3-10-2023

Rhomberg's Integration

I_h : or $I_{h/2}$

$$\int_0^1 e^{-x}\,dx$$

I_0

$I_{0.5}$

$I_{0.25}$

$I_{0.125}$

combining integrals in Rhomberg

Simpson

⇒ OR Trapezoidal extrapolator using Rhomberg's Integration.

$m = 1, 2, 3 \ldots$

Trapezoidal Extrapolation:- $h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8} \ldots$

$I_h, I_{h/2}, \cancel{I_{h/2}}, I_{h/4}, I_{h/8}$

→ First integral we find

$$I_T^m(h) = \frac{4^m I_T^{m-1}\left(\frac{h}{2}\right) - I_T^{m-1}(h)}{4^m - 1}$$

where $I_T^0(h) = I_{h}$, $I_T^0\left(\frac{h}{2}\right) = I_{h/2}$

→ continue finding till answer 4 deci places closer to exact value

Extrapolation Table based on Trapezoidal rule.

S.size	$(O h^2)$ difference	$(O h^4)$	$O h^6$	$O h^8$
h	I_h	$I_T^1(h)$	$I_T^2(h)$	$I^3(h)$
$h/2$	$I_{h/2}$	$I_T^1(h/2)$	$I^2(h/2)$	
$h/4$	$I_{h/4}$	$I_h^1(h/4)$		
$h/8$	$I_{h/8}$			

For $I_T^1(h) = \dfrac{4 I_T^0(h/2) - I_T^0(h)}{3}$

$I_{h/2} =$

$$= \frac{1}{3}\left(4 I_T^0(h/2) - I_T^0(h)\right)$$

$I_T^0(h) = I_{h}$ ⇒ values found through trapezoidal or simpson rule

Example:- $I = \int_0^1 e^{-x} dx$ for $h = 1, 0.5, 0.25, 0.125$

$I_h = I_1 = 0.683939721.$
$I_{h/2} = I_{0.5} = 0.645235190$
$I_{h/4} = I_{0.25} = 0.635409429$
$I_{h/8} = I_{0.125} = 0.632943418$

Exact Solution:-
0.6321205588

$$I'_T(h) = \frac{1}{3}\left(4I^0_T(h/2) - I^0_T(h)\right)$$

$$= \frac{1}{3}\left[4(0.645235190) - 0.683939721\right]$$

$$= 0.632333680 = I'_T(h)$$

$$I'_T(h/2) = \frac{1}{3}\left(4I_{h/4} - I_{h/2}\right)$$

$$= \frac{1}{3}\left[0.635409429 - 0.645235190\right]$$

$= 0.6321341753 \Rightarrow$ stop as correct acc to 4 decimal places of the exact value.

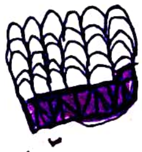
$$I'_{T\,h/48} = \frac{1}{3}\left(4\,h/8\;I_{h/4}\right)$$

$= 0.632121414 \Rightarrow$ For 5 decimal places

$$I^2_T(h) = \frac{1}{15}\left[16I'_T(h/2) - I'_T(h)\right]$$

$$= \frac{1}{15}\left[16(0.6321341753) - 0.632333680\right]$$

$I^2_T(h/2) = 0.632120563$



$\Rightarrow$ change in Table heading
start from $0h^4$ $0h^6$ $0h^8$

Simpson R Extrapolation.

$$I_S^m(h) = \frac{4^{m+1} I_S^{(m-1)}(h/2) - I_S^{(m-1)}(h)}{4^{m+1} - 1}$$

$I_h = I_{h_{0.5}} = 0.6323680$ $\quad I_{h/2} = I_{h/2} = I_{0.25} = 0.6313475$

$I_{h/4} = I_{0.125} = 0.6321415$

$$I_S^1 = \frac{4^2 I^0 (h/2) - I^0 h}{15}$$

$$= \frac{16(0.6313475) - 0.6323680}{15} = I(h)$$

$= 0.6321289$

$$I_T^2\left(\frac{h}{2}\right) = \frac{4^2 I^0\left(\frac{h}{4}\right) - I^0(h/2)}{15.}$$

$= 0.6321120$

Thursday, 5-10-2023

$\Rightarrow$ For Root Finding ~~Ar~~

$\cos x + e^x \sin x = 0$ — cannot typically find its roots.

$[a, b] \Rightarrow$ choose an interval which on a gives us a +pos val and on b a neg value.

accurate value:- when 2.1213
next iteration of 4 deci places = 2.1213
↳ repetition = correct answer.

⇒ Keep advancing forward for instance if a = 1
then check if val at a = 1.1 still gives a pos val.

⇒ Choose a Bisection Method. (divide interval by 2).

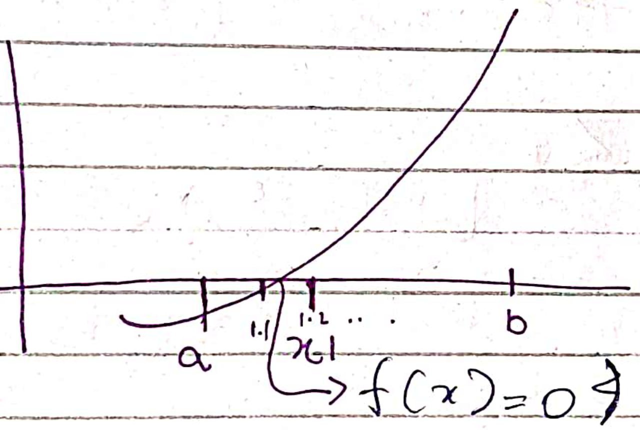
$x_1 = \frac{a+b}{2} = c$

⇒ check if c gives pos or neg value.
and we replace with either a if neg or b if pos.

⇒ $x_2 = [c, b]$. if $c < 0$

⇒ $x_2 = \frac{c+b}{2} = c_1$

If $c_1 > 0$ then $[c, c_1]$ ~ interval.



Bisection Method:-

⇒ Find the real root equation
$x^3 - x - 4 = 0$ using Bisection Method
correct upto 3 decimal places

$f(x) = x^3 - x - 4$

$f(1) = < 0$ $\qquad$ $f(2) > 0$.

↳ Bisect interval:-

DOC

Choosing an a value ~~gt er~~ tell which we have neg answer

$f(1.8)$ $f(1.7) < 0$ and $f(1.8) > 0$

$x_1 = \frac{1.7 + 1.8}{2} = 1.75$

$\Rightarrow$ Check if x_1 neg or pos:- $(1.75)^3 - 1.75 - 4 < 0$.

New Interval:- $[1.75, 1.8]$

$x_2 = \frac{1.75 + 1.8}{2} = 1.775$

$(1.775)^3 - 1.775 - 4 < 0$ so replace it with 1.75

New interval = $[1.775, 1.8]$

$\Rightarrow x_3 = \frac{1.775 + 1.8}{2} = 1.7875$

$f(1.7875) < 0$
interval now:- $[1.7875, 1.8]$

$\Rightarrow x_4 = \frac{1.7875 + 1.8}{2} = 1.79375$

$f(x_4) < 0 \Rightarrow [1.79375, 1.8]$

$\Rightarrow x_5 = \frac{1.79375 + 1.8}{2} = 1.796875$

$f(x_5) > 0 \Rightarrow [1.79375, 1.796875]$

$\Rightarrow x_6 = \frac{1.79375 + 1.796875}{2} = 1.7953125$

$f(x_6) < 0 \Rightarrow [1.7953125, 1.796875]$

$\Rightarrow x_7 = \frac{(1.793125 + 1.796875)}{2} = 1.79609375$

$f(x_7) < 0 = [1.79609375, 1.796875]$.

$x8 = (1.79609375 + 1.796875)/2$
$= 1.796484375 \Rightarrow$ value repeated
so answer

Q) Find the positive root of $f(x) = x^3 - 6x^2 + 11x - 6 = 0$

$f(0) < 0 = -6$ $\quad$ $f(0.8)$
$f(1.1) > 0 = 1.71$ $\quad$ $f(0.8) = -18.128$

$x1 = \frac{1.1 + 0.8}{2} = 0.95$

$f(0.95) = -0.107625 < 0$
$= [0.95, 1.1]$

$x2 = \frac{0.95 + 1.1}{2} = 1.025$

$f(1.025) > 0$
$= [0.95, 1.025] = 0.9875$
$x3 = 0.9875$ $\quad f(x3) < 0$

$= [0.9875, 1.025]$

$x4 = \frac{0.9875 + 1.025}{2} = 1.00625$

Regular Falsi Method:-

let $f(x)=0$

$[x_1, x_2]$

$f(x_1) < 0$ $\quad$ $f(x_2) > 0$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$x_3 = c \Rightarrow$ check if pos (replace x_2)
or neg (replace x_1)

if $(x_3) < 0 = [c, x_2]$

Solve the equation by using Regular Falsi Meth
$xe^x = \cos x$ correct upto 4 decimal place

-0.0532221 $\quad$ $[0.5, 0.6]$ $\quad$ 0.267935

$f(0.5) < 0$ $\quad$ $f(0.6) > 0$

$f[0.5, 0.6]$

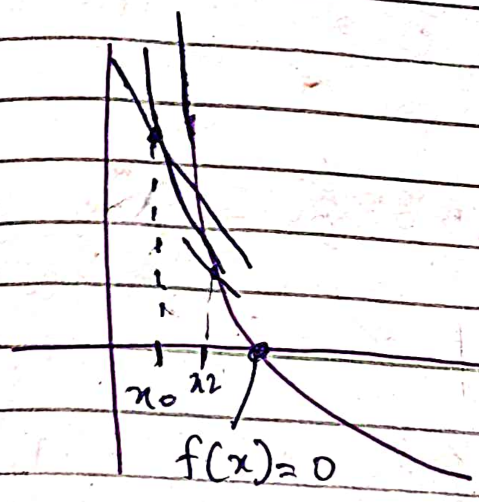
$$x_3 = \frac{x_1 f(x_2) - x_2 (f x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0.5(0.267935) - 0.6(0.053221)}{0.267935 + 0.0532221}$$

$= 0.3177102421$

Root finding (best accuracy)

Newton Raphson Method:



↳ draw tangent on $x = x_0$
↳ do the same for $x = x_1$
↳ will gradually approach $f(x)=0$

$$\frac{y_0 - y_1}{x_0 - x_1} = \frac{dy}{dx} = f'(x)$$

$$y_0 - y_1 = f'(x_1)(x_0 - x_1)$$

(Assume $y = 0$)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

1- Let $f(x) = 0$
and find x_0 that approaches towards 0 (using trial and error)

Example :- Find the real root of equation by Newton Raphson Method correct up to 4 decimal places.

⇒ choose an ~~interval~~ x_0 that approaches 0.

$f(x) = x^3 - 3x + 1$ $\quad 0.127 \Rightarrow 0.3$

⇒ Choose $x_0 = 0.3$? $f(0.3) = 0.127$

$f'(x) = 3x^2 - 3$

$f'(0.3) = -2.73$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

min 6 dec

$$x_1 = 0.3 - \frac{0.127}{-2.73} = 0.346201465$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.346201465 - \frac{0.04838}{-2.6397}$$

0.00

$$= 0.3472961$$

$$f(x_2) = 0.0000000647228991$$

$$f'(x_2) = -2.63815624$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} =$$

$$= 0.3472964$$

$\Rightarrow$ Answer same till ~~2d~~ 5 deci places

Fixed point Iteration Method:-

$$y = f(x)$$

$$x^3 - x^2 + 1 = 0.$$

How to make x the subject:-

1) $x^3 - x^2 + 1 = 0$

$\hookrightarrow \underset{\phi(x)}{x} = \sqrt[3]{x^2 - 1}$

2) $x(x^2 - x) + 1 = 0$

$\hookrightarrow \underset{\phi(x)}{x} = \frac{-1}{x^2 - x}$

$$3)\ \overset{\phi(x)}{\cancel{x}} = \sqrt{x^3+1}.$$

$|\phi'(x)| < 1.$ => can be true for all values of x or 1 or 2 or none.

$x_{n+1} = \phi(x_n)$

if $n=0 \longrightarrow$ $x_1 = \phi(x_0)$, $n=1$ $x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$

1) $y = f(x) = 0$

2) Choose an interval where $f(a) < 0$ $f(b) > 0$

=> 3) Choose $\phi(x)$ that meets the condition $|\phi'(x)| < 1$.

Example:- Find the real root of the equation $x^3 - 9x + 1 = 0$ upto 3 decimal places.

$f(2) < 0$ and $f(3) > 0$

↳ choose any x value in between this interval. for e.g $x_0 = 2.7$

-> Make x the subject using all possible ways:-

1) $x^3 = 9x - 1$

$x = \sqrt[3]{9x-1}$

let $x = \phi(x)$

2) $x = \dfrac{x^3+1}{9}$

3) $x(x^2-9)+1=0$

$$x = \frac{1}{9-x^2}$$

let $x = \phi(x)$

from 1) $\phi(x) = (9x-1)^{1/3}$

$$\phi'(x) = \frac{9}{3}(9x-1)^{-2/3}$$

$$= \frac{3}{(9x-1)^{2/3}}$$

$$|\phi'(x_0)| = \frac{3}{(9(2.7)-1)^{2/3}} = 0.368 < 1$$

3) $\phi(x) = \dfrac{1}{9-x^2}$

$$\phi'(x) = -(9-x^2)^{-2}(-2x)$$

$$\phi'(x) = \frac{2x}{(9-x^2)^2}$$

Cannot choose as condition not sati.

$$\phi'(x_0) = \frac{2(2.7)}{(9-(2.7)^2)^2} = 1.84 > 1 \ \times$$

For 2) $\dfrac{x^3+1}{9}$ $\phi'(x) = \dfrac{3x^2+1}{9} = 2.54$

$\Rightarrow (9x-1)^{1/3}$

For 1) $|\varphi'(x_0)| = 0.367748$

$x_{n+1} = \varphi(x_n)$ → repeat until answer repeats till 3 deci. places

$\varphi(x_0) = x_1 = (9(2.7)-1)^{1/3} = 2.8561782$

$x_2 = (9(2.8561782)-1)^{1/3} = 2.912494808$

$x_3 = (9(2.912494808)-1)^{1/3} = 2.932277281$

$x_4 = (9(x_3)-1)^{1/3} = 2.939163364$

$x_5 = (9(x_4)-1)^{1/3} = 2.941552787$

$x_6 = (9(x_5)-1)^{1/3} = 2.942380993$

$x_7 = (9(x_6-1))^{1/3} = 2.942667952$

$x_8\ (9(x_7)-1)^{1/3} = 2.942767366 = 3$ deci. places same

$f(x) = x^3 + x^2 - 1 =$

$f(0) = -1 < 0$ $\qquad f(1) > 0 = 1$

$x_0 = 0.6$

1) $x^3 = \cancel{x^2}\, 1 + x^2$

$x = \sqrt[3]{1+x^2}$

2) $x^2 = 1 - x^3$

$x = \sqrt{1-x^3}$

3) $x^2(x+1) - 1 = 0$

$x^2 = \dfrac{1}{x+1}$ $\qquad x = \sqrt{\dfrac{1}{x+1}}$

For 1) X $\qquad$ For ~~2)~~ ✓

For 3) ✓ conditions met $|\varphi(x)|$

Secant Method:-

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Example:- Find the real root of the equation $x^3 - 5x + 1 = 0$ that lies in the interval $[0, 1]$ Perform 4 iterations of the secant Method:-

$x_0 = 0$ $\qquad$ $x_1 = 1$

For $n = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 (f(x_0))}{f(x_1) - f(x_0)}$$

★ Cannot use this method if $x_0 = x_1$

$x_2 = \dfrac{0(1) - 1(\cancel{0}1)}{-3-1} = \dfrac{-1}{-4} = \dfrac{1}{4} = 0.25$

$= [\cancel{1,0.0625}]$ $[0.25, 1]$

$x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$\boxed{f(x_2) = (-15/64)}$

$= \dfrac{1(-15/64) - 0.25(-3)}{\left(\frac{-15}{64} + 3\right)}$

$= 0.186440678$ [

$x_4 = \cancel{x_2 f(x}$

$x_{3+1}^{n} = \dfrac{x_2(f(x_3)) - x_3(f(x_2)}{f(x_3) - f(x_2)}$

$= \dfrac{\cancel{(-15/64)}\ 0.25(\cancel{-15}/0.0742773117) - 0.186440678(-15/64)}{0.0742773117 + 15/64}$

$= 0.2017362562$

$\dfrac{x_{n-1} f(x_n) - x_n(f x_{n-1})}{f(x_n) - f(x_{n-1})}$

$\dfrac{x(n-1)(f(x_n) - x_n(f x_{n-1})}{f(x_n) - f(x_{n-1})}$

System of linear equations

Tuesday
17-10-2023

Number of iterations / errors.
via Bisection Method
[a,b].

$$\frac{b-a}{2^N} = \text{number of iterations/intervals}$$

$$= \frac{b-a}{2^N} \leq \varepsilon$$ {where ε is number of decimal places

$$\frac{b-a}{\varepsilon} = 2^N$$

$$\log\left(\frac{b-a}{\varepsilon}\right) \leq \log 2^N$$

$$\log(b-a) - \log(\varepsilon) \leq N\log 2$$

$$= \frac{\log(b-a) - \log(\varepsilon)}{\log N} \leq N.$$

interval [less than 4 greater to 3

Example:- Find the min number of iterations needed by bisection method to approximate the root $x = 3$ of $x^3 - 6x^2 + 11x - 6 = 0$ with error tolerance of 10^{-3}.

$$f(x) = x^3 - 6x^2 + 11x - 6.$$

$$= [2.5, 4]\ \cancel{3.6}$$

$$N \geq \frac{\log(4-2.5) - \log(10^{-3})}{\log 2}$$

$$N \geq 10.55$$

$$N = 11$$

$$\frac{\log(b-a) - \log(\varepsilon)}{\log 2}$$

Q. 9, 3.1?

$b-a$

$$N \geq \frac{\log(3.1-2.9) - \log(10^{-3})}{\log 2}$$

$N \geq 7.64 = 8$

$$= N \geq \frac{\log(0.6) - \log(10^{-3})}{\log 2}$$

$N \geq 9.22 = 10$

$$\frac{y-y_0}{x-x_0} = f'(x_0)$$

$y - y_0 = f'(x_0)(x - x_0)$

$y = f(x) = 0$

$y_0 = f(x_0)$

$$x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$$

$x_0 = c \Rightarrow$ in fixed point iteration

$\hookrightarrow |\phi'(x_0)| < 1$ converge else diverge

$$\frac{b-a}{2^N} = \leq \varepsilon \Rightarrow \text{no of decimal places}$$

$$\frac{b-a}{2^N} \leq \varepsilon$$

System of linear Equations.

(Q-)

$\Rightarrow$ Find root of $\cos x = 3x - 1$ correct upto 4 decimal places by using iterative Method.

~~cos x~~

Q-2) Approximate the positive square root of 2, choose $x_0 = 1.5$
$x_1 = 1$

4 iterations.

Q-1)

$$x_0 = 0.6$$

$$\cos x - 3x + 1 = 0$$

$$\frac{\cos x + 1}{3} = x$$

$$f(x) = -\sin x \qquad f$$

$$= -\sin(0.6)$$

$$|f'(x)| < 1 \checkmark$$

$$x = \frac{\cos x + 1}{3}$$

x_1

$$f(\cancel{x_0}) = 0.608445205$$

$$\cancel{f}\ x_2 = \frac{\cos(x_1) + 1}{3} = 0.60684\,5906$$

$$x_3 = \frac{\cos(x_2) + 1}{3} = 0.6071502719$$

$$x_4 = \frac{\cos(x_3) + 1}{3} = 0.6070924013$$

$$x_5 = 0.6071$$

$\sqrt{2} = x$

$= \cancel{\cos}\ x_6 = \cos(x_5)+1 = 0.6071$

3

pos

(Q-2) $x_0 = 1.5$ $x_1 = 1$

$n = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$f(x_0) = \sqrt[2]{1.5} = \sqrt{6}/2$

$f(x_1) = \sqrt[2]{1} = 1$

$$c = \frac{1.5(1) - 1(\sqrt{6}/2)}{1 - \sqrt{6}/2} = -1.224745$$

$= 1.5(\sqrt{2}) -$

$x^2 = 2$ $f(x)$ $x^2 - 2 = 0$

$[\overset{-}{1}, \overset{+}{1.5}]$

$x_0 = 1$ values

x_0 prev x ignored ~~x_0~~ when new value found in secant method

if x_0, x_1 and (x_2 in 2nd iteration) then new interval $[x_1, x_2]$.

19-10-202

Thursday

System of Linear Equations

Echelon Form

⇒ Find Partial pivot

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

- Partial Pivot: 1^{st} column greatest value row → shift to 1^{st} row.

$\bullet$ ⇒ Pivot

0 below

0 values = 0

$$\left[\begin{array}{ccc|c} \bullet & 2 & 1 & 4 \\ 0 & \bullet & 2 & 1 \\ 0 & 0 & 5 & 2 \end{array}\right]$$

$5x^3 = 2$

Example:- $x + x_2 + x_3 = 3$

$4x_1 + 3x_2 + 4x_3 = 8$

$9x_1 + 3x_2 + 4x_3 = 7$

Solve using partial pivot method.

Step 1: Convert to a matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 7 \end{bmatrix}$$

$$\quad A \qquad\qquad x \qquad b$$

Step 2: $[A|b] = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 9 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 7 \end{bmatrix}$

$\frac{b-a}{2N} \leq \varepsilon$

Step 3: Choose Partial pivot: $\frac{b-a}{\varepsilon} \leq N$

$\Rightarrow$ Interchange $R_1 \leftrightarrow R_3$

$$= \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 4 & 3 & 4 & 8 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$R_2 \rightarrow R_2 - 4R_3$:

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -1 & 0 & -4 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$\Rightarrow$ ~~B~~ $R_3 \rightarrow 9R_3 - R_1$

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -1 & 0 & -4 \\ 0 & 6 & 5 & 20 \end{array}\right]$$

$24 + 20$

can solve directly too or remove R_3 b

$R_3 \rightarrow R_3 + 6R_2$:

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & -4 \end{array}\right]$$

$= 9x_1 + 3x_2 + 4x_3 = 7$

$\cancel{+}x_2 = \cancel{+}4$

$5x_3 = -4$

$x_3 = \frac{-4}{5}$

$9x_1 = 7 - 12$

Total pivot Method

$|a_{11}| = ?$ (Positive)

$\max(|a_{1k}|) = c$

if $|a_{11}| = c$

$|a_{21}| = ?$ $\max(|a_{2k}|) = c$

$|a_{21}| = c$

$\Rightarrow |a_{31}| = ?$ $\max |a_{3k}| = c$

$|a_{31}| = c$

Example:- (Total pivoting)

Choose total pivot

$3x_1 - 4x_2 + 5x_3 = -1$

$-3x_1 + 2x_2 + x_3 = 1$

$6x_1 + 8x_2 - x_3 = 35$

$$[A|b] = \left[\begin{array}{ccc|c} 3 & -4 & 5 & -1 \\ -3 & 2 & 1 & 1 \\ 6 & 8 & -1 & 35 \end{array}\right]$$

$|a_{11}| = 3$ $\max(|a_{1k}|) = 5$

$|a_{11}| \neq c$

$|a_{21}| = 3$ $\max(|a_{2k}|) = 3$

$|a_{21}| = c$

$|a_{31}| = 6$ $\max(|a_{3k}| = 8)$

$|a_{31}| \neq c$

Total pivot row:- 2^{nd} row.

(continued)

FAST
Foundation for Advancement of Science & Technology

Rate of convergence, root finding methods:

1. Bisection 2. Regular Falsi, 3. Secant 4. Newton Raphson

1. Bisection Method:- $x = (a+b)/2$

in terms of $x = (x_{n-1} + x_n)/2$.

1. Let $f(x) = 0$ α exact root of $f(x)$.

$\Rightarrow f(\alpha) = 0$

$\boxed{\begin{array}{l} e_{n+1} = A e_n^K \\ K = \text{convergence} \\ \text{for e.g} \\ K=1 \text{ liner} \\ K=2 \text{ quadratic} \end{array}}$

✱ $x_n = \alpha + e_n$ { where e_n is the error ~~and is~~ and n is the iteration number. }

$x_{n+1} = \alpha + e_{n+1}$

$$x_{n+1} = \frac{x_{n-1} + x_n}{2}$$

Substitute in ✱:

$$\alpha + e_{n+1} = \frac{\alpha + e_{n-1} + \alpha + e_n}{2}$$

$$\alpha + e_{n+1} = \frac{2\alpha}{2} + \frac{e_{n-1} + e_n}{2}$$

$$\alpha + e_{n+1} = \alpha + (e_{n-1} + e_n)/2$$

$$e_{n+1} = \frac{e_{n-1} + e_n}{2} = \frac{1}{2}(e_{n-1} + e_n)$$

$$= \frac{e_n}{2}\left(1 + \frac{e_{n-1}}{e_n}\right) \Rightarrow$$ { $1 + \frac{e_{n-1}}{e_n} \Rightarrow$ minor so neglect this ratio or make equal to A }

$e_{n+1} = A e_n^1$ { where $A = 1 + \frac{e_{n-1}}{e_n}$ from above }

$A e_n^k = A e_n^1$

$k = 1 \Rightarrow 1$ mean order of convergence is linear.

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Rate of convergence: Newton Raphson

Let $f(x) = 0$ and α exact root of $f(x)$

$\hookrightarrow f(\alpha) = 0$

1) $x_n = \alpha + e_n$ 2) $x_{n+1} = \alpha + e_{n+1}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

⇒ substituting x_{n+1} and x_n:

$$\cancel{\alpha} + e_{n+1} = \cancel{\alpha} + e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

⇒ Show: $e_{n+1} = A e_n^k$

$$= e_{n+1} = e_n - f(\alpha + e_n) / f'(\alpha + e_n)$$

⇒ Apply Taylor's Theorem:

$$e_n - \frac{\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots\right]}{\left[f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots\right]}$$

* Keep on neglecting higher order error for e.g $\frac{e_n^2}{2!}$ and so on...

$$\Rightarrow e_{n+1} = e_n - \frac{e_n f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$

⇒ insert $f(\alpha) = 0$ and neglect higher higher order values

⇒ Taking LCM:

$$e_{n+1} = \frac{\cancel{e_n f'(\alpha)} + e_n^2 f''(\alpha) - \cancel{e_n f'(\alpha)}}{f'(\alpha) + e_n f''(\alpha)}$$

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{f'(\alpha) + e_n f''(\alpha)} = \frac{e_n^2 f''(\alpha)}{f'(\alpha)\left[1 + \frac{e_n f''(\alpha)}{f'(\alpha)}\right]}$$

denominator of

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Applying Binomial Theorem:

(first term same, power values above in numerator)

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{f'(\alpha)} \left[1 + \frac{e_n f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

change in power

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{f'(\alpha)} \left[1 - e_n \frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$e_{n+1} = A e_n^2 \quad \text{so} \quad k = 2$$

$$A e^k{}_n = A e^2{}_n$$

- **Rate of convergence of Regular Falsi Method:**

Let $f(x) = 0$ and α exact root of $f(x)$

so $f(\alpha) = 0$

$$x_n = \alpha + e_n$$

$$x_{n+1} = \alpha + e_{n+1}$$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1}(f(x_n))}{f(x_{n+1}) - f(x_n)}$$

$$\alpha + e_{n+2} = \frac{(\alpha + e_n) f(\alpha + e_{n+1}) - (\alpha + e_{n+1}) f(\alpha + e_n)}{f(\alpha + e_{n+1}) - f(\alpha + e_n)}$$

$$= \frac{\alpha f(\alpha + e_{n+1}) + e_n f(\alpha + e_{n+1}) - \alpha f(\alpha + e_n) - e_{n+1} f(\alpha + e_n)}{f(\alpha + e_{n+1}) - f(\alpha + e_n)}$$

take α common.

$$= \frac{\alpha\left(f(\alpha + e_{n+1}) - f(\alpha + e_n)\right) + e_n f(\alpha + e_{n+1}) - e_{n+1}(f(\alpha + e_n))}{f(\alpha + e_{n+1}) - f(\alpha + e_n)}$$

$$\alpha + e_{n+2} = \alpha + \frac{e_n f(\alpha + e_{n+1}) - e_{n+1} f(\alpha + e_n)}{f(\alpha + e_{n+1}) - f(\alpha + e_n)}$$

Applying Taylor's Theorem:

$$e_{n+2} = e_n \left[f(\alpha) + e_{n+1} f'(\alpha) + \frac{e_{n+1}^2}{2!} f''(\alpha + \dots \right]$$

$$- e_{n+1} \left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \dots \right]$$

= insert $f(\alpha) = 0$)

$$e_{n+2} = \frac{e_n e_{n+1} f''(\alpha)}{2\left[f'(\alpha + \frac{f''(\alpha)}{2}(e_{n+1} + e_n)\right)}$$

$$e_{n+2} = M e_n e_{n+1} \rightarrow 1$$

$$e_{n+1} = A e_n^k \qquad \left(\frac{e_{n+1}}{A}\right)^{1/k} = e_n$$

$$e_{n+2} = A e_{n+1}^k$$

$$M e_n e_{n+1} = A e_{n+1}^k$$

$$M \left(\frac{e_{n+1}}{A}\right)^{1/k} \cdot e_{n+1}^1 = A e_{n+1}^k$$

$$M \cdot A^{-1/k} (e_{n+1})^{\frac{1}{k}+1} = A e_{n+1}^k$$

$$(e_{n+1})^{1/k} (e_{n+1})^1$$

from α: $\frac{1}{k} + 1 = k$ $\qquad 1 + k = k^2$

$$k^2 - k - 1 = 0$$

$k = 1.618$ (only take pos)

$\hookrightarrow$ super quadratic rate of convergence

* Fixed point iteration method => own.

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Thursday 26-10-2023

LU - Decomposition Method:-

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad X = B$$

* Solve eq 2 first
* substitue the found values above in eq 1.

$A = LU$ $\qquad$ $LUX = B$.

$\Rightarrow$ Let $UX = Y \rightarrow$ 1 eq $\qquad$ $LY = B \rightarrow$ eq 2

$\Rightarrow$ Assume that $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\rightarrow$ lower tri matrix $\qquad$ $\rightarrow$ upper tri matrix

$$Q \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\Rightarrow \text{''} = \begin{bmatrix} l_{11}U_{11} + 0 + 0 & l_{11}U_{12} & l_{11}U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + l_{22}U_{22} & l_{21}U_{13} + l_{22}U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & + l_{31}U_{13} + l_{32}U_{23} + l_{33}U_{33} \end{bmatrix}$$

$a_{11} = l_{11}U_{11}$	$a_{12} = l_{11}U_{12}$	$a_{13} = l_{11}U_{13}$
$a_{21} = l_{21}U_{11}$	$a_{22} = l_{21}U_{12} + l_{22}U_{22}$	$a_{23} = l_{21}U_{13} + l_{22}U_{23}$
$a_{31} = l_{31}U_{11}$	$a_{32} = l_{31}U_{12} + l_{32}U_{22}$	$a_{33} = l_{31}U_{13} + l_{32}U_{23} + l_{33}U_{33}$

$\boxed{\text{Dolittle's Method}} = L_{ii} = 1 \Rightarrow$ lower tri matrix

Crout's Method $= U_{ii} = 1 \Rightarrow$ upper tri matrix

e.g U_{11} U_{12} U_{13} ..

upper tri matrix (convert)!

$x_1 + x_2 + x_3 = 1$

$4x_1 + 3x_2 - x_3 = 6$

$3x_1 + 5x_2 + 3x_3 = 4$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}_{B}$$

By Crout's Method:-

= Upper triangular matrix:-

=> consider diagonal entries of U matrix to be 1

$$= \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

=> also make changes in matrix $L \times U$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}U_{12} & l_{11}U_{13} \\ l_{21} & l_{21}U_{12} + l_{22} & l_{21}U_{13} + l_{22}U_{23} \\ l_{31} & l_{31}U_{12} + l_{32} & l_{31}U_{13} + l_{32}U_{23} + l_{33} \end{bmatrix}$$

$l_{11} = 1$

$l_{21} = 4$

$l_{31} = 3$

$l_{11}U_{12} = 1$

$U_{12} = 1$

$l_{21}U_{12} + l_{22} = 3$

$4(1) + l_{22} = 3$

$l_{22} = -1$

$l_{31}U_{12} + l_{32} = 5$

$3(1) + l_{32} = 5$

$l_{32} = 2$

$l_{11}U_{13} = 1$

$U_{13} = 1$

$l_{21}U_{13} + l_{22}U_{23} = -1$

$4(1) + (-1)(U_{23}) = -1$

$U_{23} = 5$

$l_{31}U_{13} + l_{32}U_{23} + l_{33} = 3$

$3(1) + 2(5) + l_{33} = 3$

$l_{33} = -10$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = L \quad U$

$LUX = B \qquad UX = Y$ — i, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$LY = B$ — ii

$\Rightarrow$ from eq ii $\longrightarrow$ $LY = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

$\boxed{y_1} = 1$, $\boxed{4y_1} - y_2 = 6$, $3y_1 + 2y_2 - 10y_3 = 4$

$\boxed{y_2} = -2 \qquad \boxed{y_3} = -1/2$

$\Rightarrow$ From eq i $\quad Ux = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \cancel{-1/2}\ -2 \\ -1/2 \end{bmatrix}$$

$x_3 = -1/2 \qquad x_2 + 5x_3 = -2 \qquad x_1 + x_2 + x_3 = 1$

$x_2 = -1/2 \qquad x_1 = 2$

Q.) 4/x

$3x_1 - 4x_2 + 5x_3 = -1$

$-3x_1 + 2x_2 + x_3 = 1$

$6x_1 + 8x_2 - x_3 = 35$

$$\begin{bmatrix} 3 & -4 & 5 \\ -3 & 2 & 1 \\ 6 & 8 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 35 \end{bmatrix}$$

X

$$\begin{bmatrix} 3 & -4 & 5 \\ -3 & 2 & 1 \\ 6 & 8 & -1 \end{bmatrix} = \begin{array}{c} u_{11} \\ \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \end{array}$$

$$= \cancel{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & & \\ l_{31} & & \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 5 \\ -3 & 2 & 1 \\ 6 & 8 & -1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 3 & -4 & 5 \\ -3 & 2 & 1 \\ 6 & 8 & -1 \end{bmatrix}} \cancel{\begin{bmatrix} u_{11} & u_{12} & u_{13} \end{bmatrix}}$$

$u_{11} = 3 \quad u_{12} = -4$

$\boxed{u_{13}} = 5$

$l_{21} u_{11} = l_{21}(3) = -3$

$\boxed{l_{21}} = -1$

$2 = l_{21} u_{12} + u_{22}$

$2 = (-1)(-4) + u_{22}$

$2 = 4 + u_{22} \rightarrow \boxed{u_{22} = -2}$

$1 = l_{21} u_{13} + u_{23}$

$1 = (-1)(5) + u_{23}$

$1 = -5 + u_{23} \Rightarrow \boxed{u_{23} = 6}$

$l_{31} u_{11} = 6$

$l_{31}(3) = 6$

$\boxed{l_{31} = 2}$

$8 = l_{32}(-4) + l_{32}(-2)$

$8 = -4l_{31} - 2l_{32}$

$8 = -8 - 2l_{32} =$