Homotopical Algebra and Homological Algebra

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1 Introduction

In algebraic topology, we have seen the following analogy between topology and homological algebra.

Topology	**Homological Algebra**
spaces	chain complexes over R
homotopies	chain homotopies
homotopy equivalences	chain homotopy equivalences
homotopy groups $\pi_n(X) \simeq [S^n, X]$	homology groups $H^n(X) \simeq [\text{“}S^n\text{”}, X]$
weak homotopy equivalences	quasi-isomorphisms
CW approximation	projective/injective resolutions
homotopy category $\mathsf{h}\mathsf{CW}$	derived category $\mathsf{D}(R)$
suspension and looping $\Sigma \dashv \Omega$	shifting $[1] \dashv [-1]$
......	......

(R is a commutative ring; [,] denotes the set of homotopy classes of maps; and “S^n” denotes the chain complex whose only non-zero term is R at its n-th place.)

Homotopical algebra is a language that unifies these two theories. It is a general theory that also applies to many other situations.

What is an ∞-category?

The language of ∞-categories is the modern language for homotopical algebra. It encodes data describing the “higher structures” of a category that can not be seen in ordinary category theory.

Roughly speaking, an ∞-category consists of a class of objects, a class of morphisms between them, and moreover, there are higher morphisms between these morphisms. For example, there are 2-morphisms between ordinary morphisms, which can be seen as “homotopies” between morphisms. There are 3-morphisms between 2-morphisms, which can be seen as “homotopies between homotopies”, and so on.

We will not give the definition for an ∞-category until a few sections later, since defining them requires some work. However, the idea of ∞-categories can be seen through the following examples of ∞-categories.

Example 1.1. Consider the category Top of topological spaces. Seen as an ∞-category, it will consist of the following data.

- Objects: topological spaces.
- 1-Morphisms: continuous maps between topological spaces.
- 2-Morphisms (between 1-morphisms): homotopies between two maps with the same source and target.
- 3-Morphisms (between 2-morphisms): homotopies between homotopies of maps.
- ...... $\triangleleft$

Example 1.2. For a commutative ring R, consider the category Ch_R of cochain complexes over R. As an ∞-category, it will consist of the following data.

- Objects: cochain complexes over R.
- 1-Morphisms: chain maps between cochain complexes.
- 2-Morphisms: chain homotopies between chain maps.
- 3-Morphisms: chain homotopies between chain homotopies.
- ...... $\triangleleft$

Example 1.3. For a topological space X, consider the fundamental groupoid of X, denoted by $\Pi(X)$. As an ∞-category, it will consist of the following data.

- Objects: points of X.
- 1-Morphisms: paths connecting two points.
- 2-Morphisms: homotopies between paths, fixing endpoints.
- 3-Morphisms: homotopies between homotopies.
- ......

Note that the associativity law $f \circ (g \circ h) = (f \circ g) \circ h$ does not hold in this example; it only holds "up to homotopy". This is one of the difficulties we will encounter in studying ∞-categories. $\triangleleft$

Homotopy categories

In many cases, ∞-categories arise as homotopy categories of ordinary categories with certain extra data. We will now demonstrate this procedure through a concrete example.

Definition 1.4. *The category* hTop *consists of*

- *Objects: topological spaces.*
- *Morphisms: homotopy classes of maps.*

A key observation is that hTop is obtained from Top by "inverting the homotopy equivalences". Let us make this precise.

Definition 1.5. *A* **category with weak equivalences** *is a pair* (C, W)*, where* C *is a category, and* $\mathsf{W} \subset \operatorname{Mor}(\mathsf{C})$ *is a class of morphisms, such that*

- *All isomorphisms of* C *are in* W*.*
- W *satisfies* **two-out-of-three***: for any diagram*

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & \bullet \end{tikzcd}$$

in C*, if two of the arrows* $f, g, g \circ f$ *are in* W*, then so is the third.*

For example, the pair $(\mathsf{Top}, \mathsf{HoEq})$ is a category with weak equivalences, where HoEq is the class of homotopy equivalences in Top.

Definition 1.6. *Let* (C, W) *be a category with weak equivalences. The* **localisation** *of* C *with respect to* W *is a category* $\mathsf{C}[\mathsf{W}^{-1}]$*, together with a functor* $\mathsf{C} \to \mathsf{C}[\mathsf{W}^{-1}]$*, with the following universal property:*

- *For any functor* $F : \mathsf{C} \to \mathsf{D}$ *sending* W *to isomorphisms, there is a unique functor* $\tilde{F} : \mathsf{C}[\mathsf{W}^{-1}] \to \mathsf{D}$*, such that the diagram*

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "F"] \arrow[dr] & & \mathsf{D} \\ & \mathsf{C}[\mathsf{W}^{-1}] \arrow[ur, dashed, "\exists!\, \tilde{F}"'] & \end{tikzcd}$$

commutes up to a natural isomorphism.

Roughly speaking, the category $\mathsf{C}[\mathsf{W}^{-1}]$ is obtained from C by inverting all the arrows in W. In fact, this idea can be formulated into an explicit construction of the localisation $\mathsf{C}[\mathsf{W}^{-1}]$.

Construction 1.7. Let (C, W) be a category with weak equivalences. Define $\mathsf{C}[\mathsf{W}^{-1}]$ to be the category with the same objects as C, with $\operatorname{Hom}_{\mathsf{C}[\mathsf{W}^{-1}]}(X, Y)$ the set of all possible sequences

$$X \to Z_1 \leftarrow Z_2 \to \cdots \leftarrow Z_n \to Y$$

in C, where all arrows going leftward are in W, quotiented by the following relations: identity arrows can be dropped; adjacent arrows pointing to the same direction can be composed; adjacent arrows pointing to different directions can also be dropped if they represent the same morphism. It is then almost obvious that our construction does give a localisation with the desired universal property.

The only problem is that $\operatorname{Hom}_{\mathsf{C}[\mathsf{W}^{-1}]}(X, Y)$ may be too large to be a set; however, we do not care about this problem for now, and it is easily overcome by switching to a larger universe. ◁

Proposition 1.8. $\mathsf{hTop} \simeq \mathsf{Top}[\mathsf{HoEq}^{-1}]$.

Proof. Not yet written ◻

In fact, we will see that localisation gives rise to higher structure. In this example, the ordinary category hTop is just the first layer of information that we get from localisation. The full information is retained in an ∞-category, which is, in this case, the ∞-category Top given in (1.1).

We are not very interested in this example, because the higher structure has a very clear description. What we are more interested in is how to invert the weak homotopy equivalences, or in homological algebra, invert the quasi-isomorphisms, and get the ∞-categorical version of the derived category.

It is painful to study localisation directly from the definitions. We will need the help of *model categories*, which are categories with weak equivalences, together with some extra structures that will help us substantially in computations related to ∞-categories. Here is our mind-map.

$$\begin{tikzcd}
\overset{\text{model category}}{(\mathsf{C}, \mathsf{W}, \mathsf{Cof}, \mathsf{Fib})} \arrow[dr, "\text{present (easier)}"] & \\
(\mathsf{C}, \mathsf{W}) \arrow[u, dashed, "\text{find a structure}"] \arrow[r, "\text{localise}", "\text{(hard to describe)}"'] & \infty\text{-category } \mathsf{C}[\mathsf{W}^{-1}] \arrow[d, "\text{forget}"] \\
& \text{ordinary category } \mathsf{C}[\mathsf{W}^{-1}]
\end{tikzcd}$$

A naive attempt on higher categories

We will now try to give a simple, but "wrong", definition of higher categories. Keeping in mind that higher categories are just categories with higher dimensional arrows, we will formulate this idea into a rigorous definition.

Definition 1.9. *A* **monoidal category** *is a category* C*, together with*

- *An object* $1 \in \mathsf{C}$*, called the* **unit**.
- *A functor* $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$*,*

such that there are natural isomorphisms

$$\begin{aligned}
a:&\quad (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z),\\
l:&\quad 1 \otimes X \xrightarrow{\sim} X,\\
r:&\quad X \otimes 1 \xrightarrow{\sim} X,
\end{aligned}$$

so that the diagrams

$$\begin{tikzcd}[column sep=small]
 & & (X \otimes Y) \otimes (Z \otimes W) \arrow[drr] & & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[urr] \arrow[dr] & & & & X \otimes (Y \otimes (Z \otimes W)) \\
 & (X \otimes (Y \otimes Z)) \otimes W \arrow[rr] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[ur] &
\end{tikzcd},$$

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr] \arrow[dr] & & X \otimes (1 \otimes Y) \arrow[dl] \\
 & X \otimes Y &
\end{tikzcd}, \qquad \begin{tikzcd}
1 \otimes 1 \arrow[d, bend right, "l"'] \arrow[d, bend left, "r"] \\
1
\end{tikzcd}$$

are commutative for any $X, Y, Z, W \in \mathsf{C}$.

The first diagram is also called the *pentagon axiom*. One can prove that even if we have more than 4 objects, the pentagon axiom ensures that the "associahedron" diagrams are commutative.

For example, the following triples $(\mathsf{C}, \otimes, 1)$ are all examples of monoidal categories:

- $(\mathsf{Set}, \times, *)$, where $*$ denotes the singleton set.
- $(\mathsf{Set}, \sqcup, \varnothing)$, where $\sqcup$ denotes disjoint union.
- (any category with products, $\times, *$), where $*$ denotes the terminal object, i.e. the empty product.
- (any category with coproducts, $\sqcup, \varnothing$), where $\varnothing$ denotes the initial object, i.e. the empty coproduct.
- $(\mathsf{Ch}_R, \otimes, R)$, where the unit R is concentrated in degree 0, and the tensor product is given by

$$(C \otimes D)^n := \bigoplus_{p+q=n} C^p \otimes_R D^q.$$

Definition 1.10. *Let* V *be a monoidal category. A* V**-enriched category** C *consists of the following data.*

- *A class of objects.*
- *For any* $X, Y \in \mathsf{C}$*, a hom-object* $\mathrm{Hom}_{\mathsf{C}}(X, Y) \in \mathsf{V}$.
- *For any* $X, Y, Z \in \mathsf{C}$*, a composition map*

$$\circ : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \otimes \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$$

  in V.

- *For any $X \in \mathsf{C}$, an identity morphism*

$$\mathbb{1}_X : 1 \to \mathrm{Hom}_{\mathsf{C}}(X,X)$$

  in V. We think of $\mathbb{1}_X$ as an "element" of $\mathrm{Hom}_{\mathsf{C}}(X,X)$, but since this object is not a set, we consider morphisms $1 \to \mathrm{Hom}_{\mathsf{C}}(X,X)$ as its "elements".

They satisfy the following conditions.

- *Composition is associative.*
- *Composing with the identity morphism gives the original morphism.*

We leave it to the reader to formulate these two axioms rigorously.

For example, an ordinary category is a category enriched over $(\mathsf{Set}, \times, *)$.

When we have a forgetful functor $\mathsf{V} \to \mathsf{Set}$ preserving the monoidal structure, we may think of a V-enriched category as a category with extra structures, as in the following examples.

Example 1.11. The category Top is enriched over itself, since for $X, Y \in \mathsf{Top}$, the set $\mathrm{Hom}_{\mathsf{Top}}(X, Y)$ can be given a natural topology, namely the compact open topology, so that composition is continuous. $\lhd$

Example 1.12. The category Ch_R is enriched over itself. For $X, Y \in \mathsf{Ch}_R$, we define a cochain complex $\mathcal{H}om(X, Y)$ by

$$\mathcal{H}om(X, Y)^n := \{f = \{f^k : X^k \to Y^{k+n}\}_{k \in \mathbb{Z}}\},$$

where f is not necessarily a chain map, and the differential is given by

$$df := d \circ f - (-1)^{\deg f} f \circ d.$$

The chain maps are the 0-cocycles of this cochain complex. Such a cocycle corresponds to a chain map from the unit $1 \in \mathsf{Ch}_R$ to this cochain complex, which is, in a sense, an "element" of this cochain complex. $\lhd$

Example 1.13. The category Cat of all (small) categories is enriched over itself. This is because for any two categories X, Y, we may form their functor category $\mathsf{Fun}(X, Y)$, whose objects are functors $X \to Y$, and morphisms are natural transformations between these functors. $\lhd$

Using the language of enriched categories, we can now give a first definition of higher categories.

Definition 1.14. *A* **strict 2-category** *is a category enriched over* Cat.

For example, Cat itself is a strict 2-category. We may think of functors as its 1-morphisms, and natural transformations as its 2-morphisms.

Any Top-enriched category can be regarded as a strict 2-category, since we may replace its hom-spaces by their fundamental groupoids. In this case, the 2-morphisms are just paths connecting the 1-morphisms in the hom-space. For example, Top itself is a strict 2-category. Its 2-morphisms are homotopies between maps.

However, for a topological space X, the fundamental groupoid $\Pi(X)$ is NOT a strict 2-category. This is because, as we have mentioned, the composition in $\Pi(X)$ is not strictly associative. This indicates that our definition is too strict, and is somehow "wrong".

Definition 1.15. *Inductively, we define a* **strict n-category** *as a category enriched over the category of $(n-1)$-categories.*

This gives rise to a definition of an ∞-category, which is too strict and will not be used in the future.

Definition 1.16. *A* **strict ω-category** *is a sequence*

$$\mathsf{C}_1 \hookrightarrow \mathsf{C}_2 \hookrightarrow \mathsf{C}_3 \hookrightarrow \cdots,$$

where C_n is a strict n-category, such that C_{n-1} is obtained from C_n by discarding all the n-arrows.

(n,r)-categories

Although we have not defined n-categories and ∞-categories in general, we now have some intuitive ideas about what they are, and we can talk about them in a semi-rigorous way.

Definition 1.17. *Let $0 \leq r \leq n$ and $n > 0$. An* **(n,r)-category** *is an n-category in which all $r+1, r+2, \dots, n$-morphisms are invertible.*

Since we are not only talking about strict n-categories, "invertible" actually means "invertible up to a higher homotopy", or in other words, "having a homotopy inverse". Let us look at some examples.

- An ordinary category is a $(1,1)$-category.
- An ordinary groupoid is a $(1,0)$-category.
- The strict n-categories defined above are (n,n)-categories. A strict ω-category is an (∞,∞)-category.
- $(\infty,0)$-categories are called ∞-groupoids. They are very important objects, since they correspond to homotopy types. In fact, there is a theory called Homotopy Type Theory (HoTT), which aims to rebuild the foundations of mathematics, using ∞-groupoids instead of sets as the basic building blocks.

- A category enriched over the category of (n,r)-categories is an $(n+1,r+1)$-category.
- Top is an $(\infty,1)$-category, since every homotopy is invertible. Namely, a homotopy composed with its own inverse is homotopic to the identity homotopy.
- Ch_R is an $(\infty,1)$-category, since composition of homotopies is addition of maps between cochain complexes, and for a chain homotopy, adding its additive inverse gives the zero map, which corresponds to the identity homotopy.
- For a topological space X, the fundamental groupoid $\Pi(X)$ is an example of an ∞-groupoid. It describes the homotopy type of X.

Regarding the fifth point, we may extend the notion of (n,r)-categories to the case $n<1$. In fact, we will see in the future that the correct notions are as follows.

- $(0,0)$-categories are sets.
- $(0,1)$-categories are partially ordered sets.
- $(-1,0)$-categories are either $\varnothing$ or singleton sets. In other words, they are truth values.
- $(-2,0)$-categories are singleton sets.

Higher category theorists believe that the bottom layer should be -2, and they think that it is more natural to renumber everything so that -2 becomes 0. Thus, in the skyscraper of mathematics, logic lives on the 1st floor; set theory lives on the 2nd floor; and category theory lives on the 3rd floor.

From this viewpoint, it seems more natural to consider all the floors as a whole. That is possibly why homotopy type theorists wish to replace set theory by higher category theory as the foundation of mathematics.

2 Model categories

Model categories are categories with weak equivalences, together with some extra data, namely a class Cof of *cofibrations*, and a class Fib of *fibrations*. This extra data will help us in computations related to ∞-categories. Recall that we have a mind-map

$$\begin{tikzcd}
\overset{\text{model category}}{(\mathsf{C},\mathsf{W},\mathsf{Cof},\mathsf{Fib})} \arrow[drr, "\text{present (easier)}"] & & \\
(\mathsf{C},\mathsf{W}) \arrow[u, dashed, "\text{find a structure}"] \arrow[rr, "\text{localise}", "\text{(hard to describe)}"'] & & \infty\text{-category } \mathsf{C}[\mathsf{W}^{-1}] \arrow[d, "\text{forget}"] \\
& & \text{ordinary category } \mathsf{C}[\mathsf{W}^{-1}]
\end{tikzcd}$$

Model categories have two main advantages in such computations:

- **Homotopies between morphisms** are easily described. Normally we only know which morphisms are weak equivalences; we do not know which morphisms become homotopic after localisation. However, in model categories, we can see such homotopies, and we can even see all the higher dimensional homotopies, just like we are working with topological spaces. As a result, the localisation $\mathsf{C}[\mathsf{W}^{-1}]$ will have a good description as the *homotopy category* of C.
- **Derived functors** are very easy to compute. They will be defined later in this section.

Definition and examples

Recall from algebraic topology that a map $p : X \to Y$ of topological spaces is called a **Hurewicz fibration** if for any space A and any diagram (without the dashed arrow)

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd},$$

there exists a dashed arrow making the diagram commute. A map $i : A \to B$ is called a **Hurewicz cofibration** if for any space Y and any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y^I \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y^{\{0\}} \end{tikzcd},$$

there exists a dashed arrow making the diagram commute, where Y^I denotes the space of all maps $I \to Y$ equipped with the compact open topology.

Definition 2.1. *Let* C *be a category, and let* $J \subset \mathrm{Mor}(\mathsf{C})$ *be a class of morphisms. We say that a map* $p : X \to Y$ *in* C *has the* **right lifting property** *against* J*, if for any diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in C*, where the map* $A \to B$ *is in* J*, there exists a dashed arrow making the diagram commute. The class of all arrows* p *with this property is denoted by* $\mathrm{RLP}(J)$.

Dually, a map $i : A \to B$ *in* C *has the* **left lifting property** *against* J*, if for any diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in C*, where the map* $X \to Y$ *is in* J*, there exists a dashed arrow making the diagram commute. The class of all arrows i with this property is denoted by* $\mathrm{LLP}(J)$*.*

For example, we have by definition

$$\begin{aligned}\{\text{Hurewicz fibrations}\} &= \mathrm{RLP}\{A \times \{0\} \hookrightarrow A \times I\},\\ \{\text{Hurewicz cofibrations}\} &= \mathrm{LLP}\{Y^I \twoheadrightarrow Y^{\{0\}}\}.\end{aligned}$$

As an exercise, the reader can show that $\mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(J))) = \mathrm{RLP}(J)$, so that if we put $\mathsf{R} := \mathrm{RLP}(J)$ and $\mathsf{L} := \mathrm{LLP}(\mathrm{RLP}(J))$, then $\mathsf{L} = \mathrm{LLP}(\mathsf{R})$ and $\mathsf{R} = \mathrm{RLP}(\mathsf{L})$.

Definition 2.2. *A* **weak factorisation system** *is a triple* $(\mathsf{C}, \mathsf{L}, \mathsf{R})$*, where* C *is a category, and* $\mathsf{L}, \mathsf{R} \subset \mathrm{Mor}(\mathsf{C})$ *are two classes of morphisms, such that*

- $\mathsf{L} = \mathrm{LLP}(\mathsf{R})$ *and* $\mathsf{R} = \mathrm{RLP}(\mathsf{L})$.
- *Every morphism* $f : A \to B$ *in* C *can be factored into*

  $$A \xrightarrow{l} X \xrightarrow{r} B,$$

  where $l \in \mathsf{L}$ *and* $r \in \mathsf{R}$*. Moreover, we require that the factorisation is functorial in* f*.*

For example, let HCof, HFib and HoEq denote the class of closed Hurewicz cofibrations, the class of Hurewicz fibrations, and the class of homotopy equivalences, respectively. We will see that the triples $(\mathsf{Top}, \mathsf{HCof} \cap \mathsf{HoEq}, \mathsf{HFib})$ and $(\mathsf{Top}, \mathsf{HCof}, \mathsf{HFib} \cap \mathsf{HoEq})$ both form weak factorisation systems.

Proposition 2.3. *Let* $(\mathsf{C}, \mathsf{L}, \mathsf{R})$ *be a weak factorisation system.*

- L *and* R *contain all isomorphisms in* C*.*
- L *and* R *are closed under composition.*
- L *is preserved by pushouts and* R *is preserved by pullbacks.*

Proof. Exercise for the reader. □

Definition 2.4. *A* **model category** $(\mathsf{C}, \mathsf{W}, \mathsf{Cof}, \mathsf{Fib})$ *is a category* C *with three distinguished classes of morphisms* $\mathsf{W}, \mathsf{Cof}, \mathsf{Fib} \subset \mathrm{Mor}(\mathsf{C})$*, such that*

- C *admits all small colimits and limits.*
- (C, W) *is a category with weak equivalences.*
- $(\mathsf{C}, \mathsf{Cof} \cap \mathsf{W}, \mathsf{Fib})$ *is a weak factorisation system.*
- $(\mathsf{C}, \mathsf{Cof}, \mathsf{Fib} \cap \mathsf{W})$ *is a weak factorisation system.*

We shall now introduce some terminology.

- Morphisms in W are called *weak equivalences*.
- Morphisms in Cof are called *cofibrations*.
- Morphisms in Fib are called *fibrations*.
- Morphisms in $\mathsf{Cof} \cap \mathsf{W}$ are called *trivial cofibrations*, or *acyclic cofibrations*.
- Morphisms in $\mathsf{Fib} \cap \mathsf{W}$ are called *trivial fibrations*, or *acyclic fibrations*.
- An object $X \in \mathsf{C}$ is *cofibrant* if the map $\varnothing \to X$ is a cofibration, where $\varnothing$ denotes the initial object of C. The initial object exists because it is the empty colimit.
- An object $X \in \mathsf{C}$ is *fibrant* if the map $X \to *$ is a fibration, where $*$ denotes the terminal object of C. The terminal object exists because it is the empty limit.

Note that Cof and Fib determine each other, since in a weak factorisation system, the two classes of morphisms determine each other. Note also that cofibrations and trivial cofibrations are preserved by pushouts, and fibrations and trivial fibrations are preserved by pullbacks.

Let us look at some examples. It is often very tedious to verify the axioms of a model category, so we will present the results without giving proofs.

Example 2.5. The **Hurewicz model structure** on Top is defined as follows.

- W is the class of homotopy equivalences.
- Cof is the class of closed Hurewicz cofibrations.
- Fib is the class of Hurewicz fibrations. ◁

Example 2.6. The **Quillen model structure** on Top is defined as follows.

- W is the class of weak homotopy equivalences.
- $\mathsf{Fib} := \mathrm{RLP}\{D^n \times \{0\} \hookrightarrow D^n \times I \mid n \geq 0\}$ is the class of Serre fibrations.
- $\mathsf{Fib} \cap \mathsf{W} = \mathrm{RLP}\{S^{n-1} \hookrightarrow D^n \mid n \geq 0\}$. This is an alternative characterisation of $\mathsf{Fib} \cap \mathsf{W}$.
- Cof is determined by $\mathsf{Cof} = \mathrm{LLP}(\mathsf{Fib} \cap \mathsf{W})$. In particular, the maps $S^{n-1} \hookrightarrow D^n$ are cofibrations.

In this model category, every topological space is fibrant. All CW complexes are cofibrant, because cofibrations are preserved by pushouts, and preserved by composition and sequential colimits.

In the sequel, we will always use the Quillen model structure, instead of the Hurewicz model structure.

Example 2.7. The **projective model structure** on Ch_R is defined as follows.

- W is the class of quasi-isomorphisms.
- Fib is the class of degreewise surjections.

In this model category, every cochain complex is fibrant. Any bounded-above cochain complex consisting of projective modules, e.g. a projective resolution of an R-module, is cofibrant. ◁

Example 2.8. The **injective model structure** on Ch_R is defined as follows.

- W is the class of quasi-isomorphisms.
- Cof is the class of degreewise injections.

In this model category, every cochain complex is cofibrant. Any bounded-below cochain complex consisting of injective modules, e.g. an injective resolution of an R-module, is fibrant. ◁

As can be seen in these examples, **cofibrant-fibrant objects**, i.e. objects that are both cofibrant and fibrant, are special objects that behave like CW complexes or projective/injective resolutions. We will soon see that these objects have much better properties than others. However, it turns out that every object is weakly equivalent to a cofibrant-fibrant object.

Construction 2.9. Let C be a model category. By the axioms for a model category, for any $X \in \mathsf{C}$, we may factorise the map $\varnothing \to X$ into

$$\varnothing \xrightarrow{\in \mathsf{Cof}} QX \xrightarrow{\in \mathsf{Fib} \cap \mathsf{W}} X.$$

Then QX is a cofibrant object that is weakly equivalent to X. Moreover, Q is a functor, and is called the **cofibrant replacement** functor.

Dually, we may factorise the map $X \to *$ into

$$X \xrightarrow{\in \mathsf{Cof} \cap \mathsf{W}} RX \xrightarrow{\in \mathsf{Fib}} *.$$

Then RX is a fibrant object that is weakly equivalent to X. Moreover, R is a functor, and is called the **fibrant replacement** functor.

As an exercise, the reader can show that the objects RQX and QRX are both cofibrant-fibrant. ◁

Homotopy category

In this section, we aim to recover "homotopies" from the axioms of a model category. In topology, a homotopy between maps $X \to Y$ are given by a map

$$X \times I \to Y, \quad \text{or} \quad X \to Y^I.$$

In a model category, we define homotopies by considering objects that behave like $X \times I$ or Y^I.

Definition 2.10. *Let* C *be a model category, and let* $X \in \mathsf{C}$ *be an object.*

- *A* **cylinder object** $\mathrm{Cyl}(X)$ *for* X *is a factorisation of the codiagonal map* $\nabla_X := (\mathbb{1}, \mathbb{1}) \colon X \sqcup X \to X$ *as*

$$\begin{tikzcd} X \sqcup X \arrow[rr, "\nabla_X"] \arrow[dr, "\mathsf{Cof} \ni i"'] & & X \\ & \mathrm{Cyl}(X) \,. \arrow[ur, "p \in \mathsf{W}"'] & \end{tikzcd}$$

  The cylinder object is said to be **very good** *if moreover* $p \in \mathsf{Fib} \cap \mathsf{W}$.

- *A* **path space object** $\mathrm{Path}(X)$ *for* X *is a factorisation of the diagonal map* $\Delta_X := (\mathbb{1}, \mathbb{1}) \colon X \to X \times X$ *as*

$$\begin{tikzcd} X \arrow[rr, "\Delta_X"] \arrow[dr, "\mathsf{W} \ni i"'] & & X \times X \\ & \mathrm{Path}(X) \,. \arrow[ur, "p \in \mathsf{Fib}"'] & \end{tikzcd}$$

  The path space object is said to be **very good** *if moreover* $i \in \mathsf{Cof} \cap \mathsf{W}$.

These are objects that behave like $X \times I$ and X^I in topology, respectively. By the axioms of a model category, we can always find very good cylinder and path space objects for any object X.

Definition 2.11. *Let* $f, g \colon X \to Y$ *be two morphisms in* C.

- *A* **left homotopy** *from* f *to* g *is a morphism* $\mathrm{Cyl}(X) \to Y$*, such that the diagram*

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

  commutes. In this case, we say that f *and* g *are* **left homotopic**.

- *A* **right homotopy** *from* f *to* g *is a morphism* $X \to \mathrm{Path}(Y)$*, such that the diagram*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d] \arrow[dr, "g"] & \\ Y & \mathrm{Path}(Y) \arrow[l, "p_0"] \arrow[r, "p_1"'] & Y \end{tikzcd}$$

  commutes. In this case, we say that f *and* g *are* **right homotopic**.

In topology, left homotopies are equivalent to right homotopies. We will soon show that this is also true for cofibrant-fibrant objects, but before that, let us get familiar with the axiomatic way of doing homotopy theory through an example.

Example 2.12. If $f_0, f_1 : X \to Y$ are left homotopic maps, and $g_0, g_1 : Y \to Z$ are left homotopic maps, then $g_0 \circ f_0$ and $g_1 \circ f_1$ are left homotopic, provided that the canonical very good cylinder objects, i.e. those obtained from the functorial factorisation of the codiagonal maps of X and Y, are used.

Proof. In topology, we prove this obvious fact by considering the composition

$$X \times I \to X \times I \times I \to Y \times I \to Z,$$

where the first map is induced from the diagonal map $I \hookrightarrow I \times I$, which is a path connecting the vertices $(0,0)$ and $(1,1)$.

In the model category setting, we wish to get a series of maps

$$\mathrm{Cyl}(X) \to \mathrm{Cyl}(\mathrm{Cyl}(X)) \to \mathrm{Cyl}(Y) \to Z,$$

which gives the desired homotopy. Since we are using the canonical cylinder objects, we can regard Cyl as a functor. This gives the map $\mathrm{Cyl}(\mathrm{Cyl}(X)) \to \mathrm{Cyl}(Y)$. Thus, the only problem now is to construct the first map $\mathrm{Cyl}(X) \to \mathrm{Cyl}(\mathrm{Cyl}(X))$, as "a path connecting the vertices $(0,0)$ and $(1,1)$". We do this by lifting in the diagram

$$\begin{tikzcd}
X \sqcup X \arrow[r, "{(i_0 i_0, i_1 i_1)}"] \arrow[d, "\mathsf{Cof} \ni i"'] & \mathrm{Cyl}(\mathrm{Cyl}(X)) \arrow[d, "\in \mathsf{Fib} \cap \mathsf{W}"] \\
\mathrm{Cyl}(X) \arrow[r] \arrow[ur, dashed] & X .
\end{tikzcd}$$

The reader can now readily check that the composition map $\mathrm{Cyl}(X) \to Z$ does give a left homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$. □

Proposition 2.13. *Let $f_0, f_1 : X \to Y$ be two maps. If X is cofibrant and Y is fibrant, then f_0 and f_1 are left homotopic if and only if they are right homotopic. In this case, a homotopy exists for any cylinder object of X and for any path space object of Y.*

Proof. Not yet written □

This proposition shows that homotopy is a really nice property to work with, at least for the cofibrant-fibrant objects. As an exercise, the reader can show that under the above conditions, being homotopic is an equivalence relation for maps $X \to Y$. Thus, we can take homotopy classes of maps and define the homotopy category.

Definition 2.14. *The* **homotopy category** *of* C *is a category* $\mathrm{Ho}(\mathsf{C})$*, with*

- *Objects: the cofibrant-fibrant objects of* C.
- *Morphisms: homotopy classes of morphisms in* C.

Do not forget that we study model categories in order to study localisations. And here is the key result:

Theorem 2.15. *We have an equivalence of categories*

$$\mathrm{Ho}(\mathsf{C}) \simeq \mathsf{C}[\mathsf{W}^{-1}].$$

The proof of this theorem depends on the following fact.

Proposition 2.16 (Whitehead's Theorem)**.** *Suppose $X, Y \in \mathsf{C}$ are cofibrant-fibrant objects. Then a morphism $f : X \to Y$ is a weak equivalence if and only if it is a homotopy equivalence, i.e. it has a homotopy inverse.*

Proof. Not yet written □

Proof of (2.15). Not yet written □

Remark 2.17. We have noted that Ho(C) is the first layer of information obtained from localisation. The full information is retained in an $(\infty, 1)$-category. We will see that model categories are capable of providing such higher structures, through a construction called a **framing**. This construction is analogous to $X \times \Delta^n$ and X^{Δ^n}, in order to describe higher homotopies in a model category. We will possibly return to this point later. ◁

Derived functors

Recall that in homological algebra, derived functors are a way of passing a functor between abelian categories $F : \mathsf{A} \to \mathsf{B}$ to a functor between derived categories $\mathsf{D}(\mathsf{A}) \to \mathsf{D}(\mathsf{B})$. This is a special case of the following construction.

For a model category C, denote the subcategory of C consisting of cofibrant (resp. fibrant, cofibrant-fibrant) objects by C_{c} (resp. C_{f}, C_{cf}).

Definition 2.18. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor, where* C *is a model category and* D *is a category with weak equivalences.*

- *If F preserves weak equivalences, then F induces a functor*

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{D}).$$

This is called the **total derived functor** *of F.*

- *If $F|_{\mathsf{C}_{\mathrm{c}}}$ preserves weak equivalences, then there is a diagram*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "Q"] \arrow[d] & \mathsf{C}_{\mathrm{c}} \arrow[r, "F"] \arrow[d] & \mathsf{D} \arrow[d] \\ \mathrm{Ho}(\mathsf{C}) \arrow[r, equal] & \mathrm{Ho}(\mathsf{C}) \arrow[r, "\mathbb{L}F"] & \mathrm{Ho}(\mathsf{D}), \end{tikzcd}$$

which commutes up to natural isomorphisms, where Q is any cofibrant replacement functor of C. *The functor $\mathbb{L}F$ is called the* **left derived functor** *of F. For any $X \in \mathsf{C}$, we have the formula*

$$\mathbb{L}F(X) \simeq F(QX)$$

in $\mathrm{Ho}(\mathsf{D})$.

- *If $F|_{\mathsf{C}_\mathrm{f}}$ preserves weak equivalences, then there is a diagram*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "R"] \arrow[d] & \mathsf{C}_\mathrm{f} \arrow[r, "F"] \arrow[d] & \mathsf{D} \arrow[d] \\ \mathrm{Ho}(\mathsf{C}) \arrow[r, equal] & \mathrm{Ho}(\mathsf{C}) \arrow[r, "\mathbb{R}F"] & \mathrm{Ho}(\mathsf{D}) \end{tikzcd},$$

which commutes up to natural isomorphisms, where R is any fibrant replacement functor of C. *The functor $\mathbb{R}F$ is called the* **right derived functor** *of F. For any $X \in \mathsf{C}$, we have the formula*

$$\mathbb{R}F(X) \simeq F(RX)$$

in $\mathrm{Ho}(\mathsf{D})$.

Example 2.19. In the category of cochain complexes with the injective model structure, the cofibrant replacement functor is taking projective resolutions. Thus the left derived functors defined above coincides with the standard definition in homological algebra.

Similarly, in the category of cochain complexes with the projective model structure, the fibrant replacement functor is taking injective resolutions, which defines right derived functors. ◁

Derived adjunctions

Needs expanding

Definition 2.20. *Let C, D be two model categories, and let $(F \dashv G)$ be a pair of adjoint functors between them. Then $(F \dashv G)$ is called a* **Quillen adjunction** *if F preserves cofibrations and trivial cofibrations.*

Note that F preserves cofibrations iff G preserves trivial fibrations, and F preserves trivial cofibrations iff G preserves fibrations.

Lemma 2.21 (Ken Brown)**.** *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between two model categories. If $F|_{\mathsf{C}_\mathrm{c}}$ preserves trivial cofibrations, then $F|_{\mathsf{C}_\mathrm{c}}$ preserves weak equivalences.*

Proof. Not yet written □

Corollary 2.22. *If $(F \dashv G)$ is a Quillen adjunction between two model categories C and D, then $\mathbb{L}F$ and $\mathbb{R}G$ exist. Moreover, $(\mathbb{L}F \dashv \mathbb{R}G)$ is an adjunction between the homotopy categories* $\mathrm{Ho}(\mathsf{C})$ *and* $\mathrm{Ho}(\mathsf{D})$.

Proof. Not yet written □

Definition 2.23. *A Quillen adjunction $(F \dashv G)$ is called a* **Quillen equivalence** *if $\mathbb{L}F$ and $\mathbb{R}G$ are equivalences of categories.*

Proposition 2.24. *Let $(F \dashv G)$ be a Quillen equivalence between two model categories C and D. Let $X \in \mathsf{C}$ be a cofibrant object, and let $Y \in \mathsf{D}$ be a fibrant object. Then the counit and unit maps*

$$X \to GRFX \quad \text{and} \quad FQGY \to Y$$

are weak equivalences.

Proof. Follows directly from the definitions. □

3 Simplicial sets

Simplicial sets arise from many topics in mathematics, and they have a wide range of applications in various branches of mathematics. For our purpose, they will be used as a model for ∞-categories, as well as ordinary categories. We will see how this is done in this section.

Definition and examples

Before giving the actual definition, let us look at some examples of simplicial sets.

Example 3.1. Let X be a simplicial complex (as in topology), together with an ordering of vertices for each simplex σ, such that the inclusion of a face of σ into σ preserves the ordering of vertices. Let X_n denote the set of n-simplices of X, which may be degenerate. Then we have a series of maps

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_1 \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} X_0 \,,$$

where X_n has $(n+1)$ maps to X_{n-1}, called the *face maps*, defined by taking the $(n+1)$ faces of an n-simplex. X_n also has $(n+1)$ maps to X_{n+1}, called the *degeneracy maps*, defined by regarding an n-simplex as a degenerate $(n+1)$-simplex. This structure is called a *simplicial set*. ◁

Example 3.2. Let X be a topological space, and denote

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X),$$

where Δ^n denotes the standard n-simplex in topology. Then $\mathrm{Sing}(X)$ is a simplicial set, having the same structure as in the previous example. This construction is used to define singular (co)homology in algebraic topology. $\triangleleft$

Now we will give the formal definition of a simplicial set.

Definition 3.3. *The category $\mathbb{\Delta}$ is defined as follows.*

- *Its objects are the sets $[n] := \{0, \dots, n\}$ for all integers $n \geq 0$.*
- *The hom-set $\mathrm{Hom}_{\mathbb{\Delta}}([m],[n])$ consists of all maps from $[m]$ to $[n]$ preserving the order $\leq$.*

In the category $\mathbb{\Delta}$, there are two special classes of morphisms.

- There are n maps from $[n]$ to $[n-1]$, denoted by d^i $(0 \leq i \leq n-1)$, defined by merging the elements i and $i+1$ in $[n]$. These are called the **coface maps**.
- There are $(n+2)$ maps from $[n]$ to $[n+1]$, denoted by s^i $(0 \leq i \leq n+1)$, defined by skipping the element i in $[n+1]$. These are called the **codegeneracy maps**.

In fact, all morphisms in $\mathbb{\Delta}$ can be written as a composition of these coface and codegeneracy maps. These maps form a diagram

$$\begin{tikzcd}
{[0]} \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & {[1]} \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] & {[2]} \arrow[r, leftarrow, shift left=6] \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[r, leftarrow, shift right=6] & \cdots
\end{tikzcd}$$

in the category $\mathbb{\Delta}$.

Definition 3.4. *A* **simplicial set** *is a functor from $\mathbb{\Delta}^{\mathrm{op}}$ to* Set*, i.e. a contravariant functor from $\mathbb{\Delta}$ to* Set*. We denote the category of simplicial sets by*

$$\mathsf{sSet} := \mathsf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathsf{Set}).$$

More generally, for any category C*, a* **simplicial object** *in* C *is a functor from $\mathbb{\Delta}^{\mathrm{op}}$ to* C*, and a* **cosimplicial object** *in* C *is a functor from $\mathbb{\Delta}$ to* C*.*

Let X be a simplicial set. The set $X_n := X([n])$ is called the set of n-**simplices** of X. The maps

$$d_i : X_n \to X_{n-1} \quad \text{and} \quad s_i : X_n \to X_{n+1}, \quad \text{for } 0 \leq i \leq n,$$

induced by the morphisms d^i and s^i in the category $\mathbb{\Delta}$, are called the **face maps** and the **degeneracy maps**, respectively.

Example 3.5. We construct some examples of simplicial sets.

- The simplicial set $\Delta[n]$, as a simplicial complex, corresponds to the standard n-simplex. Its k-simplices are in 1–1 correspondence with order-preserving maps $[k] \to [n]$, where $[n]$ can be seen as the set of vertices of $\Delta[n]$.
- Note that $\Delta[\bullet]$ is a cosimplicial object in sSet.
- By removing the only non-degenerate n-simplex in $\Delta[n]$, we obtain its boundary $\partial\Delta[n]$.
- The simplicial set S^n is defined by $\Delta[n]/\partial\Delta[n]$, where the quotient is done degreewise. ◁

Proposition 3.6. *The category* sSet *admits all (small) colimits and limits, which are defined degreewise, e.g.*

$$(X \times Y)_n := X_n \times Y_n.$$

Proof. Exercise for the reader. □

For example, the product $\Delta[1] \times \Delta[1]$ is a solid square, which looks like

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd},$$

with 2 non-degenerate 2-simplices and 5 non-degenerate 1-simplices, but actually it has $3 \times 3 = 9 = 5 + 4$ possibly degenerate 1-simplices in total.

Geometric realisation

In our mind, a simplicial set is thought of as a simplicial complex together with an ordering of vertices for each simplex. The idea of geometric realisation is that one can forget the ordering and get a topological space. In fact, geometric realisation is a much more general construction which can not only give topological spaces, but any kind of objects we want. Let us explain this.

Proposition 3.7. *Every simplicial set can be obtained from $\varnothing$ by attaching $\Delta[n]$ along its boundary $\partial\Delta[n]$ and taking colimits.* □

In the category Top, we have the standard n-simplex Δ^n, as a topological space. Moreover, $\Delta^\bullet$ is a cosimplicial object in Top which specifies "how $\Delta[n]$ should look like in Top". Given this data, we can easily define a functor

$$|\bullet| \colon \mathsf{sSet} \to \mathsf{Top}$$

by sending $\Delta[n]$ to Δ^n, and extending it to other simplicial sets by taking colimits. This functor is called the **geometric realisation** functor.

Moreover, the functor $|\bullet|$ has a right adjoint called Sing, which is defined by

$$\operatorname{Sing} X := \operatorname{Hom}_{\mathsf{Top}}(\Delta^\bullet, X),$$

which is exactly as we defined it before.

This construction can be generalised as follows.

Construction 3.8. Let C be a category with colimits. If we specify any cosimplicial object $\Delta^\bullet$ in C, then we know "how $\Delta[n]$ should look like in C", and we can similarly define an adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "{\Delta[n]\mapsto\Delta^n}"] \arrow[r, phantom, "\perp"] & \mathsf{C} \arrow[l, bend left, "{X\mapsto \operatorname{Hom}_{\mathsf{C}}(\Delta^\bullet, X)}"] \end{tikzcd}.$$ ◁

As we will see in the future, many constructions related to simplicial sets are special cases of this construction. Here is one example.

Example 3.9. Let us take $\mathsf{C} = \mathsf{Ch}_R$, the category of cochain complexes of R-modules, where R is a ring. Take

$$\Delta^n := \quad \Big(\cdots \to 0 \to \underset{(-n)}{R^{\oplus\binom{n+1}{n+1}}} \to \cdots \to \underset{(-1)}{R^{\oplus\binom{n+1}{2}}} \to \underset{(0)}{R^{\oplus(n+1)}} \to 0 \to \cdots\Big)$$

to be the simplicial chain complex of the standard n-simplex. This construction gives a functor $\mathsf{sSet} \to \mathsf{Ch}_R$, which computes the simplicial homology of a simplicial set. The composition

$$\mathsf{Top} \xrightarrow{\operatorname{Sing}} \mathsf{sSet} \to \mathsf{Ch}_R$$

computes the singular homology of a topological space. ◁

Let us go back to the adjunction $|\bullet| \dashv \operatorname{Sing}$. Actually, this is a Quillen equivalence between model categories.

For $0 \le i \le n$, we define the simplicial set $\Lambda_i[n]$, called a **horn**, by removing the face of $\partial\Delta[n]$ opposite to the i-th vertex.

Theorem 3.10. *The category* sSet *has a* **standard model structure***, with*

- $\mathsf{W} :=$ {*weak homotopy equivalences of topological spaces*}.
- $\mathsf{Cof} :=$ {*injections*}.
- $\mathsf{Fib} = \mathrm{RLP}\{\Lambda_i[n] \hookrightarrow \Delta[n] \mid 0 \le i \le n,\ n > 0\}$.
- $\mathsf{Fib} \cap \mathsf{W} = \mathrm{RLP}\{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \ge 0\}$.

The proof is rather tedious and will not be presented here. The reader is referred to [Ho, Theorem 3.6.5] for a proof.

Finally, we state without proof the following result.

Theorem 3.11. *The adjunction* $|\bullet| \dashv \mathrm{Sing}$ *is a Quillen equivalence between* sSet *and* Top.

For a proof, see [Ho, Theorem 3.6.7].

As an easy exercise, the reader can show that the adjunction is a Quillen adjunction, without using this theorem.

Categories as simplicial sets

Simplicial sets can be seen as a model for categories. For example, the simplicial set $\Delta[2]$ can be seen as a diagram

$$\begin{tikzcd} & \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur] \arrow[rr] & {} & \bullet \end{tikzcd},$$

where the double arrow indicates that the 2-simplex "witnesses" the composition of the two arrows. In an ordinary category, this diagram is just a chain of 2 arrows

$$\bullet \to \bullet \to \bullet .$$

As another example, the simplicial set $\Delta[2] \times \Delta[1]$ corresponds to a diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}.$$

Definition 3.12. *Let* C *be a (small) category. The* **nerve** *of* C *is a simplicial set denoted by* $\mathrm{N}(\mathsf{C})$. *Its* n*-simplices are chains of* n *arrows*

$$\bullet \to \bullet \to \cdots \to \bullet$$

in C. *Its* 0*-th (resp.* n*-th) face map is defined by discarding the first arrow (resp. the last arrow). For* $0 < i < n$, *its* i*-th face map is defined by composing its* i*-th and* $(i+1)$*-th maps. Its degeneracy maps are defined by inserting identity morphisms.*

The reader can verify that, in the above examples, the nerve of the categories are the corresponding simplicial sets.

Remark 3.13. In fact, this is another special case of (3.8). Namely, the cosimplicial object $\Delta^\bullet$ in Cat, given by

$$\Delta^n := \quad \bullet \to \bullet \to \cdots \to \bullet$$

with n consecutive arrows, gives an adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathrm{Ho}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "\mathrm{N}"] \end{tikzcd}.$$

The right adjoint is precisely the nerve functor defined above. ◁

Note that not all simplicial sets are categories. For example, consider $\Lambda_1[2]$, with two arrows that cannot be composed. In fact, if we define the **spine** $\mathrm{Sp}(n) \subset \Delta[n]$ to consist of all vertices and the edges $[i, i+1]$ for $0 \leq i < n$, then a simplicial set X is the nerve of a category, if and only if it satisfies the lifting property

$$\begin{tikzcd} \mathrm{Sp}(n) \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

for all $n \geq 0$.

Kan complexes and quasi-categories

Kan complexes are simplicial sets that looks like topological spaces, and will be used to model ∞-groupoids.

Definition 3.14. *A* **Kan complex** *is a fibrant simplicial set. In other words, it satisfies the* **horn extension property***, that is, the lifting property*

$$\begin{tikzcd} \Lambda_i[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

for all $0 \leq i \leq n$, $n > 0$.

Example 3.15. The simplicial set $\Delta[n]$ is *not* a Kan complex if $n \geq 1$. Namely, consider the horn

$$\Lambda_0[2] \to \Delta[n],$$
$$0, 1, 2 \mapsto 0, 1, 0.$$

Then it is impossible to extend this horn to make a $\Delta[2]$. The reason is that $\Delta[n]$, seen as a category, does not have inverses of morphisms. In other words, it does not look like a groupoid, while Kan complexes must look like groupoids. $\triangleleft$

Example 3.16. For any topological space X, the simplicial set $\mathrm{Sing}\, X$ is a Kan complex, as one can show easily. Therefore, for any simplicial set S, the simplicial set $\mathrm{Sing}\, |S|$ can be used as a fibrant replacement of S. $\triangleleft$

As we have seen above, Kan complexes look like groupoids with higher structures. In a Kan complex, 1-morphisms are invertible, but only up to a 2-morphism, i.e. 2-simplex. The horn extension property ensures that all higher morphisms are also invertible, up to even higher morphisms. This is exactly what an ∞-groupoid should be.

Definition 3.17. *An* ∞**-groupoid***, or an* $(\infty, 0)$**-category***, is a Kan complex.*

For example, for a topological space X, the Kan complex $\operatorname{Sing} X$ can be seen as the "fundamental ∞-groupoid" of X.

Definition 3.18. *A* **homotopy type** *is a homotopy type of ∞-groupoids, i.e. an element of* $\mathrm{Ho}(\mathsf{sSet})$*, which is equivalent to the category* $\mathrm{Ho}(\mathsf{Top}) \simeq \mathsf{hCW}$.

Next, we wish to define $(\infty, 1)$-categories in a similar way. The following table shows that the extension of horns corresponds to properties of a category, assuming that we are considering the nerve of an ordinary category.

Horn	**Property**
$\Lambda_0[2]$	every morphism is left invertible
$\Lambda_1[2]$	composition of morphisms
$\Lambda_2[2]$	every morphism is right invertible
$\Lambda_0[3]$	every morphism is an epimorphism
$\Lambda_1[3], \Lambda_2[3]$	associativity of composition
$\Lambda_3[3]$	every morphism is a monomorphism
otherwise	(satisfied by any category)

As an exercise, the reader should verify everything in the table.

We notice that the extension of $\Lambda_0[2]$, $\Lambda_2[2]$, $\Lambda_0[3]$ and $\Lambda_3[3]$ can not be satisfied by all categories, while the extension of "inner horns" $\Lambda_i[n]$ for $0 < i < n$ describes properties that any category should satisfy.

Definition 3.19. *A* **quasi-category***, or an* $(\infty, 1)$**-category***, is a simplicial set having the lifting property*

$$\begin{tikzcd} \Lambda_i[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

for $0 < i < n$*. In other words,* **inner horns** *can be extended.*

The terminology is that quasi-categories are one of the various models for $(\infty, 1)$-categories. For this reason, we will stick to the term "quasi-categories". Some people also call quasi-categories "weak Kan complexes".

Roadmap

A number of models and tools for studying $(\infty, 1)$-categories will be used in the sequel. Since we have finished most of the definitions, it is a good point now to draw a roadmap as a preview of what we will encounter.

For a monoidal category V, let $\mathsf{Cat}_{\mathsf{V}}$ denote the category of categories enriched over V. (We ignore the set-theoretic issues here.) Let Model denote the category of model categories. Let $\mathsf{Model}_{\mathsf{V}}$ denote the category of V-enriched model categories, which we have not defined yet. Let QsCat denote the category of quasi-categories (which should really be replaced by sSet in the following diagram).

We have a diagram of categories

$$
\begin{tikzcd}
\mathsf{Model} \arrow[dd, "\text{localise}"'] \arrow[ddddr, bend right=60, "\text{Ho}"'] & \mathsf{Model}_{\mathsf{sSet}} \arrow[l, "\text{forget}"'] \arrow[dd, "\text{forget}"] & & \\
& & \mathsf{Cat}_{\mathsf{sMod}_R} \arrow[r, bend left, "\text{Dold–Kan}"] \arrow[r, phantom, "\perp_{\simeq}"] \arrow[dl, bend left, "\text{forget}"] \arrow[dl, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{Ch}_R} \text{ (dg categories)} \arrow[l, bend left] \\
\mathsf{QsCat} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp_{\simeq}"] \arrow[ddr, bend right, "h"'] & \mathsf{Cat}_{\mathsf{sSet}} \arrow[l, bend left, "\mathfrak{N}"] \arrow[ur, bend left, "\text{free}"] \arrow[dr, bend left, "|\cdot|"] \arrow[dr, phantom, "\perp_{\simeq}"] \arrow[dd, "h"'] & & \\
& & \mathsf{Cat}_{\mathsf{Top}} \arrow[ul, bend left, "\mathit{Sing}"] \arrow[dl, bend left, "h"] & \\
& \mathsf{Cat}_{\mathsf{hCW}} \arrow[d, "\pi_0"] & & \\
& \mathsf{Cat}\,. & &
\end{tikzcd}
$$

Some of the maps are easy to define, while others will be defined later. This diagram is “commutative”, in a sense which will be made precise later on. Everything above $\mathsf{Cat}_{\mathsf{hCW}}$ can be seen as models for $(\infty, 1)$-categories. Homotopy theory in these categories are called **homotopy coherent**, as opposed to **homotopy commutative**, which refers to commutative diagrams in $\mathsf{Cat}_{\mathsf{hCW}}$.

As we can see in the diagram, homological algebra, which is done in a dg (differential graded) category, is related to $(\infty, 1)$-category theory. This relationship will be studied a few sections later.

4 Models for ∞-categories

Infinity categories are difficult to study. Unlike in classical category theory, in order to understand infinity categories, one needs to work with various models, instead of a single definition.

In these notes, we will work with two models for $(\infty, 1)$-categories: quasi-categories, and categories enriched over Kan complexes. We have already defined both concepts in previous sections, and we have seen why they are able to model infinity categories. Now, we will explore the relationships between these two models. We will see that these models are equivalent, in the sense of a Quillen equivalence.

Quasi-categories

Recall from the previous section that a quasi-category is a simplicial set in which inner horns can be extended.

Let C be a quasi-category, Let $x, y \in \mathsf{C}_0$ be two points. Our first goal is to define the hom-space $\mathrm{Hom}_{\mathsf{C}}(x, y)$. Instead of being a discrete set, it should contain information describing homotopies and higher homotopies, which comprise the higher structure of an infinity category.

We now introduce some terminology for a quasi-category C.

- We say $x \in \mathsf{C}$ is an **object** of C, if $x \in \mathsf{C}_0$.
- We say $f : x \to y$ is a **morphism** of C, if $f \in \mathsf{C}_1$, and $d_0 f = x$, $d_1 f = y$.
- The **identity morphism** 1_x of an object $x \in \mathsf{C}$ refers to the degenerate 1-simplex $s_0(x)$.
- For morphisms $f : x \to y$ and $g : y \to z$, we say that a morphism $h : x \to z$ is a **composition** of g and f, if there is a 2-simplex $\sigma \in \mathsf{C}_2$ such that $d_0\sigma = g$, $d_2\sigma = f$, and $d_1\sigma = h$. This can be drawn as a diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

  in C. Note that composition is not unique in a quasi-category.
- Let $A \subset \mathsf{C}_0$ be a set of objects of C. The **full subcategory** spanned by A is the largest sub-simplicial set of C whose 0-simplices are precisely those in A.

Definition 4.1. *Let $f, g : x \to y$ be two morphisms in C. We say that f and g are* **homotopic**, *if the following equivalent conditions hold.*

- *There is a 2-simplex* $\begin{tikzcd} x \arrow[d, "1_x"'] \arrow[dr, "f"] & \\ x \arrow[r, "g"'] & y \end{tikzcd}$ *in* C.
- *There is a 2-simplex* $\begin{tikzcd} x \arrow[d, "1_x"'] \arrow[dr, "g"] & \\ x \arrow[r, "f"'] & y \end{tikzcd}$ *in* C.
- *There is a 2-simplex* $\begin{tikzcd} & y \arrow[d, "1_y"] \\ x \arrow[ur, "f"] \arrow[r, "g"'] & y \end{tikzcd}$ *in* C.
- *There is a 2-simplex* $\begin{tikzcd} & y \arrow[d, "1_y"] \\ x \arrow[ur, "g"] \arrow[r, "f"'] & y \end{tikzcd}$ *in* C.
- *There is a square* $\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "1_x"'] & y \arrow[d, "1_y"] \\ x \arrow[r, "g"'] & y \end{tikzcd}$ *in* C, *which is a map* $\Delta[1] \times \Delta[1] \to \mathsf{C}$.
- *There is a square* $\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "1_x"'] & y \arrow[d, "1_y"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$ *in* C.

Using the horn extension property for $\Lambda_1[3]$ and $\Lambda_2[3]$, one can show that all the above conditions are equivalent. We leave this as an exercise for the reader.

Proposition 4.2. *Homotopy of morphisms is an equivalence relation, and respects composition of morphisms. In particular, composition is unique up to homotopy.*

Proof. Exercise for the reader. $\square$

Definition 4.3. *Let* C *be a quasi-category. The* **homotopy category** *of* C *is an ordinary category* $\mathrm{Ho}(\mathsf{C})$*, defined as follows.*

- *Its objects are objects of* C*.*
- *Its morphisms are homotopy classes of morphisms in* C*.*

We regard $\mathrm{Ho}(\mathsf{C})$ as obtained from C by forgetting its higher structure.

Theorem 4.4 (Joyal)**.** *Let* C *be a quasi-category. Then* C *is a Kan complex if and only if* $\mathrm{Ho}(\mathsf{C})$ *is a groupoid.*

It is easy to see that if C is a Kan complex, i.e. an ∞-groupoid, then $\mathrm{Ho}(\mathsf{C})$ is a groupoid. But the converse is a non-trivial result.

Our next step is to assign a "higher structure", i.e. a homotopy type, to every hom-space $\mathrm{Hom}_{\mathsf{C}}(x, y)$, as a way to describe higher homotopies between morphisms.

Definition 4.5. *Let* C *be a quasi-category, and let* $x, y \in \mathsf{C}$ *be two objects.*

- *The simplicial set* $\mathrm{Hom}^{\triangleright}_{\mathsf{C}}(x, y)$ *is defined as follows. Its n-simplices are*
$$\mathrm{Hom}^{\triangleright}_{\mathsf{C}}(x, y)_n := \{\sigma \in \mathsf{C}_{n+1} \mid \sigma|_{\Delta\{0,\ldots,n\}} = x,\ \sigma(n+1) = y\},$$
where σ *is regarded as a map* $\Delta[n+1] \to \mathsf{C}$*, and* $\Delta\{0, \ldots, n\} \subset \Delta[n+1]$ *is the face spanned by the vertices* $0, \ldots, n$*.*
- *Dually, we define the simplicial set* $\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y)$ *by*
$$\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y)_n := \{\sigma \in \mathsf{C}_{n+1} \mid \sigma(0) = x,\ \sigma|_{\Delta\{1,\ldots,n+1\}} = y\}.$$
- *There is yet another version of the hom-space* $\mathrm{Hom}^{\square}_{\mathsf{C}}(x, y)$*, defined by*
$$\mathrm{Hom}^{\square}_{\mathsf{C}}(x, y)_n := \{\sigma : \Delta[n] \times \Delta[1] \to \mathsf{C} \mid \sigma|_{\Delta[n]\times\{0\}} = x,\ \sigma|_{\Delta[n]\times\{1\}} = y\}.$$

These three constructions do not give isomorphic simplicial sets in general. However, we will see that these three simplicial sets are homotopy equivalent Kan complexes, so that the hom-space has a well-defined homotopy type.

Note that none of these constructions can produce a category enriched over simplicial sets. The reason is that composition can not be well-defined. However, we will soon construct a simplicial category whose hom-spaces are homotopy equivalent to these ones.

Proposition 4.6. *The simplicial sets* $\mathrm{Hom}^{\triangleright}_{\mathsf{C}}(x, y)$*,* $\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y)$ *and* $\mathrm{Hom}^{\square}_{\mathsf{C}}(x, y)$ *are Kan complexes.*

Proof. For $\mathrm{Hom}^{\triangleright}_{\mathsf{C}}(x, y)$, it is not difficult to see that it has the horn extension property for the horns $\Lambda_i[n]$ for $0 < i \leq n$. Therefore, it is a quasi-category. Moreover, in its homotopy category, every morphism has a left inverse. This implies that its homotopy category is a groupoid, so that by (4.4), it is a Kan complex. The same argument shows that $\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y)$ is also a Kan complex.

For $\mathrm{Hom}^{\square}_{\mathsf{C}}(x, y)$, the proof uses the Joyal model structure on sSet, and will be presented later. □

Simplicial categories

Recall from the first section that intuitively, an $(\infty, 1)$-category can be seen as a category enriched over the category of $(\infty, 0)$-categories, which are modelled with Kan complexes. Therefore, categories enriched over Kan complexes should be another model for $(\infty, 1)$-categories. We generalise this a bit by considering categories enriched over all simplicial sets.

Definition 4.7. *A* **simplicial category** *is a category enriched over* sSet.

In a simplicial category, we regard the 0-simplices of the hom-spaces as morphisms, the 1-simplices as homotopies between morphisms, and higher dimensional simplices as higher homotopies.

For example, the category sSet is a simplicial category, equipped with the following simplicial structure on its hom-spaces.

Definition 4.8. *Let* X, Y *be two simplicial sets. The* **mapping space** $\mathrm{Map}(X, Y)$ *is the simplicial set whose* n*-simplices are maps from* $X \times \Delta[n]$ *to* Y*.*

The category Top can also be seen as a simplicial category, by taking the Sing of all its mapping spaces, equipped with the compact open topology.

It is very easy to define the homotopy category of a simplicial category.

Construction 4.9. Let C be a simplicial category. The functors

$$\mathsf{sSet} \xrightarrow{h} \mathsf{hCW} \xrightarrow{\pi_0} \mathsf{Set}$$

assign C with an hCW-enriched category $h\mathsf{C}$, and an ordinary category denoted by $\mathrm{Ho}(\mathsf{C}) := \pi_0 h\mathsf{C}$. The latter is called the **homotopy category** of C. ◁

Although the definition of an enriched category requires strict (i.e. unique) composition and strict associativity, many other things can be done in the non-strict, or “up to homotopy” way, and it is often good to think of simplicial categories in the non-strict way. Here is an example.

Definition 4.10. *A* **simplicial groupoid** *is a simplicial category whose homotopy category is a groupoid. A* **simplicial group** *is a simplicial groupoid with a unique object.*

Compare (4.4). In this definition, although composition is strict, taking the inverse is non-strict, i.e. up to homotopy.

Next, we aim to define a pair of adjoint functors

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{sSet}} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd},$$

as a conversion between quasi-categories and simplicial categories. We will see that this adjunction is a Quillen equivalence, given suitable model structures on both sides.

To define such an adjunction, by (3.8), we only need to specify what $\mathfrak{C}\Delta[n]$ is. Intuitively, it should look like a chain of n arrows

$$0 \to 1 \to \cdots \to n\,,$$

whose nerve is $\Delta[n]$. However, we need to modify it in order to allow the composition law, e.g. $(0 \to 1 \to 2) = (0 \to 2)$, to hold only up to homotopy. This motivates the following construction.

Construction 4.11. Let $n \geq 0$ be an integer. The simplicial category $\mathfrak{C}^n$ is defined as follows.

- It has $(n+1)$ objects, which we call $0, 1, \ldots, n$.
- Its hom-spaces are given by $\mathrm{Hom}(i, j) = N(P_{ij})$, where

$$P_{ij} := \begin{cases} \varnothing, & i > j, \\ \text{poset of subsets of } \{i, i+1, \ldots, j\} \text{ containing } i, j, & i \leq j, \end{cases}$$

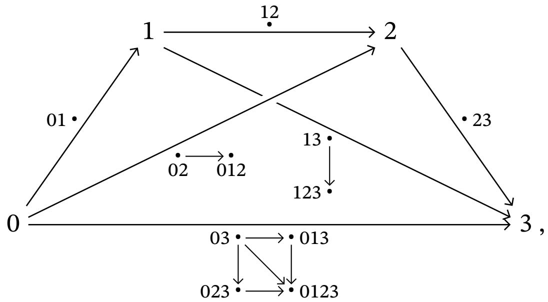
  where a poset is naturally regarded as a category. Composition is defined by union of sets. ◁

For example, $\mathfrak{C}^2$ is the simplicial category

$$\begin{tikzcd} & 1 \arrow[dr, "\bullet\, 12"] & \\ 0 \arrow[ur, "01\, \bullet"] \arrow[rr, "\overset{\bullet}{02} \longrightarrow \overset{\bullet}{012}"'] & & 2 \end{tikzcd},$$

where the composition $12 \circ 01$ gives the map $012 \colon 0 \to 2$, which is homotopic, but not equal to, the map $02 \colon 0 \to 2$ which we regard as going directly (not passing 1) from 0 to 2.

Likewise, $\mathfrak{C}^3$ is the simplicial category



where the mapping space $N(P_{03})$ is isomorphic to $\Delta[1] \times \Delta[1]$. As we can see, the 4 points in this space correspond to the 4 ways to go from 0 to 4 along the arrows, namely, $0 \to 3$, $0 \to 1 \to 3$, $0 \to 2 \to 3$, and $0 \to 1 \to 2 \to 3$. All these 4 maps from 0 to 3 are homotopic.

In general, the mapping space $\mathrm{Hom}_{\mathfrak{C}^n}(i, j)$ is the space of all ways to go from i to j along the arrows. If $i < j$, then it is isomorphic to $\Delta[1]^{j-i-1}$.

Definition 4.12. *The cosimplicial object $\mathfrak{C}^\bullet$ in $\mathsf{Cat}_{\mathsf{sSet}}$ determines an adjunction*

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{sSet}} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd} ,$$

in the sense of (3.8)*. The right adjoint $\mathfrak{N}$ is called the* **simplicial nerve** *functor, or the* **homotopy coherent nerve** *functor.*

Now we can prove that categories enriched over Kan complexes are converted to quasi-categories under this construction.

Proposition 4.13. *If C is a category enriched over Kan complexes, then $\mathfrak{N}\mathsf{C}$ is a quasi-category.*

Proof. Not yet written □

Equivalence of the two models

For a quasi-category or a simplicial category C, we have defined a mapping space $\mathrm{Hom}_{\mathsf{C}}(x, y)$ for each pair of objects x, y in C. In fact, the homotopy type of the mapping space is preserved by the above conversion between the two models.

Theorem 4.14. *We have the following.*

- *If* C *is a quasi-category, then we have a weak equivalence of simplicial sets*

$$\mathrm{Hom}^{(*)}_{\mathsf{C}}(x,y) \simeq \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(x,y)$$

for each pair of objects $x, y \in \mathsf{C}$*, where* $(*)$ *can be* $\lhd$*,* $\rhd$*, or* $\square$*.*

- *If* C *is a category enriched over Kan complexes, then we have a homotopy equivalence of Kan complexes*

$$\mathrm{Hom}_{\mathsf{C}}(x,y) \simeq \mathrm{Hom}^{(*)}_{\mathfrak{N}\mathsf{C}}(x,y)$$

for each pair of objects $x, y \in \mathsf{C}$*, where* $(*)$ *can be* $\lhd$*,* $\rhd$*, or* $\square$*.*

The proof of the theorem will be given in the next section. Note that weak equivalences between Kan complexes are homotopy equivalences, by Whitehead's theorem (2.16).

Thus, for a quasi-category C, we can define the hCW-enriched category $h\mathsf{C}$ to be $h\mathfrak{C}\mathsf{C}$. It has the same objects and the same homotopy types of mapping spaces as C. In fact, this definition works when C is any simplicial set.

Next, we describe the adjoint pair $(\mathfrak{C} \dashv \mathfrak{N})$ as a Quillen equivalence, so that the two models are really equivalent.

Definition 4.15. *Let* C, D *be two simplicial sets, or two simplicial categories. A map* $f : \mathsf{C} \to \mathsf{D}$ *is called a* **categorical equivalence***, if* $h(f) : h\mathsf{C} \to h\mathsf{D}$ *is an equivalence of* hCW*-enriched categories, i.e. the following holds.*

- f *is fully faithful, i.e. induces weak equivalences of mapping spaces.*
- f *is essentially surjective, i.e.* $\pi_0 h(f) = \mathrm{Ho}(f)$ *is an essentially surjective functor between ordinary categories.*

Remark 4.16. This defines the ∞-categorical notion of a categorical equivalence. Any reasonable property of ∞-categories should be preserved by categorical equivalences. Otherwise, we will regard that property as ill-defined. $\lhd$

From now on, by an **equivalence of ∞-categories**, or a similar phrase, we always refer to a categorical equivalence.

Recall that a functor $f : \mathsf{C} \to \mathsf{D}$ between ordinary categories is an **isofibration**, if for any $x \in \mathsf{C}$ and any isomorphism $\alpha : f(x) \to y$ in D, there exists $\tilde{\alpha} : x \to \tilde{y}$ in C such that $\alpha = f(\tilde{\alpha})$.

Theorem 4.17. *The category* sSet *has the* **Joyal model structure***, with*

- $\mathsf{W} = \{$*categorical equivalences*$\}$.
- $\mathsf{Cof} = \{$*injections*$\}$.
- *The fibrant objects are quasi-categories.*

The category $\mathsf{Cat}_{\mathsf{sSet}}$ *has the* **Bergner model structure***, with*

- $\mathsf{W} = \{$*categorical equivalences*$\}$.
- $\mathsf{Fib} = \{$*isofibrations that are fibrations on mapping spaces*$\}$.
- *The fibrant objects are categories enriched over Kan complexes.*

Using these model structures, the adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{sSet}} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

is a Quillen equivalence.

See [Lu09, Theorem 2.2.5.1].

This theorem has the following important corollary, which states that one can convert between quasi-categories and categories enriched between Kan complexes, up to a categorical equivalence.

Corollary 4.18 (Conversion between the two models)**.** *The functors*

$$\begin{tikzcd} \mathsf{QsCat} \arrow[r, bend left, "R\mathfrak{C}"] & \mathsf{Cat}_{\mathsf{Kan}} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

are inverses of each other up to categorical equivalences, where R denotes the fibrant replacement functor of sSet*.*

Proof. Let C be a quasi-category, and let D be a category enriched over Kan complexes. The unit and counit maps

$$\mathfrak{N}R\mathfrak{C}\mathsf{C} \to \mathsf{C} \quad \text{and} \quad \mathsf{D} \to \mathfrak{C}Q\mathfrak{N}\mathsf{D}$$

are categorical equivalences (2.24). Here we may take Q to be the identity functor, which implies that the functors

$$\mathfrak{N}R\mathfrak{C}\mathsf{C} \to \mathsf{C} \quad \text{and} \quad \mathsf{D} \to R\mathfrak{C}\mathfrak{N}\mathsf{D}$$

are categorical equivalences. □

Remark 4.19. We do not use the standard model structure on sSet when we study quasi-categories, and here is a reason. Let C be a quasi-category. In many examples, C will contain an initial object x, and thus, the geometric realisation of C can be contracted to x linearly. Therefore, the homotopy type of C is trivial. On the other hand, the Joyal model structure captures the internal information of a quasi-category, rather than the global homotopy type. ◁

Examples

Let us look at a few examples of $(\infty, 1)$-categories, modelled as quasi-categories. We are now able to talk about these objects in a rigorous way.

Proposition 4.20. *Let X, Y be two simplicial sets.*

- *If Y is a Kan complex, then* $\mathrm{Map}(X, Y)$ *is a Kan complex.*
- *If Y is a quasi-category, then* $\mathrm{Map}(X, Y)$ *is a quasi-category.*

Proof. Exercise for the reader. □

Example 4.21. The **quasi-category of spaces** is defined to be

$$\mathsf{S} := \mathfrak{N}(\mathsf{Kan}),$$

where Kan is the simplicial category of Kan complexes, which is enriched over Kan complexes by (4.20). ◁

Example 4.22. The Kan-enriched category QsCat is defined as follows.

- The objects are quasi-categories.
- The morphism space $\mathrm{Hom}(\mathsf{C}, \mathsf{D})$ is the maximal Kan complex in the simplicial set $\mathrm{Map}(\mathsf{C}, \mathsf{D})$.

Such a maximal Kan complex exists, since by (4.20) and (4.4), it is the sub-simplicial set spanned by all the invertible edges.

The **quasi-category of quasi-categories** is defined to be

$$\mathsf{Cat}_\infty := \mathfrak{N}(\mathsf{QsCat}).$$

Note that we have taken the maximal Kan complex, instead of a fibrant replacement. This is because the edges in the mapping space are natural transformations, and we discard those natural transformations that are not invertible, rather than inverting them.

This means that in order to get an $(\infty, 1)$-category of $(\infty, 1)$-categories, we have to discard some information. The essential reason is that $(\infty, 1)$-categories should form an $(\infty, 2)$-category. This can be seen as follows: in our model, the simplicial category of quasi-categories is naturally enriched over quasi-categories, making it an $(\infty, 2)$-category. ◁

Example 4.23. Let A be an abelian category. The simplicial category Ch_{A} of cochain complexes in A is defined as follows. Recall that for a map $f : X \to Y$ of cochain complexes (not necessarily a chain map), we defined

$$df := d \circ f - (-1)^{|f|} f \circ d,$$

where $|f| = k$ if f sends X^n to Y^{n+k}.

- The 0-simplices of $\mathrm{Hom}(X, Y)$ are those maps $f : X \to Y$ such that

$$|f| = 0 \quad \text{and} \quad df = 0.$$

  In other words, they are chain maps.
- A 1-simplex between two 0-simplices f, g is a map $a : X \to Y$ such that

$$|a| = -1 \quad \text{and} \quad da = f - g.$$

  In other words, they are chain homotopies.
- An n-simplex consists of the data $(\sigma, \sigma_0, \dots, \sigma_n)$, where each σ_i is an $(n-1)$-simplex, and $\sigma : X \to Y$ is a map satisfying

$$|\sigma| = -n \quad \text{and} \quad d\sigma = \sigma_0 - \sigma_1 + \sigma_2 - \cdots \pm \sigma_n.$$

  The σ_i are the faces of σ, and we require that the faces of the σ_i are compatible with each other. In other words, an n-simplex is a map

$$C_\bullet^{\text{cell}}(\Delta^n) \to \mathcal{H}om(X, Y)$$

  from the cellular chain complex of Δ^n to the chain complex $\mathcal{H}om(X, Y)$ defined in (1.12).

We define

$$\mathsf{K}_\infty(\mathsf{A}) := \mathfrak{N}(\mathsf{Ch}_\mathsf{A})$$

to be the **quasi-category of cochain complexes** in A. If Ch_A has a suitable model structure, e.g. if A is the category of modules over a ring, then we define

$$\mathsf{D}_\infty(\mathsf{A}) := \mathfrak{N}(\mathsf{Ch}_{\mathsf{A},\mathrm{cf}})$$

to be the **derived quasi-category** of A. ◁

Remark 4.24. We have seen that localising a category with weak equivalences gives rise to ∞-categories. It turns out that if C is a model category, together with a simplicial enrichment, satisfying some extra compatibility conditions, then we have an equivalence of quasi-categories

$$\mathsf{C}[\mathsf{W}^{-1}] \simeq \mathfrak{N}(\mathsf{C}_{\mathrm{cf}}).$$

Such a simplicial structure can be found in all the examples that we have seen. Therefore, we have

$$\mathsf{S} \simeq \mathsf{sSet}[\mathsf{W}^{-1}] \quad \text{and} \quad \mathsf{D}_\infty(\mathsf{A}) \simeq \mathsf{Ch}_\mathsf{A}[\mathsf{W}^{-1}],$$

as examples of ∞-categorical localisations. These will be made precise in later sections. ◁

5 Grothendieck construction

Assume we have a functor $f : X \to S$ between two categories. Then for any $s \in S$, we may take the fibre $X_s := f^{-1}(s)$, which is a subcategory of X. This gives a rule assigning every object of S a category. We might expect that this assignment, i.e. taking the fibres, gives a functor $S \to \mathsf{Cat}$. Indeed, this is true provided that f is a "fibration", in a sense that will be made precise soon. This is called the **Grothendieck construction**. Schematically, this means that we have a correspondence

$$\begin{tikzcd} X \arrow[d, "\text{fibration}"'] & \iff & \mathsf{Cat} \\ S & & S \arrow[u, "\text{take fibres}"'] \end{tikzcd}$$

between "fibrations" over S and functors from S to Cat.

Let us look at an example arising from algebraic geometry.

Example 5.1. Let Sch denote the category of schemes. For a scheme X, let QCoh_X be the category of quasi-coherent sheaves on X. This defines a functor

$$\mathsf{QCoh}_{(-)} : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Cat}.$$

If we look carefully, it is not a functor in the usual sense, since the composition law

$$f^* \circ g^* = (g \circ f)^*$$

only holds up to a natural isomorphism, i.e. a 2-morphism in Cat. We call such a "functor" a 2-**functor**.

Applying the Grothendieck construction to this 2-functor, we should get a "fibration" which we denote by

$$\mathsf{QCoh} \to \mathsf{Sch}^{\mathrm{op}},$$

whose fibre over $X \in \mathsf{Sch}^{\mathrm{op}}$ is the category QCoh_X. Indeed, we may construct the category QCoh of all quasi-coherent sheaves as follows.

- The objects are pairs $(X, \mathcal{F})$, where $\mathcal{F}$ is a quasi-coherent sheaf on the scheme X.
- A morphism from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ consists of a map of schemes $f : Y \to X$, together with a map of $\mathcal{O}_Y$-modules $f^*\mathcal{F} \to \mathcal{G}$, or equivalently, a map of $\mathcal{O}_X$-modules $\mathcal{F} \to f_*\mathcal{G}$.

The natural forgetful functor $\mathsf{QCoh} \to \mathsf{Sch}^{\mathrm{op}}$ is the "fibration" that we wished to construct. This is an example of a *cocartesian fibration*, which we will define soon. ◁

For ordinary categories

We will study four kinds of fibrations of categories, which correspond to functors to Cat according to the following table, where Gpd denotes the category of groupoids.

Fibration		**Functor**
left fibration $X \to S$	$\Longleftrightarrow$	$S \to \mathsf{Gpd}$
right fibration $X \to S$	$\Longleftrightarrow$	$S^{\mathrm{op}} \to \mathsf{Gpd}$
cocartesian fibration $X \to S$	$\Longleftrightarrow$	$S \to \mathsf{Cat}$
cartesian fibration $X \to S$	$\Longleftrightarrow$	$S^{\mathrm{op}} \to \mathsf{Cat}$

We start with left fibrations, which Grothendieck originally called "categories cofibred in groupoids". By the table above, for any $s \in S$, the fibre X_s should be a groupoid, and for any morphism $s \to s'$ in S, we should have a "transport map" $X_s \to X_{s'}$.

In order to define the transport map, we require that the map $p : X \to S$ satisfies the following properties.

- Transport of objects: for any $x \in X$ and any morphism $\alpha : s \to s'$ in S, where $s := p(x)$, there exists a morphism $\tilde{\alpha} : x \to x'$ in X such that $p(\tilde{\alpha}) = \alpha$. Pictorially, this means that the lifting problem

$$\begin{tikzcd} x \arrow[d, dotted, no head] & \\ s \arrow[r] & s' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} x \arrow[r, dashed] \arrow[d, dotted, no head] & x' \arrow[d, dotted, no head] \\ s \arrow[r] & s' \end{tikzcd}$$

  has a solution.

- Transport of morphisms: the lifting problem

$$\begin{tikzcd} x \arrow[r] \arrow[d] & x' \\ y \arrow[r] \arrow[d, dotted, no head] & y' \arrow[d, dotted, no head] \\ s \arrow[r] & s' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} x \arrow[r] \arrow[d] & x' \arrow[d, dashed] \\ y \arrow[r] \arrow[d, dotted, no head] & y' \arrow[d, dotted, no head] \\ s \arrow[r] & s' \end{tikzcd}$$

  has a *unique* solution.

It is easy to see that if these properties are satisfied, then the transport map $X_s \to X_{s'}$ is well-defined up to a natural isomorphism. Namely, one chooses an arbitrary way to transport the objects, and then the morphisms can be transported uniquely.

We define left fibrations by reformulating these axioms.

Definition 5.2. *A functor $p : X \to S$ is a* **left fibration***, if it satisfies the following transport axioms.*

- *The lifting problem*

$$\begin{tikzcd} \Lambda_0[1] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[1] \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

  has a solution.
- *The lifting problem*

$$\begin{tikzcd} \Lambda_0[2] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[2] \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

  has a unique solution.

In these diagrams, the notations $\Lambda_0[n]$ *and* $\Delta[n]$ *represent their corresponding ordinary categories. Precisely, they are* $\mathrm{Ho}(\Lambda_0[n])$ *and* $\mathrm{Ho}(\Delta[n])$*.*

As an exercise, the reader can verify that this definition is equivalent to the two transport axioms given above it.

Remark 5.3. The uniqueness in the second axiom is actually a consequence of the lifting property for $\Lambda_0[3]$, and thus will be replaced by the latter in the ∞-categorical definition. ◁

If we take the map $\Delta[2] \to S$ to be the constant map at s, we see immediately that every morphism in the fibre X_s has a left inverse, and hence X_s is a groupoid.

The discussion above readily implies the following.

Theorem 5.4 (Grothendieck construction)**.** *A left fibration between small categories* $X \to S$ *gives rise to a 2-functor* $S \to \mathsf{Gpd}$ *by taking the fibres. Conversely, every 2-functor* $S \to \mathsf{Gpd}$ *corresponds to a left fibration* $X \to S$ *in this manner.*

Proof. We only need to prove the converse. However, the construction of such a left fibration is done in the same way as in (5.1). □

Dually, a functor $p : X \to S$ is a **right fibration** if $p^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a left fibration. Under the Grothendieck construction, they correspond to functors $S^{\mathrm{op}} \to \mathsf{Gpd}$.

For left fibrations

Our next goal is to define Grothendieck construction for ∞-categories. As an application, we will give a proof for the fact that

$$\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y) \simeq \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(x, y) \simeq \mathrm{Hom}^{\triangleright}_{\mathsf{C}}(x, y)$$

for objects x, y of a quasi-category C.

Definition 5.5. *A map of simplicial sets* $p : X \to S$ *is called a* **left fibration**, *if the lifting problem*

$$\begin{tikzcd} \Lambda_i[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

has a solution for all $0 \leq i < n$.

Dually, p *is called a* **right fibration** *if the above lifting problem has a solution for all* $0 < i \leq n$.

This definition is a natural generalisation of the corresponding notion for ordinary categories.

Proposition 5.6. *Let* X *be a simplicial set. The map* $X \to *$ *is a left fibration, if and only if* X *is a Kan complex.*

Proof. The "if" part is trivial. For the converse, suppose that $X \to *$ is a left fibration. Then X is a quasi-category. Moreover, every morphism in the ordinary category $\mathrm{Ho}(X)$ admits a left inverse, so that $\mathrm{Ho}(X)$ is a groupoid. By (4.4), X must be a Kan complex. □

Let S be a simplicial set, and let $\mathrm{sSet}_{/S}$ denote the over-category, which consists of

- The objects are maps $X \to S$.
- A morphism from a map $X \to S$ to a map $Y \to S$ is a map $X \to Y$ that fits into a commutative triangle

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S\,. \end{tikzcd}$$

Following Lurie [Lu09], we will define a pair of adjoint functors

$$\begin{tikzcd} \mathrm{sSet}_{/S} \arrow[r, bend left, "\mathrm{St}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_{\mathrm{sSet}}(\mathfrak{C}S, \mathrm{sSet})\,, \arrow[l, bend left, "\mathrm{Un}"] \end{tikzcd}$$

where $\mathrm{Fun}_{\mathrm{sSet}}$ denotes the (ordinary) category of simplicially enriched functors, and we have the following.

- The functor St is called the **straightening functor**. Restricted to the left fibrations, it will give the Grothendieck construction.
- The functor Un is called the **unstraightening functor**. Restricted to the functors $\mathfrak{C}S \to \mathsf{Kan}$, or equivalently $S \to \mathfrak{N}(\mathsf{Kan}) =: \mathsf{S}$, it will give the other direction of the construction.

- The adjunction (St $\dashv$ Un) will be a Quillen equivalence, given suitable model structures on both categories.

Now, we begin the construction.

Definition 5.7. *Let X, Y be simplicial sets. Their* **join** *is the simplicial set $X \star Y$, whose n simplices are*

$$(X \star Y)_n := \left\{ (f, \sigma_0, \sigma_1) \;\middle|\; \begin{array}{c} f : \Delta[n] \to \Delta[1] \\ \sigma_0 : f^{-1}(0) \to X \\ \sigma_1 : f^{-1}(1) \to Y \end{array} \right\},$$

with the natural face and degeneracy maps.

In other words, $X \star Y$ is obtained from $X \sqcup Y$ by adjoining all possible arrows (and higher dimensional arrows) pointing from X to Y. For example, we have

$$\Delta[n] \star \Delta[m] \simeq \Delta[n+m+1].$$

As special notations, write

$$X^{\triangleright} := X \star \{\infty\} \quad \text{and} \quad X^{\triangleleft} := \{\infty\} \star X,$$

where $\{\infty\}$ denotes a singleton set, and the point ∞ is called the **cone point**.

Construction 5.8. Let $X \to S$ be a map of simplicial sets. We construct the functor

$$\mathrm{St}_S X : \mathfrak{C}S \to \mathsf{sSet}$$

as follows. For $s \in S$, define

$$(\mathrm{St}_S X)(s) := \mathrm{Hom}_M(\infty, s),$$

where the simplicial category M is defined by

$$M := \mathfrak{C}(X^{\triangleleft} \sqcup_X S).$$

In other words, this construction is done through the following procedure. First, adjoint a point at infinity ∞ on the left of X, giving $X^{\triangleleft}$. Next, for each $s \in S$, crush the fibre X_s to a point. Roughly speaking, the resulting category is M, which is a "straightened" version of $X^{\triangleleft}$. At this point, the morphism space $\mathrm{Hom}_M(\infty, s)$ should still preserve information about X_s, since it was made out of all morphisms from ∞ to X_s. Indeed, we will see that it is weakly homotopy equivalent to X_s if $X \to S$ is a left fibration. $\triangleleft$

Construction 5.9. Let $F : \mathfrak{C}S \to \mathsf{sSet}$ be a functor of simplicial categories. We construct a left fibration

$$\mathrm{Un}_S F \to S,$$

so that Un_S becomes a right adjoint of St_S. Namely, we define

$$(\mathrm{Un}_S F)_n := \left\{ (\sigma, f) \;\middle|\; \begin{array}{l} \sigma : \Delta[n] \to S \\ f : \mathrm{St}_S\, \sigma \to F \end{array} \right\},$$

with a natural map to S. Its n-simplices are just "maps from $\Delta[n]$ to F", but precisely speaking, here $\Delta[n]$ should be replaced by its Grothendieck construction, which is $\mathrm{St}_S\, \Delta[n]$. This motivates the above construction. ◁

By the constructions, we see that St_S and Un_S form an adjunction

$$\begin{tikzcd} \mathsf{sSet}_{/S} \arrow[r, bend left, "\mathrm{St}_S"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{sSet}) \arrow[l, bend left, "\mathrm{Un}_S"] \end{tikzcd}.$$

We will sometimes omit the subscript S and simply write St and Un.

Example 5.10. Let us consider the case $S = \{*\}$. In this case, the Grothendieck construction converts between fibrations onto $\{*\}$ and their fibres. We expect that the adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathrm{St}_{\{*\}}"] \arrow[r, phantom, "\perp"] & \mathsf{sSet} \arrow[l, bend left, "\mathrm{Un}_{\{*\}}"] \end{tikzcd}$$

should be an equivalence.

By definition, We have

$$\mathrm{St}_{\{*\}}\, \Delta[n] \simeq \mathrm{Hom}_{\mathfrak{C}(\Delta[n]/\Delta[n-1])}(0, *),$$

where $\Delta[n-1] \hookrightarrow \Delta[n]$ as the 0-th face $\{1, \ldots, n\}$, and $\Delta[n]/\Delta[n-1]$ denotes the simplicial set obtained from $\Delta[n]$ by crushing the 0-th face $\Delta[n-1]$ to a point, and we denote this point by $*$.

Let $Q^\bullet$ denote the cosimplicial simplicial set defined by

$$Q^n := \mathrm{Hom}_{\mathfrak{C}(\Delta[n]/\Delta[n-1])}(0, *).$$

Then the adjunction $(\mathrm{St}_{\{*\}} \dashv \mathrm{Un}_{\{*\}})$ is given by $Q^\bullet$ via the construction (3.8). With careful calculation (which we omit here; see [Lu09, Remark 2.2.2.6]), one sees that there is a homeomorphism

$$|Q^n| \simeq \Delta^n$$

of topological spaces, which is compatible with the coface maps (but not the codegeneracy maps). It follows that for any $X \in \mathsf{sSet}$, there is a homeomorphism

$$|\mathrm{St}_{\{*\}} X| \simeq |X|,$$

so that $\mathrm{St}_{\{*\}} X$ and X are weakly equivalent.

This calculation also implies that $\mathrm{St}_{\{*\}}$ preserves cofibrations and trivial cofibrations, so that $(\mathrm{St}_{\{*\}} \dashv \mathrm{Un}_{\{*\}})$ is a Quillen equivalence, i.e., equivalence up to homotopy. $\triangleleft$

For a general $S \in \mathsf{sSet}$, we have the following result.

Theorem 5.11. *The category* $\mathsf{sSet}_{/S}$ *has the* **covariant model structure***, with*

- W *is the class of maps* $X \to Y$ *such that the induced map*

$$X^{\triangleleft} \coprod_X S \to Y^{\triangleleft} \coprod_Y S$$

  is a categorical equivalence.
- $\mathsf{Cof} = \{$*injections*$\}$.
- $\mathsf{Fib} \subset \{$*left fibrations*$\}$.
- *The fibrant objects are left fibrations over* S.

The category $\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{sSet})$ *has the* **projective model structure***, with*

- $\mathsf{W} = \{$*pointwise weak equivalences*$\}$.
- $\mathsf{Fib} = \{$*pointwise fibrations*$\}$.
- Cof *is determined by the lifting property.*

Using these model structures, the adjunction

$$\begin{tikzcd} \mathsf{sSet}_{/S} \arrow[r, bend left, "\mathrm{St}_S"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{sSet}) \arrow[l, bend left, "\mathrm{Un}_S"] \end{tikzcd}$$

is a Quillen equivalence.

See [Lu09, Theorem 2.2.1.2].

Corollary 5.12 (Grothendieck construction)**.** *The functors*

$$\begin{tikzcd} \mathsf{LFib}_{/S} \arrow[r, bend left, "R\,\mathrm{St}_S"] & \mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{Kan}) \arrow[l, bend left, "\mathrm{Un}_S"] \end{tikzcd}$$

are inverses of each other up to weak equivalences, where

- R *denotes the fibrant replacement functor on* $\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{sSet})$.
- $\mathsf{LFib}_{/S}$ *is the category of left fibrations over* S*, as a full subcategory of* $\mathsf{sSet}_{/S}$.
- *The weak equivalences on* $\mathsf{LFib}_{/S}$ *are fibrewise homotopy equivalences.*

- *The weak equivalences on* $\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{Kan})$ *are pointwise homotopy equivalences.*

Proof. Let $X \to S$ be a left fibration, and let $F : \mathfrak{C}S \to \mathsf{Kan}$ be a functor. The unit and counit maps

$$\mathrm{Un}_S\, R\, \mathrm{St}_S\, X \to X \quad \text{and} \quad F \to \mathrm{St}_S\, Q\, \mathrm{Un}_S\, F$$

are categorical equivalences (2.24). Here we may take Q to be the identity functor, which implies that the functors

$$\mathrm{Un}_S\, R\, \mathrm{St}_S\, X \to X \quad \text{and} \quad F \to R\, \mathrm{St}_S\, \mathrm{Un}_S\, F$$

are weak equivalences. The statement on weak equivalences on $\mathsf{LFib}_{/S}$ will follow from (5.14) below. □

Remark 5.13. Dually, for right fibrations, we have the **contravariant model structure** on $\mathsf{sSet}_{/S}$, and a Quillen equivalence

$$\begin{tikzcd} \mathsf{sSet}_{/S} \arrow[r, bend left, "\mathrm{St}_S"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S^{\mathrm{op}}, \mathsf{sSet}) \arrow[l, bend left, "\mathrm{Un}_S"] \end{tikzcd},$$

where $\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S^{\mathrm{op}}, \mathsf{sSet})$ is equipped with the projective model structure. ◁

Now we can prove the most desired property of this construction, which states that the Grothendieck construction gives back the fibres of a left fibration.

Proposition 5.14. *We have the following.*

- *Let $X \to S$ be a left fibration. then for any $s \in S$, there is a weak homotopy equivalence*
$$(\mathrm{St}_S\, X)(s) \simeq X_s.$$
- *Let $F : \mathfrak{C}S \to \mathsf{Kan}$ be a functor. then for any $s \in S$, there is a homotopy equivalence of Kan complexes*
$$(\mathrm{Un}_S\, F)_s \simeq F(s).$$

Proof. Not yet written □

Our goal now is to use the Grothendieck construction to study the hom-spaces of a quasi-category.

Let C be a quasi-category, and let K be a simplicial set. Let $p : K \to \mathsf{C}$ be a map of simplicial sets, which we regard as a commutative diagram in C.

Definition 5.15. *The* **over-category** $\mathsf{C}_{/K}$ *is a quasi-category, with n-simplices*

$$(\mathsf{C}_{/K})_n := \{\sigma : \Delta[n] \star K \to \mathsf{C} \mid \sigma|_K = p\}.$$

Dually, the **under-category** $\mathsf{C}_{K/}$ *is a quasi-category, with n-simplices*

$$(\mathsf{C}_{K/})_n := \{\sigma : K \star \Delta[n] \to \mathsf{C} \mid \sigma|_K = p\}.$$

The following is an immediate consequence of (5.14).

Corollary 5.16. *Let C be a quasi-category, and let $x, y \in \mathsf{C}$. Then*

$$\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y) \simeq (\mathrm{St}_{\mathsf{C}}\, \mathsf{C}_{x/})(y). \qquad \square$$

Moreover, we have the following.

Proposition 5.17. *Let C be a quasi-category, and let $x, y \in \mathsf{C}$. Then*

$$(\mathrm{St}_{\mathsf{C}}\, \mathsf{C}_{x/})(y) \simeq \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(x, y).$$

Proof. Not yet written $\square$

Combining these two equivalences, we see that

$$\mathrm{Hom}^{\triangleleft}_{\mathsf{C}}(x, y) \simeq \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(x, y).$$

A dual argument, involving right fibrations, shows that

$$\mathrm{Hom}^{\triangleright}_{\mathsf{C}}(x, y) \simeq \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(x, y).$$

Therefore, we have proved the fact that C and $\mathfrak{C}\mathsf{C}$ have weakly homotopy equivalent mapping spaces, as our first application of the Grothendieck construction.

For cocartesian fibrations

Recall that we have a table of Grothendieck constructions.

Fibration		**Functor**
left fibration $X \to S$	$\Longleftrightarrow$	$S \to \mathsf{Gpd}$
right fibration $X \to S$	$\Longleftrightarrow$	$S^{\mathrm{op}} \to \mathsf{Gpd}$
cocartesian fibration $X \to S$	$\Longleftrightarrow$	$S \to \mathsf{Cat}$
cartesian fibration $X \to S$	$\Longleftrightarrow$	$S^{\mathrm{op}} \to \mathsf{Cat}$

Now, we sketch the construction for cocartesian and cartesian fibrations.

Let $p : X \to S$ be a functor between ordinary categories. A morphism $f : x \to y$ in X is said to be p-**cocartesian**, if we have a "cocartesian square"

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, dotted, no head] \arrow[dr, phantom, "\urcorner", near end] & y \arrow[d, dotted, no head] \\ p(x) \arrow[r, "p(f)"] & p(y)\,. \end{tikzcd}$$

Precisely speaking, the map $x \to y$ has the following universal property. For any commutative diagram without the dashed arrow,

$$\begin{tikzcd}
x \arrow[r, "f"'] \arrow[rr, "g"] \arrow[d, dotted, no head] & y \arrow[r, dashed, "\exists!\,\tilde{h}"'] \arrow[d, dotted, no head] & z \arrow[d, dotted, no head] \\
p(x) \arrow[r, "p(f)"'] \arrow[rr, "p(g)"] & p(y) \arrow[r, "h"'] & p(z)
\end{tikzcd}$$

there exists a unique morphism $\tilde{h} : y \to z$, such that $\tilde{h}f = g$ and $p(\tilde{h}) = h$.

We reformulate this definition as follows.

Definition 5.18. *Let $p : X \to S$ be a functor between ordinary categories. A morphism $f : x \to y$ in X is said to be p-***cocartesian***, if the lifting problem*

$$\begin{tikzcd}
\Lambda_0[2] \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta[2] \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

has a unique solution, whenever the edge $[0,1]$ in $\Lambda_0[2]$ is sent to f in X.

This definition generalises naturally to quasi-categories.

Definition 5.19. *Let $p : X \to S$ be a functor between quasi-categories. A morphism $f : x \to y$ in X is said to be p-***cocartesian***, if the lifting problem*

$$\begin{tikzcd}
\Lambda_0[n] \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

has a solution whenever the edge $[0,1]$ in $\Lambda_0[n]$ is sent to f in X.

As before, the uniqueness condition is dropped, as it is replaced by higher lifting properties.

Definition 5.20. *Let $p : X \to S$ be a functor between ordinary categories. Then p is called a* **cocartesian fibration***, if X admits "pushouts" from S:*

$$\begin{tikzcd}
x \arrow[d, dotted, no head] & \\
s \arrow[r] & s'
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
x \arrow[r, dashed] \arrow[d, dotted, no head] \arrow[dr, phantom, "\urcorner" very near end] & x' \arrow[d, dotted, no head] \\
s \arrow[r] & s' .
\end{tikzcd}$$

Precisely speaking, for any $x \in X$ and any morphism $\alpha : s \to s'$ in S, where $s := p(x)$, there exists a p-cocartesian morphism $\tilde{\alpha} : x \to x'$ in X, such that $p(\tilde{\alpha}) = \alpha$.

For quasi-categories, we have an extra requirement.

Definition 5.21. *Let $p : X \to S$ be a functor between quasi-categories. Then p is called a* **cocartesian fibration***, if*

- *The lifting problem*

$$\begin{tikzcd} \Lambda_i[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

  has a solution if $0 < i < n$. That is, p is an **inner fibration***.*
- *X admits "pushouts" from S:*

$$\begin{tikzcd} x \arrow[d, dotted, no head] & \\ s \arrow[r] & s' \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} x \arrow[r, dashed] \arrow[d, dotted, no head] & x' \arrow[d, dotted, no head] \\ s \arrow[r] & s' \end{tikzcd} .$$

The first condition is a general requirement for fibration-like maps of quasi-categories, and it is satisfied by any functor between ordinary categories.

Proposition 5.22. *A left fibration $X \to S$ is equivalently a cocartesian fibration such that every edge of X is cocartesian.* □

Cocartesian fibrations provide an obvious way to define transport functors between the fibres, which implies the following theorem.

Theorem 5.23 (Grothendieck construction)**.** *A cocartesian fibration $p : X \to S$ of ordinary categories gives rise to a 2-functor $S \to \mathsf{Cat}$ by taking the fibres. Conversely, every 2-functor $S \to \mathsf{Cat}$ corresponds to a cocartesian fibration $X \to S$ in this manner.* □

For quasi-categories, one needs a "marked" version of the straightening and unstraightening construction. We define sSet^+ to be the category of simplicial sets with some marked edges, such that all degenerate edges are marked. The morphisms in sSet^+ are required to send marked edges to marked edges.

For a simplicial set S, let $S^\sharp$ denote the marked version of S in which all edges are marked, and let $S^\flat$ denote the marked version of S in which only the degenerate edges are marked. We denote $(\mathsf{sSet}^+)_{/S^\sharp}$ by $\mathsf{sSet}^+_{/S}$.

Theorem 5.24. *There exists a Quillen equivalence*

$$\begin{tikzcd} \mathsf{sSet}^+_{/S} \arrow[r, bend left, "\mathrm{St}^+_S"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{sSet}^+) \arrow[l, bend left, "\mathrm{Un}^+_S"] \end{tikzcd},$$

given suitable model structures on both categories. Moreover,

- Un_S^+ *sends functors to* $\mathsf{QsCat} \simeq (\mathsf{sSet}^+)_{\mathrm{cf}}$ *to cocartesian fibrations.*
- St_S^+ *gives back the fibres for cocartesian fibrations, up to a categorical equivalence.*

See [Lu09, Theorem 3.2.0.1].

Corollary 5.25 (Grothendieck construction)**.** *The functors*

$$\begin{tikzcd} \mathsf{CocFib}_{/S} \arrow[r, bend left, "R\,\mathrm{St}_S^+"] & \mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{QsCat}) \arrow[l, bend left, "\mathrm{Un}_S^+"] \end{tikzcd}$$

are inverses of each other up to weak equivalences, where

- *R denotes the fibrant replacement functor on* $\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{sSet}^+)$.
- $\mathsf{CocFib}_{/S}$ *is the category of cocartesian fibrations over S.*
- *The weak equivalences on* $\mathsf{CocFib}_{/S}$ *are fibrewise categorical equivalences.*
- *The weak equivalences on* $\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}S, \mathsf{QsCat})$ *are pointwise categorical equivalences.* □

Example 5.26. Let $S = \mathsf{Cat}_\infty$ be the category of quasi-categories, and let $F := (-)^\flat \in \mathrm{Fun}(S, \mathsf{sSet}^+)$ be the inclusion functor. The cocartesian fibration

$$\mathrm{Un}_S^+ F =: \mathcal{Z} \to \mathsf{Cat}_\infty$$

is a **universal fibration**, in that every cocartesian fibration is equivalent to its pullback. Namely, let $X \to T$ be a cocartesian fibration. Then there exists a **classifying map**

$$f := \mathrm{St}_T^+ X : T \to \mathsf{Cat}_\infty,$$

so that

$$X \simeq f^*\mathcal{Z}.$$

Roughly speaking, this is because X and $f^*\mathcal{Z}$ both are cocartesian fibrations over S, and they have the same fibres. ◁

6 Model categories and localisation

In the introduction, we mentioned that if (C, W) is a category with weak equivalences, then we get an ∞-category $\mathsf{C}[\mathsf{W}^{-1}]$ by inverting the arrows in W. If moreover, the pair (C, W) comes from a model structure, which is usually the case, then this ∞-category can be described more easily.

In this section, we aim to reformulate this idea rigorously, and we will see in the following sections how the model structure is going to help us.

Localising an infinity category

First of all, we describe the process of localising a category with weak equivalences (C, W) to get an ∞-category $\mathsf{C}[\mathsf{W}^{-1}]$. We generalise the situation by also allowing C to be an ∞-category.

Definition 6.1. *Let* C *be a quasi-category, and let* W *be a set of edges (i.e., 1-simplices) of* C*. The* **localisation** $\mathsf{C}[\mathsf{W}^{-1}]$*, if it exists, is a quasi-category defined by the following universal property:*

- *For any quasi-category* D*, and any diagram without the dashed arrow*

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "F"] \arrow[dr] & & \mathsf{D} \\ & \mathsf{C}[\mathsf{W}^{-1}], \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

  such that F *sends* W *to invertible edges in* D*, there exists a 'unique' dashed arrow making the diagram commute.*

The 'uniqueness' should be formulated in an ∞*-categorical way: we require that*

$$\mathrm{Fun}(\mathsf{C}[\mathsf{W}^{-1}], \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}^{\mathsf{W}\mapsto \mathrm{eq}}(\mathsf{C}, \mathsf{D})$$

is an equivalence of quasi-categories, where the right hand side denotes the full subcategory of $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ *spanned by those functors sending* W *to invertible edges.*

As with all things determined by universal properties, the localisation is unique up to an equivalence, if it exists.

Remark 6.2. In fact, the localisation is unique up to an equivalence, which is unique up to a higher equivalence, which is unique up to an even higher equivalence, and so on. In ordinary category theory, we often hear people say that an object is 'unique up to a unique isomorphism'. This is a low dimensional truncation of our notion of uniqueness, because no (non-trivial) higher equivalences exist in an ordinary category. ◁

To prove the existence of such a localisation, we need to recall the notion of marked simplicial sets.

Definition 6.3. *A* **marked simplicial set** *is a pair* (X, E)*, where* X *is a simplicial set, and* E *is a set of edges of* X *which contains all degenerate edges. The edges in* E *are called the* **marked edges**.

Recall that a map of marked simplicial sets from (X, E) to (Y, F) is a map of simplicial sets $f : X \to Y$, such that $f(E) \subset F$. As we have mentioned in the previous section, the category sSet^+ of all marked simplicial sets carries a model structure, which we shall describe now.

Definition 6.4. *Let X be a simplicial set.*

- *The marked simplicial set $X^\sharp$ is the marked version of X in which all edges are marked.*
- *The marked simplicial set $X^\flat$ is the marked version of X in which only the degenerate edges are marked.*
- *If X is a quasi-category, then the marked simplicial set $X^\natural$ is the marked version of X in which the invertible edges are marked.*

Theorem 6.5. *The category* sSet^+ *has the* **(co)cartesian model structure***, in which*

- *The cofibrations are the injections.*
- *The weak equivalences are the maps $f : X \to Y$ such that for any quasi-category Z, we have an equivalence of quasi-categories*

$$\mathrm{Hom}_{\mathsf{sSet}^+}(Y, Z^\natural) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{sSet}^+}(X, Z^\natural),$$

where both sides are full subcategories of the mapping spaces of simplicial sets.

- *The fibrant objects are those which are isomorphic to $X^\natural$ for some quasi-category X.*

Proof. See [Lu09, Proposition 3.1.3.7]. □

The name of the model structure may seem a little strange. In fact, it is a special case of the two model structures on $\mathsf{sSet}^+_{/S}$ when $S = \{*\}$. These two model structures were used in the formulation of Grothendieck construction for (co)cartesian fibrations in the last section.

The existence of localisations follows as an immediate consequence.

Corollary 6.6. *Let* C *be a quasi-category, and let* W *be a set of edges of* C*. Then* $\mathsf{C}[\mathsf{W}^{-1}]$ *exists.*

Proof. Without loss of generality, we may assume that W contains all degenerate edges. Then (C, W) is a marked simplicial set. Let $\mathsf{C}[\mathsf{W}^{-1}] := R(\mathsf{C}, \mathsf{W})$ be its fibrant replacement, which is a fibrant object, and thus a quasi-category. Then for any quasi-category D, we have

$$\mathrm{Hom}_{\mathsf{sSet}^+}\big(\mathsf{C}[\mathsf{W}^{-1}],\ \mathsf{D}^\natural\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{sSet}^+}\big((\mathsf{C}, \mathsf{W}),\ \mathsf{D}^\natural\big),$$

as desired. □

This proof is terribly abstract, and it does not give us a clue on how to work with it in concrete examples. However, there is a construction given by Dwyer and Kan [DK] which is much more explicit.

Construction 6.7. Let (C, W) be a category with weak equivalences, in the ordinary sense. The simplicial category $\mathsf{L}(\mathsf{C}, \mathsf{W})$ is constructed as follows. It has the same objects as C, and for $X, Y \in \mathsf{C}$, the simplicial set $\mathrm{Hom}_{\mathsf{L}(\mathsf{C},\mathsf{W})}(X, Y)$ is defined by

$$\mathrm{Hom}_{\mathsf{L}(\mathsf{C},\mathsf{W})}(X,Y)_n := \left\{ \begin{tikzcd} & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \cdots & \bullet \arrow[l] \arrow[d] \arrow[ddr] & \\ & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \cdots & \bullet \arrow[l] \arrow[d] \arrow[dr] & \\ X \arrow[uur] \arrow[ur] \arrow[dr] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \\ & \bullet & \bullet \arrow[l] \arrow[r] & \cdots & \bullet \arrow[l] \arrow[ur] & \end{tikzcd} \right\} \Big/ \sim ,$$

where

- There are $n + 1$ rows and any number of columns. Each bullet represents an object of C.
- The horizontal arrows may go to the left or the right, but the arrows in the same column must go in the same direction.
- The arrows going downwards or leftwards are in W.
- The equivalence relation $\sim$ is generated by composition of adjacent horizontal arrows in the same direction.

The simplicial category $\mathsf{L}(\mathsf{C}, \mathsf{W})$ is called the **hammock localisation** of (C, W), since the above diagram looks like a hammock. ◁

By this construction, we easily see that the homotopy category of $\mathsf{L}(\mathsf{C}, \mathsf{W})$ is the ordinary category which we denoted by $\mathsf{C}[\mathsf{W}^{-1}]$ in (1.7).

Theorem 6.8. *Let* (C, W) *be an ordinary category with weak equivalences. Then there is a weak equivalence of marked simplicial sets*

$$(\mathsf{C}, \mathsf{W}) \simeq (\mathfrak{N} R \mathsf{L}(\mathsf{C}, \mathsf{W}))^{\natural},$$

where R denotes the fibrant replacement functor from $\mathsf{Cat}_{\mathsf{sSet}}$ *to* $\mathsf{Cat}_{\mathsf{Kan}}$. *In particular, there is an equivalence of quasi-categories*

$$\mathsf{C}[\mathsf{W}^{-1}] \simeq \mathfrak{N} R \mathsf{L}(\mathsf{C}, \mathsf{W}).$$

Proof. See [Hi13, §1.2]. □

As a corollary, the homotopy category of $\mathsf{C}[\mathsf{W}^{-1}]$ is equivalent to the category defined in (1.7).

Simplicial model categories

The above idea will only work with model categories together with a compatible simplicial structure. They are called *simplicial model categories*.

Defining the compatibility conditions between the model structure and the simplicial structure will require some preliminaries in monoidal category theory and enriched category theory.

For simplicity, we will only be concerned about monoidal categories that are *symmetric*, i.e., the tensor product is commutative.

Definition 6.9. *A* **symmetric monoidal category** *consists of*

- *A monoidal category* V.
- *A natural equivalence*

$$T_{X,Y} : X \otimes Y \to Y \otimes X$$

for $X, Y \in \mathsf{V}$*,*

such that

- $T \circ T = \mathbb{1}$.
- *For any* $X, Y, Z \in \mathsf{V}$*, the diagram*

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r] \arrow[d] & (Y \otimes X) \otimes Z \arrow[r] & Y \otimes (X \otimes Z) \arrow[d] \\
X \otimes (Y \otimes Z) \arrow[r] & (Y \otimes Z) \otimes X \arrow[r] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

commutes.

It can be shown that the last axiom above is enough to ensure that all possible ways to form the tensor product of finitely many objects will give a unique result up to a unique isomorphism. The essential reason is that Cat is a 2-category and has no higher structure more than the level 2. This will be explained in later sections.

For example, all the examples of monoidal categories that we have mentioned above are symmetric.

Example 6.10. Let us give an example of a non-symmetric monoidal category. Consider a non-commutative ring R, and consider the category of bimodules over R, with its monoidal structure given by the tensor product of bimodules. Then this category is a non-symmetric monoidal category. ◁

Definition 6.11. *A symmetric monoidal category* V *is said to be* **closed***, if there exists an* **internal hom** *functor*

$$(-)^{(-)} : \mathsf{V} \times \mathsf{V}^{\mathrm{op}} \to \mathsf{V},$$

and a natural isomorphism

$$\mathrm{Hom}_{\mathsf{V}}(X \otimes Y,\ Z) \simeq \mathrm{Hom}_{\mathsf{V}}(X,\ Z^Y)$$

for $X, Y, Z \in \mathsf{V}$*. In short, the tensor product has a right adjoint.*

To avoid awkward notation, we sometimes write $\mathrm{Hom}_{\mathsf{V}}(X, Y)$ for Y^X.

Example 6.12. Here are some examples and non-examples of closed symmetric monoidal categories.

- The symmetric monoidal category $(\mathsf{Set}, \times)$ is closed, with internal hom given by the powering of sets.
- The symmetric monoidal category $(\mathsf{sSet}, \times)$ is closed, with internal hom given by the mapping space of simplicial sets (4.8).
- The symmetric monoidal category $(\mathsf{Top}, \times)$ is *not* closed. This is why algebraic topologists like to use the category of *compactly generated spaces*, which is closed with the $\times$ monoidal structure, with internal hom given by the mapping space with compact open topology. $\triangleleft$

Definition 6.13. *Let* V *be a closed symmetric monoidal category, and let* C *be a category enriched over* V*.*

- C *is* **tensored** *over* V*, if there exists a functor*

  $$- \otimes - :\ \mathsf{V} \times \mathsf{C} \to \mathsf{C},$$

  with a natural isomorphism

  $$\mathrm{Hom}_{\mathsf{C}}(A \otimes X,\ Y) \simeq \mathrm{Hom}_{\mathsf{V}}\big(A,\ \mathrm{Hom}_{\mathsf{C}}(X, Y)\big)$$

  for $A \in \mathsf{V}$ *and* $X, Y \in \mathsf{C}$*, where* $\mathrm{Hom}_{\mathsf{V}}$ *denotes the internal hom of* V*.*
- C *is* **cotensored***, or* **powered***, over* V*, if there exists a functor*

  $$(-)^{(-)} :\ \mathsf{C} \times \mathsf{V}^{\mathrm{op}} \to \mathsf{C},$$

  with a natural isomorphism

  $$\mathrm{Hom}_{\mathsf{C}}(X,\ Y^A) \simeq \mathrm{Hom}_{\mathsf{V}}\big(A,\ \mathrm{Hom}_{\mathsf{C}}(X, Y)\big)$$

  for $A \in \mathsf{V}$ *and* $X, Y \in \mathsf{C}$*, where* $\mathrm{Hom}_{\mathsf{V}}$ *denotes the internal hom of* V*.*

Example 6.14.

- As is easily seen, the categories Set, Top and sSet are tensored and cotensored over Set.
- The simplicial categories Top and sSet are tensored and cotensored over sSet.

Now we are ready to define a simplicial model category.

Definition 6.15. *Let C be a simplicial category. The* **underlying category** C_0 *of C is an ordinary category, obtained by regarding the 0-simplices as arrows, and discarding all higher simplices.*

Definition 6.16. *A* **simplicial model category** *consists of*

- *A simplicial category C.*
- *A model structure on the underlying category C_0 of C,*

such that

- *C is tensored and cotensored over sSet.*
- *For every cofibration $i\colon A \to B$ in sSet, and any cofibration $j\colon X \to Y$ in C_0, the obvious map which we denote by*

$$i \,\Box\, j\colon A \otimes Y \coprod_{A \otimes X} B \otimes X \to B \otimes Y$$

  is a cofibration in C_0, and is trivial whenever i or j is trivial.

The last criterion above may seem somewhat technical, but it is a very natural requirement if we consider the example

$$\begin{aligned} &\mathsf{C} = \mathsf{V} = \mathsf{sSet}, \\ &A = \{0\} \subset \Delta[1], \quad B = \Delta[1], \\ &X = \partial\Delta[2], \qquad\quad Y = \Delta[2]. \end{aligned}$$

The purpose of this criterion is to ensure that the model structures on C and on V are compatible.

For example, the categories Top and sSet are simplicial model categories.

An interesting point about simplicial model categories is that there are two ways to obtain an ∞-category from them.

- First, a simplicial model category is a simplicial category, which gives rise to an ∞-category.
- Second, any category with weak equivalences can be localised to obtain an ∞-category.

Surprisingly, these two ∞-categories turn out to be equivalent. Therefore, in order to study the localised ∞-category, one only needs to look at the simplicial structure of the model category, which is often quite easy to describe.

Theorem 6.17 (Dwyer–Kan)**.** *Let C be a simplicial model category. Then C_{cf} is enriched over Kan complexes, and there is an equivalence of quasi-categories*

$$\mathsf{C}[\mathsf{W}^{-1}] \simeq \mathfrak{N}\mathsf{C}_{\mathrm{cf}}.$$

Proof. See [Lu, Theorem 1.3.4.20]. $\square$

Example 6.18. As was mentioned in (4.24), this theorem implies that we have equivalences

$$\mathsf{S} \simeq \mathsf{sSet}[\mathsf{W}^{-1}] \quad \text{and} \quad \mathsf{D}_\infty(\mathsf{A}) \simeq \mathsf{Ch}_\mathsf{A}[\mathsf{W}^{-1}]. \qquad \lhd$$

7 Limits and adjunctions

In infinity categories, homotopy equivalent objects are equivalent by definition. Therefore, the only well-defined notion of (co)limits is that of homotopy (co)limits. It is very difficult to compute homotopy (co)limits directly from the definitions. However, model categories will give us great help in such computations.

Colimits and limits

The simplest colimit is the empty colimit, that is, the initial object. In ordinary category theory, the initial object is characterised by the property that it admits a unique morphism to any other object.

The ∞-categorical way of saying something is unique is to say that all possible choices form a contractible space, i.e. a contractible Kan complex.

Definition 7.1. *Let C be a quasi-category. An object $x \in \mathsf{C}$ is called an* **initial object***, if for any $y \in \mathsf{C}$, the mapping space $\mathrm{Hom}_\mathsf{C}(x, y)$ is contractible.*

The notation $\mathrm{Hom}_\mathsf{C}(x, y)$ refers to any one of

$$\mathrm{Hom}^{\lhd}_\mathsf{C}(x, y), \quad \mathrm{Hom}^{\rhd}_\mathsf{C}(x, y), \quad \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(x, y), \quad \text{etc.},$$

which all have the same homotopy type.

Remark 7.2. An initial object of C is equivalently an initial object of the hCW-enriched category $h\mathsf{C}$. In fact, most of the notions defined in this section will be equivalent to the corresponding notions for hCW-enriched categories. $\lhd$

One should expect that initial objects are unique if they exist. As before, uniqueness means being a contractible space.

Proposition 7.3. *Let C be a quasi-category.*

- *An object $x \in \mathsf{C}$ is an initial object, if and only if the left fibration*

$$\mathsf{C}_{x/} \to \mathsf{C}$$

  is a trivial fibration.

- *The full subcategory spanned by the initial objects is either empty, or a contractible Kan complex.*

Proof. Not yet written

Colimits are nothing but initial objects of under-categories.

Definition 7.4. *Let C be a quasi-category, K a simplicial set, and let $f : K \to \mathsf{C}$ be a diagram. A* **colimit** *of f is an initial object of $\mathsf{C}_{K/}$.*

We immediately deduce the following.

Corollary 7.5. *Let C be a quasi-category, K a simplicial set, and let $f : K \to \mathsf{C}$ be a diagram.*

- *A map $\bar{f} : K^{\triangleright} \to \mathsf{C}$ is a colimit of f, if and only if the induced left fibration*

$$\mathsf{C}_{K^{\triangleright}/} \to \mathsf{C}_{K/}$$

  is a trivial fibration.
- *The category of colimits of f is either empty, or a contractible Kan complex.*

Proof. We have an isomorphism of simplicial sets

$$\mathsf{C}_{K^{\triangleright}/} \simeq (\mathsf{C}_{K/})_{x/},$$

where x denotes the image of the cone point of $K^{\triangleright}$. Everything else is clear. □

Remark 7.6. The natural map

$$\mathsf{C}_{K^{\triangleright}/} \to \mathsf{C}_{x/}$$

is always a trivial fibration, as can be shown by verifying the lifting property, by transfinite induction on the number of simplices of K. Details are left to the reader. ◁

Colimits are not computable via this definition. We need to deduce some of their properties to make them computable.

Proposition 7.7. *Let C be a quasi-category, and let $\{x_\alpha\}$ be a collection of objects in C. An object $x \in \mathsf{C}$ is a coproduct of the objects x_α, if and only if for any $y \in \mathsf{C}$, the induced map*

$$\mathrm{Hom}_{\mathsf{C}}(x, y) \to \prod_{\alpha} \mathrm{Hom}_{\mathsf{C}}(x_\alpha, y)$$

is a homotopy equivalence.

Proof. The right hand side is equivalent to

$$\prod_\alpha (\mathsf{C}_{x_\alpha/})_y \simeq (\mathsf{C}_{\{x_\alpha\}/})_y,$$

where the subscript y means taking the fibre of the map to C. The left hand side is equivalent to

$$(\mathsf{C}_{x/})_y \simeq ((\mathsf{C}_{\{x_\alpha\}/})_{x/})_y$$

by (7.6). Thus x is a coproduct, if and only if the left fibration

$$(\mathsf{C}_{\{x_\alpha\}/})_{x/} \to \mathsf{C}_{\{x_\alpha\}/}$$

is a trivial fibration, if and only if it is a categorical equivalence (since categorical equivalences are weak homotopy equivalences), if and only if their fibres are equivalent, by the next lemma. □

Lemma 7.8. *Let S be a simplicial set, and let $X, Y \in \mathsf{sSet}_{/S}$ be two left fibrations over S. Let $f : X \to Y$ be a map in $\mathsf{sSet}_{/S}$. Then f is a categorical equivalence if and only if f induces weak equivalences on each fibre.*

Proof. Not yet written □

This result on coproducts is a special case of a general theorem, which we state below.

Let K be a category enriched over Kan complexes. Then the category of simplicially enriched functors

$$\mathsf{Fun}_{\mathsf{sSet}}(\mathsf{K}, \mathsf{sSet})$$

carries two model structures:

- The **projective model structure**, with weak equivalences and fibrations defined pointwise, and cofibrations defined by the lifting property.
- The **injective model structure**, with weak equivalences and cofibrations defined pointwise, and fibrations defined by the lifting property.

The constant functor

$$\mathrm{const} : \mathsf{sSet} \to \mathsf{Fun}_{\mathsf{sSet}}(\mathsf{K}, \mathsf{sSet})$$

is right Quillen with respect to the projective model structure, and left Quillen with respect to the injective model structure. Therefore, the colimit functor

$$\mathrm{colim} : \mathsf{Fun}_{\mathsf{sSet}}(\mathsf{K}, \mathsf{sSet}) \to \mathsf{sSet}$$

is left Quillen with respect to the projective model structure, as it is the left adjoint of the constant functor. The limit functor

$$\lim : \mathsf{Fun}_{\mathsf{sSet}}(\mathsf{K}, \mathsf{sSet}) \to \mathsf{sSet}$$

is right Quillen with respect to the injective model structure, as it is the right adjoint of the constant functor.

Definition 7.9. *Let* K *be a category enriched over Kan complexes.*

- *The* **homotopy colimit** *functor*

$$\mathrm{hocolim}\colon\ \mathrm{Ho}(\mathrm{Fun}_{\mathsf{sSet}}(\mathsf{K},\mathsf{sSet})) \to \mathrm{Ho}(\mathsf{sSet})$$

  is the left derived functor of the left Quillen functor

$$\mathrm{colim}\colon\ \mathrm{Fun}_{\mathsf{sSet}}(\mathsf{K},\mathsf{sSet}) \to \mathsf{sSet},$$

  where $\mathrm{Fun}_{\mathsf{sSet}}(\mathsf{K},\mathsf{sSet})$ *is equipped with the projective model structure.*

- *The* **homotopy limit** *functor*

$$\mathrm{holim}\colon\ \mathrm{Ho}(\mathrm{Fun}_{\mathsf{sSet}}(\mathsf{K},\mathsf{sSet})) \to \mathrm{Ho}(\mathsf{sSet})$$

  is the right derived functor of the right Quillen functor

$$\lim\colon\ \mathrm{Fun}_{\mathsf{sSet}}(\mathsf{K},\mathsf{sSet}) \to \mathsf{sSet},$$

  where $\mathrm{Fun}_{\mathsf{sSet}}(\mathsf{K},\mathsf{sSet})$ *is equipped with the injective model structure.*

Example 7.10. The homotopy pushout

$$\{*\} \underset{\{*,*\}}{\sqcup} \{*\}$$

is the homotopy type of S^1, as is familiar in topology. A proof will be given later. $\lhd$

We can now use homotopy limits of simplicial sets to define homotopy (co)limits in categories enriched over Kan complexes.

Let K be a category enriched over Kan complexes. We denote by $\mathsf{K}^{\rhd}$ the Kan-enriched category obtained by adjoining a terminal object ∞, with

$$\mathrm{Hom}_{\mathsf{K}^{\rhd}}(x,\infty) = \{*\} \quad (x \in \mathsf{K}^{\rhd}), \qquad \mathrm{Hom}_{\mathsf{K}^{\rhd}}(\infty,x) = \varnothing \quad (x \in \mathsf{K}).$$

Let $\mathsf{K}^{\lhd}$ be defined dually.

Definition 7.11. *Let* C *and* K *be categories enriched over Kan complexes, and let* $f\colon \mathsf{K} \to \mathsf{C}$ *be a functor.*

- *A* **homotopy colimit** *of* f *is a functor*

$$\bar{f}\colon \mathsf{K}^{\rhd} \to \mathsf{C},$$

  such that $\bar{f}|_{\mathsf{K}} = f$*, and for any* $y \in \mathsf{C}$*, the induced map*

$$\mathrm{Hom}_{\mathsf{C}}(\bar{f}(\infty),y) \to \underset{k\in\mathsf{K}}{\mathrm{holim}}\,\mathrm{Hom}_{\mathsf{C}}(f(k),y)$$

  is a weak homotopy equivalence.

- *A* **homotopy limit** *of f is a functor*

$$\bar{f}\colon \mathsf{K}^{\triangleleft} \to \mathsf{C},$$

such that $\bar{f}|_{\mathsf{K}} = f$, and for any $y \in \mathsf{C}$, the induced map

$$\mathrm{Hom}_{\mathsf{C}}(y, \bar{f}(\infty)) \to \operatorname*{holim}_{k \in \mathsf{K}} \mathrm{Hom}_{\mathsf{C}}(y, f(k))$$

is a weak homotopy equivalence.

This definition can be seen as the definition of ∞-(co)limits in the model of categories enriched over Kan complexes. The following theorem states that (co)limits in quasi-categories coincide with homotopy (co)limits in this sense.

Theorem 7.12. *Let C and K be categories enriched over Kan complexes, and let $f\colon \mathsf{K} \to \mathsf{C}$ be a functor. Then the functor between quasi-categories*

$$\mathfrak{N}f\colon \mathfrak{N}\mathsf{K} \to \mathfrak{N}\mathsf{C}$$

has a (co)limit, if and only if f has a homotopy (co)limit, and in this case, they coincide.

See [Lu09, Theorem 4.2.4.1].

Of course, this also means that if $f\colon \mathsf{K} \to \mathsf{C}$ is a functor of quasi-categories, then its (co)limits are computed by the homotopy (co)limit of the functor

$$R\mathfrak{C}f\colon R\mathfrak{C}\mathsf{K} \to R\mathfrak{C}\mathsf{C}$$

between Kan-enriched categories. This is because (co)limits are preserved by categorical equivalences.

Example 7.13. When $\mathsf{K} = \Lambda_0[2]$, one has an isomorphism $\mathsf{K}^{\triangleright} \simeq \Delta[1] \times \Delta[1]$. The colimit of a diagram $\mathsf{K} \to \mathsf{C}$ is called a (homotopy) **pushout**, which is a square in C, i.e. a map $\Delta[1] \times \Delta[1] \to \mathsf{C}$. We denote pushouts by the usual notation $y \sqcup_x z$. By the theorem, we have

$$\mathrm{Hom}_{\mathsf{C}}\big(y \underset{x}{\sqcup} z, w\big) \simeq \mathrm{Hom}_{\mathsf{C}}(y, w) \underset{\mathrm{Hom}_{\mathsf{C}}(x,w)}{\times} \mathrm{Hom}_{\mathsf{C}}(z, w)$$

for any $w \in \mathsf{C}$, where the right hand side denotes a homotopy pullback of Kan complexes. ◁

As a corollary, we have the following criterion that establishes certain ordinary pushouts as homotopy pushouts.

Proposition 7.14. *Let C be a simplicial model category. The ordinary pushout*

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ z \arrow[r] \arrow[ur, phantom, "\lrcorner"] & y \underset{x}{\sqcup} z \end{tikzcd}$$

coincides with the homotopy pushout, if the map $x \to y$ is a cofibration.

The proof is a standard argument in model category theory which we omit here.

This gives an easy way to compute homotopy pushouts in a simplicial model category. Namely, to compute $y \sqcup_x z$, one finds a weak equivalence $y \to y'$, such that the composition $x \to y \to y'$ is a cofibration. Then the homotopy pushout $y \sqcup_x z$ is given by the ordinary pushout $y' \sqcup_x z$. For example, in Top, one has the homotopy pushout

$$\{*\} \coprod_X \{*\} \simeq \Sigma X.$$

Remark 7.15. For simplicial model categories, the ∞-categorical (co)limits coincide with the derived functors of the (co)limit functors, provided that the projective and injective model structures exist. ◁

Finally, we mention a formula that computes homotopy (co)limits directly. Readers unfamiliar with (co)ends may skip to the next subsection.

Proposition 7.16. *Let $f : \mathsf{K} \to \mathsf{C}$ be a functor of* Kan*-enriched categories. Assume that* C *is tensored and cotensored over* sSet. *For example, this is always the case if* C *is a simplicial model category.*

- *The homotopy colimit of f is computed by the simplicially enriched coend*

$$\operatorname{hocolim} f \simeq \int^{k \in \mathsf{K}} \mathfrak{N}(\mathsf{K}_{k/}) \otimes f(k),$$

  where $\otimes$ denotes the simplicial tensoring on C.
- *The homotopy limit of f is computed by the simplicially enriched end*

$$\operatorname{holim} f \simeq \int_{k \in \mathsf{K}} f(k)^{\mathfrak{N}(\mathsf{K}_{/k})},$$

  where the powering denotes the simplicial powering on C.

See for example [Ri].

Example 7.17. Let G be a topological group acting on a space X. Then the **homotopy quotient** $X/^{\mathrm{h}}G$ is defined by

$$X/^{\mathrm{h}}G := (X \times EG)/G,$$

where EG denotes the universal G-bundle over BG. This is exactly the homotopy colimit of the corresponding functor

$$\mathsf{B}G \to \mathsf{Top},$$

where $\mathsf{B}G$ is the simplicial groupoid with a single object, whose mapping space is $\operatorname{Sing} G$. ◁

Adjoint functors

In classical category theory, adjoint functors have played a very important role. For example, colimits and limits are special cases of adjoint functors. For ∞-categories, there is also a homotopical version of adjoint functors, which we will now study.

Definition 7.18. *Let* C, D *be quasi-categories, and let*

$$F\colon \mathsf{C} \to \mathsf{D}, \quad G\colon \mathsf{D} \to \mathsf{C}$$

be two functors. We say that F and G are a pair of **adjoint functors** *if there is an equivalence of functors*

$$\mathrm{Hom}_{\mathsf{D}}(F-,-) \simeq \mathrm{Hom}_{\mathsf{C}}(-,G-)\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{S}.$$

Precisely, we mean that the functors

$$\mathrm{Hom}_{\mathfrak{C}\mathsf{D}}(F-,-) \quad \textit{and} \quad \mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(-,G-)$$

are weakly equivalent in the category

$$\mathrm{Fun}_{\mathsf{sSet}}(\mathfrak{C}(\mathsf{C}^{\mathrm{op}} \times \mathsf{D}), \mathsf{sSet}),$$

equipped with the projective model structure.

In order to understand the properties of adjoint functors, we will formulate another two equivalent definitions.

Definition 7.19. *Let* C *be a quasi-category.*

- *The quasi-category*
$$\mathcal{P}(\mathsf{C}) := \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{S})$$
is called the category of **presheaves** *on* C*, where* $\mathsf{S} := \mathfrak{N}(\mathsf{Kan})$.
- *The map*
$$\begin{aligned} Y\colon \mathsf{C} &\to \mathcal{P}(\mathsf{C}), \\ x &\mapsto h_x := \mathrm{Hom}_{\mathsf{C}}(-,x) \end{aligned}$$
is called the **Yoneda embedding**. *Precisely, this map is defined by the simplicially enriched functor*
$$\begin{aligned} \mathfrak{C}(\mathsf{C} \times \mathsf{C}^{\mathrm{op}}) &\to \mathsf{Kan}, \\ (x,y) &\mapsto R\,\mathrm{Hom}_{\mathfrak{C}\mathsf{C}}(y,x) \end{aligned}$$
by passing to the adjoint, where R denotes the fibrant replacement.
- *A presheaf on* C *is called* **representable** *if it is equivalent to* h_x *for some* $x \in \mathsf{C}$.

Via the Grothendieck construction, presheaves on C are equivalent to right fibrations over C. Namely, given $F \in \mathcal{P}(\mathsf{C})$, the right fibration

$$\mathrm{Un}_{\mathsf{C}}\, F \to \mathsf{C}$$

has fibre (equivalent to) $F(x)$ at $x \in \mathsf{C}$.

Definition 7.20. *Let C be a quasi-category. A right fibration $X \to \mathsf{C}$ is* **representable**, *if the presheaf*

$$\mathrm{St}_{\mathsf{C}}\, X \in \mathcal{P}(\mathsf{C})$$

is representable.

By definition, right fibrations of the form $\mathrm{Un}_{\mathsf{C}}\, h_x$ are representable.

Dually, a left fibration $p\colon X \to \mathsf{C}$ is representable if the right fibration $p^{\mathrm{op}}\colon X^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$ is representable.

Proposition 7.21. *Let C be a quasi-category, and let $p\colon \tilde{\mathsf{C}} \to \mathsf{C}$ be a right fibration.*

- *p is representable, if and only if $\tilde{\mathsf{C}}$ has a terminal object.*
- *An object $x \in \mathsf{C}$ represents p, if and only if there exists a terminal object $\tilde{x}$ of $\tilde{\mathsf{C}}$, with $p(\tilde{x}) = x$.*

In particular, the full subcategory of C spanned by the representing objects of p is either empty or a contractible Kan complex.

Proof. Not yet written □

We can now give the equivalent definitions of adjoint functors.

Proposition 7.22. *Let C, D be two quasi-categories. Then a pair of adjoint functors between C and D is equivalent to a left fibration*

$$p\colon X \to \mathsf{C}^{\mathrm{op}} \times \mathsf{D},$$

such that

- *p is* **left representable**: *for any $x \in \mathsf{C}$, the slice*

  $$p|_{x\times\mathsf{D}}\colon p^{-1}(x \times \mathsf{D}) \to x \times \mathsf{D}$$

  is a representable left fibration.
- *p is* **right representable**: *for any $y \in \mathsf{D}$, the slice*

  $$p|_{\mathsf{C}^{\mathrm{op}}\times y}\colon p^{-1}(\mathsf{C}^{\mathrm{op}} \times y) \to \mathsf{C}^{\mathrm{op}} \times y$$

  is a representable left fibration.

In fact, p is given by the Grothendieck construction of the functor

$$\mathrm{Hom}_{\mathsf{D}}(F-,-) \simeq \mathrm{Hom}_{\mathsf{C}}(-,G-) \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{S}.$$

Sketch of Proof. Not yet written ☐

Proposition 7.23. *Let C, D be two quasi-categories. Then a pair of adjoint functors between C and D is equivalent to a map*

$$p \colon M \to \Delta[1],$$

such that

- *There are categorical equivalences $p^{-1}(0) \simeq \mathsf{C}$ and $p^{-1}(1) \simeq \mathsf{D}$.*
- *p is a cocartesian fibration.*
- *p is a cartesian fibration.*

In fact, M is given by $\mathfrak{N}\mathsf{M}$, where the Kan-enriched category M is defined by

- *Objects: pairs $(x, 0)$ for $x \in \mathsf{C}$ and $(y, 1)$ for $y \in \mathsf{D}$.*
- *Morphism spaces:*

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{M}}((x,0),(y,0)) &:= \mathrm{Hom}_{R\mathfrak{C}\mathsf{C}}(x,y),\\
\mathrm{Hom}_{\mathsf{M}}((x,1),(y,1)) &:= \mathrm{Hom}_{R\mathfrak{C}\mathsf{D}}(x,y),\\
\mathrm{Hom}_{\mathsf{M}}((x,0),(y,1)) &:= \mathrm{Hom}_{R\mathfrak{C}\mathsf{D}}(Fx,y) \simeq \mathrm{Hom}_{R\mathfrak{C}\mathsf{C}}(x,Gy),\\
\mathrm{Hom}_{\mathsf{M}}((x,1),(y,0)) &:= \varnothing.
\end{aligned}$$

Sketch of Proof. Not yet written ☐

Proposition 7.24. *Let K, C be quasi-categories. Suppose that every functor $\mathsf{K} \to \mathsf{C}$ has a colimit. Then the constant functor*

$$\mathrm{const} \colon \mathsf{C} \to \mathrm{Fun}(\mathsf{K}, \mathsf{C})$$

admits a left adjoint, which is the colimit functor.

Proof. Not yet written ☐

Remark 7.25. Let C, D be simplicial model categories, and let $(F \dashv G)$ be a Quillen adjunction between them. Then F and G give rise to a pair of adjoint functors between the quasi-categories $\mathfrak{N}(\mathsf{C}_{\mathrm{cf}})$ and $\mathfrak{N}(\mathsf{D}_{\mathrm{cf}})$. ◁

8 Stable categories

In topology, we have homotopy pushout and pullback diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd},$$

as has been shown in the last section. In a stable category, we instead have the pushout-pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d] \\ * \arrow[r] & X[1] \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd},$$

which means that Σ and Ω are inverse to each other. For example, we will see that for cochain complexes, the functors Σ and Ω coincide with the shifting functors $[1]$ and $[-1]$, and thus, the category of cochain complexes is an example of a stable category.

For any category with finite limits, we will describe a *stabilisation* procedure, that turns the category into a stable one. For example, the stabilisation of the category of topological spaces will be the category of spectra, which we will define below.

From now on, when referring to quasi-categories, we will change our terminology to "$(\infty, 1)$-**categories**", or "∞-**categories**" for short. This means that we are not only studying one particular model; we are also talking about the behaviour of an intrinsic notion of $(\infty, 1)$-categories, which is independent of models. However, we only provide rigorous formulations for quasi-categories.

Stable categories

Definition 8.1. *An ∞-category C is said to be* **pointed***, if it has a zero object $0 \in \mathsf{C}$, i.e. an object that is both initial and terminal.*

Definition 8.2. *Let C be a pointed ∞-category.*

- *A* **triangle** *in C is a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}.$$

- *A* **cofibre sequence** *in C is a triangle that is also a pushout square. In this case, we say that Z is the* **homotopy cofibre** *of the map $X \to Y$.*
- *A* **fibre sequence** *in C is a triangle that is also a pullback square. In this case, we say that X is the* **homotopy fibre** *of the map $Y \to Z$.*

Definition 8.3. *A* **stable ∞-category** *is a pointed ∞-category C, such that*

- *Every morphism in C has a homotopy cofibre and a homotopy fibre.*
- *A triangle in C is a cofibre sequence if and only if it is a fibre sequence.*

Definition 8.4. *Let C be a pointed ∞-category. The functors $\Sigma\colon \mathsf{C} \to \mathsf{C}$ and $\Omega\colon \mathsf{C} \to \mathsf{C}$ are defined by*

$$\Sigma X := 0 \sqcup_X 0 \quad \textit{and} \quad \Omega X := 0 \times_X 0.$$

As one might expect, Σ and Ω are adjoint to each other, since for any $X, Y \in \mathsf{C}$, one has

$$\mathrm{Hom}_{\mathsf{C}}(\Sigma X, Y) \simeq \{*\} \times_{\mathrm{Hom}_{\mathsf{C}}(X,Y)} \{*\} \simeq \mathrm{Hom}_{\mathsf{C}}(X, \Omega Y).$$

If C is stable, then Σ and Ω are inverse to each other. In this case, we denote

$$X[1] := \Sigma X \quad \text{and} \quad X[-1] := \Omega X.$$

For a non-negative integer n, we denote

$$X[n] := \Sigma^n X \quad \text{and} \quad X[-n] := \Omega^n X.$$

Proposition 8.5. *Let C be a pointed ∞-category that admits pushouts and pullbacks. Then the following are equivalent.*

- *Σ is an equivalence.*
- *Ω is an equivalence.*
- *C is stable.*
- *A square in C is a pushout square, if and only if it is a pullback square.*

Proof. Not yet written $\square$

Definition 8.6. *Let C, D be stable ∞-categories. A functor $f\colon \mathsf{C} \to \mathsf{D}$ is said to be* **exact**, *if it preserves the zero object, and preserves (co)fibre sequences.*

Homotopy category of a stable category

Our goal in this subsection is to prove the following theorem.

Theorem 8.7. *Let C be a stable ∞-category. Then the homotopy category* $\mathrm{Ho}(\mathsf{C})$ *carries a natural structure of a triangulated category.*

Proof. First, we need to show that $\mathrm{Ho}(\mathsf{C})$ is an additive category.

- $\mathrm{Ho}(\mathsf{C})$ *has finite coproducts.*

  By (7.7), a coproduct in C gives rise to a coproduct in $\mathrm{Ho}(\mathsf{C})$. By definition, C has an initial object. Thus it suffices to show that for any $X, Y \in \mathsf{C}$, the coproduct $X \sqcup Y$ exists. We form the pushout diagram

$$\begin{tikzcd} X[-1] \arrow[r, "0"] \arrow[d] \arrow[dr, phantom, "\ulcorner \quad \lrcorner"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

  in C. Then Z is the coproduct $X \sqcup Y$, since

$$\begin{aligned} Z &\simeq \operatorname{cofibre}(X[-1] \xrightarrow{0} Y) \\ &\simeq \operatorname{cofibre}(X[-1] \to 0) \sqcup \operatorname{cofibre}(0 \to Y) \\ &\simeq X \sqcup Y, \end{aligned}$$

  where the second step uses the fact that taking the cofibre commutes with colimits, as can be checked by the definition of homotopy colimits.

- $\mathrm{Ho}(\mathsf{C})$ *is an additive category.*

  For any $X, Y \in \mathsf{C}$, note that

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(X[1], Y) &\simeq \mathrm{Hom}_{\mathsf{C}}(0, Y) \underset{\mathrm{Hom}_{\mathsf{C}}(X,Y)}{\times} \mathrm{Hom}_{\mathsf{C}}(0, Y) \\ &\simeq \Omega \, \mathrm{Hom}_{\mathsf{C}}(X, Y). \end{aligned}$$

  Therefore,

$$\pi_0 \, \mathrm{Hom}_{\mathsf{C}}(X, Y) \simeq \pi_1 \, \mathrm{Hom}_{\mathsf{C}}(X[-1], Y) \simeq \pi_2 \, \mathrm{Hom}_{\mathsf{C}}(X[-2], Y) \simeq \cdots$$

  is an abelian group. We leave it to the reader to check that composition is bilinear.

- *For a map $f \colon X \to Y$, what is $-f$?*

  A map from X to Y can be seen as a diagram

$$\begin{tikzcd} X[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \, . \end{tikzcd}$$

  If we swap the two 0's, the map will reverse its sign. Precisely, if this square corresponds to the map f, then the transpose of this square will correspond to $-f$. This is because the group structure is defined by the identification

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \simeq \{*\} \underset{\mathrm{Hom}_{\mathsf{C}}(X[-1],Y)}{\times} \{*\} \simeq \Omega \, \mathrm{Hom}_{\mathsf{C}}(X[-1], Y),$$

  and if the two $\{*\}$'s are swapped, then the loop space will have its loops reversed.

Next, we define a triangulated structure on Ho(C), and we verify the axioms of a triangulated category.

- *Distinguished triangles.*

  We define a triangle $X \to Y \to Z \to X[1]$ in Ho(C) to be a distinguished triangle, if it is isomorphic to one that can be lifted to a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & W \end{tikzcd}$$

  in C, where W is equivalent to $X[-1]$ since the outer square is a pushout-pullback.

- (TR 1)

  - *Distinguished triangles are stable under isomorphisms.*
  - *For any object X, the following triangle is distinguished:*

$$X \xrightarrow{1_X} X \to 0 \to X[1].$$

  - *Every map $X \to Y$ extends to a distinguished triangle*

$$X \to Y \to Z \to X[1].$$

  These properties are all obvious.

- (TR 2) *If*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

  is a distinguished triangle, so is

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1].$$

  To prove this, let us form the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\quad\lrcorner"] & W \arrow[d] \\ & 0 \arrow[r] & V \end{tikzcd}$$

  in C. The two outer rectangles establish equivalences $W \simeq X[1]$ and $V \simeq Y[1]$. The map $W \to V$ in the diagram is $f[1]$, since it is induced by the map of diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd} \longrightarrow \begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

in C. However, after transposing the diagram to match the definition of distinguished triangles, the map $W \to V$ becomes $-f[1]$.

- (TR 3) *Given a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

  without the dashed arrow, where the two rows are distinguished triangles, there exists a dashed arrow making the diagram commute.

  This is because taking cofibres in C is functorial.

- (TR 4) *The octahedral axiom.*

  We omit the proof here, but it is not difficult.

We conclude that Ho(C) is a triangulated category. □

Spectra and stable homotopy theory

In this subsection, we review the classical theory of topological spectra and stable homotopy theory, as a model for the stabilisation procedure that will apply to general stable ∞-categories.

Stable homotopy theory is concerned with properties of a topological space that stabilise after applying the suspension functor sufficiently many times. For example, the **stable homotopy groups** of a pointed topological space X are defined by

$$\pi_n^{\mathrm{s}}(X) := \operatorname*{colim}_{k \to +\infty} \pi_{n+k}(\Sigma^k X)$$

for all $n \in \mathbb{Z}$. For example, the Freudenthal suspension theorem states that

$$\pi_n^{\mathrm{s}}(S^0) \simeq \pi_{2n+2}(S^{n+2}) \simeq \pi_{2n+3}(S^{n+3}) \simeq \cdots .$$

Topological spectra contain the data that encodes these stabilised properties of a topological space.

Definition 8.8. *A* **spectrum** *E consists of*

- *A series of pointed topological spaces*

$$E_0, E_1, E_2, \dots .$$

- *For each $n \geq 0$, a map*

$$\sigma_n : \Sigma E_n \to E_{n+1}.$$

A **map of spectra** f *between two spectra* E *and* F *consists of a map*

$$f_n : E_n \to F_n$$

for each $n \geq 0$*, such that* $\sigma_n \circ \Sigma f_n = f_{n+1} \circ \sigma_n$ *for all* $n \geq 0$.

The category of spectra is denoted by Sp.

Example 8.9. Let X be a pointed topological space. The **suspension spectrum** $\Sigma^\infty X$ is defined by

$$(\Sigma^\infty X)_n := \Sigma^n X,$$

with $\sigma_n := 1_{\Sigma^{n+1}X}$. The purpose of the notation Σ^∞ is to signify that we are looking for the properties of $\Sigma^n X$ as $n \to \infty$.

The **sphere spectrum** $\mathbb{S}$ is defined by

$$\mathbb{S} := \Sigma^\infty S^0,$$

so that $\mathbb{S}_n \simeq S^n$. ◁

Definition 8.10. *The* **stable homotopy groups** *of a spectrum* E *are defined by*

$$\pi_n^{\mathrm{s}}(E) := \operatorname*{colim}_{k \to +\infty} \pi_n(E_k)$$

for all $n \in \mathbb{Z}$.

For example, we have

$$\pi_n(\Sigma^\infty X) \simeq \pi_n^{\mathrm{s}}(X)$$

for any pointed topological space X.

Note that the suspension functor Σ and the shifting functor $[1]$ induce the same maps on stable homotopy groups. We will see that they are equivalent up to homotopy, given a certain model structure on Sp.

Definition 8.11. *A spectrum* E *is called an* **Ω-spectrum***, if the maps*

$$E_n \to \Omega E_{n+1}$$

corresponding to σ_n *are weak equivalences for all* $n \geq 0$.

In this case, we write $\Omega^\infty E := E_0$. For any $n \geq 0$, we have $\pi_n(E) \simeq \pi_n(E_0)$.

Theorem 8.12. *The category* Sp *has the* **stable model structure***, with*

- $\mathsf{W} = \{$*maps that induce isomorphisms of all stable homotopy groups*$\}$.
- $\mathsf{Cof} = \{$*degreewise cofibrations*$\}$.
- *The fibrant objects are precisely the* Ω*-spectra.*

Moreover, the adjoint pair

$$\begin{tikzcd} \mathsf{Sp} \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathsf{Sp} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

is a Quillen equivalence, and their derived functors are isomorphic to the shifting functors:

$$\mathbb{L}\Sigma \simeq [1] \quad \textit{and} \quad \mathbb{R}\Omega \simeq [-1].$$

Corollary 8.13. *The ∞-category of spectra, presented by the model category of spectra, is a stable ∞-category.*

Proof. It suffices to check the existence of homotopy (co)fibres. However, by definition, every model category admits arbitrary (co)limits. $\square$

As a result, the homotopy category $\mathrm{Ho}(\mathsf{Sp})$ has a triangulated structure.

Spectra can also be viewed from another perspective, namely, as representing objects of generalised cohomology theories.

Definition 8.14. *A* **generalised cohomology theory** *is a family of functors*

$$\widetilde{E}^n : \mathsf{Top}_*^{\mathrm{op}} \to \mathsf{Ab},$$

where $n \in \mathbb{Z}$, and Top_* denotes the category of pointed topological spaces, satisfying the following axioms.

- (Homotopy) *If $f : X \to Y$ is a weak homotopy equivalence, then $\widetilde{E}^\bullet(f)$ is an isomorphism.*
- (Exactness) *A homotopy cofibre sequence $X \to Y \to Z$ induces an exact sequence*
$$\widetilde{E}^\bullet(Z) \to \widetilde{E}^\bullet(Y) \to \widetilde{E}^\bullet(X).$$
- (Suspension) *There is a natural isomorphism*
$$\widetilde{E}^{\bullet+1}(\Sigma X) \simeq \widetilde{E}^\bullet(X).$$
- (Additivity) *$\widetilde{E}^\bullet$ takes coproducts to products:*
$$\widetilde{E}^\bullet\left(\bigvee_\alpha X_\alpha\right) \simeq \prod_\alpha \widetilde{E}^\bullet(X_\alpha).$$

For example, reduced singular cohomology with coefficients in any abelian group is a generalised cohomology theory. K-theory is also a generalised cohomology theory.

Theorem 8.15 (Brown)**.** *Every generalised cohomology theory $\widetilde{E}^\bullet$ is represented by a spectrum E, in the sense that*

$$\widetilde{E}^n(X) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathsf{Sp})}(\Sigma^\infty X[-n], E)$$

for any topological space X.

Therefore, the category of spectra, which we regard as a "stabilisation" of the category of topological spaces, is equivalent (in the homotopical sense) to the category of generalised cohomology theories of topological spaces.

Stabilisation

Let C be an ∞-category with finite limits. The goal of this subsection is to construct a stable ∞-category $\mathrm{Sp}(\mathsf{C})$ of *spectrum objects* of C, together with a functor

$$\Omega^\infty : \mathrm{Sp}(\mathsf{C}) \to \mathsf{C},$$

which is universal in the sense that for any stable ∞-category D, every functor $\mathsf{D} \to \mathsf{C}$ preserving finite limits will factor through the functor Ω^∞ uniquely.

Before we give the general construction, let us introduce another perspective from which topological spectra can be viewed.

Let E be an Ω-spectrum. Let $\mathsf{Top}_*^{\mathrm{fin}}$ be the category of pointed topological spaces which are homotopy equivalent to finite CW complexes. Then we may define a functor

$$F : \mathsf{Top}_*^{\mathrm{fin}} \to \mathsf{Top}_*,$$

sending S^n to the space E_n. To define such a functor, notice that such a functor sends a pushout square to a pullback square:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & S^{n+1} \arrow[ul, phantom, "\lrcorner"] \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} E_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & E_{n+1} \end{tikzcd} .$$

If we require that F sends any pushout square to a pullback square, then F will be uniquely defined (up to an equivalence), assuming that the spaces E_n can be delooped arbitrarily many times, which is the case since we assumed that E is an Ω-spectrum. (The reader may prove this as an exercise.)

The requirement that F should send a pushout square to a pullback square is analogous to the excision property of classical homology theories. Therefore, spectra may be seen as "homology theories" of topological spaces, with values in topological spaces.

In general, we will define a *spectrum object* of C to be a "homology theory" of topological spaces, with values in C.

Definition 8.16. *Let C and D be ∞-categories.*

- *Suppose C has pushouts. A functor $F : \mathsf{C} \to \mathsf{D}$ is* **excisive** *if it sends pushout squares to pullback squares.*
- *Suppose C has a zero object. A functor $F : \mathsf{C} \to \mathsf{D}$ is* **reduced** *if it preserves the zero object.*

We denote by

$$\mathrm{Fun}_*(\mathsf{C}, \mathsf{D}) \quad \textit{and} \quad \mathrm{Exc}_*(\mathsf{C}, \mathsf{D})$$

the categories of reduced functors and reduced excisive functors from C to D, respectively, as full subcategories of $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$.

Recall that the category of spaces was defined to be $\mathsf{S} := \mathfrak{N}(\mathsf{Kan})$.

Definition 8.17. *The category of* **finite spaces***, denoted by* $\mathsf{S}^{\mathrm{fin}}$*, is defined to be the full subcategory of* S *consisting of Kan complexes whose geometric realisations are equivalent to finite CW complexes. Let*

$$\mathsf{S}^{\mathrm{fin}}_* := (\mathsf{S}^{\mathrm{fin}})_{\{*\}/}$$

be the category of pointed finite spaces.

Definition 8.18. *Let* C *be an* ∞*-category. The category of* **spectrum objects** *of* C *is defined by*

$$\mathrm{Sp}(\mathsf{C}) := \mathrm{Exc}_*(\mathsf{S}^{\mathrm{fin}}_*, \mathsf{C}).$$

The functor

$$\Omega^\infty : \ \mathrm{Sp}(\mathsf{C}) \to \mathsf{C}$$

is defined by evaluating at $S^0 \in \mathsf{S}^{\mathrm{fin}}_*$.

For example, the above discussion suggests that we should have

$$\mathsf{Sp} \simeq \mathrm{Sp}(\mathsf{S}_*) \simeq \mathrm{Sp}(\mathsf{Top}_*).$$

Remark 8.19. In fact, a reduced excisive functor $E : \mathsf{S}^{\mathrm{fin}}_* \to \mathsf{C}$ is determined by the objects $E_n := E(S^n)$, together with induced equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$. Therefore, one can show that $\mathrm{Sp}(\mathsf{C})$ is equivalent to the homotopy limit

$$\mathrm{Sp}(\mathsf{C}) \simeq \lim(\cdots \to \mathsf{C} \xrightarrow{\Omega} \mathsf{C} \xrightarrow{\Omega} \mathsf{C}). \qquad \triangleleft$$

Proposition 8.20. *Let* C *and* D *be* ∞*-categories.*

- *The category* $\mathrm{Sp}(\mathsf{C})$ *is stable if* C *admits finite limits.*
- *More generally, the category* $\mathrm{Exc}_*(\mathsf{C}, \mathsf{D})$ *is stable if* C *admits finite colimits and* D *admits finite limits.*

Proof. Not yet written

Proposition 8.21. *Let* C *be an* ∞*-category with finite limits. Then* C *is stable if and only if the functor* $\Omega^\infty : \ \mathrm{Sp}(\mathsf{C}) \to \mathsf{C}$ *is an equivalence.*

Proof. Not yet written $\square$

Proposition 8.22. *Let* C *and* D *be* ∞*-categories.*

- *If* C *is stable and* D *has finite limits, then the functor* $\Omega^\infty : \ \mathrm{Sp}(\mathsf{C}) \to \mathsf{C}$ *induces an equivalence*

$$\mathrm{Fun}^{\mathrm{l.ex.}}(\mathsf{C}, \mathrm{Sp}(\mathsf{D})) \simeq \mathrm{Fun}^{\mathrm{l.ex.}}(\mathsf{C}, \mathsf{D}),$$

where $\mathrm{Fun}^{\mathrm{l.ex.}}$ *denotes the category of* **left exact** *functors, i.e. functors that preserve finite limits.*

- *More generally, if* C *is pointed and has finite limits, and* D *has finite colimits, then the functor* $\Omega^\infty : \mathrm{Sp}(\mathsf{D}) \to \mathsf{D}$ *induces an equivalence*

$$\mathrm{Exc}_*(\mathsf{C}, \mathrm{Sp}(\mathsf{D})) \simeq \mathrm{Exc}_*(\mathsf{C}, \mathsf{D}).$$

Proof. Not yet written ☐

Therefore, we have proved the following.

Theorem 8.23. *There is a pair of adjoint functors*

$$\begin{tikzcd} \mathsf{Cat}_\infty^{\mathrm{stable}} \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_\infty^{\mathrm{fin.lim.}} \arrow[l, bend left, "\mathrm{Sp}"] \end{tikzcd},$$

where

- $\mathsf{Cat}_\infty^{\mathrm{stable}}$ *denotes the subcategory of* Cat_∞ *consisting of stable* ∞*-categories and exact functors between them.*
- $\mathsf{Cat}_\infty^{\mathrm{fin.lim.}}$ *denotes the subcategory of* Cat_∞ *consisting of* ∞*-categories having finite limits and left exact functors between them.*
- i *denotes the inclusion functor, and* Sp *is the functor constructed above, called the* **stabilisation functor**. ☐

In fact, this can be seen as an adjunction of $(\infty, 2)$-categories, since we have shown the equivalence of mapping spaces as $(\infty, 1)$-categories, not just as homotopy types.

9 Dold–Kan correspondence

The following two sections will be focused on the study of homological algebra. Our goal is to establish homological algebra as a special case of homotopical algebra. We will construct and study the ∞-category of chain complexes, and the ∞-categorical version of the derived category.

In this section, we start from the Dold–Kan correspondence, which is a classical result in homological algebra that establishes an equivalence between chain complexes in an abelian category A and simplicial objects in A. It gives an equivalence of (ordinary) categories

$$\mathsf{Ch}(\mathsf{A})_{\geq 0} \xrightarrow[\sim]{\mathrm{DK}} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{A}),$$

where $\mathsf{Ch}(\mathsf{A})_{\geq 0}$ denotes the full subcategory of $\mathsf{Ch}(\mathsf{A})$ consisting of chain complexes that terminate at the 0th term. This equivalence will relate the homotopy theory of chain complexes to the homotopy theory of simplicial sets.

t-structures

Definition 9.1. *Let* C *be a triangulated category. A* **t-structure** *on* C *consists of two full subcategories*

$$\mathsf{C}^{\leq 0},\ \mathsf{C}^{\geq 0} \subset \mathsf{C},$$

satisfying the following axioms: if we denote

$$\mathsf{C}^{\leq n} := \mathsf{C}^{\leq 0}[n] \quad \text{and} \quad \mathsf{C}^{\geq n} := \mathsf{C}^{\geq 0}[n]$$

for $n \in \mathbb{Z}$*, then*

- $\mathsf{C}^{\leq 0}$ *and* $\mathsf{C}^{\geq 0}$ *are stable under isomorphisms.*
- $\mathsf{C}^{\leq 0} \subset \mathsf{C}^{\leq 1}$ *and* $\mathsf{C}^{\geq 1} \subset \mathsf{C}^{\geq 0}$.
- *If* $X \in \mathsf{C}^{\leq 0}$ *and* $Y \in \mathsf{C}^{\geq 1}$*, then* $\mathrm{Hom}_{\mathsf{C}}(X, Y) = 0$.
- *For every* $X \in \mathsf{C}$*, there exists a distinguished triangle*

  $$X^{\leq 0} \to X \to X^{\geq 1},$$

  such that $X^{\leq 0} \in \mathsf{C}^{\leq 0}$ *and* $X^{\geq 1} \in \mathsf{C}^{\geq 1}$.

For example, if $\mathsf{C} = \mathsf{Ch}(\mathsf{A})$ is the category of cochain complexes in an abelian category A, then we may take $\mathsf{C}^{\leq 0}$ to be those cochain complexes X with $X_n = 0$ for all $n > 0$. Similarly we can define $\mathsf{C}^{\geq 0}$. Then the first three axioms are immediate, and for the fourth one, we may take

$$X^{\leq 0} := \tau^{\leq 0} X \quad \text{and} \quad X^{\geq 1} := \tau^{\geq 1} X,$$

where $\tau^{\leq n}$ and $\tau^{\geq n}$ are the truncation functors, defined by

$$\begin{aligned}
\tau^{\leq n} X &:= \big(\cdots \to X^{n-2} \xrightarrow{d_{n-2}} X^{n-1} \xrightarrow{d_{n-1}} \ker d_n \to 0 \to \cdots\big), \\
\tau^{\geq n} X &:= \big(\cdots \to 0 \to \operatorname{coker} d_{n-1} \xrightarrow{d_n} X^{n+1} \xrightarrow{d_{n+1}} X^{n+2} \to \cdots\big).
\end{aligned}$$

Remark 9.2. If $X \in \mathsf{C}^{\leq 0}$ and $Y \in \mathsf{C}^{\geq 1}$, we require that $\mathrm{Hom}_{\mathsf{C}}(X, Y) = 0$, but not $\mathrm{Hom}_{\mathsf{C}}(Y, X) = 0$, and here is a justification. Let us look at $\mathsf{C} = \mathsf{Ch}(\mathsf{A})$ as an example. In this case, although the only chain map from Y to X is zero, there may exist nonzero chain homotopies and higher homotopies. ◁

Notation 9.3. Sometimes it is more convenient to use homological indexing instead of cohomological indexing. Therefore, we introduce the following notation. For a triangulated category C with a t-structure, we denote

$$\mathsf{C}_{\geq n} := \mathsf{C}^{\leq -n} \quad \text{and} \quad \mathsf{C}_{\leq n} := \mathsf{C}^{\geq -n}.$$

Using this notation, we have $\mathsf{C}_{\geq n} = \mathsf{C}_{\geq 0}[-n]$, etc. ◁

Definition 9.4. *Let* C *be a stable* ∞*-category. A* **t-structure** *on* C *is a t-structure on the triangulated category* $\mathrm{Ho}(\mathsf{C})$.

An important property of a t-structure is that it establishes $\mathsf{C}^{\geq n}$ as a *localisation* of C, in the sense that it is equivalent to $\mathsf{C}[\mathsf{W}^{-1}]$ for some collection W of arrows in C.

Definition 9.5. *A functor $f : \mathsf{C} \to \mathsf{D}$ between ∞-categories is called a* **localisation functor***, if it has a fully faithful right adjoint.*

Remark 9.6. This definition of localisation is narrower than what we called localisations before, i.e. functors obtained by inverting some of the arrows. In fact, if f is a localisation functor in this new sense, then f is obtained by inverting all morphisms in C which are sent to equivalences in D. For a proof of this fact, see [Lu09, Proposition 5.2.7.12]. ◁

Theorem 9.7. *Let C be a stable ∞-category with a t-structure, and let $n \in \mathbb{Z}$. Then there exists a localisation functor*

$$\tau^{\geq n} : \mathsf{C} \to \mathsf{C}^{\geq n},$$

called the **truncation functor***, which is left adjoint to the inclusion functor.*

Proof. Not yet written □

As a corollary, the subcategory $\mathsf{C}^{\geq n}$ of C is stable under limits which exist in C. Dually, $\mathsf{C}^{\leq n}$ is stable under colimits.

Remark 9.8. One might expect that C is the stabilisation of the subcategory $\mathsf{C}_{\geq 0} = \mathsf{C}^{\leq 0}$. However, by (8.19), this is true if and only if

$$\mathsf{C} \simeq \lim_{n \to +\infty} \mathsf{C}_{\geq -n}.$$

For example, this is true for the category of chain complexes, but not for the category of bounded chain complexes.

Definition 9.9. *Let C be a stable ∞-category with a t-structure. The* **heart** *of C is defined to be*

$$\mathsf{C}^{\heartsuit} := \mathsf{C}^{\geq 0} \cap \mathsf{C}^{\leq 0} \subset \mathsf{C}.$$

We define a family of functors

$$\pi_n : \mathsf{C} \to \mathsf{C}^{\heartsuit}$$

by $\pi_0 := \tau^{\geq 0}\tau^{\leq 0} \simeq \tau^{\leq 0}\tau^{\geq 0}$, and $\pi_n := \pi_0 \circ [-n]$.

For a proof of $\tau^{\geq 0}\tau^{\leq 0} \simeq \tau^{\leq 0}\tau^{\geq 0}$, see [Lu, Proposition 1.2.1.10].

For example, if $\mathsf{C} = \mathsf{Ch}(\mathsf{A})$, then $\mathsf{C}^{\heartsuit} \simeq \mathsf{A}$, and π_n takes the $(-n)$-th cohomology group, i.e. the n-th homology group, of a chain complex.

Dold–Kan correspondence

Let A be an abelian category. In this section, we will define a functor

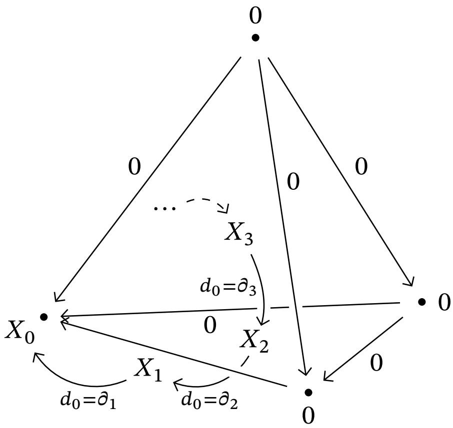
$$\mathrm{DK}\colon \mathsf{Ch}(\mathsf{A})_{\geq 0} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{A}),$$

and we will prove that this functor is an equivalence of categories.

Construction 9.10. Let A be an abelian category, and let $X_\bullet \in \mathsf{Ch}(\mathsf{A})_{\geq 0}$. The simplicial object

$$\mathrm{DK}_\bullet(X) \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{A})$$

is represented by the following picture.



A formal definition may go as follows.

- For every $n \geq 0$, the space of n-simplices is given by

$$\mathrm{DK}_n(X) := \bigoplus_{\substack{\alpha\colon [n]\to[k] \\ \text{surjective}}} X_k.$$

- For every morphism $\beta\colon [n'] \to [n]$ in Δ, the induced map

$$\beta^*\colon \mathrm{DK}_n(X) \to \mathrm{DK}_{n'}(X)$$

  is given by the maps

$$f_{\alpha,\alpha'}\colon X_k \to X_{k'}$$

  for surjective maps $\alpha\colon [n] \to [k]$ and $\alpha'\colon [n'] \to [k']$ in Δ, where

$$f_{\alpha,\alpha'} := \begin{cases} 1_{X_k}, & \text{if } k' = k \text{ and } \begin{smallmatrix} [n'] & \longrightarrow & [n] \\ \downarrow & & \downarrow \\ [k'] & \longrightarrow & [k] \end{smallmatrix} \text{ commutes,} \\ \partial_k, & \text{if } k' = k-1 \text{ and } \begin{smallmatrix} [n'] & \longrightarrow & [n] \\ \downarrow & & \downarrow \\ [k'] & \xrightarrow{d^0} & [k] \end{smallmatrix} \text{ commutes,} \\ 0, & \text{otherwise,} \end{cases}$$

where $\partial_k : X_k \to X_{k-1}$ denotes the differential, and $d^0 : [k-1] \to [k]$ is the map sending $i \in [k-1]$ to $i+1 \in [k]$. ◁

By construction, every non-degenerate simplex in $\mathrm{DK}_n(X)$ can be written as the sum of a nonzero element in X_n and a degenerate n-simplex.

Example 9.11. Let X be the chain complex whose only nonzero term is a $\mathbb{Z}$ at the n-th place. Then

$$\mathrm{DK}_\bullet(X) \simeq \mathbb{Z}\Delta[n]/\mathbb{Z}\partial\Delta[n],$$

where $\mathbb{Z}(-) : \mathsf{sSet} \to \mathsf{sAb}$ denotes the free functor. This is quite obvious if we consider the picture above. ◁

Next, we construct an inverse to the functor DK, sending a simplicial object to its corresponding chain complex.

Definition 9.12. *Let X be a simplicial object in A.*

- *The* **simplicial chain complex** *of X is the chain complex*

  $$C_\bullet(X) \in \mathsf{Ch}(\mathsf{A})_{\geq 0},$$

  given by

  $$C_n(X) := X_n \quad \text{and} \quad \partial_n := \sum_{i=0}^{n} (-1)^i d_i.$$

- *The* **normalised chain complex** *of X, or the* **inverse Dold–Kan construction** *of X, is the chain complex*

  $$\reflectbox{DK}_\bullet(X) \in \mathsf{Ch}(\mathsf{A})_{\geq 0},$$

  given by

  $$\reflectbox{DK}_n(X) := \bigcap_{0<i\leq n} \ker\big(d_i : X_n \to X_{n-1}\big) \quad \text{and} \quad \partial_n := d_0.$$

It might look as if $\reflectbox{DK}(X)$ contains less information than X. However, we will prove that the lost information is inessential, and can be recovered from the group structure of X (referring to the case $\mathsf{A} = \mathsf{Ab}$, of course).

Theorem 9.13. *The functors*

$$\begin{tikzcd} \mathsf{Ch}(\mathsf{A})_{\geq 0} \arrow[r, bend left, "\mathrm{DK}"] \arrow[r, phantom, "\simeq"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{A}) \arrow[l, bend left, "\reflectbox{DK}"] \end{tikzcd}$$

are inverse to each other, up to a natural isomorphism.

Proof. Not yet written □

For stable ∞-categories

There is an analogous result of the Dold–Kan correspondence in the context of stable ∞-categories. Namely, for a stable ∞-category C, we will define the category $\mathsf{Ch}(\mathsf{C})_{\geq 0}$ of *filtered objects* in C, and the result states that

$$\mathsf{Ch}(\mathsf{C})_{\geq 0} \simeq \mathrm{Fun}(\mathfrak{N}\Delta^{\mathrm{op}}, \mathsf{C})$$

as ∞-categories.

Let I be a linearly ordered set (typically, one takes $I = \mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$). Denote

$$I^{[1]} := \{(i,j) \mid i \leq j \in I\},$$

with $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$.

Definition 9.14. *Let C be a pointed ∞-category, and let I be a linearly ordered set. An I-**complex** in C is a functor*

$$F : \mathfrak{N}(I^{[1]}) \to \mathsf{C},$$

such that

- *For any $i \in I$, we have $F(i,i) \simeq 0$.*
- *For any $i \leq j \leq k$, the diagram*

$$\begin{tikzcd} F(i,j) \arrow[r] \arrow[d] & F(i,k) \arrow[d] \\ 0 \simeq F(j,j) \arrow[r] \arrow[ur, phantom, "\lrcorner"] & F(j,k) \end{tikzcd}$$

is a pushout square, i.e. a cofibre sequence.

Let $\mathsf{Ch}(I, \mathsf{C})$ denote the full subcategory of $\mathrm{Fun}(\mathfrak{N}(I^{[1]}), \mathsf{C})$ spanned by all the I-complexes.

This terminology is justified by the following example.

Example 9.15. Let C be a stable ∞-category, and let $X \in \mathbb{Z}, \mathsf{C}$ be a $\mathbb{Z}$-complex in C. Then for any $n \in \mathbb{Z}$, we have a diagram

$$\begin{tikzcd} X(n-1,n) \arrow[r] \arrow[d] & X(n-1,n+1) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X(n,n+1) \arrow[r, "\partial"] \arrow[ur, phantom, "\lrcorner"] & X(n-1,n)[1] \end{tikzcd}$$

in C. Therefore, if we set

$$X_n := X(n-1,n)[-n],$$

then we obtain a chain complex

$$\cdots \to X_1 \xrightarrow{\partial} X_0 \xrightarrow{\partial} X_{-1} \to \cdots$$

in Ho(C). The fact that $\partial^2 = 0$ follows from the equalities

$$\begin{aligned}
\partial^2 &= \big(X(n, n+1) \to X(n-1, n)[1] \to X(n-2, n-1)[2]\big) \\
&= \big(X(n, n+1) \to X(n-2, n)[1] \to X(n-1, n)[1] \to X(n-2, n-1)[2]\big) \\
&= \big(X(n, n+1) \to X(n-2, n)[1] \xrightarrow{0} X(n-2, n-1)[2]\big) \\
&= 0
\end{aligned}$$

in Ho(C). ◁

Remark 9.16. If I has a least object $-\infty$, then

$$\mathsf{Ch}(I, \mathsf{C}) \simeq \mathrm{Fun}(I \setminus \{-\infty\}, \mathsf{C}),$$

since every I-complex X is determined by the values $X(-\infty, i)$ for all $i \in I \setminus \{-\infty\}$. Namely, one has

$$X(i, j) \simeq \mathrm{cofibre}\big(X(-\infty, i) \to X(-\infty, j)\big)$$

for all $i \leq j$ in I. ◁

Example 9.17. Let us consider the case

$$I = \mathbb{Z}_{\geq 0} \cup \{-\infty\} \quad \text{and} \quad \mathsf{C} = \mathsf{Ch}(\mathsf{Ab}).$$

Then by the previous example, an I-complex in C can be seen as a chain complex of chain complexes in Ab. Let X be an I-complex, which is a diagram

$$\begin{tikzcd}
0 \arrow[r] & X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X(-\infty, 1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X(-\infty, 2) \arrow[r] \arrow[d] & \cdots \\
& 0 \arrow[r] & X_1[1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X(0, 2) \arrow[r] \arrow[d] & \cdots \\
& & 0 \arrow[r] & X_2[2] \arrow[r] \arrow[d] & \cdots \\
& & & 0 \arrow[r] & \cdots
\end{tikzcd}$$

in C. The upper-left pushout square implies that we must have

$$X(-\infty, 1) \simeq \mathrm{cofibre}\big(\partial : X_1 \to X_0\big).$$

However, as we will show in the next section, the homotopy cofibre in the category of chain complexes is equivalent to the mapping cone

$$\mathrm{cone}(\partial) \simeq X_0 \oplus X_1[1],$$

where $\simeq$ denotes an isomorphism of graded abelian groups. Therefore, the first row of the diagram may be seen as a sequence

$$0 \to X_0 \to X_0 \oplus X_1[1] \to X_0 \oplus X_1[1] \oplus X_2[2] \to \cdots$$

of graded abelian groups, in which each term is equipped with a twisted differential. Therefore, an I-complex in C may also be seen as a filtered chain complex of abelian groups. $\triangleleft$

In homological algebra, a filtered chain complex produces a spectral sequence, which computes the homology of the total complex. This generalises to stable ∞-categories as well, establishing a further connection between homological algebra and stable homotopy theory.

Definition 9.18. *Let C be a stable ∞-category. A* **filtered object** *in C is a functor*

$$X : \mathfrak{N}(\mathbb{Z}) \to \mathsf{C}.$$

We may regard X as a $\mathbb{Z}$-complex by first obtaining a $(\mathbb{Z} \cup \{-\infty\})$-complex, and then restricting to $\mathbb{Z}^{[1]} \subset (\mathbb{Z} \cup \{-\infty\})^{[1]}$.

Let C be a stable ∞-category with a t-structure. Suppose that the heart $\mathsf{C}^{\heartsuit}$ is equivalent to $\mathfrak{N}(\mathsf{A})$ for an abelian category A.

Let X be a filtered object in C. Write

$$E^r_{p,q} := \operatorname{im}\Big(\pi_{p+q}X(p-r,p) \to \pi_{p+q}X(p-1,p+r-1)\Big) \in \mathsf{A}$$

for all $r \geq 1$ and $p, q \in \mathbb{Z}$. Define

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

by the commutative diagram

$$\begin{tikzcd}
\pi_{p+q}X(p-r,p) \arrow[r] \arrow[d, "\delta"] & \pi_{p+q}X(p-1,p+r-1) \arrow[d, "\delta"] \\
\pi_{p+q-1}X(p-2r,p-r) \arrow[r] & \pi_{p+q-1}X(p-r-1,p-1)\,,
\end{tikzcd}$$

where δ denotes the connecting morphism in the long exact sequence

$$\cdots \to \pi_n X(i,j) \to \pi_n X(i,k) \to \pi_n X(j,k) \xrightarrow{\delta} \pi_{n-1}X(i,j) \to \cdots$$

for any $i \leq j \leq k$.

Theorem 9.19. *The pair*

$$(E^r_{p,q}, d^r)$$

is a spectral sequence in A.

Moreover, if $X(n) \simeq 0$ for $n \ll 0$, and if we assume that $\mathsf{C}_{\leq 0}$ is stable under $\mathbb{Z}$-indexed colimits, then the spectral sequence converges:

$$E^r_{p,q} \Rightarrow \pi_{p+q} \operatorname{colim}\big(X : \mathfrak{N}(\mathbb{Z}) \to \mathsf{C}\big)$$

See [Lu, Propositions 1.2.2.7 and 1.2.2.14].

There is also a version of Dold–Kan correspondence for stable ∞-categories.

Theorem 9.20. *Let C be a stable ∞-category. Then*

$$\mathrm{Fun}(\mathfrak{N}\mathbb{Z}_{\geq 0}, \mathsf{C}) \simeq \mathrm{Fun}(\mathfrak{N}\Delta^{\mathrm{op}}, \mathsf{C}).$$

See [Lu, Theorem 1.2.4.1].

10 Homological algebra

In this section, our main goal is to study the category of cochain complexes. We will construct the ∞-category of cochain complexes, and then, we will invert the quasi-isomorphisms in it, obtaining an ∞-categorical version of the derived category.

Nerve of a dg category

Definition 10.1. *Let R be a commutative ring. A* **dg category** *over R is a category enriched over the symmetric monoidal category* $(\mathsf{Ch}_R, \otimes)$

The abbreviation "dg" stands for "differential graded". Cochain complexes are also known as dg vector spaces (or modules).

Definition 10.2. *Let C be a dg category over R.*

- *The* **underlying category** *of C is obtained from C by applying the functor*

$$Z^0 : (\mathsf{Ch}_R, \otimes) \to (\mathsf{Set}, \times),$$

  taking the set of 0-cocycles of a cochain complex.
- *The* **homotopy category** *of C is obtained from C by applying the functor*

$$H^0 : (\mathsf{Ch}_R, \otimes) \to (\mathsf{Set}, \times),$$

  taking the 0-th cohomology of a cochain complex.

For example, if $\mathsf{C} = \mathsf{Ch}_R$, which is enriched over itself, as described in (1.12), then the underlying category of C is just the ordinary category of cochain complexes Ch_R, and its homotopy category is the ordinary category hCh_R. The 0-coboundaries are exactly the chain homotopies.

We now wish to define a quasi-category $\mathfrak{N}_{\mathrm{dg}}(\mathsf{C})$, such that the homotopies in this quasi-category are the chain homotopies, and the higher homotopies are higher chain homotopies, etc.

A simple idea is to consider the sequence of adjunctions

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{sSet}} \arrow[l, bend left, "\mathfrak{N}"] \arrow[r, bend left, "\text{free}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{sMod}_R} \arrow[l, bend left, "\text{forget}"] \arrow[r, bend left, "\rotatebox{180}{DK}"] \arrow[r, phantom, "\simeq"] & \mathsf{Cat}_{(\mathsf{Ch}_R)_{\geq 0}} \arrow[l, bend left, "\mathrm{DK}"] \arrow[r, bend left, "i_{\geq 0}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{Ch}_R} \arrow[l, bend left, "\tau_{\geq 0}"] \end{tikzcd},$$

and one may define

$$\mathfrak{N}_{\mathrm{dg}} := \mathfrak{N} \circ \text{forget} \circ \mathrm{DK} \circ \tau_{\geq 0} \colon \mathsf{Cat}_{\mathsf{Ch}_R} \to \mathsf{sSet},$$

and will see soon that the image lies in QsCat.

This construction does give the correct quasi-category, but there is a more direct way to construct this quasi-category. We will first define $\mathfrak{N}_{\mathrm{dg}}$ in the more direct way, and then we will prove that the two constructions are equivalent.

Construction 10.3. Let C be a dg category. The quasi-category $\mathfrak{N}_{\mathrm{dg}}(\mathsf{C})$ is constructed as follows. Its n-simplices are of the form

$$(\{X_i\}_{i\in[n]}, \{f_I\}_{I\subset[n],|I|\geq 2}),$$

where

- $X_0, \dots, X_n$ are objects of C.
- For $0 \leq i_0 < \cdots < i_{m+1} \leq n$,

  $$f_{i_0\cdots i_{m+1}} \in \mathrm{Hom}_{\mathsf{C}}(X_{i_0}, X_{i_{m+1}})_m,$$

  such that

  $$df_{i_0\cdots i_{m+1}} = \sum_{j=1}^{m} (-1)^{m+1-j} \big(f_{i_0\cdots\hat{i_j}\cdots i_{m+1}} - f_{i_j\cdots i_{m+1}} \circ f_{i_0\cdots i_j}\big).$$

For example,

- A 0-simplex is just an object $X \in \mathsf{C}$.
- A 1-simplex connecting the objects $X, Y \in \mathsf{C}$ is a morphism

  $$f_{01} \in \mathrm{Hom}_{\mathsf{C}}(X,Y)_0 \quad \text{such that} \quad df_{01} = 0.$$

  In other words, f_{01} is just a morphism in the underlying category of C.
- A 2-simplex

  $$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

  is a chain homotopy

  $$\alpha := f_{012} \in \mathrm{Hom}_{\mathsf{C}}(X,Z)_1 \quad \text{such that} \quad d\alpha = g \circ f - h.$$

We leave it to the reader to define the face and degeneracy maps in $\mathfrak{N}_{\mathrm{dg}}(\mathsf{C})$, and verify that it is a quasi-category. ◁

Proposition 10.4. *Let* C *be a dg category. Then for any* $X, Y \in \mathsf{C}$*, there is an isomorphism of simplicial sets*

$$\mathrm{Hom}^{\triangleright}_{\mathfrak{N}_{\mathrm{dg}}(\mathsf{C})}(X, Y) \simeq \mathrm{DK}(\tau_{\geq 0} \mathrm{Hom}_{\mathsf{C}}(X, Y)).$$

Proof. Not yet written □

Corollary 10.5. *Let* C *be a dg category. Then there is a categorical equivalence*

$$\mathfrak{N}_{\mathrm{dg}}(\mathsf{C}) \simeq \mathfrak{N} \circ \mathrm{forget} \circ \mathrm{DK} \circ \tau_{\geq 0}(\mathsf{C}). \qquad \square$$

Theorem 10.6. *The category* $\mathsf{Cat}_{\mathsf{Ch}_R}$ *has the* **Tabuada model structure***, with*

- *A weak equivalence is a functor inducing an equivalence of homotopy categories, and inducing quasi-isomorphisms on all mapping spaces.*
- *A fibration is a functor inducing an isofibration of homotopy categories, and inducing (degreewise) surjections on all mapping spaces.*

The functor

$$\mathfrak{N}_{\mathrm{dg}} : \mathsf{Cat}_{\mathsf{Ch}_R} \to \mathsf{sSet}$$

is a right Quillen functor with respect to this model structure, and the Joyal model structure on sSet.

The ∞-category of cochain complexes

Definition 10.7. *Let* A *be an abelian category. The category*

$$\mathfrak{N}_{\mathrm{dg}}(\mathsf{Ch}(\mathsf{A}))$$

is called the **∞-category of cochain complexes** *in* A*, where* $\mathsf{Ch}(\mathsf{A})$ *is seen as a dg category over* $\mathbb{Z}$.

In this section, we take a closer look at this particular ∞-category, and in particular, the pushouts and pullbacks in this category. We will show that it is a stable ∞-category.

Proposition 10.8. *Let* $f : X \to Y$ *be a morphism in the category* sAb *of simplicial abelian groups. Then* f *is a fibration of simplicial sets, if and only if the map*

$$\rotatebox[origin=c]{180}{DK}(f) : \rotatebox[origin=c]{180}{DK}(X) \to \rotatebox[origin=c]{180}{DK}(Y)$$

is a degreewise surjection of chain complexes.

Proof. Not yet written □

Corollary 10.9. *Every simplicial abelian group is a Kan complex.* □

Corollary 10.10. *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

be an ordinary pushout diagram in $\mathsf{Ch}(\mathsf{A})$. *If f is a degreewise split injection, then this diagram is a homotopy pushout diagram, i.e. a pushout diagram in* $\mathfrak{N}_{\mathrm{dg}}(\mathsf{Ch}(\mathsf{A}))$.

Proof. Not yet written □

This corollary enables us to construct homotopy cofibres of cochain complexes in an explicit way.

For $X \in \mathsf{Ch}(\mathsf{A})$, let $X \otimes D^1$ denote the object of $\mathsf{Ch}(\mathsf{A})$ defined by

$$(X \otimes D^1)^n := X^{n+1} \oplus X^n \quad \text{and} \quad d^n := \begin{pmatrix} -d^{n+1} & 0 \\ 1 & d^n \end{pmatrix}.$$

This is called the **cone** over X, and is always quasi-isomorphic to 0, i.e. exact. In fact, if $\mathsf{A} = \mathsf{Mod}_R$ for a commutative ring R, then $X \otimes D^1$ is the tensor product of X and the cochain complex

$$D^1 := \left(\cdots \to 0 \to R \xrightarrow{1} R \to 0 \to \cdots\right).$$

Definition 10.11. *Let $f : X \to Y$ be a morphism in* $\mathsf{Ch}(\mathsf{A})$. *The* **mapping cone** *of f is a cochain complex* $\mathrm{cone}(f)$*, defined by the ordinary pushout*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X \otimes D^1 \arrow[d] \\ Y \arrow[r] & \mathrm{cone}(f)\,. \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Explicitly, one has

$$\mathrm{cone}(f)^n \simeq X^{n+1} \oplus Y^n \quad \textit{and} \quad d^n := \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

Proposition 10.12. *Let $f : X \to Y$ be a morphism in* $\mathsf{Ch}(\mathsf{A})$. *Then the mapping cone of f is the homotopy cofibre of f:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ Y \arrow[r] & \mathrm{cone}(f)\,. \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

In particular, the suspension of X is homotopy equivalent to the shifted complex $X[1]$, i.e. one has the homotopy pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ 0 \arrow[r] & X[1] \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Proof. This follows immediately from (10.10). □

Theorem 10.13. *The category $\mathfrak{N}_{\mathrm{dg}}(\mathsf{Ch}(\mathsf{A}))$ is stable.*

Proof. Not yet written □

11 Operads

Not yet written

12 Associative algebras

Not yet written

13 Little disks and factorisation algebras

Not yet written

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