Derangement

Introduction: Derangement is an interesting thing is Mathematics. Suppose, some objects are arranged ~~ordered~~ in an order. By the term derangement, it means, ~~if~~ ~~we~~ the total number of permutations possible in which any objects can not be placed to their previous/initial order. let's say, A, B and C three objects are,

A B C

by permuting we find, they are not placed in previous order.

$$\left.\begin{matrix} BCA \\ CAB \end{matrix}\right\} \to 2$$

Here, ABC, BAC, ~~CAB~~ CBA, ACB are not possible because at least ~~of~~ one objects are placed ~~just~~ like previous order.

Calculation/Derivation: The calculation of derangement is not so ~~a~~ ~~simp~~ easy. If there are n ~~ob~~ number of objects, the number of permutation of these ~~number~~ objects n!

If we can calculate all permutations of the objects having at least one object is placed in previous order, and ~~calculate~~ subtract the number from number of all permutations ($n!$), we will get the number of derangement (let say D_n). So,

D_n = number of all permutations without any condition — number of all permutations having at least one object is placed in previous order.

~~at~~

let say, A_i is the set of elements in which ith element is in its original position.

Now,

$$D_n = n! - n(A_1 \cup A_2 \cup A_3 \cdots\cdots \cup A_n)$$

Again from inclusion exclusion principal we get,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$\therefore n(A_1 \cup A_2 \cup A_3 \cdots \cup A_n) = \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \cdots\cdots (-1)^{n-1} n(A_i \cap A_j \cap A_k \cdots \cap A_n)$$

Here, $\sum n(A_i) = {}^nC_1 \times (n-1)!$; its because if we ~~take~~ place only one object in its previous position, the remaining objects can be placed in $(n-1)!$ ways.

Again, $\sum n(A_i \cap A_j) = {}^nC_2 \times (n-2)!$; its because if we place two objects in their previous position, the remaining objects can be placed in $(n-2)!$ also here ${}^nC_2$ means we can combine 2 objects by ${}^nC_2$ ways.

Similarly we get,

$$n(A_1 \cup A_2 \cup A_3 \cdots \cup A_n) = {}^nC_1 \times (n-1)! - {}^nC_2 \times (n-2)! + {}^nC_3 \times (n-3)! - \cdots \cdots (-1)^{n-1}\, {}^nC_n \times 1$$

$$= \frac{n!}{1!\,(n-1)!} \times (n-1)! - \frac{n!}{2!\,(n-2)!} \times (n-2)! + \frac{n!}{3!\,(n-3)!} \times (n-3)! - \cdots \cdots (-1)^{n-1} \frac{n!}{n! \times 1!} \times 1$$

$$= n! \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \cdots (-1)^{n-1} \frac{1}{n!} \right]$$

$$\therefore D_n = n! - n! \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \cdots (-1)^{n-1} \frac{1}{n!} \right]$$

$$= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots \cdots (-1)^{n} \frac{1}{n!} \right]$$

This is the formula for calculating dearrangement.

Conclusion: Derangement is an important thing in permutation and combination in number theory. By Derangement it become easier to solve ~~the~~ many problems based on one to one function, letter envelop problems, hats problems etc which were difficult to solve based on general formulae.