## Answers to Selected Problems in Electrodynamics by Griffiths

I spent a semester coaching a student through Electrodynamics. We did tedious homework problems from the Griffiths textbook. I decided to thresh-out the calculations and type the complete solutions. Caveat emptor tho!

**solutions by:** a.benedict balbuena

> **Problem 1: 1.29**
>
> Calculate the line integral of the function $\mathbf{v} = x^2\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$ from the origin to the point $(1,1,1)$ by three different routes:
>
> (a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.
>
> (b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$.
>
> (c) The direct straight line.
>
> (d) What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b)?

**Solution:**

Before we start, we should consider the nature of the vector field. That is, we ask "Is it conservative"? (A field is conservative if its curl is zero.) Hence, we compute

$$\nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 2y & y^2 \end{vmatrix} = \left(\frac{\partial(y^2)}{\partial y} - \frac{\partial(2y)}{\partial z}\right)\hat{\mathbf{x}} - \left(\frac{\partial(y^2)}{\partial x} - \frac{\partial(x^2)}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial(2y)}{\partial x} - \frac{\partial(x^2)}{\partial y}\right)\hat{\mathbf{z}} = 2y\hat{\mathbf{x}} \neq \mathbf{0}$$

The non-zero curl implies that the line integral $\int \mathbf{v}\cdot d\mathbf{l}$ is **path-dependent** ($I_a \neq I_b \neq I_c$) and the integral over a **closed loop is non-zero** ($I_{loop} = 1 \neq 0$).

The line integral of the vector function $\mathbf{v} = x^2\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$ is given by:

$$\int_P \mathbf{v}\cdot d\mathbf{l} = \int_P (v_x dx + v_y dy + v_z dz) = \int_P (x^2 dx + 2y dy + y^2 dz)$$

We proceed with the line integrals:

(a) 1. **Segment 1 ($C_1$): $(0,0,0) \to (1,0,0)$.**
$y = 0, z = 0 \implies dy = 0, dz = 0$. $x$ goes from 0 to 1.

$$I_1 = \int_{C_1} x^2 dx + 2y dy + y^2 dz = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

2. **Segment 2 ($C_2$): $(1,0,0) \to (1,1,0)$.**
$x = 1, z = 0 \implies dx = 0, dz = 0$. $y$ goes from 0 to 1.

$$I_2 = \int_{C_2} x^2 dx + 2y dy + y^2 dz = \int_0^1 2y dy = \left[y^2\right]_0^1 = 1$$

3. **Segment 3 ($C_3$): $(1,1,0) \to (1,1,1)$.**
   $x = 1, y = 1 \implies dx = 0, dy = 0$. $z$ goes from 0 to 1.

$$I_3 = \int_{C_3} x^2 dx + 2y dy + y^2 dz = \int_0^1 (1)^2 dz = [z]_0^1 = 1$$

The total integral for path (a) is:

$$I_a = I_1 + I_2 + I_3 = \frac{1}{3} + 1 + 1 = \frac{7}{3}$$

(b)

1. **Segment 1 ($C_1'$): $(0,0,0) \to (0,0,1)$.**
   $x = 0, y = 0 \implies dx = 0, dy = 0$. $z$ goes from 0 to 1.

$$I_1' = \int_{C_1'} x^2 dx + 2y dy + y^2 dz = \int_0^1 (0)^2 dz = 0$$

2. **Segment 2 ($C_2'$): $(0,0,1) \to (0,1,1)$.**
   $x = 0, z = 1 \implies dx = 0, dz = 0$. $y$ goes from 0 to 1.

$$I_2' = \int_{C_2'} x^2 dx + 2y dy + y^2 dz = \int_0^1 2y dy = \left[y^2\right]_0^1 = 1$$

3. **Segment 3 ($C_3'$): $(0,1,1) \to (1,1,1)$.**
   $y = 1, z = 1 \implies dy = 0, dz = 0$. $x$ goes from 0 to 1.

$$I_3' = \int_{C_3'} x^2 dx + 2y dy + y^2 dz = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

The total integral for path (b) is:

$$I_b = I_1' + I_2' + I_3' = 0 + 1 + \frac{1}{3} = \frac{4}{3}$$

(c) The path $C_d$ is the straight line from $(0,0,0)$ to $(1,1,1)$. It can be parameterized by:

$$\mathbf{l}(t) = (t, t, t), \quad 0 \le t \le 1$$

This gives $x = t, y = t, z = t$, and $dx = dt, dy = dt, dz = dt$.

$$I_c = \int_{C_d} (x^2 dx + 2y dy + y^2 dz) = \int_0^1 (t^2 dt + 2t dt + t^2 dt)$$

$$I_c = \int_0^1 (2t^2 + 2t) dt = \left[\frac{2t^3}{3} + \frac{2t^2}{2}\right]_0^1 = \left[\frac{2t^3}{3} + t^2\right]_0^1 = \left(\frac{2(1)^3}{3} + (1)^2\right) - 0 = \frac{2}{3} + 1 = \frac{5}{3}$$

(d) The closed loop $C_{loop}$ goes out along path (a) and back along the reverse of path (b), denoted as $-C_b$. The integral is:

$$I_{loop} = \oint_{C_{loop}} \mathbf{v} \cdot d\mathbf{l} = \int_{C_a} \mathbf{v} \cdot d\mathbf{l} + \int_{-C_b} \mathbf{v} \cdot d\mathbf{l} = \int_{C_a} \mathbf{v} \cdot d\mathbf{l} - \int_{C_b} \mathbf{v} \cdot d\mathbf{l}$$

$$I_{loop} = I_a - I_b = \frac{7}{3} - \frac{4}{3} = \frac{3}{3} = 1$$

**Problem 2: 1.3**

1. Calculate the surface integral of the function $\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$, over the square on the xy-plane with vertices (0,0), (2,0), (2,2), (0,2). For consistency, let "upward" be the positive direction.

2. Does the surface integral depend only on the boundary line for this function?

3. What is the total flux over the *closed* surface of the cube with side-length 2 over the first octant having the origin as one of its vertices? [Note: For the closed surface, the positive direction is "outward," and hence "down," for the bottom face.]

**Solution:**

1. The surface integral of the function $\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ over the specified square is calculated as follows:

   **Surface definition:** The square lies on the xy-plane, so $z = 0$. The boundaries are $0 \leq x \leq 2$ and $0 \leq y \leq 2$.

   **Area Vector:** The area vector $d\mathbf{a}$ for an "upward" pointing surface on the xy-plane is $d\mathbf{a} = \hat{\mathbf{z}}\, dx\, dy$.

   **Integrand:** The dot product $\mathbf{v} \cdot d\mathbf{a}$ on the surface (where $z = 0$) is:

   $$\mathbf{v} \cdot d\mathbf{a} = y(z^2-3)\, dx\, dy = y(0-3)\, dx\, dy = -3y\, dx\, dy$$

   We proceed with the integration:

   $$\begin{aligned}\iint_S \mathbf{v} \cdot d\mathbf{a} &= \int_0^2 \int_0^2 -3y\, dx\, dy \\ &= \int_0^2 [-3yx]_0^2\, dy \\ &= \int_0^2 -6y\, dy \\ &= [-3y^2]_0^2 \\ &= -12\end{aligned}$$

   The surface integral over the square is -12.

2. No, the surface integral does not depend only on the boundary line for this function.

   A surface integral of a vector field $v$ depends only on its boundary line if the field is solenoidal, meaning its divergence is zero ($\nabla \cdot \mathbf{v} = 0$). Let's calculate the divergence of $v$v:

   $$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(x+2) + \frac{\partial}{\partial z}(y(z^2-3)) \\ &= 2z + 0 + 2yz \\ &= 2z(1+y)\end{aligned}$$

   Since $\nabla \cdot \mathbf{v} \neq 0$, the field is not solenoidal, and the value of its surface integral depends on the specific surface chosen, not just its boundary.

3. To find the total flux over the closed surface of the cube, we use the Divergence Theorem: $\oiint_S \mathbf{v} \cdot d\mathbf{a} = \iiint_V (\nabla \cdot \mathbf{v})\, dV$.

   The cube is defined by $0 \leq x \leq 2$, $0 \leq y \leq 2$, and $0 \leq z \leq 2$.

   As calculated above, $\nabla \cdot \mathbf{v} = 2z(1+y)$.

   We now, proceed with the integration:

$$
\begin{aligned}
Flux &= \int_0^2 \int_0^2 \int_0^2 2z(1+y)\, dx\, dy\, dz \\
&= \int_0^2 \int_0^2 [2z(1+y)x]_0^2\, dy\, dz \\
&= \int_0^2 \int_0^2 4z(1+y)\, dy\, dz \\
&= \int_0^2 [4z(y + \frac{y^2}{2})]_0^2\, dz \\
&= \int_0^2 4z(2+2)\, dz = \int_0^2 16z\, dz \\
&= [8z^2]_0^2 = 32
\end{aligned}
$$

   The total flux over the closed surface of the cube is 32.

**Problem 3: 1.31**

Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

**Solution:**

**Problem 4: 1.32**

Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0,0,0)$, $\mathbf{b} = (1,1,1)$, and each of the following three paths:

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$;

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$;

(c) the parabolic path $z = x^2, y = x$.

**Solution:**

The **Fundamental Theorem for Gradients** states that the line integral of the gradient of a scalar function is path-independent and is equal to the difference in the function's value at the endpoints:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

In the context of this problem, we are given the scalar function: $T = x^2 + 4xy + 2yz^3$ with $\mathbf{a} = (0,0,0)$ and $\mathbf{b} = (1,1,1)$.

First, we compute the right-hand side of the theorem:

$$T(\mathbf{a}) = T(0,0,0) = (0)^2 + 4(0)(0) + 2(0)(0)^3 = 0$$
$$T(\mathbf{b}) = T(1,1,1) = (1)^2 + 4(1)(1) + 2(1)(1)^3 = 1+4+2 = 7$$

Thus, the expected result is

$$T(\mathbf{b}) - T(\mathbf{a}) = 7 - 0 = 7$$

Recall that the gradient is $\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$ with the problem giving us $\frac{\partial T}{\partial x} = 2x + 4y$, $\frac{\partial T}{\partial y} = 4x + 2z^3$, and $\frac{\partial T}{\partial z} = 6yz^2$.
So the line integral is:

$$\int (\nabla T) \cdot d\mathbf{l} = \int (2x+4y)dx + (4x+2z^3)dy + (6yz^2)dz$$

We calculate the line integral for each of the given paths:

$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

1. **Segment 1 ($C_1$):** $(0,0,0) \to (1,0,0)$. $(y=0, z=0 \implies dy=0, dz=0)$

$$I_1 = \int_0^1 (2x + 4(0))dx = \int_0^1 2x\,dx = [x^2]_0^1 = 1$$

2. **Segment 2 ($C_2$):** $(1,0,0) \to (1,1,0)$. $(x=1, z=0 \implies dx=0, dz=0)$

$$I_2 = \int_0^1 (4(1) + 2(0)^3)dy = \int_0^1 4\,dy = [4y]_0^1 = 4$$

3. **Segment 3 ($C_3$):** $(1,1,0) \to (1,1,1)$. $(x=1, y=1 \implies dx=0, dy=0)$

$$I_3 = \int_0^1 (6(1)z^2)dz = \int_0^1 6z^2\,dz = [2z^3]_0^1 = 2$$

$$I_a = I_1 + I_2 + I_3 = 1+4+2 = \mathbf{7}$$

$(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

1. **Segment 1 ($C_1'$):** $(0,0,0) \to (0,0,1)$. $(x=0, y=0 \implies dx=0, dy=0)$

$$I_1' = \int_0^1 (6(0)z^2)dz = 0$$

2. **Segment 2 ($C_2'$):** $(0,0,1) \to (0,1,1)$. $(x=0, z=1 \implies dx=0, dz=0)$

$$I_2' = \int_0^1 (4(0) + 2(1)^3)dy = \int_0^1 2\,dy = [2y]_0^1 = 2$$

3. **Segment 3 ($C_3'$):** $(0,1,1) \to (1,1,1)$. $(y=1, z=1 \implies dy=0, dz=0)$

$$I_3' = \int_0^1 (2x + 4(1))dx = \int_0^1 (2x+4)\,dx = [x^2+4x]_0^1 = (1+4) - 0 = 5$$

$$I_b = I_1' + I_2' + I_3' = 0 + 2 + 5 = \mathbf{7}$$

**The parabolic path $z = x^2, y = x$:** We parameterize the path by $x$ from 0 to 1.

$$y = x \implies dy = dx$$

$$z = x^2 \implies dz = 2x\,dx$$

Substitute into the integral:

$$\begin{aligned}
I_c &= \int_0^1 \left[(2x + 4x)dx + (4x + 2(x^2)^3)dx + (6(x)(x^2)^2)(2x\,dx)\right] \\
&= \int_0^1 [6x + (4x + 2x^6) + 12x^6]\,dx \\
&= \int_0^1 (10x + 14x^6)\,dx \\
&= \left[5x^2 + \frac{14x^7}{7}\right]_0^1 \\
&= \left[5x^2 + 2x^7\right]_0^1 \\
&= (5(1)^2 + 2(1)^7) - 0 \\
&= 5 + 2 \\
&= \mathbf{7}
\end{aligned}$$

Since $\int(\nabla T) \cdot d\mathbf{l} = 7$ for all three paths, and $T(\mathbf{b}) - T(\mathbf{a}) = 7$, the **Fundamental Theorem for Gradients** is verified.

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = 7 = T(\mathbf{b}) - T(\mathbf{a})$$

> **Problem 5: 1.33**
>
> Test the divergence theorem for the function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$. Take as your volume the cube with side-length 2 over the first octant having the origin as one of its vertices

**Solution:**

The **Divergence Theorem** states:

$$\oiint_S \mathbf{v} \cdot d\mathbf{a} = \iiint_V (\nabla \cdot \mathbf{v})\,dV$$

We must calculate both sides of the equation independently and show that they are equal.

Here, our vector function is $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$ with the volume being a cube defined by $0 \leq x \leq 2$, $0 \leq y \leq 2$, and $0 \leq z \leq 2$.

**The Volume Integral (Right-Hand Side):** First, we calculate the divergence of $\mathbf{v}$:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx) = y + 2z + 3x$$

Next, we integrate this divergence over the volume of the cube:

$$\iiint_V (\nabla \cdot \mathbf{v})\, dV = \int_0^2 \int_0^2 \int_0^2 (3x + y + 2z)\, dx\, dy\, dz$$

We evaluate the integral:

1. **Integrate with respect to x:**

$$\int_0^2 (3x + y + 2z)\, dx = \left[\frac{3x^2}{2} + yx + 2zx\right]_0^2 = \left(\frac{3(4)}{2} + 2y + 4z\right) - 0 = 6 + 2y + 4z$$

2. **Integrate with respect to y:**

$$\int_0^2 (6 + 2y + 4z)\, dy = \left[6y + y^2 + 4zy\right]_0^2 = (12 + 4 + 8z) - 0 = 16 + 8z$$

3. **Integrate with respect to z:**

$$\int_0^2 (16 + 8z)\, dz = \left[16z + 4z^2\right]_0^2 = (32 + 16) - 0 = 48$$

**The value of the volume integral is 48.**

**Part 2: The Surface Integral (Left-Hand Side):** We calculate the flux $\oiint_S \mathbf{v} \cdot d\mathbf{a}$ through the six faces of the cube. The area vector $d\mathbf{a}$ points outward.

1. **Front Face ($x = 2$):** $d\mathbf{a} = \hat{\mathbf{x}}\, dy\, dz$. $\mathbf{v} \cdot d\mathbf{a} = (xy)\, dy\, dz = (2y)\, dy\, dz$.

$$\iint_{S_1} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 2y\, dy\, dz = \int_0^2 [y^2]_0^2\, dz = \int_0^2 4\, dz = [4z]_0^2 = 8$$

2. **Back Face ($x = 0$):** $d\mathbf{a} = -\hat{\mathbf{x}}\, dy\, dz$. $\mathbf{v} \cdot d\mathbf{a} = -(xy)\, dy\, dz = -(0 \cdot y)\, dy\, dz = 0$.

$$\iint_{S_2} \mathbf{v} \cdot d\mathbf{a} = 0$$

3. **Right Face ($y = 2$):** $d\mathbf{a} = \hat{\mathbf{y}}\, dx\, dz$. $\mathbf{v} \cdot d\mathbf{a} = (2yz)\, dx\, dz = (4z)\, dx\, dz$.

$$\iint_{S_3} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 4z\, dx\, dz = \int_0^2 [4zx]_0^2\, dz = \int_0^2 8z\, dz = [4z^2]_0^2 = 16$$

4. **Left Face ($y = 0$):** $d\mathbf{a} = -\hat{\mathbf{y}}\, dx\, dz$. $\mathbf{v} \cdot d\mathbf{a} = -(2yz)\, dx\, dz = -(2 \cdot 0 \cdot z)\, dx\, dz = 0$.

$$\iint_{S_4} \mathbf{v} \cdot d\mathbf{a} = 0$$

5. **Top Face ($z = 2$):** $d\mathbf{a} = \hat{\mathbf{z}}\, dx\, dy$. $\mathbf{v} \cdot d\mathbf{a} = (3zx)\, dx\, dy = (6x)\, dx\, dy$.

$$\iint_{S_5} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 6x\, dx\, dy = \int_0^2 [3x^2]_0^2\, dy = \int_0^2 12\, dy = [12y]_0^2 = 24$$

6. **Bottom Face ($z = 0$):** $d\mathbf{a} = -\hat{\mathbf{z}}\,dx\,dy$. $\mathbf{v} \cdot d\mathbf{a} = -(3zx)\,dx\,dy = -(3 \cdot 0 \cdot x)\,dx\,dy = 0$.

$$\iint_{S_6} \mathbf{v} \cdot d\mathbf{a} = 0$$

The total surface integral is the sum of the flux through all six faces:

$$\oiint_S \mathbf{v} \cdot d\mathbf{a} = 8 + 0 + 16 + 0 + 24 + 0 = 48$$

**The value of the surface integral is 48.**

Since the values are equal, the **Divergence Theorem is verified** for the given function and volume.

> **Problem 6: 1.34**
>
> Test Stokes' theorem for the function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$, using the triangular area defined by the vertices (0,0,0), (0,2,0), (0,0,2).

**Solution:**

Stokes' Theorem states:

$$\oint_C \mathbf{v} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$$

To test the theorem, we must calculate both the line integral around the boundary of the surface and the surface integral of the curl of the function, and then show that they are equal.

**The Line Integral (Left-Hand Side):** The line integral is $\oint \mathbf{v} \cdot d\mathbf{l} = \oint (xy)dx + (2yz)dy + (3zx)dz$.

We choose the counter-clockwise path $C$: $(0,0,0) \xrightarrow{C_1} (0,2,0) \xrightarrow{C_2} (0,0,2) \xrightarrow{C_3} (0,0,0)$.

1. **Path $C_1$:** $(0,0,0) \to (0,2,0)$
   $x = 0, z = 0 \implies dx = 0, dz = 0$. $y$ varies from 0 to 2.

$$I_1 = \int_{C_1} (0)dx + (2y(0))dy + (3(0)x)dz = \int_0^2 0\,dy = 0$$

2. **Path $C_2$:** $(0,2,0) \to (0,0,2)$
   $x = 0 \implies dx = 0$. The line equation is $y + z = 2$, so $y = 2 - z$ and $dy = -dz$. $z$ varies from 0 to 2 (since the integral is from $y = 2$ to $y = 0$, we integrate $z$ from 0 to 2 and use the substitution $dy = -dz$ to maintain the sign).

$$I_2 = \int_{C_2} (0)dx + (2yz)dy + (3zx)dz = \int_{z=0}^{2} 2(2-z)z(-dz) + 3z(0)dz$$

$$I_2 = \int_0^2 (-4z + 2z^2)dz = \left[-2z^2 + \frac{2z^3}{3}\right]_0^2 = \left(-2(4) + \frac{2(8)}{3}\right) - 0 = -8 + \frac{16}{3} = -\frac{24}{3} + \frac{16}{3} = -\frac{8}{3}$$

3. **Path $C_3$:** $(0,0,2) \to (0,0,0)$
   $x=0, y=0 \implies dx=0, dy=0$. $z$ varies from 2 to 0.

$$I_3 = \int_{C_3} (0)dx + (2(0)z)dy + (3zx)dz = \int_2^0 0\,dz = 0$$

The total line integral is:

$$\oint_C \mathbf{v}\cdot d\mathbf{l} = I_1 + I_2 + I_3 = 0 - \frac{8}{3} + 0 = -\frac{8}{3}$$

**The Surface Integral (Right-Hand Side):** First, calculate the curl of $\mathbf{v}$:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3zx \end{vmatrix}$$

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\,(0-2y) - \hat{\mathbf{y}}\,(3z-0) + \hat{\mathbf{z}}\,(0-x) = -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}$$

Next, determine the differential area vector $d\mathbf{a}$. The surface $S$ is in the $yz$-plane, so $x=0$. By the right-hand rule, the counter-clockwise path $C$ requires the normal vector to point in the positive $x$-direction:

$$d\mathbf{a} = \hat{\mathbf{x}}\,dy\,dz$$

The integrand is:

$$(\nabla \times \mathbf{v})\cdot d\mathbf{a} = (-2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}})\cdot(\hat{\mathbf{x}}\,dy\,dz) = -2y\,dy\,dz$$

We integrate over the triangular region defined by $y=0$, $z=0$, and $y+z=2$.

$$\iint_S (\nabla \times \mathbf{v})\cdot d\mathbf{a} = \int_{z=0}^{2}\int_{y=0}^{2-z} (-2y)\,dy\,dz$$

1. **Inner integral (w.r.t. $y$):**

$$\int_0^{2-z} -2y\,dy = [-y^2]_0^{2-z} = -(2-z)^2$$

2. **Outer integral (w.r.t. $z$):**

$$\int_0^2 -(2-z)^2\,dz$$

   Using the substitution $u = 2-z$, $du = -dz$:

$$= \int_{u=2}^{0} -u^2(-du) = \int_2^0 u^2 du = \left[\frac{u^3}{3}\right]_2^0 = 0 - \frac{2^3}{3} = -\frac{8}{3}$$

   (Alternatively: $\int_0^2 -(2-z)^2\,dz = \left[\frac{(2-z)^3}{3}\right]_0^2 = \left(\frac{(2-2)^3}{3}\right) - \left(\frac{(2-0)^3}{3}\right) = 0 - \frac{8}{3} = -\frac{8}{3}$)

The total surface integral is:

$$\iint_S (\nabla \times \mathbf{v})\cdot d\mathbf{a} = -\frac{8}{3}$$

Since the results for the line integral and the surface integral are equal, $\oint_C \mathbf{v}\cdot d\mathbf{l} = \iint_S (\nabla\times\mathbf{v})\cdot d\mathbf{a}$, meaning **Stokes' theorem is verified for this scenario**.

**Problem 7: 1.35**

Check Corollary 1 (the integral $\oint(\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used) by using the function $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$ and boundary defined by the path $\beta = (0,0,0) \to (0,1,0) \to (0,1,1) \to (0,0,1) \to (0,0,0)$, but integrating over the five faces of the unit cube in the first octant where the face of the unit cube give by the boundary path $\beta$ is open.

**Solution:**

The corollary states that for a given boundary line, the surface integral of the curl of a vector field is the same for all surfaces that share that boundary.

We verify this by showing that two different calculations yield the same result:

1. The line integral around the boundary path $C$ (equal to the surface integral over the simple square surface $S_{yz}$ in the $yz$-plane, by Stokes' theorem).
2. The surface integral over the more complex surface $S_{cube}$ (the five faces of the open cube) which shares the same boundary $C$.

**The Line Integral $\oint_C \mathbf{v} \cdot d\mathbf{l}$:** This is the flux through the simple surface $S_{yz}$ bounded by $C$. The line integral is $\oint_C \mathbf{v} \cdot d\mathbf{l} = \oint_C (2xz + 3y^2)dy + (4yz^2)dz$.

1. **Path $C_1$:** $(0,0,0) \to (0,1,0)$. $x = 0, z = 0 \implies dx = 0, dz = 0.$ $y : 0 \to 1.$

$$I_1 = \int_0^1 (2(0)(0) + 3y^2)dy = \int_0^1 3y^2 dy = [y^3]_0^1 = 1$$

2. **Path $C_2$:** $(0,1,0) \to (0,1,1)$. $x = 0, y = 1 \implies dx = 0, dy = 0.$ $z : 0 \to 1.$

$$I_2 = \int_0^1 (4(1)z^2)dz = \int_0^1 4z^2 dz = \left[\frac{4z^3}{3}\right]_0^1 = \frac{4}{3}$$

3. **Path $C_3$:** $(0,1,1) \to (0,0,1)$. $x = 0, z = 1 \implies dx = 0, dz = 0.$ $y : 1 \to 0.$

$$I_3 = \int_1^0 (2(0)(1) + 3y^2)dy = \int_1^0 3y^2 dy = [y^3]_1^0 = 0 - 1 = -1$$

4. **Path $C_4$:** $(0,0,1) \to (0,0,0)$. $x = 0, y = 0 \implies dx = 0, dy = 0.$ $z : 1 \to 0.$

$$I_4 = \int_1^0 (4(0)z^2)dz = 0$$

Summing the results:

$$\oint_C \mathbf{v} \cdot d\mathbf{l} = I_1 + I_2 + I_3 + I_4 = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}$$

**Target Value:** $\frac{4}{3}$.

**The Surface Integral $\iint_{S_{cube}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$:** First, calculate the curl of $\mathbf{v}$:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix} = \hat{\mathbf{x}}\left(4z^2 - 2x\right) - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}\left(2z - 0\right)$$

$$\nabla \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}$$

The boundary path $C$ is traversed counter-clockwise when viewed from the positive $x$-axis. By the right-hand rule, the normal vectors $d\mathbf{a}$ for the five faces must point generally in the **positive $x$-direction** (outward normals for this open cube "bucket").

1. **Front Face ($S_1$): $x = 1$. $d\mathbf{a} = \hat{\mathbf{x}}\, dy\, dz$.**

$$\iint_{S_1} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 (4z^2 - 2x)\big|_{x=1}\, dy\, dz = \int_0^1 \int_0^1 (4z^2 - 2)\, dy\, dz$$

$$= \int_0^1 (4z^2 - 2)\, dz = \left[\frac{4z^3}{3} - 2z\right]_0^1 = \frac{4}{3} - 2 = -\frac{2}{3}$$

2. **Right Face ($S_2$): $y = 1$. $d\mathbf{a} = \hat{\mathbf{y}}\, dx\, dz$.**

$$\iint_{S_2} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \iint_{S_2} \left((4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}\right) \cdot (\hat{\mathbf{y}}\, dx\, dz) = 0$$

3. **Left Face ($S_3$): $y = 0$. $d\mathbf{a} = -\hat{\mathbf{y}}\, dx\, dz$.**

$$\iint_{S_3} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$$

4. **Top Face ($S_4$): $z = 1$. $d\mathbf{a} = \hat{\mathbf{z}}\, dx\, dy$.**

$$\iint_{S_4} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 (2z)\big|_{z=1}\, dx\, dy = \int_0^1 \int_0^1 2\, dx\, dy = 2$$

5. **Bottom Face ($S_5$): $z = 0$. $d\mathbf{a} = -\hat{\mathbf{z}}\, dx\, dy$.**

$$\iint_{S_5} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 (-2z)\big|_{z=0}\, dx\, dy = 0$$

Summing the flux over the five faces:

$$\iint_{S_{cube}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = -\frac{2}{3} + 0 + 0 + 2 + 0 = \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$$

Since we got the same result, the corollary to Stokes' theorem is verified: the flux of the curl depends only on the boundary line, not on the particular surface used.

**Problem 8: 1.59**

Verify the divergence theorem for the vector field

$$\mathbf{v} = r^2 \sin\theta\,\hat{r} + 4r^2 \cos\theta\,\hat{\theta} + r^2 \tan\theta\,\hat{\phi},$$

over the "ice-cream cone" region defined by

$$0 \le r \le R,\;\; 0 \le \phi \le 2\pi,\;\; 0 \le \theta \le \theta_0,$$

where $\theta_0 = 30^\circ = \pi/6$.

**Solution:**

The Divergence Theorem states that the total flux of a vector field through a closed surface $S$ is equal to the volume integral of the divergence of that field over the enclosed volume $V$:

$$\oiint_S \mathbf{v} \cdot d\mathbf{a} = \iiint_V (\nabla \cdot \mathbf{v})\, dV.$$

We will compute the sides of the equation separately and show them to be equal.

**The Volume Integral:**

1. **Calculate the Divergence** The divergence in spherical coordinates is:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Substituting the components:

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(r^2\sin\theta)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta(4r^2\cos\theta)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\tan\theta) \\
&= \frac{1}{r^2}(4r^3\sin\theta) + \frac{4r^2}{r\sin\theta}(\cos^2\theta - \sin^2\theta) + 0 \\
&= 4r\sin\theta + 4r\frac{\cos^2\theta - \sin^2\theta}{\sin\theta} \\
&= 4r\sin\theta + 4r\frac{\cos^2\theta}{\sin\theta} - 4r\sin\theta \\
&= 4r\frac{\cos^2\theta}{\sin\theta}
\end{aligned}$$

2. **Evaluate the Volume Integral** Using $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$:

$$\iiint_V (\nabla \cdot \mathbf{v}) \, dV = \int_0^{2\pi} \int_0^{\theta_0} \int_0^R \left( 4r \frac{\cos^2\theta}{\sin\theta} \right) (r^2 \sin\theta \, dr \, d\theta \, d\phi)$$
$$= \int_0^{2\pi} \int_0^{\theta_0} \int_0^R 4r^3 \cos^2\theta \, dr \, d\theta \, d\phi$$
$$= \left( \int_0^{2\pi} d\phi \right) \left( \int_0^R 4r^3 \, dr \right) \left( \int_0^{\theta_0} \cos^2\theta \, d\theta \right)$$
$$= (2\pi) \cdot \left[ r^4 \right]_0^R \cdot \int_0^{\theta_0} \frac{1}{2}(1 + \cos(2\theta)) \, d\theta$$
$$= (2\pi) R^4 \cdot \frac{1}{2} \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_0^{\theta_0}$$
$$= \pi R^4 \left( \theta_0 + \frac{1}{2} \sin(2\theta_0) \right)$$

Substituting $\theta_0 = \pi/6$:

$$\iiint_V (\nabla \cdot \mathbf{v}) \, dV = \pi R^4 \left( \frac{\pi}{6} + \frac{1}{2} \sin\left(\frac{\pi}{3}\right) \right) = \pi R^4 \left( \frac{\pi}{6} + \frac{1}{2} \frac{\sqrt{3}}{2} \right) = \pi R^4 \left( \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)$$

**The Surface Integral (Flux):** The surface $S$ is composed of the spherical cap $S_1$ and the conical side $S_2$.

1. **Flux through the Spherical Cap ($S_1$)** On $S_1$, $r = R$ and $d\mathbf{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$.

$$\mathbf{v} \cdot d\mathbf{a} = (R^2 \sin\theta \, \hat{r}) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}) = R^4 \sin^2\theta \, d\theta \, d\phi$$

$$\iint_{S_1} \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi} \int_0^{\theta_0} R^4 \sin^2\theta \, d\theta \, d\phi$$
$$= R^4 (2\pi) \int_0^{\theta_0} \frac{1}{2}(1 - \cos(2\theta)) \, d\theta$$
$$= \pi R^4 \left[ \theta - \frac{1}{2} \sin(2\theta) \right]_0^{\theta_0} = \pi R^4 \left( \theta_0 - \frac{1}{2} \sin(2\theta_0) \right)$$

2. **Flux through the Conical Side ($S_2$)** On $S_2$, $\theta = \theta_0$ and $d\mathbf{a} = r \, dr \, d\phi \, r \sin\theta_0 \, \hat{\theta} = r \sin\theta_0 \, dr \, d\phi \, \hat{\theta}$.

$$\mathbf{v} \cdot d\mathbf{a} = (4r^2 \cos\theta_0 \, \hat{\theta}) \cdot (r \sin\theta_0 \, dr \, d\phi \, \hat{\theta}) = 4r^3 \cos\theta_0 \sin\theta_0 \, dr \, d\phi$$

Using $\sin(2\theta_0) = 2 \sin\theta_0 \cos\theta_0$:

$$\mathbf{v} \cdot d\mathbf{a} = 2r^3 \sin(2\theta_0) \, dr \, d\phi$$

$$\iint_{S_2} \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi} \int_0^R 2r^3 \sin(2\theta_0) \, dr \, d\phi$$
$$= 2 \sin(2\theta_0) \left( \int_0^{2\pi} d\phi \right) \left( \int_0^R r^3 \, dr \right)$$
$$= 2 \sin(2\theta_0) (2\pi) \left[ \frac{r^4}{4} \right]_0^R = \pi R^4 \sin(2\theta_0)$$

3. **Total Flux** The total flux is the sum of the partial fluxes:

$$\oiint_S \mathbf{v} \cdot d\mathbf{a} = \iint_{S_1} \mathbf{v} \cdot d\mathbf{a} + \iint_{S_2} \mathbf{v} \cdot d\mathbf{a}$$
$$= \pi R^4 \left( \theta_0 - \frac{1}{2} \sin(2\theta_0) \right) + \pi R^4 \sin(2\theta_0)$$
$$= \pi R^4 \left( \theta_0 + \sin(2\theta_0) - \frac{1}{2} \sin(2\theta_0) \right)$$
$$= \pi R^4 \left( \theta_0 + \frac{1}{2} \sin(2\theta_0) \right)$$

Substituting $\theta_0 = \pi/6$:

$$\oiint_S \mathbf{v} \cdot d\mathbf{a} = \pi R^4 \left( \frac{\pi}{6} + \frac{1}{2} \sin\left(\frac{\pi}{3}\right) \right) = \pi R^4 \left( \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)$$

The calculated volume integral $\iiint_V (\nabla \cdot \mathbf{v})\, dV = \pi R^4 \left( \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)$ is equal to the surface integral $\oiint_S \mathbf{v} \cdot d\mathbf{a} = \pi R^4 \left( \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)$. Thus the Divergence Theorem is verified.

**Problem 9: 1.6**

(a) Combine Corollary 2 to the gradient theorem (the integral $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any *closed* surface) with Stokes' theorem ($\mathbf{v} = \nabla T$ in this case). Show that the result is consistent with what you already knew about second derivatives.

(b) Combine Corollary 2 to Stokes' theorem with the divergence theorem. Show that the result is consistent with what you already knew.

**Solution:**

(a) Let $\mathbf{v} = \nabla T$ for some scalar field $T$. Substituting into Corollary 2:

$$\oiint_S (\nabla \times (\nabla T)) \cdot d\mathbf{a} = 0 \tag{1}$$

To understand why this holds identically, we examine the curl of a gradient. The $x$-component is:

$$[\nabla \times (\nabla T)]_x = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial y}\right) \tag{2}$$
$$= \frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial z \partial y} \tag{3}$$

Assuming $T$ has continuous second partial derivatives, Clairaut's theorem guarantees the equality of mixed partials:

$$\frac{\partial^2 T}{\partial y \partial z} = \frac{\partial^2 T}{\partial z \partial y} \tag{4}$$

Thus $[\nabla \times (\nabla T)]_x = 0$. The same reasoning applies to the $y$- and $z$-components, yielding the vector identity:

$$\nabla \times (\nabla T) = \mathbf{0} \quad \text{for all } T \in C^2 \tag{5}$$

Therefore, the integrand in equation (1) vanishes identically, making the equality trivial but consistent with known vector calculus.

(b) Applying the divergence theorem to the vector field $\mathbf{F} = \nabla \times \mathbf{v}$:

$$\oiint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \iiint_V \nabla \cdot (\nabla \times \mathbf{v})\, dV \tag{6}$$

Corollary 2 states the left-hand side is zero for any closed surface $S$:

$$\iiint_V \nabla \cdot (\nabla \times \mathbf{v})\, dV = 0 \tag{7}$$

Since this holds for *any* volume $V$, the integrand must vanish identically (provided it is continuous):

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0 \tag{8}$$

To verify this identity directly, we expand the expression:

$$\nabla \cdot (\nabla \times \mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \tag{9}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \tag{10}$$

Expanding further:

$$= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} \tag{11}$$

$$+ \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} \tag{12}$$

Assuming continuous second partial derivatives, mixed partials cancel pairwise:

$$\frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_z}{\partial y \partial x} = 0 \tag{13}$$

$$\frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_y}{\partial z \partial x} = 0 \tag{14}$$

$$\frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_x}{\partial z \partial y} = 0 \tag{15}$$

Thus all terms cancel, confirming the identity $\nabla \cdot (\nabla \times \mathbf{v}) = 0$.

Combining Corollary 2 of Stokes' theorem with the Divergence Theorem forces us to conclude that $\nabla \cdot (\nabla \times \mathbf{v}) = 0$. This is perfectly consistent with the known vector identity, which is itself a consequence of the equality of mixed partial derivatives. The physical interpretation is that the curl of a field (its "swirl") can have no sources or sinks; it must be a solenoidal field.

**Problem 10: 1.61**

Although the gradient, divergence, and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show the following:

(*a*) $\int_V (\nabla T) d\tau = \oint_S T\, d\mathbf{a}$.
*[Hint: Let* $\mathbf{v} = cT$*, where* $\mathbf{c}$ *is a constant vector, in the divergence theorem; use the product rules.]*

(*b*) $\int_V (\nabla \times \mathbf{v}) d\tau = -\oint_S \mathbf{v} \times d\mathbf{a}$.
*[Hint: Replace* $\mathbf{v}$ *by* $(\mathbf{v} \times \mathbf{c})$ *in the divergence theorem.]*

(*c*) $\int_V [T\nabla^2 U + (\nabla T) \cdot (\nabla U)] d\tau = \oint_S (T\nabla U) \cdot d\mathbf{a}$.
*[Hint: Let* $\mathbf{v} = T\nabla U$ *in the divergence theorem.]*

(*d*) $\int_V (T\nabla^2 U - U\nabla^2 T) d\tau = \oint_S (T\nabla U - U\nabla T) \cdot d\mathbf{a}$.
*[Comment: This is sometimes called* ***Green's second identity****; it follows from (c), which is known as* ***Green's identity****.]*

(*e*) $\oint_S \nabla T \times d\mathbf{a} = -\oint_P T\, d\mathbf{l}$.
*[Hint: Let* $\mathbf{v} = \mathbf{c}T$ *in Stokes' theorem.]*

(*f*) $\int_S [(d\mathbf{a} \times \nabla) \times \mathbf{v}] = -\oint_P \mathbf{v} \times d\mathbf{l}$.
*[Hint: Replace* $\mathbf{v}$ *by* $(\mathbf{v} \times \mathbf{c})$ *in Stokes' theorem.]*

**Solution:**

(*a*) We will prove the identity for one component (e.g., the $x$-component) and then generalize, as a vector equation is true if and only if each of its component equations is true.

Recall that the standard **Divergence Theorem** states

$$\iiint_V (\nabla \cdot \mathbf{v})\, d\tau = \oiint_S \mathbf{v} \cdot d\mathbf{a}$$

where $\mathbf{v}$ is an arbitrary vector field.

Our goal is to prove:

$$\iiint_V (\nabla T)\, d\tau = \oiint_S T\, d\mathbf{a}$$

Focusing on the $x$**-component** of the desired identity, we have

$$\left[\iiint_V (\nabla T)\, d\tau\right]_x = \left[\oiint_S T\, d\mathbf{a}\right]_x .$$

If we let $da_x$ be the $x$-component of the vector area element $d\mathbf{a}$, we can write the preceding equation as

$$\iiint_V \frac{\partial T}{\partial x}\, d\tau = \oiint_S T\, da_x .$$

To prove this, we choose a vector field $\mathbf{v}$ that we will use in the standard Divergence Theorem. Let's define a vector field $\mathbf{v}$ as

$$\mathbf{v} = T\hat{\mathbf{x}} = (T, 0, 0)$$

where $T$ is our scalar function.

Now, let's apply the Divergence Theorem to this specific $\mathbf{v}$.

(a) Calculate the divergence of $\mathbf{v}$:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x}(T) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = \frac{\partial T}{\partial x}$$

(b) Calculate the dot product $\mathbf{v} \cdot d\mathbf{a}$:

$$\mathbf{v} \cdot d\mathbf{a} = (T\hat{\mathbf{x}}) \cdot (da_x\hat{\mathbf{x}} + da_y\hat{\mathbf{y}} + da_z\hat{\mathbf{z}}) = T\, da_x$$

(c) Substitute these into the Divergence Theorem: The theorem $\iiint_V (\nabla \cdot \mathbf{v})\, d\tau = \oiint_S \mathbf{v} \cdot d\mathbf{a}$ becomes:

$$\iiint_V \frac{\partial T}{\partial x}\, d\tau = \oiint_S T\, da_x$$

This proves that the $x$-component of our target identity is true.

Similar arguments can apply for the $y$ and $z$ components:

- To prove the $y$-component, we choose $\mathbf{v} = T\hat{\mathbf{y}}$, which gives $\iiint_V \frac{\partial T}{\partial y}\, d\tau = \oiint_S T\, da_y$.
- To prove the $z$-component, we choose $\mathbf{v} = T\hat{\mathbf{z}}$, which gives $\iiint_V \frac{\partial T}{\partial z}\, d\tau = \oiint_S T\, da_z$.

We now have the three component equations:

$$(x\text{-comp}): \quad \iiint_V \frac{\partial T}{\partial x}\, d\tau = \oiint_S T\, da_x$$

$$(y\text{-comp}): \quad \iiint_V \frac{\partial T}{\partial y}\, d\tau = \oiint_S T\, da_y$$

$$(z\text{-comp}): \quad \iiint_V \frac{\partial T}{\partial z}\, d\tau = \oiint_S T\, da_z$$

We can combine these three scalar equations back into a single vector equation by multiplying each by its respective unit vector ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$) and adding them together:

$$\iiint_V \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) d\tau = \oiint_S T(da_x\hat{\mathbf{x}} + da_y\hat{\mathbf{y}} + da_z\hat{\mathbf{z}})$$

Recognizing the definitions of the gradient ($\nabla T$) and the vector area element ($d\mathbf{a}$), we obtain

$$\boxed{\int_V (\nabla T)\, d\tau = \oint_S T\, d\mathbf{a}} \tag{16}$$

This completes the proof.

For a shorter proof, we have the following:

Let $\mathbf{c}$ be an arbitrary constant vector. We construct the vector field $\mathbf{v} = T\mathbf{c}$ and compute its divergence using the product rule:

$$\nabla \cdot (T\mathbf{c}) = (\nabla T) \cdot \mathbf{c} + T(\nabla \cdot \mathbf{c}) \tag{17}$$

Since $\mathbf{c}$ is constant, $\nabla \cdot \mathbf{c} = 0$, yielding:

$$\nabla \cdot (T\mathbf{c}) = (\nabla T) \cdot \mathbf{c} \tag{18}$$

Applying the divergence theorem to the vector field $T\mathbf{c}$:

$$\int_V \nabla \cdot (T\mathbf{c})\, d\tau = \oint_S (T\mathbf{c}) \cdot d\mathbf{a} \tag{19}$$

Substituting equation (2) into the left side and expanding the right side:

$$\int_V (\nabla T) \cdot \mathbf{c}\, d\tau = \oint_S T(\mathbf{c} \cdot d\mathbf{a}) \tag{20}$$

Since $\mathbf{c}$ is constant, we can factor it out:

$$\mathbf{c} \cdot \int_V \nabla T\, d\tau = \mathbf{c} \cdot \oint_S T\, d\mathbf{a} \tag{21}$$

This can be rewritten as:

$$\mathbf{c} \cdot \left( \int_V \nabla T\, d\tau - \oint_S T\, d\mathbf{a} \right) = 0 \tag{22}$$

Since this equality holds for *every* arbitrary constant vector $\mathbf{c}$, the vector in parentheses must be the zero vector. Therefore:

$$\boxed{\int_V (\nabla T)\, d\tau = \oint_S T\, d\mathbf{a}} \tag{23}$$

This second proof is essentially a compressed version of the first one, where setting $\mathbf{c} = \hat{\mathbf{x}}$ in the second proof gives exactly the x-component proof in the first one. The arbitrary vector in the second proof covers what is done separately for $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

($b$) Recall the standard **Divergence Theorem**:

$$\iiint_V (\nabla \cdot \mathbf{F})\, d\tau = \oiint_S \mathbf{F} \cdot d\mathbf{a}$$

where $\mathbf{F}$ is an arbitrary, well-behaved vector field.

The identity to be proven is:

$$\iiint_V (\nabla \times \mathbf{v})\, d\tau = -\oiint_S \mathbf{v} \times d\mathbf{a}$$

This is another vector generalization of the Divergence Theorem. The vector equation is true if and only if each of its component equations is true. The strategy is to prove the identity for an arbitrary component (e.g., the x-component) using the standard Divergence Theorem, and then argue that the same logic applies to the other components.

The x-component of the left-hand side is:

$$\left[\iiint_V (\nabla \times \mathbf{v})\, d\tau\right]_x = \iiint_V (\nabla \times \mathbf{v})_x\, d\tau = \iiint_V \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) d\tau$$

The x-component of the right-hand side is:

$$\left[-\oiint_S \mathbf{v} \times d\mathbf{a}\right]_x = -\oiint_S (\mathbf{v} \times d\mathbf{a})_x = -\oiint_S (v_y da_z - v_z da_y) = \oiint_S (v_z da_y - v_y da_z)$$

So, our goal is to prove the following scalar equation for the x-component:

$$\iiint_V \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) d\tau = \oiint_S (v_z da_y - v_y da_z)$$

To do this, we must choose a vector field $\mathbf{F}$ in the standard Divergence Theorem. Let's define a vector field $\mathbf{F}$ as:

$$\mathbf{F} = (0, v_z, -v_y)$$

Applying the Divergence Theorem to this specific $\mathbf{F}$:

(a) Calculate the divergence of $\mathbf{F}$:

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
&= \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(v_z) + \frac{\partial}{\partial z}(-v_y) \\
&= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}
\end{aligned}$$

(b) Calculate the dot product $\mathbf{F} \cdot d\mathbf{a}$:

$$\begin{aligned}
\mathbf{F} \cdot d\mathbf{a} &= (F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}) \cdot (da_x\hat{\mathbf{x}} + da_y\hat{\mathbf{y}} + da_z\hat{\mathbf{z}}) \\
&= F_x da_x + F_y da_y + F_z da_z \\
&= (0)da_x + (v_z)da_y + (-v_y)da_z \\
&= v_z da_y - v_y da_z
\end{aligned}$$

(c) Substitute these into the Divergence Theorem: The theorem $\iiint_V (\nabla \cdot \mathbf{F})\, d\tau = \oiint_S \mathbf{F} \cdot d\mathbf{a}$ becomes:

$$\iiint_V \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) d\tau = \oiint_S (v_z da_y - v_y da_z)$$

This proves that the x-component of our target identity is true.

A similar argument holds for the process for the $y$ and $z$ components:

- To prove the $y$-component, we choose $\mathbf{F} = (-v_z, 0, v_x)$, which gives $\iiint_V (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})\, d\tau = \oiint_S (v_x da_z - v_z da_x)$.
- To prove the $z$-component, we choose $\mathbf{F} = (v_y, -v_x, 0)$, which gives $\iiint_V (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\, d\tau = \oiint_S (v_y da_x - v_x da_y)$.

Since we have proven the identity for all three components, we can combine them back into a single vector equation:

$$\iiint_V (\nabla \times \mathbf{v})\, d\tau = \oiint_S (v_z da_y - v_y da_z)\hat{\mathbf{x}} + (v_x da_z - v_z da_x)\hat{\mathbf{y}} + (v_y da_x - v_x da_y)\hat{\mathbf{z}}$$

The right-hand side is equal to

$$-\oiint_S [(v_y da_z - v_z da_y)\hat{\mathbf{x}} + (v_z da_x - v_x da_z)\hat{\mathbf{y}} + (v_x da_y - v_y da_x)\hat{\mathbf{z}}]$$

which is the definition of $-\oiint_S \mathbf{v} \times d\mathbf{a}$.

Therefore, we have proven the identity:

$$\iiint_V (\nabla \times \mathbf{v})\, d\tau = -\oiint_S \mathbf{v} \times d\mathbf{a}$$

$(c)$ This identity, known as **Green's first identity**, is a direct consequence of the **Divergence Theorem**, which we state as

$$\iiint_V (\nabla \cdot \mathbf{v})\, d\tau = \oiint_S \mathbf{v} \cdot d\mathbf{a}$$

where $\mathbf{v}$ is an arbitrary, well-behaved vector field, $V$ is a volume, and $S$ is the closed surface that bounds $V$.

Our goal is to prove:

$$\iiint_V \left[T\nabla^2 U + (\nabla T) \cdot (\nabla U)\right] d\tau = \oiint_S (T\nabla U) \cdot d\mathbf{a}$$

The strategy is to choose a specific vector field to substitute into the Divergence Theorem, such that the theorem transforms into the desired identity.

We strategically choose the vector field $\mathbf{v}$ to match the right-hand side of the identity:

$$\mathbf{v} = T\nabla U$$

Substitute $\mathbf{v}$ into both sides of the theorem.

- **Surface Integral (RHS):**

$$\oiint_S \mathbf{v} \cdot d\mathbf{a} = \oiint_S (T\nabla U) \cdot d\mathbf{a}$$

  This matches the desired right-hand side.

- **Volume Integral (LHS):**

$$\iiint_V (\nabla \cdot \mathbf{v})\, d\tau = \iiint_V \nabla \cdot (T\nabla U)\, d\tau$$

Use the vector calculus product rule for the divergence of a scalar times a vector, $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + (\nabla f) \cdot \mathbf{A}$, with $f = T$ and $\mathbf{A} = \nabla U$:

$$\nabla \cdot (T\nabla U) = T(\nabla \cdot (\nabla U)) + (\nabla T) \cdot (\nabla U)$$

We recognize that the divergence of the gradient, $\nabla \cdot (\nabla U)$, is the **Laplacian operator**, $\nabla^2 U$:

$$\nabla \cdot (T\nabla U) = T\nabla^2 U + (\nabla T) \cdot (\nabla U)$$

Substituting this expanded form back into the volume integral yields:

$$\iiint_V \nabla \cdot (T\nabla U)\, d\tau = \iiint_V \left[T\nabla^2 U + (\nabla T) \cdot (\nabla U)\right] d\tau$$

By the Divergence Theorem, the volume integral equals the surface integral, thus proving:

$$\iiint_V \left[T\nabla^2 U + (\nabla T) \cdot (\nabla U)\right] d\tau = \oiint_S (T\nabla U) \cdot d\mathbf{a}$$

($d$) This identity, known as **Green's second identity**, can be proven by starting with Green's first identity (which was proven in the previous problem) and then using a symmetry argument.

The starting point for our proof is Green's first identity, which is a direct consequence of the Divergence Theorem:

$$\iiint_V [T\nabla^2 U + (\nabla T) \cdot (\nabla U)] d\tau = \oiint_S (T\nabla U) \cdot d\mathbf{a} \qquad (*)$$

This identity holds for any two well-behaved scalar functions $T$ and $U$.

Since the identity holds for any pair of scalar functions, it must also hold if we simply swap the roles of $T$ and $U$ everywhere in the equation. Let's write down the identity with $T$ and $U$ interchanged:

$$\iiint_V [U\nabla^2 T + (\nabla U) \cdot (\nabla T)] d\tau = \oiint_S (U\nabla T) \cdot d\mathbf{a} \qquad (**)$$

Now, we subtract the second identity $(**)$ from the first identity $(*)$:

- **Subtracting the Volume Integrals (Left-Hand Side):**

$$\iiint_V \left([T\nabla^2 U + (\nabla T) \cdot (\nabla U)] - [U\nabla^2 T + (\nabla U) \cdot (\nabla T)]\right) d\tau$$

The dot product is commutative, meaning $(\nabla T) \cdot (\nabla U) = (\nabla U) \cdot (\nabla T)$. Therefore, these two terms cancel each other out when we subtract. This leaves us with:

$$\iiint_V (T\nabla^2 U - U\nabla^2 T) d\tau$$

- **Subtracting the Surface Integrals (Right-Hand Side):**

$$\oiint_S (T\nabla U)\cdot d\mathbf{a} - \oiint_S (U\nabla T)\cdot d\mathbf{a}$$

Since integration is a linear operation, we can combine the integrands:

$$\oiint_S (T\nabla U - U\nabla T)\cdot d\mathbf{a}$$

Equating the results of the subtractions from the left and right sides, we arrive at Green's second identity:

$$\iiint_V (T\nabla^2 U - U\nabla^2 T)d\tau = \oiint_S (T\nabla U - U\nabla T)\cdot d\mathbf{a}$$

This completes the proof.

$(e)$ This identity is a vector generalization of Stokes' Theorem.

Recall the standard **Stokes' Theorem**:

$$\iint_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_P \mathbf{v}\cdot d\mathbf{l}$$

where $\mathbf{v}$ is an arbitrary, well-behaved vector field, $S$ is an open surface, and $P$ is the closed path that forms the boundary of $S$.

We will apply the standard Stokes' Theorem to a cleverly chosen vector field to prove the vector generalization.

Let's define a vector field $\mathbf{v}$ using the scalar function $T$ from the problem statement and an arbitrary constant vector $\mathbf{c}$:

$$\mathbf{v} = T\mathbf{c}$$

Since $T$ is a scalar and $\mathbf{c}$ is a constant vector, $\mathbf{v}$ is a valid vector field. We substitute this choice of $\mathbf{v}$ into both sides of Stokes' Theorem.

- **The Line Integral (RHS):**

$$\oint_P \mathbf{v}\cdot d\mathbf{l} = \oint_P (T\mathbf{c})\cdot d\mathbf{l} = \mathbf{c}\cdot\left(\oint_P T\,d\mathbf{l}\right)$$

(Since $\mathbf{c}$ is a constant vector, it can be pulled outside the integral.)

- **The Surface Integral (LHS):**

$$\iint_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \iint_S (\nabla\times(T\mathbf{c}))\cdot d\mathbf{a}$$

Using the vector product rule $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) + (\nabla f)\times\mathbf{A}$ with $f = T$ and $\mathbf{A} = \mathbf{c}$ (where $\nabla\times\mathbf{c} = \mathbf{0}$):

$$\nabla\times(T\mathbf{c}) = (\nabla T)\times\mathbf{c}$$

Substituting this back:

$$\iint_S ((\nabla T)\times\mathbf{c})\cdot d\mathbf{a}$$

Using the scalar triple product identity $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$, we let $\mathbf{A} = \nabla T$, $\mathbf{B} = \mathbf{c}$, and $\mathbf{C} = d\mathbf{a}$:

$$\iint_S (d\mathbf{a} \times \nabla T) \cdot \mathbf{c} = \left( \iint_S d\mathbf{a} \times \nabla T \right) \cdot \mathbf{c}$$

(Again, pulling the constant vector $\mathbf{c}$ outside the integral.)

Equating the transformed sides of Stokes' Theorem:

$$\left( \iint_S d\mathbf{a} \times \nabla T \right) \cdot \mathbf{c} = \mathbf{c} \cdot \left( \oint_P T \, d\mathbf{l} \right)$$

Rearranging the terms:

$$\left( \iint_S d\mathbf{a} \times \nabla T - \oint_P T \, d\mathbf{l} \right) \cdot \mathbf{c} = 0$$

Since this equation must hold for *any* arbitrary constant vector $\mathbf{c}$, the vector quantity in parentheses must be the zero vector:

$$\iint_S d\mathbf{a} \times \nabla T - \oint_P T \, d\mathbf{l} = \mathbf{0}$$

$$\iint_S d\mathbf{a} \times \nabla T = \oint_P T \, d\mathbf{l}$$

Finally, using the anti-commutative property of the cross product ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$), we can write $\mathbf{da} \times \nabla T = -\nabla T \times d\mathbf{a}$:

$$-\iint_S \nabla T \times d\mathbf{a} = \oint_P T \, d\mathbf{l}$$

Multiplying by $-1$ yields the desired result:

$$\iint_S \nabla T \times d\mathbf{a} = -\oint_P T \, d\mathbf{l}$$

(*f*) We begin with the following integral identity, which is itself a vector generalization of Stokes' Theorem (proven in a previous exercise):

$$\int_S (\nabla T) \times d\mathbf{a} = -\oint_P T \, d\mathbf{l}$$

Using the anti-commutative property of the cross product ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$), we can rewrite the left side:

$$\int_S -(d\mathbf{a} \times \nabla T) = -\oint_P T \, d\mathbf{l}$$

Multiplying by $-1$, we get:

$$\int_S (d\mathbf{a} \times \nabla) T = \oint_P T \, d\mathbf{l} \qquad (*)$$

This vector identity holds for any well-behaved scalar function $T$.

Since the identity $(*)$ holds for any scalar function, we can apply it to each of the Cartesian components $(v_x, v_y, v_z)$ of our vector field $\mathbf{v}$:

$$\begin{aligned}
\text{For } T = v_x : &\quad \int_S (d\mathbf{a} \times \nabla) v_x = \oint_P v_x \, d\mathbf{l} \\
\text{For } T = v_y : &\quad \int_S (d\mathbf{a} \times \nabla) v_y = \oint_P v_y \, d\mathbf{l} \\
\text{For } T = v_z : &\quad \int_S (d\mathbf{a} \times \nabla) v_z = \oint_P v_z \, d\mathbf{l}
\end{aligned}$$

We construct a new vector identity by taking the cross product of each of the component equations with its corresponding unit vector ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$) and then summing the results. We bring the constant unit vectors inside the integrals:

$$\begin{aligned}
\int_S [(d\mathbf{a} \times \nabla) v_x] \times \hat{\mathbf{x}} &= \oint_P v_x (d\mathbf{l} \times \hat{\mathbf{x}}) \\
\int_S [(d\mathbf{a} \times \nabla) v_y] \times \hat{\mathbf{y}} &= \oint_P v_y (d\mathbf{l} \times \hat{\mathbf{y}}) \\
\int_S [(d\mathbf{a} \times \nabla) v_z] \times \hat{\mathbf{z}} &= \oint_P v_z (d\mathbf{l} \times \hat{\mathbf{z}})
\end{aligned}$$

Summing these three new vector equations yields:

$$\int_S \sum_{i=x,y,z} [(d\mathbf{a} \times \nabla) v_i] \times \hat{\mathbf{i}} = \oint_P \sum_{i=x,y,z} v_i (d\mathbf{l} \times \hat{\mathbf{i}})$$

We then simplify both sides:

- **Right-Hand Side:** The integrand simplifies as:

$$\sum_{i=x,y,z} v_i (d\mathbf{l} \times \hat{\mathbf{i}}) = d\mathbf{l} \times (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}) = d\mathbf{l} \times \mathbf{v}$$

- **Left-Hand Side:** The sum can be shown to be equivalent to the vector triple product:

$$\sum_{i=x,y,z} [(d\mathbf{a} \times \nabla) v_i] \times \hat{\mathbf{i}} = (d\mathbf{a} \times \nabla) \times \mathbf{v}$$

Substituting the simplified integrands back into our summed equation gives:

$$\int_S (d\mathbf{a} \times \nabla) \times \mathbf{v} = \oint_P d\mathbf{l} \times \mathbf{v}$$

Finally, using the anti-commutative property on the right-hand side ($d\mathbf{l} \times \mathbf{v} = -\mathbf{v} \times d\mathbf{l}$), we arrive at the desired identity:

$$\int_S [(d\mathbf{a} \times \nabla) \times \mathbf{v}] = -\oint_P \mathbf{v} \times d\mathbf{l}$$

**Problem 11: 1.62**

The integral $\mathbf{a} = \int_S d\mathbf{a}$ is sometimes called the **vector area** of the surface $S$. If $S$ happens to be *flat*, then $|\mathbf{a}|$ is the *ordinary* (scalar) area, obviously.

1. Find the vector area of a hemispherical bowl of radius $R$.
2. Show that $\mathbf{a} = \mathbf{0}$ for any *closed* surface.
3. Show that $\mathbf{a} = \int_S d\mathbf{a}$ is the same for all surfaces sharing the same boundary.
4. Show that $\mathbf{a} \equiv \int_S d\mathbf{a} = \frac{1}{2}\oint \mathbf{r} \times d\mathbf{l}$, where the integral is around the boundary line.
5. Show that for any constant vector $\mathbf{c}$, we have

$$\oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} = \mathbf{a} \times \mathbf{c}.$$

**Solution:**

1. We want to calculate the vector area $\mathbf{a} = \int_S d\mathbf{a}$ for a hemispherical bowl of radius $R$. Let's place the bowl's base in the $xy$-plane, centered at the origin, with the bowl opening upward (in the $+z$ direction).

   The vector area is the surface integral of the unit normal vector over the surface:

$$\mathbf{a} = \iint_S d\mathbf{a} = \iint_S \hat{\mathbf{n}}\, dA$$

   For a sphere, the outward normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$.

   The surface element on a sphere of radius $R$ is $dA = R^2 \sin\theta\, d\theta\, d\phi$.

   The unit normal vector is $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$.

   The vector surface element is $d\mathbf{a} = \hat{\mathbf{r}}\, dA = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$.

   The hemisphere is defined by $0 \le \theta \le \pi/2$ and $0 \le \phi \le 2\pi$.

   We calculate the integral component by component:

   **x-component:**[

$$\begin{aligned} a_x &= \int_0^{2\pi} \int_0^{\pi/2} (R^2 \sin\theta\cos\phi)(\sin\theta\, d\theta\, d\phi) \\ &= R^2 \left(\int_0^{2\pi} \cos\phi\, d\phi\right) \left(\int_0^{\pi/2} \sin^2\theta\, d\theta\right) \end{aligned}$$

   The integral of $\cos\phi$ over a full period is zero, so $a_x = 0$.

   **y-component:**

$$\begin{aligned} a_y &= \int_0^{2\pi} \int_0^{\pi/2} (R^2 \sin\theta\sin\phi)(\sin\theta\, d\theta\, d\phi) \\ &= R^2 \left(\int_0^{2\pi} \sin\phi\, d\phi\right) \left(\int_0^{\pi/2} \sin^2\theta\, d\theta\right) \end{aligned}$$

   The integral of $\sin\phi$ over a full period is zero, so $a_y = 0$.

**z-component:**

$$
\begin{aligned}
a_z &= \int_0^{2\pi}\int_0^{\pi/2} (R^2\cos\theta)(\sin\theta\, d\theta\, d\phi) \\
&= R^2\left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right)
\end{aligned}
$$

Evaluating the integrals:

$$
\begin{aligned}
\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta &= \int_0^{\pi/2}\frac{1}{2}\sin(2\theta)\, d\theta \\
&= \left[-\frac{1}{4}\cos(2\theta)\right]_0^{\pi/2} \\
&= \left(-\frac{1}{4}(-1)\right) - \left(-\frac{1}{4}(1)\right) \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}, .
\end{aligned}
$$

$$
\int_0^{2\pi} d\phi = 2\pi
$$

So, $a_z = R^2(2\pi)(\frac{1}{2}) = \pi R^2$.

The vector area of the hemispherical bowl is $\boxed{\mathbf{a} = \pi R^2\hat{\mathbf{z}}}$.

This makes sense: the vector area points in the "average" direction of the surface normals, and its magnitude is the area of the flat disk that forms the bowl's shadow.

2. We prove this using a component-wise application of the divergece theorem. Consider the $x$-component of $\mathbf{a}$:

$$
a_x = \oint_S da_x = \oint_S \hat{\mathbf{x}}\cdot d\mathbf{a} \tag{24}
$$

Applying the divergence theorem with the constant vector field $\mathbf{v} = \hat{\mathbf{x}}$ gives us

$$
\oint_S \hat{\mathbf{x}}\cdot d\mathbf{a} = \int_V (\nabla\cdot\hat{\mathbf{x}})\, d\tau = \int_V 0\, d\tau = 0 \tag{25}
$$

since $\nabla\cdot\hat{\mathbf{x}} = \frac{\partial}{\partial x}(1) = 0$.

Similarly, $a_y = 0$ and $a_z = 0$.

Therefore, for any closed surface we have

$$
\boxed{\mathbf{a} = \mathbf{0}}
$$

We also have a different proof that uses the preceding part.

Let $S$ be a closed surface bounding a volume $V$. From part (a), we have:

$$\int_V (\nabla T)\, d\tau = \oint_S T\, d\mathbf{a} \tag{26}$$

Choose $T = 1$ (constant everywhere). Then:

$$\nabla T = \nabla(1) = \mathbf{0} \tag{27}$$

Substituting into equation (8):

$$\int_V \mathbf{0}\, d\tau = \oint_S (1)\, d\mathbf{a} \tag{28}$$

The left side equals $\mathbf{0}$, and the right side simplifies to:

$$\mathbf{0} = \oint_S d\mathbf{a} = \mathbf{a} \tag{29}$$

3. Let $S_1$ and $S_2$ be two different open surfaces that share the same closed boundary line, $P$. Let their respective vector areas be:

$$\mathbf{a}_1 = \int_{S_1} d\mathbf{a}_1 \quad \text{and} \quad \mathbf{a}_2 = \int_{S_2} d\mathbf{a}_2$$

Our goal is to show that $\mathbf{a}_1 = \mathbf{a}_2$.

We combine these two surfaces to form a single closed surface, $S_{\text{closed}}$.

Recall that the vector area of any *closed* surface is zero (a direct consequence of the Divergence Theorem, by setting $T = 1$ in the identity $\iiint (\nabla T) d\tau = \oiint T d\mathbf{a}$):

$$\oint_{S_{\text{closed}}} d\mathbf{a} = \mathbf{0}$$

The integral over the closed surface is the sum of the integrals over its constituent parts, $S_1$ and $S_2$. We must align the normal vectors with the requirement for a closed surface: the normal vector $d\mathbf{a}$ must always point **outward**.

Let $d\mathbf{a}_1$ and $d\mathbf{a}_2$ be the normals for the open surfaces $S_1$ and $S_2$, chosen according to the right-hand rule for the boundary path $P$. When forming the closed surface, one of these normals must be reversed to point outward.

Assuming $S_1$ and $S_2$ form the top and bottom of the closed volume, the outward-pointing normal for the $S_2$ portion of the closed surface, $d\mathbf{a}_2'$, is the negative of the original normal $d\mathbf{a}_2$:

$$d\mathbf{a}_2' = -d\mathbf{a}_2$$

The integral over the closed surface is:

$$\oint_{S_{\text{closed}}} d\mathbf{a} = \int_{S_1} d\mathbf{a}_1 + \int_{S_2} d\mathbf{a}_2'$$

Substituting $d\mathbf{a}_2' = -d\mathbf{a}_2$:

$$\oint_{S_{\text{closed}}} d\mathbf{a} = \int_{S_1} d\mathbf{a}_1 - \int_{S_2} d\mathbf{a}_2$$

Using the theorem from step 3, we set the total integral to zero:

$$\int_{S_1} d\mathbf{a}_1 - \int_{S_2} d\mathbf{a}_2 = \mathbf{0}$$

Rearranging the terms gives the desired result:

$$\int_{S_1} d\mathbf{a}_1 = \int_{S_2} d\mathbf{a}_2$$

$$\mathbf{a}_1 = \mathbf{a}_2$$

This proves that the vector area depends only on the closed boundary line, $P$, not on the particular open surface that spans it.

4. We will prove the identity by showing that each of its Cartesian components ($x$, $y$, and $z$) is true and we will be using the standard **Stokes' Theorem**:

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

We first expand the right side of the equation. Given $\mathbf{r} = x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ and $d\mathbf{l} = dx\hat{\mathbf{x}}+dy\hat{\mathbf{y}}+dz\hat{\mathbf{z}}$, the cross product is:

$$\mathbf{r} \times d\mathbf{l} = (y\,dz - z\,dy)\hat{\mathbf{x}} + (z\,dx - x\,dz)\hat{\mathbf{y}} + (x\,dy - y\,dx)\hat{\mathbf{z}}$$

Thus we can write the equation for the identity as

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}(y\,dz - z\,dy)\hat{\mathbf{x}} + (z\,dx - x\,dz)\hat{\mathbf{y}} + (x\,dy - y\,dx)\hat{\mathbf{z}}$$

The $x$-component of the identity we want to prove is:

$$\int_S da_x = \frac{1}{2}\oint_P (y\,dz - z\,dy)$$

We focus on the line integral on the right. We choose a vector field $\mathbf{v}$ such that $\mathbf{v} \cdot d\mathbf{l} = y\,dz - z\,dy$. Comparing this to the general form $\mathbf{v} \cdot d\mathbf{l} = v_x dx + v_y dy + v_z dz$, we choose:

$$\mathbf{v} = 0\hat{\mathbf{x}} - z\hat{\mathbf{y}} + y\hat{\mathbf{z}}$$

Next, we find the curl of this $\mathbf{v}$:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial y}{\partial y} - \frac{\partial(-z)}{\partial z}\right) - \hat{\mathbf{y}}(\dots) + \hat{\mathbf{z}}(\dots)$$

$$= \hat{\mathbf{x}}(1 - (-1)) = 2\hat{\mathbf{x}}$$

Applying Stokes' Theorem:

$$\oint_P (y\,dz - z\,dy) = \oint_P \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_S (2\hat{\mathbf{x}}) \cdot d\mathbf{a}$$

Since $d\mathbf{a} = da_x\hat{\mathbf{x}} + da_y\hat{\mathbf{y}} + da_z\hat{\mathbf{z}}$, the dot product is $(2\hat{\mathbf{x}}) \cdot d\mathbf{a} = 2\, da_x$. So, we have:

$$\oint_P (y\, dz - z\, dy) = \int_S 2\, da_x = 2 \int_S da_x$$

Dividing by 2 gives the desired $x$-component:

$$\frac{1}{2} \oint_P (y\, dz - z\, dy) = \int_S da_x$$

The proof for the other components follows a similar argument:

**For the $y$-component,** we want to show $\int_S da_y = \frac{1}{2} \oint_P (z\, dx - x\, dz)$. We choose the vector field $\mathbf{v} = z\hat{\mathbf{x}} + 0\hat{\mathbf{y}} - x\hat{\mathbf{z}}$, whose curl is $\nabla \times \mathbf{v} = 2\hat{\mathbf{y}}$.

**For the $z$-component,** we want to show $\int_S da_z = \frac{1}{2} \oint_P (x\, dy - y\, dx)$. We choose the vector field $\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$, whose curl is $\nabla \times \mathbf{v} = 2\hat{\mathbf{z}}$.

Applying Stokes' theorem in each case proves the respective component.

Since the identity holds for the $x$, $y$, and $z$ components individually, the vector identity is true:

$$\mathbf{a} = \int_S d\mathbf{a} = \frac{1}{2} \oint_P \mathbf{r} \times d\mathbf{l}$$

5. We begin with the following integral identity, which is a corollary of Stokes' Theorem:

$$\oint_P T\, d\mathbf{l} = -\int_S (\nabla T) \times d\mathbf{a} \tag{30}$$

This theorem relates the line integral of a scalar function $T$ multiplied by the path element $d\mathbf{l}$ to the surface integral of the cross product of the gradient of $T$ and the area element $d\mathbf{a}$.

Notice that the left-hand side of our target identity, $\oint(\mathbf{c} \cdot \mathbf{r})d\mathbf{l}$, has the same form as the left-hand side of the theorem if we choose the scalar function $T$ to be:

$$T = \mathbf{c} \cdot \mathbf{r}$$

Next, we find the gradient of our chosen $T$. Since $\mathbf{c}$ is a constant vector, the gradient of its dot product with the position vector $\mathbf{r}$ is simply $\mathbf{c}$.

$$\nabla T = \nabla(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c}$$

(To verify this, let $\mathbf{c} = (c_x, c_y, c_z)$ and $\mathbf{r} = (x, y, z)$. Then $T = c_x x + c_y y + c_z z$. The gradient is $\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} = c_x\hat{\mathbf{x}} + c_y\hat{\mathbf{y}} + c_z\hat{\mathbf{z}} = \mathbf{c}$.)

Now we substitute our expressions for $T$ and $\nabla T$ back into the integral theorem:

$$\oint_P (\mathbf{c} \cdot \mathbf{r})d\mathbf{l} = -\int_S \mathbf{c} \times d\mathbf{a}$$

We simplify the surface integral on the right-hand side.

- First, use the anti-commutative property of the cross product ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$) to absorb the minus sign:

$$-\int_S \mathbf{c} \times d\mathbf{a} = \int_S d\mathbf{a} \times \mathbf{c}$$

- Since $\mathbf{c}$ is a constant vector, it can be pulled outside the integral:

$$\int_S d\mathbf{a} \times \mathbf{c} = \left(\int_S d\mathbf{a}\right) \times \mathbf{c}$$

The integral $\int_S d\mathbf{a}$ is, by definition, the **vector area a** of the surface $S$.

$$\left(\int_S d\mathbf{a}\right) \times \mathbf{c} = \mathbf{a} \times \mathbf{c}$$

By equating the results from Step 3 and Step 5, we have proven the desired identity:

$$\oint_P (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} = \mathbf{a} \times \mathbf{c}$$

**Problem 12: 1.63**

Consider the vector function

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}. \quad (1.83)$$

At every location, $\mathbf{v}$ is directed radially outward; if ever there was a function that ought to have a large positive divergence, this is it. And yet, when you actually *calculate* the divergence, you get precisely zero:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0. \quad (1.84)$$

Or you can do it in Cartesian coordinates. The plot thickens when we apply the divergence theorem to this function. Suppose we integrate over a sphere of radius $R$, centered at the origin; the surface integral is

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \int_S \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \left(\int_0^\pi \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi. \quad (1.85)$$

1. Find the divergence of the function

   $$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}.$$

   by computing it directly, as in Eq. 1.84.

2. Test your result using the divergence theorem, as in Eq. 1.85.

3. Is there a delta function at the origin, as there was for $\hat{\mathbf{r}}/r^2$?

4. What is the general formula for the divergence of $r^n\hat{\mathbf{r}}$?

5. Find the *curl* of $r^n\hat{\mathbf{r}}$. Then test your conclusion using the equation $\int_V (\nabla \times \mathbf{v}) d\tau = -\oint_S \mathbf{v} \times d\mathbf{a}$.

**Solution:**

1. We want to find the divergence of the function $\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}$. In spherical coordinates, this function has only a radial component: $v_r = 1/r$, and $v_\theta = v_\phi = 0$.

The formula for the divergence in spherical coordinates is:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Since $v_\theta$ and $v_\phi$ are zero, the last two terms vanish. We only need to compute the first term:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r)$$

The derivative of $r$ with respect to $r$ is 1.

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}(1) = \frac{1}{r^2}$$

So, the direct calculation gives a non-zero divergence for all $r \neq 0$.

2. The Divergence Theorem states $\iiint_V (\nabla \cdot \mathbf{v})\, dV = \oint_S \mathbf{v} \cdot d\mathbf{a}$. We will calculate both sides for a sphere of radius R centered at the origin.

   On the surface of the sphere, $r = R$, so $\mathbf{v} = \frac{\hat{\mathbf{r}}}{R}$. The differential area element is $d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$.

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\hat{\mathbf{r}}}{R}\right) \cdot (R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}})$$

$$= \int_0^{2\pi}\int_0^{\pi} R\sin\theta\, d\theta\, d\phi = R\left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi}\sin\theta\, d\theta\right)$$

$$= R(2\pi)[-\cos\theta]_0^{\pi} = R(2\pi)(1-(-1)) = 4\pi R$$

   From part (a), we found $\nabla \cdot \mathbf{v} = 1/r^2$. The volume element is $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

$$\iiint_V (\nabla \cdot \mathbf{v})\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(\frac{1}{r^2}\right)(r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

   The $r^2$ terms cancel out, leaving:

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \sin\theta\, dr\, d\theta\, d\phi = \left(\int_0^{R} dr\right)\left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$= (R)(2)(2\pi) = 4\pi R$$

   The surface integral ($4\pi R$) and the volume integral ($4\pi R$) are equal. The Divergence Theorem holds.

3. No, there is no delta function at the origin for this function.

   The reason a delta function was required for $\mathbf{v} = \hat{\mathbf{r}}/r^2$ was because of a **contradiction**:

   - The direct calculation of the divergence gave 0 (for $r \neq 0$).
   - The Divergence Theorem gave a non-zero flux of $4\pi$.

The only way to resolve the contradiction $\iiint 0\, dV \neq 4\pi$ is to say that the divergence is not zero at the origin. The delta function, $\nabla \cdot (\hat{\mathbf{r}}/r^2) = 4\pi\delta^3(\mathbf{r})$, is the mathematical object that fixes this discrepancy.

For our function $\mathbf{v} = \hat{\mathbf{r}}/r$, there is **no contradiction**. The direct calculation gives $\nabla \cdot \mathbf{v} = 1/r^2$, and the volume integral of this function perfectly matches the flux calculated from the surface integral. The function $1/r^2$ correctly describes the divergence everywhere, and its integral over the volume accounts for the total flux. No delta function is needed.

4. Let $\mathbf{v} = r^n\hat{\mathbf{r}}$. This means $v_r = r^n$. Using the divergence formula:

$$\nabla \cdot (r^n\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2})$$

Using the power rule for differentiation:

$$= \frac{1}{r^2}\left((n+2)r^{n+1}\right)$$

$$\nabla \cdot (r^n\hat{\mathbf{r}}) = (n+2)r^{n-1}$$

**Note the special case:** When $n = -2$, the formula gives $\nabla \cdot (r^{-2}\hat{\mathbf{r}}) = (-2+2)r^{-3} = 0$. This is the unique case where the direct calculation gives zero for $r \neq 0$, leading to the need for a delta function at the origin to account for the non-zero flux. For all other values of $n$, the calculated divergence is sufficient.

5. We want to find the curl of the vector function $\mathbf{v} = r^n\hat{\mathbf{r}}$. In spherical coordinates, the components of this vector are:

- $v_r = r^n$
- $v_\theta = 0$
- $v_\phi = 0$

The formula for the curl in spherical coordinates is:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Let's evaluate each component by substituting our values for $v_r, v_\theta, v_\phi$:

- **$\hat{\mathbf{r}}$ component:**

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \cdot 0) - \frac{\partial(0)}{\partial\phi}\right] = 0$$

- **$\hat{\theta}$ component:**

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial(r^n)}{\partial\phi} - \frac{\partial}{\partial r}(r \cdot 0)\right] = \frac{1}{r}\left[\frac{1}{\sin\theta}(0) - 0\right] = 0$$

(Since $r^n$ has no dependence on $\phi$, $\frac{\partial(r^n)}{\partial\phi} = 0$.)

- $\hat{\phi}$ **component:**
$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r \cdot 0) - \frac{\partial(r^n)}{\partial \theta}\right] = \frac{1}{r}[0 - 0] = 0$$
(Since $r^n$ has no dependence on $\theta$, $\frac{\partial(r^n)}{\partial \theta} = 0$.)

Since all three components are zero, the curl is the zero vector.

$$\nabla \times (r^n \hat{\mathbf{r}}) = \mathbf{0} \tag{31}$$

**Testing with the Integral Identity:**
We test the conclusion using the corollary

$$\int_V (\nabla \times \mathbf{v}) d\tau = -\oint_S \mathbf{v} \times d\mathbf{a}$$

We choose our volume $V$ to be a sphere of radius $R$ centered at the origin, with surface $S$. The left-hand side is the volume integral of the curl:

$$\int_V (\nabla \times \mathbf{v}) d\tau$$

From Part 1, we found $\nabla \times \mathbf{v} = \mathbf{0}$. Therefore, the volume integral is:

$$\int_V \mathbf{0}\, d\tau = \mathbf{0}$$

The right-hand side is the negative of the surface integral:

$$-\oint_S \mathbf{v} \times d\mathbf{a}$$

We evaluate the terms on the surface of the sphere ($r = R$):

- The vector field on $S$ is $\mathbf{v} = R^n \hat{\mathbf{r}}$.
- The differential area element on $S$ is $d\mathbf{a} = dA\, \hat{\mathbf{r}} = (R^2 \sin\theta\, d\theta\, d\phi)\hat{\mathbf{r}}$.

Now, we compute the cross product in the integrand:

$$\mathbf{v} \times d\mathbf{a} = (R^n \hat{\mathbf{r}}) \times (R^2 \sin\theta\, d\theta\, d\phi \cdot \hat{\mathbf{r}}) = R^{n+2} \sin\theta\, d\theta\, d\phi \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{r}})$$

Since $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \mathbf{0}$, the integrand is the zero vector everywhere on the surface.

$$\mathbf{v} \times d\mathbf{a} = \mathbf{0}$$

Therefore, the surface integral is:

$$-\oint_S \mathbf{0} = \mathbf{0}$$

Since both sides of the integral identity evaluate to the zero vector,

$$\mathbf{0} = \mathbf{0}$$

the theorem is verified for the given vector field $\mathbf{v} = r^n \hat{\mathbf{r}}$.

**Problem 13: 1.64**

We can generalize the delta function to three dimensions:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z).$$

(As always, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector, extending from the origin to the point $(x, y, z)$.) This three-dimensional delta function is zero everywhere except at $(0, 0, 0)$, where it blows up. Its volume integral is 1.
Consider

$$D(r, \epsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2 + \epsilon^2}}.$$

1. show that $D(r, \epsilon) = (3\epsilon^2/4\pi)(r^2 + \epsilon^2)^{-5/2}$
2. check that $D(0, \epsilon) \to \infty$ as $\epsilon \to 0$
3. check that $D(r, \epsilon) \to 0$, as $\epsilon \to 0$, for all $r \neq 0$
4. check that the integral of $D(r, \epsilon)$ over all space is 1.
5. If $\epsilon \to 0$, then what happens to $D(r, \epsilon)$?

**Solution:**

1. We start with the definition $D(r, \epsilon) = -\frac{1}{4\pi}\nabla^2 f(r)$, where $f(r) = (r^2 + \epsilon^2)^{-1/2}$. Using the radial part of the Laplacian in spherical coordinates:

$$\frac{df}{dr} = \frac{d}{dr}(r^2 + \epsilon^2)^{-1/2} = -\frac{1}{2}(r^2 + \epsilon^2)^{-3/2}(2r) = -r(r^2 + \epsilon^2)^{-3/2}$$

Applying the product rule:

$$\begin{aligned}
\frac{d}{dr}\left(-r^3(r^2 + \epsilon^2)^{-3/2}\right) &= (-3r^2)(r^2 + \epsilon^2)^{-3/2} + (-r^3)\left(-\frac{3}{2}(r^2 + \epsilon^2)^{-5/2}(2r)\right) \\
&= -3r^2(r^2 + \epsilon^2)^{-3/2} + 3r^4(r^2 + \epsilon^2)^{-5/2} \\
&= \frac{-3r^2(r^2 + \epsilon^2) + 3r^4}{(r^2 + \epsilon^2)^{5/2}} = \frac{-3r^4 - 3r^2\epsilon^2 + 3r^4}{(r^2 + \epsilon^2)^{5/2}} \\
&= \frac{-3r^2\epsilon^2}{(r^2 + \epsilon^2)^{5/2}}
\end{aligned}$$

Then we Complete the Laplacian and $D(r, \epsilon)$:

$$\nabla^2 f(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = \frac{1}{r^2}\left(\frac{-3r^2\epsilon^2}{(r^2 + \epsilon^2)^{5/2}}\right) = \frac{-3\epsilon^2}{(r^2 + \epsilon^2)^{5/2}}$$

$$D(r, \epsilon) = -\frac{1}{4\pi}\nabla^2 f(r) = -\frac{1}{4\pi}\left(\frac{-3\epsilon^2}{(r^2 + \epsilon^2)^{5/2}}\right) = \frac{3\epsilon^2}{4\pi}(r^2 + \epsilon^2)^{-5/2}$$

2. We have

$$D(0, \epsilon) = \frac{3\epsilon^2}{4\pi}(0^2 + \epsilon^2)^{-5/2} = \frac{3\epsilon^2}{4\pi}(\epsilon^2)^{-5/2} = \frac{3\epsilon^2}{4\pi}\epsilon^{-5} = \frac{3}{4\pi\epsilon^3}$$

The limit as $\epsilon \to 0$ is:

$$\lim_{\epsilon\to 0} D(0,\epsilon) = \lim_{\epsilon\to 0} \frac{3}{4\pi\epsilon^3} \to \infty$$

3. For a fixed $r \neq 0$:

$$\lim_{\epsilon\to 0} D(r,\epsilon) = \lim_{\epsilon\to 0} \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}} = \frac{3(0)^2}{4\pi(r^2+0)^{5/2}} = \frac{0}{4\pi r^5} = 0$$

4. Using spherical coordinates ($dV = 4\pi r^2 dr$ due to spherical symmetry):

$$I = \int D(r,\epsilon)\, dV = \int_0^\infty \left(\frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{5/2}}\right) 4\pi r^2\, dr = 3\epsilon^2 \int_0^\infty \frac{r^2}{(r^2+\epsilon^2)^{5/2}}\, dr$$

We use the substitution $r = \epsilon \tan u$, $dr = \epsilon \sec^2 u\, du$.

$$\begin{aligned} I &= 3\epsilon^2 \int_0^{\pi/2} \frac{(\epsilon\tan u)^2}{(\epsilon^2\tan^2 u + \epsilon^2)^{5/2}} (\epsilon \sec^2 u\, du) \\ &= 3\epsilon^2 \int_0^{\pi/2} \frac{\epsilon^3 \tan^2 u \sec^2 u}{(\epsilon^2 \sec^2 u)^{5/2}}\, du = 3\epsilon^2 \int_0^{\pi/2} \frac{\epsilon^3 \tan^2 u \sec^2 u}{\epsilon^5 \sec^5 u}\, du \\ &= 3 \int_0^{\pi/2} \frac{\tan^2 u}{\sec^3 u}\, du = 3 \int_0^{\pi/2} \frac{\sin^2 u / \cos^2 u}{1/\cos^3 u}\, du = 3 \int_0^{\pi/2} \sin^2 u \cos u\, du \end{aligned}$$

Let $w = \sin u$, $dw = \cos u\, du$. Limits are $w = [0,1]$.

$$I = 3 \int_0^1 w^2\, dw = 3\left[\frac{w^3}{3}\right]_0^1 = 3\left(\frac{1}{3}\right) = 1$$

5. As $\epsilon \to 0$, the function $D(r,\epsilon)$ satisfies the three defining properties of the three-dimensional Dirac delta function $\delta^3(\mathbf{r})$: it is zero everywhere except at the origin, it approaches infinity at the origin, and its integral over all space is unity.

$$\lim_{\epsilon\to 0} D(r,\epsilon) = \delta^3(\mathbf{r})$$

**Problem 14: 1.65**

1. Check Stokes' theorem for the vector function

$$\mathbf{A} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2+y^2},$$

using a circle of radius $R$ in the $xy$ plane. Diagnose the problem, and fix it by correcting the curl of $\mathbf{A}$. Use Cartesian coordinates.

2. Convert $\mathbf{A}$ to cylindrical coordinates, and repeat the previous part, this time doing everything in terms of $s$, $\phi$, and $z$.

**Solution:**

Recall that Stokes' theorem states:

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$$

The vector function is $\mathbf{A} = \frac{-y\hat{\mathbf{x}}+x\hat{\mathbf{y}}}{x^2+y^2}$, and $C$ is a circle of radius $R$ in the $xy$-plane, centered at the origin.

1.

**The Line Integral (LHS):** We parameterize the path $C$ using $x = R\cos\theta$, $y = R\sin\theta$, $d\mathbf{l} = (-R\sin\theta\, d\theta)\hat{\mathbf{x}} + (R\cos\theta\, d\theta)\hat{\mathbf{y}}$.
With the denominator being $x^2 + y^2 = R^2$, we express $\mathbf{A}$ on the Path as

$$\mathbf{A} = \frac{(-R\sin\theta)\hat{\mathbf{x}} + (R\cos\theta)\hat{\mathbf{y}}}{R^2} = \frac{-\sin\theta}{R}\hat{\mathbf{x}} + \frac{\cos\theta}{R}\hat{\mathbf{y}}$$

So

$$\mathbf{A}\cdot d\mathbf{l} = \left(\frac{-\sin\theta}{R}\right)(-R\sin\theta\, d\theta) + \left(\frac{\cos\theta}{R}\right)(R\cos\theta\, d\theta) = (\sin^2\theta + \cos^2\theta)\, d\theta = d\theta$$

Calculating the integral,

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_0^{2\pi} d\theta = 2\pi$$

**The Surface Integral (RHS):** We compute the $\hat{\mathbf{z}}$ component: $(\nabla\times\mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$.

$$\frac{\partial A_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(1)(x^2+y^2) - (x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(-1)(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus, for $(x, y) \neq (0, 0)$:

$$(\nabla\times\mathbf{A})_z = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0 \quad\Longrightarrow\quad \nabla\times\mathbf{A} = \mathbf{0}$$

If we use the result $\nabla\times\mathbf{A} = \mathbf{0}$, the surface integral is $\iint_S \mathbf{0}\cdot d\mathbf{a} = 0$, which contradicts the line integral $2\pi$. The problem is that $\mathbf{A}$ is singular (undefined) at the origin $(0, 0)$, which is included in the surface $S$. Stokes' theorem requires the field to be well-behaved throughout $S$. The non-zero line integral shows that the curl is concentrated entirely at the singularity. The correct, generalized curl that accounts for this singularity is a two-dimensional Dirac delta function:

$$\nabla\times\mathbf{A} = 2\pi\delta(x)\delta(y)\hat{\mathbf{z}}$$

Now we calculate the surface integral with the corrected curl.
The area element is $d\mathbf{a} = dx\, dy\, \hat{\mathbf{z}}$.

$$\iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \iint_S (2\pi\delta(x)\delta(y)\hat{\mathbf{z}})\cdot(dx\, dy\, \hat{\mathbf{z}}) = 2\pi\iint_S \delta(x)\delta(y)\, dx\, dy$$

Since the surface $S$ (a disk of radius $R$) includes the origin, the sifting property of the delta function ensures the integral is 1:

$$\iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = 2\pi\times 1 = 2\pi$$

The line integral ($2\pi$) is equal to the corrected surface integral ($2\pi$), verifying Stokes' theorem. The correction required recognizing the singularity at the origin as a Dirac delta function component of the curl.

2. We must first convert the vector field is $\mathbf{A} = \frac{-y\hat{\mathbf{x}}+x\hat{\mathbf{y}}}{x^2+y^2}$.

   Converting the coordinates and denominator gives us

$$x = s\cos\phi, \quad y = s\sin\phi, \quad x^2+y^2 = s^2$$

   Convert unit vectors provides

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\phi}, \quad \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\phi}$$

   Plugging into $\mathbf{A}$:

$$\begin{aligned}
\mathbf{A} &= \frac{1}{s^2}\left[-(s\sin\phi)(\cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\phi}) + (s\cos\phi)(\sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\phi})\right] \\
&= \frac{1}{s^2}\left[(-s\sin\phi\cos\phi + s\cos\phi\sin\phi)\hat{\mathbf{s}} + (s\sin^2\phi + s\cos^2\phi)\hat{\phi}\right] \\
&= \frac{1}{s^2}\left[(0)\hat{\mathbf{s}} + s(\sin^2\phi + \cos^2\phi)\hat{\phi}\right] \\
&= \frac{s}{s^2}(1)\hat{\phi} = \frac{1}{s}\hat{\phi}
\end{aligned}$$

   The vector field in cylindrical coordinates is $\mathbf{A} = \frac{1}{s}\hat{\phi}$, with components $A_s = 0$, $A_\phi = 1/s$, and $A_z = 0$.

   **The Line Integral (LHS):** The path $C$ is a circle of radius $R$ in the $xy$-plane ($s = R$, $z = 0$). The differential path element is $d\mathbf{l} = s\,d\phi\,\hat{\phi} = R\,d\phi\,\hat{\phi}$.
   On the path, $\mathbf{A} = \frac{1}{R}\hat{\phi}$.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_0^{2\pi}\left(\frac{1}{R}\hat{\phi}\right)\cdot(R\,d\phi\,\hat{\phi}) = \int_0^{2\pi}\frac{1}{R}\cdot R\,d\phi = \int_0^{2\pi} d\phi = 2\pi$$

   **The Surface Integral (RHS):** The $\hat{\mathbf{z}}$ component of the curl in cylindrical coordinates is:

$$(\nabla\times\mathbf{A})_z = \frac{1}{s}\left(\frac{\partial(sA_\phi)}{\partial s} - \frac{\partial A_s}{\partial\phi}\right)$$

   Substituting $A_s = 0$ and $A_\phi = 1/s$ for $s \neq 0$:

$$(\nabla\times\mathbf{A})_z = \frac{1}{s}\left(\frac{\partial}{\partial s}\left(s\cdot\frac{1}{s}\right) - \frac{\partial(0)}{\partial\phi}\right) = \frac{1}{s}\left(\frac{\partial(1)}{\partial s} - 0\right) = 0$$

   If we assume $\nabla\times\mathbf{A} = \mathbf{0}$ over the entire surface $S$ (a disk of radius $R$), the surface integral is $\iint_S \mathbf{0}\cdot d\mathbf{a} = 0$. This contradicts the line integral result of $2\pi$.

   The contradiction arises because the vector field $\mathbf{A} = \hat{\phi}/s$ is singular at $s = 0$. Since the line integral is non-zero, the correct curl must account for the singularity at the origin, which is enclosed by the surface. The correct expression for the curl of this specific field is:

$$\nabla\times\mathbf{A} = \frac{2\pi}{2\pi s}\delta(s)\hat{\mathbf{z}} = \frac{\delta(s)}{s}\hat{\mathbf{z}}$$

We now do the correct Surface Integral Calculation. The area element for the flat disk is $d\mathbf{a} = s\,ds\,d\phi\,\hat{\mathbf{z}}$.

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_0^{2\pi} \int_0^R \left( \frac{\delta(s)}{s} \hat{\mathbf{z}} \right) \cdot (s\,ds\,d\phi\,\hat{\mathbf{z}})$$
$$= \int_0^{2\pi} \int_0^R \frac{\delta(s)}{s} s\,ds\,d\phi$$
$$= \int_0^{2\pi} \left[ \int_0^R \delta(s)\,ds \right] d\phi$$

Since the domain of integration ($s = [0, R]$) includes the origin, the inner integral is $\int_0^R \delta(s)\,ds = 1$.

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_0^{2\pi} (1)\,d\phi = 2\pi$$

The line integral and the corrected surface integral match:

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = 2\pi \quad \text{and} \quad \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = 2\pi$$

Stokes' theorem is verified, confirming that the function's singularity at the origin is responsible for the non-zero result.

**Problem 15: 2.3**

Find the electric field a distance $z$ above one end of a straight line segment of length $L$ (Fig. 2.7) that carries a uniform line charge $\lambda$. Check that your formula is consistent with what you would expect for the case $z \gg L$.

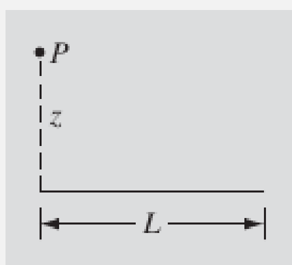



**Fig. 2.7**

**Solution:**

We establish a Cartesian coordinate system such that:

- The line segment lies on the x-axis, from $x' = 0$ to $x' = L$. We use a primed coordinate, $x'$, to denote the position of the source charge.
- The point of observation $P$ is located on the z-axis at $(0, 0, z)$.
- The position vector of the field point is $\mathbf{r} = z\hat{\mathbf{z}}$.
- An infinitesimal source element of length $dx'$ is located at position $\mathbf{r}' = x'\hat{\mathbf{x}}$.

- The charge of this element is $dq = \lambda\, dx'$.

The electric field is found by integrating the contribution from each charge element $dq$ over the entire length of the segment. The fundamental equation is the integral form of Coulomb's Law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\vec{\nabla}|^3} \vec{\nabla}$$

where $\vec{\nabla} = \mathbf{r} - \mathbf{r}'$ is the separation vector pointing from the source element to the field point. For this configuration, the separation vector is:

$$\vec{\nabla} = (z\hat{\mathbf{z}}) - (x'\hat{\mathbf{x}}) = -x'\hat{\mathbf{x}} + z\hat{\mathbf{z}}$$

Its magnitude is:

$$|\vec{\nabla}| = \sqrt{(-x')^2 + z^2} = \sqrt{x'^2 + z^2}$$

Substituting these into the integral, we obtain the expression for the total electric field:

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{-x'\hat{\mathbf{x}} + z\hat{\mathbf{z}}}{(x'^2 + z^2)^{3/2}}\, dx'$$

This vector integral can be solved by evaluating its x and z components separately:

**x-component, $E_x$:**

$$E_x = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{-x'}{(x'^2 + z^2)^{3/2}}\, dx'$$

We solve this integral using the substitution $u = x'^2 + z^2$, which implies $du = 2x'\, dx'$. The limits of integration transform as follows: when $x' = 0$, $u = z^2$; when $x' = L$, $u = L^2 + z^2$.

$$\begin{aligned}
E_x &= -\frac{\lambda}{4\pi\epsilon_0} \int_{z^2}^{L^2+z^2} \frac{1}{2} u^{-3/2}\, du \\
&= -\frac{\lambda}{8\pi\epsilon_0} \left[-2u^{-1/2}\right]_{z^2}^{L^2+z^2} \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{u}}\right]_{z^2}^{L^2+z^2} \\
&= \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{L^2 + z^2}} - \frac{1}{z}\right)
\end{aligned}$$

(Note: Since $\sqrt{L^2 + z^2} > z$, the term in parentheses is negative, ensuring $E_x < 0$. This is physically correct, as the field from the positive rod must have a component pointing away from the rod, i.e., in the negative x-direction.)

**z-component, $E_z$:**

$$E_z = \frac{\lambda z}{4\pi\epsilon_0} \int_0^L \frac{dx'}{(x'^2 + z^2)^{3/2}}$$

We solve this integral using the trigonometric substitution $x' = z\tan\theta$, which implies $dx' = z\sec^2\theta\, d\theta$. The term $(x'^2 + z^2)^{3/2}$ becomes $(z^2\tan^2\theta + z^2)^{3/2} = z^3\sec^3\theta$. The limits transform as: when $x' = 0$, $\theta = 0$; when $x' = L$, $\theta = \arctan(L/z)$.

$$\begin{aligned} E_z &= \frac{\lambda z}{4\pi\epsilon_0} \int_0^{\arctan(L/z)} \frac{z \sec^2\theta}{z^3 \sec^3\theta}\, d\theta \\ &= \frac{\lambda}{4\pi\epsilon_0 z} \int_0^{\arctan(L/z)} \cos\theta\, d\theta \\ &= \frac{\lambda}{4\pi\epsilon_0 z} \left[\sin\theta\right]_0^{\arctan(L/z)} \end{aligned}$$

To evaluate $\sin(\arctan(L/z))$, we consider a right triangle with opposite side $L$ and adjacent side $z$. The hypotenuse is $\sqrt{L^2+z^2}$, so $\sin\theta = L/\sqrt{L^2+z^2}$.

$$E_z = \frac{\lambda}{4\pi\epsilon_0 z}\left(\frac{L}{\sqrt{L^2+z^2}} - 0\right) = \frac{\lambda L}{4\pi\epsilon_0 z\sqrt{L^2+z^2}}$$

Combining the components gives the exact electric field vector at point P:

$$\boxed{\mathbf{E}(0,0,z) = \frac{\lambda}{4\pi\epsilon_0}\left[\left(\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\right)\hat{\mathbf{x}} + \left(\frac{L}{z\sqrt{L^2+z^2}}\right)\hat{\mathbf{z}}\right]}$$

**Far-Field Analysis ($z \gg L$)** We now check the behavior of this expression in the limit where the observation point is very far from the line segment. In this limit, the line should appear as a point charge $Q = \lambda L$ located at the origin, with an expected field $\mathbf{E} \approx \frac{Q}{4\pi\epsilon_0 z^2}\hat{\mathbf{z}}$.

We define the small, dimensionless parameter $\eta = L/z \ll 1$. The key term in our expression can be expanded using the binomial series $(1+\epsilon)^n \approx 1 + n\epsilon + \frac{n(n-1)}{2}\epsilon^2 + \ldots$:

$$\begin{aligned} \frac{1}{\sqrt{L^2+z^2}} &= \frac{1}{z\sqrt{1+(L/z)^2}} \\ &= \frac{1}{z}(1+\eta^2)^{-1/2} \\ &= \frac{1}{z}\left(1 - \frac{1}{2}\eta^2 + \frac{3}{8}\eta^4 - \ldots\right) \end{aligned}$$

Substituting this expansion into our component expressions:

**x-component:**

$$\begin{aligned} E_x &= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{z}\left(1 - \frac{1}{2}\frac{L^2}{z^2} + \ldots\right) - \frac{1}{z}\right) \\ &= \frac{\lambda}{4\pi\epsilon_0}\left(-\frac{L^2}{2z^3} + \ldots\right) \\ &= -\frac{\lambda L^2}{8\pi\epsilon_0 z^3} + O\left(\frac{1}{z^5}\right) \end{aligned}$$

**z-component:**

$$\begin{aligned}
E_z &= \frac{\lambda L}{4\pi\epsilon_0 z}\left(\frac{1}{z}\left(1 - \frac{1}{2}\frac{L^2}{z^2} + \dots\right)\right) \\
&= \frac{\lambda L}{4\pi\epsilon_0 z^2}\left(1 - \frac{1}{2}\frac{L^2}{z^2} + \dots\right) \\
&= \frac{\lambda L}{4\pi\epsilon_0 z^2} - \frac{\lambda L^3}{8\pi\epsilon_0 z^4} + O\left(\frac{1}{z^6}\right)
\end{aligned}$$

The leading term in the expansion for $E_z$ is $\frac{\lambda L}{4\pi\epsilon_0 z^2} = \frac{Q}{4\pi\epsilon_0 z^2}$, which is precisely the field of a point charge. The leading term for $E_x$ is of order $O(z^{-3})$, while the leading term for $E_z$ is of order $O(z^{-2})$. In the limit $z \to \infty$, the x-component becomes negligible compared to the z-component. The ratio of their magnitudes confirms this:

$$\left|\frac{E_x}{E_z}\right| \approx \frac{\lambda L^2/(8\pi\epsilon_0 z^3)}{\lambda L/(4\pi\epsilon_0 z^2)} = \frac{L}{2z}$$

Since $L/z \ll 1$, the field becomes overwhelmingly vertical, approaching the expected point-charge behavior. The formula is therefore consistent.

> **Problem 16: 2.4**
>
> Determine the electric field a distance $z$ above the center of a square loop (side length $a$) carrying a uniform line charge $\lambda$

**Solution:**

We establish a Cartesian coordinate system with its origin at the geometric center of the square loop. The geometry is defined as follows:

- The square loop lies in the $xy$-plane ($z = 0$).
- The vertices of the square are located at $(\pm a/2, \pm a/2, 0)$, such that its sides are parallel to the x and y axes.
- The point of observation, P, is located on the z-axis at the position vector $\mathbf{r} = z\hat{\mathbf{z}}$, where $z > 0$.
- The loop carries a constant, uniform line charge density $\lambda$.

The charge distribution possesses $C_4$ rotational symmetry about the z-axis and includes mirror planes at $x = 0$ and $y = 0$. For any point on the z-axis, which is the axis of symmetry, the electric field vector cannot have a component perpendicular to the axis. Any such horizontal component generated by a charge element $dq$ at a source point $\mathbf{r}'$ is precisely cancelled by the horizontal component from a corresponding element at $-\mathbf{r}'$. Consequently, the net electric field at point P must be directed purely along the z-axis.

$$\mathbf{E}(0, 0, z) = E_z(z)\hat{\mathbf{z}} \tag{32}$$

This symmetry allows us to find the total field by calculating the z-component of the field generated by one side of the loop and multiplying the result by four.

Recall that the electric field from a line charge is given by the integral:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{loop}} \frac{\lambda\, d\ell'}{|\vec{\nabla}|^3}\vec{\nabla} \tag{33}$$

where $\vec{\nabla} = \mathbf{r} - \mathbf{r'}$ is the separation vector from a source point $\mathbf{r'}$ on the loop to the field point $\mathbf{r}$. Let us consider the contribution from the side of the loop at $x' = a/2$, which extends from $y' = -a/2$ to $y' = a/2$.

- A source element is located at $\mathbf{r'} = (a/2)\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$.
- The differential line element is $d\ell' = dy'$.
- The separation vector is:

$$\vec{\nabla} = \mathbf{r} - \mathbf{r'} = (z\hat{\mathbf{z}}) - ((a/2)\hat{\mathbf{x}} + y'\hat{\mathbf{y}}) = -(a/2)\hat{\mathbf{x}} - y'\hat{\mathbf{y}} + z\hat{\mathbf{z}} \tag{34}$$

- The magnitude of the separation vector is:

$$|\vec{\nabla}| = \sqrt{(-a/2)^2 + (-y')^2 + z^2} = \sqrt{a^2/4 + y'^2 + z^2} \tag{35}$$

The differential contribution to the z-component of the electric field from this element is:

$$dE_z = \frac{\lambda\, dy'}{4\pi\epsilon_0}\frac{\vec{\nabla}\cdot\hat{\mathbf{z}}}{|\vec{\nabla}|^3} = \frac{\lambda z}{4\pi\epsilon_0}\frac{dy'}{(a^2/4 + y'^2 + z^2)^{3/2}} \tag{36}$$

To find the total contribution from this one side, we integrate with respect to $y'$ from $-a/2$ to $a/2$. Let us define the constant $k^2 = a^2/4 + z^2$ to simplify the expression.

$$E_{z,1\text{ side}} = \frac{\lambda z}{4\pi\epsilon_0}\int_{-a/2}^{a/2} \frac{dy'}{(y'^2 + k^2)^{3/2}} \tag{37}$$

This is a standard integral, whose antiderivative is $\int (x^2+k^2)^{-3/2}dx = x/(k^2\sqrt{x^2+k^2})$. Evaluating the definite integral:

$$\begin{aligned}
E_{z,1\text{ side}} &= \frac{\lambda z}{4\pi\epsilon_0}\left[\frac{y'}{k^2\sqrt{y'^2+k^2}}\right]_{-a/2}^{a/2} \\
&= \frac{\lambda z}{4\pi\epsilon_0 k^2}\left(\frac{a/2}{\sqrt{(a/2)^2+k^2}} - \frac{-a/2}{\sqrt{(a/2)^2+k^2}}\right) \\
&= \frac{\lambda z a}{4\pi\epsilon_0 k^2\sqrt{a^2/4+k^2}}
\end{aligned}$$

Substituting back the expression for $k^2$:

$$\begin{aligned}
E_{z,1\text{ side}} &= \frac{\lambda z a}{4\pi\epsilon_0(a^2/4+z^2)\sqrt{a^2/4+(a^2/4+z^2)}} \\
&= \frac{\lambda z a}{4\pi\epsilon_0(a^2/4+z^2)\sqrt{a^2/2+z^2}}
\end{aligned}$$

The total electric field is four times this contribution, as all four sides contribute equally to the z-component by symmetry.

$$E_z = 4 \times E_{z,1\text{ side}} = \boxed{\frac{\lambda a z}{\pi\epsilon_0 \left(\frac{a^2}{4} + z^2\right)\sqrt{\frac{a^2}{2} + z^2}}\hat{\mathbf{z}} = \mathbf{E}(0,0,z)} \tag{38}$$

Here are some checks we do to verify our answer:

- **Far-Field Limit ($z \gg a$):** In this limit, the denominators become $(a^2/4 + z^2) \approx z^2$ and $\sqrt{a^2/2 + z^2} \approx z$.

$$\begin{aligned} E_z &\approx \frac{\lambda a z}{\pi\epsilon_0 (z^2)(z)} \\ &= \frac{\lambda a}{\pi\epsilon_0 z^2} \end{aligned}$$

  The total charge of the loop is $Q = 4a\lambda$. The field of a point charge $Q$ at a distance $z$ is $E = \frac{Q}{4\pi\epsilon_0 z^2} = \frac{4a\lambda}{4\pi\epsilon_0 z^2} = \frac{a\lambda}{\pi\epsilon_0 z^2}$. The result is consistent.

- **At the Center ($z = 0$):** Substituting $z = 0$ into the final expression yields $\mathbf{E} = \mathbf{0}$, as required by symmetry.

- **Dimensional Analysis:** The dimensions of the expression are $\frac{[\lambda][a][z]}{[\epsilon_0][a^2][a]} = \frac{[\mathrm{C/m}][\mathrm{m}][\mathrm{m}]}{[\mathrm{C^2/(N{\cdot}m^2)}][\mathrm{m^2}][\mathrm{m}]} = \frac{\mathrm{N}}{\mathrm{C}}$, which are the correct units for an electric field.

> **Problem 17: 2.5**
>
> Find the electric field a distance $z$ above the center of a circular loop of radius $r$ (Fig. 2.9) that carries a uniform line charge $\lambda$.

**Solution:**

We establish a cylindrical coordinate system $(\rho, \phi, z)$ with its origin at the geometric center of the loop.

- The loop is situated in the $z = 0$ plane, defined by the equation $\rho = r$.
- The point of observation $P$ is located on the z-axis, with position vector $\mathbf{r} = z\hat{\mathbf{z}}$.
- A source point on the loop is parameterized by the azimuthal angle $\phi'$, with position vector $\mathbf{r}'(\phi') = r\hat{\boldsymbol{\rho}}$. In Cartesian coordinates, this is $\mathbf{r}'(\phi') = r\cos\phi'\hat{\mathbf{x}} + r\sin\phi'\hat{\mathbf{y}}$.

The electric field at a point $\mathbf{r}$ due to a continuous line charge distribution is given by the integral form of Coulomb's Law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{loop}} \frac{\lambda\, d\ell'}{|\vec{\nabla}|^3}\vec{\nabla} \tag{39}$$

where $\vec{\nabla} = \mathbf{r} - \mathbf{r}'$ is the separation vector from the source point to the field point, and $d\ell'$ is the scalar differential arc length.

The charge distribution possesses continuous rotational symmetry about the z-axis. For any field point on this axis of symmetry, the net electric field vector must be collinear with the axis.

Consider the contribution to the field $d\mathbf{E}(\phi')$ from a source element at $\mathbf{r}'(\phi')$ and the contribution $d\mathbf{E}(\phi'+\pi)$ from the diametrically opposite element at $\mathbf{r}'(\phi'+\pi) = -\mathbf{r}'(\phi')$. The respective separation vectors are $\vec{\nabla}_1 = z\hat{\mathbf{z}} - \mathbf{r}'(\phi')$ and $\vec{\nabla}_2 = z\hat{\mathbf{z}} + \mathbf{r}'(\phi')$. The sum of the fields from this pair is:

$$d\mathbf{E}_{\text{pair}} = \frac{\lambda r\, d\phi'}{4\pi\epsilon_0}\left(\frac{z\hat{\mathbf{z}} - \mathbf{r}'(\phi')}{|\vec{\nabla}_1|^3} + \frac{z\hat{\mathbf{z}} + \mathbf{r}'(\phi')}{|\vec{\nabla}_2|^3}\right)$$

Since $|\vec{\nabla}_1|^2 = |\vec{\nabla}_2|^2 = r^2 + z^2$, the denominators are equal. The horizontal components, involving $\pm\mathbf{r}'(\phi')$, cancel exactly, leaving only a vertical component. As this holds for all such pairs, the total field must be:

$$\mathbf{E}(0,0,z) = E_z(z)\hat{\mathbf{z}} \tag{40}$$

This conclusion will be confirmed by direct integration of the vector components.

The following are the components of the integral

**Separation Vector:**

$$\vec{\nabla} = \mathbf{r} - \mathbf{r}' = z\hat{\mathbf{z}} - (r\cos\phi'\hat{\mathbf{x}} + r\sin\phi'\hat{\mathbf{y}}) = -r\cos\phi'\hat{\mathbf{x}} - r\sin\phi'\hat{\mathbf{y}} + z\hat{\mathbf{z}} \tag{41}$$

**Magnitude of the Separation Vector:**

$$\begin{aligned}|\vec{\nabla}| &= \sqrt{(-r\cos\phi')^2 + (-r\sin\phi')^2 + z^2}\\ &= \sqrt{r^2(\cos^2\phi' + \sin^2\phi') + z^2}\\ &= \sqrt{r^2 + z^2}\end{aligned}$$

Note that the magnitude $|\vec{\nabla}|$ is independent of the integration variable $\phi'$.

**Differential Arc Length:**

$$d\ell' = \left|\frac{d\mathbf{r}'}{d\phi'}\right| d\phi' = |-r\sin\phi'\hat{\mathbf{x}} + r\cos\phi'\hat{\mathbf{y}}|d\phi' = r\,d\phi' \tag{42}$$

Substituting these components into the fundamental equation:

$$\mathbf{E}(z) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda(r\,d\phi')}{(r^2+z^2)^{3/2}}(-r\cos\phi'\hat{\mathbf{x}} - r\sin\phi'\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \tag{43}$$

Since all terms except the vector components of $\vec{\nabla}$ are constant with respect to $\phi'$, we can factor them out:

$$\mathbf{E}(z) = \frac{\lambda r}{4\pi\epsilon_0(r^2+z^2)^{3/2}}\int_0^{2\pi}(-r\cos\phi'\hat{\mathbf{x}} - r\sin\phi'\hat{\mathbf{y}} + z\hat{\mathbf{z}})\,d\phi' \tag{44}$$

We evaluate the integral for each vector component:

**x-component:**

$$E_x = \frac{\lambda r}{4\pi\epsilon_0(r^2+z^2)^{3/2}}\int_0^{2\pi}(-r\cos\phi')\,d\phi' = \frac{-\lambda r^2}{4\pi\epsilon_0(\dots)^{3/2}}\left[\sin\phi'\right]_0^{2\pi} = 0 \tag{45}$$

**y-component:**

$$E_y = \frac{\lambda r}{4\pi\epsilon_0(r^2+z^2)^{3/2}} \int_0^{2\pi} (-r\sin\phi')\,d\phi' = \frac{-\lambda r^2}{4\pi\epsilon_0(\dots)^{3/2}} \left[-\cos\phi'\right]_0^{2\pi} = 0 \tag{46}$$

**z-component:**

$$E_z = \frac{\lambda r}{4\pi\epsilon_0(r^2+z^2)^{3/2}} \int_0^{2\pi} z\,d\phi' = \frac{\lambda r z}{4\pi\epsilon_0(r^2+z^2)^{3/2}} \left[\phi'\right]_0^{2\pi} \tag{47}$$

$$= \frac{\lambda r z(2\pi)}{4\pi\epsilon_0(r^2+z^2)^{3/2}} = \frac{\lambda r z}{2\epsilon_0(r^2+z^2)^{3/2}} \tag{48}$$

Obtaining zero for the integrals of the x and y components provides a rigorous confirmation of the symmetry argument.

The total electric field vector is $\mathbf{E} = E_z\hat{\mathbf{z}}$. It is conventional to express the result in terms of the total charge of the loop, $Q = 2\pi r\lambda$.

$$\mathbf{E}(z) = \frac{(Q/2\pi r)rz}{2\epsilon_0(r^2+z^2)^{3/2}}\hat{\mathbf{z}} \tag{49}$$

$$\boxed{\mathbf{E}(0,0,z) = \frac{1}{4\pi\epsilon_0}\frac{Qz}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}} \tag{50}$$

Here are some checks we do to verify our answer:

- **Far-Field Limit ($z \gg a$):** The denominator can be expanded as $(r^2+z^2)^{3/2} = |z|^3(1 + r^2/z^2)^{3/2} \approx |z|^3$.

$$\mathbf{E}(z) \approx \frac{1}{4\pi\epsilon_0}\frac{Qz}{|z|^3}\hat{\mathbf{z}}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}\hat{\mathbf{z}} \quad (\text{for } z > 0)$$

  This is the expected field of a point charge $Q$ at the origin.

- **At the Center ($z = 0$):** The field vanishes, $\mathbf{E}(0) = \mathbf{0}$, as required by symmetry.

- **Dimensional Analysis:** The units of the final expression are $\frac{[\text{Charge}][\text{Length}]}{[\epsilon_0][\text{Length}^2]^{3/2}} = \frac{[\text{Charge}]}{[\epsilon_0][\text{Length}^2]}$, which correctly corresponds to the units of electric field, N/C.

> **Problem 18: 2.6**
>
> Find the electric field a distance $z$ above the center of a flat circular disk of radius $R$ (Fig. 2.10) that carries a uniform surface charge $\sigma$. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

**Solution:**

We establish a cylindrical coordinate system $(\rho, \phi, z)$ with its origin at the geometric center of the disk.

- The disk is situated in the $z = 0$ plane, occupying the region $0 \le \rho \le R$.
- The point of observation $P$ is located on the z-axis, with position vector $\mathbf{r} = z\hat{\mathbf{z}}$. We consider the case where $z > 0$.

Consider an infinitesimal charge element $dq_1$ at a source position $\mathbf{r}'_1 = (\rho', \phi', 0)$. This element produces a field $d\mathbf{E}_1$ at $P$. Now consider a second element $dq_2$ at $\mathbf{r}'_2 = (\rho', \phi' + \pi, 0)$. The separation vector from this second element to $P$ has the same magnitude and z-component as the first, but its transverse component is in the opposite direction. Consequently, the transverse components of $d\mathbf{E}_1$ and $d\mathbf{E}_2$ are equal and opposite, and their sum is purely axial. Since the entire disk can be decomposed into such pairs of elements, the total field at $P$ must be directed purely along the z-axis.

$$\mathbf{E}(0, 0, z) = E_z(z)\hat{\mathbf{z}} \tag{51}$$

To emphasize what we just did, we gave the reasons for the charge distribution to possess continuous rotational (azimuthal) symmetry about the z-axis. For any field point on this axis of symmetry, the net electric field vector must be collinear with the axis.

We can determine the total electric field by integrating the contributions from a continuum of concentric, infinitesimally thin rings.

The electric field $d\mathbf{E}$ at a distance $z$ on the axis of a circular ring of radius $\rho'$ and total charge $dq$ is a standard result:

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{z\, dq}{(\rho'^2 + z^2)^{3/2}} \hat{\mathbf{z}} \tag{52}$$

This formula already incorporates the symmetry of the ring, resulting in a purely axial field.

Now, we consider an infinitesimal ring within the disk at radius $\rho'$ with an infinitesimal width $d\rho'$:

**Area of this ring** $dA' = (2\pi\rho')\, d\rho'$.

**Charge on this ring** $dq = \sigma\, dA' = 2\pi\sigma\rho'\, d\rho'$.

We substitute the expression for $dq$ into the formula for the field of a ring and integrate with respect to the ring radius $\rho'$ from the center of the disk ($\rho' = 0$) to its edge ($\rho' = R$).

$$\begin{aligned}
\mathbf{E}(z) &= \int d\mathbf{E} \\
&= \int_0^R \frac{1}{4\pi\epsilon_0} \frac{z(2\pi\sigma\rho'\, d\rho')}{(\rho'^2 + z^2)^{3/2}} \hat{\mathbf{z}} \\
&= \frac{2\pi\sigma z}{4\pi\epsilon_0} \hat{\mathbf{z}} \int_0^R \frac{\rho'}{(\rho'^2 + z^2)^{3/2}}\, d\rho' \\
&= \frac{\sigma z}{2\epsilon_0} \hat{\mathbf{z}} \int_0^R \rho'(\rho'^2 + z^2)^{-3/2}\, d\rho'
\end{aligned}$$

To evaluate the definite integral, we use the substitution $u = \rho'^2 + z^2$, which implies $du = 2\rho'\, d\rho'$.

The limits of integration transform as follows: when $\rho' = 0$, $u = z^2$; when $\rho' = R$, $u = R^2 + z^2$.

$$\begin{aligned}
\int_0^R \rho'(\rho'^2 + z^2)^{-3/2}\, d\rho' &= \int_{z^2}^{R^2+z^2} \frac{1}{2} u^{-3/2}\, du \\
&= \frac{1}{2}\left[-2u^{-1/2}\right]_{z^2}^{R^2+z^2} \\
&= -\left[\frac{1}{\sqrt{u}}\right]_{z^2}^{R^2+z^2} \\
&= -\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}}\right) \\
&= \frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}
\end{aligned}$$

(Here we have used the fact that $z > 0$, so $\sqrt{z^2} = z$). Substituting this result back into the expression for $\mathbf{E}(z)$ yields:

$$\boxed{\mathbf{E}(z) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{z}}}$$

We can do some more computations to get consistency checks:

**Limit of an Infinite Plane ($R \to \infty$):** In the limit where the radius of the disk becomes infinite, the disk becomes an infinite plane of charge.

$$\lim_{R\to\infty} \mathbf{E}(z) = \lim_{R\to\infty} \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{z}} \tag{53}$$

As $R \to \infty$, the term $\frac{z}{\sqrt{R^2+z^2}}$ approaches zero, which gives:

$$\mathbf{E}_{\text{infinite plane}} = \frac{\sigma}{2\epsilon_0}(1-0)\hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}} \tag{54}$$

This is the well-known result for the electric field of an infinite plane of charge, which is constant in magnitude and direction, independent of the distance $z$ from the plane.

**Far-Field Limit ($z \gg R$):** In the limit where the observation point is very far from the disk, the disk should appear as a point charge $Q = \sigma\pi R^2$. The expected field is $\mathbf{E} \approx \frac{Q}{4\pi\epsilon_0 z^2}\hat{\mathbf{z}}$.

We use the binomial expansion for the term $(1+x)^n \approx 1 + nx$ for small $x$. Let $\epsilon = (R/z)^2 \ll 1$.

$$\begin{aligned}
\frac{z}{\sqrt{R^2+z^2}} &= \frac{z}{z\sqrt{1+(R/z)^2}} \\
&= (1+\epsilon)^{-1/2} \\
&\approx 1 - \frac{1}{2}\epsilon \\
&= 1 - \frac{R^2}{2z^2}
\end{aligned}$$

Substituting this approximation into our result:

$$\begin{aligned}
\mathbf{E}(z) &\approx \frac{\sigma}{2\epsilon_0}\left(1-\left(1-\frac{R^2}{2z^2}\right)\right)\hat{\mathbf{z}} \\
&= \frac{\sigma}{2\epsilon_0}\left(\frac{R^2}{2z^2}\right)\hat{\mathbf{z}} \\
&= \frac{\sigma R^2}{4\epsilon_0 z^2}\hat{\mathbf{z}}
\end{aligned}$$

Now, we substitute the total charge of the disk, $Q = \sigma\pi R^2$, which gives $\sigma R^2 = Q/\pi$:

$$\begin{aligned}
\mathbf{E}(z) &\approx \frac{Q/\pi}{4\epsilon_0 z^2}\hat{\mathbf{z}} \\
&= \frac{Q}{4\pi\epsilon_0 z^2}\hat{\mathbf{z}}
\end{aligned}$$

This matches the expected field from a point charge, confirming that our formula is consistent in the far-field limit.

**Problem 19: 2.7**

Find the electric field a distance $z$ from the center of a spherical surface of radius $R$ (Fig. 2.11) that carries a uniform charge density $\sigma$. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge $q$ on the sphere. [*Hint*: Use the law of cosines to write $\imath$ in terms of $R$ and $\theta$. Be sure to take the *positive* square root: $\sqrt{R^2 + z^2 - 2Rz} = (R - z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]

**Solution:**

We seek to determine the electric field vector $\mathbf{E}$ at a point $P$ located a distance $z$ from the center of a hollow spherical shell of radius $R$. The shell carries a total charge $q$ distributed uniformly over its surface, resulting in a constant surface charge density $\sigma = q/(4\pi R^2)$. Due to the hint and where this question occurs in the book, we will construct a solution based on the direct integration of Coulomb's Law (rather than an application of Gauss's Law).

We establish a spherical coordinate system with its origin at the center of the shell. Without loss of generality, the field point $P$ is placed on the positive z-axis, so its position vector is $\mathbf{r} = z\hat{\mathbf{z}}$. A source point on the shell is described by the vector $\mathbf{r}'$ at spherical coordinates $(R, \theta', \phi')$.

The charge distribution is spherically symmetric. For a field point on the z-axis, the system possesses azimuthal symmetry. For any infinitesimal charge element $dq$ at an azimuthal angle $\phi'$, there exists a corresponding element at $-\phi'$. When the fields from this pair are summed at $P$, their transverse components (in the xy-plane) cancel exactly due to symmetry. Therefore, the net electric field at $P$ must be directed purely along the z-axis:

$$\mathbf{E}(0, 0, z) = E_z(z)\hat{\mathbf{z}}. \tag{55}$$

This allows us to simplify the problem to the calculation of a single scalar component, $E_z$.

We first calculate the scalar potential $V(z)$, from which the electric field can be found via $E_z = -dV/dz$. The potential is given by:

$$V(z) = \frac{1}{4\pi\epsilon_0}\int_S \frac{dq}{|\vec{\nabla}|}. \tag{56}$$

The components of this integral are:

**The differential charge element:** $dq = \sigma\, dA' = \sigma R^2 \sin\theta'\, d\theta'\, d\phi'$.

**The magnitude of the separation vector:** $|\vec{\nabla}| = |\mathbf{r} - \mathbf{r}'|$, is found by the Law of Cosines:

$$|\vec{\nabla}| = \sqrt{R^2 + z^2 - 2zR\cos\theta'}.$$

Substituting these into the potential integral:

$$V(z) = \frac{\sigma R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2zR\cos\theta'}}\, d\theta'. \tag{57}$$

The integral over $\phi'$ yields a factor of $2\pi$.

$$V(z) = \frac{\sigma R^2}{2\epsilon_0} \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2zR\cos\theta'}}\, d\theta'. \tag{58}$$

We solve the remaining integral by substitution. Let $u = \cos\theta'$, so $du = -\sin\theta'\, d\theta'$. The limits of integration transform from $\theta' = [0, \pi]$ to $u = [1, -1]$.

$$\begin{aligned}
V(z) &= \frac{\sigma R^2}{2\epsilon_0} \int_1^{-1} \frac{-du}{\sqrt{R^2 + z^2 - 2zRu}} = \frac{\sigma R^2}{2\epsilon_0} \int_{-1}^{1} (R^2 + z^2 - 2zRu)^{-1/2}\, du && (59)\\
&= \frac{\sigma R^2}{2\epsilon_0} \left[ \frac{2\sqrt{R^2 + z^2 - 2zRu}}{-2zR} \right]_{-1}^{1} && (60)\\
&= -\frac{\sigma R}{2\epsilon_0 z} \left[ \sqrt{R^2 + z^2 - 2zR} - \sqrt{R^2 + z^2 + 2zR} \right] && (61)\\
&= -\frac{\sigma R}{2\epsilon_0 z} \left[ \sqrt{(R - z)^2} - \sqrt{(R + z)^2} \right]. && (62)
\end{aligned}$$

It is a mathematical identity that $\sqrt{x^2} = |x|$. Since $z > 0$, $\sqrt{(R + z)^2} = R + z$.

$$\begin{aligned}
V(z) &== -\frac{\sigma R}{2\epsilon_0 z} \left( |R - z| - (R + z) \right) && (63)\\
&= \frac{\sigma R}{2\epsilon_0 z} \left( R + z - |R - z| \right). && (64)
\end{aligned}$$

The absolute value in the expression for $V(z)$ requires a piecewise analysis for the field point being outside or inside the shell.

**(Case 1) Outside the Shell ($z > R$):** In this case, $|R - z| = z - R$. The potential becomes:

$$V(z) = \frac{\sigma R}{2\epsilon_0 z}(R + z - (z - R)) = \frac{\sigma R}{2\epsilon_0 z}(2R) = \frac{\sigma R^2}{\epsilon_0 z}.$$

Substituting $\sigma = q/(4\pi R^2)$, we get $V(z) = \frac{q}{4\pi\epsilon_0 z}$. The electric field is:

$$E_z(z) = -\frac{dV}{dz} = -\frac{d}{dz}\left( \frac{q}{4\pi\epsilon_0 z} \right) = \frac{q}{4\pi\epsilon_0 z^2}.$$

**(Case 2) Inside the Shell ($z < R$):** In this case, $|R - z| = R - z$. The potential becomes:

$$V(z) = \frac{\sigma R}{2\epsilon_0 z}(R + z - (R - z)) = \frac{\sigma R}{2\epsilon_0 z}(2z) = \frac{\sigma R}{\epsilon_0}.$$

Substituting for $\sigma$, we get $V(z) = \frac{q}{4\pi\epsilon_0 R}$. The potential is constant everywhere inside the shell. The electric field is therefore zero:

$$E_z(z) = -\frac{dV}{dz} = -\frac{d}{dz}\,(\text{constant}) = 0.$$

By generalizing from the specific point on the z-axis to any point at a radial distance $r$, we get that the electric field of a uniformly charged spherical shell is:

$$\boxed{\mathbf{E}(\mathbf{r}) = \begin{cases} \mathbf{0} & \text{for } r < R \text{ (inside)} \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} & \text{for } r > R \text{ (outside)} \end{cases}} \tag{65}$$

This result is called the Shell Theorem, which shows that for points outside the shell, the field is identical to that of a point charge $q$ at the origin. For points inside, its field is exactly zero.

We can also obtain the theorem via an application of Gauss's Law due to the high degree of spherical symmetry in the problem.

The charge distribution is spherically symmetric, meaning the charge density is a function only of the radial distance from the origin, $\rho(\mathbf{r}) = \rho(r)$. This symmetry imposes strict constraints on the form of the electric field $\mathbf{E}(\mathbf{r})$.

Consider an arbitrary point $P$ and the electric field $\mathbf{E}(P)$ at that point. If we rotate the entire system (the charged sphere and the point $P$) by any angle about any axis passing through the origin, the charge distribution remains unchanged. Since the physical laws governing electromagnetism are isotropic, the electric field vector must also remain unchanged under this rotation. The only vector field whose value at every point is invariant under all such rotations is one that is purely radial and whose magnitude depends only on the radial distance.

Therefore, the electric field must be of the form:

$$\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}} \tag{66}$$

where $r = |\mathbf{r}|$ is the distance from the origin and $\hat{\mathbf{r}}$ is the radial unit vector. This conclusion is fundamental to the application of Gauss's Law.

Gauss's Law in integral form states:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{67}$$

Given the spherical symmetry of the field, we choose our Gaussian surface to be a sphere of radius $r$ concentric with the charged shell.

For a spherical Gaussian surface of radius $r$, the differential area element is everywhere parallel to the radial unit vector: $d\mathbf{a} = da\,\hat{\mathbf{r}}$. The electric flux is:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \oint_S (E(r)\hat{\mathbf{r}}) \cdot (da\,\hat{\mathbf{r}}) \tag{68}$$

Since $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$, the integrand becomes a scalar. From our symmetry argument, the magnitude $E(r)$ is constant for a fixed radius $r$ and can be factored out of the integral:

$$\oint_S E(r)\, da = E(r) \oint_S da = E(r) \times (4\pi r^2) \tag{69}$$

This expression for the flux is valid for any concentric spherical Gaussian surface.

We now apply Gauss's Law to two distinct regions.

**(Case 1) Outside the Shell ($z > R$):** Let the field point be at a distance $r$ from the center, where $r > R$. Our Gaussian surface of radius $r$ completely encloses the charged spherical shell.

The total charge enclosed, $Q_{\text{enc}}$, is the total charge of the shell, $q$.

Applying Gauss's Law,

$$E(r) \cdot 4\pi r^2 = \frac{q}{\epsilon_0} \tag{70}$$

Solving for the field gives us the electric field vector $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$.

**(Case 2) Inside the Shell ($z < R$):** Let the field point be at a distance $r$ from the center, where $r < R$. Our Gaussian surface of radius $r$ is entirely inside the charged shell.

The Gaussian surface lies completely inside the region where the charge resides. Therefore, it encloses no charge, $Q_{\text{enc}} = 0$.

Applying Gauss' Law,

$$E(r) \cdot 4\pi r^2 = \frac{0}{\epsilon_0} = 0 \tag{71}$$

Since $4\pi r^2 \neq 0$ for any non-trivial surface, the magnitude of the electric field must be zero, $E(r) = 0$. The electric field vector is therefore the zero vector, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$.

Combining the results for both cases, the electric field of a uniformly charged spherical shell is given by the piecewise function:

$$\boxed{\mathbf{E}(\mathbf{r}) = \begin{cases} \mathbf{0} & \text{for } r < R \text{ (inside)} \\ \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{r^2} \hat{\mathbf{r}} & \text{for } r > R \text{ (outside)} \end{cases}} \tag{72}$$

**Boundary Condition Check.** The normal component of the electric field is discontinuous across the charged surface by an amount $\sigma/\epsilon_0$. At $r = R$:

$$\Delta E_\perp = E_{\text{out}}(R) - E_{\text{in}}(R) = \left( \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \right) - (0) = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma}{R^2} = \frac{\sigma}{\epsilon_0} \tag{73}$$

Our result correctly reproduces the known boundary condition.

**Problem 20: 2.8**

Use your result in the preceding problem (2.7) to find the field inside and outside a solid sphere of radius $R$ that carries a uniform volume charge density $\rho$. Express your answers in

> terms of the total charge of the sphere, $q$. Draw a graph of $|\mathbf{E}|$ as a function of the distance from the center.

**Solution:**

The solid sphere can be conceptualized as a continuum of infinitesimally thin, concentric spherical shells. The electric field of the solid sphere is the vector superposition of the fields produced by all such shells.

Recall that the preceding problem involves proving the **Shell Theorem**, which describes the electric field of a uniformly charged spherical shell of radius $r'$ and charge $dq'$:

1. For a field point at a radial distance $r > r'$ (outside the shell), the shell's electric field is identical to that of a point charge $dq'$ located at the origin:
$$d\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{dq'}{r^2}\hat{\mathbf{r}}. \tag{74}$$
2. For a field point at a radial distance $r < r'$ (inside the shell), the shell's electric field is zero:
$$d\mathbf{E}_{\text{in}} = \mathbf{0}. \tag{75}$$

The spherical symmetry of the charge distribution guarantees that the total electric field will be purely radial, i.e., $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$.

**(Case 1) Outside the Shell ($z > R$):** Let the field point $P$ be located at a radial distance $r > R$ from the center.

For any infinitesimal shell of radius $r'$ composing the solid sphere, its radius satisfies $0 \le r' \le R$. Since $r > R$, it follows that $r > r'$ for all shells.

According to the Shell Theorem (part 1), every constituent shell contributes to the electric field at $P$ as if its charge $dq'$ were a point charge at the origin. The total electric field is the integral (superposition) of these contributions:
$$\mathbf{E}(r) = \int_{\text{sphere}} d\mathbf{E}_{\text{out}} = \int_{\text{sphere}} \frac{1}{4\pi\epsilon_0}\frac{dq'}{r^2}\hat{\mathbf{r}}. \tag{76}$$

Since the term $\frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$ is constant with respect to the integration over the source variable $q'$, it can be factored out:
$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\int_{\text{sphere}} dq' = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{77}$$

Thus, for any point outside the sphere, the electric field is identical to that of a point charge $q$ located at the center.

**(Case 2) Inside the Shell ($z < R$):** Let the field point $P$ be located at a radial distance $r < R$ from the center.

We partition the solid sphere into two regions:

1. An inner solid sphere of radius $r$, consisting of all shells with radii $0 \le r' \le r$.

2. An outer thick spherical shell, consisting of all shells with radii $r < r' \leq R$.

The total field at $P$ is the superposition of the fields from these two regions.

**Contribution from the Outer Shell ($r < r' \leq R$):** For every infinitesimal shell in this region, the field point $P$ is located inside the shell ($r < r'$). According to the Shell Theorem (part 2), the electric field contribution from each of these shells is exactly zero. Therefore, the total contribution from the entire outer region is zero.

$$\mathbf{E}_{\text{outer}}(r) = \mathbf{0}. \tag{78}$$

**Contribution from the Inner Sphere ($0 \leq r' \leq r$):** For every infinitesimal shell in this region, the field point $P$ is located outside the shell ($r > r'$). According to the Shell Theorem (part 1), each of these shells contributes a field as if its charge were a point charge at the origin. The total field from this inner sphere is therefore equivalent to the field of a point charge whose magnitude is the total charge enclosed within the radius $r$, which we denote $q_{\text{enc}}(r)$.

$$\mathbf{E}_{\text{inner}}(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{enc}}(r)}{r^2} \hat{\mathbf{r}}. \tag{79}$$

The enclosed charge is calculated as:

$$q_{\text{enc}}(r) = \rho \times (\text{Volume of sphere of radius } r) = \rho \left( \frac{4}{3}\pi r^3 \right). \tag{80}$$

To express this in terms of the total charge $q$, we substitute $\rho = \frac{3q}{4\pi R^3}$:

$$q_{\text{enc}}(r) = \left( \frac{3q}{4\pi R^3} \right) \left( \frac{4}{3}\pi r^3 \right) = q\frac{r^3}{R^3}. \tag{81}$$

Substituting this expression for $q_{\text{enc}}(r)$ into the equation for $\mathbf{E}_{\text{inner}}(r)$:

$$\mathbf{E}_{\text{inner}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \left( q\frac{r^3}{R^3} \right) \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}}. \tag{82}$$

The total field inside the sphere is $\mathbf{E}(r) = \mathbf{E}_{\text{inner}}(r) + \mathbf{E}_{\text{outer}}(r) = \mathbf{E}_{\text{inner}}(r)$.

Hence, the electric field of a uniformly charged solid sphere is given by the piecewise function:

$$\boxed{\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \dfrac{qr}{R^3} \hat{\mathbf{r}} & \text{for } r \leq R \text{ (inside)} \\[2ex] \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{r^2} \hat{\mathbf{r}} & \text{for } r \geq R \text{ (outside)} \end{cases}} \tag{83}$$

The field is continuous at the boundary $r = R$, where both expressions yield $|\mathbf{E}| = \frac{q}{4\pi\epsilon_0 R^2}$.
Here is the physical behavior that we expect :

- At the center $r = 0$, we have $E(0) = 0$.
- Inside the sphere the field grows **linearly** with $r$ (exactly like a Hooke-law restoring force).

- Outside, the field falls off as $1/r^2$, identical to a point charge $q$ located at the center.
- In the far-field limit $r \gg R$, the field reproduces Coulomb's law for the total charge $q$.

If we define the normalization constant to be

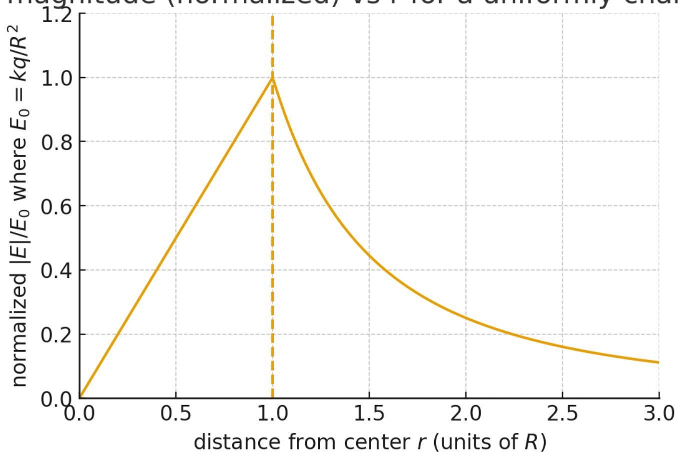
$$E_0 \equiv \frac{kq}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2},$$

then the dimensionless electric field magnitude is

$$\frac{|\mathbf{E}(r)|}{E_0} = \begin{cases} \dfrac{r}{R}, & 0 \le r \le R, \\ \dfrac{R^2}{r^2}, & r \ge R. \end{cases}$$

Graphing this dimensionless value will give a straight line rising from 0 at $r = 0$ to 1 at $r = R$, and then decaying as $1/r^2$ for $r > R$.

Electric field magnitude (normalized) vs r for a uniformly charged solid sphere




> **Problem 21: 4.10**
>
> A sphere of radius $R$ carries a polarization
>
> $$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$
>
> where $k$ is a constant and $\mathbf{r}$ is the vector from the center.
>
> 1. Calculate the bound charges $\sigma_b$ and $\rho_b$.
> 2. Find the field inside and outside the sphere.

**Solution:**

1. The polarization of a dielectric medium induces effective bound charges, described by a volume charge density $\rho_b$ and a surface charge density $\sigma_b$. These are defined as:

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{84}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \tag{85}$$

where $\hat{\mathbf{n}}$ is the outward unit normal to the dielectric surface.

**Bound Volume Charge Density, $\rho_b$:** The polarization vector is $\mathbf{P} = k\mathbf{r} = k(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})$. We compute its divergence:

$$\nabla \cdot \mathbf{P} = \nabla \cdot (k\mathbf{r}) = k(\nabla \cdot \mathbf{r}) = k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = k(1+1+1) = 3k. \tag{86}$$

The bound volume charge density is therefore a constant for any point within the sphere ($r < R$):

$$\rho_b = -3k \quad (\text{for } r < R). \tag{87}$$

For $r > R$, the polarization is zero, so $\rho_b = 0$.

**Bound Surface Charge Density, $\sigma_b$:** The surface of the dielectric is the sphere at $r = R$. The outward unit normal vector on this surface is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. The polarization evaluated on this surface is $\mathbf{P}(R) = k(R\hat{\mathbf{r}})$.

$$\sigma_b = \mathbf{P}(R) \cdot \hat{\mathbf{n}} = (kR\hat{\mathbf{r}}) \cdot (\hat{\mathbf{r}}) = kR. \tag{88}$$

The bound surface charge density is a constant over the spherical surface.

**Consistency Check:** The total bound charge for an isolated, neutral dielectric must be zero.

$$\begin{aligned} Q_{\text{total}} &= \int_V \rho_b \, dV + \oint_S \sigma_b \, dA \qquad (89)\\ &= (-3k)\left(\frac{4}{3}\pi R^3\right) + (kR)\left(4\pi R^2\right) \qquad (90)\\ &= -4\pi k R^3 + 4\pi k R^3 = 0. \qquad (91) \end{aligned}$$

Indeed, our calculation is self-consistent.

2. The electric field $\mathbf{E}$ is generated by the calculated bound charge distribution. Due to the spherical symmetry of both the volume and surface charge distributions (both are constant over their respective domains), the resulting electric field must be purely radial and its magnitude can only depend on the distance $r$ from the center, i.e., $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$. This symmetry makes the problem amenable to Gauss's Law:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}. \tag{92}$$

We choose a spherical Gaussian surface of radius $r$ concentric with the dielectric sphere.

**Field Inside the Sphere ($r < R$):** For a Gaussian surface of radius $r < R$, the enclosed charge $Q_{\text{enc}}$ consists only of the bound volume charge within that radius. The surface charge at $R$ is outside the Gaussian surface and, by the Shell Theorem, produces no field for $r < R$.

$$Q_{\text{enc}}(r) = \int_{V_r} \rho_b \, dV = (-3k)\left(\frac{4}{3}\pi r^3\right) = -4\pi k r^3. \tag{93}$$

Applying Gauss's Law:

$$\begin{aligned} \oint_S \mathbf{E} \cdot d\mathbf{a} = E(r) \cdot (4\pi r^2) = \frac{Q_{\text{enc}}}{\epsilon_0} &= \frac{-4\pi k r^3}{\epsilon_0} \qquad (94)\\ \implies E(r) = -\frac{kr}{\epsilon_0}. \qquad (95) \end{aligned}$$

In vector form, the electric field inside the sphere is:

$$\mathbf{E}(\mathbf{r}) = -\frac{k}{\epsilon_0}\mathbf{r} \quad (\text{for } r < R). \tag{96}$$

**Field Outside the Sphere ($r > R$):** For a Gaussian surface of radius $r > R$, the enclosed charge is the total bound charge of the entire sphere (volume plus surface), which we have shown to be zero.

$$Q_{\text{enc}} = Q_{\text{total}} = 0. \tag{97}$$

Applying Gauss's Law:

$$E(r) \cdot (4\pi r^2) = \frac{0}{\epsilon_0} = 0 \tag{98}$$

$$\implies E(r) = 0. \tag{99}$$

The electric field outside the sphere is zero.

$$\mathbf{E}(\mathbf{r}) = \mathbf{0} \quad (\text{for } r > R). \tag{100}$$

The field inside the sphere is a non-uniform field that points radially inward, opposing the polarization. The field is zero everywhere outside the sphere, consistent with the net neutrality of the isolated dielectric object.

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\frac{k}{\epsilon_0}\mathbf{r}, & r < R \\ \mathbf{0}, & r > R \end{cases} \tag{101}$$

**Problem 22: 4.11**

A short cylinder, of radius $a$ and length $L$, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis.

1. Find the bound charge
2. Sketch the electric field for (i) $L \gg a$, (ii) $L \ll a$, and (iii) for $L \approx a$.

**Solution:**

We consider a cylinder of radius $a$ and length $L$, with its axis along the z-axis and its geometric center at the origin. The cylinder carries a uniform, "frozen-in" polarization $\mathbf{P} = P\hat{\mathbf{z}}$ for $s \leq a$ and $|z| \leq L/2$, and $\mathbf{P} = \mathbf{0}$ elsewhere. The electric field $\mathbf{E}$ is generated by the bound charge distribution derived from $\mathbf{P}$.

1. The bound charge densities are given by the constitutive relations:

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{102}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \tag{103}$$

**Volume Bound Charge, $\rho_b$:** Since the polarization vector $\mathbf{P}$ is uniform (a constant vector field) within the dielectric, its divergence is identically zero.

$$\nabla \cdot \mathbf{P} = \frac{\partial P_s}{\partial s} + \frac{1}{s}\frac{\partial P_\phi}{\partial \phi} + \frac{\partial P_z}{\partial z} = \frac{\partial (0)}{\partial s} + \frac{1}{s}\frac{\partial (0)}{\partial \phi} + \frac{\partial P}{\partial z} = 0. \tag{104}$$

Thus, the bound volume charge density is zero everywhere: $\rho_b = 0$.

**Surface Bound Charge, $\sigma_b$:** The bound surface charge is evaluated on the three distinct surfaces of the cylinder:

**Top Face ($z = +L/2$):** The outward unit normal is $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$.

$$\sigma_{b,\text{top}} = (P\hat{\mathbf{z}}) \cdot (\hat{\mathbf{z}}) = P. \tag{105}$$

**Bottom Face ($z = -L/2$):** The outward unit normal is $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$.

$$\sigma_{b,\text{bottom}} = (P\hat{\mathbf{z}}) \cdot (-\hat{\mathbf{z}}) = -P. \tag{106}$$

**Side Wall ($s = a$):** The outward unit normal is the radial vector $\hat{\mathbf{s}}$.

$$\sigma_{b,\text{side}} = (P\hat{\mathbf{z}}) \cdot (\hat{\mathbf{s}}) = 0. \tag{107}$$

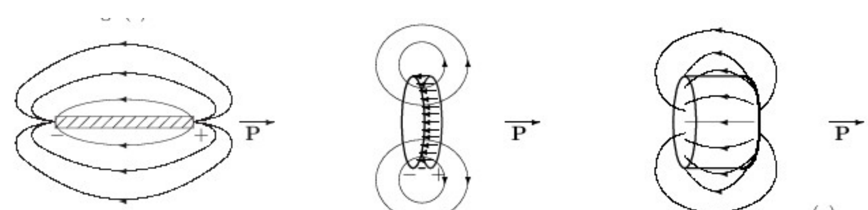
The polarized cylinder is therefore electrostatically equivalent to two oppositely charged disks. The total bound charge is $Q_b = P(\pi a^2) - P(\pi a^2) = 0$, as required for a neutral dielectric.

2. We just need to sketch the electric fields for the given cases.

$L \gg a$**:** The ends look like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$.

$L \ll a$**:** It should look like a circular parallel-plate capacitor. Field is nearly uniform inside while we have a nonuniform "fringing field" at the edges.

$L \approx a$**:** See image below




(a) Like a dipole (b) Like a parallel-plate capacitor (c)

**Optional calculations for the Electric Field Analysis:** Because all bound charge resides on the two end faces, the electric field of the polarized cylinder is exactly the same as the field produced by two circular disks of radius $a$ carrying uniform surface charge densities. We have the top disk at $z = +L/2$ with $+\sigma$ where $\sigma = P$,and the bottom disk at $z = -L/2$ with $-\sigma$ where $\sigma = P$.

Therefore we can compute the field by superposing the fields of these two disks. We will first derive the axial field (the field on the cylinder axis), as this is most useful for discussing the three regimes and for drawing field sketches. (Off-axis field can be found by integrating ring elements but is not required here.)

Recall (by integrating concentric ring elements or via a solid-angle argument) that a circular disk of radius $a$ in the plane $z = 0$ with uniform surface charge $\sigma$ produces an axial field at point $z > 0$:

$$E_{\text{disk}}(z) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right), \qquad \text{(pointing in the } +z \text{ direction).} \tag{108}$$

Then, we treat the end faces as two uniformly charged disks and superpose their fields. In particular, place the two disks at $z = \pm L/2$, with

$$\text{top disk: } z_1 = +L/2, \quad \sigma_1 = +P, \qquad \text{bottom disk: } z_2 = -L/2, \quad \sigma_2 = -P.$$

Let $z$ be the coordinate of the field point on the axis. Define

$$s_1 = |z - z_1| = \left| z - \frac{L}{2} \right|, \qquad s_2 = |z - z_2| = \left| z + \frac{L}{2} \right|.$$

After proper sign bookkeeping, the axial field (positive direction $= +z$) is

$$E_z(z) = \frac{P}{2\varepsilon_0} \left[ \left( \operatorname{sgn}\!\left( z + \frac{L}{2} \right) - \frac{z + \frac{L}{2}}{\sqrt{(z + \frac{L}{2})^2 + a^2}} \right) - \left( \operatorname{sgn}\!\left( z - \frac{L}{2} \right) - \frac{z - \frac{L}{2}}{\sqrt{(z - \frac{L}{2})^2 + a^2}} \right) \right],$$

where $\operatorname{sgn}(x)$ is the sign function $x/|x|$ for $x \neq 0$. This formula is valid for all $z$ and reduces to the usual piecewise expressions for regions inside and outside the cylinder.

At $z = 0$ (the midplane), the field is directed from the $+P$ face toward the $-P$ face. Its magnitude is

$$E_z(0) = \frac{P}{\varepsilon_0} \left( 1 - \frac{L/2}{\sqrt{(L/2)^2 + a^2}} \right).$$

This expression is especially convenient for analyzing limiting regimes.

**Long Cylinder, $L \gg a$.:** For $L \gg a$, the two charged disks are widely separated. Near the midplane each disk looks like a distant small source, and their fields almost cancel. Using the expansion

$$\sqrt{s^2 + a^2} \approx s\left( 1 + \frac{a^2}{2s^2} \right), \quad 1 - \frac{s}{\sqrt{s^2 + a^2}} \approx \frac{a^2}{2s^2}, \quad s = \frac{L}{2},$$

we find

$$E_z(0) \approx \frac{2Pa^2}{\varepsilon_0 L^2} \ll \frac{P}{\varepsilon_0}.$$

*Sketch:* Field lines emerge from the $+P$ face and spread radially outward before returning to the $-P$ face far away. The field is very weak in the central region (near $z = 0$), with strong fringing near each end only.

**Flat Disk ("Pancake"), $L \ll a$.:** For $L \ll a$, the two faces form an approximate parallel-plate capacitor. Between the faces, the field is nearly uniform:

$$E_{\text{between}} \approx \frac{P}{\varepsilon_0},$$

directed from the $+P$ face toward the $-P$ face. Using the center formula with $L/2 \ll a$ gives

$$E_z(0) \approx \frac{P}{\varepsilon_0}.$$

*Sketch:* Inside, field lines are almost straight and uniform, similar to a capacitor field. Outside, the field bulges out near the rim (fringing) and is weak elsewhere.

**Intermediate Case, $L \approx a$.:** When $L$ and $a$ are comparable, the field between the disks is neither negligible nor perfectly uniform. For instance, if $L = a$, then $s = L/2 = a/2$, giving

$$E_z(0) = \frac{P}{\varepsilon_0}\left(1 - \frac{1/2}{\sqrt{5/4}}\right) = \frac{P}{\varepsilon_0}\left(1 - \frac{1}{\sqrt{5}}\right) \approx 0.553\,\frac{P}{\varepsilon_0}.$$

*Sketch:* Field lines connect the $+P$ and $-P$ faces but curve noticeably outward. The central field is of order $(0.5)P/\varepsilon_0$, and fringing extends roughly one cylinder radius outward.

The on-axis field smoothly transitions from nearly uniform ($P/\varepsilon_0$) when $L \ll a$ to almost vanishing when $L \gg a$, with intermediate strength for $L \approx a$.

**Problem 23: 4.12**

Calculate the potential of a uniformly polarized sphere directly from the equation

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{r}}}{r^2} d\tau'.$$

**Solution:**

We want to obtain the electric potential $V(\mathbf{r})$ at an arbitrary field point $\mathbf{r}$ due to a sphere of radius $R$, centered at the origin, which carries a uniform polarization $\mathbf{P}$. The potential is given by the integral:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\, d\tau' \tag{109}$$

where $\mathbf{r}'$ is the position vector of a source point within the sphere, $\boldsymbol{\mathfrak{r}} = \mathbf{r} - \mathbf{r}'$ is the separation vector, and the integral is over the volume $\mathcal{V}$ of the sphere.

Without loss of generality, we align the z-axis with the direction of the uniform polarization, $\mathbf{P} = P\hat{\mathbf{z}}$. By rotational symmetry, the potential can only depend on the distance $r$ from the origin and the polar angle $\theta$ relative to the z-axis. We can therefore place the field point $\mathbf{r}$ on the z-axis ($\theta = 0$) to simplify the geometry of the integral, and then generalize the result.

- **Field Point:** $\mathbf{r} = r\hat{\mathbf{z}}$.
- **Source Point:** $\mathbf{r}'$ has spherical coordinates $(r', \theta', \phi')$.
- **Separation Vector Magnitude:** By the law of cosines, $|\boldsymbol{\mathfrak{r}}| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'}$.
- **Integrand Dot Product:**

$$\mathbf{P}\cdot\hat{\boldsymbol{\mathfrak{r}}} = (P\hat{\mathbf{z}})\cdot\frac{r\hat{\mathbf{z}} - \mathbf{r}'}{|\boldsymbol{\mathfrak{r}}|} = \frac{P(r - r'\cos\theta')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} \tag{110}$$

The potential integral becomes:

$$V(r) = \frac{P}{4\pi\epsilon_0}\int_0^R\int_0^\pi\int_0^{2\pi} \frac{r - r'\cos\theta'}{(r^2 + r'^2 - 2rr'\cos\theta')^{3/2}}\, r'^2 \sin\theta'\, d\phi'\, d\theta'\, dr' \tag{111}$$

The integral over $\phi'$ is trivial, yielding a factor of $2\pi$.

$$V(r) = \frac{P}{2\epsilon_0} \int_0^R r'^2 \left[ \int_0^\pi \frac{r - r' \cos\theta'}{(r^2 + r'^2 - 2rr' \cos\theta')^{3/2}} \sin\theta' \, d\theta' \right] dr' \tag{112}$$

Let the inner integral over $\theta'$ be denoted by $I(r, r')$.

For the angular integral $I(r, r')$, let $u = \cos\theta'$, so $du = -\sin\theta' \, d\theta'$.

$$I(r, r') = \int_{-1}^1 \frac{r - r'u}{(r^2 + r'^2 - 2rr'u)^{3/2}} \, du \tag{113}$$

This integral can be evaluated by parts. Let $A = r^2 + r'^2$ and $B = 2rr'$. Note that $\frac{d}{du}(A - Bu)^{-1/2} = \frac{B}{2}(A - Bu)^{-3/2}$.

$$\begin{aligned}
I(r, r') &= \frac{2}{B} \int_{-1}^1 (r - r'u) \frac{d}{du} \left( (A - Bu)^{-1/2} \right) du && (114) \\
&= \frac{2}{B} \left[ \frac{r - r'u}{\sqrt{A - Bu}} \right]_{-1}^1 - \frac{2}{B} \int_{-1}^1 \frac{-r'}{\sqrt{A - Bu}} du && (115) \\
&= \frac{1}{rr'} \left( \frac{r - r'}{|r - r'|} - 1 \right) + \frac{1}{r^2 r'} \left( |r - r'| - (r + r') \right) && (116)
\end{aligned}$$

We must now consider two cases for the absolute value:

$r > r'$**:** In this case, $|r - r'| = r - r'$.

$$I(r, r') = \frac{1}{rr'}(1 - 1) + \frac{1}{r^2 r'}(r + r' - (r - r')) = 0 + \frac{2r'}{r^2 r'} = \frac{2}{r^2}$$

$r < r'$**:** Here, $|r - r'| = r' - r$.

$$I(r, r') = \frac{1}{rr'}(-1 - 1) + \frac{1}{r^2 r'}(r + r' - (r' - r)) = \frac{-2}{rr'} + \frac{2r}{r^2 r'} = 0$$

The angular integral $I(r, r')$ thus has a simple piecewise form:

$$I(r, r') = \begin{cases} 2/r^2, & \text{if } r' < r \\ 0, & \text{if } r' > r \end{cases} \tag{117}$$

We substitute this result back into the expression for $V(r)$. We have two cases for the calculation based on the field point $r$:

**Outside the Sphere (**$r > R$**):** In this case, $r$ is always greater than the integration variable $r'$ (since $0 \le r' \le R$). So we only use the first part of our result for $I(r, r')$.

$$V(r) = \frac{P}{2\epsilon_0} \int_0^R r'^2 \left( \frac{2}{r^2} \right) dr' = \frac{P}{\epsilon_0 r^2} \int_0^R r'^2 dr' = \frac{P}{\epsilon_0 r^2} \left[ \frac{r'^3}{3} \right]_0^R = \frac{PR^3}{3\epsilon_0 r^2} \tag{118}$$

**Inside the Sphere ($r < R$):** In this case, the integration variable $r'$ will be both smaller and larger than the fixed value $r$. We must split the integral:

$$V(r) = \frac{P}{2\epsilon_0}\left(\int_0^r r'^2 I(r,r')dr' + \int_r^R r'^2 I(r,r')dr'\right) \tag{119}$$

$$= \frac{P}{2\epsilon_0}\left(\int_0^r r'^2 \left(\frac{2}{r^2}\right) dr' + \int_r^R r'^2 (0) dr'\right) = \frac{P}{\epsilon_0 r^2}\int_0^r r'^2 dr' = \frac{Pr}{3\epsilon_0} \tag{120}$$

We found the potential for a field point on the z-axis ($\mathbf{r} = r\hat{\mathbf{z}}$). We generalize this result to an arbitrary field point $\mathbf{r} = (r, \theta, \phi)$ by replacing $r$ with $r$ and $Pr$ with $Pr\cos\theta = \mathbf{P}\cdot\mathbf{r}$.

- **Inside ($r \le R$):** $V(r) = \frac{Pr}{3\epsilon_0}$ becomes $\boxed{V(\mathbf{r}) = \frac{Pr\cos\theta}{3\epsilon_0}} = \frac{\mathbf{P}\cdot\mathbf{r}}{3\epsilon_0}$.

- **Outside ($r \ge R$):** $V(r) = \frac{PR^3}{3\epsilon_0 r^2}$ becomes $\boxed{V(\mathbf{r}) = \frac{PR^3\cos\theta}{3\epsilon_0 r^2}}$. The total dipole moment is $\mathbf{p} = \mathbf{P}(\frac{4}{3}\pi R^3)$. The potential can be written as $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$.

The final result for an arbitrary field point $\mathbf{r}$ is:

$$\boxed{V(\mathbf{r}) = \begin{cases} \dfrac{\mathbf{P}\cdot\mathbf{r}}{3\epsilon_0}, & r \le R \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, & r \ge R \end{cases}} \tag{121}$$

where $\mathbf{p} = \frac{4}{3}\pi R^3\mathbf{P}$ is the total electric dipole moment of the sphere. The external potential is that of a perfect point dipole at the origin. The internal potential corresponds to a uniform electric field $\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}} = -\frac{\mathbf{P}}{3\epsilon_0}$.

**Alternative Solution:** We seek to calculate the electric potential $V(\mathbf{r})$ at an arbitrary field point $\mathbf{r}$ due to a sphere of radius $R$, centered at the origin, which carries a uniform polarization $\mathbf{P}$. The potential is given by the integral:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\,d\tau' \tag{122}$$

where $\mathbf{r}'$ is the position vector of a source point within the sphere, $\boldsymbol{\mathfrak{r}} = \mathbf{r} - \mathbf{r}'$ is the separation vector, and the integral is over the volume $\mathcal{V}$ of the sphere.

Since the polarization $\mathbf{P}$ is uniform (constant in magnitude and direction) throughout the sphere, we can factor it out of the integral using the linearity of the dot product and the integral operator:

$$V(\mathbf{r}) = \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\,d\tau'\right\} \tag{123}$$

We now focus on the term in the curly brackets. Let us call this vector quantity $\mathbf{A}(\mathbf{r})$. Recall that the electric field $\mathbf{E}_\rho$ produced by a volume charge distribution $\rho(\mathbf{r}')$ is given by:

$$\mathbf{E}_\rho(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\,d\tau' \tag{124}$$

If we consider a hypothetical solid sphere of radius $R$ filled with a uniform *charge* density $\rho_0$, its electric field would be:

$$\mathbf{E}_{\rho_0}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_0 \hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\,d\tau' = \rho_0 \left\{ \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathfrak{r}}}}{\mathfrak{r}^2}\,d\tau' \right\} = \rho_0 \mathbf{A}(\mathbf{r}) \tag{125}$$

Therefore, the vector integral $\mathbf{A}(\mathbf{r})$ *is precisely the electric field of a uniformly charged solid sphere, divided by the constant charge density* $\rho_0$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{E}_{\rho_0}(\mathbf{r})}{\rho_0} \tag{126}$$

Also recall that the electric field of a solid sphere of radius $R$ with uniform charge density $\rho_0$ is a standard result from electrostatics, typically derived using Gauss's Law. The total charge is $q = \rho_0(\frac{4}{3}\pi R^3)$. The field is:

$$\mathbf{E}_{\rho_0}(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{qr}{R^3}\hat{\mathbf{r}} = \dfrac{\rho_0 r}{3\epsilon_0}\hat{\mathbf{r}} = \dfrac{\rho_0}{3\epsilon_0}\mathbf{r}, & r \le R \text{ (inside)} \\ \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} = \dfrac{\rho_0 R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}}, & r \ge R \text{ (outside)} \end{cases} \tag{127}$$

Anymore, we can evaluate the potential $V(\mathbf{r}) = \mathbf{P}\cdot\mathbf{A}(\mathbf{r})$ by dividing the field $\mathbf{E}_{\rho_0}$ by $\rho_0$ and taking the dot product with $\mathbf{P}$. We have two cases for this calculation:

**Outside the Sphere ($r > R$):**

$$\mathbf{A}_{\text{out}}(\mathbf{r}) = \frac{\mathbf{E}_{\rho_0,\text{out}}(\mathbf{r})}{\rho_0} = \frac{1}{\rho_0}\left(\frac{\rho_0 R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}}\right) = \frac{R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}} \tag{128}$$

The potential is:

$$V_{\text{out}}(\mathbf{r}) = \mathbf{P}\cdot\mathbf{A}_{\text{out}}(\mathbf{r}) = \mathbf{P}\cdot\left(\frac{R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}}\right) = \frac{R^3}{3\epsilon_0 r^2}(\mathbf{P}\cdot\hat{\mathbf{r}}) \tag{129}$$

The total dipole moment of the sphere is $\mathbf{p} = \mathbf{P} \times \text{Volume} = \mathbf{P}(\frac{4}{3}\pi R^3)$. We can rewrite the potential as:

$$V_{\text{out}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\frac{4}{3}\pi R^3 \mathbf{P}}{r^2}\cdot\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \tag{130}$$

**Inside the Sphere ($r < R$):**

$$\mathbf{A}_{\text{in}}(\mathbf{r}) = \frac{\mathbf{E}_{\rho_0,\text{in}}(\mathbf{r})}{\rho_0} = \frac{1}{\rho_0}\left(\frac{\rho_0}{3\epsilon_0}\mathbf{r}\right) = \frac{1}{3\epsilon_0}\mathbf{r} \tag{131}$$

The potential is:

$$V_{\text{in}}(\mathbf{r}) = \mathbf{P}\cdot\mathbf{A}_{\text{in}}(\mathbf{r}) = \mathbf{P}\cdot\left(\frac{1}{3\epsilon_0}\mathbf{r}\right) = \frac{\mathbf{P}\cdot\mathbf{r}}{3\epsilon_0} \tag{132}$$

Therefore, the potential of a uniformly polarized sphere is:

$$\boxed{V(\mathbf{r}) = \begin{cases} \dfrac{\mathbf{P}\cdot\mathbf{r}}{3\epsilon_0}, & r \le R \text{ (inside)} \\ \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, & r \ge R \text{ (outside)} \end{cases}} \tag{133}$$

where $\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}$ is the total electric dipole moment of the sphere. The external potential is that of a perfect point dipole at the origin, and the internal potential corresponds to a uniform electric field $\mathbf{E}_{\text{in}} = -\nabla V_{\text{in}} = -\frac{\mathbf{P}}{3\epsilon_0}$.

**Problem 24: 4.13**

A very long cylinder, of radius $a$, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis.

1. Find the electric field inside the cylinder.

2. Show that the field outside the cylinder can be expressed in the form

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}].$$

**Solution:**

Let the infinitely long cylinder have radius $a$ and its axis along the $z$-axis. Let the polarization $\mathbf{P}$ be uniform and directed along a fixed transverse unit vector $\hat{\mathbf{u}}$ (say, along the $x$-axis):

$$\mathbf{P} = P\,\hat{\mathbf{u}}.$$

We model the polarized cylinder as the superposition of two infinite, uniformly charged cylinders of equal and opposite volume charge densities $\pm\rho$, each of radius $a$, whose axes are displaced by a small vector $\mathbf{d} = d\,\hat{\mathbf{u}}$ in the direction of $\mathbf{P}$. The idea is that the pair has zero net charge but a nonzero dipole moment density corresponding to the polarization. In the limit $d \to 0$ while keeping

$$\rho\,\mathbf{d} = \mathbf{P},$$

the superposition reproduces the macroscopic polarization.

From Gauss's law, for an infinite cylinder of radius $a$ and uniform charge density $\rho$, centered on the $z$-axis, the electric field is purely radial:

$$\mathbf{E}_{\text{cyl}}(s) = \begin{cases} \dfrac{\rho}{2\varepsilon_0}\,\mathbf{s}, & s < a, \\ \\ \dfrac{\rho a^2}{2\varepsilon_0}\,\dfrac{\mathbf{s}}{s^2}, & s > a, \end{cases}$$

where $\mathbf{s}$ is the transverse position vector from the cylinder axis and $s = |\mathbf{s}|$.

Let the $+$ cylinder (charge $+\rho$) be centered at $\mathbf{r}_+ = \frac{1}{2}\mathbf{d}$ and the $-$ cylinder (charge $-\rho$) at $\mathbf{r}_- = -\frac{1}{2}\mathbf{d}$. For an observation point at transverse position $\mathbf{s}$ measured from the midpoint (the origin), define

$$\mathbf{s}_+ = \mathbf{s} - \tfrac{1}{2}\mathbf{d}, \qquad \mathbf{s}_- = \mathbf{s} + \tfrac{1}{2}\mathbf{d},$$

with magnitudes $s_\pm = |\mathbf{s}_\pm|$.

The total field is the vector sum

$$\mathbf{E}(\mathbf{s}) = \mathbf{E}_+(\mathbf{s}_+) + \mathbf{E}_-(\mathbf{s}_-),$$

where $\mathbf{E}_\pm$ are the fields due to each cylinder, using the appropriate inside/outside formula.

**Field inside the polarized cylinder ($s < a$):** For $s$ small enough that both $\mathbf{s}_+$ and $\mathbf{s}_-$ are within the interior of each cylinder, use the inside-field expression for both:

$$\mathbf{E}(\mathbf{s}) = \frac{\rho}{2\varepsilon_0}(\mathbf{s}_+ - \mathbf{s}_-) = \frac{\rho}{2\varepsilon_0}\big[(\mathbf{s} - \tfrac{1}{2}\mathbf{d}) - (\mathbf{s} + \tfrac{1}{2}\mathbf{d})\big] = -\frac{\rho}{2\varepsilon_0}\,\mathbf{d}.$$

Using $\rho\,\mathbf{d} = \mathbf{P}$, this gives the *uniform internal field*

$$\boxed{\mathbf{E}_{\text{in}} = -\frac{\mathbf{P}}{2\varepsilon_0}, \qquad (s < a).}$$

**Field outside the polarized cylinder ($s > a$):** For $s$ large enough that both field points lie outside the individual cylinders, use the outside-field formula:

$$\mathbf{E}(\mathbf{s}) = \frac{\rho a^2}{2\varepsilon_0}\left(\frac{\mathbf{s}_+}{s_+^2} - \frac{\mathbf{s}_-}{s_-^2}\right). \tag{1}$$

We expand to first order in $\mathbf{d}$. Let $\mathbf{s}_\pm = \mathbf{s} \mp \frac{1}{2}\mathbf{d}$ and $s_\pm^2 = s^2 \mp \mathbf{s}\cdot\mathbf{d} + O(d^2)$. For any small $\boldsymbol{\delta}$,

$$\frac{\mathbf{s} + \boldsymbol{\delta}}{|\mathbf{s} + \boldsymbol{\delta}|^2} = \frac{\mathbf{s}}{s^2} + \frac{\boldsymbol{\delta}}{s^2} - 2\,\frac{\mathbf{s}(\mathbf{s}\cdot\boldsymbol{\delta})}{s^4} + O(\delta^2).$$

Applying this with $\boldsymbol{\delta}_+ = -\frac{1}{2}\mathbf{d}$ and $\boldsymbol{\delta}_- = +\frac{1}{2}\mathbf{d}$, we find:

$$\begin{aligned}
\frac{\mathbf{s}_+}{s_+^2} - \frac{\mathbf{s}_-}{s_-^2} &= \left[\frac{\mathbf{s}}{s^2} + \frac{\boldsymbol{\delta}_+}{s^2} - 2\frac{\mathbf{s}(\mathbf{s}\cdot\boldsymbol{\delta}_+)}{s^4}\right] - \left[\frac{\mathbf{s}}{s^2} + \frac{\boldsymbol{\delta}_-}{s^2} - 2\frac{\mathbf{s}(\mathbf{s}\cdot\boldsymbol{\delta}_-)}{s^4}\right] + O(d^2) \\
&= \frac{\boldsymbol{\delta}_+ - \boldsymbol{\delta}_-}{s^2} - 2\,\frac{\mathbf{s}\big[(\mathbf{s}\cdot\boldsymbol{\delta}_+) - (\mathbf{s}\cdot\boldsymbol{\delta}_-)\big]}{s^4} + O(d^2) \\
&= -\frac{\mathbf{d}}{s^2} + 2\,\frac{\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^4} + O(d^2).
\end{aligned}$$

Substituting this into (1):

$$\mathbf{E}(\mathbf{s}) = \frac{\rho a^2}{2\varepsilon_0}\left[\frac{2\,\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^4} - \frac{\mathbf{d}}{s^2}\right] + O(d^2).$$

Using $\rho\,\mathbf{d} = \mathbf{P}$, we obtain

$$\mathbf{E}(\mathbf{s}) = \frac{a^2}{2\varepsilon_0}\left[\frac{2\,\mathbf{s}(\mathbf{s}\cdot\mathbf{P})}{s^4} - \frac{\mathbf{P}}{s^2}\right] + O(d^2).$$

Expressing $\hat{\mathbf{s}} = \mathbf{s}/s$,

$$\frac{2\,\mathbf{s}(\mathbf{s}\cdot\mathbf{P})}{s^4} - \frac{\mathbf{P}}{s^2} = \frac{1}{s^2}\big[\,2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}\big].$$

Hence, the *external field* is

$$\boxed{\mathbf{E}_{\text{out}}(\mathbf{r}) = \frac{a^2}{2\varepsilon_0 s^2}\big[\,2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}\big], \qquad (s > a).}$$

These two results agree with the physical picture: the inside field is uniform and opposite to $\mathbf{P}$, while the outside field decays as $1/s^2$ and has the angular structure characteristic of a dipole sheet extended along the cylinder axis.

**Alternative solution:** We choose Cartesian and cylindrical coordinates so that the cylinder axis is the $z$-axis and the polarization $\mathbf{P}$ lies in the $x$-direction:

$$\mathbf{P} = P\,\hat{\mathbf{x}}, \qquad P = \text{constant}.$$

Because the cylinder is infinitely long and the sources (bound charges) are translation-invariant in $z$, all fields are independent of $z$. By linearity and symmetry the potential (and fields) depend only on the transverse (planar) coordinates $(s, \phi)$ and have an angular dependence proportional to $\cos\phi$ (the same symmetry as $P_x$). We therefore reduce the problem to a two-dimensional Laplace problem in the transverse plane with azimuthal dependence $\cos\phi$.

The bound volume charge density is

$$\rho_b = -\nabla\cdot\mathbf{P} = 0$$

because $\mathbf{P}$ is uniform. The only bound charge is a surface charge on the curved surface $s = a$:

$$\sigma_b(\phi) = \mathbf{P}\cdot\hat{\mathbf{n}} = P\cos\phi,$$

where $\hat{\mathbf{n}} = \hat{\mathbf{s}}$ is the outward normal of the cylinder surface and $\phi$ is the usual azimuthal angle measured from the $x$-axis.

Hence the problem reduces to finding the electrostatic potential produced by the surface charge $\sigma_b(\phi) = P\cos\phi$ on the infinite cylindrical surface $s = a$.

Because the problem is independent of $z$ and has $\cos\phi$ angular dependence, seek a scalar potential $V(s, \phi)$ of the separable form

$$V(s, \phi) = R(s)\cos\phi.$$

In the transverse plane Laplace's equation (in cylindrical coordinates, with no $z$-dependence) is

$$\frac{1}{s}\frac{d}{ds}\Big(s\frac{dR}{ds}\Big) - \frac{1}{s^2}R = 0.$$

This is the Bessel/regular equation for angular index $m = 1$. Its general solution is

$$R(s) = As^1 + Bs^{-1},$$

so the general $\cos\phi$-mode potentials are

$$V_{\text{in}}(s, \phi) = As\cos\phi, \qquad (0 \le s \le a)$$

$$V_{\text{out}}(s, \phi) = \frac{B}{s}\cos\phi, \qquad (s \ge a).$$

Regularity at the axis $s = 0$ requires no singular term inside, so the interior solution contains only the $s^1$ term. Vanishing at infinity of the potential requires the $s$ term outside vanish, so the exterior solution is the $1/s$ term. Thus the two-parameter form above is correct.

Two boundary conditions at the surface $s = a$ determine $A$ and $B$:

**(i) Continuity of potential:**

$$V_{\text{in}}(a, \phi) = V_{\text{out}}(a, \phi) \quad \Rightarrow \quad Aa\cos\phi = \frac{B}{a}\cos\phi$$

so

$$B = Aa^2. \tag{1}$$

**(ii) Discontinuity of normal electric displacement / relation to surface charge:**

The normal component of the electric field is $E_s = -\partial_s V$. The boundary condition coming from Gauss' law across a surface charge gives

$$\varepsilon_0\big(E_s^{\text{out}} - E_s^{\text{in}}\big) = \sigma_b(\phi).$$

Substitute $E_s = -\partial_s V$ and evaluate the radial derivatives:

$$\partial_s V_{\text{in}}(s, \phi) = A\cos\phi, \qquad \partial_s V_{\text{out}}(s, \phi) = -\frac{B}{s^2}\cos\phi.$$

Thus at $s = a$

$$\varepsilon_0\Big(-\frac{B}{a^2}\cos\phi - A\cos\phi\Big) = \sigma_b(\phi) = P\cos\phi.$$

Careful with signs: $E_s^{\text{out}} = -\partial_s V_{\text{out}} = -(-B/s^2\cos\phi) = +\frac{B}{s^2}\cos\phi$, and $E_s^{\text{in}} = -\partial_s V_{\text{in}} = -A\cos\phi$. Using the original (standard) form $\varepsilon_0(E_s^{\text{out}} - E_s^{\text{in}}) = \sigma_b$ yields

$$\varepsilon_0\Big(\frac{B}{a^2}\cos\phi - (-A\cos\phi)\Big) = P\cos\phi,$$

i.e.

$$\varepsilon_0\big(A + \frac{B}{a^2}\big) = P. \tag{2}$$

(One can check sign consistency by direct substitution; the result below agrees.)

Substitute (1) into (2):

$$\varepsilon_0\big(A + \frac{Aa^2}{a^2}\big) = \varepsilon_0(2A) = P \quad \Rightarrow \quad A = \frac{P}{2\varepsilon_0}.$$

Hence from (1)

$$B = Aa^2 = \frac{Pa^2}{2\varepsilon_0}.$$

**Interior field.** The interior potential is therefore

$$V_{\text{in}}(s, \phi) = \frac{P}{2\varepsilon_0}\, s\cos\phi.$$

Observe that $s\cos\phi = x$ (Cartesian $x$-coordinate). Thus

$$V_{\text{in}}(\mathbf{r}) = \frac{P}{2\varepsilon_0}\, x.$$

The electric field is $\mathbf{E} = -\nabla V$, so

$$\boxed{\mathbf{E}_{\text{in}} = -\frac{P}{2\varepsilon_0}\,\hat{\mathbf{x}}.}$$

Recalling $\mathbf{P} = P\hat{\mathbf{x}}$, we may write this in the vector form valid for any orientation of the uniform transverse $\mathbf{P}$:

$$\boxed{\mathbf{E}_{\text{in}} = -\frac{\mathbf{P}}{2\varepsilon_0} \qquad (s < a).}$$

Thus the interior field is uniform, opposite to the polarization, and of magnitude $P/(2\varepsilon_0)$.

**Exterior potential and field.** The exterior potential is

$$V_{\text{out}}(s,\phi) = \frac{B}{s}\cos\phi = \frac{Pa^2}{2\varepsilon_0}\cdot\frac{\cos\phi}{s}.$$

Write $\cos\phi = \hat{\mathbf{s}}\cdot\hat{\mathbf{x}} = (\hat{\mathbf{s}}\cdot\mathbf{P})/P$ and note $\hat{\mathbf{s}}\,(\hat{\mathbf{s}}\cdot\mathbf{P}) = (\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}}$. It is convenient to express the potential and field in coordinate-free vector form. First observe the scalar identity in the transverse plane:

$$\frac{\cos\phi}{s} = \frac{\hat{\mathbf{s}}\cdot\hat{\mathbf{x}}}{s} = \frac{\hat{\mathbf{s}}\cdot\mathbf{P}}{P}\cdot\frac{1}{s} = \frac{\mathbf{P}\cdot\hat{\mathbf{s}}}{Ps}.$$

Thus

$$V_{\text{out}}(\mathbf{r}) = \frac{a^2}{2\varepsilon_0}\frac{\mathbf{P}\cdot\hat{\mathbf{s}}}{s}.$$

Now compute the electric field in the transverse plane from $V_{\text{out}}$. Because everything is $z$-independent, the full three-dimensional field has no $z$-component outside (only transverse components). We compute $\mathbf{E}_{\text{out}} = -\nabla_\perp V_{\text{out}}$. Use the vector identity (valid in the transverse plane)

$$\nabla_\perp\Big(\frac{\mathbf{P}\cdot\hat{\mathbf{s}}}{s}\Big) = -\frac{1}{s^2}\big[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}\big],$$

which can be checked by writing $\hat{\mathbf{s}} = \mathbf{s}/s$ and differentiating componentwise or by converting to Cartesian components. (A brief check: in Cartesian coordinates $x = s\cos\phi,\ y = s\sin\phi$, and the expression reduces to standard derivatives of $x/s^2$ etc.; the identity is algebraically equivalent to the gradient of $x/s^2$.)

Therefore

$$\mathbf{E}_{\text{out}}(\mathbf{r}) = -\nabla_\perp V_{\text{out}} = -\frac{a^2}{2\varepsilon_0}\nabla_\perp\Big(\frac{\mathbf{P}\cdot\hat{\mathbf{s}}}{s}\Big) = \frac{a^2}{2\varepsilon_0 s^2}\big[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}\big].$$

Hence we obtain the desired exterior field representation:

$$\boxed{\mathbf{E}_{\text{out}}(\mathbf{r}) = \frac{a^2}{2\varepsilon_0 s^2}\big[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}\big], \qquad (s \geq a).}$$

**Comments and checks.**

- The interior result $\mathbf{E}_{\text{in}} = -\mathbf{P}/(2\varepsilon_0)$ is uniform and independent of $s$, as required by the translational symmetry along the axis and the uniformity of the polarization.

- On the surface $s = a$ the radial component of $\mathbf{E}$ exhibits the correct jump to produce the surface charge $\sigma_b = P\cos\phi$: the radial (normal) component of $\varepsilon_0\mathbf{E}$ discontinuity equals $\sigma_b$.
- If we choose $\mathbf{P}$ in an arbitrary transverse direction, the formulas are unchanged by replacing $\hat{\mathbf{x}}$ by $\mathbf{P}/P$ because the problem is linear in $\mathbf{P}$.
- In particular, for points on the $x$-axis ($\hat{\mathbf{s}} = \hat{\mathbf{x}}$) the exterior field reduces to

$$\mathbf{E}_{\text{out}} = \frac{a^2}{2\varepsilon_0 s^2}(2P\hat{\mathbf{x}} - \mathbf{P}) = \frac{a^2 P}{2\varepsilon_0 s^2}\hat{\mathbf{x}},$$

i.e. $\mathbf{E}_{\text{out}}$ points along $\mathbf{P}$ on the forward symmetry axis as expected.

> **Problem 25: 4.14**
>
> When you polarize a neutral dielectric, the charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges $\sigma_b$ and $\rho_b$. Prove from Eqs. 4.11 ($\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$) and 4.12 ($\rho_b \equiv -\nabla\cdot\mathbf{P}$) that the total bound charge vanishes.

**Solution:**

We are given a finite volume $\mathcal{V}$ of a dielectric material, bounded by a closed surface $\mathcal{S}$. The material has a polarization vector field $\mathbf{P}(\mathbf{r})$, which is assumed to be zero everywhere outside of $\mathcal{V}$. The polarization gives rise to a bound volume charge density $\rho_b$ within the material and a bound surface charge density $\sigma_b$ on its boundary. These are defined as:

$$\rho_b \equiv -\nabla\cdot\mathbf{P} \tag{134}$$

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \tag{135}$$

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector to the surface $\mathcal{S}$.

Our objective is to prove that the total bound charge, $Q_{\text{bound}}$, which is the sum of the total volume charge and the total surface charge, is identically zero:

$$Q_{\text{bound}} = \int_{\mathcal{V}} \rho_b\, d\tau + \oint_{\mathcal{S}} \sigma_b\, da = 0 \tag{136}$$

**Proof** We begin by substituting the definitions of $\rho_b$ and $\sigma_b$ from Equations (134) and (135) into the expression for the total bound charge:

$$Q_{\text{bound}} = \int_{\mathcal{V}} (-\nabla\cdot\mathbf{P})\, d\tau + \oint_{\mathcal{S}} (\mathbf{P}\cdot\hat{\mathbf{n}})\, da \tag{137}$$

The first term is the volume integral of the divergence of the vector field $\mathbf{P}$. The second term is the surface integral of the normal component of $\mathbf{P}$ over the surface that bounds the volume.

Recall that the **Divergence Theorem** says that for any continuously differentiable vector field $\mathbf{F}$ defined on a volume $\mathcal{V}$ with boundary $\mathcal{S}$, the following relation holds:

$$\int_{\mathcal{V}} (\nabla\cdot\mathbf{F})\, d\tau = \oint_{\mathcal{S}} (\mathbf{F}\cdot\hat{\mathbf{n}})\, da \tag{138}$$

The theorem equates the volume integral of the divergence of a vector field to the flux of that field through the closed surface bounding the volume.

Applying the Divergence Theorem to our polarization vector field $\mathbf{P}$, with $\mathbf{F} = \mathbf{P}$, gives:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{P})\, d\tau = \oint_{\mathcal{S}} (\mathbf{P} \cdot \hat{\mathbf{n}})\, da \tag{139}$$

Now, we can substitute this result into our expression for the total bound charge:

$$Q_{\text{bound}} = -\left(\int_{\mathcal{V}} (\nabla \cdot \mathbf{P})\, d\tau\right) + \left(\oint_{\mathcal{S}} (\mathbf{P} \cdot \hat{\mathbf{n}})\, da\right) \tag{140}$$

$$= -\left(\oint_{\mathcal{S}} (\mathbf{P} \cdot \hat{\mathbf{n}})\, da\right) + \left(\oint_{\mathcal{S}} (\mathbf{P} \cdot \hat{\mathbf{n}})\, da\right) \tag{141}$$

The two terms are identical in magnitude and opposite in sign. Therefore, they sum to zero:

$$Q_{\text{bound}} = 0 \tag{142}$$

This completes the proof.

**Physical Interpretation.** The polarization of a neutral dielectric represents a microscopic re-arrangement of charge (a stretching of dipoles or alignment of existing ones), not the creation of new charge. Any charge that is moved away from a point within the volume (creating a net charge density $\rho_b$) must accumulate elsewhere, either at another point within the volume or at the surface (contributing to $\sigma_b$). The Divergence Theorem provides the rigorous mathematical framework to show that this charge rearrangement is perfectly balanced, ensuring that the net charge of the isolated object remains zero, as required by the principle of charge conservation.

# References