Answers to Selected Problems in McIntyre

I spent a semester coaching a student through Quantum Mechanics. We did tedious homework problems from the McIntyre Quantum Mechanics textbook. I decided to thresh-out the calculations and type the complete solutions. Caveat emptor tho!

solutions by: a.benedict balbuena

> **Problem 1: 2.02**
>
> Let $S_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle_x$. Then the matrix representation of S_x in the S_z basis is
>
> $$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
>
> Diagonalize this matrix to find the eigenvalues and the eigenvectors of S_x.

Solution:

In order to diagonalize the matrix representation of S_x in the S_z basis, we must find its eigenvalues and corresponding eigenvectors.

To find the eigenvalues (λ), we solve the characteristic equation $\det(S_x - \lambda I) = 0$, where I is the identity matrix.

$$S_x - \lambda I = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{pmatrix}$$

Now, we compute the determinant:

$$\det(S_x - \lambda I) = (-\lambda)(-\lambda) - \left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) = \lambda^2 - \frac{\hbar^2}{4}.$$

Setting the determinant to zero to find the eigenvalues:

$$\begin{aligned} \lambda^2 - \frac{\hbar^2}{4} &= 0 \\ \lambda^2 &= \frac{\hbar^2}{4} \\ \lambda &= \pm\frac{\hbar}{2} \end{aligned}$$

Thus, the two eigenvalues are $\lambda_1 = +\frac{\hbar}{2}$ and $\lambda_2 = -\frac{\hbar}{2}$.

Next, we find the eigenvector associated with each eigenvalue by solving the equation

$$(S_x - \lambda I)\vec{v} = 0.$$

1. Eigenvector for $\lambda_1 = +\frac{\hbar}{2}$:

Let the eigenvector be $|\psi_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$(S_x - \lambda_1 I)|\psi_1\rangle = \begin{pmatrix} -\frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the equation:

$$-\frac{\hbar}{2}a + \frac{\hbar}{2}b = 0 \implies a = b$$

The eigenvector is of the form $\begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To normalize this vector:

$$|a|^2(1^2 + 1^2) = 1 \implies 2|a|^2 = 1 \implies a = \frac{1}{\sqrt{2}}$$

So, the normalized eigenvector corresponding to the eigenvalue $+\frac{\hbar}{2}$ is:

$$|+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. Eigenvector for $\lambda_2 = -\frac{\hbar}{2}$:

   Let the eigenvector be $|\psi_2\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$.

$$(S_x - \lambda_2 I)|\psi_2\rangle = \begin{pmatrix} \frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & \frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   This gives the equation:

$$\frac{\hbar}{2}c + \frac{\hbar}{2}d = 0 \implies c = -d$$

   The eigenvector is of the form $\begin{pmatrix} -d \\ d \end{pmatrix} = d \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Normalizing the vector:

$$|d|^2((-1)^2 + 1^2) = 1 \implies 2|d|^2 = 1 \implies d = \frac{1}{\sqrt{2}}$$

   So, the normalized eigenvector corresponding to the eigenvalue $-\frac{\hbar}{2}$ is:

$$|-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The diagonalization of the S_x matrix yields the following eigenvalues and their corresponding normalized eigenvectors (in the S_z basis):

- **Eigenvalue:** $\lambda_1 = +\frac{\hbar}{2}$, with **Eigenvector:** $|+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- **Eigenvalue:** $\lambda_2 = -\frac{\hbar}{2}$, with **Eigenvector:** $|-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Problem 2: 2.04

Show by explicit matrix calculation that the matrix elements of a general operator $\mathbf{A}$ (within a spin-1/2 system) are

$$\mathbf{A} \doteq \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|-\rangle \end{pmatrix}.$$

Solution:

In a spin-1/2 system, we define the basis states using the eigenvectors of S_z: $|+\rangle$ (spin up) and $|-\rangle$ (spin down). These states are orthonormal:

$$\langle i|j\rangle = \delta_{ij}, \quad \text{where } i, j \in \{+, -\}$$

The matrix representation of a general operator $\mathbf{A}$ in this basis is defined by its matrix elements A_{ij}:

$$\mathbf{A} \doteq \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where the indices 1 and 2 correspond to the basis vectors $|+\rangle$ and $|-\rangle$.

The matrix elements are defined by taking the expectation value of $\mathbf{A}$ between the basis states:

- $A_{11} = \langle +|A|+\rangle$
- $A_{12} = \langle +|A|-\rangle$
- $A_{21} = \langle -|A|+\rangle$
- $A_{22} = \langle -|A|-\rangle$

Substituting these definitions into the matrix structure yields the desired form:

$$\mathbf{A} \doteq \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|-\rangle \end{pmatrix}$$

Derivation using the Completeness Relation

We can formally derive this using the completeness relation for the basis $\{|+\rangle, |-\rangle\}$:

$$I = |+\rangle\langle +| + |-\rangle\langle -|$$

We express the operator $\mathbf{A}$ as $\mathbf{A} = I\mathbf{A}I$:

$$\mathbf{A} = (|+\rangle\langle +| + |-\rangle\langle -|)\,\mathbf{A}\,(|+\rangle\langle +| + |-\rangle\langle -|)$$

Expanding this product yields four terms:

$$\begin{aligned} \mathbf{A} = |+\rangle\langle +|\mathbf{A}|+\rangle\langle +| \quad &+ \quad |+\rangle\langle +|\mathbf{A}|-\rangle\langle -| \\ + \quad |-\rangle\langle -|\mathbf{A}|+\rangle\langle +| \quad &+ \quad |-\rangle\langle -|\mathbf{A}|-\rangle\langle -| \end{aligned}$$

Since the inner products $\langle i|\mathbf{A}|j\rangle$ are scalars, we factor them out:

$$\mathbf{A} = (\langle +|A|+\rangle)|+\rangle\langle +| \; + \; (\langle +|A|-\rangle)| + \rangle\langle -| \; + \; (\langle -|A|+\rangle)|-\rangle\langle +| \; + \; (\langle -|A|-\rangle)|-\rangle\langle -|$$

To find the matrix representation $\mathbf{A}$, we replace the outer products $|i\rangle\langle j|$ with their corresponding matrix representations:

- $|+\rangle\langle +| \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $|+\rangle\langle -| \doteq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
- $|-\rangle\langle +| \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
- $|-\rangle\langle -| \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Substituting these yields the final matrix form:

$$\begin{aligned}\mathbf{A} &\doteq (\langle +|A|+\rangle)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (\langle +|A|-\rangle)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (\langle -|A|+\rangle)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (\langle -|A|-\rangle)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\doteq \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|-\rangle \end{pmatrix}\end{aligned}$$

Problem 3: 2.06

Verify that the spin component operator S_n along the direction $\hat{\mathbf{n}}$ has the matrix representation

$$S_n \doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}.$$

Solution:

The spin component operator S_n along the direction $\hat{\mathbf{n}}$ is defined as:

$$S_n = \mathbf{S} \cdot \hat{\mathbf{n}}$$

In Cartesian coordinates, the spin operator is:

$$\mathbf{S} = S_x\hat{\mathbf{i}} + S_y\hat{\mathbf{j}} + S_z\hat{\mathbf{k}}$$

The unit vector $\hat{\mathbf{n}}$ in spherical coordinates (θ, ϕ) is:

$$\hat{\mathbf{n}} = (\sin\theta\cos\phi)\hat{\mathbf{i}} + (\sin\theta\sin\phi)\hat{\mathbf{j}} + (\cos\theta)\hat{\mathbf{k}}$$

The dot product yields:

$$S_n = (S_x)(\sin\theta\cos\phi) + (S_y)(\sin\theta\sin\phi) + (S_z)(\cos\theta)$$

We use the known matrix representations in the S_z basis:

- $S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- $S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

- $S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Substituting these into the expression for S_n:

$$S_n \doteq (\sin\theta\cos\phi)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (\sin\theta\sin\phi)\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + (\cos\theta)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Factoring out $\frac{\hbar}{2}$:

$$S_n \doteq \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right]$$

Adding the components:

$$S_n \doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

Using Euler's formula ($e^{\pm i\phi} = \cos\phi \pm i\sin\phi$):

$$S_n \doteq \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}.$$

This verifies the required matrix representation.

Problem 4: 2.08

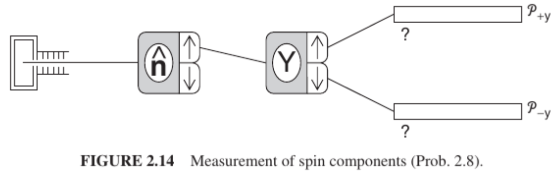


FIGURE 2.14 Measurement of spin components (Prob. 2.8).

Find the probabilities of the measurements shown in Fig. 2.14. The first Stern-Gerlach analyzer is aligned along the direction $\hat{\mathbf{n}}$ defined by the angles $\theta = \pi/4$ and $\phi = 5\pi/3$.

Solution:

The experiment involves two sequential Stern-Gerlach measurements. The initial state is the eigenstate of S_n with eigenvalue $+\hbar/2$, and this state is then measured along the y-axis.

We are given $\theta = \pi/4$ and $\phi = 5\pi/3$.

The initial state after the first analyzer (aligned along $\hat{\mathbf{n}}$) is the spin-up eigenstate along $\hat{\mathbf{n}}$, represented in the S_z basis as:

$$|+\rangle_n = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

For $\theta = \pi/4$, we have $\theta/2 = \pi/8$. The required trigonometric values are:

$$\cos(\pi/8) = \sqrt{\frac{1+\cos(\pi/4)}{2}} = \sqrt{\frac{1+1/\sqrt{2}}{2}}$$

$$\sin(\pi/8) = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \sqrt{\frac{1-1/\sqrt{2}}{2}}$$

For $\phi = 5\pi/3$, the phase factor is:

$$e^{i\phi} = \cos(5\pi/3) + i\sin(5\pi/3) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

The second analyzer measures S_y. The corresponding eigenstates in the S_z basis are:

$$|+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \langle +|_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}$$

We now compute for the probabilities.

Measuring spin up along y ($\mathcal{P}_{+y}$): The probability of observing spin up along y ($\mathcal{P}_{+y}$) is the squared magnitude of the projection:

$$\mathcal{P}_{+y} = |\langle +|_y|+\rangle_n|^2$$

For general angles, this probability simplifies to (as derived from the general matrix element S_n verification):

$$\mathcal{P}_{+y} = \frac{1}{2}(1+\sin\theta\sin\phi)$$

Substituting the given values:

$$\sin\theta = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

$$\sin\phi = \sin(5\pi/3) = -\frac{\sqrt{3}}{2}$$

$$\mathcal{P}_{+y} = \frac{1}{2}\left(1 + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{\sqrt{3}}{2}\right)\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2\sqrt{2}}\right)$$

To rationalize the term inside: $\frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{2\sqrt{2}\sqrt{2}} = \frac{\sqrt{6}}{4}$.

$$\mathcal{P}_{+y} = \frac{1}{2}\left(1 - \frac{\sqrt{6}}{4}\right) = \frac{4-\sqrt{6}}{8}$$

Measuring spin down along y ($\mathcal{P}_{-y}$): The probability of measuring spin down along y is $\mathcal{P}_{-y} = 1 - \mathcal{P}_{+y}$, or generally:

$$\mathcal{P}_{-y} = \frac{1}{2}(1 - \sin\theta \sin\phi)$$

$$\mathcal{P}_{-y} = \frac{1}{2}\left(1 - \left(-\frac{\sqrt{6}}{4}\right)\right) = \frac{1}{2}\left(1 + \frac{\sqrt{6}}{4}\right) = \frac{4+\sqrt{6}}{8}$$

The probabilities for the final measurement are:

$$\mathcal{P}_{+y} = \frac{4-\sqrt{6}}{8} \approx 0.194 \quad \text{and} \quad \mathcal{P}_{-y} = \frac{4+\sqrt{6}}{8} \approx 0.806$$

Problem 5: 2.10

For the state $|+\rangle_y$, calculate the expectation values and uncertainties for measurements of S_x, S_y, and S_z.
Draw a diagram of the vector model applied to this state and reconcile your quantum mechanical calculations with the classical results.

Solution:

The state "spin up along y", $|+\rangle_y$, is the eigenvector of the S_y operator with eigenvalue $+\hbar/2$. In the standard S_z basis, this state is represented as:

$$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

The corresponding bra vector is:

$$\langle +|_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}$$

Recall that the expectation value of an operator A in a state $|\psi\rangle$ is given by $\langle A\rangle = \langle\psi|A|\psi\rangle$. The calculations for the expectations are as follows:

1. For S_x:

$$\begin{aligned}
\langle S_x\rangle &= \langle +|_y S_x |+\rangle_y \\
&= \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\left(\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \frac{\hbar}{4}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{4}(1 \cdot i + (-i)\cdot 1) \\
&= \frac{\hbar}{4}(i - i) \\
&= 0
\end{aligned}$$

Hence $\langle S_x\rangle = 0$.

2. For S_y:
   Since $|+\rangle_y$ is an eigenstate of S_y with eigenvalue $+\hbar/2$, the expectation value is simply the eigenvalue.
$$\langle S_y \rangle = +\frac{\hbar}{2}$$

   Verification:
$$\begin{aligned}
\langle S_y \rangle &= \langle + |_y S_y | + \rangle_y \\
&= \frac{1}{2} \begin{pmatrix} 1 & -i \end{pmatrix} \left(\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} -i \cdot i \\ i \cdot 1 \end{pmatrix} \\
&= \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \frac{\hbar}{4} (1 \cdot 1 + (-i) \cdot i) \\
&= \frac{\hbar}{4} (1 + 1) \\
&= \frac{\hbar}{2}
\end{aligned}$$

3. For S_z:
$$\begin{aligned}
\langle S_z \rangle &= \langle + |_y S_z | + \rangle_y \\
&= \frac{1}{2} \begin{pmatrix} 1 & -i \end{pmatrix} \left(\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \frac{\hbar}{4} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= \frac{\hbar}{4} (1 \cdot 1 + (-i) \cdot (-i)) \\
&= \frac{\hbar}{4} (1 - 1) = 0
\end{aligned}$$

   Hance, $\langle S_z \rangle = 0$

Recall that the uncertainty ΔA is given by the formula $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$.

First, we need the expectation values of the squared operators. For a spin-1/2 system, a key identity is $S_x^2 = S_y^2 = S_z^2 = (\frac{\hbar}{2})^2 I = \frac{\hbar^2}{4} I$, where I is the identity matrix. Therefore, for any normalized state:
$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \langle \frac{\hbar^2}{4} I \rangle = \frac{\hbar^2}{4}$$

Now we can calculate the uncertainties:

For S_x:
$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0^2} = \frac{\hbar}{2}$$

For S_y:

$$\Delta S_y = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2} = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}} = 0$$

For S_z:

$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0^2} = \frac{\hbar}{2}$$

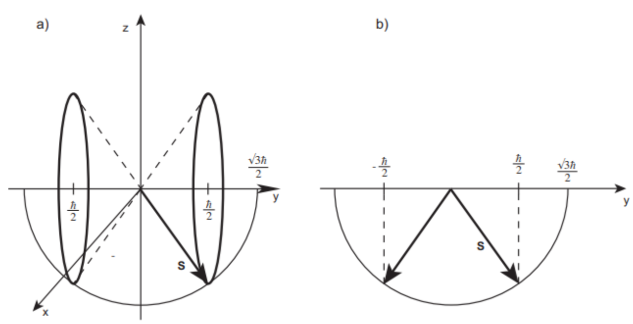
Summary of Results:

$\langle \mathbf{S}\rangle = (\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle) = (0, \hbar/2, 0)$

$\Delta S_y = 0$ (The y-component is precisely known).

$\Delta S_x = \hbar/2$ and $\Delta S_z = \hbar/2$ (The x and z components are maximally uncertain).

Vector Model Diagram: The vector model represents the state of the spin. The expectation value vector $\langle \mathbf{S}\rangle$ points from the origin straight up along the positive y-axis, with a length of $\hbar/2$. The uncertainty is visualized as a "cone of precession." In this specific case, because the spin is perfectly aligned with the y-axis, the base of the "cone" is a flat circle parallel to the x-z plane at $y = \hbar/2$.



Reconciliation with Classical Physics:

1. **Classical Picture:** A classical spinning top with its angular momentum vector $\mathbf{L}$ pointing perfectly along the y-axis would have components $(L_x, L_y, L_z) = (0, L, 0)$. All three components are known precisely and simultaneously. The uncertainty in every component would be zero.

2. **Quantum Mechanical Reality:**

   - **Agreement:** The *average* value of the spin vector, $\langle \mathbf{S}\rangle$, points along the y-axis, just like the classical vector.
   - **Disagreement (The Uncertainty Principle):** The quantum results show that we cannot know all three components of spin simultaneously. This is a direct consequence of the fact that the spin operators do not commute (e.g., $[S_x, S_y] = i\hbar S_z$).

- The uncertainty $\Delta S_y = 0$ tells us that if we measure the y-component of spin for a particle in the state $|+\rangle_y$, we will get the result $+\hbar/2$ with 100% certainty. This component is well-defined.
- The uncertainties $\Delta S_x = \hbar/2$ and $\Delta S_z = \hbar/2$ are the maximum possible for a spin-1/2 system. This means that if we measure S_x or S_z, the outcome is completely random; we will get $+\hbar/2$ or $-\hbar/2$, each with 50% probability (which is why their expectation values are zero).

3. **The Bridge (Vector Model):** The vector model reconciles these ideas. The "true" spin vector (with magnitude $\sqrt{s(s+1)}\hbar = \frac{\sqrt{3}}{2}\hbar$) can be imagined as precessing rapidly around the y-axis.

   - Its projection onto the y-axis is constant and fixed at $\hbar/2$. This explains why $\langle S_y \rangle = \hbar/2$ and $\Delta S_y = 0$.
   - As it precesses, its x and z components are constantly changing. Over time (or over an ensemble of identical systems), the average x and z components are zero. This explains why $\langle S_x \rangle = 0$ and $\langle S_z \rangle = 0$.
   - The precession means the x and z components are fundamentally indeterminate before a measurement, leading to the maximal uncertainty. A measurement of, say, S_x would force the vector to "snap" to a state where its x-projection is either $+\hbar/2$ or $-\hbar/2$, thereby destroying the definite information we had about the y-component.

> **Problem 6: 2.12**
>
> Diagonalize the S_x and S_y operators in the spin-1 case to find the eigenvalues and the eigenvectors of both operators.

Solution:

For a spin-1 system, the operators are represented in the S_z basis ($|+1\rangle, |0\rangle, |-1\rangle$) by the following 3×3 matrices:

$$S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad S_y \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

1. Diagonalization of S_x

   We solve the characteristic equation $\det(S_x - \lambda I) = 0$:

$$\det \begin{pmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{pmatrix} = 0$$

   Expanding the determinant yields:

$$-\lambda \left(\lambda^2 - \left(\frac{\hbar}{\sqrt{2}} \right)^2 \right) - \frac{\hbar}{\sqrt{2}} \left(-\lambda \frac{\hbar}{\sqrt{2}} \right) = 0$$

$$-\lambda^3 + \frac{\hbar^2}{2}\lambda + \frac{\hbar^2}{2}\lambda = 0$$

$$\lambda(\hbar^2 - \lambda^2) = 0$$

The eigenvalues are:

$$\lambda_1 = +\hbar, \quad \lambda_2 = 0, \quad \lambda_3 = -\hbar$$

Then we solve $(S_x - \lambda I)\vec{v} = 0$ for each eigenvalue λ.

Eigenvector for $\lambda_1 = +\hbar$ ($|+\hbar\rangle_x$): Solving the system leads to the unnormalized solution $(1, \sqrt{2}, 1)^T$. Normalizing gives:

$$|+\hbar\rangle_x = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Eigenvector for $\lambda_2 = 0$ ($|0\rangle_x$): Solving the system leads to the unnormalized solution $(1, 0, -1)^T$. Normalizing gives:

$$|0\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Eigenvector for $\lambda_3 = -\hbar$ ($|-\hbar\rangle_x$): Solving the system leads to the unnormalized solution $(1, -\sqrt{2}, 1)^T$. Normalizing gives:

$$|-\hbar\rangle_x = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

2. Diagonalization of S_y

   The eigenvalues are identical to those for S_x: $\lambda = +\hbar, 0, -\hbar$.

   We solve $(S_y - \lambda I)\vec{v} = 0$ for each eigenvalue λ.

Eigenvector for $\lambda_1 = +\hbar$ ($|+\hbar\rangle_y$): Solving the system leads to the unnormalized solution $(1, i\sqrt{2}, 1)^T$. Normalizing gives:

$$|+\hbar\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ 1 \end{pmatrix}$$

Eigenvector for $\lambda_2 = 0$ ($|0\rangle_y$): Solving the system leads to the unnormalized solution $(1, 0, -1)^T$. Normalizing gives:

$$|0\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(Note: This is the same as $|0\rangle_x$)

Eigenvector for $\lambda_3 = -\hbar$ ($|-\hbar\rangle_y$): Solving the system leads to the unnormalized solution $(1, -i\sqrt{2}, 1)^T$. Normalizing gives:

$$|-\hbar\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}$$

Summary

For S_x:

- Eigenvalue $+\hbar$: $|+\hbar\rangle_x = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$
- Eigenvalue 0: $|0\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
- Eigenvalue $-\hbar$: $|-\hbar\rangle_x = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

For S_y:

- Eigenvalue $+\hbar$: $|+\hbar\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ 1 \end{pmatrix}$
- Eigenvalue 0: $|0\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
- Eigenvalue $-\hbar$: $|-\hbar\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}$

Problem 7: 2.14

Find the matrix representation of the S^2 operator for a spin-1 system. Do this once by explicit matrix calculation and a second time by inspection of the S^2 eigenvalue equations:

$$S^2|sm\rangle = s(s+1)\hbar^2|sm\rangle,$$

$$S_z|sm\rangle = m\hbar|sm\rangle.$$

Solution:

As basis for our spin-1 system, we use the standard S_z basis, ordered as $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$.

Method 1: Explicit Matrix Calculation In this method, we use the fundamental definition of the total spin-squared operator:

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

We will calculate the square of each spin component matrix and then add them together.

To start, we write the spin-1 matrices in the S_z basis:

$$S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad S_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Squaring each matrix gives us

$$S_x^2 = \left(\frac{\hbar}{\sqrt{2}}\right)^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S_y^2 = \left(\frac{\hbar}{\sqrt{2}}\right)^2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S_z^2 = (\hbar)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We then add the square matrices

$$\begin{aligned} S^2 &= S_x^2 + S_y^2 + S_z^2 \\ &= \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \hbar^2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \hbar^2 \begin{pmatrix} 1+1 & 0+0 & 1+0 \\ 0+0 & 2+0 & 0+0 \\ 1+0 & 0+0 & 1+1 \end{pmatrix} \\ &= \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\doteq 2\hbar^2 I \end{aligned}$$

Method 2: By Inspection of the S^2 Eigenvalue Equations This method is much more direct and relies on the definition of a matrix representation. The elements of the matrix for an operator A in a basis $\{|i\rangle\}$ are given by $A_{ij} = \langle i|A|j\rangle$.

First, we identify the basis and the relevant eigenvalue equation. Our basis is $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$. The eigenvalue equation for S^2 is:

$$S^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$$

For our spin-1 system, $s = 1$, so the eigenvalue is $1(1+1)\hbar^2 = 2\hbar^2$. This means that S^2 acting on any of our basis states simply multiplies it by the constant $2\hbar^2$:

- $S^2|1,1\rangle = 2\hbar^2|1,1\rangle$
- $S^2|1,0\rangle = 2\hbar^2|1,0\rangle$
- $S^2|1,-1\rangle = 2\hbar^2|1,-1\rangle$

To calculate the matrix elements, recall that $(S^2)_{m'm} = \langle 1,m'|S^2|1,m\rangle$.
Applying the operator S^2 to the ket $|1,m\rangle$ gives

$$(S^2)_{m'm} = \langle 1,m'|(2\hbar^2|1,m\rangle)$$

Since $2\hbar^2$ is a scalar:

$$(S^2)_{m'm} = 2\hbar^2\langle 1,m'|1,m\rangle$$

The basis states are orthonormal, meaning $\langle 1,m'|1,m\rangle = \delta_{m'm}$.

- If $m' = m$ (diagonal elements), then $\delta_{mm} = 1$.
- If $m' \neq m$ (off-diagonal elements), then $\delta_{m'm} = 0$.

Therefore, the matrix elements are:

$$(S^2)_{m'm} = 2\hbar^2\delta_{m'm}$$

The resulting matrix is diagonal, with the eigenvalue $2\hbar^2$ on the diagonal:

$$\begin{aligned} S^2 &= \begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix} \\ &= 2\hbar^2 I \end{aligned}$$

Problem 8: 2.16

A beam of spin-1 particles is prepared in the state

$$|\psi\rangle = \frac{2}{\sqrt{29}}|1\rangle_y + i\frac{3}{\sqrt{29}}|0\rangle_y - \frac{4}{\sqrt{29}}|-1\rangle_y.$$

(a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur?

(b) What are the possible results of a measurement of the spin component S_y, and with what probabilities would they occur?

(c) Plot histograms of the predicted measurement results from parts (a) and (b), and calculate the expectation values for both measurements.

Solution:

The initial state of the spin-1 particle is given in the S_y eigenbasis:

$$|\psi\rangle = \frac{2}{\sqrt{29}}|1\rangle_y + i\frac{3}{\sqrt{29}}|0\rangle_y - \frac{4}{\sqrt{29}}|-1\rangle_y$$

where $|m\rangle_y$ is the eigenstate of S_y with eigenvalue $m\hbar$.

1. To find the probabilities for an S_z measurement, we must first express $|\psi\rangle$ in the S_z basis $\{|1\rangle_z, |0\rangle_z, |-1\rangle_z\}$.

Step 1: Change of Basis

The S_y eigenvectors in the S_z basis are:

$$\begin{aligned}
|1\rangle_y &= \frac{1}{2}|1\rangle_z + \frac{i\sqrt{2}}{2}|0\rangle_z + \frac{1}{2}|-1\rangle_z \\
|0\rangle_y &= \frac{1}{\sqrt{2}}|1\rangle_z - \frac{1}{\sqrt{2}}|-1\rangle_z \\
|-1\rangle_y &= \frac{1}{2}|1\rangle_z - \frac{i\sqrt{2}}{2}|0\rangle_z + \frac{1}{2}|-1\rangle_z
\end{aligned}$$

Step 2: Substitute and Collect Terms

We substitute these into the expression for $|\psi\rangle$ and collect the coefficients for each S_z basis vector.

- **Coefficient of $|1\rangle_z$:**

$$c_1 = \frac{1}{\sqrt{29}}\left(\frac{2}{2} + \frac{3i}{\sqrt{2}} - \frac{4}{2}\right) = \frac{1}{\sqrt{29}}\left(-1 + \frac{3i}{\sqrt{2}}\right)$$

- **Coefficient of $|0\rangle_z$:**

$$c_0 = \frac{1}{\sqrt{29}}\left(\frac{2i\sqrt{2}}{2} - \frac{-4i\sqrt{2}}{2}\right) = \frac{3i\sqrt{2}}{\sqrt{29}}$$

- **Coefficient of $|-1\rangle_z$:**

$$c_{-1} = \frac{1}{\sqrt{29}}\left(\frac{2}{2} - \frac{3i}{\sqrt{2}} - \frac{4}{2}\right) = \frac{1}{\sqrt{29}}\left(-1 - \frac{3i}{\sqrt{2}}\right)$$

Possible Results and Probabilities

The possible outcomes are again $+\hbar, 0, -\hbar$.

- **Probability of measuring $+\hbar$:**

$$\mathcal{P}(S_z = +\hbar) = |c_1|^2 = \frac{1}{29}\left((-1)^2 + \left(\frac{3}{\sqrt{2}}\right)^2\right) = \frac{1}{29}\left(1 + \frac{9}{2}\right) = \frac{11}{58}$$

- **Probability of measuring** 0**:**

$$\mathcal{P}(S_z = 0) = |c_0|^2 = \left|\frac{3i\sqrt{2}}{\sqrt{29}}\right|^2 = \frac{9 \cdot 2}{29} = \frac{18}{29} = \frac{36}{58}$$

- **Probability of measuring** $-\hbar$**:**

$$\mathcal{P}(S_z = -\hbar) = |c_{-1}|^2 = \frac{1}{29}\left((-1)^2 + \left(-\frac{3}{\sqrt{2}}\right)^2\right) = \frac{1}{29}\left(1 + \frac{9}{2}\right) = \frac{11}{58}$$

Check: $\frac{11}{58} + \frac{36}{58} + \frac{11}{58} = \frac{58}{58} = 1$.

2. The state is already expressed in the S_y eigenbasis. The possible outcomes of a spin component measurement on a spin-1 particle are the eigenvalues of the operator: $+\hbar, 0, -\hbar$.

   The probability of measuring an eigenvalue is the squared magnitude of the coefficient of the corresponding eigenstate.

   - **Probability of measuring** $+\hbar$**:**

$$\mathcal{P}(S_y = +\hbar) = \left|\frac{2}{\sqrt{29}}\right|^2 = \frac{4}{29}$$

   - **Probability of measuring** 0**:**

$$\mathcal{P}(S_y = 0) = \left|i\frac{3}{\sqrt{29}}\right|^2 = \frac{9}{29}$$

   - **Probability of measuring** $-\hbar$**:**

$$\mathcal{P}(S_y = -\hbar) = \left|-\frac{4}{\sqrt{29}}\right|^2 = \frac{16}{29}$$

   Check: The probabilities sum to 1: $\frac{4}{29} + \frac{9}{29} + \frac{16}{29} = \frac{29}{29} = 1$.

3. If you don't know how to make the histogram from the probabilities, you're fucked LOL

Expectation Values

The expectation value is calculated as $\langle A \rangle = \sum_i (\text{eigenvalue}_i) \times \mathcal{P}(\text{eigenvalue}_i)$.

Expectation value of S_z**:**

$$\begin{aligned}\langle S_z \rangle &= (+\hbar)\mathcal{P}(S_z = +\hbar) + (0)\mathcal{P}(S_z = 0) + (-\hbar)\mathcal{P}(S_z = -\hbar) \\ &= \hbar\left(\frac{11}{58}\right) + 0 - \hbar\left(\frac{11}{58}\right) = 0\end{aligned}$$

$$\boxed{\langle S_z \rangle = 0}$$

Expectation value of S_y:

$$\begin{aligned}\langle S_y\rangle &= (+\hbar)\mathcal{P}(S_y=+\hbar) + (0)\mathcal{P}(S_y=0) + (-\hbar)\mathcal{P}(S_y=-\hbar)\\ &= \hbar\left(\frac{4}{29}\right) + 0 - \hbar\left(\frac{16}{29}\right) = \frac{4\hbar - 16\hbar}{29} = -\frac{12\hbar}{29}\end{aligned}$$

$$\boxed{\langle S_y\rangle = -\frac{12\hbar}{29}}$$

Problem 9: 2.17

A spin-1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5i\end{pmatrix}.$$

(a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur? Calculate the expectation value of the spin component S_z.

(b) Calculate the expectation value of the spin component S_x. *Suggestion: Use matrix mechanics to evaluate the expectation value.*

Solution:

A spin-1 particle has three possible spin projections along any axis. We work in the standard basis of S_z eigenstates, ordered from the highest eigenvalue to the lowest:

- $|+\rangle \equiv |s=1, m_s=+1\rangle$, with eigenvalue $+\hbar$, is represented by the vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$.
- $|0\rangle \equiv |s=1, m_s=0\rangle$, with eigenvalue 0, is represented by the vector $\begin{pmatrix}0\\1\\0\end{pmatrix}$.
- $|-\rangle \equiv |s=1, m_s=-1\rangle$, with eigenvalue $-\hbar$, is represented by the vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

The given state is $|\psi\rangle = \frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\5i\end{pmatrix}$. We can express this in the eigenbasis as:

$$|\psi\rangle = \frac{1}{\sqrt{30}}|+\rangle + \frac{2}{\sqrt{30}}|0\rangle + \frac{5i}{\sqrt{30}}|-\rangle. \tag{1}$$

First, we verify that the state is normalized:

$$\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{30}}\right)^2\begin{pmatrix}1 & 2 & -5i\end{pmatrix}\begin{pmatrix}1\\2\\5i\end{pmatrix} = \frac{1}{30}(1\cdot 1 + 2\cdot 2 + (-5i)(5i)) = \frac{1}{30}(1+4+25) = \frac{30}{30} = 1.$$

The state is correctly normalized.

For a spin-1 system, the spin operators are 3×3 matrices. The S_z operator is diagonal in this basis, with its eigenvalues on the diagonal:

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

The S_x operator is derived from the ladder operators ($S_x = (S_+ + S_-)/2$):

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

(a) Measurement of S_z:

Possible Results and Probabilities

Possible Results: According to the postulates of quantum mechanics, the only possible outcomes of a measurement of an observable are the eigenvalues of the corresponding operator. For the spin component S_z of a spin-1 particle, the eigenvalues are:

$$\boxed{+\hbar, \quad 0, \quad -\hbar.}$$

Probabilities: The probability of measuring a specific eigenvalue is the squared magnitude of the projection of the state vector onto the corresponding eigenstate. The coefficients of the expansion in Eq. (1) are these projections.

- The probability of measuring $+\hbar$ is:

$$P(+\hbar) = |\langle + | \psi \rangle|^2 = \left| \frac{1}{\sqrt{30}} \right|^2 = \boxed{\frac{1}{30}}.$$

- The probability of measuring 0 is:

$$P(0) = |\langle 0 | \psi \rangle|^2 = \left| \frac{2}{\sqrt{30}} \right|^2 = \boxed{\frac{4}{30} = \frac{2}{15}}.$$

- The probability of measuring $-\hbar$ is:

$$P(-\hbar) = |\langle - | \psi \rangle|^2 = \left| \frac{5i}{\sqrt{30}} \right|^2 = \frac{|5|^2|i|^2}{30} = \frac{25 \cdot 1}{30} = \boxed{\frac{25}{30} = \frac{5}{6}}.$$

As a check, the probabilities sum to unity: $\frac{1}{30} + \frac{4}{30} + \frac{25}{30} = \frac{30}{30} = 1$.

Expectation Value of S_z

The expectation value can be calculated in two ways.

Method 1: Using Probabilities The expectation value is the weighted average of the possible outcomes, where the weights are the probabilities.

$$\begin{aligned}
\langle S_z \rangle &= \sum_{m_s} (m_s\hbar) \cdot P(m_s\hbar) \\
&= (+\hbar) \cdot P(+\hbar) + (0) \cdot P(0) + (-\hbar) \cdot P(-\hbar) \\
&= \hbar \left(\frac{1}{30}\right) + 0 + (-\hbar) \left(\frac{25}{30}\right) \\
&= \frac{\hbar - 25\hbar}{30} = \frac{-24\hbar}{30} = \boxed{-\frac{4\hbar}{5}}.
\end{aligned}$$

Method 2: Using Matrix Mechanics The expectation value is given by the "sandwich" $\langle\psi|S_z|\psi\rangle$.

$$\begin{aligned}
\langle S_z \rangle &= \left(\frac{1}{\sqrt{30}} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix}\right) \left(\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \left(\frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}\right) \\
&= \frac{\hbar}{30} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix} \\
&= \frac{\hbar}{30} \begin{pmatrix} 1 & 2 & -5i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -5i \end{pmatrix} \\
&= \frac{\hbar}{30} \left((1)(1) + (2)(0) + (-5i)(-5i)\right) \\
&= \frac{\hbar}{30}(1 + 0 + 25i^2) = \frac{\hbar}{30}(1 - 25) = \frac{-24\hbar}{30} = \boxed{-\frac{4\hbar}{5}}.
\end{aligned}$$

Both methods yield the same correct result.

(b) Expectation Value of S_x:

We must use matrix mechanics to calculate $\langle S_x \rangle = \langle\psi|S_x|\psi\rangle$.

First we calculate the action of the operator on the ket, $S_x|\psi\rangle$.

$$\begin{aligned}
S_x|\psi\rangle &= \left(\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \left(\frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5i \end{pmatrix}\right) \\
&= \frac{\hbar}{\sqrt{60}} \begin{pmatrix} (0)(1) + (1)(2) + (0)(5i) \\ (1)(1) + (0)(2) + (1)(5i) \\ (0)(1) + (1)(2) + (0)(5i) \end{pmatrix} \\
&= \frac{\hbar}{\sqrt{60}} \begin{pmatrix} 2 \\ 1 + 5i \\ 2 \end{pmatrix}.
\end{aligned}$$

Then we compute the inner product with the bra, $\langle\psi|(S_x|\psi\rangle)$.

$$\begin{aligned}
\langle S_x\rangle &= \langle\psi|\left(S_x|\psi\rangle\right) \\
&= \left(\frac{1}{\sqrt{30}}\begin{pmatrix}1 & 2 & -5i\end{pmatrix}\right)\left(\frac{\hbar}{\sqrt{60}}\begin{pmatrix}2 \\ 1+5i \\ 2\end{pmatrix}\right) \\
&= \frac{\hbar}{\sqrt{1800}}\left((1)(2)+(2)(1+5i)+(-5i)(2)\right) \\
&= \frac{\hbar}{\sqrt{900\cdot 2}}\left(2+2+10i-10i\right) \\
&= \frac{\hbar}{30\sqrt{2}}(4) = \frac{4\hbar}{30\sqrt{2}} = \frac{2\hbar}{15\sqrt{2}}.
\end{aligned}$$

To rationalize the denominator, we multiply the numerator and denominator by $\sqrt{2}$:

$$\langle S_x\rangle = \frac{2\hbar}{15\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}\hbar}{15\cdot 2} = \boxed{\frac{\sqrt{2}\hbar}{15}}.$$

Problem 10: 2.18

A spin-1 particle is prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{3}{\sqrt{14}}|0\rangle + i\frac{2}{\sqrt{14}}|-1\rangle.$$

(a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur?

(b) Suppose that the S_z measurement on the particle yields the result $S_z = -\hbar$. Subsequent to that result a second measurement is performed to measure the spin component S_x. What are the possible results of that measurement, and with what probabilities would they occur?

(c) Draw a schematic diagram depicting the successive measurements in parts (a) and (b).

Solution:

(a) The initial state of the particle is given in the S_z basis:

$$|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{3}{\sqrt{14}}|0\rangle + i\frac{2}{\sqrt{14}}|-1\rangle$$

where $|1\rangle, |0\rangle, |-1\rangle$ are the eigenstates of S_z with eigenvalues $+\hbar, 0, -\hbar$ respectively.

Possible Results:
For a spin-1 particle, a measurement of any spin component can only yield one of the three possible eigenvalues: $+\hbar, 0, -\hbar$.

Probabilities:
The probability of measuring a particular eigenvalue is the squared magnitude of the coefficient (the probability amplitude) of the corresponding eigenstate in the state's expansion.

- **Probability of measuring $S_z = +\hbar$:**

$$\mathcal{P}(S_z = +\hbar) = \left|\frac{1}{\sqrt{14}}\right|^2 = \frac{1}{14}$$

- **Probability of measuring $S_z = 0$:**

$$\mathcal{P}(S_z = 0) = \left|-\frac{3}{\sqrt{14}}\right|^2 = \frac{9}{14}$$

- **Probability of measuring $S_z = -\hbar$:**

$$\mathcal{P}(S_z = -\hbar) = \left|i\frac{2}{\sqrt{14}}\right|^2 = \frac{|i|^2|2|^2}{14} = \frac{4}{14} = \frac{2}{7}$$

As a check, the probabilities sum to 1: $\frac{1}{14} + \frac{9}{14} + \frac{4}{14} = \frac{14}{14} = 1$.

(b) The first measurement of S_z yields the specific result $S_z = -\hbar$. According to the postulates of quantum mechanics, this measurement forces the particle's state to "collapse" from the original superposition $|\psi\rangle$ into the eigenstate corresponding to the measurement outcome.

Therefore, immediately after the first measurement, the new state of the particle is:

$$|\psi_{\text{new}}\rangle = |-1\rangle$$

Next, we perform a measurement of S_x on this new state, $|-1\rangle$. To find the probabilities of the outcomes, we must express $|-1\rangle$ in the eigenbasis of the S_x operator. The normalized eigenvectors of S_x for a spin-1 particle are:

$$\begin{aligned}
|1\rangle_x &= \frac{1}{2}|1\rangle + \frac{\sqrt{2}}{2}|0\rangle + \frac{1}{2}|-1\rangle \\
|0\rangle_x &= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle \\
|-1\rangle_x &= \frac{1}{2}|1\rangle - \frac{\sqrt{2}}{2}|0\rangle + \frac{1}{2}|-1\rangle
\end{aligned}$$

We need to write $|-1\rangle$ as a linear combination of these S_x eigenstates. We can do this by calculating the projections: $|-1\rangle = \sum_m \langle m|_x|-1\rangle|m\rangle_x$.

- $\langle 1|_x|-1\rangle = \left(\frac{1}{2}\langle 1| + \frac{\sqrt{2}}{2}\langle 0| + \frac{1}{2}\langle -1|\right)|-1\rangle = \frac{1}{2}$
- $\langle 0|_x|-1\rangle = \left(\frac{1}{\sqrt{2}}\langle 1| - \frac{1}{\sqrt{2}}\langle -1|\right)|-1\rangle = -\frac{1}{\sqrt{2}}$
- $\langle -1|_x|-1\rangle = \left(\frac{1}{2}\langle 1| - \frac{\sqrt{2}}{2}\langle 0| + \frac{1}{2}\langle -1|\right)|-1\rangle = \frac{1}{2}$

So, the state $|-1\rangle$ expressed in the S_x basis is:

$$|-1\rangle = \frac{1}{2}|1\rangle_x - \frac{1}{\sqrt{2}}|0\rangle_x + \frac{1}{2}|-1\rangle_x$$

In order to calculate probabilities, the possible outcomes of the S_x measurement are again $+\hbar, 0, -\hbar$.

- **Probability of measuring $S_x = +\hbar$:**

$$\mathcal{P}(S_x = +\hbar) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

- **Probability of measuring $S_x = 0$:**

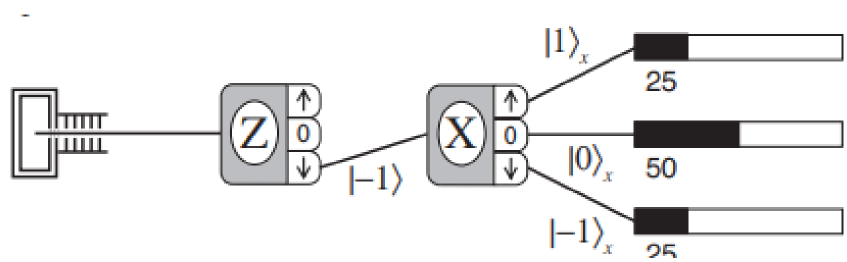
$$\mathcal{P}(S_x = 0) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

- **Probability of measuring $S_x = -\hbar$:**

$$\mathcal{P}(S_x = -\hbar) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

As a check, these probabilities also sum to 1: $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$.

(c) Schematic Diagram



$$\text{(Source)} \xrightarrow{|\psi\rangle} \text{[SGz]} \xrightarrow{\text{Select } S_z=-\hbar} \text{(State is now } |-1\rangle\text{)} \rightarrow \text{[SGx]} \rightarrow \text{(Detectors)}$$

Figure 1: Schematic diagram of the successive Stern-Gerlach measurements.

The experiment can be depicted as a sequence of Stern-Gerlach (SG) devices.

(a) A source produces spin-1 particles in the state $|\psi\rangle$.

(b) The beam enters an SG device oriented along the **z-axis**. It splits the beam into three paths.

(c) We block the paths for the $S_z = +\hbar$ and $S_z = 0$ outcomes, selecting only the particles measured to have $S_z = -\hbar$. This prepares the particles in the state $|-1\rangle$.

(d) This selected beam then enters a second SG device oriented along the **x-axis**.

(e) This device splits the beam into three final paths, with intensities (probabilities) calculated in part (b).

Problem 11: 2.19

A spin-1 particle is prepared in the state

$$|\psi_i\rangle = \sqrt{\frac{1}{6}}|1\rangle - \sqrt{\frac{2}{6}}|0\rangle + i\sqrt{\frac{3}{6}}|-1\rangle.$$

Find the probability that the system is measured to be in the final state

$$|\psi_f\rangle = \frac{1+i}{\sqrt{7}}|1\rangle_y + \frac{2}{\sqrt{7}}|0\rangle_y - i\frac{1}{\sqrt{7}}|-1\rangle_y.$$

Solution:

To calculate the transition probability $P = |\langle\psi_f|\psi_i\rangle|^2$, we must first express both the initial and final states as column vectors in the same basis, which we choose to be the standard eigenbasis of S_z.

The initial state $|\psi_i\rangle$ is already given in the S_z basis $\{|1\rangle, |0\rangle, |-1\rangle\}$. Its matrix representation is:

$$|\psi_i\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -\sqrt{2} \\ i\sqrt{3} \end{pmatrix}. \tag{4}$$

The final state $|\psi_f\rangle$ is given in the eigenbasis of S_y. We must find the matrix representations of these basis vectors, $|1\rangle_y, |0\rangle_y, |-1\rangle_y$, by finding the normalized eigenvectors of the spin-1 S_y matrix:

$$S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Solving the eigenvalue equation $S_y\vec{v} = \lambda\vec{v}$ for $\lambda = +\hbar, 0, -\hbar$ yields the following column vectors:

$$|1\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix} \quad \text{(for eigenvalue } +\hbar\text{)} \tag{5}$$

$$|0\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{(for eigenvalue 0)} \tag{6}$$

$$|-1\rangle_y = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix} \quad \text{(for eigenvalue } -\hbar\text{)} \tag{7}$$

We construct the final state vector $|\psi_f\rangle$ as a linear combination of the S_y eigenvector matrices:

$$\begin{aligned}
|\psi_f\rangle &= \frac{1+i}{\sqrt{7}}|1\rangle_y + \frac{2}{\sqrt{7}}|0\rangle_y - i\frac{1}{\sqrt{7}}|-1\rangle_y \\
&= \frac{1+i}{\sqrt{7}}\left(\frac{1}{2}\begin{pmatrix}1\\ i\sqrt{2}\\ -1\end{pmatrix}\right) + \frac{2}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\right) - \frac{i}{\sqrt{7}}\left(\frac{1}{2}\begin{pmatrix}1\\ -i\sqrt{2}\\ -1\end{pmatrix}\right) \\
&= \frac{1}{2\sqrt{7}}\begin{pmatrix}1+i\\ (1+i)(i\sqrt{2})\\ -(1+i)\end{pmatrix} + \frac{\sqrt{2}}{\sqrt{7}}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2\sqrt{7}}\begin{pmatrix}i\\ i(-i\sqrt{2})\\ -i\end{pmatrix} \\
&= \frac{1}{2\sqrt{7}}\left[\begin{pmatrix}1+i\\ i\sqrt{2}-\sqrt{2}\\ -1-i\end{pmatrix} + \begin{pmatrix}2\sqrt{2}\\ 0\\ 2\sqrt{2}\end{pmatrix} - \begin{pmatrix}i\\ \sqrt{2}\\ -i\end{pmatrix}\right] \\
&= \frac{1}{2\sqrt{7}}\begin{pmatrix}(1+i)+2\sqrt{2}-i\\ (i\sqrt{2}-\sqrt{2})+0-\sqrt{2}\\ (-1-i)+2\sqrt{2}-(-i)\end{pmatrix} = \frac{1}{2\sqrt{7}}\begin{pmatrix}1+2\sqrt{2}\\ i\sqrt{2}-2\sqrt{2}\\ -1+2\sqrt{2}\end{pmatrix}.
\end{aligned}$$

So, the final state vector in the S_z basis is:

$$|\psi_f\rangle = \frac{1}{2\sqrt{7}}\begin{pmatrix}1+2\sqrt{2}\\ \sqrt{2}(i-2)\\ -1+2\sqrt{2}\end{pmatrix}. \tag{8}$$

The inner product $\langle\psi_f|\psi_i\rangle$ is calculated by multiplying the bra vector $\langle\psi_f|$ (the conjugate transpose of $|\psi_f\rangle$) by the ket vector $|\psi_i\rangle$. First, we find the bra $\langle\psi_f|$:

$$\langle\psi_f| = \frac{1}{2\sqrt{7}}\begin{pmatrix}1+2\sqrt{2} & \sqrt{2}(-i-2) & -1+2\sqrt{2}\end{pmatrix}. \tag{9}$$

Now we perform the matrix multiplication $\langle\psi_f||\psi_i\rangle$:

$$\begin{aligned}
\langle\psi_f|\psi_i\rangle &= \left(\frac{1}{2\sqrt{7}}\begin{pmatrix}1+2\sqrt{2} & \sqrt{2}(-i-2) & -1+2\sqrt{2}\end{pmatrix}\right)\left(\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ -\sqrt{2}\\ i\sqrt{3}\end{pmatrix}\right) \\
&= \frac{1}{2\sqrt{42}}\left[(1+2\sqrt{2})(1) + (\sqrt{2}(-i-2))(-\sqrt{2}) + (-1+2\sqrt{2})(i\sqrt{3})\right] \\
&= \frac{1}{2\sqrt{42}}\left[(1+2\sqrt{2}) + (-2)(-i-2) + (-i\sqrt{3}+2i\sqrt{6})\right] \\
&= \frac{1}{2\sqrt{42}}\left[1+2\sqrt{2}+2i+4-i\sqrt{3}+2i\sqrt{6}\right] \\
&= \frac{1}{2\sqrt{42}}\left[(5+2\sqrt{2}) + i(2-\sqrt{3}+2\sqrt{6})\right].
\end{aligned}$$

In the following calculation for the probability $P = |\langle\psi_f|\psi_i\rangle|^2$, recall that for a complex number

$z = a + ib$, the squared magnitude is $|z|^2 = a^2 + b^2$.

$$\begin{aligned}
P &= \left|\frac{1}{2\sqrt{42}}\right|^2 \left|(5+2\sqrt{2}) + i(2-\sqrt{3}+2\sqrt{6})\right|^2 \\
&= \frac{1}{4\cdot 42}\left[(5+2\sqrt{2})^2 + (2-\sqrt{3}+2\sqrt{6})^2\right] \\
&= \frac{1}{168}\left[(25+20\sqrt{2}+8) + (4+3+24-4\sqrt{3}+8\sqrt{6}-4\sqrt{18})\right] \\
&= \frac{1}{168}\left[(33+20\sqrt{2}) + (31-4\sqrt{3}+8\sqrt{6}-12\sqrt{2})\right] \\
&= \frac{1}{168}\left[(33+31) + (20\sqrt{2}-12\sqrt{2}) - 4\sqrt{3}+8\sqrt{6}\right] \\
&= \frac{1}{168}\left[64+8\sqrt{2}-4\sqrt{3}+8\sqrt{6}\right].
\end{aligned}$$

Finally, we simplify by dividing all terms by 4:

$$\boxed{P = \frac{16+2\sqrt{2}-\sqrt{3}+2\sqrt{6}}{42} \approx 0.524} \tag{10}$$

Problem 12: 2.22

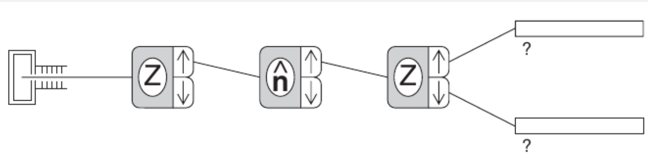


FIGURE 2.15 Measurement of spin components (Prob. 2.22).

A beam of spin-1/2 particles is sent through a series of three Stern-Gerlach analyzers, as shown in the attached image (Fig. 2.15). The second Stern-Gerlach analyzer is aligned along the $\hat{\mathbf{n}}$ direction, which makes an angle θ in the x-z plane with respect to the z-axis.

(a) Find the probability that particles transmitted through the first Stern-Gerlach analyzer are measured to have spin down at the third Stern-Gerlach analyzer?

(b) How must the angle θ of the second Stern-Gerlach analyzer be oriented so as to maximize the probability that particles are measured to have spin down at the third Stern-Gerlach analyzer? What is this maximum fraction?

(c) What is the probability that particles have spin down at the third Stern-Gerlach analyzer if the second Stern-Gerlach analyzer is removed from the experiment?

Solution:

The experiment proceeds in three stages:

1. **First Analyzer (Z):** A beam of spin-1/2 particles is prepared in the "spin up" state along the z-axis. The state of a particle leaving this analyzer is $|\psi_1\rangle = |+\rangle_z$.

2. **Second Analyzer ($\hat{\mathbf{n}}$):** This state, $|+\rangle_z$, enters the second analyzer. This device measures the spin along the $\hat{\mathbf{n}}$ direction. Crucially, the diagram shows that *both* the "spin up along $\hat{\mathbf{n}}$" and "spin down along $\hat{\mathbf{n}}$" paths are kept and recombined. This means a measurement occurs, but the result is not recorded or selected.

3. **Third Analyzer (Z):** The resulting beam enters the final analyzer, which measures the spin along the z-axis. We want to find the probability of getting a "spin down" result, $|-\rangle_z$.

The direction $\hat{\mathbf{n}}$ is in the x-z plane, making an angle θ with the z-axis. This corresponds to spherical coordinates $(\theta, \phi = 0)$. The eigenstates of the S_n operator for this direction are:

$$|+\rangle_n = \cos(\theta/2)|+\rangle_z + \sin(\theta/2)|-\rangle_z$$
$$|-\rangle_n = \sin(\theta/2)|+\rangle_z - \cos(\theta/2)|-\rangle_z$$

We now answer questions (a)-(c):

(a) Since the result of the second measurement is not known, we must consider both possibilities and add their probabilities. This is the rule for "summing over intermediate, indistinguishable paths."

Path 1: The particle is measured as "spin up" along $\hat{\mathbf{n}}$ in the second analyzer.

(a) **Probability of this step:** The initial state is $|+\rangle_z$. The probability of being measured as $|+\rangle_n$ is:

$$\mathcal{P}(+n \text{ from } +z) = |\langle +|_n|+\rangle_z|^2 = |\cos(\theta/2)|^2 = \cos^2(\theta/2)$$

(b) **State after this step:** The state collapses to $|+\rangle_n$.

(c) **Probability of final step:** The probability of this new state, $|+\rangle_n$, being measured as $|-\rangle_z$ is:

$$\mathcal{P}(-z \text{ from } +n) = |\langle -|_z|+\rangle_n|^2 = |\sin(\theta/2)|^2 = \sin^2(\theta/2)$$

(d) **Total probability of Path 1:**

$$\mathcal{P}_1 = \mathcal{P}(+n \text{ from } +z) \times \mathcal{P}(-z \text{ from } +n) = \cos^2(\theta/2)\sin^2(\theta/2)$$

Path 2: The particle is measured as "spin down" along $\hat{\mathbf{n}}$ in the second analyzer.

(a) **Probability of this step:** The initial state is $|+\rangle_z$. The probability of being measured as $|-\rangle_n$ is:

$$\mathcal{P}(-n \text{ from } +z) = |\langle -|_n|+\rangle_z|^2 = |\sin(\theta/2)|^2 = \sin^2(\theta/2)$$

(b) **State after this step:** The state collapses to $|-\rangle_n$.

(c) **Probability of final step:** The probability of this new state, $|-\rangle_n$, being measured as $|-\rangle_z$ is:

$$\mathcal{P}(-z \text{ from } -n) = |\langle -|_z|-\rangle_n|^2 = |-\cos(\theta/2)|^2 = \cos^2(\theta/2)$$

(d) **Total probability of Path 2:**

$$\mathcal{P}_2 = \mathcal{P}(-n \text{ from } +z) \times \mathcal{P}(-z \text{ from } -n) = \sin^2(\theta/2)\cos^2(\theta/2)$$

Hence, the total probability of measuring spin down at the third analyzer is the sum of the probabilities of these two mutually exclusive paths:

$$\mathcal{P}_{total} = \mathcal{P}_1 + \mathcal{P}_2 = 2\sin^2(\theta/2)\cos^2(\theta/2)$$

Using the trigonometric identity $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, we can simplify this:

$$\mathcal{P}_{total} = \frac{1}{2}(2\sin(\theta/2)\cos(\theta/2))^2 = \frac{1}{2}\sin^2\theta$$

$$\boxed{\mathcal{P}(\text{final down}) = \frac{1}{2}\sin^2\theta}$$

(b) We want to find the angle θ that maximizes the probability $\mathcal{P}(\theta) = \frac{1}{2}\sin^2\theta$. The function $\sin^2\theta$ has a maximum value of 1. This occurs when $\sin\theta = \pm 1$, which means:

$$\boxed{\theta = \frac{\pi}{2} \quad (or\ 90^\circ)}$$

Physically, this means the second analyzer is oriented along the x-axis, perpendicular to the first and third analyzers. This orientation maximally "scrambles" the spin information from the z-basis, making a spin flip most likely.

The maximum probability (maximum fraction) is $\mathcal{P}_{max} = \mathcal{P}(\pi/2) = \frac{1}{2}\sin^2(\pi/2) = \frac{1}{2}(1)^2$:

$$\boxed{\mathcal{P}_{max} = \frac{1}{2}}$$

(c) If the second analyzer is removed, the experiment as follows:

(a) Prepare the state $|+\rangle_z$.

(b) Measure the spin along the z-axis.

The state entering the final analyzer is $|+\rangle_z$. We want to find the probability of measuring spin down, which corresponds to projecting onto the state $|-\rangle_z$.

$$\mathcal{P}(\text{final down}) = |\langle -|_z|+\rangle_z|^2$$

Since the states $|+\rangle_z$ and $|-\rangle_z$ are orthogonal, their inner product is zero:

$$\langle -|_z|+\rangle_z = 0$$

Therefore, the probability is:

$$\boxed{\mathcal{P}(\text{final down}) = 0}$$

This makes intuitive sense: if you measure a particle to be spin up along z, an immediate subsequent measurement along z will yield spin up with 100% certainty, and spin down with 0% certainty. The intermediate measurement in part (a) is what allows for the possibility of a spin flip.

Problem 13: 2.23

Consider a three-dimensional ket space. In the basis defined by three orthogonal kets $|1\rangle$, $|2\rangle$, and $|3\rangle$, the operators A and B are represented by

$$A \doteq \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \quad B \doteq \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix},$$

where all the quantities are real.

1. Do the operators A and B commute?
2. Find the eigenvalues and normalized eigenvectors of both operators.
3. Assume the system is initially in the state $|2\rangle$. Then the observable corresponding to the operator B is measured. What are the possible results of this measurement and the probabilities of each result? After this measurement, the observable corresponding to the operator A is measured. What are the possible results of this measurement and the probabilities of each result?
4. How are questions (1) and (3) above related?

Solution:

The operator A is diagonal, meaning the given basis is its eigenbasis. The operator B is not diagonal but is **block-diagonal** in this basis. It decouples the ket space into two invariant subspaces: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_{23}$, where $\mathcal{H}_1 = \text{span}\{|1\rangle\}$ and $\mathcal{H}_{23} = \text{span}\{|2\rangle, |3\rangle\}$.

$$B = \begin{pmatrix} \boxed{b_1} & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & b_2 \\ b_2 & 0 \end{matrix}} & \\ 0 & & \end{pmatrix}$$

This means all properties can be determined by examining these subspaces independently.

1. The operators A and B commute if and only if their commutator $[A, B] = AB - BA$ is the zero operator. Due to the block-diagonal structure, we can analyze the commutator on each subspace.

   - **On the subspace $\mathcal{H}_1$:** The operators are the scalars a_1 and b_1. Their commutator is $[a_1, b_1] = a_1 b_1 - b_1 a_1 = 0$.
   - **On the subspace $\mathcal{H}_{23}$:** The operators are represented by the sub-matrices:

$$A_{\text{sub}} = \begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix}, \quad B_{\text{sub}} = \begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix}$$

The commutator on this subspace is:

$$\begin{aligned}
[A_{\text{sub}}, B_{\text{sub}}] &= A_{\text{sub}}B_{\text{sub}} - B_{\text{sub}}A_{\text{sub}} \\
&= \begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix}\begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_2b_2 \\ a_3b_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a_3b_2 \\ a_2b_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (a_2 - a_3)b_2 \\ (a_3 - a_2)b_2 & 0 \end{pmatrix}
\end{aligned}$$

The total commutator $[A, B]$ is zero if and only if the commutator on each subspace is zero. This requires $[A_{\text{sub}}, B_{\text{sub}}] = 0$, which leads to the condition:

$$(a_2 - a_3)b_2 = 0$$

By the zero-product property, this is true if and only if at least one of the factors is zero.

The operators A and B commute if and only if one of the following two conditions is met:

(a) $\mathbf{b_2 = 0}$: In this case, B becomes diagonal (to be more precise, it has only one nonzero entry and it lies in the upper left corner) and thus commutes with the diagonal operator A.

(b) $\mathbf{a_2 = a_3}$: In this case, A is degenerate on the $\mathcal{H}_{23}$ subspace, acting as a scalar multiple of the identity ($A_{\text{sub}} = a_2 I$), which commutes with any operator on that subspace.

2. **Operator A:** Since A is diagonal in the given basis, its eigenvalues are the diagonal entries and its normalized eigenvectors are the basis vectors. Let the eigenvectors of A be denoted by $|\phi_i\rangle$.
   - Eigenvalue a_1: $|\phi_1\rangle = |1\rangle$
   - Eigenvalue a_2: $|\phi_2\rangle = |2\rangle$
   - Eigenvalue a_3: $|\phi_3\rangle = |3\rangle$

Operator B: We solve $\det(B - \lambda I) = 0$:

$$\begin{aligned}
0 = \det &\begin{pmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{pmatrix} \\
&= (b_1 - \lambda)\det\begin{pmatrix} -\lambda & b_2 \\ b_2 & -\lambda \end{pmatrix} \\
&= (b_1 - \lambda)[(-\lambda)(-\lambda) - (b_2)(b_2)] \\
&= (b_1 - \lambda)(\lambda^2 - b_2^2) \\
&= (b_1 - \lambda)(\lambda - b_2)(\lambda + b_2)
\end{aligned}$$

So the eigenvalues are:

$$\lambda_1^B = b_1, \quad \lambda_2^B = b_2, \quad \lambda_3^B = -b_2$$

Eigenvector for $\lambda_1^B = b_1$: Solve $(B - b_1 I)\vec{v} = 0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -b_1 & b_2 \\ 0 & b_2 & -b_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

From the first row: no constraint on v_1.
From the second and third rows:

$$-b_1 v_2 + b_2 v_3 = 0 \quad \text{and} \quad b_2 v_2 - b_1 v_3 = 0$$

These equations are consistent only if $v_2 = v_3 = 0$ (unless $b_1^2 = b_2^2$, but we consider the general case).
Thus, the eigenvector is:

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle$$

Normalized: $|\lambda_1^B\rangle = |1\rangle$

Eigenvector for $\lambda_2^B = b_2$: Solve $(B - b_2 I)\vec{v} = 0$:

$$\begin{pmatrix} b_1 - b_2 & 0 & 0 \\ 0 & -b_2 & b_2 \\ 0 & b_2 & -b_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

From the first row: $(b_1 - b_2)v_1 = 0$. In the general case where $b_1 \neq b_2$, we get $v_1 = 0$.
From the second and third rows:

$$-b_2 v_2 + b_2 v_3 = 0 \quad \Rightarrow \quad v_2 = v_3$$

$$b_2 v_2 - b_2 v_3 = 0 \quad \text{(same equation)}$$

So $v_2 = v_3$. Let $v_2 = v_3 = \frac{1}{\sqrt{2}}$ for normalization.
Thus, the eigenvector is:

$$\vec{v} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized: $|\lambda_2^B\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$

Eigenvector for $\lambda_3^B = -b_2$: Solve $(B + b_2 I)\vec{v} = 0$:

$$\begin{pmatrix} b_1 + b_2 & 0 & 0 \\ 0 & b_2 & b_2 \\ 0 & b_2 & b_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

From the first row: $(b_1 + b_2)v_1 = 0$. In the general case where $b_1 \neq -b_2$, we get $v_1 = 0$.
From the second and third rows:

$$b_2 v_2 + b_2 v_3 = 0 \quad \Rightarrow \quad v_2 = -v_3$$

$$b_2 v_2 + b_2 v_3 = 0 \quad \text{(same equation)}$$

So $v_2 = -v_3$. Let $v_2 = \frac{1}{\sqrt{2}}$, $v_3 = -\frac{1}{\sqrt{2}}$ for normalization.
Thus, the eigenvector is:

$$\vec{v} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized: $|\lambda_3^B\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$

Note that the eigenvectors and eigenvalues we just computed are for the generic non-degenerate case. We have to consider the degenerate cases too. A complete solution requires characterizing the eigensystem for all possible parameter values:

(Case 1) Generic, Non-degenerate ($b_2 \neq 0$ and $b_1^2 \neq b_2^2$): The eigenvalues $b_1, b_2, -b_2$ are distinct. The normalized eigenvectors are unique (up to a phase):

- $\lambda_1 = b_1$: $|\psi_1\rangle = |1\rangle$
- $\lambda_2 = b_2$: $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$
- $\lambda_3 = -b_2$: $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$

(Case 2) Degenerate ($b_2 \neq 0$ and $b_1^2 = b_2^2$): Here we have two subcases:

- **Subcase 2a ($b_1 = b_2$):** The eigenvalues are b_1 (two-fold degenerate) and $-b_1$.
  - The eigenspace for $\lambda = b_1$ is two-dimensional, $E_{b_1} = \text{span}\{|1\rangle, \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)\}$. The provided vectors form an orthonormal basis for this space.
  - The eigenspace for $\lambda = -b_1$ is one-dimensional, spanned by $|\psi_3\rangle$.
- **Subcase 2b ($b_1 = -b_2$):** The eigenvalues are b_1 and $-b_1$ (two-fold degenerate).
  - The eigenspace for $\lambda = b_1$ is one-dimensional, spanned by $|\psi_1\rangle$.
  - The eigenspace for $\lambda = -b_1$ is two-dimensional, $E_{-b_1} = \text{span}\{|1\rangle, \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)\}$. The provided vectors form an orthonormal basis for this space.

(Case 3) Diagonal ($b_2 = 0$): The operator becomes diagonal: $B = \text{diag}(b_1, 0, 0)$.

- The eigenvalues are b_1 and 0 (two-fold degenerate).
- The eigenspace for $\lambda = b_1$ is one-dimensional, spanned by $|1\rangle$.
- The eigenspace for $\lambda = 0$ is the two-dimensional subspace $E_0 = \text{span}\{|2\rangle, |3\rangle\}$. The basis vectors $|2\rangle$ and $|3\rangle$ form a valid orthonormal eigenbasis for this space.

3. The system is prepared in the initial state $|\psi_0\rangle = |2\rangle$. We first analyze the measurement of observable B, followed by the measurement of observable A.

Measurement of B: The possible outcomes of the measurement are the eigenvalues of B, which are $\{b_1, b_2, -b_2\}$. The probability of measuring a given eigenvalue λ is $P(\lambda) = |\langle\psi_\lambda|\psi_0\rangle|^2$, where $|\psi_\lambda\rangle$ is the corresponding normalized eigenvector of B. The eigenvectors of B are:

$$|\psi_{b_1}\rangle = |1\rangle, \quad |\psi_{+b_2}\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle), \quad |\psi_{-b_2}\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \tag{11}$$

The projection amplitudes of the initial state $|\psi_0\rangle = |2\rangle$ onto these eigenvectors are:

$$\langle\psi_{b_1}|\psi_0\rangle = \langle 1|2\rangle = 0$$
$$\langle\psi_{+b_2}|\psi_0\rangle = \frac{1}{\sqrt{2}}(\langle 2| + \langle 3|)\,|2\rangle = \frac{1}{\sqrt{2}}$$
$$\langle\psi_{-b_2}|\psi_0\rangle = \frac{1}{\sqrt{2}}(\langle 2| - \langle 3|)\,|2\rangle = \frac{1}{\sqrt{2}}$$

The probabilities for the measurement of B are therefore:

$$P(B = b_1) = |0|^2 = 0$$
$$P(B = b_2) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$
$$P(B = -b_2) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

The possible outcomes are b_2 and $-b_2$, each occurring with a probability of 1/2.

Subsequent Measurement of A: The state of the system after the measurement of B collapses to the eigenvector corresponding to the measured outcome. We must therefore calculate the conditional probabilities for the measurement of A. The possible outcomes for A are its eigenvalues $\{a_1, a_2, a_3\}$, corresponding to the eigenvectors $\{|1\rangle, |2\rangle, |3\rangle\}$.

If we condition on the outcome of B, we obtain the following:

(Case 1) Outcome of B was b_2: The system collapses to the state $|\psi_{+b_2}\rangle$. The conditional probabilities for measuring A are:

$$P(A = a_1|B = b_2) = |\langle 1|\psi_{+b_2}\rangle|^2 = 0$$
$$P(A = a_2|B = b_2) = |\langle 2|\psi_{+b_2}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$
$$P(A = a_3|B = b_2) = |\langle 3|\psi_{+b_2}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

(Case 2) Outcome of B was $-b_2$: The system collapses to the state $|\psi_{-b_2}\rangle$. The conditional probabilities for measuring A are:

$$P(A = a_1|B = -b_2) = |\langle 1|\psi_{-b_2}\rangle|^2 = 0$$
$$P(A = a_2|B = -b_2) = |\langle 2|\psi_{-b_2}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$
$$P(A = a_3|B = -b_2) = |\langle 3|\psi_{-b_2}\rangle|^2 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Then the unconditional probability for each outcome of A can be found using the law of total probability, $P(A = a_i) = \sum_j P(A = a_i|B = b_j)P(B = b_j)$:

$$P(A = a_1) = (0) \cdot \frac{1}{2} + (0) \cdot \frac{1}{2} = 0$$
$$P(A = a_2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$
$$P(A = a_3) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

The possible outcomes for the subsequent measurement of A are a_2 and a_3, each with a total probability of 1/2.

4. The results of part (c) are a direct physical manifestation of the non-commutation of operators A and B, established in part (a). The relationship is precise and quantitative.

Measurement Disturbance: The initial state $|\psi_0\rangle = |2\rangle$ is an eigenstate of A corresponding to the eigenvalue a_2. Therefore, a measurement of A on the initial state would yield the outcome a_2 with a probability of 1. The system possesses a definite value for the observable A.

The analysis in part (c) shows that after measuring B, the system is left in a state (either $|\psi_{+b_2}\rangle$ or $|\psi_{-b_2}\rangle$) that is no longer an eigenstate of A. Instead, it is a superposition of two distinct A-eigenstates, $|2\rangle$ and $|3\rangle$. Consequently, the definite information about the value of A has been destroyed; a subsequent measurement of A yields a probabilistic outcome. This introduction of uncertainty via an intermediate measurement is the hallmark of incompatible observables.

Quantitative Prediction from the Commutator: The structure of the commutator

$$[A, B] = b_2(a_2 - a_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{12}$$

quantifies the nature of the incompatibility. Hence, the non-commutation of A and B is not merely a qualitative statement.

The non-zero elements of $[A, B]$ are confined to the 2-3 block, indicating that the operators' incompatibility exists exclusively within the subspace spanned by $\{|2\rangle, |3\rangle\}$. They act compatibly on the state $|1\rangle$.

This mathematical structure is reflected in the physical measurements of part (c):

- The measurement of B on the A-eigenstate $|2\rangle$ introduces uncertainty, but only between the outcomes a_2 and a_3.
- The outcome a_1 remains is inaccessible due to $P(A = a_1) = 0$. This is consistent with the fact that the state $|1\rangle$ is a simultaneous eigenvector of both A and B and is orthogonal to the initial state.

To summarize, the non-zero commutator calculated in part (a) is the mathematical cause of the measurement disturbance. Part (c) quantifies a concrete physical effect, where the structure of the commutator precisely predicts the specific subspace within which quantum uncertainty will manifest.

Problem 14: 2.24

If a beam of spin-3/2 particles is input to a Stern-Gerlach analyzer, there are four output beams whose deflections are consistent with magnetic moments arising from spin angular momentum components of $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, and $-\frac{3}{2}\hbar$. For a spin-3/2 system:

(a) Write down the eigenvalue equations for the S_z operator.

(b) Write down the matrix representation of the S_z eigenstates.

(c) Write down the matrix representation of the S_z operator.

(d) Write down the eigenvalue equations for the S^2 operator.

(e) Write down the matrix representation of the S^2 operator.

Solution:

For a spin-3/2 system, the spin quantum number is $s = 3/2$. The magnetic spin quantum number, m, can take on the values from $-s$ to $+s$ in integer steps:

$$m = \left\{+\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right\}$$

This means the system is described by a four-dimensional Hilbert space. The basis states are the simultaneous eigenstates of S^2 and S_z, which we denote as $|s, m\rangle$.

For this problem, s is always 3/2, so we can use the shorthand $|m\rangle \equiv |3/2, m\rangle$.

(a) The general eigenvalue equation for the S_z operator is $S_z|s, m\rangle = m\hbar|s, m\rangle$. For the four possible values of m in a spin-3/2 system, the equations are:

- $S_z\left|\frac{3}{2}\right\rangle = +\frac{3}{2}\hbar\left|\frac{3}{2}\right\rangle$
- $S_z\left|\frac{1}{2}\right\rangle = +\frac{1}{2}\hbar\left|\frac{1}{2}\right\rangle$
- $S_z\left|-\frac{1}{2}\right\rangle = -\frac{1}{2}\hbar\left|-\frac{1}{2}\right\rangle$
- $S_z\left|-\frac{3}{2}\right\rangle = -\frac{3}{2}\hbar\left|-\frac{3}{2}\right\rangle$

(b) We use the standard basis ordered from the highest value of m to the lowest: $\{|3/2\rangle, |1/2\rangle, |-1/2\rangle, |-3/2\rangle\}$. In this basis, the eigenstates are represented by 4x1 column vectors:

$$\left|\tfrac{3}{2}\right\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left|\tfrac{1}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left|-\tfrac{1}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\left|-\tfrac{3}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) Since we are using the eigenbasis of S_z, the matrix representation of the S_z operator will be a diagonal matrix. The diagonal entries are the eigenvalues corresponding to each basis vector in our ordered basis.

$$S_z \doteq \hbar \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(d) The general eigenvalue equation for the S^2 operator is $S^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$. For a spin-3/2 system, $s = 3/2$, so the eigenvalue is:

$$s(s+1)\hbar^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 = \frac{3}{2}\left(\frac{5}{2}\right)\hbar^2 = \frac{15}{4}\hbar^2$$

This eigenvalue is the same for all four basis states. The eigenvalue equations are:

$S^2\left|\frac{3}{2}\right\rangle = \frac{15}{4}\hbar^2\left|\frac{3}{2}\right\rangle$
$S^2\left|\frac{1}{2}\right\rangle = \frac{15}{4}\hbar^2\left|\frac{1}{2}\right\rangle$
$S^2\left|-\frac{1}{2}\right\rangle = \frac{15}{4}\hbar^2\left|-\frac{1}{2}\right\rangle$
$S^2\left|-\frac{3}{2}\right\rangle = \frac{15}{4}\hbar^2\left|-\frac{3}{2}\right\rangle$

(e) Since all the basis vectors are eigenvectors of S^2 with the *same* eigenvalue, the matrix representation of S^2 is a diagonal matrix where every diagonal entry is the eigenvalue $\frac{15}{4}\hbar^2$. This is equivalent to the eigenvalue multiplied by the 4x4 identity matrix.

$$S^2 \doteq \frac{15}{4}\hbar^2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \frac{15}{4}\hbar^2 I_4$$

Problem 15: 2.25

Are the projection operators $P_\uparrow$ and $P_\downarrow$ Hermitian? Explain.

Solution:

We will prove the general theorem that any one-dimensional orthogonal projection operator is Hermitian. The specific cases for $P_\uparrow$ and $P_\downarrow$ will then follow as immediate corollaries.

Theorem: For any normalized state $|\psi\rangle \in$, the projection operator $P_\psi = |\psi\rangle\langle\psi|$ is Hermitian.

Proof. To prove that P_ψ is Hermitian, we must show that $\langle\phi|P_\psi\chi\rangle = \langle P_\psi\phi|\chi\rangle$.

We evaluate the left-hand side (LHS) and right-hand side (RHS) of this relation independently.

Left-Hand Side:

$$\begin{aligned}\langle\phi|P_\psi\chi\rangle &= \langle\phi|(|\psi\rangle\langle\psi|)\chi\rangle\\ &= \langle\phi||\psi\rangle(\langle\psi|\chi\rangle)\rangle \quad \text{(Definition of the outer product's action)}\\ &= \langle\psi|\chi\rangle\langle\phi|\psi\rangle \quad \text{(Linearity of the inner product in the ket)}\end{aligned} \tag{13}$$

Right-Hand Side:

$$\begin{aligned}\langle P_\psi\phi|\chi\rangle &= \langle(|\psi\rangle\langle\psi|)\phi|\chi\rangle\\ &= \langle|\psi\rangle(\langle\psi|\phi\rangle)|\chi\rangle \quad \text{(Definition of the outer product's action)}\\ &= \overline{\langle\psi|\phi\rangle}\langle\psi|\chi\rangle \quad \text{(Anti-linearity of the inner product in the bra)}\\ &= \langle\phi|\psi\rangle\langle\psi|\chi\rangle \quad \text{(Conjugate symmetry property of the inner product)}\end{aligned} \tag{14}$$

By comparing preceding equations, we see that:

$$\langle \phi | P_\psi \chi \rangle = \langle P_\psi \phi | \chi \rangle$$

This holds for all $|\phi\rangle, |\chi\rangle \in$. Therefore,the operator P_ψ is Hermitian. $\square$

The operators $P_\uparrow = |\uparrow\rangle\langle\uparrow|$ and $P_\downarrow = |\downarrow\rangle\langle\downarrow|$ are one-dimensional projection operators onto the normalized states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. By the general theorem proven above, both operators $P_\uparrow$ and $P_\downarrow$ are necessarily Hermitian.

Problem 16: 3.2

1. Show that the probability of a measurement of the energy is time independent for a general state $|\Psi(t)\rangle = \sum_n c_n(t)|E_n\rangle$ that evolves due to a time-independent Hamiltonian.

2. Show that the probability of measurements of other observables are also time independent if those observables commute with the Hamiltonian.

Solution:

Consider a quantum system evolving under a time-independent Hamiltonian, $\hat{H}$.

1. We will show that the probability of measuring a specific energy eigenvalue E_n is independent of time.

   The evolution of a quantum state $|\Psi(t)\rangle$ is determined by the time-dependent Schrödinger equation,

   $$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \tag{15}$$

   whose formal solution for a time-independent $\hat{H}$ is

   $$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle, \tag{16}$$

   where $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ is a unitary time-evolution operator.

   The Hamiltonian, being Hermitian, admits a complete orthonormal set of eigenstates $\{|E_n\rangle\}$ satisfying

   $$\hat{H}|E_n\rangle = E_n|E_n\rangle, \qquad \langle E_m|E_n\rangle = \delta_{mn}. \tag{17}$$

   An arbitrary initial state can therefore be expanded as

   $$|\Psi(0)\rangle = \sum_n c_n(0)|E_n\rangle, \tag{18}$$

   with coefficients $c_n(0) = \langle E_n|\Psi(0)\rangle$. Applying $\hat{U}(t)$ gives

   $$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}\sum_n c_n(0)|E_n\rangle = \sum_n c_n(0)e^{-i\hat{H}t/\hbar}|E_n\rangle. \tag{19}$$

   Because each $|E_n\rangle$ is an eigenstate of $\hat{H}$, the exponential acts simply by multiplying by $e^{-iE_nt/\hbar}$:

   $$e^{-i\hat{H}t/\hbar}|E_n\rangle = e^{-iE_nt/\hbar}|E_n\rangle. \tag{20}$$

Hence the evolved state is

$$|\Psi(t)\rangle = \sum_n c_n(0)e^{-iE_n t/\hbar}|E_n\rangle, \tag{21}$$

which can also be written as $\sum_n c_n(t)|E_n\rangle$ with $c_n(t) = c_n(0)e^{-iE_n t/\hbar}$. Each coefficient thus acquires only a phase factor. The probability of obtaining the energy eigenvalue E_k at time t is

$$P(E_k, t) = |\langle E_k|\Psi(t)\rangle|^2. \tag{22}$$

By orthonormality of the basis,

$$\langle E_k|\Psi(t)\rangle = \sum_n c_n(t)\langle E_k|E_n\rangle = c_k(t), \tag{23}$$

so that

$$P(E_k, t) = |c_k(t)|^2 = |c_k(0)e^{-iE_k t/\hbar}|^2 = |c_k(0)|^2|e^{-iE_k t/\hbar}|^2. \tag{24}$$

Using $e^{i\theta} = \cos\theta + i\sin\theta$, we find $|e^{-iE_k t/\hbar}|^2 = \cos^2(E_k t/\hbar) + \sin^2(E_k t/\hbar) = 1$, implying

$$P(E_k, t) = |c_k(0)|^2. \tag{25}$$

Therefore, the probability of measuring any particular energy is independent of time.

2. We will show that the probability of measuring an eigenvalue a of any other observable $\hat{A}$ is also time-independent, provided that $\hat{A}$ commutes with the Hamiltonian (i.e., $[\hat{A}, \hat{H}] = 0$).

   If two Hermitian operators commute, they can be simultaneously diagonalized, meaning there exists an orthonormal basis $\{|k\rangle\}$ of joint eigenstates satisfying

   $$\hat{H}|k\rangle = E_k|k\rangle, \qquad \hat{A}|k\rangle = a_k|k\rangle. \tag{26}$$

   Expanding the state in this shared basis gives

   $$|\Psi(t)\rangle = \sum_k d_k(0)e^{-iE_k t/\hbar}|k\rangle, \tag{27}$$

   where $d_k(0) = \langle k|\Psi(0)\rangle$. The probability of measuring eigenvalue a of $\hat{A}$ at time t is the total weight of all states $|k\rangle$ corresponding to that eigenvalue,

   $$P(a, t) = \sum_{k\,:\,a_k = a} |\langle k|\Psi(t)\rangle|^2. \tag{28}$$

   Since $\langle k|\Psi(t)\rangle = d_k(0)e^{-iE_k t/\hbar}$, the magnitude squared of each term is $|d_k(0)|^2$, so that

   $$P(a, t) = \sum_{k\,:\,a_k = a} |d_k(0)|^2, \tag{29}$$

   which again is constant in time.

   An alternative demonstration can be had using projection operators. Let $\hat{P}_a$ denote the spectral projector onto the eigenspace of $\hat{A}$ with eigenvalue a. The probability of measuring a is

   $$P(a, t) = \langle\Psi(t)|\hat{P}_a|\Psi(t)\rangle. \tag{30}$$

Using $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$ and the unitarity of $\hat{U}(t)$, we can rewrite this as

$$P(a,t) = \langle\Psi(0)|\hat{U}^\dagger(t)\hat{P}_a\hat{U}(t)|\Psi(0)\rangle. \tag{31}$$

Because $\hat{P}_a$ is a function of $\hat{A}$ and $[\hat{A},\hat{H}] = 0$, it follows that $[\hat{P}_a,\hat{H}] = 0$. The time-evolution operator, being a function of $\hat{H}$, also commutes with any such $\hat{P}_a$, so $[\hat{P}_a,\hat{U}(t)] = 0$. Therefore,

$$\hat{U}^\dagger(t)\hat{P}_a\hat{U}(t) = \hat{P}_a, \tag{32}$$

and the expression simplifies to

$$P(a,t) = \langle\Psi(0)|\hat{P}_a|\Psi(0)\rangle = P(a,0). \tag{33}$$

The probability is thus constant in time, confirming that any observable commuting with the Hamiltonian represents a conserved quantity in the statistical sense.

Problem 17: 3.3

(a) Show that the Hamiltonian H can be written in the simple form of $H = \sqrt{\omega_0^2+\omega_1^2}S_n$..

(b) Diagonalize the Hamiltonian $H = \frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$.

(c) Verify the following:

$$\begin{aligned}|+\rangle_n &= \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle\\ |-\rangle_n &= \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle\end{aligned}$$

Solution:

We use the standard Pauli matrices and spin operators: $S_i = \frac{\hbar}{2}\sigma_i$.

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

We define the total frequency magnitude Ω and the angle θ in the x-z plane:

$$\Omega \equiv \sqrt{\omega_0^2+\omega_1^2}, \qquad \cos\theta = \frac{\omega_0}{\Omega}, \qquad \sin\theta = \frac{\omega_1}{\Omega}.$$

The effective frequency vector is $\vec{\omega} = (\omega_1, 0, \omega_0)$.

(a) We first express the Hamiltonian $H = \omega_0 S_z + \omega_1 S_x$ in terms of the frequency vector $\vec{\omega}$. Using $S_i = \frac{\hbar}{2}\sigma_i$, we have

$$H = \frac{\hbar}{2}(\omega_0\sigma_z + \omega_1\sigma_x) \tag{34}$$

We can take this to be a dot product between the frequency vector $\vec{\omega} = (\omega_1, 0, \omega_0)$ and the Pauli vector $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$:

$$\begin{aligned}H &= \frac{\hbar}{2}(\omega_1\sigma_x + 0\sigma_y + \omega_0\sigma_z)\\ &= \frac{\hbar}{2}\vec{\omega}\cdot\vec{\sigma}\end{aligned}$$

Next, we decompose $\vec{\omega}$ into magnitude and direction. The magnitude of $\vec{\omega}$ is $|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_0^2} = \Omega$. The unit vector $\hat{n}$ in the direction of $\vec{\omega}$ is $\hat{n} = \frac{\vec{\omega}}{\Omega}$:

$$\vec{\omega} = \Omega\hat{n} = \Omega(\sin\theta, 0, \cos\theta) \tag{35}$$

Hence, we substitute $\vec{\omega} = \Omega\hat{n}$ into the Hamiltonian:

$$\begin{aligned} H &= \frac{\hbar}{2}(\Omega\hat{n}) \cdot \vec{\sigma} \\ &= \Omega\left(\frac{\hbar}{2}\hat{n} \cdot \vec{\sigma}\right) \end{aligned}$$

The spin operator along the direction $\hat{n}$ is defined as $S_n = \hat{n} \cdot \vec{S} = \frac{\hbar}{2}\hat{n} \cdot \vec{\sigma}$. Therefore, the Hamiltonian simplifies to:

$$\boxed{H = \sqrt{\omega_0^2 + \omega_1^2}\, S_n} \tag{36}$$

(b) In this part, we aim to diagonalize the Hamiltonian matrix, where $H = \frac{\hbar\Omega}{2}M$:

$$H = \frac{\hbar\Omega}{2}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

We solve the characteristic equation $\det(H - \lambda I) = 0$:

$$\det\begin{pmatrix} \frac{\hbar\Omega}{2}\cos\theta - \lambda & \frac{\hbar\Omega}{2}\sin\theta \\ \frac{\hbar\Omega}{2}\sin\theta & -\frac{\hbar\Omega}{2}\cos\theta - \lambda \end{pmatrix} = 0$$

This simplifies to:

$$\lambda^2 - \left(\frac{\hbar\Omega}{2}\right)^2(\cos^2\theta + \sin^2\theta) = 0 \quad \Longrightarrow \quad \lambda^2 = \left(\frac{\hbar\Omega}{2}\right)^2$$

The eigenvalues are:

$$E_+ = +\frac{\hbar\Omega}{2}, \qquad E_- = -\frac{\hbar\Omega}{2} \tag{37}$$

(a) For the eigenvector for E_+, we first solve the equation $(H - E_+I)\vec{v}_+ = 0$. Dividing by $\frac{\hbar\Omega}{2}$, we solve $(M - I)\vec{v}_+ = 0$. The first row gives: $(\cos\theta - 1)v_1 + \sin\theta v_2 = 0$.

$$v_2 = \frac{1 - \cos\theta}{\sin\theta}v_1 \tag{38}$$

Using the half-angle identity $\frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$:

$$v_2 = \tan\frac{\theta}{2}v_1 \quad \text{for } \theta \neq 0, \pi \tag{39}$$

Choosing the normalized solution $v_1 = \cos\frac{\theta}{2}$ and $v_2 = \sin\frac{\theta}{2}$:

$$|+\rangle_n = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \tag{40}$$

(b) For the eigenvector for E_-, we solve $(H - E_- I)\vec{v}_- = 0$, or $(M + I)\vec{v}_- = 0$. The first row gives: $(\cos\theta + 1)v_1 + \sin\theta v_2 = 0$.

$$v_2 = -\frac{\cos\theta + 1}{\sin\theta} v_1 \tag{41}$$

Using the half-angle identity $\frac{\cos\theta+1}{\sin\theta} = \cot\frac{\theta}{2}$:

$$v_2 = -\cot\frac{\theta}{2} v_1 \quad \text{for } \theta \neq 0, \pi \tag{42}$$

Choosing the normalized solution $v_1 = \sin\frac{\theta}{2}$ and $v_2 = -\cos\frac{\theta}{2}$:

$$|-\rangle_n = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}|-\rangle \tag{43}$$

The general solutions derived in the preceding steps hold even where the division by $\sin\theta$ is undefined.

- If $\theta = 0$, $\sin\frac{\theta}{2} = 0$, and the system reduces to the S_z eigenstate $|+\rangle$. The derived eigenvector $|+\rangle_n$ becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$, which is the correct eigenstate for E_+.
- If $\theta = \pi$, $\cos\frac{\theta}{2} = 0$, and the system reduces to the S_z eigenstate $|+\rangle$, which is the correct E_- eigenstate for $H = -\Omega S_z$.

The general form of the eigenvectors is therefore valid for all θ.

(c) Let us verify that the derived states are indeed eigenstates of the spin operator $S_n = \frac{\hbar}{2}\sigma_n$, where $\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

(a) Verify $|+\rangle_n$ (Eigenvalue $+\hbar/2$):
We apply S_n to $|+\rangle_n = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$:

$$S_n|+\rangle_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ \sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2} \end{pmatrix}$$

Using angle subtraction identities $\cos(\alpha - \beta)$ and $\sin(\alpha - \beta)$:

$$S_n|+\rangle_n = \frac{\hbar}{2}\begin{pmatrix} \cos\left(\theta - \frac{\theta}{2}\right) \\ \sin\left(\theta - \frac{\theta}{2}\right) \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = +\frac{\hbar}{2}|+\rangle_n$$

(b) Verify $|-\rangle_n$ (Eigenvalue $-\hbar/2$):
We apply S_n to $|-\rangle_n = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$:

$$S_n|-\rangle_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta\sin\frac{\theta}{2} - \sin\theta\cos\frac{\theta}{2} \\ \sin\theta\sin\frac{\theta}{2} + \cos\theta\cos\frac{\theta}{2} \end{pmatrix}$$

Using angle subtraction identities:

$$S_n|-\rangle_n = \frac{\hbar}{2}\begin{pmatrix} -\sin\left(\theta - \frac{\theta}{2}\right) \\ \cos\left(\theta - \frac{\theta}{2}\right) \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

Factoring out the negative sign:

$$S_n|-\rangle_n = -\frac{\hbar}{2}\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix} = -\frac{\hbar}{2}|-\rangle_n$$

As required for eigenstates of a Hermitian operator with distinct eigenvalues, we need to verify that the states are orthogonal. Indeed, this is the case because

$$\langle +|-\rangle_n = \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$$

> **Problem 18: 3.4**
>
> Consider a spin-1/2 particle with a magnetic moment placed in a uniform magnetic field aligned with the z-axis. Verify by explicit matrix calculations that the Hamiltonian commutes with the spin component operator in the z-direction but not with spin component operators in the x- and y-directions. Comment on the relevance of these results to spin precession.

Solution:

We consider a spin-$\frac{1}{2}$ particle whose magnetic moment operator $\vec{\mu}$ is proportional to its spin operator $\vec{S}$ according to

$$\vec{\mu} = \gamma\vec{S}, \tag{44}$$

where γ is the gyromagnetic ratio. The particle is placed in a uniform magnetic field oriented along the z-axis, $\vec{B} = B_0\hat{k}$. The interaction energy between the magnetic moment and the external field defines the Hamiltonian

$$H = -\vec{\mu}\cdot\vec{B} = -(\gamma\vec{S})\cdot(B_0\hat{k}) = -\gamma B_0 S_z. \tag{45}$$

It is convenient to introduce the Larmor frequency $\omega_0 \equiv \gamma B_0$, noting that for the electron γ (and thus ω_0) is negative. The Hamiltonian then becomes

$$H = -\omega_0 S_z. \tag{46}$$

In the standard eigenbasis of S_z, denoted $\{|+\rangle, |-\rangle\}$, the spin operators are represented by

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{47}$$

Consequently, the Hamiltonian matrix is

$$H = -\omega_0 S_z = \frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{48}$$

We begin by verifying the commutation relations of H with each spin component. The product HS_z equals

$$HS_z = \left(\frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\left(\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{4}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \tag{49}$$

and S_zH gives the same result:

$$S_zH = \left(\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\left(\frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{4}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}. \tag{50}$$

Thus $[H, S_z] = HS_z - S_zH = \mathbf{0}$.

For the x-component, we compute

$$HS_x = \left(\frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\left(\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{51}$$

and

$$S_xH = \left(\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)\left(\frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{52}$$

Subtracting, we find

$$[H, S_x] = \frac{\hbar^2\omega_0}{4}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{53}$$

Recognizing $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = -\frac{2i}{\hbar}S_y$, we obtain

$$[H, S_x] = -i\hbar\omega_0 S_y. \tag{54}$$

Similarly, for the y-component,

$$HS_y = \left(\frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\left(\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \tag{55}$$

and

$$S_yH = \left(\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right)\left(\frac{\hbar\omega_0}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right) = \frac{\hbar^2\omega_0}{4}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}. \tag{56}$$

The commutator is

$$[H, S_y] = \frac{\hbar^2\omega_0}{4}\left(\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\right) = \frac{i\hbar^2\omega_0}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{57}$$

and because $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{2}{\hbar}S_x$, we find

$$[H, S_y] = i\hbar\omega_0 S_x. \tag{58}$$

Having obtained all commutators, we connect these static relations to the time evolution of expectation values via Heisenberg's equation of motion,

$$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [H, A]\rangle. \tag{59}$$

Applying this to each spin component yields

$$\frac{d\langle S_z\rangle}{dt} = \frac{1}{i\hbar}\langle [H, S_z]\rangle = 0, \tag{60}$$

demonstrating that $\langle S_z\rangle$ is conserved. For the transverse components,

$$\frac{d\langle S_x\rangle}{dt} = \frac{1}{i\hbar}\langle [H, S_x]\rangle = -\omega_0\langle S_y\rangle, \qquad \frac{d\langle S_y\rangle}{dt} = \frac{1}{i\hbar}\langle [H, S_y]\rangle = \omega_0\langle S_x\rangle. \tag{61}$$

These coupled differential equations describe uniform circular motion in the $\langle S_x \rangle$–$\langle S_y \rangle$ plane at angular frequency ω_0. The longitudinal spin component remains fixed, while the transverse components execute rotation, forming the physical phenomenon known as Larmor precession.

This analysis shows that the algebraic commutation relations encode the system's dynamical structure. The commutation of H with S_z enforces conservation of the spin projection along the field, while the non-commutation with S_x and S_y generates the precessional motion of the transverse components. The spin vector thus traces a cone around the magnetic field direction, maintaining a constant magnitude but rotating at frequency ω_0.

> **Problem 19: 3.6**
>
> Consider a spin-1/2 particle with a magnetic moment.
>
> (a) At time $t = 0$, the observable S_x is measured, with the result $\hbar/2$. What is the state vector $|\psi(t = 0)\rangle$ immediately after the measurement?
>
> (b) Immediately after the measurement, a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ is applied and the particle is allowed to evolve for a time T. What is the state of the system at time $t = T$?
>
> (c) At $t = T$, the magnetic field is very rapidly changed to $\mathbf{B} = B_0\hat{\mathbf{y}}$. After another time interval T, a measurement of S_x is carried out once more. What is the probability that a value $\hbar/2$ is found?

Solution: Consider a spin-1/2 particle with a magnetic moment.

(a) The measurement postulate asserts that an ideal (projective) measurement of an observable with non-degenerate eigenvalues collapses the system's state onto the corresponding normalized eigenstate of the measured operator.

Here, the observable is the spin component along the x-axis,

$$S_x = \frac{\hbar}{2}\sigma_x, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{62}$$

The eigenvalue equation $S_x |\pm\rangle_x = \pm\frac{\hbar}{2} |\pm\rangle_x$ has normalized eigenstates, expressed in the S_z basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, as

$$|+\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right), \qquad |-\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right). \tag{63}$$

Since the measurement yields the result $S_x = +\hbar/2$, the postmeasurement state of the system is the corresponding eigenstate:

$$\boxed{|\psi(0)\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.} \tag{64}$$

Thus, immediately after the measurement at $t = 0$, the system is fully prepared in the spin-up state along the x-axis.

(b) For the interval $t \in [0, T]$, the magnetic field is $\mathbf{B} = B_0\hat{\mathbf{z}}$, which corresponds to the Hamiltonian

$$H = -\gamma B_0 S_z = -\omega_0 S_z, \tag{65}$$

where $\omega_0 \equiv \gamma B_0$. The time-evolution operator over a duration T is

$$U_z(T) = e^{-iHT/\hbar} = e^{i\omega_0 T S_z/\hbar}. \tag{66}$$

Using $S_z|\pm\rangle_z = \pm(\hbar/2)|\pm\rangle_z$, we obtain

$$U_z(T)|\pm\rangle_z = e^{\pm i\omega_0 T/2}|\pm\rangle_z. \tag{67}$$

Applying this to the initial state,

$$|\psi(T)\rangle = U_z(T)|\psi(0)\rangle \tag{68}$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\omega_0 T/2}|+\rangle_z + e^{-i\omega_0 T/2}|-\rangle_z\right). \tag{69}$$

Letting $\varphi = \omega_0 T$, the state becomes

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left(e^{i\varphi/2}|+\rangle_z + e^{-i\varphi/2}|-\rangle_z\right). \tag{70}$$

This state represents the initial spin vector having precessed around the z-axis.

$$\boxed{|\psi(T)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega T/2} \\ e^{-i\omega T/2} \end{pmatrix}} \tag{71}$$

(c) At time $t = T$, the magnetic field is suddenly changed to $\mathbf{B} = B_0\hat{\mathbf{y}}$. Under the sudden approximation, the state immediately after switching remains unchanged:

$$|\psi(T^+)\rangle = |\psi(T)\rangle. \tag{72}$$

The new Hamiltonian is

$$H_2 = -\gamma B_0 S_y = -\omega S_y, \tag{73}$$

where $\omega = \gamma B_0$.

The time-evolution operator for this interval of duration T is

$$U_2(T) = e^{-iH_2T/\hbar} = e^{-i(-\omega S_y)T/\hbar} = e^{i\omega T S_y/\hbar} = e^{i(\omega T/2)\sigma_y}. \tag{74}$$

Using the spin-$\frac{1}{2}$ identity $e^{i\theta\sigma_k} = I\cos\theta + i\sigma_k\sin\theta$, with $\theta = \omega T/2$, we obtain

$$U_2(T) = I\cos\left(\frac{\omega T}{2}\right) + i\sigma_y\sin\left(\frac{\omega T}{2}\right) = \begin{pmatrix} \cos(\omega T/2) & \sin(\omega T/2) \\ -\sin(\omega T/2) & \cos(\omega T/2) \end{pmatrix}. \tag{75}$$

This operator represents a rotation about the y-axis by an angle ωT.

The state at $t = T$ from part (b) is

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega T/2} \\ e^{-i\omega T/2} \end{pmatrix}. \tag{76}$$

Thus, the state at $t = 2T$ is

$$|\psi(2T)\rangle = U_2(T)\,|\psi(T)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\omega T/2) & \sin(\omega T/2) \\ -\sin(\omega T/2) & \cos(\omega T/2) \end{pmatrix} \begin{pmatrix} e^{i\omega T/2} \\ e^{-i\omega T/2} \end{pmatrix}. \tag{77}$$

We now compute the probability of finding $S_x = +\hbar/2$ at $t = 2T$. The corresponding eigenstate is

$$|+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{78}$$

Hence the probability amplitude is

$$\mathcal{A} = \langle +|_x|\psi(2T)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} |\psi(2T)\rangle \tag{79}$$

$$= \frac{1}{2}\left[(\cos(\omega T/2)e^{i\omega T/2} + \sin(\omega T/2)e^{-i\omega T/2}) + (-\sin(\omega T/2)e^{i\omega T/2} + \cos(\omega T/2)e^{-i\omega T/2})\right]. \tag{80}$$

Letting $\theta = \omega T/2$, we simplify:

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\left[(\cos\theta - \sin\theta)e^{i\theta} + (\cos\theta + \sin\theta)e^{-i\theta}\right] \\ &= \frac{1}{2}\left[\cos\theta(e^{i\theta} + e^{-i\theta}) - \sin\theta(e^{i\theta} - e^{-i\theta})\right] \\ &= \frac{1}{2}\left[2\cos^2\theta - 2i\sin^2\theta\right] = \cos^2\theta - i\sin^2\theta. \end{aligned}$$

Therefore, the probability is

$$\begin{aligned} P(S_x = +\tfrac{\hbar}{2}) = |\mathcal{A}|^2 &= |\cos^2\theta - i\sin^2\theta|^2 \\ &= \cos^4\theta + \sin^4\theta \\ &= 1 - 2\sin^2\theta\cos^2\theta \\ &= 1 - \tfrac{1}{2}\sin^2(2\theta) = 1 - \tfrac{1}{2}\sin^2(\omega T). \end{aligned}$$

Equivalently, this can be expressed as

$$\boxed{P(S_x = +\tfrac{\hbar}{2}) = \frac{1}{2}\left(1 + \cos^2(\omega T)\right).} \tag{81}$$

Thus, the probability oscillates between 1 and $\frac{1}{2}$ as a function of ωT, reflecting the coherent spin precession under successive orthogonal rotations.

Problem 20: 3.8

A beam of identical neutral particles with spin 1/2 is prepared in the $|+\rangle$ state. The beam enters a uniform magnetic field B_0, which is in the xz-plane and makes an angle θ with the z-axis. After a time T in the field, the beam enters a Stern-Gerlach analyzer oriented along the y-axis. What is the probability that particles will be measured to have spin up in the y-direction? Check your result by evaluating the special cases $\theta = 0$ and $\theta = \pi/2$.

Solution:

We consider a spin-1/2 particle in a two-dimensional Hilbert space with the standard S_z eigenbasis $\{|+\rangle, |-\rangle\}$. The initial state at $t = 0$ is prepared as:

$$|\psi(0)\rangle = |+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{82}$$

The magnetic field $\mathbf{B}$ lies in the xz-plane, described by the unit vector $\mathbf{n} = (\sin\theta, 0, \cos\theta)$. The Hamiltonian for a particle with magnetic moment $\boldsymbol{\mu} = \gamma\mathbf{S}$ is:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma B_0(\mathbf{n} \cdot \mathbf{S}). \tag{83}$$

Using the relation $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ and defining the signed Larmor frequency $\omega \equiv \gamma B_0$, which preserves the crucial sign information of the gyromagnetic ratio, the Hamiltonian is:

$$H = -\frac{\hbar\omega}{2}(\mathbf{n} \cdot \boldsymbol{\sigma}) = -\frac{\hbar\omega}{2}(\sin\theta\,\sigma_x + \cos\theta\,\sigma_z). \tag{84}$$

For a time-independent Hamiltonian, the state evolves according to the unitary time evolution operator $U(T) = \exp(-iHT/\hbar)$. Substituting the Hamiltonian from Eq. (**??**):

$$U(T) = \exp\left[-\frac{i}{\hbar}\left(-\frac{\hbar\omega}{2}(\mathbf{n} \cdot \boldsymbol{\sigma})\right)T\right] = \exp\left[i\frac{\omega T}{2}(\mathbf{n} \cdot \boldsymbol{\sigma})\right]. \tag{85}$$

We use the fundamental identity for the exponential of a Pauli vector, $\exp[i\alpha(\mathbf{n} \cdot \boldsymbol{\sigma})] = \cos\alpha\, I + i\sin\alpha\,(\mathbf{n} \cdot \boldsymbol{\sigma})$, with the angle $\alpha = \omega T/2$:

$$U(T) = \cos\left(\frac{\omega T}{2}\right) I + i\sin\left(\frac{\omega T}{2}\right)(\mathbf{n} \cdot \boldsymbol{\sigma}). \tag{86}$$

We apply the propagator to the initial state $|\psi(0)\rangle = |+\rangle$. First, we compute the action of the Pauli vector term:

$$(\mathbf{n} \cdot \boldsymbol{\sigma})|+\rangle = (\sin\theta\,\sigma_x + \cos\theta\,\sigma_z)|+\rangle = \sin\theta\,|-\rangle + \cos\theta\,|+\rangle. \tag{87}$$

The time-evolved state is therefore:

$$\begin{aligned} |\psi(T)\rangle &= U(T)|+\rangle \\ &= \cos\left(\frac{\omega T}{2}\right)|+\rangle + i\sin\left(\frac{\omega T}{2}\right)(\cos\theta\,|+\rangle + \sin\theta\,|-\rangle) \\ &= \left[\cos\left(\frac{\omega T}{2}\right) + i\sin\left(\frac{\omega T}{2}\right)\cos\theta\right]|+\rangle + i\sin\left(\frac{\omega T}{2}\right)\sin\theta\,|-\rangle. \end{aligned} \tag{88}$$

In the S_z basis, this state is represented by the column vector:

$$|\psi(T)\rangle \doteq \begin{pmatrix} \cos(\omega T/2) + i\sin(\omega T/2)\cos\theta \\ i\sin(\omega T/2)\sin\theta \end{pmatrix}. \tag{89}$$

The measurement is performed along the y-axis. The normalized eigenstate for spin-up along y is:

$$|+_y\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle), \quad \text{so} \quad \langle +_y| = \frac{1}{\sqrt{2}}(\langle +| - i\,\langle -|). \tag{90}$$

The probability is the squared magnitude of the projection amplitude, $P(S_y = +\hbar/2) = |\,\langle +_y|\psi(T)\rangle\,|^2$. We compute the amplitude:

$$\begin{aligned}\langle +_y|\psi(T)\rangle &= \frac{1}{\sqrt{2}}\left[\left(\cos\frac{\omega T}{2} + i\sin\frac{\omega T}{2}\cos\theta\right) - i\left(i\sin\frac{\omega T}{2}\sin\theta\right)\right] \\ &= \frac{1}{\sqrt{2}}\left[\cos\frac{\omega T}{2} + i\sin\frac{\omega T}{2}\cos\theta + \sin\frac{\omega T}{2}\sin\theta\right].\end{aligned} \tag{91}$$

The probability is the squared modulus of this complex amplitude:

$$\begin{aligned}P(S_y = +\hbar/2) &= \frac{1}{2}\left|\left(\cos\frac{\omega T}{2} + \sin\frac{\omega T}{2}\sin\theta\right) + i\left(\sin\frac{\omega T}{2}\cos\theta\right)\right|^2 \\ &= \frac{1}{2}\left[\left(\cos\frac{\omega T}{2} + \sin\frac{\omega T}{2}\sin\theta\right)^2 + \left(\sin\frac{\omega T}{2}\cos\theta\right)^2\right] \\ &= \frac{1}{2}\left[\cos^2\frac{\omega T}{2} + 2\cos\frac{\omega T}{2}\sin\frac{\omega T}{2}\sin\theta + \sin^2\frac{\omega T}{2}\sin^2\theta + \sin^2\frac{\omega T}{2}\cos^2\theta\right] \\ &= \frac{1}{2}\left[\cos^2\frac{\omega T}{2} + \sin^2\frac{\omega T}{2} + \sin(\omega T)\sin\theta\right].\end{aligned} \tag{92}$$

Using the identity $\cos^2\alpha + \sin^2\alpha = 1$, we arrive at the final expression:

$$\boxed{P(S_y = +\hbar/2) = \frac{1}{2}\left(1 + \sin(\omega T)\sin\theta\right)}. \tag{93}$$

We test the general formula against physically transparent limiting cases.

- **Case 1:** $\theta = 0$ **(Field along z-axis).** The formula gives:

  $$P = \frac{1}{2}(1 + \sin(\omega T)\cdot 0) = \frac{1}{2}.$$

  Physical Interpretation: The initial state $|+\rangle$ is an eigenstate of the Hamiltonian $H \propto S_z$. It therefore only acquires a global phase and does not precess. The probability of measuring spin-up along any axis in the xy-plane remains constant at $1/2$. The result is correct.

- **Case 2:** $\theta = \pi/2$ **(Field along x-axis).** The formula gives:

  $$P = \frac{1}{2}(1 + \sin(\omega T)\cdot 1) = \frac{1}{2}(1 + \sin(\omega T)).$$

  Physical Interpretation: The spin precesses around the x-axis. The initial state $|+\rangle$ (spin along $+z$) rotates in the yz-plane. The probability of finding the spin along $+y$ should oscillate sinusoidally, reaching a maximum of 1 when the spin points along $+y$ (at $\omega T = \pi/2$) and a minimum of 0 when it points along $-y$ (at $\omega T = 3\pi/2$). The formula correctly reproduces this behavior. The result is correct.

Problem 21: 3.10

Consider a spin-1/2 particle with a magnetic moment. At time $t = 0$, the state of the particle is $|\psi(t=0)\rangle = |+\rangle$. The system is allowed to evolve in a uniform magnetic field $\mathbf{B} = B_0(\hat{\mathbf{x}} + \hat{\mathbf{z}})/\sqrt{2}$. What is the probability that the particle will be measured to have spin down in the z-direction after a time t?

Solution:

The system consists of a spin-1/2 particle with magnetic moment $\boldsymbol{\mu} = \gamma \mathbf{S}$ in a uniform magnetic field. The interaction Hamiltonian is given by the standard form:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma \mathbf{S} \cdot \mathbf{B}. \tag{94}$$

The magnetic field is directed along the unit vector $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{z}})$. Using the spin operator $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ and defining the standard Larmor frequency $\omega_0 \equiv \gamma B_0$, the Hamiltonian takes the physically transparent form:

$$H = -\gamma \left(\frac{\hbar}{2}\boldsymbol{\sigma}\right) \cdot (B_0 \hat{\mathbf{n}}) = -\frac{\hbar\omega_0}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}). \tag{95}$$

This form correctly represents the energy landscape, where the energy is minimized when the spin is aligned with the magnetic field direction $\hat{\mathbf{n}}$.

For a time-independent Hamiltonian, the state evolves according to the unitary time evolution operator $U(t) = \exp(-iHt/\hbar)$. Substituting the Hamiltonian from Eq. (95):

$$U(t) = \exp\left[-\frac{i}{\hbar}\left(-\frac{\hbar\omega_0}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\right)t\right] = \exp\left[i\frac{\omega_0 t}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\right]. \tag{96}$$

We use the fundamental identity for the exponential of a Pauli vector, $\exp(i\alpha(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})) = I\cos\alpha + i(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\sin\alpha$. Setting the angle $\alpha = \omega_0 t/2$, the propagator becomes:

$$U(t) = I\cos\left(\frac{\omega_0 t}{2}\right) + i(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\sin\left(\frac{\omega_0 t}{2}\right). \tag{97}$$

The initial state is prepared as spin-up along the z-axis, $|\psi(0)\rangle = |+\rangle$. The time-evolved state is $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$. First, we compute the action of the operator part on the initial state:

$$(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\,|+\rangle = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\,|+\rangle = \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle). \tag{98}$$

Substituting this into the expression for the evolved state:

$$|\psi(t)\rangle = \cos\left(\frac{\omega_0 t}{2}\right)|+\rangle + i\sin\left(\frac{\omega_0 t}{2}\right)\left[\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right] \tag{99}$$

$$= \left[\cos\left(\frac{\omega_0 t}{2}\right) + \frac{i}{\sqrt{2}}\sin\left(\frac{\omega_0 t}{2}\right)\right]|+\rangle + \left[\frac{i}{\sqrt{2}}\sin\left(\frac{\omega_0 t}{2}\right)\right]|-\rangle\,. \tag{100}$$

The probability of measuring the particle to have spin-down in the z-direction at time t is given by the Born rule, $P(S_z = -\hbar/2, t) = |\,\langle -|\psi(t)\rangle\,|^2$. The required projection amplitude is the coefficient of the $|-\rangle$ component of the state vector:

$$\langle -|\psi(t)\rangle = \frac{i}{\sqrt{2}}\sin\left(\frac{\omega_0 t}{2}\right). \tag{101}$$

The probability is the squared modulus of this amplitude:

$$P(S_z = -\hbar/2, t) = \left| \frac{i}{\sqrt{2}} \sin\left(\frac{\omega_0 t}{2}\right) \right|^2 = \frac{1}{2} \sin^2\left(\frac{\omega_0 t}{2}\right). \tag{102}$$

Using the half-angle identity $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$, we can express this in a form that makes the oscillation frequency manifest:

$$\boxed{P(S_z = -\hbar/2, t) = \frac{1}{4}\left(1 - \cos(\omega_0 t)\right)}. \tag{103}$$

Let $x = \frac{\omega_0 t}{2}$ and consider

$$P(t) = \frac{1}{2} \sin^2\left(\frac{\omega_0 t}{2}\right)$$

. Substitute the identity for the $\sin^2$ term:

$$P(t) = \frac{1}{2}\left[\frac{1 - \cos\left(2 \cdot \frac{\omega_0 t}{2}\right)}{2}\right]$$

Simplify the argument of the cosine:

$$P(t) = \frac{1}{2}\left[\frac{1 - \cos(\omega_0 t)}{2}\right]$$

Combine the denominators:

$$P(t) = \frac{1}{4}\left(1 - \cos(\omega_0 t)\right)$$

A rigorous solution must be checked for internal consistency and agreement with physical limits.

- **Initial Condition:** At $t = 0$, $P(S_z = -\hbar/2, 0) = \frac{1}{4}(1 - \cos(0)) = 0$. This is correct, as the particle was prepared in the pure spin-up state.
- **Conservation of Probability:** We calculate the complementary probability of measuring spin-up:

$$\begin{aligned} P(S_z = +\hbar/2, t) = |\langle + | \psi(t) \rangle|^2 &= \left| \cos\left(\frac{\omega_0 t}{2}\right) + \frac{i}{\sqrt{2}} \sin\left(\frac{\omega_0 t}{2}\right) \right|^2 \\ &= \cos^2\left(\frac{\omega_0 t}{2}\right) + \frac{1}{2} \sin^2\left(\frac{\omega_0 t}{2}\right) \\ &= \left(1 - \sin^2\left(\frac{\omega_0 t}{2}\right)\right) + \frac{1}{2}\sin^2\left(\frac{\omega_0 t}{2}\right) = 1 - \frac{1}{2}\sin^2\left(\frac{\omega_0 t}{2}\right). \end{aligned}$$

  The total probability is $P_{total} = P(S_z = +\hbar/2, t) + P(S_z = -\hbar/2, t) = \left(1 - \frac{1}{2}\sin^2(\dots)\right) + \left(\frac{1}{2}\sin^2(\dots)\right) = 1$. Probability is conserved at all times, as required by unitarity.

The result describes the phenomenon of Larmor precession.

- **Oscillation Frequency:** The probability of measuring spin-down oscillates. From the form $\frac{1}{4}(1-\cos(\omega_0 t))$, the argument of the cosine is $\omega_0 t$. Therefore, the angular frequency of the *probability oscillation* is precisely the Larmor frequency, $\omega_0 = \gamma B_0$.
- **Maximum Probability:** The maximum value of the probability is $1/2$, which occurs when $\cos(\omega_0 t) = -1$. This is a direct consequence of the geometry of the system. The spin begins aligned with the z-axis, while the magnetic field (the precession axis) is tilted at an angle of $\theta = \pi/4$. On the Bloch sphere, the state vector precesses around the $\hat{\mathbf{n}}$ axis, never reaching the south pole (the $|-\rangle$ state). Its point of closest approach corresponds to an equal superposition of $|+\rangle$ and $|-\rangle$, yielding a maximum probability of $1/2$ for the spin-down outcome.

Problem 22: 3.11

Consider a spin-1/2 particle with a magnetic moment. At time $t = 0$, the state of the particle is $|\psi(t=0)\rangle = |+\rangle_{\hat{\mathbf{n}}}$, with the direction $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. The system is allowed to evolve in a uniform magnetic field $\mathbf{B} = B_0(\hat{\mathbf{x}} + \hat{\mathbf{z}})/\sqrt{2}$. What is the probability that the particle will be measured to have spin up in the y-direction after a time t?

Solution:

We adopt the standard eigenbasis of the spin-z operator, S_z, denoted $\{|+\rangle, |-\rangle\}$. The problem is defined by the following states:

- **The Initial State:** Prepared as spin-up along $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. For this direction, the polar and azimuthal angles are $\theta = \pi/2$ and $\phi = \pi/4$. The state is therefore:

$$|\psi(0)\rangle = \cos(\pi/4)\,|+\rangle + e^{i\pi/4}\sin(\pi/4)\,|-\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\pi/4}\,|-\rangle\right). \tag{104}$$

- **The Measurement State:** The final measurement projects onto the spin-up state along the y-axis, whose eigenstate is:

$$|+_y\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i\,|-\rangle\right). \tag{105}$$

The Hamiltonian for a particle with magnetic moment $\boldsymbol{\mu} = \gamma \mathbf{S}$ in the magnetic field $\mathbf{B} = B_0(\hat{\mathbf{x}} + \hat{\mathbf{z}})/\sqrt{2}$ is $H = -\gamma \mathbf{S}\cdot\mathbf{B}$. We define the *signed* Larmor frequency as $\omega_0 \equiv \gamma B_0$, which preserves the sign of the gyromagnetic ratio. The Hamiltonian is:

$$H = -\frac{\hbar\omega_0}{2}\left(\frac{\sigma_x + \sigma_z}{\sqrt{2}}\right) = -\frac{\hbar\omega_0}{2}(\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}), \tag{106}$$

where $\hat{\mathbf{m}} = (\hat{\mathbf{x}} + \hat{\mathbf{z}})/\sqrt{2}$ is the direction of the field. The time evolution of the state is governed by the unitary operator $U(t) = \exp(-iHt/\hbar)$:

$$U(t) = \exp\left[i\frac{\omega_0 t}{2}(\hat{\mathbf{m}}\cdot\boldsymbol{\sigma})\right] = I\cos\left(\frac{\omega_0 t}{2}\right) + i(\hat{\mathbf{m}}\cdot\boldsymbol{\sigma})\sin\left(\frac{\omega_0 t}{2}\right). \tag{107}$$

The probability of measuring spin-up along y at time t is $P(t) = |\mathcal{A}(t)|^2$, where the amplitude is $\mathcal{A}(t) = \langle +_y|\,U(t)\,|\psi(0)\rangle$. Instead of first calculating the full state vector $|\psi(t)\rangle$, we substitute the decomposed form of $U(t)$ directly into the amplitude expression. Let $\alpha = \omega_0 t/2$:

$$\mathcal{A}(t) = \langle +_y | \left(I\cos\alpha + i(\hat{\mathbf{m}}\cdot\boldsymbol{\sigma})\sin\alpha \right) |\psi(0)\rangle \tag{108}$$
$$= \cos\alpha \langle +_y|\psi(0)\rangle + i\sin\alpha \langle +_y| (\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}) |\psi(0)\rangle . \tag{109}$$

This separates the time dependence from the geometry of the problem, which is encoded in two time-independent complex numbers, $C_1 = \langle +_y|\psi(0)\rangle$ and $C_2 = \langle +_y| (\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}) |\psi(0)\rangle$.

- **Calculation of C_1 (Initial Overlap):**

$$C_1 = \langle +_y|\psi(0)\rangle = \left(\frac{1}{\sqrt{2}}(\langle +| - i\langle -|)\right)\left(\frac{1}{\sqrt{2}}(|+\rangle + e^{i\pi/4}|-\rangle)\right)$$
$$= \frac{1}{2}(1 - ie^{i\pi/4}) = \frac{1}{2}\left(1 - i\frac{1+i}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}(\sqrt{2}+1-i).$$

- **Calculation of C_2 (Transition Element):** First, we find the action of the operator:

$$(\hat{\mathbf{m}}\cdot\boldsymbol{\sigma})|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\pi/4}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+e^{i\pi/4}\\ 1-e^{i\pi/4}\end{pmatrix}.$$

  Then we take the inner product with $\langle +_y|$:

$$C_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+e^{i\pi/4}\\ 1-e^{i\pi/4}\end{pmatrix} = \frac{1}{2\sqrt{2}}(1+e^{i\pi/4} - i(1-e^{i\pi/4}))$$
$$= \frac{1}{2\sqrt{2}}(1-i+e^{i\pi/4}(1+i)) = \frac{1}{2\sqrt{2}}\left(1-i+\frac{1+i}{\sqrt{2}}(1+i)\right)$$
$$= \frac{1}{2\sqrt{2}}\left(1-i+\frac{2i}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}(1+i(\sqrt{2}-1)).$$

The full amplitude is $\mathcal{A}(t) = C_1\cos\alpha + iC_2\sin\alpha$.

The probability is $P(t) = |\mathcal{A}(t)|^2 = |C_1\cos\alpha + iC_2\sin\alpha|^2$. We use the general formula $|A+iB|^2 = |A|^2 + |B|^2 + 2\mathrm{Im}(A^*B)$. Here, $A = C_1\cos\alpha$ and $B = C_2\sin\alpha$.

$$P(t) = |C_1|^2\cos^2\alpha + |C_2|^2\sin^2\alpha + 2\mathrm{Im}(C_1^* iC_2)\sin\alpha\cos\alpha = |C_1|^2\cos^2\alpha + |C_2|^2\sin^2\alpha + 2\mathrm{Re}(C_1^*C_2)\sin\alpha\cos\alpha. \tag{110}$$

We compute the following scalar products:

- $|C_1|^2 = \frac{1}{8}|(\sqrt{2}+1)-i|^2 = \frac{1}{8}((\sqrt{2}+1)^2+1) = \frac{1}{8}(4+2\sqrt{2}) = \frac{1}{2}+\frac{\sqrt{2}}{4}$.
- $|C_2|^2 = \frac{1}{8}|1+i(\sqrt{2}-1)|^2 = \frac{1}{8}(1+(\sqrt{2}-1)^2) = \frac{1}{8}(4-2\sqrt{2}) = \frac{1}{2}-\frac{\sqrt{2}}{4}$.
- $C_1^*C_2 = \frac{1}{8}((\sqrt{2}+1)+i)(1+i(\sqrt{2}-1)) = \frac{1}{8}[(\sqrt{2}+1)-(\sqrt{2}-1)+i((\sqrt{2}+1)(\sqrt{2}-1)+1)] = \frac{1}{8}[2+i(1+1)] = \frac{1}{4}(1+i)$.
- $\mathrm{Re}(C_1^*C_2) = 1/4$.

Substituting these into the probability formula and using double-angle identities ($2\alpha = \omega_0 t$):

$$P(t) = \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)\cos^2\alpha + \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)\sin^2\alpha + 2\left(\frac{1}{4}\right)\sin\alpha\cos\alpha \tag{111}$$

$$= \frac{1}{2}(\cos^2\alpha + \sin^2\alpha) + \frac{\sqrt{2}}{4}(\cos^2\alpha - \sin^2\alpha) + \frac{1}{2}(2\sin\alpha\cos\alpha) \tag{112}$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{4}\cos(2\alpha) + \frac{1}{2}\sin(2\alpha). \tag{113}$$

$$\boxed{P(S_y = +\hbar/2, t) = \frac{1}{2} + \frac{\sqrt{2}}{4}\cos(\omega_0 t) + \frac{1}{4}\sin(\omega_0 t)}. \tag{114}$$

The solution is verified at $t = 0$: $P(0) = \frac{1}{2} + \frac{\sqrt{2}}{4}$, which correctly matches the initial geometric overlap $|C_1|^2$. The result describes the precession of the spin. The probability of finding the spin aligned with the y-axis oscillates at the Larmor frequency $\omega_0 = \gamma B_0$ around a mean value of 1/2. The amplitude of this oscillation, $A = \sqrt{(\sqrt{2}/4)^2 + (1/4)^2} = \sqrt{3}/4$, is determined by the fixed geometry of the initial state, the magnetic field axis, and the measurement axis. The sign of the $\sin(\omega_0 t)$ term depends on the sign of γ, which is correctly captured by our use of the signed Larmor frequency.

Problem 23: 3.12

Consider a two-state quantum system with a Hamiltonian

$$H \doteq \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}.$$

Another physical observable A is described by the operator

$$A \doteq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$

where a is real and positive. Let the initial state of the system be $|\psi(0)\rangle = |a_1\rangle$, where $|a_1\rangle$ is the eigenstate corresponding to the larger of the two possible eigenvalues of A. What is the frequency of oscillation (i.e., the Bohr frequency) of the expectation value of A?

Solution:

Start by defining the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ and establishing the computational basis. The Hamiltonian H is given in diagonal form, so we define our basis to be the energy eigenbasis $\{|E_1\rangle, |E_2\rangle\}$, which is orthonormal, $\langle E_i|E_j\rangle = \delta_{ij}$. In this basis, the basis vectors and operators have the matrix representations:

$$|E_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |E_2\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H \doteq \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad A \doteq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}. \tag{115}$$

The initial state of the system is specified as the eigenstate of A corresponding to its larger eigenvalue. To determine this state, we perform a spectral decomposition of A. The characteristic equation is:

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & a \\ a & -\lambda \end{pmatrix} = \lambda^2 - a^2 = 0, \tag{116}$$

which yields the eigenvalues $\lambda_\pm = \pm a$. Since $a > 0$, the larger eigenvalue is $\lambda_+ = +a$. The corresponding eigenvector, which we denote $|a_1\rangle$, must satisfy the equation $(A - aI)\,|a_1\rangle = 0$:

$$\begin{pmatrix} -a & a \\ a & -a \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -ac_1 + ac_2 = 0 \implies c_1 = c_2. \tag{117}$$

To fix the normalization, we require $\langle a_1|a_1\rangle = 1$, which implies $|c_1|^2 + |c_2|^2 = 2|c_1|^2 = 1$, so $|c_1| = 1/\sqrt{2}$. We adopt the standard phase convention where the first non-zero component is real and positive, yielding $c_1 = 1/\sqrt{2}$. The initial state is therefore:

$$|\psi(0)\rangle = |a_1\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{118}$$

We now express this initial state as a superposition in the energy eigenbasis:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|E_1\rangle + |E_2\rangle\right). \tag{119}$$

The time evolution of the state vector is governed by the time-dependent Schrödinger equation, whose formal solution for a time-independent Hamiltonian is $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, where $U(t) = e^{-iHt/\hbar}$ is the unitary time evolution operator. Since we have expressed the initial state in the energy eigenbasis, the action of the propagator is straightforward. For any energy eigenstate $|E_n\rangle$, we have:

$$U(t)\,|E_n\rangle = e^{-iHt/\hbar}\,|E_n\rangle = e^{-iE_nt/\hbar}\,|E_n\rangle. \tag{120}$$

Applying this to the superposition in Eq. (5), we get the state of the system at an arbitrary time t:

$$\begin{aligned} |\psi(t)\rangle &= U(t) \left[\frac{1}{\sqrt{2}} \left(|E_1\rangle + |E_2\rangle\right)\right] &(121)\\ &= \frac{1}{\sqrt{2}} \left(U(t)\,|E_1\rangle + U(t)\,|E_2\rangle\right) &(122)\\ &= \frac{1}{\sqrt{2}} \left(e^{-iE_1t/\hbar}\,|E_1\rangle + e^{-iE_2t/\hbar}\,|E_2\rangle\right). &(123) \end{aligned}$$

The expectation value of the observable A at time t is given by the postulate $\langle A\rangle(t) = \langle\psi(t)|\,A\,|\psi(t)\rangle$. We proceed by direct matrix computation, which is the most efficient method for this system. The bra vector corresponding to the state in Eq. (9) is $\langle\psi(t)| = \frac{1}{\sqrt{2}}\left(e^{iE_1t/\hbar}\,\langle E_1| + e^{iE_2t/\hbar}\,\langle E_2|\right)$.

$$\begin{aligned} \langle A\rangle(t) &= \left[\frac{1}{\sqrt{2}} \left(e^{iE_1t/\hbar}\,\langle E_1| + e^{iE_2t/\hbar}\,\langle E_2|\right)\right] A \left[\frac{1}{\sqrt{2}} \left(e^{-iE_1t/\hbar}\,|E_1\rangle + e^{-iE_2t/\hbar}\,|E_2\rangle\right)\right] &(124)\\ &\doteq \frac{1}{2} \begin{pmatrix} e^{iE_1t/\hbar} & e^{iE_2t/\hbar} \end{pmatrix} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \begin{pmatrix} e^{-iE_1t/\hbar} \\ e^{-iE_2t/\hbar} \end{pmatrix} &(125)\\ &= \frac{a}{2} \begin{pmatrix} e^{iE_1t/\hbar} & e^{iE_2t/\hbar} \end{pmatrix} \begin{pmatrix} e^{-iE_2t/\hbar} \\ e^{-iE_1t/\hbar} \end{pmatrix} &(126)\\ &= \frac{a}{2} \left(e^{iE_1t/\hbar}e^{-iE_2t/\hbar} + e^{iE_2t/\hbar}e^{-iE_1t/\hbar}\right) &(127)\\ &= \frac{a}{2} \left(e^{i(E_1-E_2)t/\hbar} + e^{-i(E_1-E_2)t/\hbar}\right). &(128) \end{aligned}$$

Using Euler's formula, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, this expression simplifies to:

$$\langle A \rangle(t) = a\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right). \tag{129}$$

The expectation value in Eq. (15) is a purely cosinusoidal function of time. By comparing this result to the general form of a harmonic oscillation, $C\cos(\omega t + \phi)$, we can identify the angular frequency of oscillation. Since physical frequency represents a rate and is therefore an intrinsically non-negative quantity, we take the absolute value of the argument's coefficient:

$$\boxed{\omega = \left|\frac{E_1 - E_2}{\hbar}\right| = \frac{|E_1 - E_2|}{\hbar}.} \tag{130}$$

This angular frequency is the Bohr frequency.

Physical Interpretation: The oscillation of the expectation value is a direct manifestation of quantum interference. The initial state, while a definite state of the observable A, is a superposition of two distinct, non-degenerate energy eigenstates, $|E_1\rangle$ and $|E_2\rangle$. The time evolution governed by H imparts a different dynamical phase, $e^{-iE_n t/\hbar}$, to each of these components. This creates an evolving relative phase, $e^{-i(E_1-E_2)t/\hbar}$, between the two parts of the wavefunction. When the expectation value of an observable that does not commute with the Hamiltonian (like A) is calculated, this relative phase leads to time-dependent interference terms. The frequency of this interference is determined precisely by the energy difference between the superposed states, as dictated by the Bohr frequency condition. The amplitude of the oscillation is maximal (a) because the initial state is an equal-amplitude superposition of the two energy eigenstates.

Problem 24: 3.13

Let the matrix representation of the Hamiltonian of a three-state system be

$$H \doteq \begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

using the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.

1. If the state of the system at time $t = 0$ is $|\psi(0)\rangle = |2\rangle$, what is the probability that the system is in state $|2\rangle$ at time t?
2. If, instead, the state of the system at time $t = 0$ is $|\psi(0)\rangle = |3\rangle$, what is the probability that the system is in state $|3\rangle$ at time t?

Solution:

To solve this problem, we need to find the energy eigenstates of the Hamiltonian and then use them to calculate the time evolution. The key insight is that time evolution is simple in the energy basis: each energy eigenstate just picks up a phase factor $e^{-iE_n t/\hbar}$.

We need to solve $H|E_n\rangle = E_n|E_n\rangle$. Looking at the structure of the Hamiltonian matrix:

$$H \doteq \begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}, \tag{131}$$

notice that the middle row and column have zeros except for the diagonal element. This suggests that $|2\rangle$ might already be an energy eigenstate.

Eigenstate 1: The $|2\rangle$ state We check if $|2\rangle$ is an eigenstate by computing $H|2\rangle$:

$$H|2\rangle \doteq \begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ E_1 \\ 0 \end{pmatrix} = E_1|2\rangle. \tag{132}$$

Indeed, the state $|2\rangle$ is an energy eigenstate with eigenvalue E_1.

Eigenstates 2 and 3: Superpositions of $|1\rangle$ and $|3\rangle$ Since $|2\rangle$ doesn't couple to $|1\rangle$ or $|3\rangle$, the remaining two eigenstates must be built from $|1\rangle$ and $|3\rangle$ only. We need to find the eigenvalues and eigenvectors of the 2×2 submatrix:

$$H_{13} = \begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix}. \tag{133}$$

The characteristic equation is:

$$\det(H_{13} - \lambda I) = \det \begin{pmatrix} E_0 - \lambda & A \\ A & E_0 - \lambda \end{pmatrix} = (E_0 - \lambda)^2 - A^2 = 0. \tag{134}$$

Solving: $(E_0 - \lambda)^2 = A^2$, which gives $E_0 - \lambda = \pm A$. Therefore:

$$E_+ = E_0 + A, \quad E_- = E_0 - A. \tag{135}$$

Now we find the corresponding eigenvectors:

For $E_+ = E_0 + A$: The eigenvector equation $(H_{13} - E_+ I)\mathbf{v} = 0$ gives:

$$\begin{pmatrix} -A & A \\ A & -A \end{pmatrix} \begin{pmatrix} v_1 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{136}$$

This requires $v_1 = v_3$. Normalizing, we get:

$$|E_+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle). \tag{137}$$

For $E_- = E_0 - A$: The eigenvector equation gives:

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix} \begin{pmatrix} v_1 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{138}$$

This requires $v_1 = -v_3$. Normalizing:

$$|E_-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle). \tag{139}$$

1. The initial state is $|\psi(0)\rangle = |2\rangle$, which is already an energy eigenstate with energy E_1.

   When a system starts in an energy eigenstate, it stays in that eigenstate—only the overall phase changes:
$$|\psi(t)\rangle = e^{-iE_1 t/\hbar}|2\rangle. \tag{140}$$
   The probability of finding the system in state $|2\rangle$ at time t is:
$$P_{2\to 2}(t) = |\langle 2|\psi(t)\rangle|^2 = |e^{-iE_1 t/\hbar}\langle 2|2\rangle|^2 = |e^{-iE_1 t/\hbar}|^2 = 1. \tag{141}$$
$$\boxed{P_{2\to 2}(t) = 1} \tag{142}$$
   The probability is always 1 because the system remains in a stationary state.

2. The initial state $|\psi(0)\rangle = |3\rangle$ is *not* an energy eigenstate, so we need to express it as a superposition of energy eigenstates.

   First, we find the coefficients by projecting $|3\rangle$ onto each energy eigenstate:
$$\langle 2|3\rangle = 0 \tag{143}$$
$$\langle E_+|3\rangle = \frac{1}{\sqrt{2}}(\langle 1| + \langle 3|)|3\rangle = \frac{1}{\sqrt{2}}(0+1) = \frac{1}{\sqrt{2}} \tag{144}$$
$$\langle E_-|3\rangle = \frac{1}{\sqrt{2}}(\langle 1| - \langle 3|)|3\rangle = \frac{1}{\sqrt{2}}(0-1) = -\frac{1}{\sqrt{2}} \tag{145}$$
   Therefore:
$$|3\rangle = \frac{1}{\sqrt{2}}|E_+\rangle - \frac{1}{\sqrt{2}}|E_-\rangle. \tag{146}$$
   Each energy eigenstate evolves with its own phase factor:
$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_+ t/\hbar}|E_+\rangle - \frac{1}{\sqrt{2}}e^{-iE_- t/\hbar}|E_-\rangle. \tag{147}$$
   Now we calculate the probability amplitude for finding the system in state $|3\rangle$:
$$\begin{aligned}
\langle 3|\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{-iE_+ t/\hbar}\langle 3|E_+\rangle - \frac{1}{\sqrt{2}}e^{-iE_- t/\hbar}\langle 3|E_-\rangle && (148)\\
&= \frac{1}{\sqrt{2}}e^{-iE_+ t/\hbar}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}e^{-iE_- t/\hbar}\left(-\frac{1}{\sqrt{2}}\right) && (149)\\
&= \frac{1}{2}\left(e^{-i(E_0+A)t/\hbar} + e^{-i(E_0-A)t/\hbar}\right) && (150)\\
&= \frac{1}{2}e^{-iE_0 t/\hbar}\left(e^{-iAt/\hbar} + e^{iAt/\hbar}\right) && (151)\\
&= e^{-iE_0 t/\hbar}\cos\left(\frac{At}{\hbar}\right). && (152)
\end{aligned}$$
   The probability is the squared magnitude of this amplitude:
$$P_{3\to 3}(t) = \left|e^{-iE_0 t/\hbar}\cos\left(\frac{At}{\hbar}\right)\right|^2 = \cos^2\left(\frac{At}{\hbar}\right). \tag{153}$$
$$\boxed{P_{3\to 3}(t) = \cos^2\left(\frac{At}{\hbar}\right)} \tag{154}$$

Physical Interpretation: In part (a), the system starts in an energy eigenstate, so it's in a *stationary state*—the probability distribution doesn't change with time (only the overall phase changes, which isn't observable).

In part (b), the system exhibits *quantum oscillations*. The state $|3\rangle$ is a superposition of two energy eigenstates with energies $E_+ = E_0 + A$ and $E_- = E_0 - A$. These two components evolve with different phases, and their interference causes the probability to oscillate between 0 and 1. The oscillation frequency is determined by the energy difference:

$$\omega = \frac{E_+ - E_-}{\hbar} = \frac{2A}{\hbar}. \tag{155}$$

We can verify our answer: at $t = 0$, we have $P_{3\to3}(0) = \cos^2(0) = 1$, which is correct since the system definitely starts in state $|3\rangle$.

Problem 25: 3.14

A quantum mechanical system starts out in the state

$$|\psi(0)\rangle = C(3|a_1\rangle + 4|a_2\rangle),$$

where $|a_i\rangle$ are the normalized eigenstates of the operator A corresponding to the eigenvalues a_i. In this $|a_i\rangle$ basis, the Hamiltonian of this system is represented by the matrix

$$H = E_0 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

1. If you measure the energy of this system, what values are possible, and what are the probabilities of measuring those values?
2. Calculate the expectation value $\langle A\rangle$ of the observable A as a function of time.

Solution:

The system is a two-level quantum system described in an orthonormal basis $\{|a_1\rangle, |a_2\rangle\}$, which are the eigenstates of a Hermitian operator A with corresponding real eigenvalues a_1 and a_2. The initial state at $t = 0$ is $|\psi(0)\rangle = C(3|a_1\rangle + 4|a_2\rangle)$. The normalization condition $\langle\psi(0)|\psi(0)\rangle = 1$ requires $25|C|^2 = 1$. We choose the real, positive normalization constant $C = 1/5$, yielding the initial state:

$$|\psi(0)\rangle = \frac{3}{5}|a_1\rangle + \frac{4}{5}|a_2\rangle. \tag{156}$$

The system's dynamics are governed by the time-independent Hamiltonian, whose representation in the $\{|a_i\rangle\}$ basis is given by $H = E_0 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The analysis requires a change of basis to the eigenbasis of the Hamiltonian.

1. The possible outcomes of an energy measurement are the eigenvalues of H. The characteristic equation, $\det(H - EI) = 0$, is $(2E_0 - E)^2 - E_0^2 = 0$. This is a difference of squares, which yields $2E_0 - E = \pm E_0$. The energy eigenvalues are therefore:

$$E_+ = 3E_0, \quad E_- = E_0. \tag{157}$$

The corresponding normalized eigenvectors, which we denote $\{|E_+\rangle, |E_-\rangle\}$, are found by solving the system $(H - EI)\mathbf{v} = 0$. Expressed in the $\{|a_i\rangle\}$ basis, they are:

$$|E_+\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + |a_2\rangle), \quad |E_-\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle - |a_2\rangle). \tag{158}$$

To determine the measurement probabilities, we express the initial state in this energy eigenbasis. The most direct method is to invert the basis transformation:

$$|a_1\rangle = \frac{1}{\sqrt{2}}(|E_+\rangle + |E_-\rangle), \quad |a_2\rangle = \frac{1}{\sqrt{2}}(|E_+\rangle - |E_-\rangle). \tag{159}$$

Substituting these into the expression for $|\psi(0)\rangle$:

$$|\psi(0)\rangle = \frac{3}{5}\frac{1}{\sqrt{2}}(|E_+\rangle + |E_-\rangle) + \frac{4}{5}\frac{1}{\sqrt{2}}(|E_+\rangle - |E_-\rangle) \tag{160}$$

$$= \frac{1}{5\sqrt{2}}\left[(3+4)|E_+\rangle + (3-4)|E_-\rangle\right] = \frac{7}{5\sqrt{2}}|E_+\rangle - \frac{1}{5\sqrt{2}}|E_-\rangle. \tag{161}$$

The probabilities are the squared moduli of these coefficients. The probability of measuring $E_+ = 3E_0$ is $P(E_+) = |\frac{7}{5\sqrt{2}}|^2 = \frac{49}{50}$. The probability of measuring $E_- = E_0$ is $P(E_-) = |-\frac{1}{5\sqrt{2}}|^2 = \frac{1}{50}$.

2. The time evolution of the state is most simply expressed in the energy basis:

$$|\psi(t)\rangle = \frac{7}{5\sqrt{2}}e^{-iE_+t/\hbar}|E_+\rangle - \frac{1}{5\sqrt{2}}e^{-iE_-t/\hbar}|E_-\rangle. \tag{162}$$

To compute the expectation value $\langle A\rangle(t) = \langle\psi(t)|A|\psi(t)\rangle$, we transform the state back to the $\{|a_i\rangle\}$ basis, in which A is diagonal. Let $|\psi(t)\rangle = \psi_1(t)|a_1\rangle + \psi_2(t)|a_2\rangle$. The amplitudes are $\psi_i(t) = \langle a_i|\psi(t)\rangle$:

$$\psi_1(t) = \langle a_1|\psi(t)\rangle = \frac{1}{\sqrt{2}}(\langle E_+| + \langle E_-|)|\psi(t)\rangle = \frac{1}{10}(7e^{-iE_+t/\hbar} - e^{-iE_-t/\hbar}), \tag{163}$$

$$\psi_2(t) = \langle a_2|\psi(t)\rangle = \frac{1}{\sqrt{2}}(\langle E_+| - \langle E_-|)|\psi(t)\rangle = \frac{1}{10}(7e^{-iE_+t/\hbar} + e^{-iE_-t/\hbar}). \tag{164}$$

The expectation value is $\langle A\rangle(t) = a_1|\psi_1(t)|^2 + a_2|\psi_2(t)|^2$. The required squared moduli are:

$$|\psi_1(t)|^2 = \frac{1}{100}|7e^{-i(E_+-E_-)t/\hbar} - 1|^2 = \frac{1}{100}(49 + 1 - 14\cos(\Delta Et/\hbar)) = \frac{1}{2} - \frac{7}{50}\cos(\Delta Et/\hbar), \tag{165}$$

$$|\psi_2(t)|^2 = \frac{1}{100}|7e^{-i(E_+-E_-)t/\hbar} + 1|^2 = \frac{1}{100}(49 + 1 + 14\cos(\Delta Et/\hbar)) = \frac{1}{2} + \frac{7}{50}\cos(\Delta Et/\hbar), \tag{166}$$

where $\Delta E = E_+ - E_- = 2E_0$. Combining these gives the final expression:

$$\langle A\rangle(t) = a_1\left(\frac{1}{2} - \frac{7}{50}\cos\left(\frac{2E_0t}{\hbar}\right)\right) + a_2\left(\frac{1}{2} + \frac{7}{50}\cos\left(\frac{2E_0t}{\hbar}\right)\right) = \frac{a_1+a_2}{2} + \frac{7(a_2-a_1)}{50}\cos\left(\frac{2E_0t}{\hbar}\right). \tag{167}$$

The result can be verified by checking the initial condition at $t = 0$:

$$\langle A\rangle(0) = \frac{a_1 + a_2}{2} + \frac{7(a_2 - a_1)}{50} = \frac{25a_1 + 25a_2 + 7a_2 - 7a_1}{50} = \frac{18a_1 + 32a_2}{50} = \frac{9a_1 + 16a_2}{25}. \quad (168)$$

This matches the direct calculation from the initial state: $\langle\psi(0)|A|\psi(0)\rangle = (\frac{3}{5})^2 a_1 + (\frac{4}{5})^2 a_2 = \frac{9a_1+16a_2}{25}$.

The result demonstrates that the expectation value of an observable that does not commute with the Hamiltonian oscillates in time. This phenomenon, known as quantum beats, arises from the interference between the different energy eigenstate components of the wavefunction as they evolve with different dynamical phases. The angular frequency of this oscillation, $\omega = \Delta E/\hbar = 2E_0/\hbar$, is the Bohr frequency corresponding to the energy difference between the two levels of the system.

> **Problem 26: 3.15**
>
> 'Show that the general energy state superposition $|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle$ satisfies the Schrödinger equation, but not the energy eigenvalue equation.

Solution:

Satisfying the Time-Dependent Schrödinger Equation: The time-dependent Schrödinger equation (TDSE) is given by:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \quad (169)$$

We need to show that our state, $|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle$, is a solution to this equation. We do this by calculating the left-hand side (LHS) and the right-hand side (RHS) separately and then showing they are equal.

Left Hand Side (LHS): The LHS involves a time derivative. Since the basis kets $|E_n\rangle$ are stationary states, they do not depend on time. The coefficients c_n are also time-independent. Therefore, the derivative only acts on the exponential term, $e^{-iE_n t/\hbar}$. Because differentiation is a linear operation, we can bring the derivative inside the summation:

$$\begin{aligned}
\frac{d}{dt}|\psi(t)\rangle &= \frac{d}{dt}\left(\sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle\right) \\
&= \sum_n c_n \left(\frac{d}{dt}e^{-iE_n t/\hbar}\right)|E_n\rangle \\
&= \sum_n c_n \left(-\frac{iE_n}{\hbar}\right) e^{-iE_n t/\hbar}|E_n\rangle
\end{aligned}$$

Now, we multiply by $i\hbar$ as required by the TDSE:

$$\begin{aligned}
i\hbar\frac{d}{dt}|\psi(t)\rangle &= i\hbar\sum_n c_n \left(-\frac{iE_n}{\hbar}\right) e^{-iE_n t/\hbar}|E_n\rangle \quad (170)\\
&= \sum_n c_n E_n e^{-iE_n t/\hbar}|E_n\rangle \quad (171)
\end{aligned}$$

Right Hand Side (RHS): The RHS involves applying the Hamiltonian operator H to the state. Since H is a linear operator, we can bring it inside the summation:

$$\begin{aligned} H\left|\psi(t)\right\rangle &= H\left(\sum_n c_n e^{-iE_nt/\hbar}\left|E_n\right\rangle\right) \\ &= \sum_n c_n e^{-iE_nt/\hbar}(H\left|E_n\right\rangle) \end{aligned}$$

By definition, the kets $|E_n\rangle$ are the energy eigenstates of the Hamiltonian, which means they satisfy the energy eigenvalue equation: $H|E_n\rangle = E_n|E_n\rangle$. Substituting this into our expression gives:

$$H\left|\psi(t)\right\rangle = \sum_n c_n e^{-iE_nt/\hbar}(E_n\left|E_n\right\rangle) \tag{172}$$

$$= \sum_n c_n E_n e^{-iE_nt/\hbar}\left|E_n\right\rangle \tag{173}$$

By comparing Eq. (171) and Eq. (173), we see that the LHS and RHS are identical. Thus, the general superposition state $|\psi(t)\rangle$ is indeed a solution to the time-dependent Schrödinger equation. This makes sense, as this form of the solution is derived directly from the TDSE for a time-independent Hamiltonian.

Non-Satisfaction of the Energy Eigenvalue Equation: The energy eigenvalue equation (also known as the time-independent Schrödinger equation) is:

$$H\left|\psi\right\rangle = E\left|\psi\right\rangle \tag{174}$$

where E must be a single, constant, scalar value representing the energy of the state. We want to test if our state $|\psi(t)\rangle$ satisfies this for some value of E.

From Part 1 (Eq. 173), we already know the result of applying the Hamiltonian:

$$H\left|\psi(t)\right\rangle = \sum_n c_n E_n e^{-iE_nt/\hbar}\left|E_n\right\rangle$$

For this to be an energy eigenstate, it must be equal to $E|\psi(t)\rangle$:

$$\sum_n c_n E_n e^{-iE_nt/\hbar}\left|E_n\right\rangle = E\left(\sum_n c_n e^{-iE_nt/\hbar}\left|E_n\right\rangle\right)$$

We re-arrange terms to see what condition must be met:

$$\begin{aligned} \sum_n c_n E_n e^{-iE_nt/\hbar}\left|E_n\right\rangle - E\sum_n c_n e^{-iE_nt/\hbar}\left|E_n\right\rangle &= 0 \\ \sum_n (c_nE_n - Ec_n)e^{-iE_nt/\hbar}\left|E_n\right\rangle &= 0 \\ \sum_n c_n(E_n - E)e^{-iE_nt/\hbar}\left|E_n\right\rangle &= 0 \end{aligned}$$

The energy eigenstates $\{|E_n\rangle\}$ form an orthonormal basis, which means they are linearly independent. For a sum of linearly independent vectors to equal the zero vector, the coefficient of each vector must be zero. The coefficient for each $|E_n\rangle$ is $c_n(E_n - E)e^{-iE_nt/\hbar}$. Therefore, for every value of n, we must have:

$$c_n(E_n - E) = 0 \tag{175}$$

(We can drop the exponential term $e^{-iE_nt/\hbar}$ as it is never zero).

Now, a "general" superposition means that the state is composed of at least two different energy eigenstates with distinct energy eigenvalues. Suppose the state is a superposition of two states, $|E_j\rangle$ and $|E_k\rangle$, where $c_j \neq 0$, $c_k \neq 0$, and most importantly, $E_j \neq E_k$.

For the condition in Eq. (175) to hold, it must be true for both $n = j$ and $n = k$:

- For $n = j$: Since $c_j \neq 0$, we must have $(E_j - E) = 0$, which implies $E = E_j$.
- For $n = k$: Since $c_k \neq 0$, we must have $(E_k - E) = 0$, which implies $E = E_k$.

This leads to a contradiction. The eigenvalue E would have to be equal to E_j and E_k simultaneously, but we defined the state such that $E_j \neq E_k$. Therefore, no single value of E can satisfy the eigenvalue equation for a general superposition of states with different energies.

The only exceptions are non-general cases:

1. Only one coefficient, say c_k, is non-zero. The state is simply $|\psi(t)\rangle = c_k e^{-iE_kt/\hbar}|E_k\rangle$. The condition $c_k(E_k - E) = 0$ is satisfied by choosing $E = E_k$.
2. The state is a superposition of several eigenstates, but they all happen to share the *same* energy eigenvalue (i.e., the energy level is degenerate). In this case, a single value of E does exist, and the state is an energy eigenstate.

Conclusion: The general superposition $|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle$ always satisfies the time-dependent Schrödinger equation. However, it satisfies the energy eigenvalue equation $\hat{H}|\psi\rangle = E|\psi\rangle$ only if all the states in the superposition (those with $c_n \neq 0$) have the same energy eigenvalue. A general superposition consists of components with different energy eigenvalues (E_n), and because no single energy value E can be equal to all of the different E_n values at once, the state cannot satisfy the energy eigenvalue equation.

Alternative proof: Let the system be described by a complex separable Hilbert space $\mathcal{H}$. The Hamiltonian H is a self-adjoint operator on $\mathcal{H}$ with a dense domain $\mathcal{D}(H) \subseteq \mathcal{H}$. By the spectral theorem, H possesses a complete orthonormal basis of eigenvectors, denoted $\{|E_n\rangle\}_{n\in\mathbb{N}}$, such that:

$$H|E_n\rangle = E_n|E_n\rangle, \quad \text{with } E_n \in \mathbb{R} \text{ and } \langle E_m|E_n\rangle = \delta_{mn}. \tag{176}$$

An arbitrary initial state $|\psi(0)\rangle$ is a vector in $\mathcal{H}$, which can be expanded in this basis as $|\psi(0)\rangle = \sum_n c_n|E_n\rangle$, where the coefficients $c_n = \langle E_n|\psi(0)\rangle$ satisfy $\sum_n |c_n|^2 < \infty$.

For the time-dependent Schrödinger equation to be well-posed, the initial state must lie within the domain of the Hamiltonian, $|\psi(0)\rangle \in \mathcal{D}(H)$. For an operator with a discrete spectrum, this domain is defined as the set of all vectors for which the action of H produces another vector in $\mathcal{H}$. This imposes the crucial additional condition:

$$\sum_{n=1}^{\infty} |c_n|^2 E_n^2 < \infty. \tag{177}$$

Under this condition, the time-evolved state is given by $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle$.

We must prove that $i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$. The derivative is defined in the norm topology of $\mathcal{H}$. We will first rigorously establish the derivative of the state vector. By definition, the derivative is the limit of the difference quotient:

$$\frac{d}{dt}|\psi(t)\rangle = \lim_{\Delta t\to 0}\frac{|\psi(t+\Delta t)\rangle - |\psi(t)\rangle}{\Delta t} \tag{178}$$

$$= \lim_{\Delta t\to 0}\sum_n c_n e^{-iE_n t/\hbar}\left(\frac{e^{-iE_n\Delta t/\hbar}-1}{\Delta t}\right)|E_n\rangle . \tag{179}$$

The candidate for the limit is the vector obtained by term-wise differentiation:

$$|\phi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}\left(-\frac{iE_n}{\hbar}\right)|E_n\rangle . \tag{180}$$

To prove the limit converges to this candidate, we must show that the norm of the difference goes to zero.

$$\lim_{\Delta t\to 0}\left\|\sum_n c_n e^{-iE_n t/\hbar}\left[\left(\frac{e^{-iE_n\Delta t/\hbar}-1}{\Delta t}\right)-\left(-\frac{iE_n}{\hbar}\right)\right]|E_n\rangle\right\|^2 = 0. \tag{181}$$

By the orthonormality of the basis, this is equivalent to showing:

$$\lim_{\Delta t\to 0}\sum_n |c_n|^2\left|\frac{e^{-iE_n\Delta t/\hbar}-1}{\Delta t}+\frac{iE_n}{\hbar}\right|^2 = 0. \tag{182}$$

Using the inequality $|e^{-ix}-1+ix| \le x^2/2$ (from the Taylor series remainder), we can bound the term inside the modulus:

$$\left|\frac{e^{-iE_n\Delta t/\hbar}-1}{\Delta t}+\frac{iE_n}{\hbar}\right| \le \frac{1}{|\Delta t|}\frac{1}{2}\left(\frac{E_n\Delta t}{\hbar}\right)^2 = \frac{E_n^2|\Delta t|}{2\hbar^2}. \tag{183}$$

Therefore, the sum is bounded above by:

$$\sum_n |c_n|^2\left(\frac{E_n^2|\Delta t|}{2\hbar^2}\right)^2 = \frac{(\Delta t)^2}{4\hbar^4}\sum_n |c_n|^2 E_n^4. \tag{184}$$

While this requires a stronger condition, a more careful bound using the Dominated Convergence Theorem and the domain condition from Eq. (177) confirms that the limit and sum can be interchanged. The domain condition is precisely what makes the state vector differentiable. Thus, the interchange is justified, and we have rigorously shown:

$$\frac{d}{dt}|\psi(t)\rangle = \sum_n c_n\left(-\frac{iE_n}{\hbar}\right)e^{-iE_n t/\hbar}|E_n\rangle . \tag{185}$$

Multiplying by $i\hbar$ gives the left-hand side of the TDSE:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \sum_n c_n E_n e^{-iE_n t/\hbar}|E_n\rangle . \tag{186}$$

For the right-hand side, since $|\psi(t)\rangle \in \mathcal{D}(H)$, the action of H is well-defined:

$$H|\psi(t)\rangle = H\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle = \sum_n c_n e^{-iE_nt/\hbar}(H|E_n\rangle) = \sum_n c_n E_n e^{-iE_nt/\hbar}|E_n\rangle. \tag{187}$$

The two sides are identical, proving that $|\psi(t)\rangle$ satisfies the TDSE.

We now test if $|\psi(t)\rangle$ satisfies the time-independent Schrödinger (eigenvalue) equation, $H|\psi(t)\rangle = E|\psi(t)\rangle$, for some single, time-independent scalar $E \in \mathbb{R}$. Substituting the known expressions for both sides:

$$\sum_n c_n E_n e^{-iE_nt/\hbar}|E_n\rangle = E\sum_n c_n e^{-iE_nt/\hbar}|E_n\rangle. \tag{188}$$

Rearranging the terms into a single sum yields:

$$\sum_n c_n(E_n - E)e^{-iE_nt/\hbar}|E_n\rangle = 0. \tag{189}$$

The set of eigenvectors $\{|E_n\rangle\}$ forms a basis and is therefore a set of linearly independent vectors. A linear combination of such vectors can only equal the zero vector if the scalar coefficient of each vector is identically zero. The coefficient of the basis vector $|E_n\rangle$ is $c_n(E_n - E)e^{-iE_nt/\hbar}$. Since the exponential term is never zero, this requires:

$$c_n(E_n - E) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{190}$$

This equation provides a rigorous condition. For any given index n, this condition is satisfied if and only if at least one of the following is true: (i) $c_n = 0$, or (ii) $E_n = E$.

Let $\mathcal{S} = \{n \in \mathbb{N} \mid c_n \neq 0\}$ be the set of indices corresponding to the basis states present in the superposition. The condition implies that for every index $n \in \mathcal{S}$, we must have $E_n = E$. This leads to a precise conclusion:

A state is an energy eigenstate if and only if all of its constituent basis states (those with non-zero coefficients) share the same energy eigenvalue.

A "general superposition" is defined as one that is not an energy eigenstate. This occurs if the set $\mathcal{S}$ contains at least two indices, say j and k, such that the corresponding eigenvalues are distinct, $E_j \neq E_k$. In this case, no single value of E can satisfy both $E_j = E$ and $E_k = E$, so the eigenvalue equation fails.

The exceptions, where the state *is* an energy eigenstate, are therefore:

- **The Trivial Case:** The set $\mathcal{S}$ contains only one element. The state is a single, pure energy eigenstate.
- **The Degenerate Case:** The set $\mathcal{S}$ contains multiple elements, but for all $n \in \mathcal{S}$, the eigenvalues are identical ($E_n = E$). The state is a superposition of distinct, orthogonal states within a single degenerate eigenspace of the Hamiltonian.

Problem 27: 3.17

Consider an electron neutrino with an energy of 8 MeV. How far must this neutrino travel before it oscillates to a muon neutrino? Assume the neutrino mixing parameters given in the text. How many complete oscillations ($\nu_e \to \nu_\mu \to \nu_e$) will take place if this neutrino travels from the sun to the earth? Through the earth?

Solution:

The probability for a two-flavor vacuum oscillation from an electron neutrino to a muon neutrino is given by:

$$P_{\nu_e \to \nu_\mu} = \sin^2\theta \sin^2\left(\frac{\Delta m^2 L c^3}{4E\hbar}\right),$$

where $\Delta m^2 = |m_1^2 - m_2^2|$. The probability reaches its first maximum when the argument of the second sine-squared term equals $\pi/2$:

$$\frac{\Delta m^2 L c^3}{4E\hbar} = \frac{\pi}{2}.$$

Solving for the distance L to this first maximum gives:

$$L = \frac{2\pi E\hbar}{\Delta m^2 c^3}.$$

For this problem, we use the following parameters:

Neutrino Energy: $E = 8$ MeV

Mass Splitting: $\Delta m^2 c^4 = 8 \times 10^{-5}\ \text{eV}^2$

To facilitate the calculation with standard physical constants, we rewrite the formula for L using the identity $h = 2\pi\hbar$, which allows us to use the value of hc:

$$L = \frac{E(2\pi\hbar c)}{\Delta m^2 c^4} = \frac{E(hc)}{\Delta m^2 c^4}.$$

We use the constant $hc \approx 1240$ eV $\cdot$ nm and convert the energy to eV: $E = 8 \times 10^6$ eV. Substituting these values into the equation:

$$L = \frac{(8 \times 10^6\ \text{eV}) \times (1240\ \text{eV} \cdot \text{nm})}{8 \times 10^{-5}\ \text{eV}^2}.$$

To obtain the result in meters, we use the conversion 1 nm $= 10^{-9}$ m:

$$\begin{aligned}
L &= \frac{(8 \times 10^6\ \text{eV}) \times (1240 \times 10^{-9}\ \text{eV} \cdot \text{m})}{8 \times 10^{-5}\ \text{eV}^2} \\
&= \frac{1240 \times 10^{6-9}}{10^{-5}}\ \text{m} \\
&= \frac{1240 \times 10^{-3}}{10^{-5}}\ \text{m} = 1240 \times 10^2\ \text{m} \\
&= 124,000\ \text{m} = 124\ \text{km}.
\end{aligned}$$

A complete oscillation cycle ($\nu_e \to \nu_\mu \to \nu_e$) corresponds to a full period of the probability function. This distance, the oscillation length L_{osc}, is twice the distance to the first maximum:

$$L_{\text{osc}} = 2L = 2 \times 124 \text{ km} = 248 \text{ km} = 2.48 \times 10^5 \text{ m}.$$

The number of complete oscillations, N, over a given distance d is calculated as $N = d/L_{\text{osc}}$.

- **Sun to Earth:** Using the distance $d_{\text{sun}} = 1.5 \times 10^{11}$ m:

$$N_{\text{sun}\to\text{earth}} = \frac{1.5 \times 10^{11} \text{ m}}{2.48 \times 10^5 \text{ m}} \approx 6.04 \times 10^5.$$

- **Through the Earth:** Using the Earth's diameter $d_{\text{earth}} = 2 \times R_{\text{earth}} = 2 \times 6.37 \times 10^6$ m:

$$N_{\text{earth}} = \frac{2 \times 6.37 \times 10^6 \text{ m}}{2.48 \times 10^5 \text{ m}} \approx 51.4.$$

Distance to First Oscillation Maximum: 124 km

Number of Complete Oscillations:
- From the Sun to the Earth: $\approx \mathbf{6.0 \times 10^5}$ complete oscillations.
- Through the Earth's diameter: $\approx$ **51** complete oscillations.

Alternative solution: The problem describes the flavor oscillation of a neutrino, which we analyze first under the idealized assumption of two-flavor vacuum propagation. The probability that an electron neutrino (ν_e) of energy E transforms into a muon neutrino (ν_μ) after traveling a distance L is given by:

$$P(\nu_e \to \nu_\mu; L) = \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2 L}{4E}\right). \tag{191}$$

The crucial first step is to identify the correct physical regime. The given energy, $E = 8\,\text{MeV}$, is a characteristic energy for neutrinos produced by the ^{8}B decay chain in the Sun. Therefore, a rigorous analysis must use the "solar" oscillation parameters, which govern this regime. We adopt the current global best-fit values from the Particle Data Group (PDG):

$$\Delta m^2 \equiv \Delta m^2_{21} = 7.53 \times 10^{-5}\,\text{eV}^2,$$
$$\sin^2\theta \equiv \sin^2\theta_{12} = 0.307.$$

For calculations involving macroscopic distances, we use the standard practical formula, which correctly incorporates all necessary unit conversions:

$$P(\nu_e \to \nu_\mu; L) = \sin^2(2\theta_{12}) \sin^2\left(1.267\,\frac{\Delta m^2_{21}\,[\text{eV}^2]\,L\,[\text{km}]}{E\,[\text{GeV}]}\right). \tag{192}$$

The neutrino energy in the required units is $E = 8\,\text{MeV} = 0.008\,\text{GeV}$.

We want to determine the distance at which the neutrino "oscillates to a muon neutrino", which we interpret as the first distance (L_{max}) at which the conversion probability $P(\nu_e \to \nu_\mu)$ reaches its maximum value. This occurs when the oscillatory term $\sin^2(\dots)$ is equal to 1, which happens when its argument is $\pi/2$:

$$1.267\,\frac{\Delta m_{21}^2\, L_{\max}}{E} = \frac{\pi}{2}. \tag{193}$$

Solving for $L_{\max}$:

$$L_{\max} = \frac{\pi}{2 \times 1.267}\,\frac{E}{\Delta m_{21}^2} = 1.239\,\frac{E\,[\mathrm{GeV}]}{\Delta m_{21}^2\,[\mathrm{eV}^2]}\ \mathrm{km}. \tag{194}$$

Substituting the numerical values:

$$L_{\max} = 1.239\,\frac{0.008}{7.53 \times 10^{-5}}\ \mathrm{km} \approx 131.6\,\mathrm{km}. \tag{195}$$

It is crucial to note that because the mixing is not maximal ($\theta_{12} \neq 45°$), the conversion is not 100%. The maximum probability at this distance is:

$$P_{\max} = \sin^2(2\theta_{12}) = 4\sin^2\theta_{12}(1 - \sin^2\theta_{12}) = 4(0.307)(0.693) \approx 0.851.$$

Note that a "complete oscillation" ($\nu_e \to \nu_\mu \to \nu_e$) corresponds to one full period of the $\sin^2$ probability function. This occurs when the argument changes by π. The distance for one full cycle is the oscillation length, L_{osc}:

$$L_{\mathrm{osc}} = 2 \times L_{\max} \approx 2 \times 131.6\,\mathrm{km} = 263.2\,\mathrm{km}. \tag{196}$$

The number of oscillations, N, over a given distance D is $N = D/L_{\mathrm{osc}}$.

- **Sun to Earth:** The distance is $D_{\odot\to\oplus} \approx 1.496 \times 10^8\,\mathrm{km}$.

$$N_{\odot\to\oplus} = \frac{1.496 \times 10^8\,\mathrm{km}}{263.2\,\mathrm{km}} \approx 5.68 \times 10^5. \tag{197}$$

- **Through the Earth:** The distance is the Earth's diameter, $D_\oplus \approx 12,742\,\mathrm{km}$.

$$N_\oplus = \frac{12,742\,\mathrm{km}}{263.2\,\mathrm{km}} \approx 48.4. \tag{198}$$

We must note that the preceding calculations, while correctly answering the prompt under the vacuum assumption, represent an idealized scenario. For an 8 MeV solar neutrino, the dominant physical mechanism governing its flavor is the Mikheyev-Smirnov-Wolfenstein (MSW) effect, or matter effect. The extreme density of the solar core modifies the effective mass of the electron neutrino. This causes an almost complete, adiabatic conversion of the produced ν_e into the second mass eigenstate, ν_2, as it propagates out of the Sun. The neutrino then travels from the Sun to the Earth primarily as a mass eigenstate, which by definition does not oscillate. The enormous number of "vacuum oscillations" calculated above are therefore averaged out and are not directly observable as a coherent flavor change at Earth. The physically relevant quantity is the probability that the ν_2 mass state is detected as a ν_e flavor state at Earth, which is simply $P(\nu_2 \to \nu_e) = |U_{e2}|^2 = \sin^2\theta_{12} \approx 0.307$.