Classical Mechanics Formula Sheet

This is a sheet with the most important formulae for MATH 228. These are formulae that you should know. They are the formulae that you will use as the starting point for almost any Classical Mechanics problem.

Bold letters $\mathbf{a}, \mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\Gamma}$ stand for vectors.

I won't always write down what each letter means.

Newtons Three Laws

If there is no force, velocity is constant.

$$\mathbf{F} = m\mathbf{a}$$

For every force, there is an equal and opposite force.

$$\begin{aligned} \mathbf{v} &= \frac{d}{dt}\mathbf{s} \qquad & \mathbf{s} &= \int \mathbf{v}\, dt \\ \mathbf{a} &= \frac{d}{dt}\mathbf{v} \qquad & \mathbf{v} &= \int \mathbf{a}\, dt \\ \mathbf{a} &= \frac{d^2}{dt^2}\mathbf{s} & & \end{aligned}$$

Momentum, Kinetic Energy

$$\begin{aligned} \mathbf{p} &= m\mathbf{v} \\ \frac{d}{dt}\mathbf{p} &= \mathbf{F} \end{aligned}$$

If there is no external force, momentum is conserved.

$$K.E. = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$$

Force $\times$ time = change in momentum
Force $\times$ distance = change in energy = work done

An *elastic* collision is one which conserves total kinetic energy.
In an *inelastic* collision kinetic energy is lost (converted to heat etc.).
Both types of collision conserve momentum.

Centre of Mass

$$\begin{aligned} M &= \sum_i m_i \\ x_{CM} &= \frac{1}{M}\sum_i m_i x_i \qquad etc \\ \mathbf{s_{CM}} &= \frac{1}{M}\sum_i m_i \mathbf{s_i} \end{aligned}$$

$$
\begin{aligned}
M &= \int\!\!\int\!\!\int_R \rho(x,y,z)\; dx\, dy\, dz \\
x_{CM} &= \frac{1}{M}\int\!\!\int\!\!\int_R \rho(x,y,z)\; x\; dx\, dy\, dz \\
y_{CM} &= \frac{1}{M}\int\!\!\int\!\!\int_R \rho(x,y,z)\; y\; dx\, dy\, dz \\
z_{CM} &= \frac{1}{M}\int\!\!\int\!\!\int_R \rho(x,y,z)\; z\; dx\, dy\, dz \\
\mathbf{s}_{\mathbf{CM}} &= \frac{1}{M}\int\!\!\int\!\!\int_R \rho(x,y,z)\; \mathbf{s}\; dx\, dy\, dz
\end{aligned}
$$

Polar Coordinates

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
dx\; dy &\to r\; dr\; d\theta
\end{aligned}
$$

$$
KE = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)
$$

Potential Energy

For a conservative force

$$
\begin{aligned}
curl\; \mathbf{F} &= \mathbf{0} \\
\mathbf{F} &= -grad\; V = -\nabla V \\
V &= -\int \mathbf{F}\cdot d\mathbf{s}
\end{aligned}
$$

$$
\begin{aligned}
V &= mgh && \text{uniform gravity field} \\
V &= \tfrac{1}{2}Kx^2 && \text{spring} \\
V &= -\frac{GM_1M_2}{|r_{12}|} && \text{gravitational potential energy}
\end{aligned}
$$

Circular orbits

Keeping a body in a circular orbit requires a radial force

$$
F = -r\omega^2 = -\frac{v^2}{r}\; .
$$

The minus sign means that the force is directed towards the centre orbit.

Angular Momentum

$$\begin{aligned}
J &= mrv_\theta = mr^2\dot{\theta} \\
\Gamma &= rF_\theta \\
\mathbf{J} &= m\mathbf{r}\times\mathbf{v} \\
\mathbf{J} &= I\boldsymbol{\omega} \\
\frac{d}{dt}\mathbf{J} &= \boldsymbol{\Gamma} \\
\boldsymbol{\Gamma} &= \mathbf{r}\times\mathbf{F} \\
KE &= \frac{1}{2}I\omega^2
\end{aligned}$$

Rate at which radius vector sweeps out area $= \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\frac{J}{m} = \frac{1}{2}\hat{J}$.

Effective Potential

The effective potential for a body in a central potential is

$$V_{eff}(r) = V(r) + \frac{J^2}{2mr^2}$$

Total energy is

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

Maximum and minimum radius when

$$E = V_{eff}(r)$$

Gravitational Force

There is an attractive force $F = \dfrac{GM_1M_2}{r_{12}^2}$ between two masses.

The vector equation for the force exerted *by* M_1 *on* M_2 is

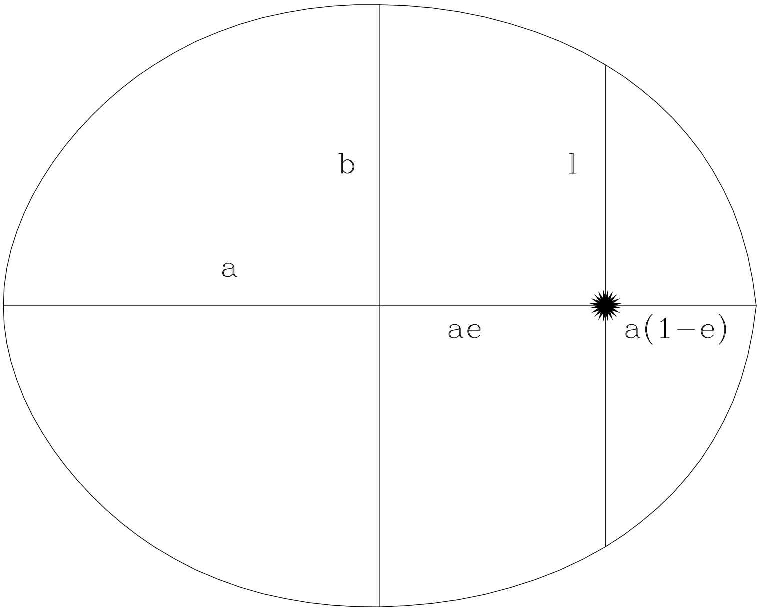
$$\mathbf{F} = \frac{GM_1M_2}{|\mathbf{s_1}-\mathbf{s_2}|^3}(\mathbf{s_1}-\mathbf{s_2})$$

Gravitational Orbits

$$r = \frac{l}{1+e\cos\theta}$$

(true for any conic section: circle, ellipse, parabola, hyperbola).

Ellipse; elliptical orbits



$$
\begin{aligned}
r_{min} &= a(1-e) \\
r_{max} &= a(1+e) \\
l &= a(1-e^2) \\
b &= a\sqrt{(1-e^2)} \\
A &= \pi ab
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{2\pi}{(GM)^{1/2}} a^{3/2} \propto a^{3/2} \\
E &= -\frac{GMm}{2a} \\
J &= m\sqrt{GMa(1-e^2)}
\end{aligned}
$$

Period T and Total Energy E don't depend on eccentricity e.

In Solar system problems, if we use years and AUs as units,

$$GM_{Sun} \equiv \mu_{Sun} = 4\pi^2 \ AU^3 y^{-2}$$

(because the Earth moves through exactly 2π radians per year).

Moment of Inertia

$$
\begin{aligned}
I_{zz} &= \int\int\int \rho(x,y,z)[x^2+y^2] \, dx \, dy \, dz \\
I_{xy} = I_{yx} &= -\int\int\int \rho(x,y,z) xy \, dx \, dy \, dz
\end{aligned}
$$

Rolling

A rolling wheel rotates at the rate

$$\dot{\theta} = \omega = -\frac{v}{R}$$

if there is no sliding.

Pseudo Forces

Coriolis $F = -2m\,\boldsymbol{\omega} \times \mathbf{v}$.

Centrifugal $F = -m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s})$.

If spin is about z-axis

$$\begin{aligned} m\ddot{x} &= F_x + 2m\omega\dot{y} + m\omega^2 x \\ m\ddot{y} &= F_y - 2m\omega\dot{x} + m\omega^2 y \\ m\ddot{z} &= F_z \end{aligned}$$

Euler's equations

$$\begin{aligned} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z = 0, \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z = 0, \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y = 0, \end{aligned}$$

describe the changes in $\boldsymbol{\omega}$, the angular velocity vector of a rotating object, when there is no external torque acting. I_{ij} are components of the inertia matrix.

Euler's equations apply in a co-rotating coordinate system fixed to the body. The axes x, y, z must be chosen to be the principal axes of the inertia tensor. ω is the angular velocity vector in this co-rotating frame.