COLLEGE ADMISSIONS WITH LIMITED STUDENT APPLICATIONS

AUBREY CLARK

Abstract. This paper studies the college admissions problem with limited student applications. Theorem 1 shows that no stable matching procedure exists that makes it a dominant strategy for each student to state his true preferences. Theorem 2 shows that under any student-optimal stable matching procedure, if each college must truthfully state its preferences then every Nash equilibrium yields a stable outcome with respect to the true preferences.

1. Introduction

For the college admissions problem Roth (1985, 1982) has shown that any student-optimal matching procedure makes it a dominant strategy for each student to state his true preferences and that when college preferences are responsive no stable matching procedure exists that makes it a dominant strategy for each college to state its true preferences. For the marriage problem Roth (1984) has shown that when an M-optimal stable matching procedure is used, every Nash equilibrium in which each man truthfully reveals his preferences yields a stable outcome with respect to the true preferences.

This paper studies the preference revelation game of the college admissions problem induced by some stable matching procedure where students are limited to stating rank order lists of length k. I prove two theorems about this game. Theorem 1 shows that no stable matching procedure exists that makes it a dominant strategy for each student to state his true preferences. This contrasts with the results in Roth (1985, 1982) which show that any student-optimal matching procedure makes it a dominant strategy for each student to state his true preferences. Theorem 2 shows that under any student-optimal stable matching procedure, if colleges must truthfully state their preferences then every Nash equilibrium yields a stable outcome with respect to the true preferences. Theorem 2 is an adaptation of the main result in Roth (1984).

2. The college admissions problem with limited student applications

2.1. The college admissions problem.

The agents of the *college admissions problem* consist of two disjoint sets $C = \{c_1, \cdots, c_n\}$, the "colleges", and $S =$

$\{s_1, \cdots, s_m\}$, the "students". Each college c has a *quota* which is a natural number q_c. The quota of college c is the maximum number of places it has for students. Each college c in C has strict preferences $P(c)$ (i.e. a strict total order) over the set $S \cup \{c\}$; that is, for any two students s and s', $sP(c)s'$ means that student s is preferred by college c to student s', $sP(c)c$ means that college c prefers to have one of its places filled by student s rather than have it unfilled, and $cP(c)s'$ means that college c prefers to have one of its places unfilled rather than have it filled by student s'. Each student s in S has similarly defined strict preferences $P(s)$ over $C \cup \{s\}$. A college c is *unacceptable* for student s if $sP(s)c$; likewise a student s is *unacceptable* for college c if $cP(c)s$. Each college c in addition to having preferences $P(c)$ over individual students has preferences $R^*(c)$ (i.e. a weak total order) over entering classes (subsets of no more than q_c elements of S). We require that for any two students s and s' (1) $sP(c)s'$ if and only if $\{s\}R^*(c)\{s'\}$ and not $\{s'\}R^*(c)\{s\}$; (2) $sP(c)c$ if and only if $\{s\}R^*(c)\emptyset$ and not $\emptyset R^*(c)\{s\}$; and (3) $cP(c)s$ if and only if $\emptyset R^*(c)\{s\}$ and not $\{s\}R^*(c)\emptyset$.

2.2. **Rank order lists.** A *rank order list* for college c is a tuple $Q(c)$ such that each component of $Q(c)$ is an element of $S \cup \{c\}$ and no two components are the same. Similarly, a *rank order list* for student s is a tuple $Q(s)$ such that each component of $Q(s)$ is an element of $C \cup \{s\}$ and no two components are the same.

A rank order list $Q(c)$ *agrees* with preferences $P(c)$ if the i'th component of $Q(c)$ is preferred under $P(c)$ to the j'th component of $Q(c)$ whenever $i < j$. A similar definition applies to a rank order list $Q(s)$ of a student s.

The *length* of a rank order list is its number of components.

We will abuse notation and let $P(c)$ denote the rank order list of college c that agrees with the preferences $P(c)$ and is of length $m + 1$. Similarly we will let $P(s)$ denote the rank order list of student s that agrees with the preferences $P(s)$ and is of length $n + 1$.

For example, if the set of students is $S = \{s_1, s_2, s_3\}$ and college c's preference relation $P(c)$ is given by $s_1P(c)s_2$, $s_2P(c)c$, and $cP(c)s_3$ then college c's rank order list is $P(c) = (s_1, s_2, c, s_3)$. The length of the rank order list $P(c)$ is 4.

2.3. **Matchings and stable matchings.** A *matching* for the college admissions problem is a correspondence $x : C \cup S \to C \cup S$ such that for each college c $x(c)$ is a subset of S and $\mid x(c) \mid\leq q_c$, for each student s $x(s)$ is a subset consisting of a single element of C or $x(s) = \{s\}$, and student s is assigned to college c at x (i.e. $\{c\} = x(s)$) if and only if c is assigned s at x (i.e. s is in $x(c)$). That is, a matching assigns a subset of the students to a subset of the places and leaves the rest of the students and places unmatched.

Each college and student has preferences over matchings defined as follows. Given two matchings x and y college c likes x at least as much as y if and only if $x(c)R^*(c)y(c)$ and student s prefers x to y if and only if $x(s)P(s)y(s)$. We will abuse notation and write $xR^*(c)y$ to denote that college c likes matching x at least as much as matching y. Similarly we will write $xP(s)y$ to denote that student s prefers matching x to matching y.

A matching x is said to be *individually irrational* if x assigns some student to an unacceptable college or x assigns some college to an unacceptable student. A matching x is called *unstable* if either it is individually irrational or there exist a college c and a student s such that $cP(s)x(s)$ and either $sP(c)\sigma$ for some σ in $x(c)$ or $sP(c)c$ and $\mid x(c) \mid < q_c$. A matching is called *stable* if it is not unstable.

2.4. **Matching procedures and preference revelation games.** A *matching procedure* for the college admissions problem is a function h from the set of all tuples of rank order lists $\mathbf{Q} = (Q(c_1), \cdots, Q(c_n), Q(s_1), \cdots, Q(s_m))$ to the set of matchings. A *stable matching procedure* is a matching procedure that maps each tuple of rank order lists $\mathbf{Q}$ to a matching that is stable with respect to the preferences corresponding to $\mathbf{Q}$.

If h is a matching procedure then the *preference revelation game* induced by h is the non-cooperative game $(C \cup S, \{A_i\}_{i \in C \cup S}, \{R^*(c), P(s)\}_{c \in C, s \in S})$ where: $C \cup S$ is the set of players; A_i is agent i's action set: for each college c A_c is the set of all rank order lists $Q(c)$ of length $m+1$ and for each student s A_s is the set of all rank order lists $Q(s)$ of length $n+1$; and $P(s)$ and $R^*(c)$ are student s's and college c's preferences over matchings. An action profile in this game is a tuple of rank order lists denoted $\mathbf{Q} = (Q(c_1), \cdots, Q(c_n), Q(s_1), \cdots, Q(s_m))$; the outcome associated with this action profile is the matching $h(\mathbf{Q})$.

Let $1 \leq k \leq n+1$. If h is a matching procedure then the preference revelation game induced by h with *student rank order lists of length* k is the preference revelation game induced by h in which each student s's strategy set A_s is the set of rank order lists $Q(s)$ of length k. When $k = n+1$ the preference revelation game induced by a matching procedure h and the preference revelation game induced by the matching procedure h with student rank order lists of length k are the same.

Suppose student s's true preferences are $P(s)$. Consider a preference revelation game induced by a matching procedure h with student rank order lists of length k. Suppose that student s reports a rank order list $Q(s)$. We will say that student s *truthfully reveals* his preferences if the first k components of the rank order lists $P(s)$ and $Q(s)$ are equal. A similar definition applies to colleges.

This definition is right. I would like to prove a theorem that says no student ever has an incentive to put a more desirable college above a less desirable college.

I think proving such a theorem will be fairly easy. It should be true for the deferred acceptance algorithm. Might have to define a property of matching procedures that gives this. It is true for the deferred acceptance algorithm because by interchanging the two things you can only do better.

3. Incentives and stability

Theorem 1. *There is no stable matching procedure that always makes it a dominant strategy for each student to truthfully reveal his preferences in the induced preference revelation game of the college admissions problem with student rank order lists of length k.*

Proof. The proof constructs an example in which truthful revelation is not a dominant strategy for some student for any stable matching procedure. Consider the college admissions problem with colleges $C = \{c_1, c_2\}$, students $S = \{s_1, s_2\}$, and preferences given by

$$\begin{aligned} P(s_1) &= (c_1, c_2, s_1) \\ P(s_2) &= (c_1, c_2, s_2) \\ P(c_1) &= (s_1, s_2, c_1) \\ P(c_2) &= (s_2, s_1, c_2). \end{aligned}$$

Let h be a stable matching procedure for this college admissions problem. Suppose students are limited to submitting rank order lists of length $k = 1$. If the agents truthfully reveal their preferences, i.e.

$$\begin{aligned} Q(s_1) &= (c_1) \\ Q(s_2) &= (c_1) \\ Q(c_1) &= (s_1, s_2, c_1) \\ Q(c_2) &= (s_2, s_1, c_2) \end{aligned}$$

then the unique stable matching is

$$\begin{gathered} s_1, s_2, c_2 \\ c_1, s_2, c_2. \end{gathered}$$

This is clearly the case because any matching that does not match s_1 and c_1 is unstable and, given this, s_2 and c_2 must be unmatched because student s_2 finds college c_2 unacceptable according to $Q(s_2)$. The matching procedure h must select this matching when the agents truthfully reveal their preferences.

If student s_2 instead states the rank order list $Q'(s_2) = (c_2)$ (and so does not truthfully reveal his preferences) while the other agents state the same rank order lists as before then the unique stable matching would be

$$\begin{array}{c} s_1, s_2 \\ c_1, c_2 \end{array}$$

which student s_2 prefers. The matching procedure h must select this matching when the agents state these preferences. Thus it is not a dominant strategy for student s_2 to truthfully reveal his preferences. □

Theorem 2. *Consider a preference revelation game for the college admissions problem induced by the student-optimal stable matching procedure with student rank order lists of length k. In the game obtained by restricting each college to truthfully reveal its preferences every Nash equilibrium outcome is stable with respect to the true preferences.*

Proof. Consider a college admissions problem. We will consider the "deferred acceptance algorithm" as defined in Gale and Shapley (1962) - there it is shown that this algorithm is a student-optimal stable matching procedure. We will denote this matching procedure by g. Consider the preference revelation game induced by g with student rank order lists of length k and suppose that for each college c the rank order list $P(c)$ is the only member of its action set, i.e., for each college c $A_c = \{P(c)\}$. Let $\mathbf{Q}$ be an action profile of this game and let x denote the matching $g(\mathbf{Q})$. Suppose that x is unstable with respect to the true preferences. We will show that $\mathbf{Q}$ is not a Nash equilibrium.

Suppose that x is not individually rational. It cannot be that x assigns some college c an unacceptable student s because x is stable with respect to the stated preferences and each college states their preferences truthfully. So it must be that x assigns some student s to an unacceptable college c. This implies that $\mathbf{Q}$ is not a Nash equilibrium because by stating the rank order list whose first component is s student s can guarantee he in not assigned to any college and he prefers this to $x(s)$.

Suppose that there exist a college c and a student s such that $cP(s)x(s)$ and either $sP(c)\sigma$ for some σ in $x(c)$ or $sP(c)c$ and $\mid x(c) \mid< q_c$. Then student s must not have applied to college c. Let $Q'(s)$ denote a rank order list in A_s whose first component is college c. Let $\mathbf{Q}'$ denote the action profile that is equal to $\mathbf{Q}$ in all components not corresponding to student s and equal to $Q'(s)$ in the component corresponding to student s. Then student s applies to college c. Except for the additional application by student s, college c gets a subset of the applications it got before. So s is assigned to college c by the matching $g(\mathbf{Q}')$ which means the deviation is profitable for him. This implies that $\mathbf{Q}$ is not a Nash equilibrium. $\square$

References

D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):pp. 9–15, 1962. 5

A. E. Roth. The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7(4):pp. 617–628, 1982. 1

A. E. Roth. Misrepresentation and stability in the marriage problem. *Journal of Economic Theory*, 34(2):383 – 387, 1984. 1

A. E. Roth. The college admissions problem is not equivalent to the marriage problem. *Journal of Economic Theory*, 36(2):277 – 288, 1985. 1