

Chapter 5

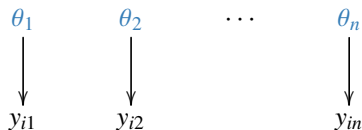
- 5.1 Lead-in to hierarchical models
- 5.2 Exchangeability (useful concept)
- 5.3 Bayesian analysis of hierarchical models (we use Stan/brms for computation)
- 5.4 Hierarchical normal model (we use Stan/brms for computation)
- 5.5 Example: parallel experiments in eight schools (useful discussion on benefits of hierarchical model)
- 5.6 Meta-analysis (can be skipped)
- 5.7 Weakly informative priors for hierarchical variance parameters

Hierarchical model

- In simple model: posterior for the parameters
- In hierarchical model: posterior for the prior parameters

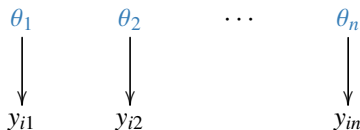
Hierarchical model

- Example: CVD treatment effectiveness
 - in hospital j the survival probability is θ_j
 - observations y_{ij} tell whether patient i survived in hospital j

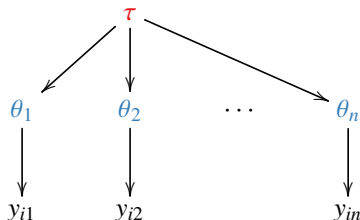


Hierarchical model

- Example: CVD treatment effectiveness
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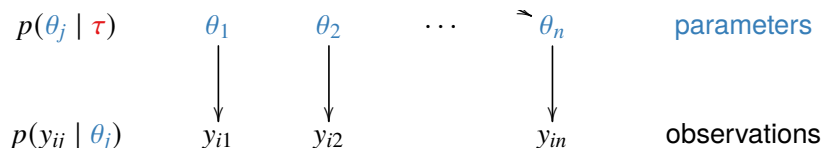
- sensible to assume that θ_j are similar



- natural to think that θ_j have common population distribution
- θ_j is not directly observed and the population distribution is unknown

Hierarchical model: terms

Level 1: observations given parameters $p(y_{ij} \mid \theta_j)$



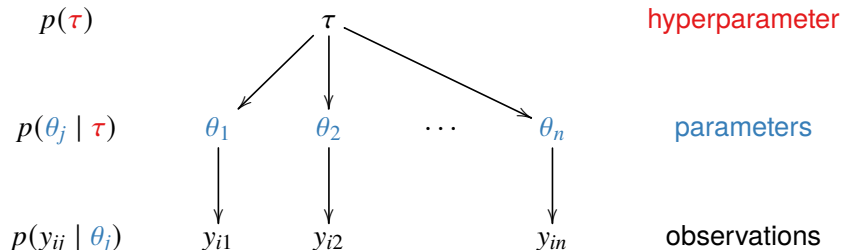
Joint posterior

$$\begin{aligned} p(\theta, \tau \mid y) &\propto p(y \mid \theta, \tau) p(\theta, \tau) \\ &\propto p(y \mid \theta) p(\theta \mid \tau) p(\tau) \end{aligned}$$

Hierarchical model: terms

Level 1: observations given parameters $p(y_{ij} \mid \theta_j)$

Level 2: parameters given hyperparameters $p(\theta_j \mid \tau)$

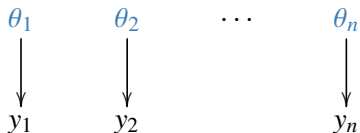


Joint posterior

$$\begin{aligned} p(\theta, \tau \mid y) &\propto p(y \mid \theta, \tau) p(\theta, \tau) \\ &\propto p(y \mid \theta) p(\theta \mid \tau) p(\tau) \end{aligned}$$

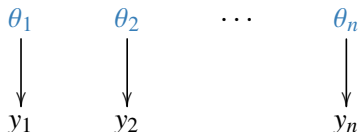
Compare

- "Separate model" (model with separate/independent effects)

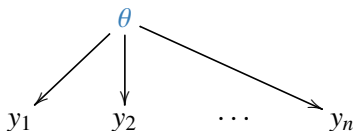


Compare

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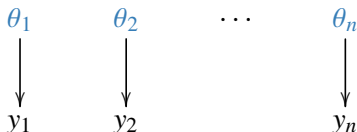


- "Joint model" (model with a common effect / pooled model)

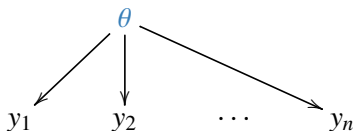


Compare

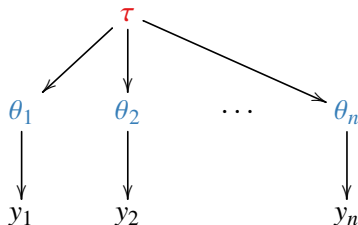
- "Separate model" (model with separate/independent effects)



- "Joint model" (model with a common effect / pooled model)



- Hierarchical model

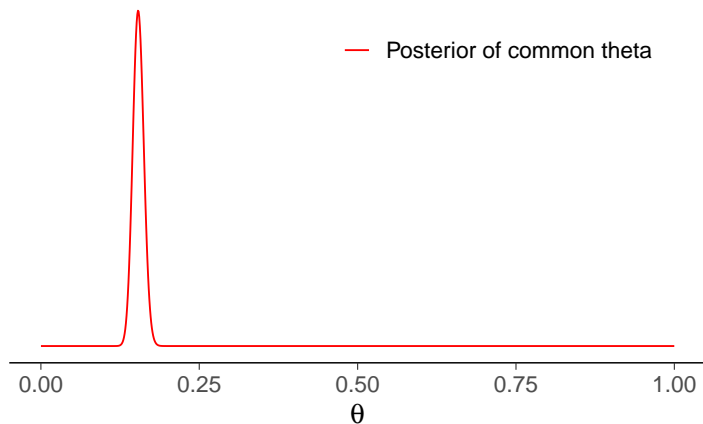


Hierarchical binomial model: rats

- Medicine testing
- Type F344 female rats in control group given placebo
 - count how many get endometrial stromal polyps
 - familiar binomial model example

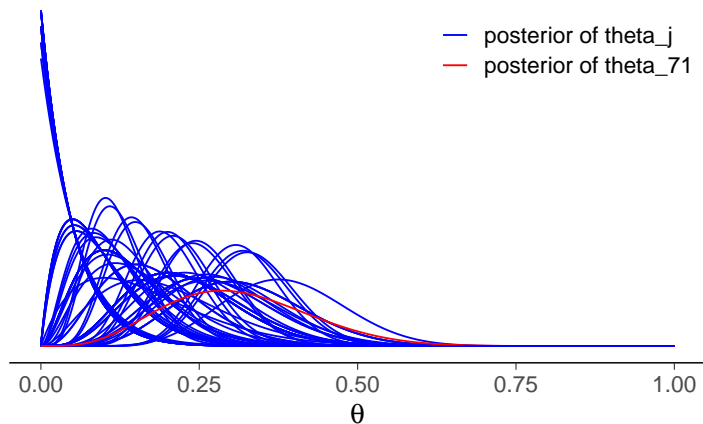
Hierarchical binomial model: rats

Pooled model



Hierarchical binomial model: rats

Separate model



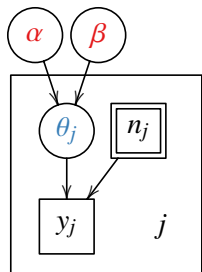
Hierarchical binomial model: rats

- Hierarchical binomial model for rats
prior parameters α and β are unknown

$$\theta_j \mid \alpha, \beta \sim \text{Beta}(\theta_j \mid \alpha, \beta)$$

$$y_j \mid n_j, \theta_j \sim \text{Bin}(y_j \mid n_j, \theta_j)$$

- Joint posterior $p(\theta_1, \dots, \theta_J, \alpha, \beta \mid y)$
 - multiple parameters

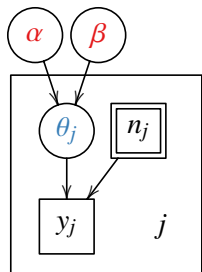


Hierarchical binomial model: rats

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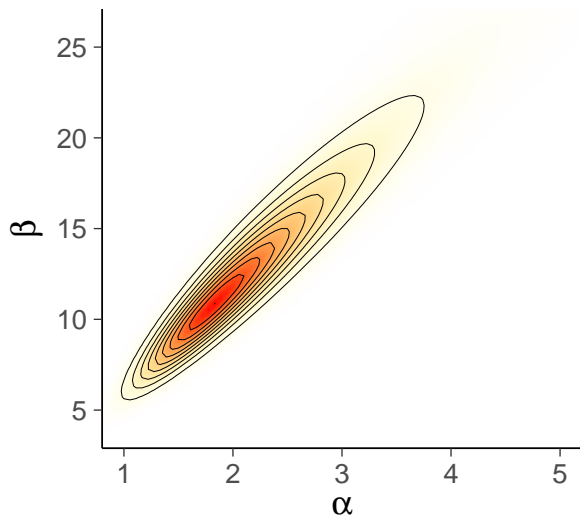
- Joint posterior $p(\theta_1, \dots, \theta_J, \alpha, \beta \mid y)$
 - multiple parameters
 - factorize $\prod_{j=1}^J p(\theta_j \mid \alpha, \beta, y) p(\alpha, \beta \mid y)$

Hierarchical binomial model: rats

- Population prior $\text{Beta}(\theta_j \mid \alpha, \beta)$
- Hyperprior $p(\alpha, \beta)$?
 - α, β both affect the location and scale
 - BDA3 has $p(\alpha, \beta) \propto (\alpha + \beta)^{-5/2}$
 - diffuse prior for location and scale (BDA3 p. 110)
- demo5_1

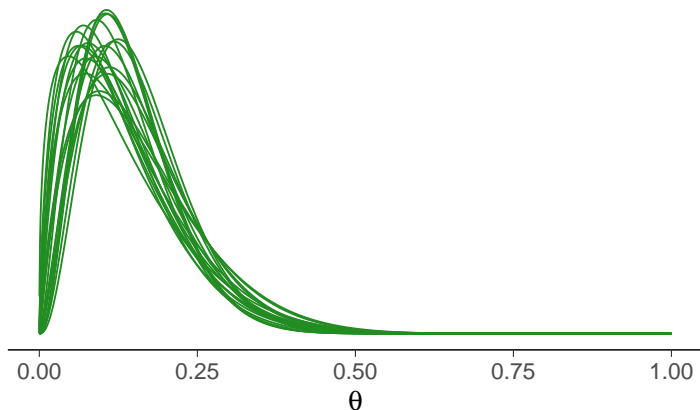
Hierarchical binomial model: rats

The marginal of α and β



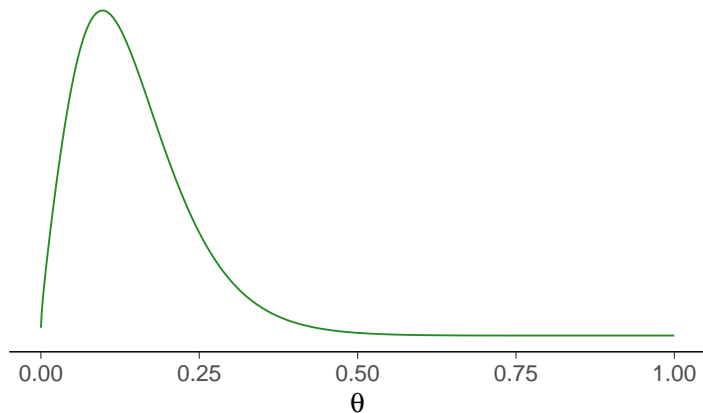
Hierarchical binomial model: rats

Beta(α, β) given posterior draws of α and β



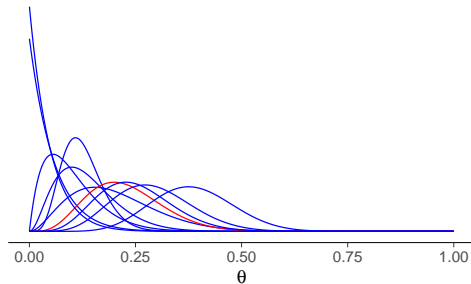
Hierarchical binomial model: rats

Population distribution (prior) for θ_j



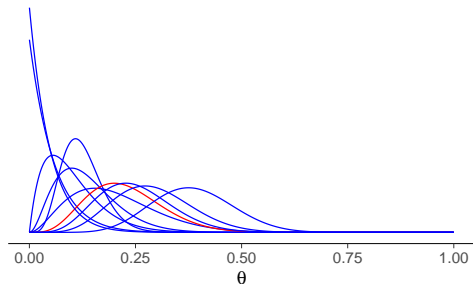
Hierarchical binomial model: rats

Separate model

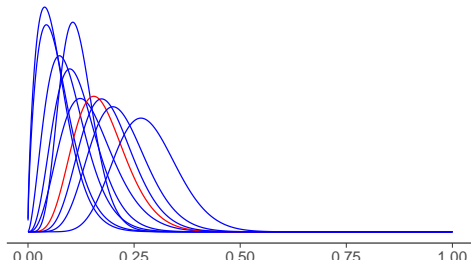


Hierarchical binomial model: rats

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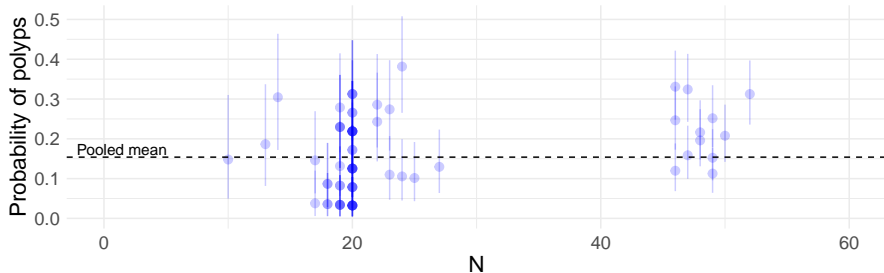


Hierarchical model

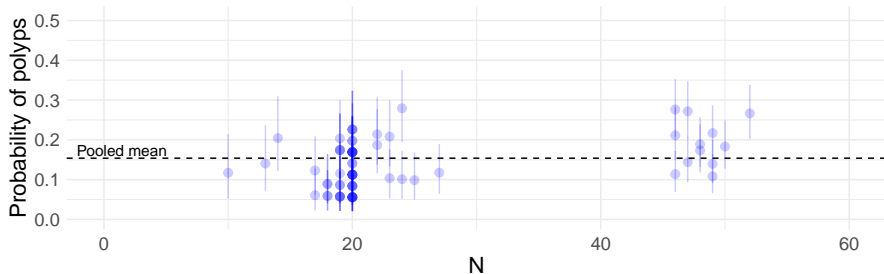


Hierarchical model and group size: Rats

Separate

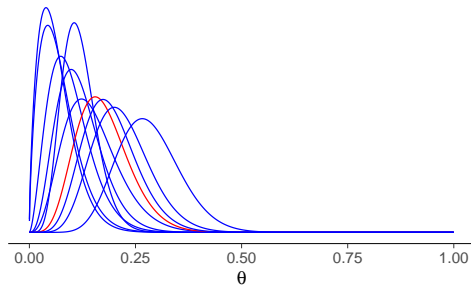


Hierarchical

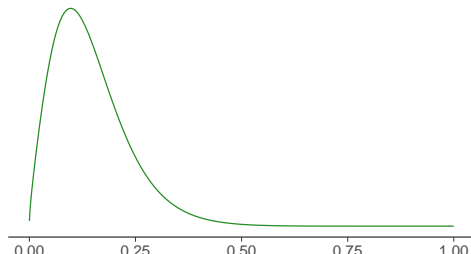


Hierarchical binomial model: rats

Hierarchical model



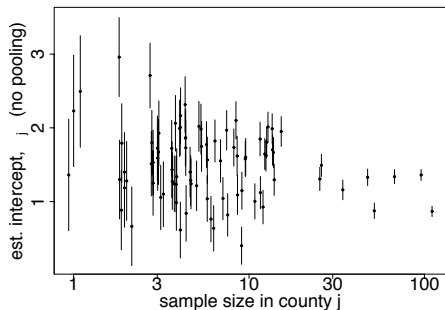
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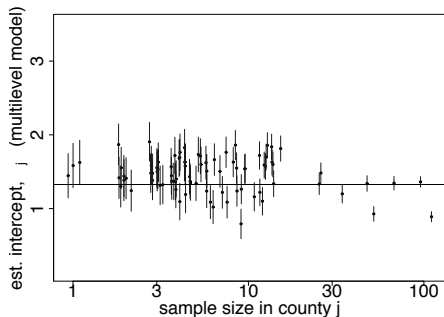
Hierarchical model and group size: Radon

919 home radon levels in 85 counties in Minnesota:

Separate



Hierarchical



Diet effect on chicken weights (at age 12 days)

- A typical treatment effect analysis
- Models
 - a separate model, in which each diet is modeled individually
 - a pooled model, in which all measurements are combined and there is no distinction between diets
 - a hierarchical model

Stan vs brms

Stan

```
model {  
  // Priors  
  for (diet in 1:N_diets) {  
    mu_diet[diet] ~ normal(mu_0, sd);  
  }  
  mu_0 ~ normal(0, 1);  
  sd ~ normal(0, 1);  
  sigma ~ normal(0, 1);  
  
  // Observation model  
  for (obs in 1:N_observations) {  
    weight[obs] ~ normal(mu_diet[diet_idx[obs]], sigma);  
  }  
}
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Stan vs brms

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}
```

brms formula

```
brm(weight ~ 1 + (1 | Diet),
```

Stan vs brms

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Stan vs brms

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  sigma ~ normal(0, 1);  
  
  // Observation model  
  weight ~ normal(mu_diet[diet_idx], sigma);  
}
```

brms formula

```
brm(weight ~ 1 + (1 | Diet), data=Chick12,
```

Stan vs brms

Stan without loops

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  sigma ~ normal(0, 1);  
  
  // Observation model  
  weight ~ normal(mu_diet[diet_idx], sigma);  
}
```

brms formula

```
brm(weight ~ 1 + (1 | Diet), data=Chick12,  
    prior=c(prior(normal(0,1), class="Intercept"), # p(mu_0)  
            prior(normal(0,1), class="sd"),        # p(tau)  
            prior(normal(0,1), class="sigma"))      # p(sigma))
```

Stan vs brms

Stan without loops

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Stan vs brms

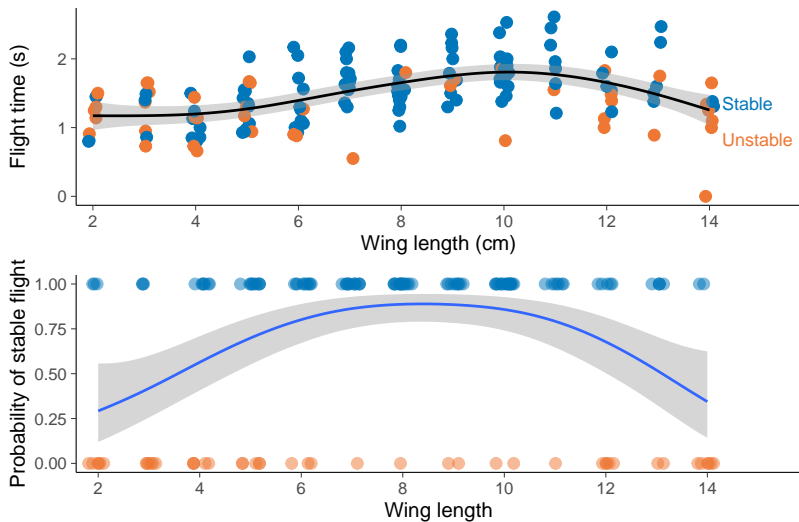
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brms formula

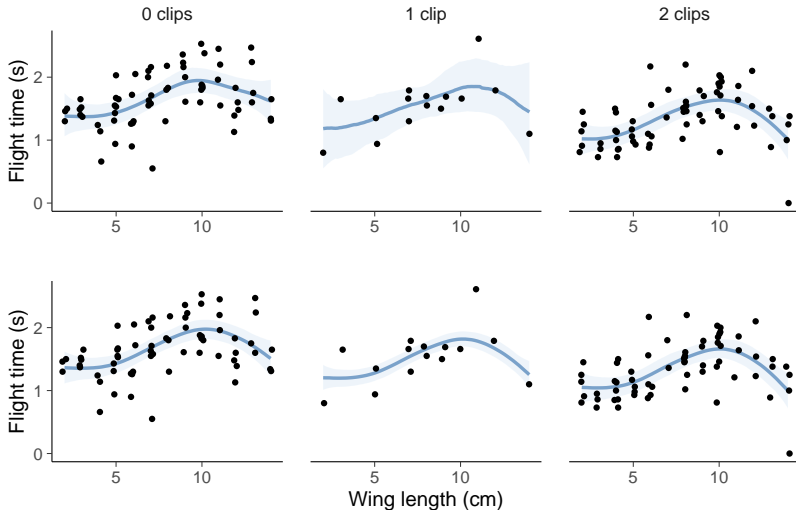
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Paper helicopters



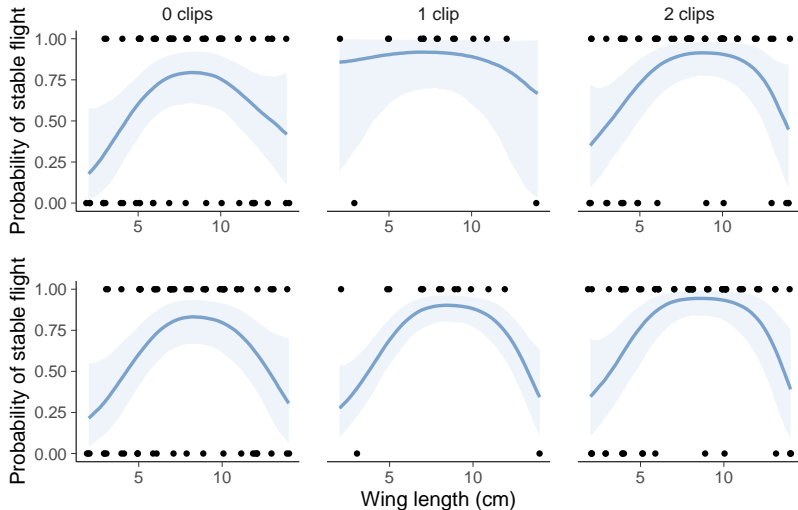
Paper helicopters: flight time

Separate model vs. hierarchical model



Paper helicopters: stability

Separate model vs. hierarchical model



Paper helicopters: brms

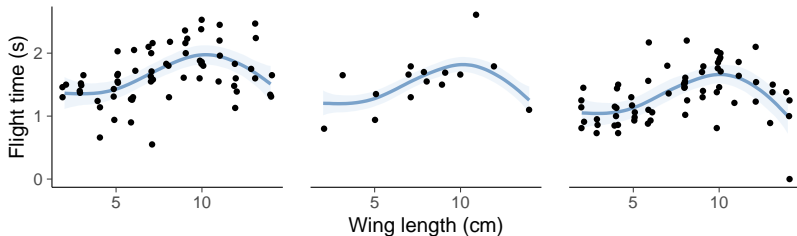
Flight time

```
flight_time ~ s(wing_length) + s(wing_length, by = nclips)
```

Paper helicopters: brms

Flight time

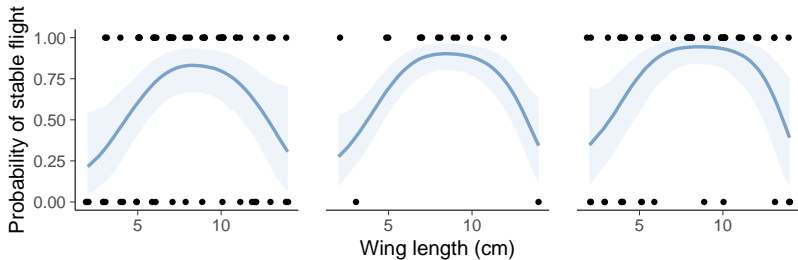
```
flight_time ~ s(wing_length) + s(wing_length, by = nclips)
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Paper helicopters: brms

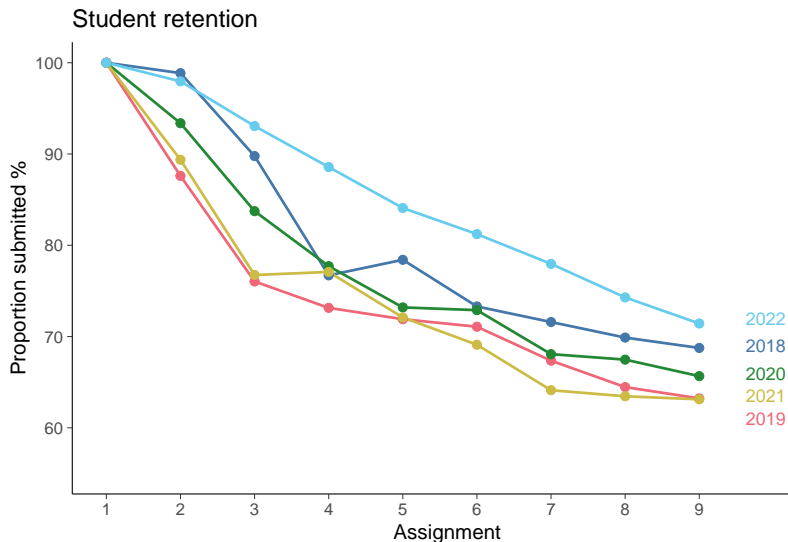
Stability

```
stable_flight ~ s(wing_length) + s(wing_length, by = nclips),  
family = bernoulli()
```

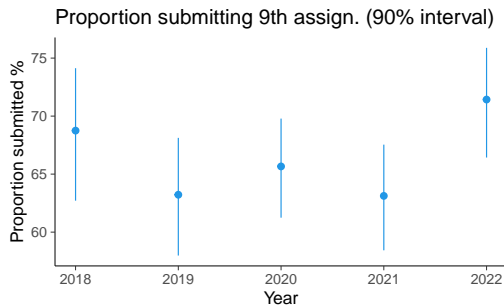


Student retention

Was year 2022 better than earlier year?

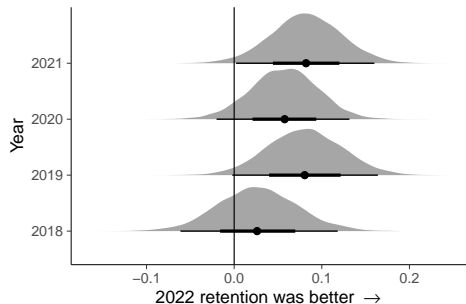
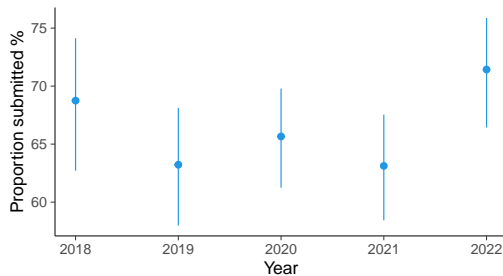


Student retention separate model



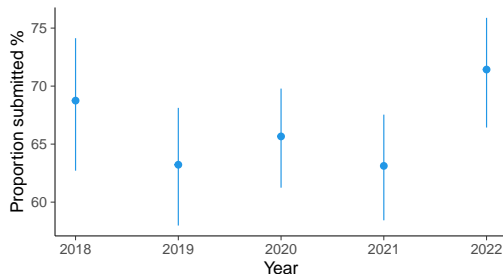
Student retention separate model

Proportion submitting 9th assign. (90% interval)

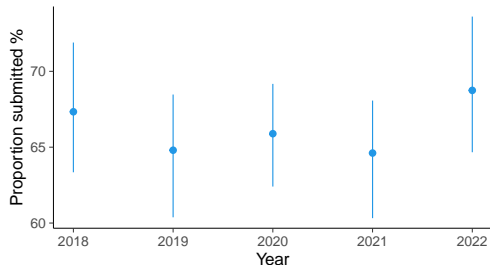


Student retention separate vs hierarchical model

Proportion submitting 9th assign. (90% interval)

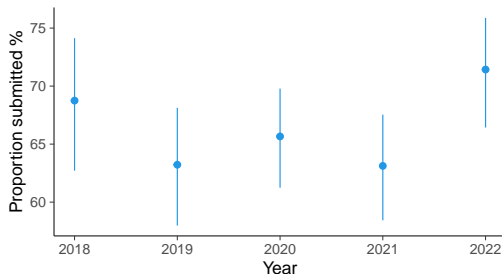


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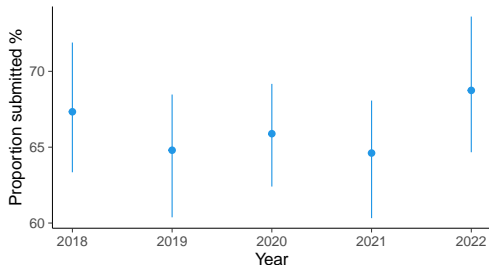


Student retention separate vs hierarchical model

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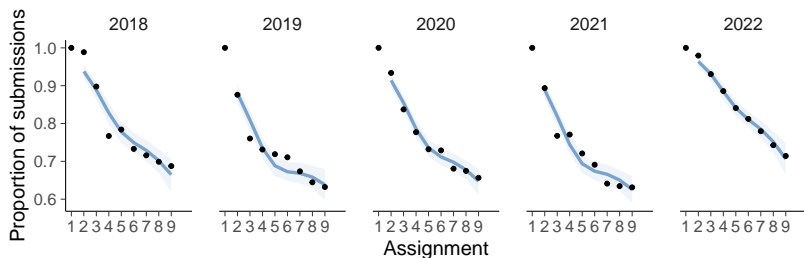


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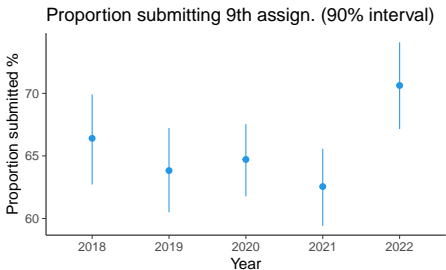
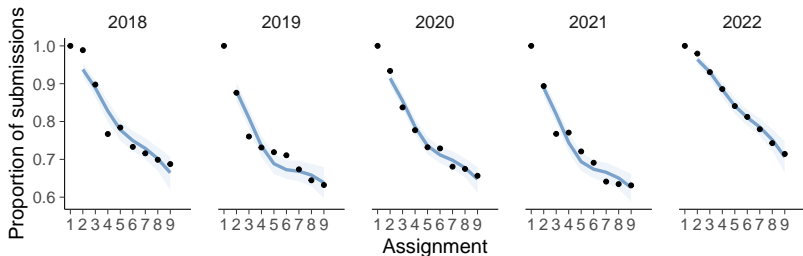
```
nstudents | trials(nstudents1) ~ 1 + (1 | year), family=binomial()
```

Student retention latent spline model

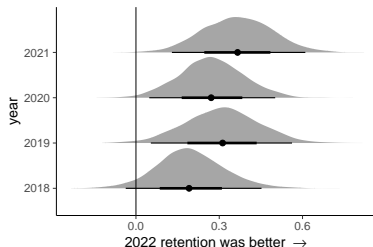
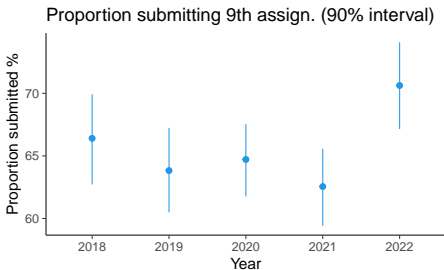
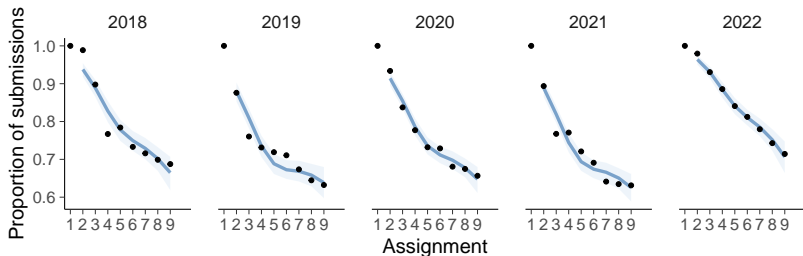


```
nstudents | trials(nstudents1) ~ s(assignment, k=4) + (assignment | year),  
family=binomial()
```

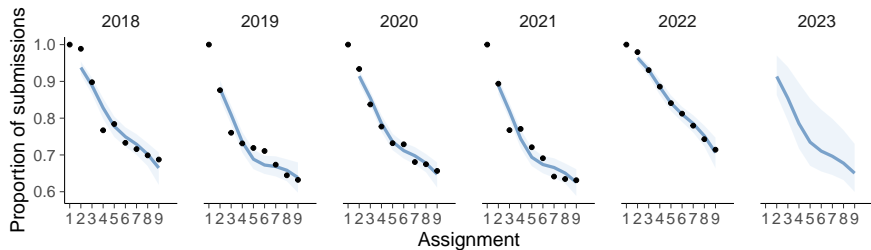
Student retention latent spline model



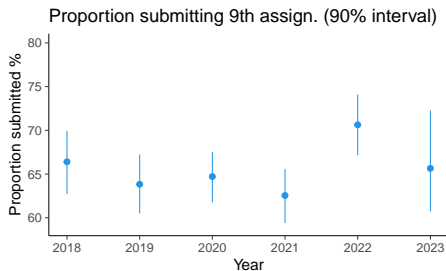
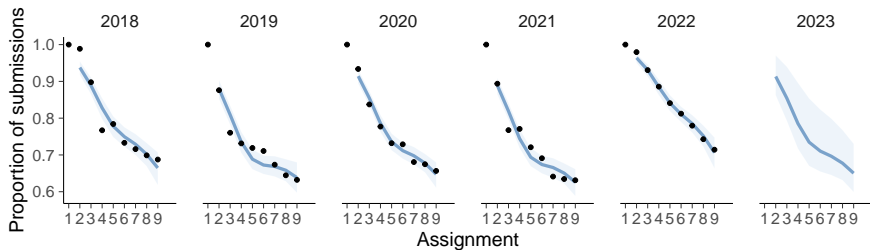
Student retention latent spline model



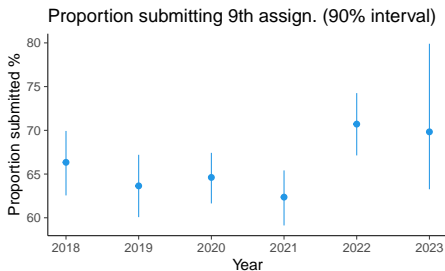
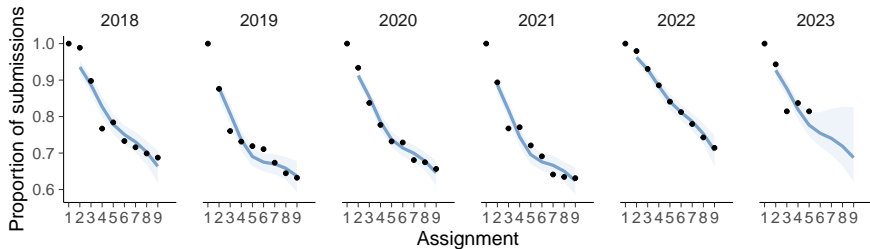
Student retention latent spline model, year 2023?



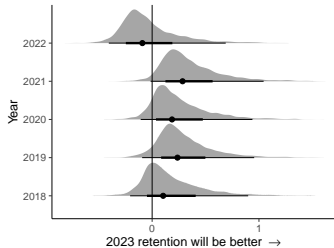
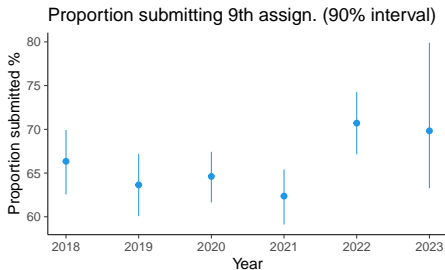
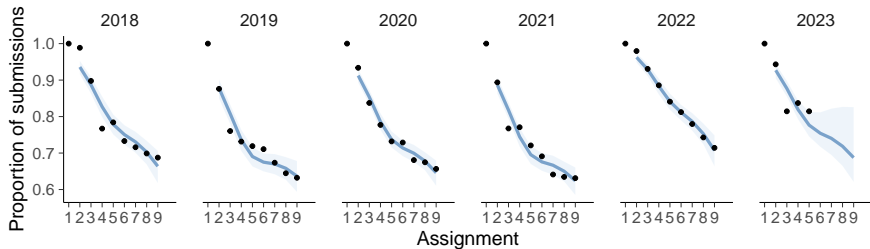
Student retention latent spline model, year 2023?



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Student retention latent spline model, year 2023?



Centered vs non-centered parameterization

HMC divergences are more likely when using hierarchical models

Centered parameterization

Hierarchical model code from the course demos

```
data {  
  int<lower=0> N;           // number of observations  
  int<lower=0> K;           // number of groups  
  array[N] int<lower=1, upper=K> x; // discrete group indicators  
  vector[N] y;             // real valued observations  
}
```

Centered parameterization

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  int<lower=0> K;           // number of groups  
  array[N] int<lower=1, upper=K> x; // discrete group indicators  
  vector[N] y;             // real valued observations  
}  
parameters {  
  real mu0;                // prior mean  
  real<lower=0> sigma0;     // prior std constrained to be pos.  
  vector[K] mu;            // group means  
  real<lower=0> sigma;      // common std constrained to be pos.  
}
```

Centered parameterization

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  vector[K] mu;            // group means  
  real<lower=0> sigma;     // common std constrained to be pos.  
}  
  
model {  
  mu0 ~ normal(10, 10);    // weakly informative prior  
  sigma0 ~ normal(0, 10);  // weakly informative prior  
  mu ~ normal(mu0, sigma0); // population prior with unknown param.  
  sigma ~ lognormal(0, .5); // weakly informative prior  
  y ~ normal(mu[x], sigma); // observation model  
}
```

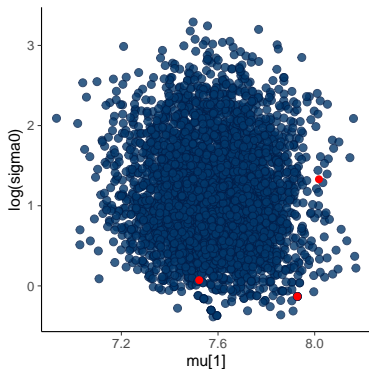
Centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

Centered parameterization

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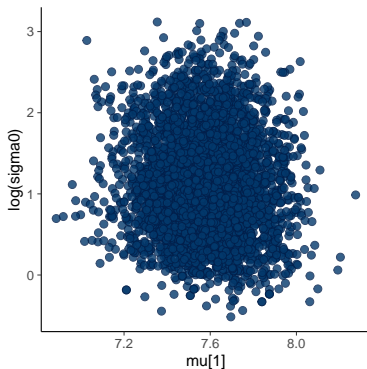
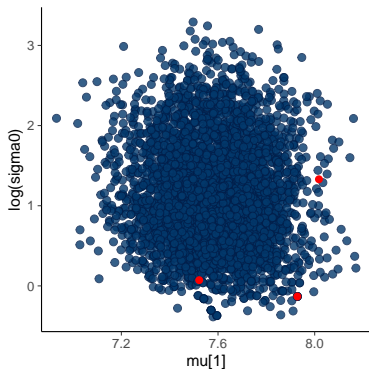
A few divergences that are not clustered.



Centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

And decreasing step size a little helps.



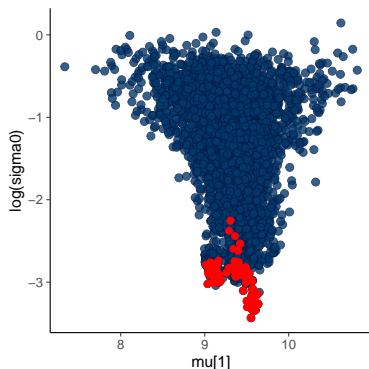
Centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

Centered parameterization

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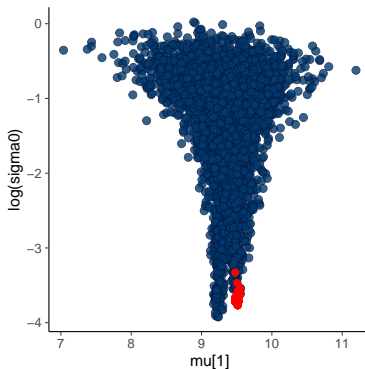
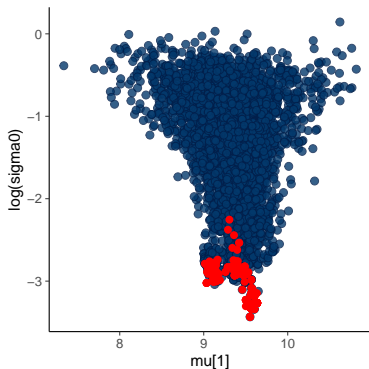
Many divergences that are clustered.



Centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

And decreasing step size doesn't remove the problem.



Non-centered parameterization

Transformation

```
parameters {  
  real mu0; // prior mean  
  real<lower=0> sigma0; // prior std constrained to be pos.  
  vector[K] z; // latent variable  
  real<lower=0> sigma; // common std constrained to be pos.  
}  
  
transformed parameters {  
  vector[K] mu = mu0 + sigma0 * z; // group means  
}  
  
model {  
  mu0 ~ normal(10, 10); // weakly informative prior  
  sigma0 ~ normal(0, 10); // weakly informative prior  
  z ~ normal(0, 1); // unit normal  
  sigma ~ lognormal(0, .5); // weakly informative prior  
  y ~ normal(mu[x], sigma); // observation model  
}
```

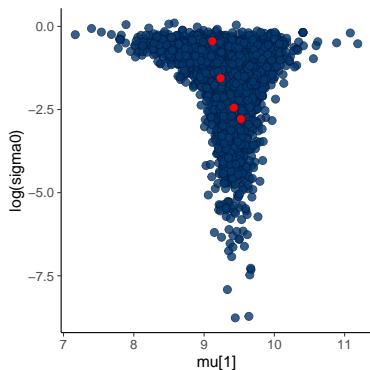
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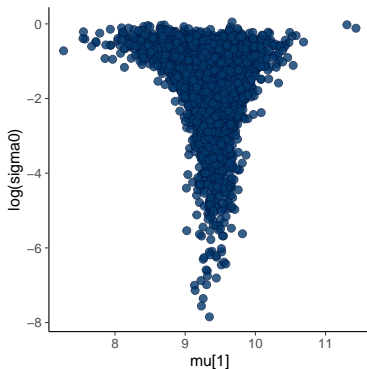
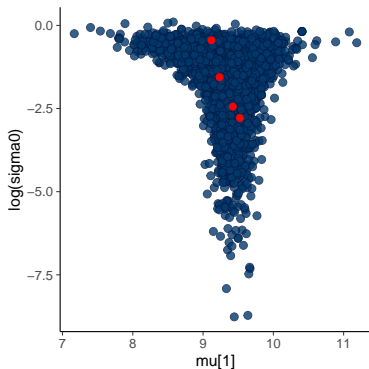
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Non-centered parameterization

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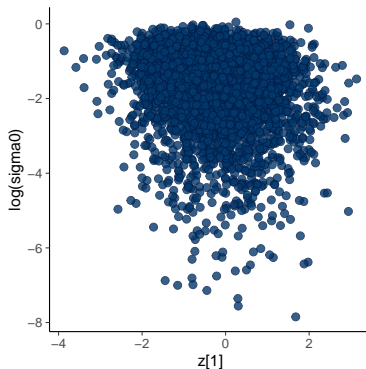
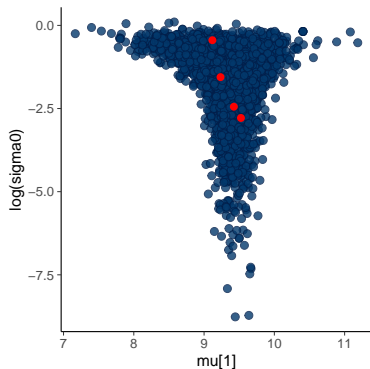
And decreasing step size a little helps.



Non-centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

Because we're actually sampling z and not μ



Non-centered parameterization

No free lunch

- non-centered parameterization is good when likelihood is weak
- non-centered parameterization is bad when likelihood is strong

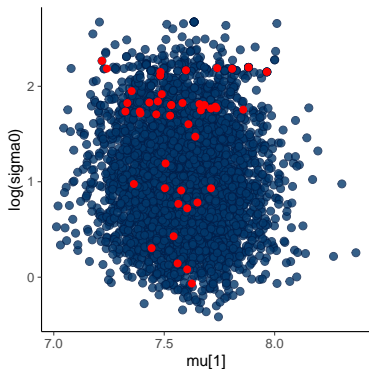
Non-centered parameterization

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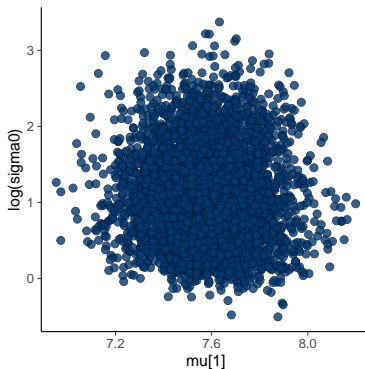
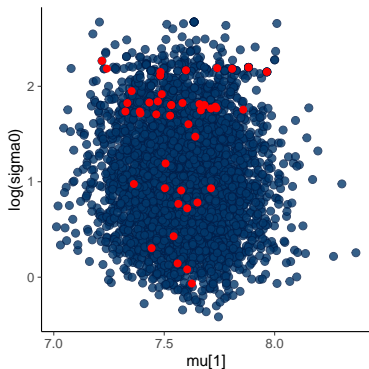
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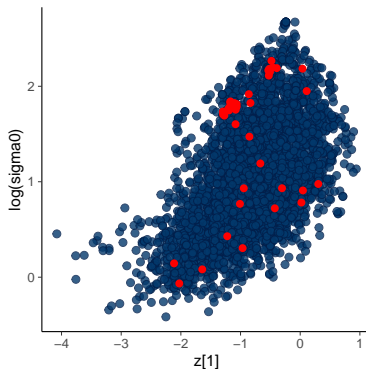
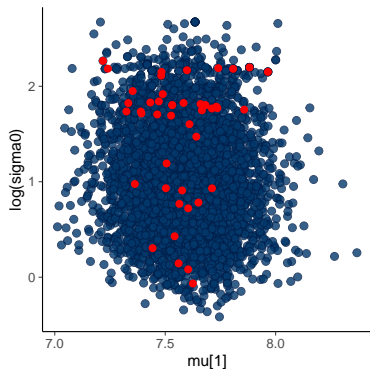
But decreasing step size a lot helps.



Non-centered parameterization

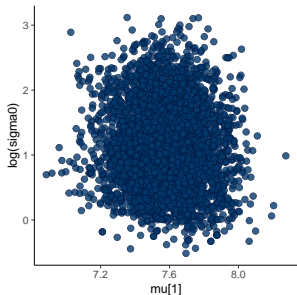
First data with many observations per group: 3 summer months with each having 71 observations.

Now the posterior for z is problematic.

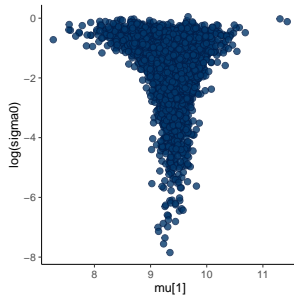
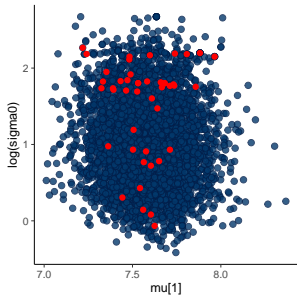
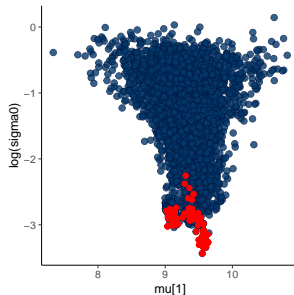


Centered vs. non-centered parameterization

Strong likelihood



Weak likelihood



brms and rstanarm

- brms and rstanarm use non-centered parameterization
 - as hierarchical models and Bayesian inference is most useful when likelihood is weak
- There can be need for both centered and non-centered parameterization in the same model
 - automation not easy, but research goes on

Exchangeability

- Justifies why we can use
 - a joint model for data
 - a joint prior for a set of parameters
- Less strict than independence

Exchangeability

- *Exchangeability*: Parameters $\theta_1, \dots, \theta_J$ (or observations y_1, \dots, y_J) are exchangeable if the joint distribution p is invariant to the permutation of indices $(1, \dots, J)$
- e.g.

$$p(\theta_1, \theta_2, \theta_3) = p(\theta_2, \theta_3, \theta_1)$$

- Exchangeability implies symmetry: If there is no information which can be used *a priori* to separate θ_j from each other, we can assume exchangeability. ("Ignorance implies exchangeability")

Exchangeability

- Exchangeability does not mean that the results of the experiments could not be different
 - e.g. if we know that the experiments have been in two different laboratories, and we know that the other laboratory has better conditions for the rats, but we do not know which experiments have been made in which laboratory
 - a priori experiments are exchangeable
 - model could have unknown parameter for the laboratory with a conditional prior for rats assumed to come from the same place (clustering model)

Exchangeability and additional information

- Example: bioassay
 - y_i number of dead animals are not exchangeable alone

Exchangeability and additional information

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Exchangeability and additional information

- Example: bioassay
 - y_i number of dead animals are not exchangeable alone
 - x_i dose is additional information
 - (x_i, y_i) exchangeable and logistic regression was used

$$p(\alpha, \beta \mid y, n, x) \propto \prod_{i=1}^n p(y_i \mid \alpha, \beta, n_i, x_i) p(\alpha, \beta)$$

Hierarchical exchangeability

- Example: hierarchical rats example
 - all rats not exchangeable

Hierarchical exchangeability

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 - all rats not exchangeable
 - in a single laboratory rats exchangeable

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 - laboratories exchangeable

Hierarchical exchangeability

- Example: hierarchical rats example
 - all rats not exchangeable
 - in a single laboratory rats exchangeable
 - laboratories exchangeable
 - → hierarchical model

Partial or conditional exchangeability

- Conditional exchangeability
 - if y_i is connected to an additional information x_i , so that y_i are not exchangeable, but (y_i, x_i) exchangeable use joint model or conditional model $(y_i \mid x_i)$.

Partial or conditional exchangeability

- Conditional exchangeability
 - if y_i is connected to an additional information x_i , so that y_i are not exchangeable, but (y_i, x_i) exchangeable use joint model or conditional model $(y_i \mid x_i)$.
- Partial exchangeability
 - if the observations can be grouped (a priori), then use hierarchical model

Exchangeability

- The simplest form of the exchangeability (but not the only one) for the parameters θ conditional independence

$$p(x_1, \dots, x_J \mid \theta) = \prod_{j=1}^J p(x_j \mid \theta)$$

Exchangeability - Counter example

- A six sided die with probabilities $\theta_1, \dots, \theta_6$
 - without additional knowledge $\theta_1, \dots, \theta_6$ exchangeable
 - due to the constraint $\sum_{j=1}^6 \theta_j$, parameters are not independent and thus joint distribution can not be presented as iid

Exchangeability

- See more examples in the BDA3 notes - Exchangeability vs. independence