Thanks zzq for his improvement on my original observation.

Given a multiplicative function f, we'd like to calculate its prefix sum $F(n) = \sum_{i=1}^{n} f(i)$.

For any multiplicative function g, there exists another multiplicative function h satisfying that $f = g * h$ where $*$ denotes the Dirichlet convolution. Since they are multiplicative, it is enough to expand and check their values at prime powers: $f(p^e) = \sum_{i=0}^{e} g(p^i)h(p^{e-i})$ where p is prime. Specifically, $f(p) = g(1)h(p) + g(p)h(1) = g(p) + h(p)$. Denote the prefix sum of g as G, i.e., $G(n) = \sum_{i=1}^{n} g(i)$. We have $F(n) = \sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} \sum_{j|i} h(j)g(\frac{i}{j}) = \sum_{j=1}^{n} \sum_{k=1}^{\lfloor\frac{n}{j}\rfloor} h(j)g(k) = \sum_{j=1}^{n} h(j)G(\lfloor\frac{n}{j}\rfloor)$. If there is a multiplicative function g such that the corresponding h satisfies that $h(p) = 0$ at primes (or, equivalently, $f(p) = g(p)$), to calculate F we only need to consider those j which is a powerful number. (Explanation: if j is not powerful, $h(j)$ is zero, thus they do not contribute to the sum and we can safely ignore them). Since there are $O(\sqrt{n})$ powerful numbers, if $G(n)$ can be calculated in time complexity $O(n^\alpha)$, $F(n)$ can be calculated in time complexity $O(max(\sqrt{n}, n^\alpha \cdot \frac{\zeta(2\alpha)\zeta(3\alpha)}{\zeta(6\alpha)}))$ where ζ is the Riemann zeta function. In practice, it runs very fast.

A few examples:

1. $f(p^e) = p$: we can let $g(x) = x$. The final time complexity is $O(\sqrt{n})$.
2. $f(p^e) = e + 1$: we can let $g(x) = \sigma_0(x)$, the divisor counting function. The final time complexity is $O(\sqrt{n})$.
3. $f(p^e) = p^e - 1$: we can let $g(x) = \varphi(x)$, the Euler's totient function. The final time complexity is $O(n^{\frac{2}{3}})$.

Note that if we let the multiplicative function $g(p^e) = f(p)^e$, the above method converts the problem "summing a multiplicative function" into a problem "summing a completely multiplicative function". We can just focus on the latter to solve the class of these problems.