What is the relationship between the length of the filars and the period of a bifilar pendulum?

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1 Introduction

A bifilar pendulum is widely used to measure the moment of inertia of objects, which quantifies the amount of torque required to cause a specific angular acceleration along an axis. By observing the minute irregularities in the moment of inertia of the Earth, geologists can measure seismic events in remote locations, changes in the Earth's magnetic inclination and the effect of tides on the Earth's gravitational acceleration (Davison, 1894). Due to its high accuracy, bifilar pendulums are now used to measure the moment of inertia of complex geometries, such as aircrafts and ships that would otherwise be infeasible to solve by hand (Jardin and Mueller, 2007). Since the moment of inertia is a critical parameter to aerodynamic and hydrodynamic stability, understanding the dynamics of a bifilar pendulum can lead to better designs in aircrafts, enhancing the safety in transportation systems.

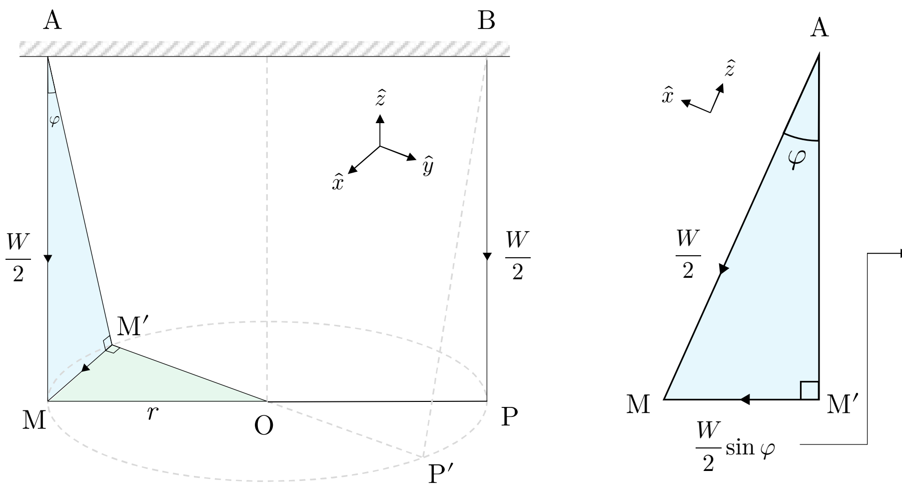
Although the general equation for bifilar pendulums have been well established, Klopsteg (1930) acknowledged the difficulty of determining the centre of mass of the test object, Kane and Tseng (1967) determined that minor inequalities of filar lengths will exhibit strong non-linear effects and chaotic side-sway motion, while Denman (1992) observed that torsional oscillations and vibrations in the filars cause significant deviations in the amplitude and period of bifilar pendulums. Given the wide range of unfulfilled assumptions and the variability of experimental results, the formulation of an accurate relationship will better aid real-life problems.

2 Framework

The following section aims to derive the equation of the bifilar pendulum, suspended by a rod (Amrozia and Muhammad, 2017; Then, 1965; Wouter, 2016):

$$T = \frac{2\pi l_{rod}}{r}\sqrt{\frac{l}{12g}} \tag{1}$$

Consider a bifilar pendulum with a rod MP suspended by filars AM and BP, disturbed from its equilibrium position MP to M′P′ and allowed to oscillate about point O along the $\hat{z}$ axis:



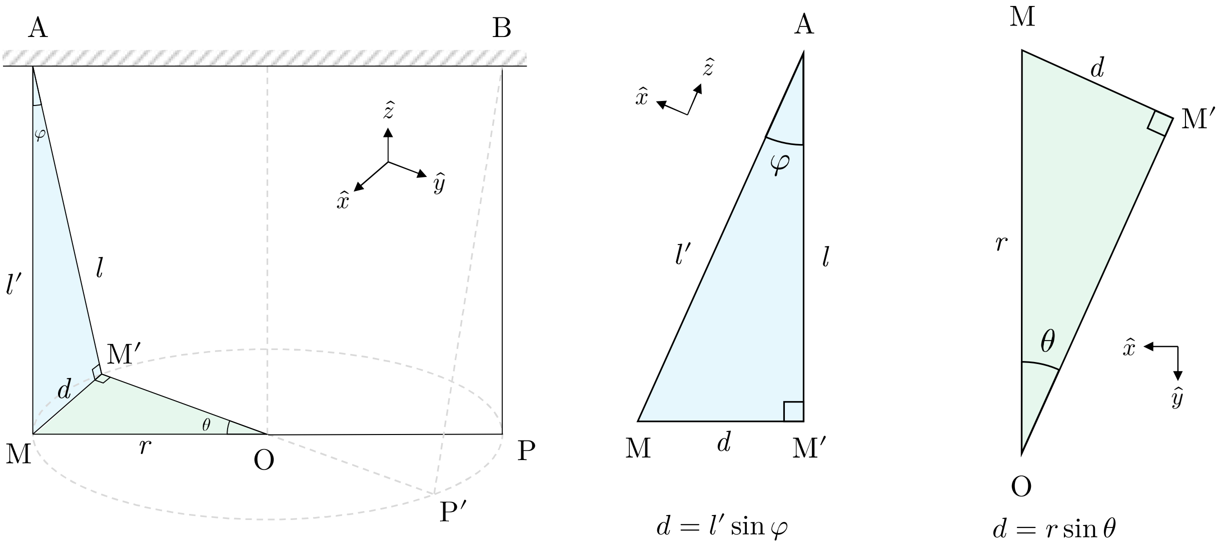
For small angles ($\varphi \to 0$)
Total restoring torque

$$\begin{aligned} \tau &\approx -rF \\ \tau &\approx -r(2\left(\frac{W}{2}\sin\varphi\right)) \\ \tau &\approx -r(2\left(\frac{mg}{2}\sin\varphi\right)) \\ \tau &\approx -mgr\sin\varphi \end{aligned} \tag{2}$$

Figure 1. The decomposition of forces of a bifilar pendulum disturbed from its equilibrium position MP.

The total restoring torque due to the weight of the rod is obtained in Equation 2. To better represent the motion of the rod, the relationship between φ and θ is found to be:

For small angles $(\theta \to 0, \varphi \to 0)$:



$$d = l' \sin\varphi$$
$$\text{As } \varphi \to 0, l' \approx l$$
$$d \approx l \sin\varphi$$

$$d = r\sin\theta$$
$$\text{As } \theta \to 0, \sin\theta \approx \theta$$
$$d \approx r\theta$$

$$\sin\varphi \approx \frac{r\theta}{l} \tag{3}$$

Figure 2. The approximate relationship between φ and θ using trigonometric identities.

Substituting Equation 3 into Equation 2, the total restoring torque expressed in terms of θ is:

$$\tau = -\frac{mgr^2\theta}{l} \tag{4}$$

Substituting Equation 4 into Newton's Second Law of Angular motion yields:

$$\tau = I\ddot{\theta}$$
$$\ddot{\theta} + \frac{mgr^2}{Il}\theta = 0 \tag{5}$$

Solving the second-order differential equation in Equation 5 with respect to time t, with initial angle $\theta_i = \theta_0$ and initial angular velocity $\dot{\theta}_i = 0$, the solution is:

$$\theta = \theta_0 \cos\left(\sqrt{\frac{mgr^2}{Il}}t\right) \tag{6}$$

Therefore, the period of the cosine curve in Equation 6 is:

$$T = \frac{2\pi}{\sqrt{\frac{mgr^2}{Il}}} = \frac{2\pi}{r}\sqrt{\frac{Il}{mg}} \tag{7}$$

The moment of inertia of a slender cylindrical rod rotating about its centre of mass, along the $\hat{z}$ axis is given by (Weisstein, 1996):

$$I = \frac{1}{12}m{l_{rod}}^2 \tag{8}$$

Substituting (8) into (7) and simplifying the equation yields the final relationship:

$$T = \frac{2\pi l_{rod}}{r}\sqrt{\frac{l}{12g}} \tag{9}$$

There are several required assumptions to support Equation 9:

1. The angle θ is sufficiently small, such that the vertical translational kinetic energy is negligible when compared to the rotational kinetic energy, otherwise the small angle approximation will not hold, and increase the error in the period by the order of $O(\theta^3)$.
2. There is no damping, specifically the loss in rotational kinetic energy, such as work done to the rod due to air resistance, otherwise θ will decrease over time, affecting T.
3. The system has a high degree of symmetricity, namely the filars AM and BP are parallel to each other, have the same length l and aligned to the $\hat{z}$ axis; the top AB and rod MP are parallel to each other, have the same length r and aligned to the $\hat{x}$-$\hat{y}$ plane. The axis of rotation should also pass through the midpoint and centre of mass of rod MP, otherwise slight deviations can amplify chaotic motion (Kane and Tseng, 1967).
4. There are no external forces that can cause lateral or vertical motion, ensuring stability.
5. The filars are assumed to be non-rigid, inextensible and have no torsional rigidity (Uhler, 1923). Microscopically, the strong intermolecular attraction between molecules of a particularly rigid filar may cause resistance to deformation, causing movements to require additional tension, resulting in damped responses and unpredictable motion.
6. The filars are massless, otherwise it may cause the centre of mass of the test subject to be shifted upwards, decreasing the effective filar length l and increase the mass m, overall decreasing the period of the oscillations (Karlin and Maday, 1985).

Additionally, Equation 9 can be linearised by expressing it in the slope-intercept form:

$$\underbrace{T}_{\substack{\text{Plotted}\\\text{on y-axis}}} = \underbrace{\frac{2\pi l_{rod}}{r}\sqrt{\frac{1}{12g}}}_{\substack{\textbf{Slope of}\\\textbf{graph}}}\underbrace{\sqrt{l}}_{\substack{\text{Plotted}\\\text{on x-axis}}} + \underbrace{0}_{\substack{\textbf{y-}\\\textbf{intercept}}} \tag{10}$$

When T is plotted against $\sqrt{l}$, the expected slope and y-intercept is $\frac{2\pi l_{rod}}{r}\sqrt{\frac{1}{12}}$ and 0 respectively.

3 Variables

Type	Name	Symbol	Description
Independent	Length of filars	l	The distance between the top AB and the centre of mass of rod MP. This is measured using a metre ruler and a series of adjustments.
Dependent	Period	T	The time taken for the rod to oscillate one period. This is measured by extracting the temporal changes in θ from a recording and deriving its period T by using Python scripts in Appendix 17.3 and 17.4.
Controlled	Radius of gyration	r	The distance between the axis of rotation and the filars. It is kept constant by securing the filar to a specific location on the rod using tape.
Controlled	Elastic strain energy	U	One of the mechanical properties of the filar. Although Assumption 5 is hard to be fulfilled, by using the same filars, the straining effect of the string is kept constant and therefore predictable.
Controlled	Initial angle of gyration	θ_i	The initial angle of which the rod is disturbed from its equilibrium. To satisfy Assumption 1, θ_i is kept constant at π radians by arranging the filars AM and BP to briefly come in contact each other.

4 Pilot Study

As the percentage error in the period of the equation of the bifilar pendulum ($T = \frac{2\pi l_{rod}}{r}\sqrt{\frac{l}{12g}}$) is $\frac{\Delta T}{T_0}$, the denominator T_0 must be sufficiently large to reduce the percentage error in T. There are three ways in which the range of the other variables can be adjusted to produce the least error:

1. Increasing the length of the rod (l_{rod})
2. Increasing the length of the filars (l)
3. Decreasing radius of gyration (r)

As explored in Assumption 2, the loss in horizontal rotational kinetic energy should be minimised. Hence, to minimise the drag, a long slender rod with low cross-sectional area, of length $l_{rod} = 0.1958 \pm 0.0001$m and diameter $d_{rod} = 0.00678 \pm 0.00001$m has been selected as the test object.

However, if the radius of gyration (r) is too low, the percentage error $\frac{\Delta r}{r_0}$ will increase. Hence, to better assess the effects of l and r on the percentage error, a 2D heatmap is created:

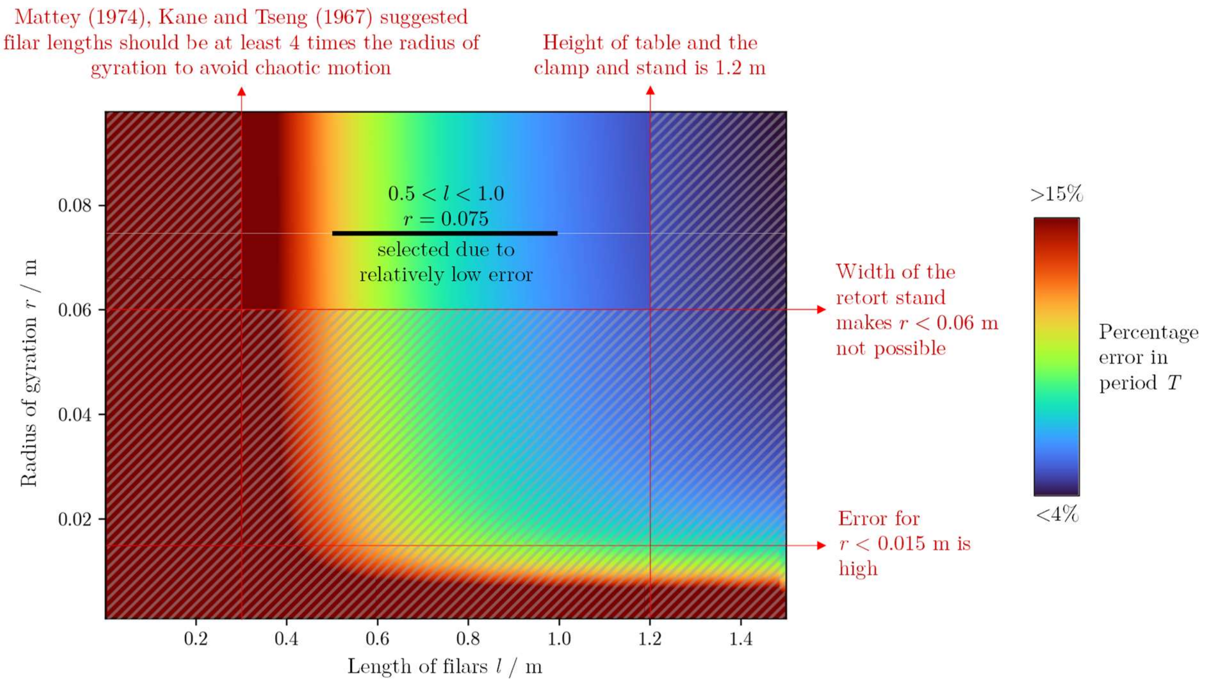


Figure 3. A heatmap of the percentage error in T for $0 < l < 1.5$ s and $0 < r < \frac{l_{rod}}{2}$ m.[1]

As observed in Fig. 3, due to experimental setup limitations, the radius of gyration r is selected to be 0.075m. As for the range of the independent variable, length of filars l, it has been decided to perform 7 regular intervals between $0.5 < l < 1.0$ m.

Moreover, because $T \propto \sqrt{l}$, when $l \to \infty$, $\frac{\mathrm{d}T}{\mathrm{d}l} \to 0$, meaning that for increasing filar lengths, the corresponding magnitude of change in T is asymptotically decreasing. Hence, in order to avoid insignificant changes in ΔT, the length of filars l is controlled to be below 1.0m, as demonstrated in Figure 4.

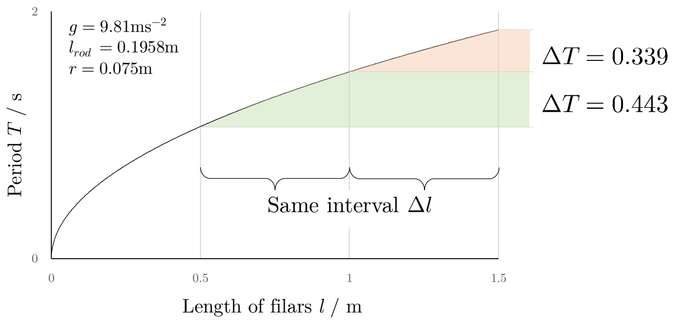


Figure 4. A graph of T against l.

[1] The relevant formulae and code used to generate the visualisation can be found in Appendix 17.1 and 17.2.

5 Apparatus

2x Clamp and stand	**3x** Metallic rod	**1x** Metre ruler ($\pm$0.001m)
2x G-clamp	**2x** Cotton yarn	**1x** Vernier scale ($\pm$0.0001m)
	1x Camera	**1x** Micrometer ($\pm$0.00001m)

6 Procedure and Precautions

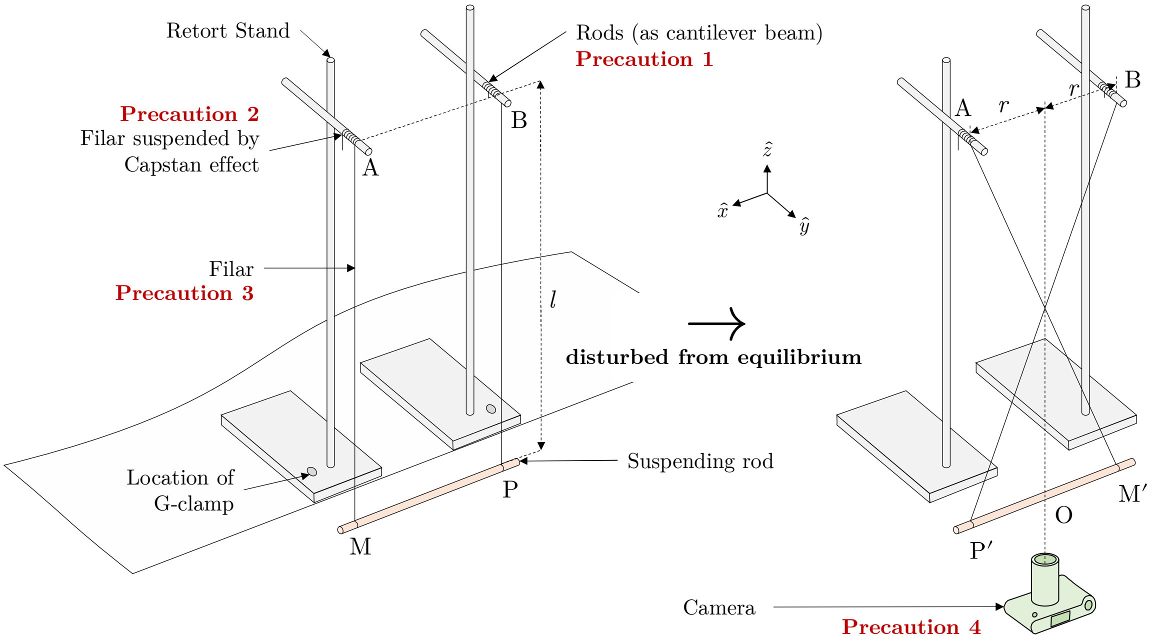


Figure 4. A diagram showing the setup of the experiment as rod MP is disturbed from its equilibrium.

1. Prepare and setup the apparatus accordingly to Figure 4.
2. Rotate rod MP clockwise such that filar AM comes in contact with filar BP, forming an angle of gyration θ of π.
3. Release rod M′P′ and allow it to rotate freely about O along the $\hat{z}$ axis.
4. Start the recording when the angle of gyration θ decreases to a small magnitude.
5. Stop the recording when the number of oscillations reaches 50.
6. Adjust the length of the filars l to the intervals by rotating the capstan at A and B.
7. Repeat steps 1-6 for three trials.

In order to reduce the possible sources of error and minimise the risk of accidents, and to satisfy assumptions such as Assumption 3, there are several precautions marked in red in Figure 4:

1. Ensure that the top AB is level and aligned along the $\hat{x}$ axis.
2. Ensure a high number of wraps around the capstan, increasing the loading force limit.
3. Ensure that filars AM and BP have the same length l and are correctly aligned along the $\hat{z}$ axis, especially during the adjustment of it in Step 6.
4. Ensure that the optical centre of the camera is colinear with O, the intersection of AM′ and BP′, such that the distance between the midpoint of AB has the same radius r.

7 Ethical, safety and environmental concerns

Since the only waste product in the experiment is the filars, which are biodegradable, there are no significant environmental or ethical concerns. Regarding safety concerns, there is a very small risk where the filar at A and B loses traction, causing rod MP to fall and cause injury, but the possibility of such has already been minimised by increasing the wrap count around the capstan.

8 Raw Data Table

Measured Length of Filars			**Period**	
	Trial 1	Trial 2	Trial 3	Half Range
l_{raw} m $\Delta_{\text{instrument}} = 0.001\text{m}$	$T_1 \pm \Delta T_1$ $\text{s} \pm \text{s}$ $\Delta_{\text{instrument}} = 0.02\text{s}$	$T_2 \pm \Delta T_2$ $\text{s} \pm \text{s}$ $\Delta_{\text{instrument}} = 0.02\text{s}$	$T_3 \pm \Delta T_3$ $\text{s} \pm \text{s}$ $\Delta_{\text{instrument}} = 0.02\text{s}$	$T \pm \Delta T$ $\text{s} \pm \text{s}$ $\Delta_{\text{instrument}} = 0.02\text{s}$
0.494	1.07 ± 0.02	1.07 ± 0.02	1.07 ± 0.02	1.07 ± 0.03
0.594	1.15 ± 0.02	1.15 ± 0.02	1.15 ± 0.02	1.15 ± 0.03
0.654	1.22 ± 0.02	1.23 ± 0.03	1.27 ± 0.03	1.24 ± 0.07
0.734	1.29 ± 0.03	1.31 ± 0.02	1.33 ± 0.03	1.31 ± 0.06
0.814	1.38 ± 0.02	1.38 ± 0.03	1.38 ± 0.02	1.38 ± 0.05
0.894	1.41 ± 0.04	1.45 ± 0.02	1.43 ± 0.02	1.43 ± 0.07
0.974	1.48 ± 0.02	1.55 ± 0.03	1.52 ± 0.03	1.52 ± 0.08

9 Qualitative Observation

During the experiment, the period is observed to increase as the length of filars increase. However, several assumptions outlined in the framework appears to be violated.

10 Processed Data Table

$$\delta f(n, \dots) = \sqrt{\sum\nolimits_n \left(\frac{\partial f}{\partial n}\delta n\right)^2} \qquad (\quad)$$

$$l = l_1 - \frac{1}{2}l_2 + \frac{1}{2}l_3$$

$$\delta\sqrt{l} = \sqrt{\left(\frac{\delta l_1}{2\sqrt{l_1 - \frac{1}{2}l_2 + \frac{1}{2}l_3}}\right)^2 + \left(-\frac{\delta l_1}{4\sqrt{l_1 - \frac{1}{2}l_2 + \frac{1}{2}l_3}}\right)^2 + \left(\frac{\delta l_1}{4\sqrt{l_1 - \frac{1}{2}l_2 + \frac{1}{2}l_3}}\right)^2}$$

11 Graph

12 Introduction

13 Conclusion

14 Sources of error and limitations

15 Improvement and Extensions

16 References

17 Appendix

17.1 Formula for the percentage error in the period

Given the formula for a bifilar pendulum suspended by a rod:

$$T = \frac{2\pi l_{rod}}{r}\sqrt{\frac{l}{12g}} \qquad (\;)$$

The error in T is therefore:

$$\delta T = \sqrt{\left(\frac{\partial T}{\partial l_{rod}}\delta l_{rod}\right)^2 + \left(\frac{\partial T}{\partial r}\delta r\right)^2 + \left(\frac{\partial T}{\partial l}\delta l\right)^2}$$

$$\delta T = \sqrt{\left(\frac{2\pi}{r}\sqrt{\frac{l}{12g}}\delta l_{rod}\right)^2 + \left(-\frac{2\pi l_{rod}}{r^2}\sqrt{\frac{l}{12g}}\delta r\right)^2 + \left(\frac{\pi l_{rod}}{r}\sqrt{\frac{1}{12gl}}\delta l\right)^2} \qquad (\;)$$

17.2 Code for pilot study

Using the equation obtained in Appendix 17.1, the following code creates Fig. 3 and Fig. 4:

Filename: `pilot.py`

Language: `Python 3.8.5`

Packages used:

- `numpy 1.8.5`
- `pandas 1.1.2`
- `matplotlib 2.2.5`

```
'''
This script does the following:
- calculate T for 0.001 < l < 1.500, with increments of 0.001
- save the raw data to a CSV file.
- for 0.001 < l < 1.500, with increments of 0.001:
    - for 0.0010 < r < 0.0979, with increments of 0.0001:
        - calculate the percentage error in T
        - plot on a 2D axis with the 'turbo' colormap
- store the graph in a PNG image.
'''

import math
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

g = 9.81

def getT(l_rod, r, l):
    return 2*math.pi*l_rod/r*math.sqrt(l/12/g)

def getTerr(l_rod, δl_rod, r, δr, l, δl):
    return math.sqrt(
        math.pow(δl_rod*((2*math.pi/r) * (math.sqrt(l/12/g))),2) +
        math.pow(δr*((2*math.pi*l_rod/math.pow(r,2)) * (math.sqrt(l/12/g))),2) +
        math.pow(δl*((2*math.pi*l_rod/r) * (.5/math.sqrt(l/12/g))),2)
    )

df0 = pd.DataFrame([(l, getT(.1958, .075, l)) for l in np.arange(.001, 1.500, .001)], columns=['l', 'T'])
df0.to_csv('output1/pilot.csv', index=False)

data = []
```

```
l_rod, δl_rod = .1958, .0001
δl = .001
δr = .001
for l in np.arange(.001, 1.500, .001):
    for r in np.arange(.001, .0979, .0001):
        pctδT = min(getTerr(l_rod, δl_rod, r, δr, l, δl) / getT(l_rod, r, l), .15)
        data.append([l, r, pctδT])

df = pd.DataFrame(data, columns=['l', 'r', 'pctδT'])
plt.rcParams['font.family'] = 'Latin Modern Roman'
plt.scatter(x=df['l'],y=df['r'],c=df['pctδT'], cmap='turbo')
plt.xlim([.001, 1.500])
plt.ylim([.001, .0979])
plt.savefig('output1/pilot.png', dpi=300)
```

17.3 Code for parsing raw video

Filename: `parse.py`

Language: `Python 3.8.5`

Packages used:

- `rich 9.10.0`
- `numpy 1.8.5`
- `pandas 1.1.2`
- `opencv-python 4.5.3.56`

```
'''
This script does the following:
- read the list of videos from a specific folder
- for each video, process each frame:
    - for each frame, convert the frame to greyscale
    - perform canny edge detection
    - perform probabilistic hough line transform
    - for each possible location of the rod:
        - select the longest line
        - calculate the length of the rod (in pixels) using Pythagoras\
        - calculate the angle of the rod (in radians) using atan2
    - aggregate the temporal changes of the rod
- store the data in multiple CSV files for further processing
'''

from rich.progress import Progress
import numpy as np
import pandas as pd
import cv2
import os

class Line:
    def __init__(self, line):
        self.p1 = np.array(line[0][:2]).astype(int)
        self.p2 = np.array(line[0][2:4]).astype(int)

    def getLen(self):
        self.length = np.sqrt(np.sum((self.p1 - self.p2) ** 2, axis=0))
        return self

    def getAngle(self):
        self.angle = np.arctan2(
            self.p2[1] - self.p1[1],
            self.p2[0] - self.p1[0] if self.p1[0] < self.p2[0] else self.p1[0] - self.p2[0]
        )
        return self
```

```python
with Progress() as progress:
    files = os.listdir('src')
    task0 = progress.add_task(f"[green]Processing files...", total=len(files))
    for filename in files: # process each video in the specified folder
        progress.update(task0, advance=1, description=f"[green]Processing {filename}...")
        capture = cv2.VideoCapture(os.path.join("src", filename))

        totalframecount = int(capture.get(cv2.CAP_PROP_FRAME_COUNT))
        task1 = progress.add_task(f"[green]Parsing frames...", total=totalframecount)
        vals, framecount = [], 0
        while 1:
            success, frame = capture.read()
            if not success:
                break

            image = cv2.cvtColor(frame, cv2.COLOR_BGR2GRAY)
            edges = cv2.Canny(image, 100, 100, L2gradient=True)
            lines = cv2.HoughLinesP(edges, 1, np.pi/180, 100, maxLineGap=200)

            if lines is not None:
                l = Line(sorted(lines, key=lambda x:Line(x).getLen().length, reverse=True)[0]).getLen().getA
ngle()
                vals.append([framecount, l.length, l.angle])

            progress.update(task1, advance=1)
            framecount += 1

        df = pd.DataFrame(vals, columns=['frame', 'length', 'angle'])
        df.to_csv(os.path.join('output', f'{os.path.splitext(filename)[0]}.csv'), index=False)
```

17.4 Code for parsing raw data

Filename: `analyse.py`[2]

Language: `Python 3.8.5`

Packages used:

- `rich 9.10.0`
- `numpy 1.8.5`
- `pandas 1.1.2`
- `scipy 1.18.5`

```python
'''
This script does the following:
- read raw data from a specific folder
- for each raw data
    - smoothen noisy data using a non-uniform Savitzky-Golay filter
    - obtain the peaks of the smoothened data
    - for each peak:
        - calculate the average time
        - calculate the difference in time
        - calculate the average length of the rod (in pixels)
        - calculate the average angle of the rod (in radians)
- aggregate all processed data
- store the data in one CSV file for further processing
'''

import os
import numpy as np
import pandas as pd
from scipy.signal import argrelextrema
from rich.progress import Progress
```

[2] The code for the non-uniform Savitzy-Golay filter is obtained from https://dsp.stackexchange.com/a/64313.

```
def non_uniform_savgol_filter(x, y, window, polynom):
    half_window = window // 2
    polynom += 1

    A = np.empty((window, polynom))
    tA = np.empty((polynom, window))
    t = np.empty(window)
    y_smoothed = np.full(len(y), np.nan)

    for i in range(half_window, len(x) - half_window, 1):
        for j in range(0, window, 1):
            t[j] = x[i + j - half_window] - x[i]

        for j in range(0, window, 1):
            r = 1.0
            for k in range(0, polynom, 1):
                A[j, k] = r
                tA[k, j] = r
                r *= t[j]

        tAA = np.matmul(tA, A)
        tAA = np.linalg.inv(tAA)
        coeffs = np.matmul(tAA, tA)

        y_smoothed[i] = 0
        for j in range(0, window, 1):
            y_smoothed[i] += coeffs[0, j] * y[i + j - half_window]

        if i == half_window:
            first_coeffs = np.zeros(polynom)
            for j in range(0, window, 1):
                for k in range(polynom):
                    first_coeffs[k] += coeffs[k, j] * y[j]
        elif i == len(x) - half_window - 1:
            last_coeffs = np.zeros(polynom)
            for j in range(0, window, 1):
                for k in range(polynom):
                    last_coeffs[k] += coeffs[k, j] * y[len(y) - window + j]

    for i in range(0, half_window, 1):
        y_smoothed[i] = 0
        x_i = 1
        for j in range(0, polynom, 1):
            y_smoothed[i] += first_coeffs[j] * x_i
            x_i *= x[i] - x[half_window]

    for i in range(len(x) - half_window, len(x), 1):
        y_smoothed[i] = 0
        x_i = 1
        for j in range(0, polynom, 1):
            y_smoothed[i] += last_coeffs[j] * x_i
            x_i *= x[i] - x[-half_window - 1]

    return y_smoothed

df2s = []
with Progress() as progress:
    files = os.listdir('output')
    task = progress.add_task(f"[green]Processing files...", total=len(files))
    for filename in files:
        progress.update(task, advance=1, description=f'[green]Processing {filename}...')
        df = pd.read_csv(os.path.join("output", filename))
        smoothed = non_uniform_savgol_filter(df.frame.values, df.angle.values, 31, 5)
        idx = argrelextrema(smoothed, np.greater, order=5)[0]

        df1 = pd.DataFrame(columns=["frame", "length", "turning_smoothed"])
```

```python
        df1['frame'] = df.iloc[idx].frame
        df1['length'] = df.iloc[idx].length
        df1['turning_smoothed'] = smoothed[idx]

        df1l, df2l = df1.values.tolist(), []
        for i, j in enumerate(df1l[:-1]):
            r0, r1 = df1l[i], df1l[i+1]
            df2l.append([
                (r1[0] + r0[0]) / 2 / 60,
                (r1[0] - r0[0]) / 60,
                (r1[1] + r0[1]) / 2,
                (r1[2] + r1[2]) / 2
            ])

        identifier = os.path.splitext(filename)[0]
        df2s.append(pd.DataFrame(df2l, columns=[
            f"{identifier}_avg_frame",
            f"{identifier}_frame_diff",
            f"{identifier}_avg_length",
            f"{identifier}_avg_turning_smoothed"
        ]))

df2 = pd.concat(df2s, axis=1)
df2.to_csv(os.path.join("output1", "all.csv"), index=False)
```

17.5 Other

null