

Assigned. Oct. 20 Due. Nov. 2Kangyan Xu

## 2. Randomized matrix multiplication

$B \in \mathbb{R}^{m \times N}$ ,  $C \in \mathbb{R}^{N \times n} \rightarrow A = BC \in \mathbb{R}^{m \times n}$ . However,  $N$  is very large and multiplication takes a long time.  
 $b_i$  and  $c_i^T$  denote the columns of  $B$  and rows of  $C^T$  respectively.

$T$  be a random variable that takes the value  $i \in \{1, 2, \dots, N\}$  with probability  $p_i$ .

We shall estimate  $A$  by drawing  $T(t) \sim T$  i.i.d. at random for  $t = 1, 2, \dots, r$  and calculating

$$\hat{A} = \frac{1}{r} \sum_{t=1}^r \frac{1}{P(T=t)} b_{T(t)} c_{T(t)}^T$$

(i) Provide an upper-bound on  $\frac{E[\|\hat{A} - A\|]}{\|B\| \|C\|}$ .

A: let  $asr = \frac{1}{2} (\text{stable rank}(B) + \text{stable rank}(C))$

Suppose the number  $r$  of samples satisfies  $r \geq \varepsilon^{-2} \cdot asr \cdot \log(mn)$

Then: 
$$\frac{E[\|\hat{A} - A\|]}{\|B\| \|C\|} \leq 2\varepsilon + \frac{2}{3}\varepsilon^2$$

(ii) What is a good choice for  $p_i$ ?

A:  $b_{i,j}$  denotes the  $j$ th column of  $B$ ,  $C_{j,:}$  denotes the  $j$ th row of  $C$ .

$$P_i = \frac{\|b_{:,i}\|^2 + \|c_{i,:}\|^2}{\|B\|_F^2 + \|C\|_F^2}$$

(iii) If we want  $\frac{E[\|\hat{A} - A\|]}{\|B\| \|C\|} \leq \varepsilon$ , how large do we have to pick  $r$ ?

A: let  $\varepsilon = 2\varepsilon + \frac{2}{3}\varepsilon^2 \Rightarrow \varepsilon = \frac{\sqrt{4+6\varepsilon}-3}{2}$ , then  $r \geq \varepsilon^{-2} \cdot asr \cdot \log(mn)$ .

(iv) What is the computational cost of forming this approximation as a function of  $\varepsilon, m$  and  $n$ ?

A: It's  $O\left(\left(\frac{\sqrt{4+6\varepsilon}-3}{2}\right)^2 \cdot asr \cdot mn \log(mn)\right)$  arithmetic operations.

## 2. Randomized SVD

Platform: Jupyter Notebook & Python3

### a) Report c (number of drawn columns/rows)

```
In [7]: # Problem 1 - Average run time
report(10,0.01)
report(10,0.05)
report(10,0.1)

The SVD run time is 2.2242400646209717
r= 10, err= 0.01, c= 183.6, error_col= 0.009558938912926827, error_row= 0.009489391882537595, run_time= 238.22850205898285
The SVD run time is 2.2752034664154053
r= 10, err= 0.05, c= 18.1, error_col= 0.038262605342132425, error_row= 0.042179575003914796, run_time= 8.122266960144042
The SVD run time is 2.2338473796844482
r= 10, err= 0.1, c= 12.3, error_col= 0.08353758683451033, error_row= 0.06897917476516016, run_time= 2.1976898431777956
```

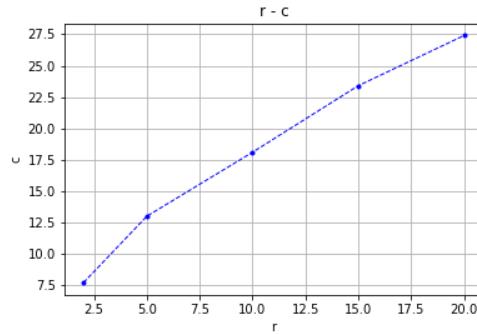
When the error is small, randomized SVD costs much time.

If the error is large enough, the run time of randomized SVD is similar to the normal SVD.

### b) Draw c as a function of r

```
In [8]: report(2,0.05)
report(5,0.05)
report(15,0.05)
report(20,0.05)

The SVD run time is 2.1245272159576416
r= 2, err= 0.05, c= 7.7, error_col= 0.03831811238166542, error_row= 0.0398126293496358, run_time= 5.290650820732116
The SVD run time is 2.1845254898071289
r= 5, err= 0.05, c= 13.0, error_col= 0.04388581597303616, error_row= 0.041495079692628806, run_time= 7.790314197540283
The SVD run time is 3.6482925415039062
r= 15, err= 0.05, c= 23.4, error_col= 0.040586322420607696, error_row= 0.04241996826399202, run_time= 9.251989006996155
The SVD run time is 4.4005935192108154
r= 20, err= 0.05, c= 27.4, error_col= 0.0457498022511705, error_row= 0.043168630999497276, run_time= 8.316781711578368
```



Here can find Least-squares best fits:  $c = 1.07205 r + 6.77073$

## Code parts:

```
In [1]: # Homework_3 / Oct. 2020 / Kangyan Xu
```

```
%pylab inline

import numpy as np
import time
from scipy.linalg import orth
from scipy.sparse.linalg import svds
from numpy.linalg import norm

m = 1000
n = 100000
```

```
In [2]: # calculate A
```

```
def mat_A(r):

    np.random.seed(1) # set fixed random seed

    # create orthonormal matrices X_m*m & Y_n*m
    X = orth(np.random.random(size=(m,m)))
    Y = orth(np.random.random(size=(n,m)))

    # create matrix D_m*m
    temp = np.zeros(m)
    for i in range(0,r):
        temp[i] = r-(i+1)+1
    for i in range(r,m):
        temp[i] = 0.004
    D = np.diag(temp)

    # A = XDY_trans_m*n
    A = np.matmul(np.matmul(X,D),Y.T)
    return A
```

```
In [3]: # run time using SVD on A
```

```
def normal_svd(A,r):

    start_time = time.time()
    # singular vectors corresponding to the top r singular values
    Ur,_,Vr_t = svds(A, k=r)

    SVD_time = time.time()-start_time

    print("The SVD run time is %.16f" %SVD_time)

    return Ur,Vr_t.T
```

```
In [4]: # for the matrix A draw c columns
```

```
def Ur_hat(A,c,r):
    # probability for picking columns pi
    pi = norm(A,2, axis=0)**2 / norm(A, 'fro')**2

    cols = np.random.choice(n,c,p=pi)
    B_col = A[:,cols] / np.sqrt(c*pi[cols]).reshape((1,c))

    Ur_hat,_,_ = svds(B_col, k=r)

    return Ur_hat

def Vr_hat(A,c,r):
    # probability for picking rows pj
    pj = norm(A,2, axis=1)**2 / norm(A, 'fro')**2

    rows = np.random.choice(m,c,p=pj)
    B_row = A[rows,:] / np.sqrt(c*pj[rows]).reshape((c,1))

    _,_,Vr_hat_t = svds(B_row, k=r)

    return Vr_hat_t.T
```

```
In [5]: # Compute errors using power method

def error_power_method(U,U_hat,size):
    # error = ||Ur^Ur^T-UrUrT||
    iteration = 15
    vec = np.random.rand(size) # random vector

    for i in range(iteration):
        # efficiently calculate MAT*vec
        vec_1 = np.dot(U_hat,np.dot(U_hat.T,vec))-np.dot(U,np.dot(U.T,vec))
        vec_2 = np.dot(U_hat,np.dot(U_hat.T,vec_1))-np.dot(U,np.dot(U.T,vec_1))

        vec_2_norm = np.linalg.norm(vec_2)
        vec = vec_2 / vec_2_norm

    vec_1 = np.dot(U_hat,np.dot(U_hat.T,vec))-np.dot(U,np.dot(U.T,vec))
    vec_2 = np.dot(U_hat,np.dot(U_hat.T,vec_1))-np.dot(U,np.dot(U.T,vec_1))
    error = np.dot(vec_1,vec_2)/(np.dot(vec_1,vec_1))
    error = np.sqrt(error)

return error
```

```
In [6]: # Problem 1 - Average run time

def report(r, err):

    c_sum = 0
    error_col_sum = 0
    error_row_sum = 0

    A = mat_A(r)
    Ur,Vr = normal_svd(A,r)

    start_time = time.time()

    for i in range(10):

        for c in range(r+1,350):
            Urhat = Ur_hat(A,c,r)
            Vrhat = Vr_hat(A,c,r)
            error_col = error_power_method(Ur,Urhat,m)
            error_row = error_power_method(Vr,Vrhat,n)

            if (error_col <= err) and (error_row <= err):
                break

            c_sum += c
            error_col_sum += error_col
            error_row_sum += error_row

        run_time = time.time()-start_time

    print('r= {}, err= {}, c= {}, error_col= {}, error_row= {}, run_time= {}'.format(r, err, c_sum/10,
                                                                                   error_col_sum/10, error_row_sum/10, run_time/10))
```

*end.*