Thanks zzq for his improvement on my original observation.

Given a multiplicative function $f$, we'd like to calculate its prefix sum $F(n) = \sum_{i=1}^{n} f(i)$.

For any multiplicative function $g$, there exists another multiplicative function $h$ satisfying that $f = g * h$ where $*$ denotes the Dirichlet convolution. Since they are multiplicative, it is enough to expand and check their values at prime powers: $f(p^e) = \sum_{i=0}^{e} g(p^i)h(p^{e-i})$ where $p$ is prime. Specifically, $f(p) = g(1)h(p) + g(p)h(1) = g(p) + h(p)$. Denote the prefix sum of $g$ as $G$, i.e., $G(n) = \sum_{i=1}^{n} g(i)$. We have $F(n) = \sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} \sum_{j|i} h(j)g(\frac{i}{j}) = \sum_{j=1}^{n} \sum_{k=1}^{\lfloor \frac{n}{j} \rfloor} h(j)g(k) = \sum_{j=1}^{n} h(j)G(\lfloor \frac{n}{j} \rfloor)$. If there is a multiplicative function $g$ such that the corresponding $h$ satisfies that $h(p) = 0$ at primes (or, equivalently, $f(p) = g(p)$), to calculate $F$ we only need to consider those $j$ which is a powerful number. (Explanation: if $j$ is not powerful, $h(j)$ is zero, thus they do not contribute to the sum and we can safely ignore them). Since there are $O(\sqrt{n})$ powerful numbers, if $G(n)$ can be calculated in time complexity $O(n^{\alpha})$, $F(n)$ can be calculated in time complexity $O(max(\sqrt{n}, n^{\alpha} \cdot \frac{\zeta(2\alpha)\zeta(3\alpha)}{\zeta(6\alpha)}))$ where $\zeta$ is the Riemann zeta function. In practice, it runs very fast.

A few examples:

1. $f(p^e) = p$: we can let $g(x) = x$. The final time complexity is $O(\sqrt{n})$.
2. $f(p^e) = e + 1$: we can let $g(x) = \sigma_0(x)$, the divisor counting function. The final time complexity is $O(\sqrt{n})$.
3. $f(p^e) = p^e - 1$: we can let $g(x) = \varphi(x)$, the Euler's totient function. The final time complexity is $O(n^{\frac{2}{3}})$.