Linear Algebra 2020.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vector Space: ① add
② Multiply C, ($C \in R$)

Column space $C(A)$	r
Null Space $N(A)$	$n-r$
$C(A^T)$	r
$N(A^T)$	$m-r$.

Linear Transformations

$T(c_1 x_1 + c_2 x_2) = c_1 T(x_1) + c_2 T(x_2)$

$C(A^T) \perp N(A)$

Projection of b on a

$$\hat{b} = \frac{a \cdot a^T}{\|a\|^2} \cdot b = P \cdot b$$

(projection Matrix) $P = P^2$

$\min \| Ax - b \|^2$

$x^* = (A^T A)^{-1} A^T b$

GRAM-SCHMIDT

$$q_1 = \frac{a}{\|a\|}$$

$$q_2 = \frac{b - (q_1^T b) q_1}{\| b - (q_1^T b) q_1 \|}$$

$$q_3 = \frac{c - (q_1^T c) q_1 - (q_2^T c) q_2}{\| c - (q_1^T c) q_1 - (q_2^T c) q_2 \|}$$

$\vdots$

$$q_{k+1} = \frac{a_{k+1} - (q_1^T a_{k+1}) q_1 - \cdots - (q_k^T a_{k+1}) q_k}{\| a_{k+1} - \sum_{i=1}^{k} (q_i^T a_{k+1}) q_i \|}$$

$A = Q \cdot R$ ($Q^{-1} = Q^T$, R is upper diagonal)

$A \cdot [b_1 \vdots b_2 \vdots b_3 \vdots \cdots] = [Ab_1 \vdots Ab_2 \vdots Ab_3 \vdots \cdots]$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \cdot B = \begin{bmatrix} a_1 B \\ a_2 B \\ a_3 B \\ \vdots \end{bmatrix}$$

$A \cdot x_p + \underbrace{A \cdot x_N}_{0} = b$

x_p — particular solution.

x_N — all the x such that $Ax = 0$.

Area of triangle : $\frac{1}{2} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Volumn = $\det(a_1\ a_2\ a_3)$

$AS = S\Lambda$ $\quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Lambda = S^{-1} A S$ $\quad A^k x = \lambda^k x$

$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$

$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

If A has n distinct eigen values then it is diagonalizable.

$(e^{\lambda t} \cdot y)' = \lambda e^{\lambda t} y$

Cayley-Hamilton Theorem

if $|A - \lambda I| = P(\lambda)$, $P(A) = 0$.

$$e^{At} = I + \frac{At}{1!} + \cdots + \frac{A^n t^n}{n!} + \cdots$$

$\Delta(\lambda) = \lambda^2 - tr(A) \cdot \lambda + \det(A)$

Similarity of Mattrices:

$B = M^{-1} A M$

$$\begin{bmatrix} r(t) \\ w(t) \end{bmatrix} = e^{At} \cdot \begin{bmatrix} r_0 \\ w_0 \end{bmatrix}, \quad e^{At} = S \cdot \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \cdot S^{-1}$$

$A = LU$

L — Lower tri with 1 on diagonal

U — Upper tri

$n \geq m = r$

$\underbrace{(AA^T)(AA^T)^{-1}}_{\text{right inverse}} = I_{m \times m}$

$m \geq n$

$\underbrace{(B^T B)^{-1} B^T}_{\text{left inverse}} B = I_{n \times n}$

Positive Define (semi ≥ 0)

$A_{n \times n}$, $A = A^T$. A is P.D. iff

(i) $\lambda(A) > 0$

(ii) $x^T A x > 0$, every $x \in R^n$ ($x \neq 0$)

$A_{n \times n}$, $A = A^T$, A is P.D. iff

(i) $x^T A x > 0 \quad \forall x \in R^n$ ($x \neq 0$)

$\Leftrightarrow$ (ii) $\lambda_i(A) > 0$

$\Leftrightarrow$ (iii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \cdots & & & a_{nn} \end{bmatrix}$

$\Delta_1 = \det(a_{11}) > 0$

$\Delta_2 = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0$

$\vdots \qquad \vdots$

Singular value decomposition SVD

$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}^T$

$U \cdot U^T = I \quad V \cdot V^T = I$

$$\Sigma = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} \sigma_1 & & 0 \\ 0 & & \sigma_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array} \right] \begin{array}{l} \}\, r \\ \}\, m-r \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{r} & \underbrace{\qquad}_{n-r} \end{array} \end{array}$$

U: eigen vectors of AA^T

V: eigen vectors of $A^T A$

$\sigma_1^2 \cdots \sigma_r^2$: eigen values of AA^T, $A^T A$

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Pseudo Inverse:

$$A^+ = V \cdot \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Norms of vectors & Matrices.

$x \in R^n \to R$ Vector.

1. $\|x\| \geq 0$
2. $\|a \cdot x\| = |a| \cdot \|x\|$, any $a \in R$, $x \in R^n$
3. $\|x+y\| \leq \|x\| + \|y\|$ (Triangle)

$A_{m \times n} \to R$ Matrix

1. $\|A\| \geq 0$
2. $\|\alpha A\| = |\alpha| \cdot \|A\|$. any $a \in R$
3. $\|A+B\| \leq \|A\| + \|B\|$ (Triangle)

Spectral Norm.

$$\|A\| \overset{def}{=} \max_x \frac{\|Ax\|_2}{\|x\|_2}$$

Fact I. $\underset{vec}{\|Ax\|} \leq \underset{Mat}{\|A\|} \cdot \underset{vec}{\|x\|}$

Fact II. $\|AB\| \leq \|A\| \cdot \|B\|$

$A(x+\partial x) = b + \partial b$

$$\frac{\|\partial x\|}{\|x\|} \leq \|A^{-1}\| \cdot \|A\| \cdot \frac{\|\partial b\|}{\|b\|}$$

condition number of $A = \|A^{-1}\| \cdot \|A\|$

$$\|A^{-1}\| \cdot \|A\| = \frac{|\lambda_{max}(A)|}{|\lambda_{min}(A)|}$$

$\|A\|^2 = \lambda_{max}(A^T A)$

$A = \underset{\text{Lower diagonal}}{L} + \underset{\text{diagonal}}{D} + \underset{\text{upper diagonal}}{U}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

* Jacobi $D^{-1}(-L-U)$
* Gauss-Seidel $(D+L)^{-1}(-U)$
* SOR $\lambda_{max} = \max \lambda_{Jacobi}$

$$\omega_{opt} = \frac{2(1-\sqrt{1-\mu^2_{max}})}{\mu^2_{max}}$$

Gershgorin Circle.

The eigen values of A are in the union of the circles: $|\lambda - a_{ii}| \leq \underset{\text{(radius)}}{\sum_{j \neq i} |a_{ij}|}$

Frobenius norm

$$\|A\| = \sum_{ij} a_{ij}^2$$

$\|A^T\| = A$

$\|A\|_F = \sqrt{trA^T A} = \sqrt{\Sigma \sigma_i^2}$

$X_{k+1} = (I + \varepsilon A) X_k - \varepsilon b$ Iteration

Converges iff $|\lambda(I+\varepsilon A)| < 1$

CHOLESKY FACTORIZATION

$A = LU$ (lower diag, upper diag)

KRONECKER PRODUCTS ⊗

$I_{m \times m} \otimes I_{n \times n} = I_{(m \cdot n) \times (m \cdot n)}$

$(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$

$(A \otimes B)^T = A^T \otimes B^T$

A, B symmetric $\Rightarrow A \otimes B$ symmetric

A, B non-singular $\Rightarrow (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

A, B orthonal ($AA^T = I$) $\Rightarrow A \otimes B$ orthogonal

A SVD: $U_A \Sigma_A V_A^T$

B SVD: $U_B \Sigma_B V_B^T$

$A \otimes B$ SVD: $(U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A^T \otimes V_B^T)$

rank $(A \otimes B) = (\text{rank} A) \cdot (\text{rank} B)$

$A \sim \lambda_i$ $B \sim \mu_j$

$A \otimes B \sim \lambda_i \mu_j$ e-vec $(x_i \otimes z_j)$

$tr(A \otimes B) = (trA) \cdot (trB)$

$A_{n \times n}$ $B_{m \times m}$

$\det(A \otimes B) = (\det A)^m (\det B)^n$

Markov

has $\lambda = 1$ and $|\lambda_i| < 1$

$\exists \pi_e : \pi_e P = \pi_e$

$J = M^{-1} A M$
