1 Finte Groups

1.A

For a group with three elements, there must be an a in the group and it is not the identity element. And it's inverse is in this group. So now we have three elements.

	e	a_1	a_2
e	e	a_1	a_2
a_1	a_1	a_1^2	a_1a_2
a_2	a_2	a_2a_1	a_2^2

by our argument, a_2 should be the inverse of a_1, so

	e	a_1	a_1^{-1}
e	e	a_1	a_2
a_1	a_1	a_1^2	e
a_1^{-1}	a_1^{-1}	e	a_1^{-2}

$a_1^2 \neq a_1$, or by cancellation law, $a_1^2a_1^{-1} = a_1a^{-1} \to a_1 = e$ and this is contradict to the requirement a is not e.

so the two choices left are $a_1^2=e$ or $a_1^2=a_1^{-1}$, but the first equation means $a_1^{-1}=a_1$, then there is only one element whicn is a_1. so $a_1^2=a_1^{-1}$ and

$$a_1^{-2}=(a_1^2)^{-1}=(a_1^{-1})^{-1}=a_1 \tag{61}$$

so the multiplication table is

	e	a_1	a_1^{-1}
e	e	a_1	a_2
a_1	a_1	a_1^2	e
a_1^{-1}	a_1^{-1}	e	a_1^{-2}

1.B

There must be two element a_1, a_1^{-1} different from e. the different groups are determined by the forth element.

$a_2a_1=e$, $a_2a_1=a_1$ and $a_2a_1=a_2$ are impossible

So $a_2a_1=a_1^{-1}$, it is a 4 elements cyclic group, and this is the only possible group.

1.C

Howard had showed $D_3=D_0\oplus D_2$(1.103), for D_n , it should contain a subrepresentation D_3 map only three states and leave other invariant, soD_n is reducible.

1.D

First we show the irreducible representation of a finite group g must be finite, and then use Shur's lemma to find the relation.

1. For a irreducible representation D_1, consider a vector $v\in D_1$. The set $Gv=\{gv: v\in G\}$ is the orbit of v, and it must be finite since G is finite. $\mathbb{C}[G]v$ is the span of the vector in orbit, and $\mathbb{C}[G]v$ must be finite dimensional. Since $\mathbb{C}[G]v$ is a nontrivial subrepresentation of V, and V is irreducible, therefore it must be all of V. This proves V is finite dimensional.

$$\begin{aligned} AD_1(g)=D_2(g)A=SD_1(g)S^{-1}A \\ \Rightarrow S^{-1}AD_1(g)=D_1(g)S^{-1}A \end{aligned} \tag{62}$$

By Shur's lemma, $S^{-1}A\propto I$ so that $A\propto S$

1.E.

If we fixed one vertex, there will be 3 rotation actions clockwise and 3 counter-clokwise. Also, one can rotate by 180° about the axes. There are 4 vertices, so there are 12 rotation element. There should be 4 conjugacy classes. If we label them, it will be elements of permutation group

$$\begin{gathered} (1)(234),(2)(134),(3)(124),(4)(123) \\ (1)(243),(2)(134),(3)(142),(4)(132) \\ (1)(2)(3)(4) \\ (12)(34),(13)(24),(14)(23) \end{gathered} \tag{63}$$

The first 6 permutations have 1 3-cycle and 1 1-cycle.$k_3=1,k_1=1$

The characters can be calculated in matrix form:

$$\begin{gathered} (4)(123):\begin{pmatrix} 0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \\ TrD((4)(123))=1 \\ (4)(132):\begin{pmatrix} 0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1 \end{pmatrix} \\ TrD((4)(123))=1 \\ \cdots\cdots \end{gathered} \tag{64}$$

They all equal to 1, since they are in one conjugate class.

The number of different permutations in the conjugacy class is

$$\frac{4!}{3^{k_3}k_3!1^{k_1}1!}=8 \tag{65}$$

which is indeed 8

The last three permutation has 2 2-cycle, $k_2=2$

$$\frac{4!}{2^2 2!}=3 \tag{66}$$

The character is

$$\begin{gathered}(12)(34):\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\\ TrD((12)(34))=0\end{gathered} \tag{67}$$

The identity is 1.

So the conjugacy classes of this group are

$$\begin{gathered}\{(1)(234),(2)(134),(3)(124),(4)(123)\\ (1)(243),(2)(134),(3)(142),(4)(132)\}\\ \{(1)(2)(3)(4)\}\\ \{(12)(34),(13)(24),(14)(23)\}\end{gathered} \tag{68}$$

1.F

$$\begin{gathered}(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)=\\ (r_{11},r_{12},r_{21},r_{22},r_{31},r_{32},r_{41},r_{42})\end{gathered} \tag{69}$$

The first index label mass, the second label the x or y component. The 4 dimensional space transforms under S_4 by the representation D_4 , since permutate these balls will give the same result. The second index label spatial coordinate, and rotate the space will also give same result, so the symmetry group is D_2 representation of S_4

S_4 has 1 identity, which is identity matrix,

2 2-cycles, which are

$$() \tag{70}$$

Chapter 2

2.A.

$$A^2=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix} \tag{71}$$

so

$$\begin{gathered}A^{2n}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\\ A^{2n+1}=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\end{gathered} \tag{72}$$

$$\begin{aligned}
&\Sigma_{n=0}^{\infty}\frac{(i\alpha X)^n}{n!}\\
&=\Sigma_{k=0}^{\infty}[\frac{(i\alpha X)^{2k}}{(2k)!}+\frac{(i\alpha X)^{2k+1}}{(2k+1)!}]\\
&=\Sigma_{k=0}^{\infty}[\frac{(i\alpha)^{2k}}{(2k)!}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}+\frac{(i\alpha)^{2k+1}}{(2k+1)!}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}]\\
&=\Sigma_{k=0}^{\infty}[\begin{pmatrix}\frac{(i\alpha)^{2k}}{(2k)!}&0&\frac{(i\alpha)^{2k+1}}{(2k+1)!}\\0&0&0\\\frac{(i\alpha)^{2k+1}}{(2k+1)!}&0&\frac{(i\alpha)^{2k}}{(2k)!}\end{pmatrix}]\\
&=\begin{pmatrix}\Sigma_{k=0}^{\infty}\frac{(-1)^k(\alpha)^{2k}}{(2k)!}&0&\Sigma_{k=0}^{\infty}\frac{i(-1)^k(\alpha)^{2k+1}}{(2k+1)!}\\0&0&0\\\Sigma_{k=0}^{\infty}\frac{i(-1)^k(\alpha)^{2k+1}}{(2k+1)!}&0&\Sigma_{k=0}^{\infty}\frac{(-1)^k(\alpha)^{2k}}{(2k)!}\end{pmatrix}\\
&=\begin{pmatrix}cos\alpha&0&i\sin\alpha\\0&0&0\\i\sin\alpha&0&cos\alpha\end{pmatrix}
\end{aligned}\tag{73}$$

2.B

$$\begin{aligned}
e^{-i\epsilon Y}Xe^{i\epsilon Y}&=[1+(-i\epsilon Y)+(-i\epsilon Y)^2/2+\ldots]X[1+(i\epsilon Y)+(i\epsilon Y)^2/2+\ldots]\\
&=X+i\epsilon[X,Y]+\ldots+(i\epsilon)^k\Sigma_{i=0}^{i=k}\frac{(-1)^i}{i!(k-i)!}Y^iXY^{k-i}+\ldots
\end{aligned}\tag{74}$$

$$\begin{aligned}
&1/2((-i\epsilon)^2Y^2X+(i\epsilon)^2XY^2)-(i\epsilon)^2YXY\\
&=1/2((-i\epsilon)^2Y[Y,X]+(i\epsilon)^2[X,Y]Y+2(i\epsilon)^2YXY)-(i\epsilon)^2YXY\\
&=(i\epsilon)^2/2[Y,[Y,X]]
\end{aligned}\tag{75}$$

We use induction. Assume

$$\begin{aligned}
&\frac{(-1)^n}{n!}Y^nX=(-1)^n/n![Y,[Y,\ldots.[Y,X]]]\to k\,Ys-the\ rest\ terms\\
&((i\epsilon)^k\Sigma_{i=0}^{i=k}\frac{(-1)^i}{i!(k-i)!}Y^iXY^{k-i}=(-i\epsilon)^k/(n)![Y,[Y,\ldots.[Y,X]]]\to k\,Ys)
\end{aligned}\tag{76}$$

$$\begin{aligned}
&\frac{(-1)^{n+1}}{(n+1)!}Y^{n+1}X=\frac{-Y}{n+1}\frac{(-1)^n}{(n)!}([Y,[Y,\ldots.[Y,X]]]\to n\,Ys-the\ rest\ terms)\\
&=(-1)^{n+1}/(n+1)![Y,[Y,\ldots.[Y,X]]]\to(n+1)\,Ys\\
&+[Y,[Y,\ldots.[Y,X]]]\frac{(-1)^nY}{(n+1)!}\to n\,Ys\ in\ bracket-\frac{-1}{n+1}the\ rest\ terms\\
&=(-1)^{n+1}/(n+1)![Y,[Y,\ldots.[Y,X]]]\to(n+1)\,Ys\\
&+[Y,[Y,\ldots.[Y,X]]]\frac{(-1)^{n+1}Y}{(n+1)!}\to n\,Ys\ in\ bracket-\frac{-Y}{n+1}the\ rest\ terms
\end{aligned}\tag{77}$$

$$\begin{aligned}
&[Y,[Y,\ldots.[Y,X]]]\frac{(-1)^{n+1}Y}{(n+1)!}\to n\,Ys\ in\ bracket-\frac{-Y}{n+1}the\ rest\ terms\\
&=(\Sigma_{i=0}^{i=n}\frac{(-1)^i}{i!(k-i)!}Y^iXY^{k-i})\frac{-Y}{n+1}-\frac{-Y}{n+1}\Sigma_{i=0}^{i=n-1}\frac{(-1)^i}{i!(k-i)!}Y^iXY^{n-i}\\
&=-1/(n+1)(\Sigma_{i=0}^{i=n}\frac{(-1)^i}{i!(k-i)!}Y^iXY^{n+1-i}-\Sigma_{i=1}^{i=n}\frac{(-1)^{i-1}}{(i-1)!(k+1-i)!}Y^iXY^{n+1-i})\\
&=-1/(n+1)[(-1)^n/(n)!XY^{n+1}+\Sigma_{i=1}^{i=n}(\frac{(-1)^i}{i!(n-i)!}+\frac{(-1)^i}{(i-1)!(n+1-i)!})Y^iXY^{n+1-i}]\\
&=-1/(n+1)[(-1)^n/(n)!XY^{n+1}+\Sigma_{i=1}^{i=n}(\frac{(-1)^i}{(i-1)!(n-i)!})(\frac{n+1-i+i}{i(n+1-i)})Y^iXY^{n+1-i}]\\
&=-1/(n+1)[(-1)^n/(n)!XY^{n+1}+\Sigma_{i=1}^{i=n}(\frac{(-1)^i}{(i-1)!(n-i)!})(\frac{n+1}{i(k+1-i)})Y^iXY^{k+1-i}]\\
&=-\Sigma_{i=0}^{i=n}(\frac{(-1)^i}{(i)!(n+1-i)!})Y^iXY^{k-i}
\end{aligned}\tag{78}$$

(all the k should be n)

$$[A,[A,\ldots.[A,B]]] \to k\ As = B \tag{79}$$

So

$$e^{i\alpha A}Be^{-i\alpha A} = e^{i\alpha B} \tag{80}$$

2.C

$$\begin{gathered}
K = e^{i\alpha_a X_a}e^{i\beta X_b} - 1 \\
= (1 + i\alpha_a X_a + 1/2(i\alpha_a X_a)^2 + 1/6(i\alpha_a X_a)^3 + \ldots) \\
(1 + i\beta_b X_b + 1/2(i\beta_b X_b)^2 + 1/6(i\beta_b X_b)^3 + \ldots) - 1 \\
= i\alpha_a X_a + i\beta_b X_b + (i\alpha_a X_a)(i\beta_b X_b) \\
+1/2(i\alpha_a X_a)^2 + 1/2(i\beta_b X_b)^2 + 1/6(i\alpha_a X_a)^3 + 1/6(i\beta_b X_b)^3 \\
+(i\alpha_a X_a)(1/2(i\beta_b X_b)^2) + (1/2(i\alpha_a X_a)^2)(i\beta_b X_b) + \ldots
\end{gathered} \tag{81}$$

$$\begin{gathered}
i\delta_a X_a = K - 1/2K^2 + 1/3K^3 + \ldots \\
= [i\alpha_a X_a + i\beta_b X_b + (i\alpha_a X_a)(i\beta_b X_b) \\
+1/2(i\alpha_a X_a)^2 + 1/2(i\beta_b X_b)^2 + 1/6(i\alpha_a X_a)^3 + 1/6(i\alpha_a X_a)^3 \\
+(i\alpha_a X_a)(1/2(i\beta_b X_b)^2) + (i\beta_b X_b)(1/2(i\beta_b X_b)^2)] \\
- \\
1/2[(i\alpha_a X_a + i\beta_b X_b)^2 \\
+(i\alpha_a X_a + i\beta_b X_b)((i\alpha_a X_a)(i\beta_b X_b) + 1/2(i\alpha_a X_a)^2 + 1/2(i\beta_b X_b)^2) + ((i\alpha_a X_a)(i\beta_b X_b) \\
+1/2(i\alpha_a X_a)^2 + 1/2(i\beta_b X_b)^2)(i\alpha_a X_a + i\beta_b X_b)] \\
+ \\
1/3[i\alpha_a X_a + i\beta_b X_b]^3
\end{gathered} \tag{82}$$

$$\begin{gathered}
i\delta_a X_a = K - 1/2K^2 + 1/3K^3 + \ldots \\
= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] +
\end{gathered} \tag{83}$$

Third order terms:

$$
\begin{gathered}
1/6(i\alpha_a X_a)^3 + 1/6(i\beta_b X_b)^3 + (i\alpha_a X_a)(1/2(i\beta_b X_b)^2) + (1/2(i\alpha_a X_a)^2(i\beta_b X_b) \\
+ \\
1/2[(i\alpha_a X_a + i\beta_b X_b)((i\alpha_a X_a)(i\beta_b X_b) + 1/2(i\alpha_a X_a)^2 + 1/2(i\beta_b X_b)^2) + ((i\alpha_a X_a)(i\beta_b X_b) \\
+1/2(i\alpha_a X_a)^2 + 1/2(i\beta_b X_b)^2)(i\alpha_a X_a + i\beta_b X_b)] \\
+ \\
1/3[i\alpha_a X_a + i\beta_b X_b]^3 \\
\\
= +(i\alpha_a X_a)(1/2(i\beta_b X_b)^2) + (1/2(i\alpha_a X_a)^2(i\beta_b X_b) \\
+ \\
-1/2[(i\alpha_a X_a)^2(i\beta_b X_b) + 1/2(i\alpha_a X_a)(i\beta_b X_b)^2 + (i\beta_b X_b)(i\alpha_a X_a)(i\beta_b X_b) + 1/2(i\beta_b X_b)(i\alpha_a X_a)^2 \\
+(i\alpha_a X_a)(i\beta_b X_b)(i\alpha_a X_a) + 1/2(i\beta_b X_b)^2(i\alpha_a X_a) + (i\alpha_a X_a)(i\beta_b X_b)^2 + 1/2(i\alpha_a X_a)^2(i\beta_b X_b)] \\
+ \\
+1/3(i\alpha_a X_a(i\alpha_a X_a i\beta_b X_b + i\beta_b X_b i\alpha_a X_a) + i\beta_b X_b(i\alpha_a X_a i\beta_b X_b + i\beta_b X_b i\alpha_a X_a) \\
+(i\alpha_a X_a(i\beta_b X_b)^2 + i\beta_b X_b(i\alpha_a x_a)^2) \\
\\
= +(i\alpha_a X_a)(1/2(i\beta_b X_b)^2) + (1/2(i\alpha_a X_a)^2(i\beta_b X_b) \\
+ \\
-1/2[3/2(i\alpha_a X_a)^2(i\beta_b X_b) + 3/2(i\alpha_a X_a)(i\beta_b X_b)^2 + (i\beta_b X_b)(i\alpha_a X_a)(i\beta_b X_b) + 1/2(i\beta_b X_b)(i\alpha_a X_a)^2 \\
+(i\alpha_a X_a)(i\beta_b X_b)(i\alpha_a X_a) + 1/2(i\beta_b X_b)^2(i\alpha_a X_a)] \\
+ \\
+1/3(i\alpha_a X_a(i\alpha_a X_a i\beta_b X_b + i\beta_b X_b i\alpha_a X_a) + i\beta_b X_b(i\alpha_a X_a i\beta_b X_b + i\beta_b X_b i\alpha_a X_a) \\
+(i\alpha_a X_a(i\beta_b X_b)^2 + i\beta_b X_b(i\alpha_a x_a)^2) \\
\\
= -1/6 i\alpha_a X_a(i\beta_b X_b)(i\alpha_a X_a) - 1/6 i\beta_b X_b(i\alpha_a X_a)((i\beta_b X_b) \\
+1/12(i\alpha_a X_a)^2 i\beta_b X_b + 1/12(i\beta_b X_b)^2 i\alpha_a X_a \\
+1/12 i\beta_b X_b(i\alpha_a X_a)^2 + 1/12 i\alpha_a X_a(i\beta_b X_b)^2 \\
= 1/12 i\alpha_a X_a[i\alpha_a X_a, i\beta_b X_b] + 1/12 i\beta_b X_b[i\beta_b X_b, i\alpha_a X_a] + 1/12[i\beta_b X_b, i\alpha_a X_a] i\alpha_a X_a + 1/12[i\alpha_a X_a, i\beta_b X_b] i\beta_b X_b \\
= 1/12([i\alpha_a X_a - i\beta_b X_b, [i\alpha_a X_a, i\beta_b X_b]])
\end{gathered} \tag{84}
$$

Chapter 3

3.A

The highest weight is $s+j$, we use j^- to lower the state

$$
\begin{gathered}
J^-|s,s>|j,j>= \sqrt{(s+s)(s-s+1)/2}|s,s-1>|j,j> \\
+\sqrt{(j+j)(j-j+1)/2}|s,s-1>|j,j>
\end{gathered} \tag{85}
$$

3.B

$$
\begin{gathered}
\{\sigma_i, \sigma_j\} = 2\delta_{ij} I \\
(\vec{r}\cdot\vec{\sigma})^2 = (r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)^2 = r_1^2 I + r_2^2 I + r_3^2 I = (r_1^2 + r_2^2 + r_3^2) I \\
(\vec{r}\cdot\vec{\sigma})^{2n} = (r_1^2 + r_2^2 + r_3^2)^n I \\
(\vec{r}\cdot\vec{\sigma})^{2n+1} = (r_1^2 + r_2^2 + r_3^2)^n (r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)
\end{gathered} \tag{86}
$$

$$
\begin{aligned}
e^{i\vec{r}\cdot\vec{\sigma}} &= \Sigma_{n=0}^{\infty}(\frac{(i\vec{r}\cdot\vec{\sigma})^n}{n!}) \\
&= \Sigma_{i=0}^{\infty}(\frac{(i\vec{r}\cdot\vec{\sigma})^{2i}}{2i!}) + \Sigma_{j=0}^{\infty}(\frac{(i\vec{r}\cdot\vec{\sigma})^{(2j+1)}}{(2j+1)!}) \\
&= \Sigma_{i=0}^{\infty}(\frac{(i^{2i}(r_1^2+r_2^2+r_3^2)^i I}{(2i)!}) + \Sigma_{j=0}^{\infty}(\frac{i^{2j+1}(r_1^2+r_2^2+r_3^2)^j(r_1\sigma_1+r_2\sigma_2+r_3\sigma_3))}{(2j+1)!} \\
&= \Sigma_{n=0}^{\infty}\begin{pmatrix} (r_1^2+r_2^2+r_3^2)^n(\frac{i^{2n}}{(2n!)}+\frac{i^{2n+1}r_3}{(2n+1)!}) & (r_1^2+r_2^2+r_3^2)^n i^{2n+1}(\frac{-ir_2}{(2n+1)!}+\frac{r_1}{(2n+1)!}) \\ (r_1^2+r_2^2+r_3^2)^n i^{2n+1}(\frac{ir_2}{(2n+1)!}+\frac{r_1}{(2n+1)!}) & (r_1^2+r_2^2+r_3^2)^n(\frac{i^{2n}}{(2n)!}+\frac{i^{2n+1}r_3}{(2n+1)!}) \end{pmatrix} \\
&= \Sigma_{n=0}^{\infty}(r_1^2+r_2^2+r_3^2)^n(-1)^n\begin{pmatrix} (\frac{1}{(2n!)}+\frac{ir_3}{(2n+1)!}) & i(\frac{-ir_2}{(2n+1)!}+\frac{r_1}{(2n+1)!}) \\ i(\frac{ir_2}{(2n+1)!}+\frac{r_1}{(2n+1)!}) & (\frac{1}{(2n)!}-\frac{ir_3}{(2n+1)!}) \end{pmatrix} \\
&= \Sigma_{n=0}^{\infty}(r_1^2+r_2^2+r_3^2)^n\frac{(-1)^n}{(2n!)}\begin{pmatrix} (1+\frac{ir_3}{(2n+1)}) & i(\frac{-ir_2+r_1}{(2n+1)}) \\ i(\frac{ir_2+r_1}{(2n+1)}) & (1-\frac{ir_3}{(2n+1)}) \end{pmatrix}
\end{aligned} \tag{87}
$$

$$
= \cos\left(\sqrt{r_1^2+r_2^2+r_3^2}\right)I + \begin{pmatrix} \frac{ir_3}{\sqrt{r_1^2+r_2^2+r_3^2}}\sin\left(\sqrt{r_1^2+r_2^2+r_3^2}\right) & \frac{r_2+ir_1}{\sqrt{r_1^2+r_2^2+r_3^2}}\sin\left(\sqrt{r_1^2+r_2^2+r_3^2}\right) \\ (\frac{-r_2+ir_1}{\sqrt{r_1^2+r_2^2+r_3^2}}\sin\left(\sqrt{r_1^2+r_2^2+r_3^2}\right) & -\frac{ir_3}{\sqrt{r_1^2+r_2^2+r_3^2}}\sin\left(\sqrt{r_1^2+r_2^2+r_3^2}\right) \end{pmatrix} \tag{88}
$$

3.D.

$$
\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{89}
$$

3.E.

$$
\begin{aligned}
\sigma_a\sigma_b &= \delta_{ab} + i\epsilon_{abc}\sigma_c \\
[\sigma_a]_{ij}[\sigma_b]_{jk} &= \delta_{ik}\delta_{ab} + i\epsilon_{abc}[\sigma_c]_{ik}
\end{aligned} \tag{90}
$$

(a)

$$
\begin{aligned}
(\sigma_a)_{ij}\delta_{xy}(\sigma_b)_{jk}(\eta_c)_{yz} - (\sigma_b\eta_c)(\sigma_a) &= (\delta_{ab} + i\epsilon_{abd}\sigma_d)\eta_c - (\delta_{ab} - i\epsilon_{abd}\sigma_d)\eta_c \\
&= 2i\epsilon_{abd}\sigma_d\eta_c
\end{aligned} \tag{91}
$$

(b)

$$
\begin{aligned}
&= tr(\sigma_a\sigma_c(\eta_b\eta_d + \eta_d\eta_b)) = tr((\delta_{ac} + i\epsilon_{ace}\sigma_e)(2\delta_{bd})) \\
&= 4\delta_{ac}\delta_{bd}
\end{aligned} \tag{92}
$$

(c)

$$
\begin{aligned}
&\sigma_1\eta_1\sigma_2\eta_2 - \sigma_2\eta_2\sigma_1\eta_1 \\
&= \sigma_1\sigma_2\eta_1\eta_2 - \sigma_2\sigma_1\eta_2\eta_1 \\
&= -\sigma_3\eta_3 - (-i)\sigma_3(-i)\eta_3 \\
&= 0
\end{aligned} \tag{93}
$$

Chapter 4

4.A

$$
\begin{aligned}
[J_a, O_x] &= O_y[\sigma_a]_{yx}/2 \\
O_2[\sigma_1]_{21} &= 2[J_1, O_1] - O_1[\sigma_1]_{11} \\
\Rightarrow O_2 &= 2[J_1, O_1] \\
O_2 i &= 2[J_2, O_1] \\
&< 3/2, -3/2, \alpha|O_2|1, -1, \beta > \\
&=< 3/2, -3/2, \alpha|([J_1, O_1] - i[J_2, O_1])|1, -1, \beta > \\
&= \sqrt{2} < 3/2, -3/2, \alpha|([J^-, O_1]|1, -1, \beta > \\
&= \sqrt{2} < 3/2, -3/2, \alpha|J^-O_1 - O_1J^-|1, -1, \beta > \\
&= \sqrt{3} < 3/2, -1/2, \alpha|O_1|1, -1, \beta >
\end{aligned} \tag{94}
$$

4.B

4.C

Using the hint,

$$\begin{aligned}(\hat{a}_1 X_1^1 + \hat{a}_2 X_2^1 + \hat{a}_3 X_3^1)(\hat{a}_1 X_1^1 + \hat{a}_2 X_2^1 + \hat{a}_3 X_3^1)\\ = (\hat{a}_1 X_1^1)^2 + (\hat{a}_2 X_2^1)^2 + (\hat{a}_3 X_3^1)^2 +\end{aligned} \tag{95}$$

Chapter 5

5.A

If we exchange fermi we will get an extra minus sign, and if we exchange boson the sign will remain unchanged. Pion is boson, so if we interchange two pions, the sign remains unchanged. total Isospin value should be 0.

5.B

$$T_a = \Sigma_{x,\alpha,m,m'} a^\dagger_{x,m,\alpha} [J_a^{j_x}]_{mm'} a_{x,m',\alpha} \tag{96}$$

$$\begin{aligned}[T_a, a^\dagger_{N,m,\alpha}] = \Sigma_{x,\alpha,m',m''} [J_a^{j_x}]_{m'm''} [a^\dagger_{x,m',\alpha} a_{x,m'',\alpha}, a^\dagger_{N,m,\alpha}]\\ = \Sigma_{x,\alpha,m',m''} [J_a^{j_x}]_{m'm''} (a^\dagger_{x,m',\alpha} [a_{x,m'',\alpha}, a^\dagger_{N,m,\alpha}] + [a^\dagger_{x,m',\alpha}, a^\dagger_{N,m,\alpha}] a_{x,m'',\alpha})\\ = \Sigma_{x,\alpha,m',m''} [J_a^{j_x}]_{m'm''} (a^\dagger_{x,m',\alpha} [a_{x,m'',\alpha}, a^\dagger_{N,m,\alpha}])\\ = \Sigma_{x,\alpha,m,m'} [J_a^{j_x}]_{m'm''} (a^\dagger_{x,m,\alpha} \delta_{mm''} \delta_{\alpha\beta} \delta_{xN})\\ = [J_a^{j_x}]_{mm''} a^\dagger_{x,m'',\alpha}\end{aligned} \tag{97}$$

5.C

Isospin of $\pi^+ P$ is $|1,1>|1/2,1/2>$

$\pi^- P$ is $|1,-1>|1/2,1/2>$

$$<1/2,1/2|<1,1||3/2,3/2> = <1/2,1/2|<1,1||1/2,1/2>|1,1> = 1$$

$$\begin{aligned}<1,-1|<1/2,1/2||3/2,-1/2>\\ = <1,-1|<1/2,1/2|(\sqrt{\frac{2}{3}}|1/2,-1/2>|1,0> + \sqrt{\frac{1}{3}}|1/2,1/2>|1,-1>)\\ = \sqrt{\frac{1}{3}}\end{aligned} \tag{98}$$

Chapter 6

$$\begin{aligned}[H_i, [E_\alpha, E_\beta]] = H_i [E_\alpha, E_\beta] - [E_\alpha, E_\beta] H_i\\ = H_i E_\alpha E_\beta - H_i E_\beta E_\alpha - E_\alpha E_\beta H_i + E_\beta E_\alpha H_i\\ = [H_i, E_\alpha] E_\beta + E_\alpha [H_i, E_\beta] - [H_i, E_\beta] E_\alpha + E_\beta [E_\alpha, H_i]\\ = \alpha_i E_\alpha E_\beta + \beta_i E_\alpha E_\beta - \beta_i E_\beta E_\alpha - \alpha_i E_\beta E_\alpha\\ = (\alpha_i + \beta_i)[E_\alpha, E_\beta]\end{aligned} \tag{99}$$

By definition, there must be a component proportional to $E_{\alpha+\beta}$.

The form may be $k_{\alpha\beta} E_{\alpha+\beta} + \alpha_i H_i$?

$$\begin{aligned}<H_j, [E_\alpha, E_\beta]> = k\, tr[E_\alpha, E_\beta] H_j = k\, tr[H_j, E_\alpha] E_\beta\\ = k\, \alpha_j tr E_\alpha E_\beta = k\, \alpha_j \delta_{-\alpha\, \beta}\end{aligned} \tag{100}$$

When $\alpha \neq -\beta$, $<[E_\alpha, E_\beta], H_j> = 0$.

So the form is $k_{\alpha\beta} E_{\alpha+\beta}$

If $\alpha + \beta$ is not a root, $[H_i, [E_\alpha, E_\beta]] = 0$ so$[E_\alpha, E_\beta]$=0

$$\begin{aligned}
&< [E_\alpha, E_\beta], [E_\alpha, E_\beta] > \\
&=< E_a E_b, E_a E_b > - < E_a E_b, E_b E_a > + < E_b E_a, E_b E_a > - < E_b E_a, E_a E_b > \\
&= Tr(E_{-a}E_{-b}E_aE_b) - Tr(E_{-a}E_{-b}E_bE_a) + Tr(E_{-b}E_{-a}E_bE_a) - Tr(E_{-b}E_{-a}E_aE_b) \\
&= Tr(E_aE_bE_{-a}E_{-b}) - Tr(E_aE_{-a}E_{-b}E_b) + Tr(E_aE_{-b}E_{-a}E_b) - Tr(E_aE_bE_{-b}E_{-a}) \\
&= Tr(E_aE_b[E_{-a}, E_{-b}]) - Tr(E_a[E_{-a}, E_{-b}]E_b) \\
&= Tr(E_a[E_b, [E_{-a}, E_{-b}]])
\end{aligned} \tag{101}$$

and $k_{\alpha\beta}$ must satisfy Jacobi identity:

$$k_{\alpha,\,\beta} = k_{\beta,\,-(\alpha+\beta)} = k_{-(\alpha+\beta)\ ,\beta} \tag{102}$$

If $\alpha + \beta$ is not a root, for arbitrary i,

$[E_\beta, E_\alpha]$ cannot proportional to H, ? how to prove it,

6.B.Suppose that the raising lowering operators of some Lie algebra satisfy [Ea, E/3] = N Ea+/3

for some nonzero N. Calculate

$$\begin{aligned}
[E_\alpha, E_\beta] = NE_{\alpha+\beta} \\
[E_\alpha, E_{-\alpha-\beta}] = N'E_{-\beta}
\end{aligned} \tag{103}$$

We want to know N'.

$$[E_a, -1/n'[e_{-a}, e_{a+b}]] = NE_{a+b} \tag{104}$$

$$\begin{aligned}
&< E_{-\beta}, [E_\alpha, E_{-\alpha-\beta}] > \\
&=< E_{-\beta}, -[E_{-\alpha-\beta}, E_\alpha] > \\
&= -Tr(E_{-\alpha-\beta}[E_\beta, E_\alpha]) \\
&= -Tr(E^\dagger_{\alpha+\beta} N E_{\alpha+\beta}) \\
&= -N
\end{aligned} \tag{105}$$

So

$$[E_\alpha, E_{-\alpha-\beta}] = -N\ E_{-\beta} \tag{106}$$

6.C.Consider the simple Lie algebra formed by the ten matrices: for a= 1 to 3 where ()a and Ta are Pauli matrices in orthogonal spaces (see problem 3.E). Take H 1= ()3 and H2 = (J3T3 as the Cartan subalgebra. Find

$$\begin{gathered}
|1 >= |i = 1 > |x = 1 > \\
|2 >= |i = 1 > |x = 2 > \\
|3 >= |i = 2 > |x = 1 > \\
|4 >= |i = 2 > |x = 2 > \\
[\sigma_a]_{ij}[\tau_b]_{xy} \\
\begin{pmatrix} [\sigma_a]_{11}\tau_b & [\sigma_a]_{12}\tau_b \\ [\sigma_a]_{21}\tau_b & [\sigma_a]_{22}\tau_b \end{pmatrix} \\
(\sigma_a\tau_b)_{ij} = [\sigma_a]_{[i/2][j/2]}[\tau_b]_{(imod2)(jmod2)} \\
[\sigma_a]_{ij}[\tau_b]_{xy}[\sigma_a]_{jk}[\tau_b]_{yz}
\end{gathered} \tag{107}$$

(a) the weights of the four dimensional representation generated by these matrices, and

$$\begin{aligned}
\sigma_3 I &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\sigma_3\tau_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{108}$$

(b)the weights of the adjoint representation.

$$\begin{aligned}\sigma_a\sigma_b &= \delta_{ab} + i\epsilon_{abc}\sigma_c \\ [\sigma_a]_{ij}[\sigma_b]_{jk} &= \delta_{ik}\delta_{ab} + i\epsilon_{abc}[\sigma_c]_{ik}\end{aligned} \tag{109}$$

From 3.E, we know

$$[(\sigma_a), (\sigma_b\eta_c)] = 2i\epsilon_{abd}\sigma_d\eta_c \tag{110}$$

We choose the basis as

$$\sigma_1, \sigma_2, \sigma_1\tau_1, \sigma_2\tau_1, \sigma_3\tau_1, \sigma_1\tau_3, \sigma_2\tau_3, \tau_2 \tag{111}$$

and diagonalize the matrix. But we find we can get the root by direct calculation.

$$\begin{gathered}[\sigma_3\tau_3, \tau_2] = \sigma_3(-i\tau_1) - \sigma_3(i\tau_1) = -2i\sigma_3\tau_1 \\ [\sigma_3\tau_3, \sigma_3\tau_1] = i\tau_2 - (-i)\tau_2 = 2i\tau_2 \\ \Rightarrow [\sigma_3\tau_3, \sigma_3\tau_3 \pm i\tau_2] = \pm 2(\sigma_3\tau_3 \pm i\tau_2)\end{gathered} \tag{112}$$

For σ_3,

$$\begin{gathered}\sigma_\pm + \sigma_\pm\tau_3 = \sigma_1 \pm i\sigma_2, root_1 = \pm 2 \\ \sigma_\pm\tau_1 = \sigma_1\tau_1 \pm i\sigma_2\tau_1, root_1 = \pm 2 \\ \sigma_3\tau_1 \pm i\tau_2, root_1 = 0 \\ \sigma_\pm - \sigma_\pm\tau_3 = \sigma_1 \pm i\sigma_2, root_1 = \pm 2\end{gathered} \tag{113}$$

$\sigma_3\tau_3$

$$\begin{gathered}[\sigma_3\tau_3, \sigma_\pm\tau_1] \\ = \sigma_3\sigma_\pm\tau_3\tau_1 - \sigma_\pm\sigma_3\tau_1\tau_3 \\ = \sigma_3\sigma_\pm 2i\tau_2 - \sigma_\pm\sigma_3(-2i\tau_2) \\ = \{\sigma_3, \sigma_\pm\}2i\tau_2 = 0\end{gathered} \tag{114}$$

$$\begin{gathered}[\sigma_3\tau_3, \sigma_\pm - \sigma_\pm\tau_3] = \pm 2\sigma_\pm\tau_3 \mp 2\sigma_\pm \\ = \mp 2(\sigma_\pm - \sigma_\pm\tau_3)\end{gathered} \tag{115}$$

$$\begin{gathered}\sigma_\pm + \sigma_\pm\tau_3 = \sigma_1 \pm i\sigma_2, root_2 = \pm 2 \\ \sigma_\pm\tau_1 = \sigma_1\tau_1 \pm i\sigma_2\tau_1, root_2 = 0 \\ \sigma_3\tau_1 \pm i\tau_2, root_2 = \pm 2 \\ \sigma_\pm - \sigma_\pm\tau_3, root_2 = \mp 2\end{gathered} \tag{116}$$

The roots are

$$(0,0), (\pm 2, 0), (0, \pm 2), (\pm 2, \pm 2), (\pm 2, \mp 2) \tag{117}$$

Chapter 7

7.A Calculate f_{147} and f_{458} in SU(3).

We use a basis for SU(3) that $Tr(T_aT_b) = \frac{1}{2}\delta_{ab}$

$$f_{abc} = -i\lambda^{-1} Tr([T_a, T_b]T_c)$$

$$\begin{aligned}
f_{147} = -i2\,Tr([T_1,T_4]T_7) = -2i\,Tr(T_1T_4T_7 - T_4T_1T_7)\\
-\frac{i}{4}(\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} - \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix})\\
= -\frac{i}{4}(\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix})\\
= -\frac{i}{4}(\begin{pmatrix}0&0&0\\0&i&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&0&-i\end{pmatrix})\\
= -\frac{i}{4}\begin{pmatrix}0&0&0\\0&i&0\\0&0&i\end{pmatrix}\\
f_{147} = -\frac{i}{4} \times 2i = -\frac{1}{2}
\end{aligned} \tag{118}$$

$$\begin{aligned}
f_{458} = -2iTr(T_4T_5T_8 - T_5T_4T_8)\\
= -\frac{i}{4}\,Tr(\lambda_4\lambda_5\lambda_8 - \lambda_5\lambda_4\lambda_8)\\
\lambda_4\lambda_5\lambda_8\\
= \frac{1}{\sqrt{3}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\\
= \frac{1}{\sqrt{3}}\begin{pmatrix}i&0&0\\0&0&0\\0&0&-i\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\\
= \frac{1}{\sqrt{3}}\begin{pmatrix}i&0&0\\0&0&0\\0&0&2i\end{pmatrix}\\
\lambda_5\lambda_4\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\\
= \frac{1}{\sqrt{3}}\begin{pmatrix}-i&0&0\\0&0&0\\0&0&i\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\\
= \frac{1}{\sqrt{3}}\begin{pmatrix}-i&0&0\\0&0&0\\0&0&-2i\end{pmatrix} f_{458} = -\frac{i}{4}(2i\sqrt{3}) = \frac{\sqrt{3}}{2}
\end{aligned} \tag{119}$$

7.B

By block matrix multiplication rule, structure constants of T_1, T_2 andT_3 are the same as pauli marices.

We can see from the matrix element, T_1, T_2, T_3 form a $\frac{1}{2}$ spin representation direct product with Trivial representation.

From(7.12), we can see T_1, T_2, T_3 form a spin 1 rep. So it is irreducible

7.C.Show that $\lambda_2, \lambda_5$and λ_7 generate an SU(2) subalgebra of SU(3). Every representation of SU(3) must also be a representation of the subalgebra. However, the irreducible representations of SU(3) are not necessarily irreducible under the subalgebra. How does the representation generated by the Gell-Mann matrices transform under this subalgebra. That is, reduce, if necessary, the three dimensional representation into representations which are irreducible under the subalgebra and state which irreducible representations appear in the reduction. Then answer the same question for the adjoint representation of SU(3).

From calculation, we get

$$\begin{gathered}
[\lambda_2, \lambda_7] = -i\lambda_5 \\
[\lambda_2, \lambda_5] = i\lambda_7 \\
[\lambda_5, \lambda_7] = i\lambda_2
\end{gathered} \tag{120}$$

$$\begin{gathered}
\lambda_2 \to 1/2\sigma_1 \\
\lambda_5 \to 1/2\sigma_2 \\
\lambda_7 \to 1/2\sigma_3
\end{gathered} \tag{181}$$

$$\begin{gathered}
\lambda_\pm = \lambda_2 \pm i\lambda_5 \\
[\lambda_7, \lambda_\pm] = \pm\lambda_\pm
\end{gathered} \tag{182}$$

Eigenvalues of λ_7 are $-1, 1, 0$, and root is ± 1. So they are spin 1 rep

Both of them are irreducible rep.

Chapter 8

8.A.

The positive roots are$(0,2), (2,0), (2,\pm 2)$. There should be 2 vector to span the vector space. And the angle between a pair of simple root should staisfy $\pi/2 \le \theta < \pi$

The only three choices are $(0,2), (2,0)$, $(0,2), (2,-2)$ and $(2,2), (2,-2)$

The difference of simple roots is not a root. So the simple roots may be $(0,2), (2,-2)$ or $(2,2), (2,-2)$.

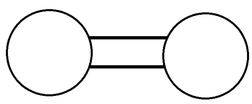
But $(2,2), (2,-2)$ can be written as sum of other simple roots. So the simple roots are $(0,2), (2,-2)$.

The fundamental weight satisfy:

$$\frac{2\alpha^j \cdot \mu^k}{\alpha^{j^2}} = \delta_{jk} \tag{183}$$

$$\begin{gathered}
2(0 + 2\mu_2^1)/4 = 1 \\
2(2\mu_1^1 - 2\mu_2^1)/8 = 0 \\
\mu^1 = (1,1)
\end{gathered} \tag{184}$$

$$\begin{gathered}
2(0 + 2\mu_2^2)/4 = 0 \\
2(2\mu_1^2 - 2\mu_2^2)/8 = 1 \\
\mu^2 = (2,0)
\end{gathered} \tag{185}$$



Where $(\alpha^1)^2 = 4, (\alpha^2)^2 = 8$

8.B

$$[\sigma_3, \sigma_3\eta_1] = 0 \tag{186}$$

$$\begin{aligned}
[\sigma_3\eta_1, \sigma_\pm\eta_1] &= \pm 2\sigma_\pm \\
[\sigma_3\eta_1, \sigma_\pm] &= \pm 2\sigma_\pm\eta_1 \\
[\sigma_3\eta_1, \sigma_\pm + \sigma_\pm\eta_1] &= \pm 2(\sigma_\pm + \sigma_\pm\eta_1) \\
[\sigma_3\eta_1, \sigma_\pm - \sigma_\pm\eta_1] &= \pm 2(-\sigma_\pm + \sigma_\pm\eta_1) \\
&= \mp 2(\sigma_\pm - \sigma_\pm\eta_1)
\end{aligned} \tag{187}$$

$$\tag{188}$$

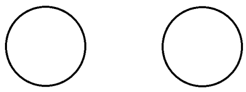
This is similar to $SO(3)$ if we change the basis to

$$\begin{aligned}
J_i &= \frac{\sigma_i + \sigma_i\eta_1}{4} \\
K_j &= \frac{\sigma_i - \sigma_i\eta_1}{4} \\
[J_i, J_j] = i\epsilon_{ijk}J_k,\ & [K_i, K_j] = i\epsilon_{ijk}K_k
\end{aligned} \tag{189}$$

$$\begin{aligned}
[J_i, K_k] \propto (1+\eta_1)&(1-\eta_1) = 0 \\
J_\pm &= J_1 \pm iJ_2 \\
K_\pm &= K_1 \pm iK_2 \\
[J_i, K_j] &= 0 \\
[J_3, J_\pm] &= \pm J_\pm \\
[K_3, K_\pm] &= \pm K_\pm
\end{aligned} \tag{190}$$

$$(\pm 1, 0), (0, \pm 1) \tag{191}$$

the positive roots are$(1,0), (0,1)$



8.C

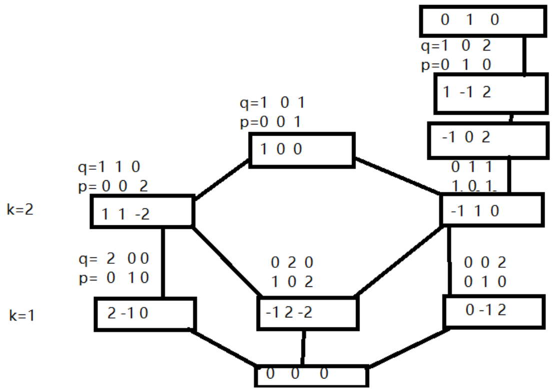
The elements of Cartan matrix satisfy:

$$A_{ij}A_{ji} = 4cos^2\theta_{\alpha_i\alpha_j}(j\ is\ not\ summation\ here) \tag{192}$$

$A_{ij}a_j^2 = A_{ji}a_i^2$ and $A_{31} = A_{13} = 0$

$$\begin{aligned}
A_{12}^2 &= 4cos^2 120^\circ = 1 \\
A_{23}^2/2 &= 4cos^2 135^\circ = 2 \\
A_{12} &= A_{21} = -1 \\
A_{23} &= 2A_{32} = -2
\end{aligned} \tag{193}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{194}$$



Chapter 9

9.A

$$\begin{gathered}
< A|A >=< \mu||E_{\alpha^2}E_{\alpha^1}E_{-\alpha^1}E_{-\alpha^2}|\mu > \\
=< \mu||E_{\alpha^2}E_{\alpha^{-1}}E_{\alpha^1}E_{\alpha^{-2}}|\mu > + < \mu||E_{\alpha^2}[E_{\alpha^1}, E_{-\alpha^1}]E_{-\alpha^2}|\mu > \\
=< \mu||[E_{\alpha^2}, E_{\alpha^{-1}}]E_{\alpha^1}E_{\alpha^{-2}}|\mu > (Not\ a\ root) + < \mu||E_{\alpha^2}(\alpha^1 \cdot H)E_{-\alpha^2}|\mu > \\
=< \mu||E_{\alpha^2}[(\alpha^1 \cdot H), E_{-\alpha^2}]|\mu > + < \mu||E_{\alpha^2}E_{-\alpha^2}(\alpha^1 \cdot \mu)|\mu > \\
= (\alpha^1 \cdot (-\alpha^2) + (\alpha^1 \cdot \mu)) < \mu||E_{\alpha^2}E_{-\alpha^2}|\mu > \\
= (\alpha^1 \cdot (-\alpha^2) + (\alpha^1 \cdot \mu))(\alpha^2 \cdot \mu)
\end{gathered} \tag{195}$$

$$< B|B >= (\alpha^2 \cdot (-\alpha^1) + \alpha^2 \cdot \mu)(\alpha^1 \cdot \mu) \tag{196}$$

$$\begin{gathered}
< A|B >=< \mu||E_{\alpha^2}E_{\alpha^1}E_{-\alpha^2}E_{-\alpha^1}|\mu > \\
=< \mu||E_{\alpha^2}E_{-\alpha^2}E_{\alpha^1}E_{-\alpha^1}|\mu > (Same\ reason\ as\ above) \\
=< \mu||(\alpha^1 \cdot H)(\alpha^2 \cdot H)|\mu > \\
= (\alpha^2 \cdot \mu)(\alpha^1 \cdot \mu)
\end{gathered} \tag{197}$$

so they are not proportional

(This is Cauchy Schwarz Inequality)

9.B

Cartan subalgebra could still taken to be $\frac{1}{2}\lambda_3\sigma_2, \frac{1}{2}\lambda_8\sigma_2$

If we write the matrix elements like in 6.C, there will be 2-d matrix at the diagonal.

We have to diagonalize σ_2

$$\begin{gathered}
S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
S^\dagger \sigma_2 S = \sigma_3
\end{gathered} \tag{198}$$

We define $T_i = T_i(\sigma_2\ if\ a = 1, 3, 4, 6, 8)$

The generators are basically the same as in $su(3)$ lie algebra. So the

$T_1\sigma_3 = 1/2diag(1, -1, -1, 1, 0, 0), T_8\sigma_2 = \frac{1}{2\sqrt{3}}diag(1, -1, 1, -1, -2, 2)$

The weight vector is like $3 \oplus \bar{3}$

The order of eigenvectors is the same as the eigenvalue above($|1 \oplus 1 >, |1 \oplus 2 >, |2 \oplus 1 > \ldots$ The first number labels λ_i, the second labels σ_2)

$$
\begin{aligned}
E_{1/2,\sqrt{3}/2}|3\oplus S^{\dagger}2> &= \frac{1}{2}(\lambda_4\sigma_3 + i\lambda_5)|3\oplus S^{\dagger}1> \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & \sigma_3+1 \\ 0 & 0 & 0 \\ \sigma_3-1 & 0 & 0 \end{pmatrix}|3\oplus 1> \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
\tag{199}
$$

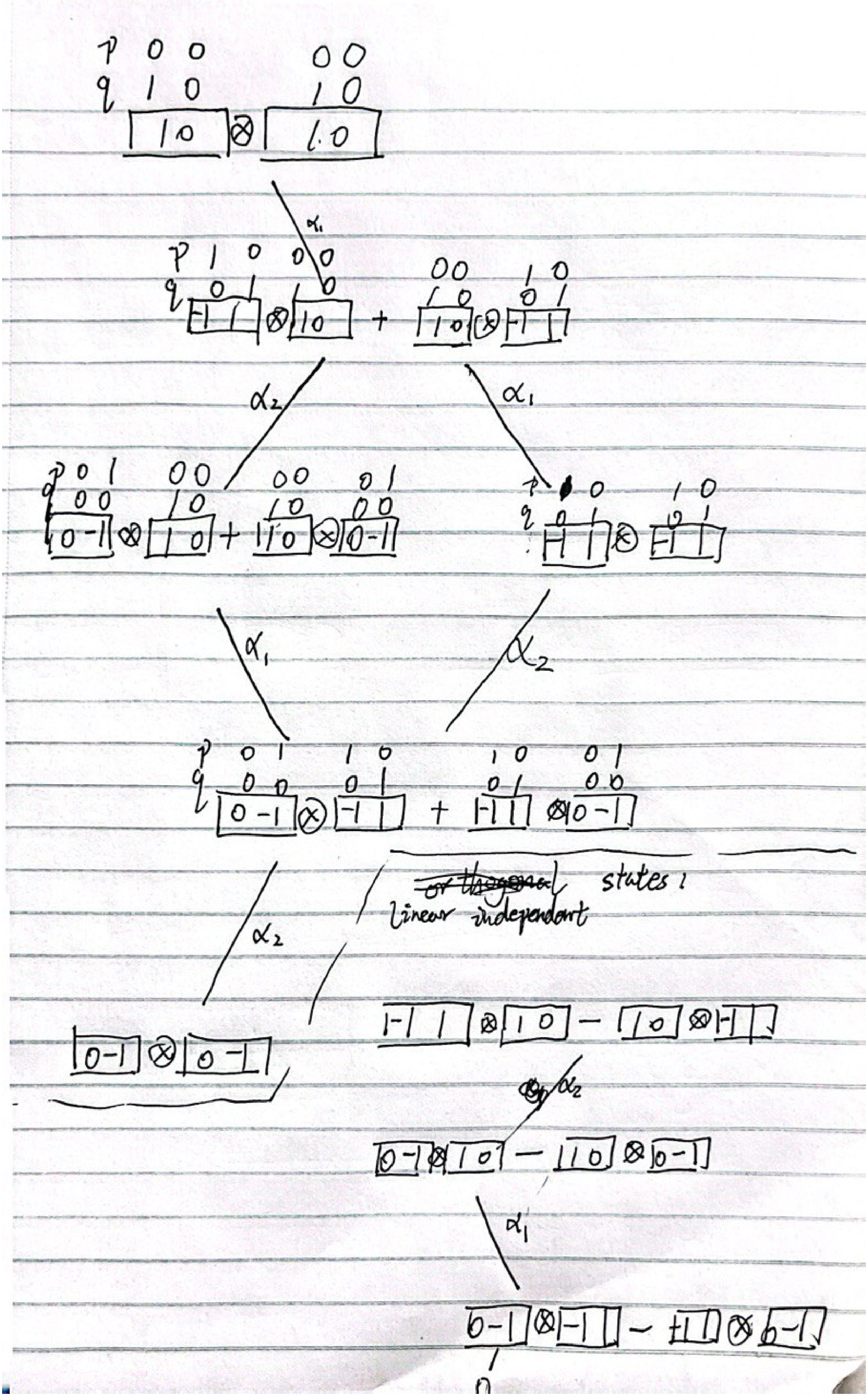
9.C

The highest weight is $(1/2+1/2, \sqrt{3}/6+\sqrt{3}/6)$

And simple roots are $\alpha^1 = (1/2, \sqrt{3}/2), \alpha^2 = (1/2, -\sqrt{3}/2)$

The cartan matrix is

$$
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{200}
$$



So this decomposes to $6 \oplus \bar{3}$

Chapter 10

10.A

This can be solved by using similar method in (10.21)~(10.29)

$$u^i v^{jk} = \frac{1}{3}(u^i v^{jk} + u^j v^{ik} + u^k v^{ij}) + \frac{1}{3}\epsilon^{ijn}\epsilon_{nlm}u^l v^{mk} + \frac{1}{3}\epsilon^{ika}\epsilon_{abc}u^b v^{cj} \tag{201}$$

(11.39) is the answer.

This expression is symmetric for jk, and if $i = j = k$, it can be easily check to be true.

if $i = j \neq k$

$$u^i v^{ik} = \frac{1}{3}(2u^i v^{ik} + u^k v^{ii}) - \frac{1}{3}\epsilon^{ika}\epsilon_{abc}u^b v^{ci} \tag{202}$$

This is true.

$$\epsilon_{nlm}u^l v^{mk} \tag{203}$$

This is 0 because a anti-symmetry tensorϵ contract with a symmetry tensor v.The dimensions of the first part are 10, and it is symmetry, so it maybe a 10. The second part is traceless, so the dimensions are 9-1=8.

10.B

Adjoint representation is the same as (1,1)

$$\begin{aligned} < u||T_a|v > &= <_a | <^b |\bar{u}^b_a v^j_i(-[T]^i_k|^k > |_j > +[T]^l_j|^i > |_l >) \\ &= \bar{u}^b_a v^j_i(-[T]^i_k)\delta^k_b\delta^a_j + \bar{u}^b_a v^j_i[T]^l_j\delta^i_b\delta^a_l \\ &= -\bar{u}^k_j v^j_i[T]^i_k + \bar{u}^i_l v^j_i[T]^l_j \\ &= -\bar{u}^k_j v^j_i[T]^i_k + \bar{u}^j_i v^k_j[T]^i_k \\ &= [T]^i_k(\bar{u}^j_i v^k_j - \bar{u}^k_j v^j_i) \end{aligned} \tag{204}$$

10.C

Highest weight state $(1, \sqrt{3}/3)$ is v^{11}. Lower this we get $2\mu - \alpha^1 = (1/2, -\sqrt{3}/6)$, the tensor comcomponentponet is $v^{12} = v^{21}$

Continuing this process

$2\mu - \alpha^1 - \alpha^2 = (0, \sqrt{3}/3)$,v^{13}

$2\mu - 2\alpha^1 = (0, -2\sqrt{3}/3)$, v^{22}

$2\mu - 2\alpha^1 - \alpha^2 = (-1/2, -\sqrt{3}/6)$, v^{23}

$2\mu - 2\alpha^1 - 2\alpha^2 = (-1, \sqrt{3}/3)$, v^{33}

Chapter 11

11.A

$$\tag{205}$$

Chapter 11

11.A

There should only be one term:

$$\bar{B}_{ij}[T_8]^j_k B^{ki} \tag{206}$$

$$B = \begin{pmatrix} B^{11} & B^{12} & B^{13} \\ B^{12} & B^{22} & B^{23} \\ B^{13} & B^{23} & B^{33} \end{pmatrix} \tag{207}$$

$$
\begin{aligned}
&\bar{B}_{i1}[T_8]^1_1 B^{1i} + \bar{B}_{i2}[T_8]^2_2 B^{2i} + \bar{B}_{i3}[T_8]^3_3 B^{3i} \\
&= \frac{\sqrt{3}}{6}(\bar{B}_{i1}B^{1i} + \bar{B}_{i2}B^{2i} - 2\bar{B}_{i3}B^{3i}) \\
&= \frac{\sqrt{3}}{6}(B^{i1\star}B^{1i} + B^{i2\star}B^{2i} - 2B^{i3\star}B^{3i}) \\
&= \frac{\sqrt{3}}{6}(|B^{11}|^2 + 2|B^{21}|^2 - |B^{31}|^2 + |B^{22}|^2 - |B^{23}|^2 - 2|B^{33}|^2)
\end{aligned} \tag{208}
$$

$$
\begin{aligned}
M_{11} = M_{33} = M_{13} = M_0 - \frac{\sqrt{3}}{3}\lambda \\
M_{12} = M_{23} = M_0 + \frac{\sqrt{3}}{6}\lambda \\
M_{22} = M_0 + \frac{\sqrt{3}}{6}\lambda
\end{aligned} \tag{209}
$$

$$
M_{12} = M_{22} \tag{210}
$$

11.B

I think that due to the spin sum, $P(\pi^0 P \to \Delta^+) = 0$,$P(K^- P \to \Sigma^{\star 0})$ is larger.

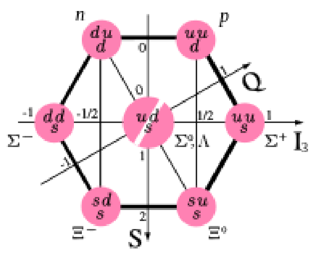
but i don't know if it is correct.

11.C

$$
Q = T_3 + Y/2 = diag(2/3, -1/3, -1/3) \tag{211}
$$

$$
\begin{aligned}
&\alpha Tr(BB^\dagger Q) + \beta Tr(B^\dagger BQ) \\
&= \alpha/3(2[BB^\dagger]^1_1 - [BB^\dagger]^2_2 - [BB^\dagger]^3_3) + \beta/3(2[B^\dagger B]^1_1 - [B^\dagger B]^2_2 - [B^\dagger B]^3_3) \\
&= \alpha/3[2(|\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}|^2 + |\Sigma^+|^2 + |p|^2) - (|\Sigma^-|^2 + |-\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}|^2 + |n|^2) \\
&-(|\Xi^-|^2 + |\Xi^0|^2 + |\frac{-2\Lambda}{\sqrt{6}}|^2)] \\
&+\beta/3(2(|\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}|^2 + |\Sigma^-|^2 + |\Xi^-|^2) - (|\Sigma^+|^2 + |-\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}}|^2 + |\Xi^0|^2) \\
&-(|p|^2 + |n|^2 + |\frac{-2\Lambda}{\sqrt{6}}|^2))
\end{aligned} \tag{212}
$$

$$
\begin{aligned}
Q_P = Q_{\Sigma^+} \\
Q_N = Q_{\Xi^0} \\
Q_{\Sigma^-} = Q_{\Xi^-}
\end{aligned} \tag{213}
$$



The triplet state can be found by rotate the coordinate anti-clockwise 120°, which means using linear combination $Q = -\frac{1}{2}H_2 - \frac{\sqrt{3}}{2}H_1, otherdirection = \frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1$

$\Sigma^0\ is\ -H_1\ \Lambda\ is\ -H_2$

New triplet state are $-\frac{\sqrt{3}}{2}\Lambda + \frac{1}{2}\Sigma^0$

And using the knowledge from adjoint rep of SU(3), the triplet root vector is

$$
\alpha[(\sqrt{3}-2)|N|^2 + 4|P|^2 - 2|\Sigma^-|^2] + \beta((\sqrt{3}-2)|N|^2 + 4|\Sigma^-|^2 - 2|P|^2) \tag{214}
$$

$$
\begin{aligned}
\mu(N) = \mu_0 + \frac{(\alpha+\beta)}{3}(\sqrt{3}-2) \\
\mu(P) = \mu_0 + 4/3\alpha - 2/3\beta \\
\mu(\Sigma^-) = \mu_0 - 2/3\alpha + 4/3\beta
\end{aligned} \tag{215}
$$

Chapter 12

D(p,q)=1/2(p+1)(q+1)(p+1+q+1)

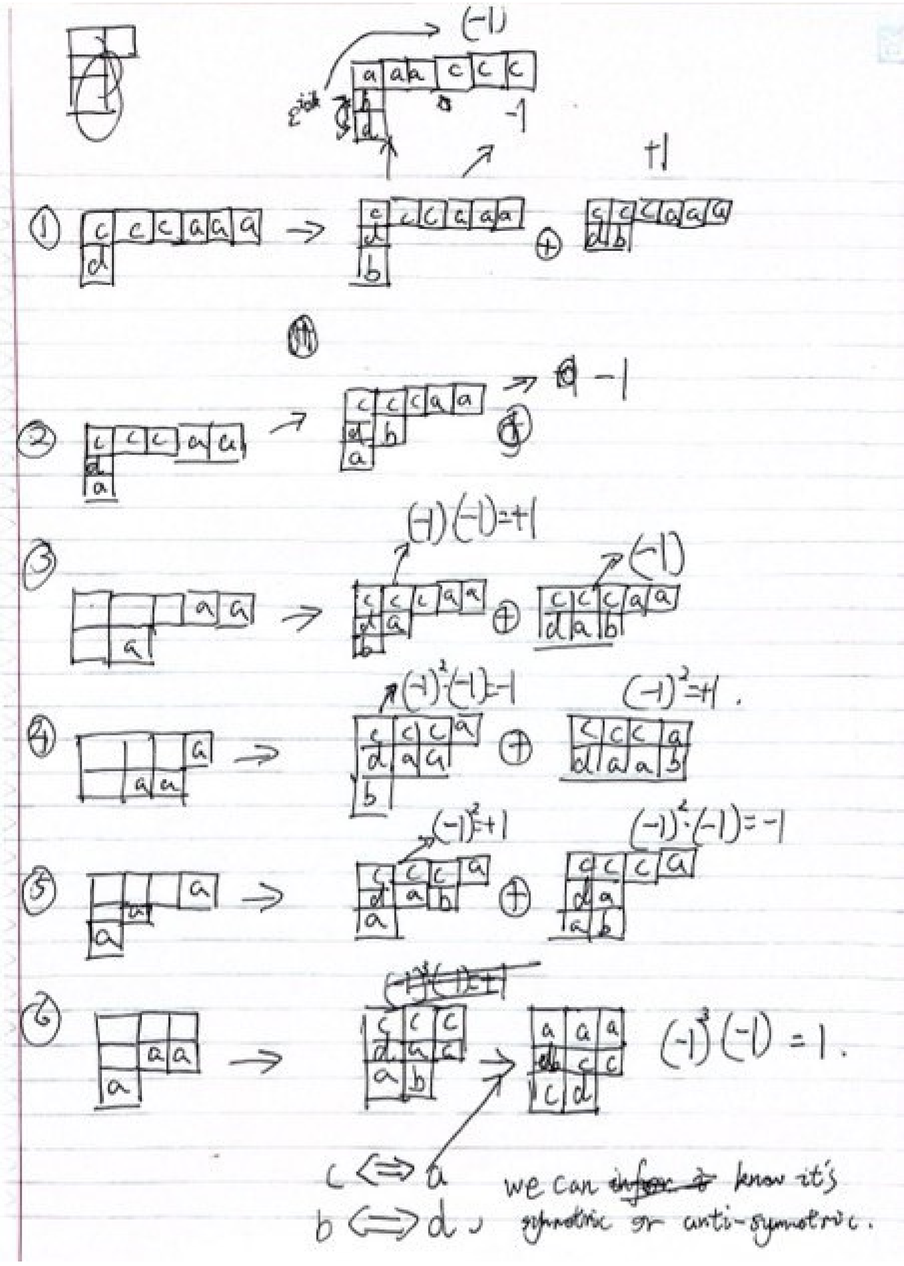
1.one column could only have one a, b or c

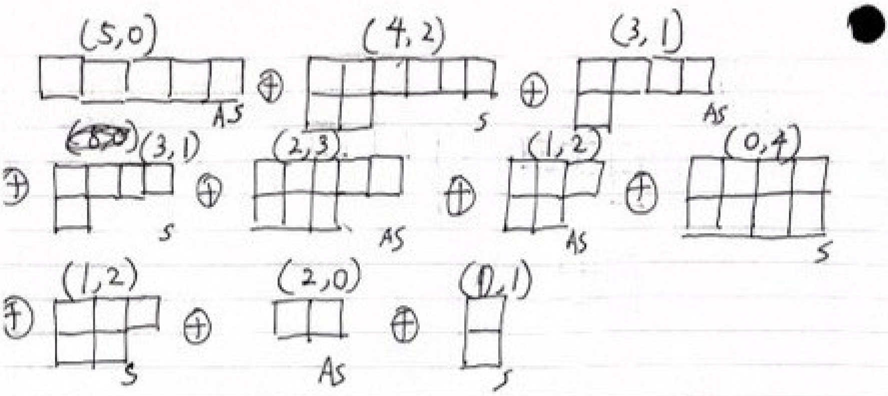
2. count from right to left, up to down, at any time(when you are counting) number of b≤ number of a
3. you should think there is a gravitation force pointing toward left and up

4.For SU(N), the numbers of element in one column equal to N, then this column can be deleted.

5. maximum numbers of elements in one column is 3

12.A

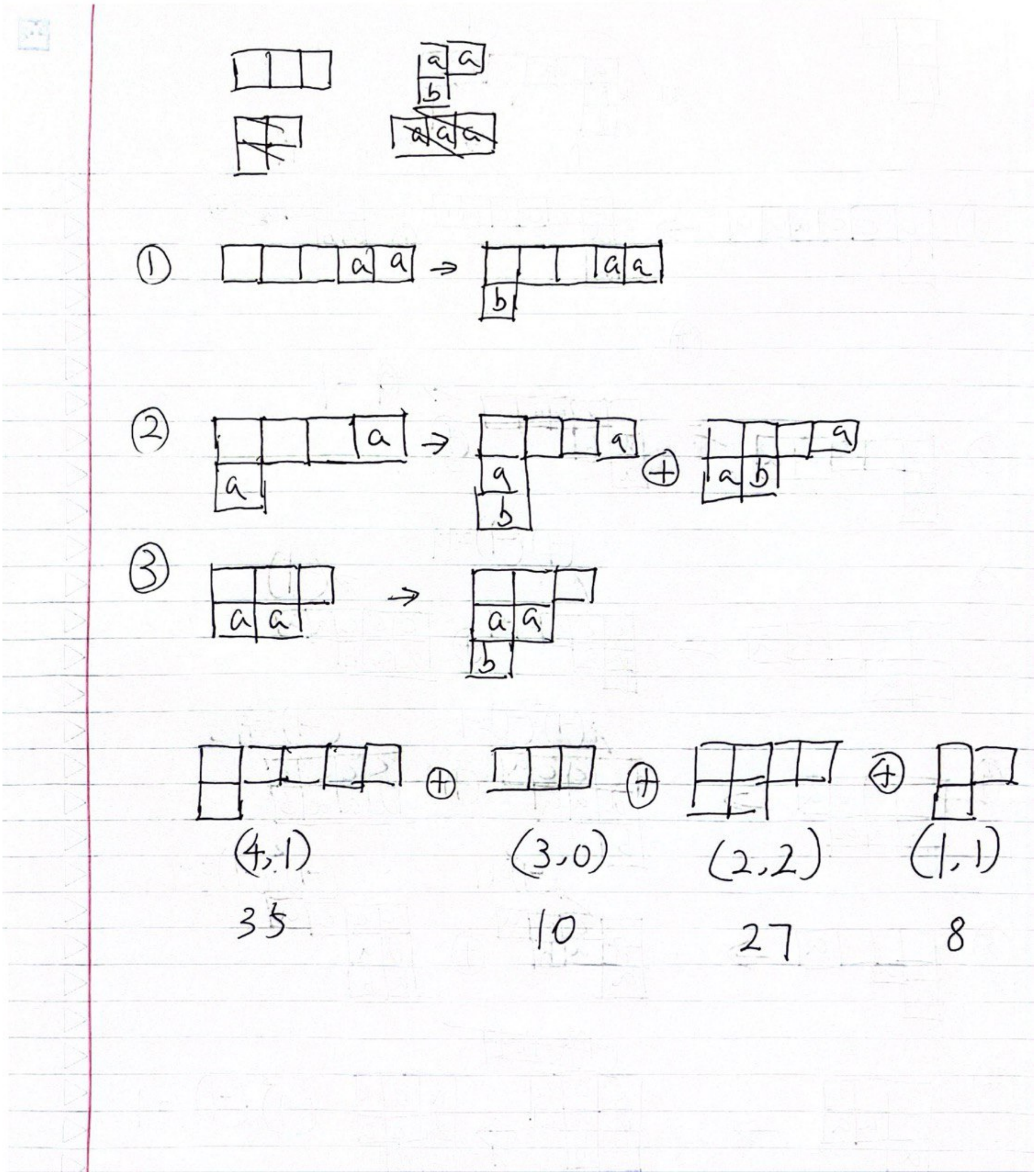




$$15 \otimes 15 = 60 \oplus \overline{42} \oplus 24 \oplus 24 \oplus 21 \oplus \overline{15} \oplus \overline{15} \oplus \overline{15} \oplus 6 + \overline{3}$$

$$15 \equiv (2,1) \qquad 15' \equiv (4,0).$$

12.B



12.C*

For SU(3), we can directly get this by young tableux.

General case : Haven't come un with a good idea.

Chapter 13

13.A

In SU(N), consider first N-2 Cartan generators and $\alpha^i = \nu^i - \nu^{i+1} for\, i = 1\, to\, N-2$

The generators we selected are just like a $su(N-1)$ lie algebra.

The fundamental representation will have $|i_1 \ldots i_m >, m \leq N-1$

13.B

Chapter 14

14.A

$$O_{ij}^k = a_i^\dagger a_j^\dagger a_k - \frac{1}{4}(\delta_{ik} a_l^\dagger a_j^\dagger a_l + \delta_{jk} a_l^\dagger a_i^\dagger a_l) \tag{216}$$

ij is symmetry and traceless:

$$\begin{aligned}\delta_k^j O_{ij}^k = a_i^\dagger a_j^\dagger a_j - \frac{1}{4}(a_l^\dagger a_j^\dagger a_l + 3 a_l^\dagger a_i^\dagger a_l) \\ = 0\end{aligned} \tag{217}$$

and O is the tensor product of 3, so O is in the $(2,1)$ representation.

So it transform like (2,1)

14.B

$$O_{11}^3 = a_1^\dagger a_1^\dagger a_3 \tag{218}$$

$$\begin{aligned}
&< 0|(b_n a_n) a_k a_j a_1^\dagger a_1^\dagger a_3 a_i^\dagger (a_l^\dagger b_l^\dagger)|0 > \\
&=< 0|(b_n a_n) a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger a_3 (a_l^\dagger b_l^\dagger)|0 > + < 0|(b_l a_l) a_k a_j a_1^\dagger a_1^\dagger \delta_{i3} (a_l^\dagger b_l^\dagger)|0 > \\
&=< 0|(b_n a_n) a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger (a_l^\dagger b_l^\dagger) a_3|0 > + < 0|(b_n a_n) a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger (\delta_{l3} b_l^\dagger)|0 > + < 0|(b_l a_l) a_k a_j a_1^\dagger a_1^\dagger \delta_{i3} (a_l^\dagger b_l^\dagger)|0 > \\
&=< 0|(b_n a_n) a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger b_3^\dagger|0 > + < 0|(b_n a_n) a_k a_j a_1^\dagger a_1^\dagger \delta_{i3} (a_l^\dagger b_l^\dagger)|0 > \\
&=< 0|a_n a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger b_n b_3^\dagger|0 > + < 0|a_n a_k a_j a_1^\dagger a_1^\dagger \delta_{i3} a_l^\dagger b_n b_l^\dagger|0 > \\
&=< 0|a_n a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger \delta_{n3}|0 > + < 0|a_n a_k a_j a_1^\dagger a_1^\dagger \delta_{i3} a_l^\dagger \delta_{nl}|0 > \\
&=< 0|a_3 a_k a_j a_1^\dagger a_1^\dagger a_i^\dagger|0 > + \delta_{i3} < 0|a_l a_k a_j a_1^\dagger a_1^\dagger a_l^\dagger|0 > \\
&=< 0|a_k a_j a_1^\dagger a_1^\dagger a_3 a_i^\dagger|0 > + \delta_{i3} < 0|a_k a_j a_l a_1^\dagger a_1^\dagger a_l^\dagger|0 > \\
&= \delta_{3i}(< 0|a_k a_j a_1^\dagger a_1^\dagger|0 > + < 0|a_k a_j a_l a_1^\dagger a_1^\dagger a_l^\dagger|0 >) \\
&= \delta_{3i}(< 0|a_k a_1^\dagger a_j a_1^\dagger|0 > + < 0|a_k \delta_{1j} a_1^\dagger|0 > + < 0|a_k a_j a_1^\dagger a_l a_1^\dagger a_l^\dagger|0 > + < 0|a_k a_j \delta_{1l} a_1^\dagger a_l^\dagger|0 >) \\
&= \delta_{3i}(< 0|a_k a_1^\dagger \delta_{1j}|0 > + \delta_{1k}\delta_{1j} + < 0|a_k a_j a_1^\dagger a_1^\dagger a_l a_l^\dagger|0 > + < 0|a_k a_j a_1^\dagger \delta_{1l} a_l^\dagger|0 > + < 0|a_k a_j \delta_{1l} a_1^\dagger a_l^\dagger|0 >) \\
&= \delta_{3i}(\delta_{1k}\delta_{1j} + \delta_{1k}\delta_{1j} + < 0|a_k a_j a_1^\dagger a_1^\dagger \delta_{ll}|0 > + < 0|a_k a_j a_1^\dagger a_1^\dagger|0 > + < 0|a_k a_j a_1^\dagger a_1^\dagger|0 >) \\
&= \delta_{3i}(\delta_{1k}\delta_{1j} + \delta_{1k}\delta_{1j} + 5 < 0|a_k a_j a_1^\dagger a_1^\dagger|0 >) \\
&= \delta_{3i}(\delta_{1k}\delta_{1j} + \delta_{1k}\delta_{1j} + 5 < 0|a_k a_1^\dagger a_j a_1^\dagger|0 > + 5 < 0|a_k \delta_{1j} a_1^\dagger|0 >) \\
&= \delta_{3i}(\delta_{1k}\delta_{1j} + \delta_{1k}\delta_{1j} + 5 < 0|a_k a_1^\dagger \delta_{1j}|0 > + 5\delta_{1j}\delta_{1k}) \\
&= \delta_{3i}(\delta_{1k}\delta_{1j} + \delta_{1k}\delta_{1j} + 5\delta_{1k}\delta_{1j} + 5\delta_{1j}\delta_{1k}) \\
&= 12\delta_{3i}\delta_{1k}\delta_{1j}
\end{aligned} \tag{219}$$

14.C

From 7.C, we know λ_2, λ_5 and λ_7 generate an SU(2) subalgebra of SU(3). And angular momentum operators also form an SU(2) algebra.

So if we can make sure one generator is in the form of Angular momentum operator, then the other is automatically satisfied.

$$\begin{aligned}
L_3 = 2Q_2 = 2a_k^\dagger [T_2]_{kl} a_l = -i a_1^\dagger a_2 + i a_2^\dagger a_1 \\
= i\frac{m\omega}{2}[-(x_1 - i\frac{p_1}{m\omega})(x_2 + i\frac{p_2}{m\omega}) + (x_2 - i\frac{p_2}{m\omega})(x_1 + i\frac{p_1}{m\omega})] \\
= i(ip_1 x_2 - ip_2 x_1) \\
= (x_1 p_2 - x_2 p_1)
\end{aligned} \tag{220}$$

14.D

$$\begin{aligned}
[Q_\alpha, a_k b_k] = [a_k^\dagger [T_\alpha]_{kl} a_l - b^\dagger [T_\alpha^\star]_{kl} b_l, a_k b_k] \\
= [a_k^\dagger [T_\alpha]_{kl} a_l, a_k] b_k - a_k [b^\dagger [T_\alpha^\star]_{kl} b_l, b_k] \\
= -a_l [T_\alpha^\star]_{lk} b_k - a_k(-b_l [T_\alpha]_{lk}) \\
= -a_l [T_\alpha^\star]_{lk} b_k - a_k(-[T_\alpha^\star]_{kl} b_l) \\
= 0
\end{aligned} \tag{221}$$

Chapter 15

this is confusing. $SU(M)SU(N)SU(MN)$

Chapter 16

$$C_2(p,q) = \Sigma_i T_i^2 = \frac{1}{3}(p^2 + pq + q^2 + 3p + 3q) \tag{222}$$

https://physics.stackexchange.com/questions/526112/what-is-the-eigenvalue-of-t2-su3-casimir or

N.~Arisaka, *On the unitary representations of SU(3)* Prog. Theor. Phys. 47 1758-1781 (1972).

16.A

$$\begin{aligned} T_a^3 T_a^{\bar{3}} &= \frac{1}{2}(T_a^2 - \frac{4}{3} - \frac{4}{3}) \\ &= -4/3 \\ &or 1/6 \end{aligned} \tag{223}$$

$$\begin{aligned} T^3 T^3 &= \frac{1}{2}(T_a^2 - \frac{4}{3} - \frac{4}{3}) \\ &= -2/3 \\ &or 1/3 \end{aligned} \tag{224}$$

$$T_a^3 T_a^{\bar{3}} - T^3 T^3 = \frac{1}{2}(T_{33a}^2 - T_{3\bar{3}a}^2) \tag{225}$$

16.B

Quix is like qq in color SU(3), so the combination of symmetry of flavor and spin must be anti-symmetric.

The state can either be anti-symmetric S_0 in spin and symmetric in flavor6 or anti-symmetric in flavor $\bar{3}$ and symmetric S_1 in spin16.A

$$\begin{aligned} T_a^3 T_a^{\bar{3}} &= \frac{1}{2}(T_a^2 - \frac{4}{3} - \frac{4}{3}) \\ &= -4/3 \\ &or 1/6 \end{aligned} \tag{226}$$

$$\begin{aligned} T^3 T^3 &= \frac{1}{2}(T_a^2 - \frac{4}{3} - \frac{4}{3}) \\ &= -2/3 \\ &or 1/3 \end{aligned} \tag{227}$$

$$T_a^3 T_a^{\bar{3}} - T^3 T^3 = \frac{1}{2}(T_{33a}^2 - T_{3\bar{3}a}^2) \tag{228}$$

16.B

Quix is like qq in color SU(3), so the combination of symmetry of flavor and spin must be anti-symmetric.

The state can either be anti-symmetric S_0 in spin and symmetric in flavor6 or anti-symmetric in flavor $\bar{3}$ and symmetric S_1 in spin

Chapter 17

Chapter 18

Chapter 19

19.A

Commutation relation between Pauli matrices are obvious.

$$[\sigma_a, \sigma_b \tau_c \eta_d] = 2i\epsilon_{abe}\sigma_e \tau_c \eta_d \tag{229}$$

other operators are the same.

$$
\begin{aligned}
&[\sigma_a\tau_b\eta_c,\sigma_d\tau_e\eta_f] \\
&=\sigma_a[\tau_b\eta_c,\sigma_d\tau_e\eta_f]+[\sigma_a,\sigma_d\tau_e\eta_f]\tau_b\eta_c \\
&=\sigma_a(\tau_b[\eta_c,\sigma_d\tau_e\eta_f]+[\tau_b,\sigma_d\tau_e\eta_f]\eta_c)+i2\epsilon_{ad\rho}\sigma_\rho\tau_e\eta_f\tau_b\eta_c \\
&=\sigma_a(\tau_b\sigma_d\tau_e 2i\epsilon_{cf\xi}\eta_\xi+\sigma_d 2i\epsilon_{be\lambda}\tau_\lambda\eta_f\eta_c)+2i\epsilon_{ad\rho}\sigma_\rho\tau_e\eta_f\tau_b\eta_c \\
&=\sigma_a\sigma_d\tau_b\tau_e 2i\epsilon_{cf\xi}\eta_\xi+\sigma_a\sigma_d\eta_f\eta_c 2i\epsilon_{be\lambda}\tau_\lambda+\tau_e\tau_b\eta_f\eta_c 2i\epsilon_{ad\rho}\sigma_\rho \\
&=(\delta_{ad}I+i\epsilon_{ad\gamma}\sigma_\gamma)(\delta_{be}I+i\epsilon_{be\theta}\tau_\theta)2i\epsilon_{cf\xi}\eta_\xi \\
&+(\delta_{ad}I+i\epsilon_{ad\gamma}\sigma_\gamma)(\delta_{fc}I+i\epsilon_{fc\mu}\eta_\mu)2i\epsilon_{be\lambda}\tau_\lambda \\
&+(\delta_{eb}I+i\epsilon_{eb\nu}\tau_\nu)(\delta_{fc}I+i\epsilon_{fc\beta}\eta_\beta)2i\epsilon_{ad\rho}\sigma_\rho \\
&=(\delta_{ad}I+i\epsilon_{ad\gamma}\sigma_\gamma)(\delta_{be}I+i\epsilon_{be\theta}\tau_\theta)2i\epsilon_{cf\xi}\eta_\xi \\
&+(\delta_{ad}I+i\epsilon_{ad\gamma}\sigma_\gamma)(\delta_{fc}I-i\epsilon_{cf\mu}\eta_\mu)2i\epsilon_{be\lambda}\tau_\lambda \\
&+(\delta_{eb}I-i\epsilon_{be\nu}\tau_\nu)(\delta_{fc}I-i\epsilon_{cf\beta}\eta_\beta)2i\epsilon_{ad\rho}\sigma_\rho \\
&=1.\ \ \delta_{ad}I\delta_{be}I2i\epsilon_{cf\lambda}\eta_\lambda+\delta_{ad}I\delta_{fc}I2i\epsilon_{be\lambda}\tau_\lambda+i\epsilon_{ad\gamma}\sigma_\gamma\delta_{be}I2i\epsilon_{cf\xi}\eta_\xi \\
&2.\ \ +0\,(all\ the\ terms\ have\ 2\ Pauli\ matrices\ are\ canceled.) \\
&3.\ \ +i\epsilon_{be\nu}\tau_\nu i\epsilon_{cf\beta}\eta_\beta 2i\epsilon_{ad\rho}\sigma_\rho \\
&=\delta_{ad}I\delta_{be}I2i\epsilon_{cf\lambda}\eta_\lambda+\delta_{ad}I\delta_{fc}I2i\epsilon_{be\lambda}\tau_\lambda+\delta_{eb}I\delta_{fc}I2i\epsilon_{ad\rho}\sigma_\rho \\
&+i\epsilon_{be\nu}\tau_\nu i\epsilon_{cf\beta}\eta_\beta 2i\epsilon_{ad\rho}\sigma_\rho
\end{aligned} \tag{230}
$$

Where we used the relation $\sigma_i\sigma_j=\delta_{jk}I+i\epsilon_{jkl}\sigma_l$

So all the generators are closed, these generators form a Lie algebra.

We choose $\sigma_3,\tau_3,\eta_3,\sigma_3\tau_3\eta_3$ to be the Cartan generators

We can get the eigenvalues and the eigenvectors from the tensor product of the basis.

$$
\begin{aligned}
111 eigenvalues &\rightarrow (1,1,1,1)=\nu_1 \\
211 &\rightarrow (-1,1,1,-1)=-\nu_4 \\
121 &\rightarrow (1,-1,1,-1)=\nu_3 \\
221 &\rightarrow (-1,-1,1,1)=-\nu_2 \\
112 &\rightarrow (1,1,-1,-1)=\nu_2 \\
212 &\rightarrow (-1,1,-1,1)=-\nu_3 \\
122 &\rightarrow (1,-1,-1,1)=\nu_4 \\
222 &\rightarrow (-1,-1,-1,-1)=-\nu_1
\end{aligned} \tag{231}
$$

roots take one weight to another, so they are

$$
\begin{gathered}
0,\pm2\nu_1,\pm2\nu_2,\pm2\nu_3,\pm2\nu_4 \\
\pm(\nu_1+\nu_2),\pm(\nu_1-\nu_2),\pm(\nu_1+\nu_3),\pm(\nu_1-\nu_3),\pm(\nu_1+\nu_4),\pm(\nu_1-\nu_4) \\
\pm(v_2+\nu_3),\pm(v_2-\nu_3),\pm(\nu_2+\nu_4),\pm(\nu_2-\nu_4) \\
,\pm(\nu_3+\nu_4),\pm(\nu_3-\nu_4)
\end{gathered} \tag{232}
$$

$the\ result\ is\ correct\ but \backslash procedure\ is\ incorrect \backslash .$ because some differences of the weight may not exist but for this problem, the result is correct. you can check another solution written by Stephen Hancock, the second part of his calculation (check) is correct.

The roots are

$$
\begin{gathered}
(0,0,0,0)(four\ generators), \\
\pm(2,2,2,2),\pm(2,2,-2,-2),\pm(2,-2,2,-2),\pm(-2,2,2,-2) \\
\pm(2,2,0,0),\pm(0,0,2,2),\pm(2,0,2,0),\pm(0,2,0,2), \\
\pm(2,0,0,2),\pm(0,2,2,0),\pm(2,0,0,-2),\pm(0,2,-2,0) \\
\pm(2,0,-2,0),\pm(0,2,0,-2),\pm(2,-2,0,0),\pm(0,0,2,-2)
\end{gathered} \tag{233}
$$

The positive roots are

$$
\begin{gathered}
(2,2,2,2),(2,2,-2,-2),(2,-2,2,-2),-(-2,2,2,-2) \\
(2,2,0,0),(0,0,2,2),(2,0,2,0),(0,2,0,2), \\
(2,0,0,2),(0,2,2,0),(2,0,0,-2),(0,2,-2,0) \\
(2,0,-2,0),(0,2,0,-2),(2,-2,0,0),(0,0,2,-2)
\end{gathered} \tag{234}
$$

It is obvious that the first and the third roots can be written as sum of positive roots.

and $(2,2,-2,-2)=(2,0,0,-2)+(0,2,-2,0),(2,0,2,0)=(2,-2,0,0)+(0,2,2,0)$

$(2,0,0,-2) = (2,-2,0,0) + (0,2,0,-2)$

$(2,2,0,0) = (2,0,-2,0) + (0,2,2,0)$

$(2,-2,0,0) = -(-2,2,2,-2) + (0,0,2,-2)$

$(0,2,0,-2) = (0,2,-2,0) + (0,0,2,-2)$

$(0,2,0,2) = (0,2,-2,0) + (0,0,2,2)$

$(2,0,-2,0) = -(-2,2,2,-2) + (0,2,0,-2)$

$(2,0,0,2) = -(-2,2,2,-2) + (0,2,2,0)$

$(0,2,2,0) = (0,2,0,-2) + (0,0,2,2)$

$$\alpha^1 = (0,0,2,2), \alpha^2 = (0,2,-2,0), \alpha^3 = (0,0,2,-2), \alpha^4 = (2,-2,-2,2) \tag{235}$$

$\theta_{\alpha^1\alpha^2} = 120^\circ = \theta_{\alpha^2\alpha^3}, \theta_{\alpha^3\alpha^4} = 135^\circ$

$|\alpha^1| = |\alpha^2| = |\alpha^3| = 2\sqrt{2}, |\alpha^4| = 4$ the simple root α^4 is longer, so it is $Sp(8)$

$$\bigcirc - \bigcirc - \bigcirc = \bullet \tag{236}$$

19.B

Commutator of $\sigma_a, \tau_a, \eta_3, \sigma_a\eta_1, \sigma_a\eta_2, \tau_a\eta_1, \tau_a\eta_2$ between themselves are in this algebra obviously.

$$[\sigma_a, \sigma_b\tau_c\eta_3] = 2i\epsilon_{abe}\sigma_e\tau_c\eta_3(other\ \ Pauli\ matrix\ are\ similar\ to\ this)$$

$$\begin{aligned} [\sigma_a\eta_1, \sigma_b\tau_c\eta_3] &= \sigma_a[\eta_1, \sigma_b\tau_c\eta_3] + [\sigma_a, \sigma_b\tau_c\eta_3]\eta_1 \\ &= \sigma_a\sigma_b\tau_c(-2i\eta_2) + 2i\epsilon_{ab\nu}\sigma_\nu\tau_c\eta_3\eta_1 \\ &= (\delta_{ab}I + i\epsilon_{ab\nu}\sigma_\nu)\tau_c(-2i\eta_2) + 2i\epsilon_{ab\nu}\sigma_\nu\tau_c i\eta_2 \\ &= \delta_{ab}I(-2i\eta_2)\tau_c \end{aligned}$$
$$[\sigma_a\eta_2, \sigma_b\tau_c\eta_3] = \delta_{ab}I(2i\eta_1)\tau_c \tag{237}$$
$$\begin{aligned} [\tau_a\eta_1, \sigma_b\tau_c\eta_3] &= \delta_{ac}I(-2i\eta_2)\sigma_b \\ [\tau_a\eta_2, \sigma_b\tau_c\eta_3] &= \delta_{ac}I(2i\eta_1)\sigma_b \end{aligned}$$
$$\begin{aligned} &[\sigma_a\tau_b\eta_3, \sigma_d\tau_e\eta_3] \\ &= \delta_{ad}I2i\epsilon_{be\lambda}\tau_\lambda + \delta_{eb}I2i\epsilon_{ad\rho}\sigma_\rho \end{aligned}$$

So the algebra is closed.

All terms commute with $\sigma_3\tau_3\eta_3, \sigma_3, \tau_3, \eta_3$ are:

We choose $\sigma_3, \tau_3, \eta_3, \sigma_3\tau_3\eta_3$ to be the Cartan generators.

The eigenvalues seem to be the same as 19.A. So we will check the roots from the generators directly.

$$\begin{aligned} \sigma_\pm &= \sigma_1 \pm i\sigma_2 \\ [\sigma_3, \sigma_\pm] &= \pm 2\sigma_\pm, [\sigma_3\tau_3\eta_3, \sigma_\pm] = \pm 2\sigma_\pm\tau_3\eta_3 \\ [\sigma_3, \sigma_\pm\tau_3\eta_3] &= \pm 2\sigma_\pm\tau_3\eta_3 \\ [\sigma_3\tau_3\eta_3, \sigma_\pm\tau_3\eta_3] &= \pm 2\sigma_\pm \end{aligned} \tag{238}$$

So the first generators can be built by

$\sigma_\pm \pm' \sigma_\pm\tau_3\eta_3$, which roots are$[H_i, \sigma_\pm \pm' \sigma_\pm\tau_3\eta_3] = \alpha_i(\sigma_\pm \pm' \sigma_\pm\tau_3\eta_3), \alpha = (\pm 2, 0, 0, \pm\pm' 2)$

Similarly, $\tau_\pm \pm' \sigma_3\tau_\pm\eta_3$,which roots are $\alpha = (0, \pm 2, 0, \pm\pm' 2)$

We consider other generators

$$\begin{aligned} [\sigma_3, \sigma_\pm\eta_{\pm'}] &= \pm 2\sigma_\pm\eta_\pm \\ [\eta_3, \sigma_\pm\eta_{\pm'}] &= \pm' 2\sigma_\pm\eta_\pm \\ [\sigma_3\tau_3\eta_3, \sigma_\pm\eta_\pm] &= 0 \end{aligned} \tag{239}$$

So the root vectors are $\alpha = (\pm 2, 0, \pm' 2, 0)$

Similarly, the root vectors of $\tau_{\pm}\eta_{\pm'}$ are $\alpha = (0, \pm 2, \pm' 2, 0)$

Now the generators left are $\sigma_3\eta_1, \sigma_3\eta_2, \tau_3\eta_1, \tau_3\eta_2, \sigma_1\tau_1\eta_3, \sigma_1\tau_2\eta_3, \sigma_2\tau_1\eta_3, \sigma_2\tau_2\eta_3$

to make them to be eigen vector of σ_3, τ_3, η_3, we have to use another linear combination of them:

$$\begin{gathered} \sigma_3\eta_{\pm}, \tau_3\eta_{\pm} \\ \sigma_{\pm}\tau_{\pm'}\eta_3 \end{gathered} \tag{240}$$

The last one is already an eigenvector, because$[\sigma_3\tau_3\eta_3, \sigma_{\pm}\tau_{\pm'}\eta_3] = 0$

The roots are $\alpha = (\pm 2, \pm' 2, 0, 0)$

$$\begin{gathered} [\sigma_3\tau_3\eta_3, \sigma_3\eta_{\pm}] = \pm 2\tau_3\eta_{\pm} \\ [\sigma_3\tau_3\eta_3, \tau_3\eta_{\pm}] = \pm 2\sigma_3\eta_{\pm} \end{gathered} \tag{301}$$

$$[\sigma_3\tau_3\eta_3, \sigma_3\eta_{\pm} \pm' \tau_3\eta_{\pm}] = \pm \pm' 2(\sigma_3\eta_{\pm} \pm' \tau_3\eta_{\pm}) \tag{302}$$

The roots of $\sigma_3\eta_{\pm} \pm' \tau_3\eta_{\pm}$ are $\alpha = (0, 0, \pm 2, \pm \pm' 2)$

Here is a table of all root

$$\begin{array}{cc} \sigma_{\pm} \pm' \sigma_{\pm}\tau_3\eta_3, & \alpha = (\pm 2, 0, 0, \pm \pm' 2) \\ \tau_{\pm} \pm' \sigma_3\tau_{\pm}\eta_3, & \alpha = (0, \pm 2, 0, \pm \pm' 2) \\ \sigma_3\eta_{\pm} \pm' \tau_3\eta_{\pm}, & \alpha = (0, 0, \pm 2, \pm \pm' 2) \\ \sigma_{\pm}\tau_{\pm'}\eta_3, & \alpha = (\pm 2, \pm' 2, 0, 0) \\ \sigma_{\pm}\eta_{\pm}, & \alpha = (\pm 2, 0, \pm' 2, 0) \\ \tau_{\pm}\eta_{\pm'}, & \alpha = (0, \pm 2, \pm' 2, 0) \end{array} \tag{303}$$

Positive roots are

$$(2,0,0,\pm 2), (0,2,0,\pm 2), (0,0,2,\pm 2), (2,\pm 2,0,0), (2,0,\pm 2,0), (0,2,\pm 2,0) \tag{304}$$

$$\begin{gathered} (2,0,0,\pm 2) = (2,-2,0,0) + (0,2,0,\pm 2), \\ (0,2,0,\pm 2) = (0,2,-2,0) + (0,0,2,\pm 2), \\ (2,0,\pm 2,0) = (2,-2,0,0) + (0,2,\pm 2,0) \end{gathered} \tag{305}$$

The roots left are

$$(0,0,2,\pm 2), (2,\pm 2,0,0), (0,2,\pm 2,0) \tag{306}$$

And

$$\begin{gathered} (0,2,2,0) = (0,2,0,2) + (0,0,2,-2) \\ (2,2,0,0) = (2,0,2,0) + (0,2,-2,0) \end{gathered} \tag{307}$$

These 4 roots cannot be written by linear combination of others.

$$\alpha^1 = (2,-2,0,0), \alpha^2 = (0,2,-2,0), \alpha^3 = (0,0,2,2), \alpha^4 = (0,0,2,-2) \tag{308}$$

$\theta_{\alpha^1\alpha^2} = 120^{\circ}$,$\theta_{\alpha^2\alpha^3} = 120^{\circ}$,$\theta_{\alpha^2\alpha^4} = 120^{\circ}$, $\theta_{\alpha^3\alpha^4} = 90^{\circ}$

The Dynkin diagram is therefore

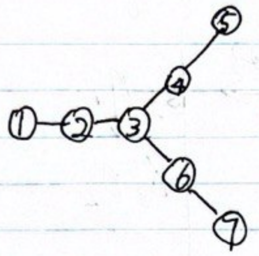
$$\begin{array}{ccccc} & & & & \bigcirc \\ \bigcirc & - & \bigcirc & < & \\ & & & & \bigcirc \end{array} \tag{309}$$

Chapter 20

20.A

If the root system is composable, then there will be two root systems that all root between them re orthogonal. Then the Dynkin diagram will be two orthogonal diagram with no edge between them. So the $\Pi - system$ is composable. This means for a decomposable $\Pi - system$,the root system is decomposable.

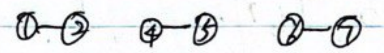
20.B



E_6'

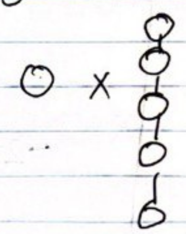
If we remove 1, 5, 7 then it will go back to E_6.

If we remove 3, it will become



And the subalgebra is $A_2 \times A_2 \times A_2$, A_2 cannot be ~~further~~ breakdown further.

If we remove 2 ~~or~~, 4, or 6, the algebra will become

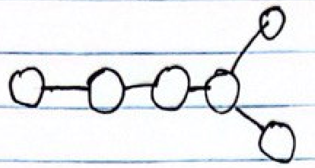


$A_1 \times A_5$. both of them cannot be further breakdown

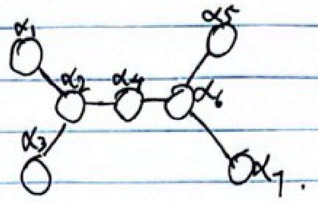
so the ~~ma~~ regular maximal subalgebra are

$A_2 \times A_2 \times A_2$, $A_1 \times A_5$,

D_6 or $SO(12)$:

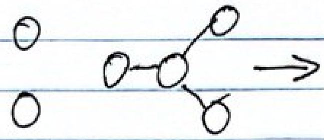


extended diagram:

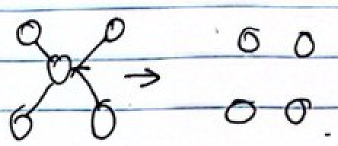


Removing α_1, α_3 α_5 α_7 will return to D_6.

(1) Removing α_2 or α_6:



$\rightarrow$ still has a ~~subalge~~ regular ~~maximum~~ maximal subalgebra.

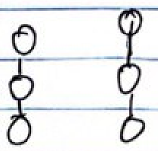


So the result is $\cancel{SU}\,su(2) \times \cancel{SU}\,su(2) \times \cancel{SO}\,so(8)$

If we further break $SO(8)$,

$su(2)^6$ $\cancel{SU(2)\times SU(2)\times SU(2)\times SU(2)\times SU(2)}$

(2). Removing α_4:



$\rightarrow A_3 \times A_3$ or ~~St~~ $su(4) \times su(4)$

Chapter 21

21.A

We choose $H_1 = \sigma_3, H_2 = \sigma_3\tau_3$ as Cartan algebra.

Root:

$$\begin{gathered}\sigma_3\tau_1 \pm i\tau_2 \\ [\sigma_3, (\sigma_3\tau_1 \pm i\tau_2)] = 0 \\ [\sigma_3\tau_3, (\sigma_3\tau_1 \pm i\tau_2)] = \pm 2(\sigma_3\tau_1 \pm i\tau_2)\end{gathered} \tag{310}$$

$$\begin{aligned}[\sigma_a\tau_1, \sigma_b\tau_3] &= \sigma_a[\tau_1, \sigma_b\tau_3] + [\sigma_a, \sigma_b\tau_3]\tau_1 \\ &= \sigma_a\sigma_b(-2i\tau_2) + 2i\epsilon_{ab\rho}\sigma_\rho(i\tau_2) \\ &= (\delta_{ab} + i\epsilon_{ab\rho}\sigma_\rho)(-2i\tau_2) + 2i\epsilon_{ab\rho}\sigma_\rho(i\tau_2) \\ &= -2\delta_{ab}i\tau_2\end{aligned} \tag{311}$$

$$[\sigma_3\tau_3, \sigma_\pm\tau_1] = 0 \tag{312}$$

$$\begin{gathered}\sigma_\pm = \sigma_1 \pm i\sigma_2 \\ \sigma_\pm \pm' \sigma_\pm\tau_3 \\ [\sigma_3, (\sigma_\pm \pm' \sigma_\pm\tau_3)] = \pm 2(\sigma_\pm \pm' \sigma_\pm\tau_3) \\ [\sigma_3\tau_3, (\sigma_\pm \pm' \sigma_\pm\tau_3)] = \pm' \pm 2(\sigma_\pm \pm' \sigma_\pm\tau_3)\end{gathered} \tag{313}$$

$E_{\pm e^1 \pm \pm' e^2} = (\sigma_\pm \pm' \sigma_\pm\tau_3), \pm e^1 \pm \pm' e^2 = (\pm 2, \pm \pm' 2)$

$E_{\pm e^1} = \sigma_\pm\tau_1, \pm\frac{1}{2}e^1 = (\pm 1, 0)$

$E_{\pm e^2} = (\sigma_3\tau_1 \pm i\tau_2), \pm\frac{1}{2}e^2 = (0, \pm 1)$

$$\begin{gathered}(2,2), (2,-2), (2,0), (0,2) \\ simple\ root: \\ (2,-2), (0,2)\end{gathered} \tag{314}$$

$$\begin{gathered}|11>, eg(1,1) \\ |21>, eg(-1,-1) \\ |12>, eg(1,-1) \\ |22>, eg(-1,1)\end{gathered} \tag{315}$$

$$\mu^2 = \frac{1}{2}(e^1 + e^2) = (1,1) \tag{316}$$

According to the book , SO(5) is pseudo-real.

$$T_a = -RT_a^\star R^{-1} \tag{317}$$

If we want R anti-commute with σ_1, σ_3, commute with σ_2, R has to be proportional to σ_2. R also has to commute with τ_2, so R is either proportional to τ_2 or R doesn't have τ matrices.

It seems that σ_2 has the correct commutation relation for every matrices. Notice that we have to be in Hermitian matrices basis. And σ_2 is antisymmetric, so so(5) is indeed pseudo-real.

21. B.

For so(2n+1) lie algebra, if we only consider M_{ab} for $a, b = 3$ to $2n+1$,

$$[M_{ab}, M_{cd}] = -i(\delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac} + \delta_{ad}M_{bc}) \tag{318}$$

The label of each commutator is still from $3\ to\ 2n+1$

$H_j = M_{2j-1,2j}$ for $j = 2$ to n

Roots will be

$$\begin{gathered}E_{\eta e^j} = \frac{1}{\sqrt{2}}(M_{2j-1,2n+1} + i\eta M_{2j,2n+1}) \\ [H_j, E_{\eta e^k}] = \eta[e^k]_j E_{\eta e^k} = \eta\delta_{jk}E_{\eta e^k}, k, j\ is\ from\ 2\ to\ n\end{gathered} \tag{319}$$

Simple roots are

$$\begin{gathered}\alpha^j = e^j - e^{j+1} for\ j = 2\ to\ n-1 \\ \alpha^n = e^n\end{gathered} \tag{320}$$

The diagram is the same diagram with out root α^1. From the Dynkin diagram, this subalgebra is indeed $so(2n-1)$

The roots could only affect $|e^2/2>$ to $|e^n/2>$

Chapter 22

22.A

Using the technique from 19.A, we can easily see there is a regular maximal subalgebra $so(2m) \times so(2n-2m)$.

We define

$$\begin{gathered}so(2m): \\ H_j = M_{2j-1,2j} \\ E_{\eta e^j+\eta' e^k} = \frac{1}{2}[M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}]\end{gathered} \tag{321}$$

Notice $\begin{gathered}E_{\eta e^j} = \frac{1}{\sqrt{2}}(M_{2j-1,2m} + i\eta M_{2j,2m}) \\ j \in N^+, j \leq m\end{gathered}$ doesn't work, that's because j=n, the comutation relation is wrong.

We define the simple roots:

$$\begin{gathered}E_{e^j-e^{j+1}} = \frac{1}{2}[M_{2j-1,2j+1} + iM_{2j,2j+1} - iM_{2j-1,2j+2} + M_{2j,2j+2}], \\ j = 1, \ldots m-1 \\ (E_{e^{m-1}-e^m} = \frac{1}{2}[M_{2m-3,2m-1} + iM_{2m-2,2m-1} - iM_{2m-3,2m} + M_{2m-2,2m}] \\ when\ j = m-1)\end{gathered} \tag{322}$$

$$\begin{gathered}E_{e^{m-1}+e^m} = \frac{1}{2}[M_{2m-3,2m-1} + iM_{2m-2,2m-1} + iM_{2m-3,2m} - M_{2m-2,2m}] \\ [H_i, E_{e^{m-1}-e^m}] = 1/2([M_{2i-1,2i}, M_{2m-3,2m-1}] + i[M_{2i-1,2i}, M_{2m-2,2m-1}] \\ -i[M_{2i-1,2i}, M_{2m-3,2m}] + [M_{2i-1,2i}, M_{2m-2,2m}]) \\ \\ = -i/2[(-\delta_{2j-1,2m-3}M_{2i,2m-1} + \delta_{im}M_{2i,2m-3}) \\ i(\delta_{2i,2m-2}M_{2i-1,2m-1} + \delta_{2i-1,2m-1}M_{2i,2m-2}) \\ -i(-\delta_{2i-1,2m-3}M_{2i,2m} - \delta_{2i,2m}M_{2i-1,2m-3}) \\ +(\delta_{2i,2m-2}M_{2i-1,2m} - \delta_{2i,2m}M_{2i-1,2m-2})] \\ \\ = -i/2[(-\delta_{i,m-1}M_{2i,2m-1} + \delta_{im}M_{2i,2m-3}) \\ +i(\delta_{i,m-1}M_{2i-1,2m-1} + \delta_{i,m}M_{2i,2m-2}) \\ +i(\delta_{i,m-1}M_{2i,2m} + \delta_{i,m}M_{2i-1,2m-3}) \\ +(\delta_{i,m-1}M_{2i-1,2m} - \delta_{i,m}M_{2i-1,2m-2})] \\ \\ = -i/2[\delta_{i,m-1}(-M_{2i,2m-1} + iM_{2i-1,2m-1} + iM_{2i,2m} + M_{2i-1,2m}) \\ +\delta_{i,m}(M_{2i,2m-3} + iM_{2i,2m-2} + iM_{2i-1,2m-3} - M_{2i-1,2m-2})] \\ \\ = -i/2[\delta_{i,m-1}(-M_{2m-2,2m-1} + iM_{2m-3,2m-1} + iM_{2m-2,2m} + M_{2m-3,2m}) \\ +\delta_{i,m}(-M_{2m-3,2m} - iM_{2m-2,2m} - iM_{2m-3,2m-1} + M_{2m-2,2m-1})] \\ \\ = 1/2[\delta_{i,m-1}(iM_{2m-2,2m-1} + M_{2m-3,2m-1} + M_{2m-2,2m} - iM_{2m-3,2m}) \\ -\delta_{i,m}(-iM_{2m-3,2m} + M_{2m-2,2m} + M_{2m-3,2m-1} + iM_{2m-2,2m-1})] \\ \\ = \delta_{i,m-1}E_{e^{m-1}-e^m} - \delta_{i,m}E_{e^{m-1}-e^m}\end{gathered} \tag{323}$$

(tell apart imaginary number i and index i, sorry for using the same label)

The matrices are in the upper left corner, so they commute with the following so(2n-2m) subalgebra.

Simiarly, we can construct so(2n-2m)

$$
\begin{gathered}
so(2n-2m): \\
H_k = M_{2k-1,2k} \\
E_{\eta e^j+\eta' e^k} = \frac{1}{2}[M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}] \\
k \in N^+, m < k \leq n
\end{gathered} \tag{324}
$$

Simple roots:

$$
\begin{gathered}
E_{e^j-e^{j+1}} = \frac{1}{2}[M_{2j-1,2j+1} + iM_{2j,2j+1} - iM_{2j-1,2j+2} + M_{2j,2j+2}], \\
j = m+1, \dots n \\
E_{e^n+e^n} = \frac{1}{2}[M_{2n-3,2n-1} + iM_{2n-2,2n-1} + iM_{2n-3,2n} - M_{2n-2,2n}]
\end{gathered} \tag{325}
$$

The roots in the subalgebra has the same form as so(2n), so it has the same effect as in so(2n). so(2n) only acts on first 2n state, so(2n-2m) acts on the other.

22.B

The existence of the subalgebra is easy to prove, just using the method in Ch 19.

This subalgebra contains a odd index so(2m) group and an even index so(2n-2m+1).

The so(2m) subalgebra is the same as 21.A

$$
\begin{gathered}
so(2m): \\
H_k = M_{2k-1,2k}, k = 1, \dots 2n \\
E_{\eta e^j+\eta' e^k} = \frac{1}{2}[M_{2j-1,2k-1} + i\eta M_{2j,2k-1} + i\eta' M_{2j-1,2k} - \eta\eta' M_{2j,2k}]
\end{gathered} \tag{326}
$$

$$
\begin{gathered}
E_{e^j-e^{j+1}} = \frac{1}{2}[M_{2j-1,2j+1} + iM_{2j,2j+1} - iM_{2j-1,2j+2} + M_{2j,2j+2}], \\
j = 1, \dots m-1 \\
(E_{e^{m-1}-e^m} = \frac{1}{2}[M_{2m-3,2m-1} + iM_{2m-2,2m-1} - iM_{2m-3,2m} + M_{2m-2,2m}] \\
when\ j = m-1) \\
E_{e^{m-1}+e^m} = \frac{1}{2}[M_{2m-3,2m-1} + iM_{2m-2,2m-1} + iM_{2m-3,2m} - M_{2m-2,2m}]
\end{gathered} \tag{327}
$$

For the rest of the algebra, we can define:

$$
\begin{gathered}
so(2n-2m+1): \\
H_l = M_{2l-1,2l}, E_{\eta e^j} = \frac{1}{\sqrt{2}}(M_{2l-1,2n+1} + i\eta M_{2l,2n+1}) \\
l = m, \dots n
\end{gathered} \tag{328}
$$

22.C*

$$
\begin{gathered}
H_1 = M_{1,2}, H_2 = M_{3,4} \\
E_{\eta e^1+\eta' e^2} = \frac{1}{2}(M_{1,3} + i\eta M_{2,3} + i\eta' M_{1,4} - \eta\eta' M_{2,4})
\end{gathered} \tag{329}
$$

The simple roots of so(4) are:

$$
E_{e^1-e^2}, E_{e^1+e^2} \tag{330}
$$

Under this simple roots, two sign change at one action, so there are two irreucibl spionr representation in it.

$$
\mu^1 = \frac{1}{2}(e^1 - e^2), \mu^2 = \frac{1}{2}(e^1 + e^2) \tag{331}
$$

$-\mu^1 = \frac{1}{2}(-e^1 + e^2)$ is the lowest weight in D^n, since they both have $\Pi_{j=1}^2 \eta_j = -1$,

$-\mu^2 = -\frac{1}{2}(e^1 + e^2)$ is the lowest weight in D^{n+1}, since $\Pi_{j-1}^2 \eta_j = 1$, so the representation is still real(or pseudo-real).

The so(3) subalgebra in this is

$$
M_{jk}\ for\ j, k \leq 3 \tag{332}
$$

There is only one generator in this subalgebra $\pm e^1$, correspond to$E_{\eta e^1} = \frac{1}{\sqrt{2}}(M_{1,3} + i\eta M_{2,3})$

In tensor product notation,

$$M_{1,3}=\frac{1}{2}\sigma_1^1$$
$$M_{2,3}=\frac{1}{2}\sigma_2^1 \tag{333}$$

In D^n,$\eta_{n+1}=-\sigma_3$

$$H_2=M_{3,4}=-\frac{1}{2}\sigma_3^1 \tag{334}$$

In D^{n+1}, $\eta_{n+1}=\sigma_3$

$$H_2=M_{3,4}=\frac{1}{2}\sigma_3^1 \tag{335}$$

Using $R=\sigma_2^1$ R is antisymmetric, so it is indeed pseudo-real.

Georgi split the $|>|>$ into irreducible representation,which get rid of one σ, so there is only one σ_a matrix.

Do we have to rising or lowering two state in $so(3,1)$?

22.D*

From Dynkin diagram, the two algebra are indeed the same.

$$O-O-O \tag{336}$$

In $su(4)$, 4 is the defining representation [1].$\mu=\nu^1$

Simple roots are

$$\alpha^1=\nu^1-\nu^2$$
$$\alpha^2=\nu^2-\nu^3 \tag{337}$$
$$\alpha^3=\nu^3-\nu^4$$

To lower the state, we have to first use $-\alpha^1$, then$-\alpha^2$, and $-\alpha^3$

in $so(6)$ spinor representation,$\mu^2=\frac{1}{2}(e^1+e^2-e^3)$,$\mu^3=\frac{1}{2}(e^1+e^2+e^3)$

And simple roots $\alpha^1=e^1-e^2,\alpha^2=e^2-e^3,\alpha^3=e^2+e^3$

For μ^3, the first root that could lower the state is $-\alpha^3$,$\frac{1}{2}(e^1-e^2-e^3)$ then it has to be $-\alpha^1$ $\frac{1}{2}(-e^1+e^2-e^3)$, and $-\alpha^2$ $\frac{1}{2}(-e^1-e^2+e^3)$, which is the lowest state. This is equivalent to [1].$1\rightarrow 3,2\rightarrow 1,3\rightarrow 2$

For $-\mu^2$, acting $-\alpha^2$ on it gives $\frac{1}{2}(e^1-e^2+e^3)$, acting $-\alpha^1$ on it gives $\frac{1}{2}(-e^1+e^2+e^3)$, acting $-\alpha^3$ on it gives $\frac{1}{2}(-e^1-e^2-e^3)$. And the last state is the lowest state.$1\rightarrow 2,2\rightarrow 1,3\rightarrow 3$

$4\otimes 4=6\oplus 10$, 6 is antisymmetric vector, 10 is symmetry. In SO(6), this is tensor product of two spinor. And 10 still contains highest weight $\mu^2+\mu^2$

https://physics.stackexchange.com/questions/258839/what-is-the-10-in-the-mathbf4-otimes-mathbf4-tensor-product-of-so6

Chapter 23

23.A

23.49

The sum of all odd terms equal to all odd terms, so we can sum them up

$$D[\Sigma_{j=0}^n[2j+1]]=\Sigma_{j=0}^n\frac{(2n+1)!}{(2j+1)!(2n-2j)!}$$
$$=D[\Sigma_{j=0}^n[2j]]=\Sigma_{j=0}^n\frac{(2n+1)!}{(2j)!(2n+1-2j)!} \tag{338}$$

$$\Sigma_{j=0}^{2n+1}\frac{(2n+1)!}{(j)!(2n+1-j)!}=(1+1)^{2n+1}=2^{2n+1} \tag{339}$$

$$D[\Sigma_{j=0}^n[2j+1]]=\Sigma_{j=0}^n\frac{(2n+1)!}{(2j+1)!(2n-2j)!}=2^{2n} \tag{340}$$

The dimension of D^{2n} and D^{2n+1} are $2^{2n+1}/2=2^n$, 1/2 is because we split the sign: $\Pi_{j=1}^{2n+1}\eta_j=\pm 1$. This is equal to what we get from right hand side.

23.50

$$Dimension\ of\ right\ hand\ side := (1+1)^{2n}/2 = 2^{2n-1} \tag{341}$$

Left hand side:2^{2n-1}

23.B*

$$\begin{aligned}
[A_j, \Gamma_l] &= [\frac{1}{2}(\Gamma_{2j-1} - i\Gamma_{2j}), \Gamma_l] \\
&= 2iM_{2j-1,l} + 2M_{2j,l} \\
[A_j^\dagger, \Gamma_l] &= 2iM_{2j-1,l} - 2M_{2j,l} \\
[T_a, \Gamma_l] &= [\Sigma_{j,l} A_j^\dagger [T_a]_{jk} A_k, \Gamma_l] \\
&= \Sigma_{j,l}(A_j^\dagger [A_k, \Gamma_l] + [A_j^\dagger, \Gamma_l] A_k) \\
&= \Sigma_{j,l}(A_j^\dagger (2iM_{2k-1,l} + 2M_{2k,l}) + (2iM_{2j-1,l} - 2M_{2j,l}) A_k) \\
agger, \Gamma_l] &= 2iM_{2j-1,l} - 2M_{2j,l} \\
[T_a, \Gamma_l] &= [\Sigma_{j,l} A_j^\dagger [T_a]_{jk} A_k, \Gamma_l] \\
&= [\Sigma_{j,l}(\frac{1}{2}\delta_{jk} + \frac{i}{2}M_{2j-1,2k-1} + \frac{1}{2}M_{2j-1,2k} - \frac{1}{2}M_{2j,2k-1} + \frac{i}{2}M_{2j,2k})[T_a]_{jk}, \Gamma_l] \\
&= [\Sigma_{j,l}(\frac{i}{2}M_{2j-1,2k-1} + \frac{1}{2}M_{2j-1,2k} - \frac{1}{2}M_{2j,2k-1} + \frac{i}{2}M_{2j,2k})[T_a]_{jk}, \Gamma_l] \\
&= \Sigma_{j,l}(\frac{i}{2}[M^{D^1}_{2j-1,2k-1}]_{ml}\Gamma_l + \frac{1}{2}[M^{D^1}_{2j-1,2k}]_{ml}\Gamma_l \\
&\quad -\frac{1}{2}[M^{D^1}_{2j,2k-1}]_{ml}\Gamma_l + \frac{i}{2}[M^{D^1}_{2j,2k}]_{ml}\Gamma_l)[T_a]_{jk} \\
&= \Sigma_{j,l}(\frac{i}{2}[M^{D^1}_{2j-1,2k-1}]_{ml}\Gamma_m + \frac{1}{2}[M^{D^1}_{2j-1,2k}]_{ml}\Gamma_m - \frac{1}{2}[M^{D^1}_{2j,2k-1}]_{ml}\Gamma_m + \frac{i}{2}[M^{D^1}_{2j,2k}]_{ml}\Gamma_m)[T_a]_{jk}
\end{aligned} \tag{342}$$

$$\tag{343}$$

$$\begin{aligned}
&= \Sigma_{j,k}(\frac{i}{2}i(\delta_{2j-1,l}\Gamma_{2k-1} - \delta_{2k-1,l}\Gamma_{2j-1}) \\
&\quad +\frac{1}{2}i(\delta_{2j-1,l}\Gamma_{2k} - \delta_{2k,l}\Gamma_{2j-1}) \\
&\quad -\frac{1}{2}i(\delta_{2j,l}\Gamma_{2k-1} - \delta_{2k-1,l}\Gamma_{2j}) \\
&\quad +\frac{i}{2}i(\delta_{2j,l}\Gamma_{2k} - \delta_{2k,l}\Gamma_{2j}))[T_a]_{jk}
\end{aligned} \tag{344}$$

When l is odd$l \to 2l-1$,

$$\begin{aligned}
&= \Sigma_{j,k}(\frac{i}{2}i(\delta_{j,l}\Gamma_{2k-1} - \delta_{k,l}\Gamma_{2j-1}) + \frac{1}{2}i(\delta_{j,l}\Gamma_{2k}) - \frac{1}{2}i(-\delta_{k,l}\Gamma_{2j}))[T_a]_{jk} \\
&= \Sigma_{j,k}((-\frac{1}{2}([T_a]_{lk}\Gamma_{2k-1} - \Gamma_{2j-1}[T_a]_{jk}) + \frac{i}{2}([T_a]_{lk}\Gamma_{2k}) + \frac{i}{2}(\Gamma_{2j}[T_a]_{jl}))
\end{aligned} \tag{345}$$

not sure about how this transforms.

23.C

The vector representation is 6 dimensions, so the indices take values from 1 to 6

$$\epsilon^{ijk}u^{ijk} = \epsilon^{ijk}\lambda\epsilon^{ijkabc}u^{abc} \tag{346}$$

Left hand side is

$$\binom{3}{3}(all\ the\ term\ appearing\ only\ one\ time) = 6(all\ the\ term\ appearing\ only\ one\ time) \tag{347}$$

Right hand side is 6×6 and one 6 is canceled.

so the factor is $\frac{1}{6} = \frac{1}{3!}$

23.D

a.

$so(14) = so(4\times 3+2)$

$$
\begin{aligned}
D^{2\cdot 3+1} &= [1]+[3]+[5]+[7] \\
D^{2\cdot 3} &= [0]+[2]+[4]+[6]
\end{aligned} \tag{348}
$$

b.

Using the result from 22.A, we know that a regular maximal subalgebra $so(4)\times so(10)$ $so(10)$ has an subalgebra $su(5)\subset so(10)$

so(10) only act on e^3 to e^7 this is correspond to $D^{2\times 2+1}=[1]+[3]+[5]$ and $D^{2\cdot 2}=[0]+[2]+[4]$

Chapter 24

24.A

We choose $1/2\sigma_3, 1/2\tau_3, 1/2\eta_3, 1/2\eta_3\rho_3, 1/2\sigma_3\tau_3\rho_3$ as cartan algebra. And we first write all generators in raising and lower operators form.

$$
\begin{gathered}
\sigma_\pm, \tau_\pm, \eta_\pm, \sigma_\pm\rho_1, \tau_\pm\rho_2, \eta_\pm\rho_3, \\
\sigma_\pm\tau_3\rho_3, \sigma_3\tau_\pm\rho_3, \sigma_\pm\tau_{\pm'}\rho_3, \\
\sigma_\pm\tau_3\rho_1, \tau_3\eta_\pm\rho_1, \tau_\pm\eta_3\rho_1, \tau_\pm\eta_{\pm'}\rho_1 \\
\eta_\pm\sigma_{\pm'}\rho_2, \eta_\pm\sigma_3\rho_2, \eta_3\sigma_\pm\rho_2 \\
\tau_3\eta_3\rho_1, \eta_3\sigma_3\rho_2, \tau_3\rho_2, \sigma_3\rho_1
\end{gathered} \tag{349}
$$

And some of the roots can be easily found using what we learnt from previous exercises:

$$
\begin{aligned}
\sigma_\pm \pm' \sigma_\pm\tau_3\rho_3, &\quad \alpha = (\pm 1, 0, 0, 0, \pm\pm' 1) \\
\tau_\pm \pm' \sigma_3\tau_\pm\rho_3, &\quad \alpha = (0, \pm 1, 0, 0, \pm\pm' 1) \\
\eta_\pm \pm' \eta_\pm\rho_3, &\quad \alpha = (0, 0, \pm 1, \pm\pm' 1, 0)
\end{aligned} \tag{350}
$$

Next we try to find root containing $\sigma_\pm\rho_1$

$$
\begin{aligned}
[1/2\eta_3\rho_3, \sigma_\pm\rho_1] \pm' [1/2\eta_3\rho_3, i\eta_3\sigma_\pm\rho_2] &= i\sigma_\pm\eta_3\rho_2 \pm' (-i\times i\,\sigma_\pm\rho_1) \\
&= \pm'(\sigma_\pm\rho_1 \pm' i\sigma_\pm\eta_3\rho_2)
\end{aligned} \tag{351}
$$

$$
\begin{aligned}
[1/2\sigma_3\tau_3\rho_3, \sigma_a\rho_1] &= [1/2\sigma_3\tau_3\rho_3, \sigma_a]\rho_1 + \sigma_a[1/2\sigma_3\tau_3\rho_3, \rho_1] \\
&= i\epsilon_{3a\mu}\sigma_\mu\tau_3\rho_3\rho_1 + i\sigma_a\sigma_3\rho_2\tau_3 \\
&= i\times i\epsilon_{3a\mu}\sigma_\mu\tau_3\rho_2 + i(\delta_{a3} + i\epsilon_{a3\mu}\sigma_\mu)\rho_2\tau_3 \\
&= i\delta_{a3}\rho_2\tau_3
\end{aligned} \tag{352}
$$

so

$$
[1/2\sigma_3\tau_3\rho_3, \sigma_\pm\rho_1] = 0 \tag{353}
$$

And

$$
[1/2\sigma_3\tau_3\rho_3, i\eta_3\sigma_a\rho_2] = -i\delta_{a3}\rho_1\tau_3 \tag{354}
$$

so that

$$
[1/2\sigma_3\tau_3\rho_3, \sigma_\pm\rho_1 \pm' i\eta_3\sigma_\pm\rho_2] = 0 \tag{355}
$$

$$
\sigma_\pm\rho_1 \pm' i\eta_3\sigma_\pm\rho_2, \alpha = (\pm 1, 0, 0, \pm' 1, 0) \tag{356}
$$

Similarly,

$$
\eta_3\tau_\pm\rho_1 \pm' i\tau_\pm\rho_2, \alpha = (0, \pm 1, 0, \pm' 1, 0) \tag{357}
$$

$$
\tag{358}
$$

The generators remaining are $\sigma_\pm\tau_{\pm'}\rho_3,\qquad \tau_\pm\eta_{\pm'}\rho_1,\qquad \eta_\pm\sigma_{\pm'}\rho_2\tau_3\eta_3\rho_1, \eta_3\sigma_3\rho_2, \tau_3\rho_2, \sigma_3\rho_1$

$$
[\eta_3\rho_3, \tau_\pm\eta_{\pm'}\rho_1 \pm'' i\sigma_{\pm'}\eta_\pm\rho_2] \tag{359}
$$

$$\begin{aligned} &[\eta_3\rho_3, \eta_a\rho_b] = \eta_3\eta_a 2i\epsilon_{3b\mu}\rho_\mu + 2i\epsilon_{3a\nu}\eta_\nu\rho_b\rho_3 \\ &= (\delta_{3a} + i\epsilon_{3a\nu}\eta_\nu)2i\epsilon_{3b\mu}\rho_\mu + i2\epsilon_{3a\nu}\eta_\nu(\delta_{3b} + i\epsilon_{b3\mu}\rho_\mu) \\ &= \delta_{3a}2i\epsilon_{3b\mu}\rho_\mu + i2\epsilon_{3a\nu}\eta_\nu\delta_{3b} \end{aligned} \tag{360}$$

$$\begin{aligned} &a \neq 3, b \neq 3, \\ &[\eta_3\rho_3, \eta_a\rho_b] \\ &= 0 \end{aligned} \tag{380}$$

$$\begin{aligned} &[\eta_3\rho_3, \tau_\pm\eta_{\pm'}\rho_1 \pm'' i\sigma_{\pm'}\eta_\pm\rho_2] \\ &= 0 \end{aligned} \tag{381}$$

The last element of root should also be 0, since it has two Pauli matrix commute at the same time and both indices are not equal. And we don't need to combine them together

$$\begin{aligned} &\tau_\pm\eta_{\pm'}\rho_1, \quad \alpha = (0, \pm 1, \pm' 1, 0, 0) \\ &\eta_\pm\sigma_{\pm'}\rho_2, \quad \alpha = (\pm' 1, 0, \pm 1, 0, 0) \end{aligned} \tag{382}$$

For same reason,

$$\sigma_\pm\tau_{\pm'}\rho_3, \alpha = (\pm 1, \pm' 1, 0, 0, 0) \tag{383}$$

$$\begin{aligned} &[1/2\eta_3\rho_3, (\tau_3\eta_3\rho_1 \pm i\tau_3\rho_2) \pm' (\sigma_3\rho_1 \pm i\eta_3\sigma_3\rho_2)] \\ &= (i\tau_3\rho_2 \pm (-i \cdot i)\eta_3\tau_3\rho_1) \pm' (i\eta_3\sigma_3\rho_2 \pm (-i \cdot i)\eta_3\sigma_3\rho_1)\pm' \\ &= \pm[(\tau_3\eta_3\rho_1 \pm i\tau_3\rho_2) \pm' (\sigma_3\rho_1 \pm i\eta_3\sigma_3\rho_2)] \end{aligned}$$

$$\begin{aligned} &[1/2\sigma_3\tau_3\rho_3, (\tau_3\eta_3\rho_1 \pm i\tau_3\rho_2) \pm' (\sigma_3\rho_1 \pm i\eta_3\sigma_3\rho_2)] \\ &= (i\sigma_3\eta_3\rho_2 \pm (-i \cdot i)\sigma_3\rho_1) \pm' (i\tau_3\rho_2 \pm (-i \cdot i)\eta_3\tau_3\rho_2) \\ &= \pm' \pm \eta_3\tau_3\rho_2 \pm' i\tau_3\rho_2 \pm \sigma_3\rho_1 + i\sigma_3\eta_3\rho_2 \\ &= \pm' \pm (\eta_3\tau_3\rho_2 \pm i\tau_3\rho_2 \pm' \sigma_3\rho_1 \pm \pm' i\sigma_3\eta_3\rho_2) \\ &= \pm \pm' [(\tau_3\eta_3\rho_1 \pm i\tau_3\rho_2) \pm' (\sigma_3\rho_1 \pm i\eta_3\sigma_3\rho_2)] \end{aligned} \tag{384}$$

$$(\tau_3\eta_3\rho_1 \pm i\tau_3\rho_2) \pm' (\sigma_3\rho_1 \pm i\eta_3\sigma_3\rho_2) = (0, 0, 0, \pm 1, \pm \pm' 1) \tag{385}$$

Cartan subalgebra: $1/2\sigma_3, 1/2\tau_3, 1/2\eta_3, 1/2\eta_3\rho_3, 1/2\sigma_3\tau_3\rho_3$

Roots:

$$\begin{aligned} &\sigma_\pm \pm' \sigma_\pm\tau_3\rho_3, && \alpha = (\pm 1, 0, 0, 0, \pm \pm' 1) \\ &\tau_\pm \pm' \sigma_3\tau_\pm\rho_3, && \alpha = (0, \pm 1, 0, 0, \pm \pm' 1) \\ &\eta_\pm \pm' \eta_\pm\rho_3, && \alpha = (0, 0, \pm 1, \pm \pm' 1, 0) \\ &\sigma_\pm\rho_1 \pm' i\eta_3\sigma_\pm\rho_2, && \alpha = (\pm 1, 0, 0, \pm' 1, 0) \\ &\eta_3\tau_\pm\rho_1 \pm' i\tau_\pm\rho_2, && \alpha = (0, \pm 1, 0, \pm' 1, 0) \\ &\tau_\pm\eta_{\pm'}\rho_1, && \alpha = (0, \pm 1, \pm' 1, 0, 0) \\ &\eta_\pm\sigma_{\pm'}\rho_2, && \alpha = (\pm' 1, 0, \pm 1, 0, 0) \\ &\sigma_\pm\tau_{\pm'}\rho_3, && \alpha = (\pm 1, \pm' 1, 0, 0, 0) \\ &(\tau_3\eta_3\rho_1 \pm i\tau_3\rho_2) \pm' (\sigma_3\rho_1 \pm i\eta_3\sigma_3\rho_2), && \alpha = (0, 0, 0, \pm 1, \pm \pm' 1) \end{aligned} \tag{386}$$

Simple roots:

$$\begin{aligned} &e^1 - e^2 : \sigma_+\tau_-\rho_3 \\ &e^2 - e^3 : \tau_+\eta_-\rho_1 \\ &e^3 - e^4 : \eta_+ - \eta_+\rho_3 \\ &e^4 - e^5 : (\tau_3\eta_3\rho_1 + i\tau_3\rho_2) - (\sigma_3\rho_1 + i\eta_3\sigma_3\rho_2) \\ &e^4 + e^5 : (\tau_3\eta_3\rho_1 + i\tau_3\rho_2) + (\sigma_3\rho_1 + i\eta_3\sigma_3\rho_2) \end{aligned} \tag{387}$$

If one su(2) is generated by $\eta_\pm(1+\rho_3)/4, \eta_3(1+\rho_3)/4$

$$\eta_\pm(1+\rho_3)/4, \alpha = (0, 0, \pm 1, \pm 1, 0) \tag{388}$$

vector that are orthogonal to it are $e^3 - e^4$ and all other $\pm e^i \pm e^j$,$i, j \neq 3, 4$

For simplicity, we choose $e^3 + e^4$ as another $su(2)$

$su(4)$ require that the angles between non-orthogonal vectors are the same.

$e^1 + e^2, e^5 - e^1, e^1 - e^2$, all the length is the same, and the angle are $-e^i \cdot e^i$

Now the subgroup $SU(2) \times SU(2) \times SU(4)$ is found.

24.B

The symmetric part of $D^5 \otimes D^5$ should be $2\mu^5$ .And from 23.5, the dimension is $10 + 126 = 136$

Chapter 25

25.A

The first one is $e^1, -e^1, e^2, -e^2$,which are orthogonal to $e^3 \& e^4$,and $e^3, -e^3, e^4, -e^4$, which is orthogonal to $e^1 \& e^2$

The second one is

$$\begin{aligned} &e^1 + e^2 + e^3 + e^4, \quad -e^1 - e^2 + e^3 + e^4 \\ &e^1 + e^2 - e^3 - e^4, \quad -e^1 - e^2 - e^3 - e^4 \end{aligned} \tag{389}$$

and

$$\begin{aligned} e^1 - e^2 + e^3 - e^4, \quad & e^1 - e^2 - e^3 + e^4 \\ -e^1 + e^2 + e^3 - e^4, \quad & -e^1 + e^2 - e^3 + e^4 \end{aligned} \tag{390}$$

25.B*

I think that's impossible, one of the roots of any so(5) subalgebra act on D^3, D^4 will change one sign, so this roots always cannot act on the state.

Chapter 26

26.A.

The Cartan subalgebra only has non-zero diagonal elements, and they satisfy $j = k, x = y$

$$[T_{\mu jj}]^{kx}_{ly} = \frac{1}{\sqrt{2}} \delta_{jk} \delta_{jl} [\sigma_\mu]_{xy} \tag{391}$$

non-diagonal elments are 0, so $\mu = 3$

$$[T_{\mu jj}]^{kx}_{kx} = \frac{1}{\sqrt{2}} \delta_{jk} \delta_{jk} [\sigma_3]_{xx} \tag{392}$$

And other off-diagonal elements will be 0, so in this case they are indeed Cartan subalgebra.

In the case

$$\begin{gathered} [T_{\mu ij}]^{kx}_{ly} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) [\sigma_\mu]_{xy} \; for \; \mu = 1 \; to \; 3, i \neq j \\ [T_{\mu ij}]^{kx}_{ly} = 0 \; when \; l = k \end{gathered} \tag{393}$$

And

$$[T_{0\mu\nu}]^{kx}_{ly} = \frac{i}{2} (\delta_{ik} \delta_{jl} - \delta_{jk} \delta_{il}) \delta_{xy} = 0 \; when \; l = k \tag{394}$$

So the Cartan subalgebra is

$$[T_{3jj}]^{kx}_{ly} = \frac{1}{\sqrt{2}} \delta_{jk} \delta_{jl} [\sigma_3]_{xy}, j = 1, \ldots n \tag{395}$$

$$[T_{3jj}]^{11}_{11} = \frac{1}{\sqrt{2}} \delta_{j1} \delta_{j1} [\sigma_3]_{11} = \frac{\delta_{j1}}{\sqrt{2}} = \frac{[e^1]_j}{\sqrt{2}} \tag{396}$$

26.B.

If three indices are all anti-symmetry,

$$\frac{1}{2} (u^a v^{bc} - u^b v^{ac} + u^c v^{ab}) \tag{397}$$

This form μ^3.

The rest of tensor product

$$\frac{1}{2} (u^a v^{bc} + u^b v^{ac} - u^c v^{ab}) \tag{398}$$

is $\mu^1 + \mu^2$