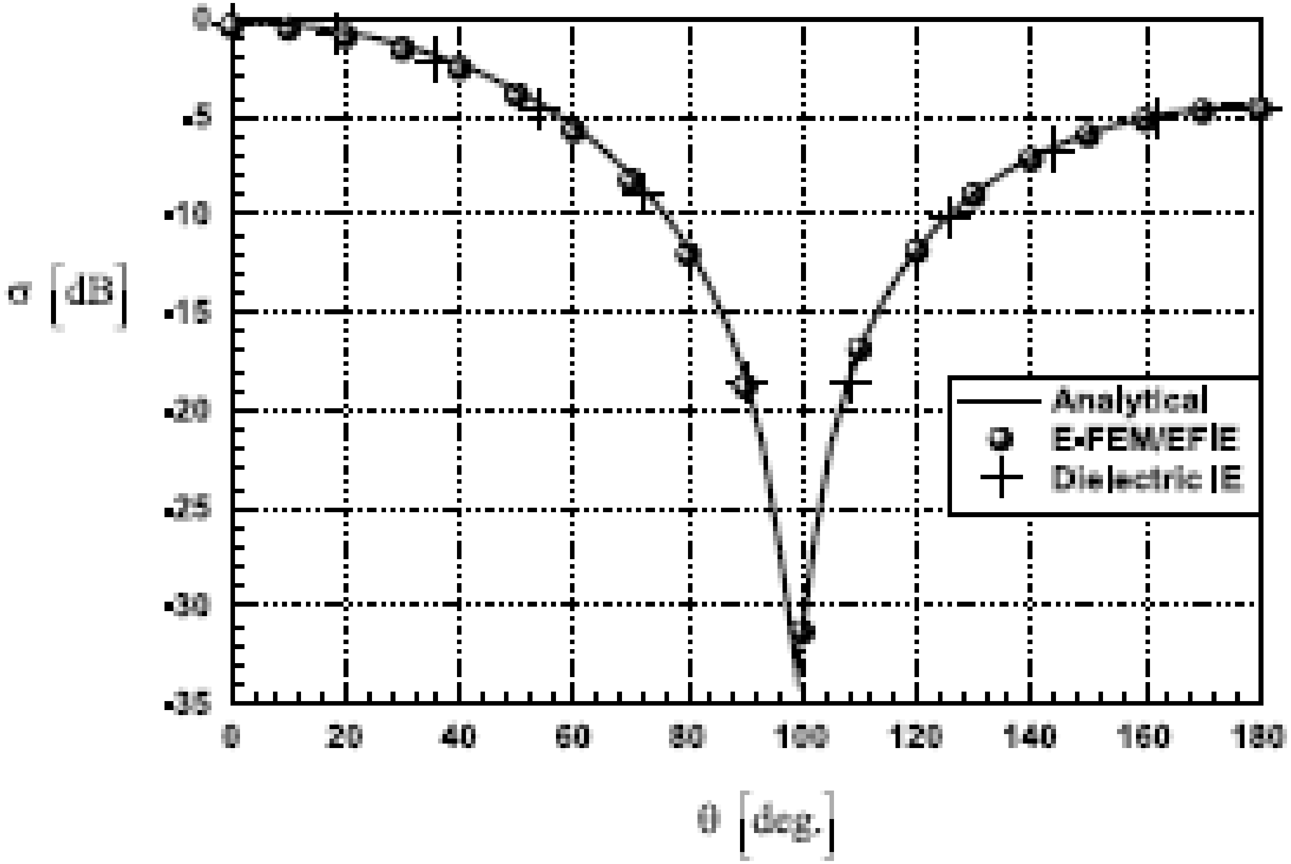
Surface Magnetic Field, $\mathbf{H}_\theta$
Dielectric Sphere





Radar Cross Section, Dielectric Sphere

$$\sigma(\hat{r}, \hat{k}) = \lim_{r \to +\infty} 4\pi r^2 \frac{|E|^2}{|E^{\mathrm{inc}}|^2}.$$



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