a.

1. let $n_0 = 2$, $c = 1$, consider $\sqrt{n} - \ln n = f(n)$

$f'(n) = \frac{1}{2\sqrt{n}} - \frac{1}{n} = \frac{\sqrt{n} - 2}{2n}$, for $n \ge n_0$, $f'(n) > 0 \Rightarrow f(n)$ is increasing when $n \ge n_0$. Besides, $\sqrt{n_0} - \ln n_0 = \sqrt{2} - \ln 2 > 0$

$\therefore f(n) > 0$ when $n \ge n_0$

$\therefore$ when $n \ge n_0$, $\ln n < \sqrt{n} \Rightarrow \ln n = O(\sqrt{n})$ #

2. ① First, show that $928 \log_{830} n = O(\ln n)$

set $n_0 = 830$, $c = 928$, consider $928 \ln n - 928 \log_{830} n$

$= 928 (\ln n - \log_{830} n) > 0$

when $n \ge n_0 \Rightarrow 928 \log_{830} n = O(\ln n)$

② Second, show that $928 \log_{830} n = \Omega(\ln n)$

set $n_0 = 3$, $c = 1$, consider $928 \log_{830} n - \ln n$

$= \underbrace{\boxed{\frac{928}{\log_e 830}}}_{>1} \log_e n - \log_e n > 0$

$\therefore 928 \log_{830} n = \Omega(\ln n)$

$\therefore$ By ①.②, $928 \log_{830} n = \Theta(\ln n)$. #

3. $\sum_{i=1}^{n} i^3 = [\frac{n(n+1)}{2}]^2 = \frac{1}{4}(n^4 + 2n^3 + n^2) = f(n)$

① Claim, $f(n) = O(n^4)$, proof: set $\begin{cases} n_0 = 2 \\ c = 1 \end{cases}$, consider $n^4 - f(n)$

$= \frac{1}{4} n^2 (\underbrace{3n^2 - 2n + 1}_{\boxed{\text{root}: 1, -\frac{1}{3}}}) > 0$ when $n \ge n_0$. $\therefore f(n) = O(n^4)$

② Claim, $f(n) = \Omega(n^4)$, proof: set $\begin{cases} n_0 = 1 \\ c = \frac{1}{4} \end{cases}$, consider $f(n) - \frac{1}{4} n^4$

$= 2n^3 + n^2 > 0$, when $n \ge n_0$ $\therefore f(n) = \Omega(n^4)$

By ①, ②, $\sum_{i=1}^{n} i^3 = \Theta(n^4)$ #

4.

Claim $(\log n)^{112} = O(n^{\frac{7}{2}-\pi})$, Let $f(n) = n^{\frac{7}{2}-\pi} - (\log n)^{112}$

pf :

consider

$$\lim_{n\to\infty} n^{\frac{7}{2}-\pi} - (\log n)^{112}$$

$$= \lim_{n\to\infty} n^{\frac{7}{2}-\pi}\left[1 - \frac{(\log n)^{112}}{n^{\frac{7}{2}-\pi}}\right]$$

$$= \lim_{n\to\infty} n^{\frac{7}{2}-\pi} - \underbrace{\left[\lim_{n\to\infty} \frac{(\log n)^{112}}{n^{\frac{7}{2}-\pi}}\right]}_{\text{L'H 112 times}} = 0$$

$$= \lim_{n\to\infty} n^{\frac{7}{2}-\pi} = \infty$$

$\therefore$ By the definition of limit, $\exists N \in \mathbb{N}$ s.t when $n \geq N$, $f(n) \geq f$ $\forall f \in \mathbb{R}$

simply choose $f > 0$, $n_0 = N$, then we conclude $(\log n)^{112} = O(n^{\frac{7}{2}-\pi})$

$\therefore n^{\pi}(\log n)^{112} = O(n^{\pi})\,O(n^{\frac{7}{2}-\pi}) = O(n^{\frac{7}{2}})$ #

5. consider $\log(n!) < \log n^n \Rightarrow \log(n!) = O(n\log n)$

$\therefore \log[(\log\log n)!] = O(\log(\log n)\log(\log\log n))$

If we let $n_0 = 2^{2^2}$

then $\log(\log n)\log(\log\log n) = O((\log\log n)^2)$

Note $O((\log\log n)^2) = O(\log n)$

$\therefore \log[(\log\log n)!] = O(\log n) \Rightarrow (\log\log n)! = O(n)$

6.

a. ① $\ln(n!) \le n\ln n \quad \Rightarrow \quad \ln(n!) = O(n\ln n)$

(for $n \ge 1$)

② $n! \ge (\lfloor \frac{n}{2} \rfloor)^{\lfloor \frac{n}{2} \rfloor} \Rightarrow \ln(n!) \ge \lfloor \frac{n}{2} \rfloor \ln \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor \left(\ln n - \ln 3 \text{ or } \ln n - \ln 2 \right)$

$= \Omega(n\ln n)$

$\therefore \ln n! = \Theta(n\ln n)$

b. ① Claim $\exists a \in \mathbb{R}^+$ s.t $\ln n! \le n\ln n - n + a\ln n = \ln \frac{n^n \cdot n^a}{2^n}$

$\because$ consider $\ln \frac{n^n \cdot n^a}{e^n} - \ln n! \le \ln \frac{(\frac{n}{2})^n n^a}{n!}$

Note for $n \in$ even $n! = \frac{n}{2} \cdot (\frac{n}{2}+1)(\frac{n}{2}-1) \cdots (\frac{n}{2}+(\frac{n}{2}-1)) \cdot 1 \cdot n$

$\le \frac{n}{2} \cdot \underbrace{(\frac{n}{2})^2 \cdot (\frac{n}{2})^2 \cdots (\frac{n}{2})^2}_{\frac{n}{2}-1} \cdot n = 2 \cdot (\frac{n}{2})^n$

$\therefore \ln \frac{(\frac{n}{2})^n n^a}{n!} \ge \ln \frac{n^a}{2}$, $\therefore$ choose $a = 1$, $n_0 \ge 2$

then $\ln \frac{n^n n^a}{2^n} - \ln n! \ge 0$

$\Rightarrow \ln n! - n\ln n + n = \Omega(\ln n)$

for odd n, $n! = (\frac{n}{2}+\frac{1}{2})(\frac{n}{2}-\frac{1}{2}) \cdots (\frac{n}{2}+(\frac{n}{2}-0.5))(\frac{n}{2}+(\frac{n}{2}+0.5)) \le (\frac{n}{2}$

which would give the same result.

② Claim $\ln n! - n\ln n + n = O(\ln n)$

$\because$ consider $\ln \frac{n^a \cdot n^n}{n! \cdot 2^n}$

b.

1. $\sqrt{n^{2.048}} = \Theta(n^{1.024})$

2. $n \log(\lfloor n \rfloor !) = \Theta(n \log(n!)) = \Theta(n^2 \ln n)$

3. $2^n = \Theta(2^n)$

4. $\Theta(n \ln n)$

5. $\Theta(n^k)$

6. $\because \ln\left((\log n)^{1.0001^n}\right) = (1.0001)^n \ln(\ln n) = \Theta\left((1.0001)^n \ln(\ln n)\right)$

$= \Theta\left((1.0001)^n\right) \quad \therefore (\log n)^{1.0001^n} = \Theta\left(2^{(1.001)^n}\right)$

$\therefore \quad 4 < 1 < 2 < 5 < 3 < 6$ #

c.

1 Apply Master Method : $n^{\log_b a} = n^{\log_9 26} \fallingdotseq n^{1.48} < n^{1.5}$

and $9 \cdot \left(\frac{n}{26}\right)^{1.5} = \frac{9}{(26)^{1.5}} n^{1.5} < n^{1.5}$

$\therefore \quad T(n) = \Theta(n^{1.5})$

2. Consider recursion tree first :

$$T(n) = 3n + T\left(\frac{2n}{3}\right) + T\left(\frac{2n}{9}\right)$$

$$= 3n + \left(3 \cdot \frac{2n}{3} + T\left(\left(\frac{2}{3}\right)^2 n\right) + T\left(\frac{2}{3} \cdot \frac{2}{9} n\right)\right)$$

$$+ \left(\frac{2n}{9} \cdot 3 + T\left(\frac{2}{3}\frac{2}{9} n\right) + T\left(\left(\frac{2}{9}\right)^2 n\right)\right)$$

$$= 3\Big(n + \underbrace{\frac{2n}{3} + \frac{2n}{9}}_{\frac{7}{9}n}\Big) + \left[T\left(\left(\frac{2}{3}\right)^2 n\right) + 2T\left(\frac{2}{3}\frac{2}{9} n\right) + T\left(\left(\frac{2}{9}\right)^2 n\right)\right]$$

$\therefore$ Guess :

$$T(n) = \Theta\left(3 \cdot \frac{1}{1 - \frac{7}{9}} n\right) = \Theta\left(\frac{27}{2} n\right)$$

proof by substitution method: Target: $\exists\, c \geq 1$ s.t. $T(n) \leq cn \quad \forall n \in \mathbb{N}^+$

for $n = 1$, $1 \leq c$ ✓

Assume for all $n < k$, $T(k) \leq ck$, consider $T(n)$

$$T(n) = 3n + T(\tfrac{2}{3}n) + T(\tfrac{2}{9}n) \leq 3n + \tfrac{2}{3}cn + \tfrac{2}{9}cn$$

$$= n \cdot (3 + \tfrac{7}{9}c)$$

take $c \geq \frac{27}{2}$, then $n(3 + \frac{7}{9}c) \leq cn$

$\therefore T(n) = \Theta(n)$ #

Problem 6:

a. 1. False, 2. True, 3. False [redacted], 4. True, 5. True

b. First, give each germ a number according to their strings by the following rules.

① character: $a \to 0$, $b \to 1$, $c \to 2$, $\vdots$, $z \to 25$; ② each number has the length 2k.

Ex: zach $\to \underbrace{25}_{z}\underbrace{00}_{a}\underbrace{02}_{c}\underbrace{07}_{h}$

Next, we give each germ an indepdent group. And we start to sort the germs by "divide and conquer".

Sort n germs = Sort $\lfloor \frac{n}{2} \rfloor$ germs + Sort $\lfloor \frac{n+1}{2} \rfloor$ germs + merge

while we are doing "merge", if the left element = the right element, we will add the right one into the left one's group and delete the right one's group.

Ex: |1|3|2|9|1| $\xrightarrow[\text{right sort}]{\text{left sort}}$ |1|2|3| + |1|9| $\xrightarrow{\text{merge}}$ |1|2|3|9|
|1|

∴ The time complexity $T(k,n)$ = (translate string into number) × (sorting)

× (some other constant-time work such as delete group)

∴ $T(k,n) = k \cdot O(n \ln n) \cdot const = O(k n \ln n)$ #.

c. I. Assume n genes are distinct. My algorithm is to divide the name of each name of genes into left-side and right-side.

For example:

① applen ⟶ app | len

② banana ⟶ ban | ana

③ appand ⟶ app | and

II. Then divide left-side and right-side into groups of identical string by the algorithm intro. in b.

For example:

① applen → app | len

② banana → ban | ana

③ appand → app | and

$\boxed{1_L, 3_L}$ $\boxed{2_L}$ $\boxed{1_R}$ $\boxed{2_R}$ $\boxed{3_R}$

III. For those groups with two or more members, consider the opposite side of them, forcing them to do I, II, III again.

For example

left		right
$\boxed{1_L, 3_L}$, $\boxed{2_L}$		$\boxed{1_R}$ $\boxed{2_R}$ $\boxed{3_R}$
Left A	Left B	

∵ number of left A > 1

∴ Force 1_R: len, 3_R: and to do I. II. III again

☆ If the length of string is divided into "1", then return $\sum_i \binom{n_i}{2}$, where n_i is the number of the member in each group. (Note $\binom{1}{2} = 0$)

Ex:

If ① ac ② ad ③ bc ⇒ left: $\boxed{1_L, 2_L}$ $\boxed{3_L}$; Right: $\boxed{1_R\ 3_R}$. $\boxed{2_R}$

left: return $\binom{2}{2} + \binom{1}{2} = 1$

Right: return $\binom{2}{2} + \binom{1}{2} = 1$

III ☆ If the length of string is not divided into "1", then return Σ (the returning value of those forced to do I, II, III again)

For example : In ① applen , ② banana , ③ appand example, one should return the value of (1_R (len) , 3_R (and)) (If there is no elements been forced to do it again, return "0".)

d. correctness : proof by contradiction :

① If there exists two germs A, B which are similar but $\begin{cases} \text{I, not counted} \\ \text{II, repeatedly counted} \end{cases}$:

I, ∃ a division s.t A, B are not in the same groups for both left and right side, which is a contradiction by problem a.

II, ∃ i, j (WLOG : $i+1<j$) s.t (A_i B_i) , (A_j B_j) form groups but are (A_{i-1}), (B_{i-1}), (A_{j-1}), (B_{j-1}) form isolated groups. (WLOG)

However, this is a contradiction since $A_{i-1} \neq B_{i-1}$ & $A_{j-1} \neq B_{j-1}$.

(A_i means the i-th character of A)

② If there exists two germs A, B not similar but counted :

In this case, there should only exist one different character of A and B to make the division continue, Clearly, it is a contradiction.

e. consider $f(a+b) = (a+b)\log(a+b) = \log(a+b)^{a+b} \geq \log(\underbrace{a^{a+b}+b^{a+b}}_{\text{since } a, b > 1})$

$\therefore f(a+b) - f(a) - f(b) = \log \frac{(a+b)^{a+b}}{a^a b^b} \geq \log \frac{a^{a+b}+b^{a+b}}{a^a b^b} = \log\left(\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \right)$

$$\begin{cases} \text{If } a \geq b ; & \log\left(\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \right) \geq \log\left(\frac{a}{b}\right)^b \geq \log 1 \geq 0 \\ \text{If } b \geq a ; & \log\left(\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \right) \geq \log\left(\frac{b}{a}\right)^a \geq \log 1 \geq 0 \end{cases}$$

$\therefore f(a+b) - f(a) - f(b) \geq 0 \Rightarrow f(a) + f(b) \leq f(a+b)$ #

choose (a,b) s.t $ad + c - b\log 2 \le ad \Rightarrow b \ge \frac{a(d-d)+c}{\log 2} = \frac{c}{\log 2}$

then $n\log n \cdot (bk\log\frac{k}{2} + adk + ck) \le n\log n\,(bk\log k + adk)$

$= bk\log k\, n\log n + a\underbrace{(dn)}_{T(1,n)}k\log k$ #

$\therefore T(k,n) = O(n\log n\, k\log k)$ #

9. Note, we know that $\max\left(\sum_i T(\lfloor\frac{k}{2}\rfloor, \alpha_{Li})\right) = T(\lfloor\frac{k}{2}\rfloor, n)$. This is because we need to spend $O(n\log n \cdot k)$ for 分組, $n\log n$ of course $\ge \alpha\log(\frac{n}{\alpha})$ for $1 \le \alpha \le n$.

Next, ∵ we assume "n" germs are all different ∴ $\exists$ at least a division s.t $\sum_i T(\lfloor\frac{k}{2}\rfloor, \alpha_{Li}) < T(\lfloor\frac{k}{2}\rfloor, n)$

or

$\sum_j T(\lceil\frac{k}{2}\rceil, \alpha_{Rj}) < T(\lceil\frac{k}{2}\rceil, n)$

Thus, we can conclude $T(k,n) < T(\lfloor\frac{k}{2}\rfloor, n) + T(\lceil\frac{k}{2}\rceil, n) + O(nk\log n)$

∴ Compare to master method, the corresponding $k^{\log_{b(n)} a(n)} < nk\log n$.

and notice that $a\cdot f(\frac{k}{b}) < 2\cdot f(\frac{k}{2}) = nk\log n$

∴ By master method III, we can conclude $T(k,n) = O(nk\log n)$ #