Problem Set 2

November 27, 2017

Problem 1

Part a Starting to contribute changes a player's payoff by $n^2 - 100 - (n-1)^2 = 2n - 99$, which is positive if and only if $n > 49.5$.

Part b There are two pure strategy Nash equilibria: one where all players contribute and the other where no player contributes. To see this, note that ceasing contribution changes a player's payoff by $(n-1)^2 - (n^2 - 100) = 101 - 2n$, which is positive if and only if $n < 50.5$. Thus, whenever $n \leq 50$, contributors have an incentive to stop contributing. Part a implies that non-contributors have an incentive to contribute when $n \geq 50$.

Problem 2

We start by computing best response correspondences. Given quantity choices of firms 2 and 3, firm 1's profit from production plan q_1 is $q_1(p(q) - c)$. When $q \leq a$, this is maximized at the critical point $q_1 = (a - q_2 - q_3 - c)/2$; otherwise it is maximized at $q_1 = 0$. Likewise for the other firms. The game is symmetric, and so we know there exists a symmetric Nash equilibrium in which all firms produce the same quantity. It is easy to show that this quantity is $(a - c)/4$. (It is a good exercise to convince yourself that the corner solutions do not arise, and also that this is the only Nash equilibrium.) The market clearing price is then $a - 3(a - c)/4$, and equilibrium profits for each firm are $((a - c)/4)^2$.

The same analysis for Cournot duopoly shows that in the symmetric Nash equilibrium, each firm produces $(a - c)/3$, which induces a market clearing price $a - 2(a - c)/4$, and firm profits $((a - c)/3)^2$. Thus, instead of merging, the two firms would be better off pooling their profits since

$$2\left(\frac{a-c}{4}\right)^2 = \frac{(a-c)^2}{8} > \frac{(a-c)^2}{9}.$$

(Recall $a > c$.)

Problem 3

In the Bertrand competition version of Problem 2, each firm chooses a price and then supplies whatever the market demands, sharing demand equally when the minimal price is set by more than one firm. It is easy to see that any strategy profile where two firms price at marginal cost and the third at least at marginal cost is a Nash equilibrium. Now consider the other types of strategy profile and show that, in each, some firm has a profitable deviation.

Problem 4

We again start by computing best response correspondences. Given quantity choices by firm 2, firm 1's profit from producing q_1 is $q_1(p(q) - c_1)$. When $q \leq a$, this is maximized at the critical point $q_1 = (a - q_2 - c_1)/2$; otherwise it is maximized at $q_1 = 0$. Likewise, $q_2 = (a - q_1 - c_2)/2$ when $q \leq a$ and otherwise $q_2 = 0$. Solving for mutual best responses gives $q_1 = (a - 2c_1 + c_2)/2$ and $q_2 = (a - 2c_2 + c_1)/2$. (Drawing out best responses and noting $a > c_1 > c_2$ shows this to be the unique Nash equilibrium.) Quantities are falling in a firm's marginal cost, and increasing (but more slowly) in their competitor's. Total quantity rises, so price falls.

Problem 5

If player 2 selects H with probability p, then player 1's expected payoff from H is $2p-(1-p)$ and from T is $-p+3(1-p)$. Thus when $p > 4/7$, H is optimal for player 1, when $p < 4/7$, T is optimal, and when $p = 4/7$ either is optimal. With player 1 selecting H with probability q, the same analysis, but now from the viewpoint of player 2, leads us to conclude that H is optimal for player 2 when $q < 1/2$, T optimal when $q > 1/2$, and otherwise either is optimal. Graphing these best response correspondences shows a unique intersection, and thus a unique Nash equilibrium, at $p = 4/7$, $q = 1/2$.

Problem 6

We are asked to verify that a strategy profile is a Nash equilibrium. As with any Nash equilibrium, this means we must show that no player has a profitable deviation. Note that if player 1 chooses a number i, then his expected payoff is $1/K$, which is independent of i. Thus, player 1 is indifferent between all his pure strategies. It follows from this that player 1 has no profitable deviation. Mutatis mutandis, same for player 2.

Problem 7

Part a There are no pure strategy Nash equilibria. This can be shown by carefully graphing the best response correspondences and noting that the graphs do not intersect, or it can be shown by considering each type of strategy profile and identifying a profitable deviation.

Part 2 There exists at least one mixed strategy Nash equilibrium because the game is symmetric. The assumption that the c.d.f. is atomless allows us to ignore the case in which both countries submit proposals and have the same bribe. As usual, a mixed strategy Nash equilibrium has the useful property that all pure strategies played with positive probability yield the same expected payoff. This implies that, in the Nash equilibrium under consideration, submitting a proposal and bribing b between 0 and $v - k$ has an expected payoff of 0, the payoff from not submitting a proposal:

$$p(F(b)(v - k - b) + (1 - F(b)(-k - b))) + (1 - p)(v - k - b) = 0.$$

Solving for $F(b)$ and using the condition $F(0) = 0$ gives $F(b) = b/(v - k)$ and $p = (v-k)/v$. This is a c.d.f. and a probability satisfying the desired conditions. Strictly speaking, we are not yet done, because we are yet to check that a player cannot profitably deviate to a pure strategy not played with positive probability. Such a pure strategy involves submitting a proposal and then bribing more than $v - k$, something strictly dominated by not submitting a proposal.