习题8.2

1、 $f(x;\theta)=\begin{cases}\frac{1}{\theta}, & 0<x<\theta \\ 0 & 其他\end{cases}$ $F(x;\theta)=\begin{cases}0, & x<0 \\ \frac{x}{\theta}, & 0\le x<\theta \\ 1, & x\ge\theta\end{cases}$

$Y=\min\{X_1,X_2,X_3\}$

$F_Y(y)=P\{Y\le y\}=P\{\min\{X_1,X_2,X_3\}\le y\}$

$=1-P\{X_1>y,X_2>y,X_3>y\}$

$=1-[(1-F_{X_1}(y))\cdot(1-F_{X_2}(y))\cdot(1-F_{X_3}(y))]$

$=1-(1-\frac{y}{\theta})^3 \quad (0<y<\theta)$

$f_Y(y)=\begin{cases}\frac{3(1-\frac{y}{\theta})^2}{\theta} & 0<y<\theta \\ 0 & 其他\end{cases}$

$E(\hat\theta)=c\int_0^\theta y f(y)\,dy=\frac{c}{4}\theta$

$c=4$ 时 $E(\hat\theta)=\theta$，此时 $\hat\theta=4\min\{X_1,X_2,X_3\}$

为 θ 的无偏估计

4、证明：$E(\hat\theta)=\theta$

$E(\hat\theta^2)=D(\hat\theta)+E^2(\hat\theta)>\theta^2$

$\therefore E(\hat\theta^2)\ne\theta^2$ $\therefore$ 不是无偏估计

5、 $E(X)=\mu=E(Y)$ $D(X)=1$ $D(Y)=4$

$\therefore E(Z)=(an+bm)\mu=\mu\Rightarrow an+bm=1$

$D(Z)=a^2n\cdot D(X)+b^2m\,D(Y)$

$=a^2n+4b^2m$

由拉格朗日乘数法 $\Rightarrow$ $a=\frac{4}{4n+m}$ $b=\frac{1}{4n+m}$ 时取最小.

2、(1) 证明：$E(X)=\int_0^\theta x f(x;\theta)\,dx=\frac{3}{4}\theta$

$\therefore E(T_1)=\frac{2}{3}[E(X_1)+E(X_2)]=\frac{2}{3}\times(\frac{3}{4}\theta+\frac{3}{4}\theta)=\theta$

$\therefore T_1$ 是 θ 的无偏估计

$Y=\max\{X_1,X_2\}$

$F_Y(y)=P\{\max\{X_1,X_2\}\le y\}=P\{X_1\le y,X_2\le y\}$

$=\int_0^y\frac{3x^2}{\theta^3}dx\cdot\int_0^y\frac{3x^2}{\theta^3}dx=\frac{y^6}{\theta^6}\quad(0<y<\theta)$

$f_Y(y)=\begin{cases}\frac{6y^5}{\theta^6} & 0<y<\theta \\ 0 & 其他\end{cases}$ $E(T_2)=\frac{7}{6}E(Y)=\frac{7}{6}\cdot\int_0^\theta y\frac{6y^5}{\theta^6}dy=\theta$

$\therefore T_2$ 是 θ 的无偏估计

(2) $D(T_1)=\frac{4}{9}[D(X_1)+D(X_2)]$

$=\frac{8}{9}\cdot[E(X^2)-E^2(X)]$

$=\frac{8}{9}\cdot[\int_0^\theta x^2\cdot\frac{3x^2}{\theta^3}dx-(\frac{3}{4}\theta)^2]$

$=\frac{1}{30}\theta^2$

$D(T_2)=\frac{49}{36}\cdot D(Y)=\frac{49}{36}[E(Y^2)-E^2(Y)]$

$=\frac{49}{36}\cdot[\int_0^\theta y^2\cdot\frac{6y^5}{\theta^6}dy-(\frac{6}{7}\theta)^2]$

$=\frac{1}{48}\theta^2<\frac{1}{30}\theta^2=D(T_1)$

$X_i \sim N(\mu_0, \sigma^2)$ $Y_i = X_i - \mu_0 \sim N(0, \sigma^2)$

6.(1) $E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i^2) = \frac{1}{n}\cdot n(\sigma^2+0) = \sigma^2$

$\therefore \hat{\sigma}^2$ 是 σ^2 的无偏估计

$\frac{Y_i}{\sigma} \sim N(0,1)$

$\hat{\sigma}^2 = \frac{1}{n}\cdot\sigma^2\cdot\sum_{i=1}^{n}\left(\frac{Y_i}{\sigma}\right)^2$

$D(\hat{\sigma}^2) = \left(\frac{\sigma^2}{n}\right)^2 2\cdot n = \frac{2\sigma^4}{n}$

$\forall \varepsilon > 0.$ $P\{|\hat{\sigma}^2 - \sigma^2| < \varepsilon\} \geq 1 - \frac{D(\hat{\sigma}^2)}{\varepsilon^2} = 1 - \frac{2\sigma^4}{n\varepsilon^2}$

$\Rightarrow \lim_{n\to\infty} P\{|\hat{\sigma}^2 - \sigma^2| < \varepsilon\} = 1$ $\Rightarrow \hat{\sigma}^2$ 是 σ^2 的一致性估计

(2) $E(\hat{\sigma}_1^2) = \sigma^2 = E(\hat{\sigma}_4^2)$ 无偏估计

而 $E(\hat{\sigma}_2^2) = \frac{n-1}{n}\sigma^2 \cdot E(\hat{\sigma}_3^2) = \frac{n}{n-1}\sigma^2$

$D(\hat{\sigma}_1^2) = \left(\frac{\sigma^2}{n-1}\right)^2\cdot 2(n-1)$ $D(\hat{\sigma}_4^2) = \frac{2\sigma^4}{n} < D(\hat{\sigma}_1^2)$

$= \frac{2\sigma^4}{n-1}$

$\therefore \hat{\sigma}_4^2$ 更有效

习题 8.4

2. σ 未知 $n=16$ $\sqrt{n}=4$ $\alpha=0.05$

$\bar{x} = 4.856$ $S = 0.1931$ $t_{1-\frac{\alpha}{2}}(16-1) = 2.1315$

$\Rightarrow$ 置信度为0.95的置信区间 $[\bar{x} - t_{1-\frac{\alpha}{2}}(16-1)\frac{S}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2}}(16-1)\cdot\frac{S}{\sqrt{n}}]$

$= [4.75, 4.96]$

3.(1) $\sigma = 3$ $\alpha = 0.01$ $z_{1-\frac{\alpha}{2}} = 2.575$

$n = 4$ $\bar{x} = 2.7$

$[\bar{x} - z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}] = [-1.16, 6.56]$

(2) $\alpha = 0.05$ $t_{1-\frac{\alpha}{2}}(4-1) = 3.1824$

$S = 2.27$

$[\bar{x} - t_{1-\frac{\alpha}{2}}(n-1)\cdot\frac{S}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}] = [-0.923, 6.324]$

4.(1) $\hat{\mu} = \bar{x} = 1000.25$

(2) $\hat{\sigma}^2 = S^2 = 6.93$

(3) $t_{1-\frac{\alpha}{2}}(12-1) = 2.201$

$\alpha = 0.05$

$[\bar{x} - t_{1-\frac{\alpha}{2}}(n-1)\cdot\frac{S}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2}}(n-1)\cdot\frac{S}{\sqrt{n}}]$

$= [998.577, 1001.923]$

(4) $\chi^2_{1-\frac{\alpha}{2}}(n-1) = 21.920$ $\chi^2_{\frac{\alpha}{2}}(n-1) = 3.816$

$\left[\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}, \frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}\right] = [3.479, 19.982]$

(5) $\sigma^2 = 9$ $z_{1-\frac{\alpha}{2}} = 1.96$

$[\bar{x} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}] = [998.553, 1001.947]$