Homework II: Models, Algorithms and Applications 3

Note for grader: On 04/05/2023, I asked Professor Duong right after class if I am able to submit the homework a day late, which is today (04/06/2023). He mentioned that he can make an exception for me, and gave me permission to do so.

I have sent the professor an email with the Homework in it because I was unable to send it through canvas, the submission was **locked**.

A

Abdallah Benelmadjat <abenelma@asu.edu>

to Duong

Apr 6, 2023, 4:16 PM (19 hours ago)

Greetings Professor,

Thank you for allowing me the opportunity yesterday to submit the homework today.
I truly appreciate it, and I cannot thank you enough for making this course enjoyable, allowing us the chance to ask questions in office hours, and learn so many diverse topics in game theory.

P.S: Here is the homework attached below, I was not able to submit it through Canvas.

Kind regards,

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Abdallah Benelmadjat

Graduate Student | M.S Energy Engineering | Modern Energy Production and Sustainable Use

School for Engineering of Matter, Transport and Energy | Arizona State University

Personal Website: https://abdallah-benelmadjat.github.io/portfolio/

The professor responded to me a day after (04/07/2023), saying the following:

D

Duong Nguyen

to me

7:18 AM (4 hours ago)

Well noted with thanks.

Best,
Duong

D

Duong Nguyen

to me

7:19 AM (4 hours ago)

PS: please upload it in the comments section under HW3 submission. Also, attach a screenshot of our conversation so that the grader is aware of this.

Thank you, I will do that. | Will do, thanks! | Will do, thank you!

So, he asked me to upload it instead in the HW3 submission comments, and link these emails so that you are aware that he gave me permission, and I won't be penalized for submitting it late.

Thank you for your understanding.

Problem I:

Does the following game have an exact potential function (1 point)?

Does it have an ordinal potential function (1 point)?

(Note that you must explain your solution, for example, how to find the potential.)

	B1	B2
A1	(3, 3)	(1, 5)
A2	(6, 1)	(4, 4)

Solution

1. Does the following game have an exact potential function?

From **lecture 25**, we have explored the concept of potential function, and have seen what is required for various statements to be proven true.

For the case of this problem, we're trying to know whether this game has an exact potential function.

Recall from **lecture 25:**

A function $\phi: S_1 \times \cdots \times S_n \to \mathbb{R}$ is an exact potential function if

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \phi(s_i, s_{-i}) - \phi(s_i', s_{-i}), \forall i, \forall s_i, s_i' \in S_i, \forall s_{-i} \in S_{-i}$$

	B1	B2
A1	(3, 3)	(1, 5)
A2	(6, 1)	(4, 4)

	B1	B2
A1	a	c
A2	b	d

(3, 1) – (6, 4) = (a, c) – (b, d) => a – b = - 3; c – d = -3

(3, 1) – (5, 4) = (a, b) – (c, d) => a – c = -2; b – d = -3

a – b = - 3; c – d = -3

a – c = -2; b – d = -3

From both, we get b - d = -2 and b - d= -3

Which leads to -2 = -3 which is impossible.

Thus, no exact potential.

2. Does the following game have an ordinal potential function?

From **lecture 25**, we have explored the concept of potential function, and have seen what is required for various statements to be proven true.

For the case of this problem, we're trying to know whether this game has an ordinal potential function.

Recall from **lecture 25:**

- A function $\phi: S_1 \times \cdots \times S_n \to \mathbb{R}$ is an ordinal potential function if
$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \ \textbf{iff} \ \phi(s_i, s_{-i}) > \phi(s_i', s_{-i}), \forall i, \forall s_i, s_i' \in S_i, \forall s_{-i} \in S_{-i}$$

We already have from the prior question:

	B1	B2
A1	(3, 3)	(1, 5)
A2	(6, 1)	(4, 4)

	B1	B2
A1	a	c
A2	b	d

If the game satisfies the following, it is having an ordinal potential function:

$$\phi(A1, B1) < \phi(A2, B1) \Rightarrow a < b$$
$$\phi(A2, B1) < \phi(A2, B2) \Rightarrow b < d$$
$$\phi(A2, B2) > \phi(A1, B2) \Rightarrow d > c$$
$$\phi(A1, B2) > \phi(A1, B1) \Rightarrow c > a$$

Let us see:

$3 < 6$

$1 < 4$

$4 > 1$

$5 > 3$

It satisfies; therefore, it has an ordinal potential function.

Problem II:

Find a pure Nash equilibrium of the congestion game given in Figure 1. Write down a table similar to the ones we study in class to express the convergence process. At the beginning all three players P1, P2, and P3 choose SAD. Note that you **only need to write down a table**. You can choose either **best** response dynamics or better response dynamics. Please explicitly state that your table expresses the best response dynamics process or better response dynamics process.

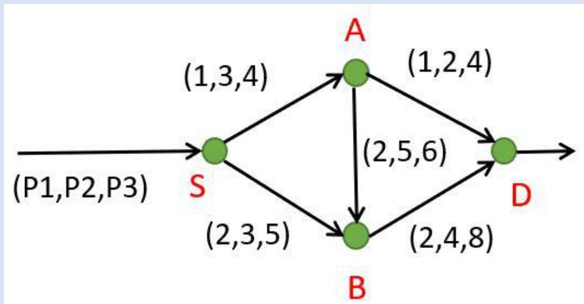


Figure 1: Problem 2

Solution

This solution follows the: Best Response Method

Firstly, with the **first iteration**, we fill out all the paths for P1, P2, P3 being the same path as detailed from the prompt. Since they're all traveling through the same route SA then AD, they take full time, which is 4 and 4, respectively.

	Path 1 path	Path 1 cost	Path 2 path	Path 2 cost	Path 3 path	Path 3 cost
Iteration 1	SAD	4+4	SAD	4+4	SAD	4+4

Second iteration,

Only Player 1 takes a different path. His two other options are either:
- SABD resulting: SA (taking the path with P2 and P3 resulting 4, then towards AB being alone resulting 2, then taking the path BD alone again with 2. Total = 4+2+2
- SBD resulting: SB being alone resulting 2, and BD alone resulting 2. Total = 2+2

We choose the best response here, since it just must be the BEST. Obviously, SBD is the best.

	Path 1 path	Path 1 cost	Path 2 path	Path 2 cost	Path 3 path	Path 3 cost
Iteration 1	SAD	4+4	SAD	4+4	SAD	4+4
Iteration 2	SBD	2+2	SAD	3+2	SAD	3+2

Third iteration,

Only Player 2 takes a different path. His two other options are either:
- SABD resulting: SA resulting 3, then towards AB resulting 2, then taking the path BD with 4. Total = 3+2+4
- SBD resulting: SB being with two players resulting 3, and BD with two players resulting 4. Total = 3+4

Or remaining same with the payoff: 3+2

We choose the best response here, since it just has to be the BEST. Obviously, remaining the same SAD is the best.

Can player 3 do better?

He has the same problem as player 2, therefore we have reached equilibrium.

There is no better option for the others, thus we have reached an optimal state for all players, and we stop here.

	Path 1 path	Path 1 cost	Path 2 path	Path 2 cost	Path 3 path	Path 3 cost
Iteration 1	SAD	4+5	SAD	4+5	SAD	4+5
Iteration 2	SBD	2+2	SAD	3+2	SAD	3+2

Problem III:

Consider a cost sharing game in Figure 2. The number associated with each edge is the cost of that edge. Each edge *e* has cost c_e, for example, $c_{AE} = 10$. Cost is split equally among all players taking edge *e*.

There are two players. Player 1 wants to go from A to E. Player 2 wants to go from B to F. A strategy for each player is the path from his source to destination (e.g., BF, BCDF for player 2). The strategy set Si of player *i* consists of all paths connecting his source node to his destination node. The cost for each player is the total cost of all edges in his chosen path $P_i \in Si$.

1) **How many pure Nash equilibria does this game have?** Justify your answer. Hint: what if both take direct paths? What if both take middle paths? What if one takes direct path and the other take middle path? (2 points)

2) Now, assume we have 3 players, P1 wants to go from A to E, and P2 and P3 want to go from B to F. **How many pure NE of this game? What are they? (2 points)**

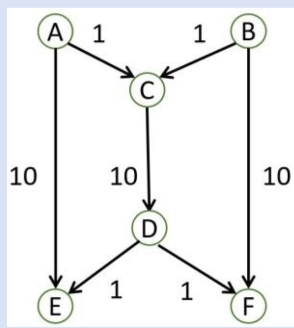


Figure 2: Problem 3

Solution

1) **How many pure Nash equilibria does this game have?**

I will be answering the relevant questions you have asked, and will give an answer to the primary question at the end:

- **What if both players take direct paths?**

We know that Player 1 wants to go from A to E. Player 2 wants to go from B to F.

If both take direct paths, then Player 1 will take the path AE.

Player 2 will take the path BF.

P1: AE (direct), c1 = 10
P2: BF (direct), c2 = 10

- **What if both take middle paths?**

We know that Player 1 wants to go from A to E. Player 2 wants to go from B to F. So, for both of them to take the middle path:

Player 1 will take the path AC, then CD, then DE → ACDE.

Player 2 will take BC then CD then DF → BCDF.

P1: AE (middle), $c1 = 1 + \frac{10}{2} + 1 = \frac{14}{2} = 7$
P2: BF (middle), $c2 = 1 + \frac{10}{2} + 1 = \frac{14}{2} = 7$

- **What if one player takes a direct path and the other takes middle path?**

We know that Player 1 wants to go from A to E. Player 2 wants to go from B to F. So, for one player to take a direct path and the other to take a middle path:

Player 1 will take the path AE.

Player 2 will take BC then CD then DF → BCDF.

P1: AE (direct), c1 = 10
P2: BF (middle), c2 = 1 + 10 +1 = 12

So, **how many pure Nash equilibria does this game have?**

Two Nash equilibria with the following strategy profile:

Strategy 1: both taking middle
P1: AE (middle), $c1 = 1 + \frac{10}{2} + 1 = \frac{14}{2} = 7$
P2: BF (middle), $c2 = 1 + \frac{10}{2} + 1 = \frac{14}{2} = 7$

Strategy 2: both taking direct paths
P1: AE (direct), c1 = 10
P2: BF (direct), c2 = 10

2) **Now, assume we have 3 players, P1 wants to go from A to E, and P2 and P3 want to go from B to F.**

- **What if all players take direct paths?**

We know that P1 wants to go from A to E, and P2 and P3 want to go from B to F.

P1: AE (direct), c1 = 10
P2: BF (direct), $c2 = \frac{10}{2} = 5$
P2: BF (direct), $c3 = \frac{10}{2} = 5$

Total cost = 20

- **What if all players take middle paths?**

P1: AE (middle), c1 = 1 + $\frac{10}{3}$ +1 = $\frac{3+10+3}{3}$ = 5.333
P2: BF (middle), c2 = $\frac{1}{2}$ + $\frac{10}{3}$ +$\frac{1}{2}$ = $\frac{13}{3}$ = 4.333
P3: BF (middle), c3 = $\frac{1}{2}$+ $\frac{10}{3}$ +$\frac{1}{2}$ = $\frac{13}{3}$ = 4.333

Total cost = 13.999

- **What if one player takes a direct path and the others takes a middle path?**

P1: AE (direct), c1 = 10
P2: BF (middle), c2 = $\frac{1}{2}$ + $\frac{10}{2}$ +$\frac{1}{2}$ = $\frac{12}{2}$ = 6
P2: BF (middle), c3 = $\frac{1}{2}$ + $\frac{10}{2}$ +$\frac{1}{2}$ = $\frac{12}{2}$ = 6

Total cost = 22

- **What if another player takes a direct path and the others takes a middle path?**

P1: AE (middle), c1 = 1 + $\frac{10}{2}$ +1 = $\frac{14}{2}$ = 7
P2: BF (middle), c2 = 1 + $\frac{10}{2}$ +1 = $\frac{14}{2}$ = 7
P2: BF (direct), c3 = 10 = 10

Total cost = 24

So, **how many pure Nash equilibria does this game have?**

I don't know for sure, but I know for certain that the strategy where all players take the middle is a NE.

Strategy 1: All players taking middle path

P1: AE (middle), c1 = 1 + $\frac{10}{3}$ +1 = $\frac{3+10+3}{3}$ = 5.333
P2: BF (middle), c2 = $\frac{1}{2}$ + $\frac{10}{3}$ +$\frac{1}{2}$ = $\frac{13}{3}$ = 4.333
P3: BF (middle), c3 = $\frac{1}{2}$+ $\frac{10}{3}$ +$\frac{1}{2}$ = $\frac{13}{3}$ = 4.333

Total cost = 13.999

Strategy 2: All players take direct path (**UNCERTAIN**)

P1: AE (direct), c1 = 10
P2: BF (direct), c2 = $\frac{10}{2}$ = 5
P2: BF (direct), c3 = $\frac{10}{2}$ = 5

Total cost = 20

Problem IV:

Consider the ferry and road problems. Now here we have 200 players. The delay on each road depends on the number of cars travelling on that road, which is 15 + 0.1N (min), where N is the number of cars on the same road.

1) See Figure 3. **Write down the cost function of each player. Is this a potential game? Is this a congestion game? What are pure Nash equilibria? What is the travel time of each player at the pure Nash equilibrium?** (2 points)

2) See Figure 4. Assume a bridge is build, to help reduce trafficc. The travel time over the bridge is 0. **Find the new Nash equilibrium and travel time of each player at the new NE**. (2 points).

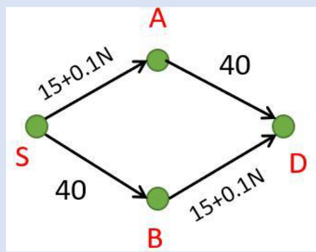


Figure 2: Problem 4, no bridge

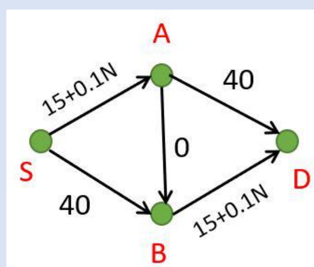


Figure 3: Problem 4, with bridge

Solution

---- Part I --

Write down the cost function of each player:

In this game, the cost function represents the delay experienced by each player when choosing a specific road.

Cost function for SAD = 15 + 0.1N + 40

Cost function for SBD = 40 + 15 + 0.1N

Is this a potential game?

The following game is a **congestion** game. As we have seen in class, congestion is also a potential game. So, we only need to prove that it is a congestion game:

Since it has a congestive component, it makes the whole game a congestion game for the following reasons:

- Payoff is higher if a player uses less of a congested resource aka **Monotonicity.**
- Payoff for each player depends only on the total amount of resources used by all players and not on which particular resources are used by each player.
- Each player has a set of strategies that correspond to how much of each resource they will use
- There is a set of resources that multiple players can use.

What are pure Nash equilibria?

There are only two possible strategies in this game: i={SAD, SBD}

To find the **Nash equilibria**, we need to find the strategy profile where no player has the incentive to change his strategy. Otherwise, if one can, it is not **Nash equilibria**.

Let us propose a strategy profile:

100 take the path {SAD}, with a cost of:

$$(15+0.1N)+40$$

Replacing N with the number of players taking the path:

$$(15+0.1*100)+40$$

$$65\text{min}$$

Cost of 100 player taking path SAD = 65min

100 take the path {SBD}, with a cost of:

$$40+(15+0.1N)$$

Replacing N with the number of players taking the path:

$$40+(15+0.1*100)$$

$$65\text{min}$$

Cost of 100 player taking path SBD = 65min

Let's see if any players have the incentive to deviate or not:

101 take the path {SAD}, with a cost of:

$$(15+0.1N)+40$$

Replacing N with the number of players taking the path:

(15+ 0.1*101)+ 40

65.1min

Cost of 100 player taking path SAD = 65.1min

99 take the path {SBD}, with a cost of:

40+ (15+ 0.1N)

Replacing N with the number of players taking the path:

40+ (15+ 0.1*99)

64.9min

Cost of 100 player taking path SBD = 64.9min

Since we need to find the strategy profile where no player has the incentive to change his strategy for it to be a **Nash equilibria**, therefore this strategy profile is a **Nash equilibria**.

With 100 taking the path SAD, and 100 taking the path SBD.

In fact, all the strategy profiles that have 100 player taking the path SAD, and 100 taking the path SBD are **Nash equilibria.**

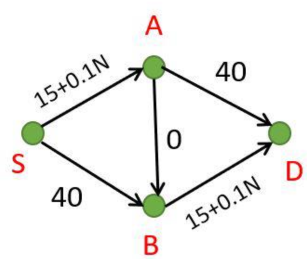
What is the travel time of each player at the pure Nash equilibrium?

Calculated above:

Cost of 100 player taking path SAD = 65min

Cost of 100 player taking path SBD = 65min

---- **Part II** ---



Find the new Nash equilibrium and travel time of each player at the new NE

A bridge was built, now there are 3 strategies:

SAD with its cost = 15 + 0.1N + 40 = 55 + 0.1N

SBD with its cost = 40 + 15 + 0.1N = 55 + 0.1N

SABD with its cost = 15 + 0.1N + 15 + 0.1N = 30 + 0.2N

Let's propose this strategy profile that all 200 players take the path of the new bridge SABD.

SABD with its cost = 30 + 0.2*200 = 70min

Can any player deviate? If not, this would be the only **Nash equilibrium**.

Let's verify:

Suppose 1 player goes a different path SAD:

Since he will not be alone from the path SA, he is counted with the other 200 players.

This would be his cost 15 + 0.1N + 40 = 15 +0.1*200 + 40 = 75min

75min is higher cost than 70min, and much higher than the cost of before the bridge was built.

There is no incentive for any player to deviate, therefore, all players taking the new path SABD is the only **Nash equilibrium.**

2)

We need to first find the social optimum cost as we have explored in **Lecture 29**.

For without bridge:

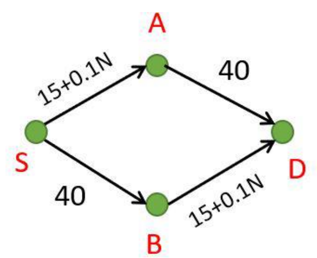


Figure 4: Problem 4, no bridge

We work with this formula:

$$POA = \frac{Total\ cost\ of\ worst\ NE}{Optimal\ Social\ Cost}$$

Total cost of NE = 100*(55 + 0.1*100) + 100*(55 + 0.1*100) = 13000

Optimal Social Cost = 100*(55 + 0.1*100) + 100*(55 + 0.1*100) = 13000

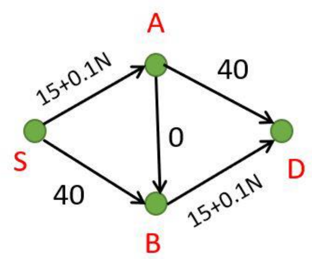
$$POA = \frac{13000}{13000}$$

Indicating that the system's efficiency loss due to selfish behavior is 0%.

Which is very efficient.

For case with bridge:

We have a different NE, which is SABD.



Total cost of NE = 200*(15+0.1*200)+200*(15+0.1*200) = 14000

Min x x(15+0.1x)+(x-200)*(15+0.1(x-200))

Optimal Social Cost = 100*(55 + 0.1*100) + 100*(55 + 0.1*100) = 13000

$$POA = \frac{14000}{13000} = 1.076 = 107.6\%$$

Indicating that the system's efficiency loss due to selfish behavior is 7.6%.

Problem V:

Consider the game shown in Figure 5. There are M players going from S to D. The delay on each road segment depends on the number of players choosing it. For example, the delay on segment SA is 15 + N 10 (min) where N indicates the number of players on that segment (e.g., if there are 50 players drive through segment SA, the delay on SA is 15 + 50/10 = 20 minutes).
1) Write and submit your (Matlab, Python, Julia) code for the best/better-response dynamics algorithm showing the convergence of this game where the x-axis is the number of iterations and the y-axis is the value of the potential function of this game for the three settings: M = 100, M = 150, and M = 200 (2 point). List one pure Nash equilibrium for each setting (i.e., list of possible paths from S to D and show how many players choose each path). (1 point)

2) (Simultaneous update) Consider M= 200. In the BRD algorithm, the players update their strategies sequentially. In other words, in each iteration, only one player updates his strategy. Repeat the same experiment, but this time, let players update their strategies simultaneously (i.e., every player updates his action (if there exists a better strategy for him) at each iteration). How are the results different from the previously discussed sequential case? Does it converge now? (3 point)

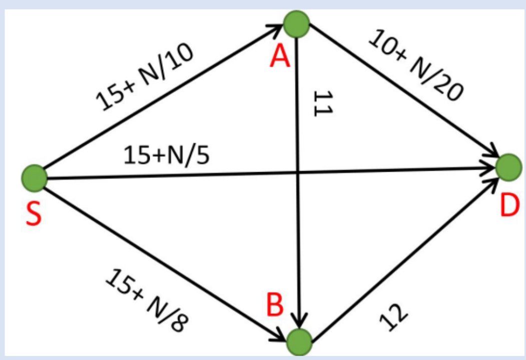


Figure 5: Problem 5

To view MATLAB task, visit (click on main.m afterwards):
https://github.com/abdallah-benelmadjat/Best-response-dynamics-algorithm-

Language used: MATLAB
MATLAB Version : R2022b
Method used: Best Response Dynamics

1) **Write and submit your (Matlab, Python, Julia) code for the best/better-response dynamics algorithm showing the convergence of this game where the x-axis is the number of iterations and the y-axis is the value of the potential function of this game for the three settings: M = 100, M = 150, and M = 200. List one pure Nash equilibrium for each setting (i.e., list of possible paths from S to D and show how many players choose each path).**

Again, code can be accessed here on GitHub: main.m

- **For M = 100**

All starting players choose SAD.

Meaning, all players go through path SA, then AD.

Results:

Iterations = 242

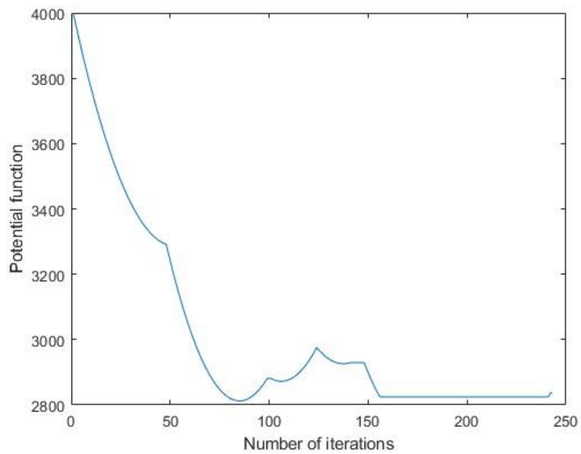


Figure 6: Convergence plot for M=100

Pathing:

SAD = 22 players with a cost of **28.3min** for each player

SD = 67 players with a cost of **28.4min** for each player

SABD = 0 players

SBD = 11 players with a cost of **28.375min** for each player

Detailed Pathing:

SA = 22

AD = 22

SD = 67

AB = 0

SB = 11

BD = 11

- **For M = 150**

All starting players choose SAD.

Meaning, all players go through path SA, then AD.

Results:

Iterations = 352

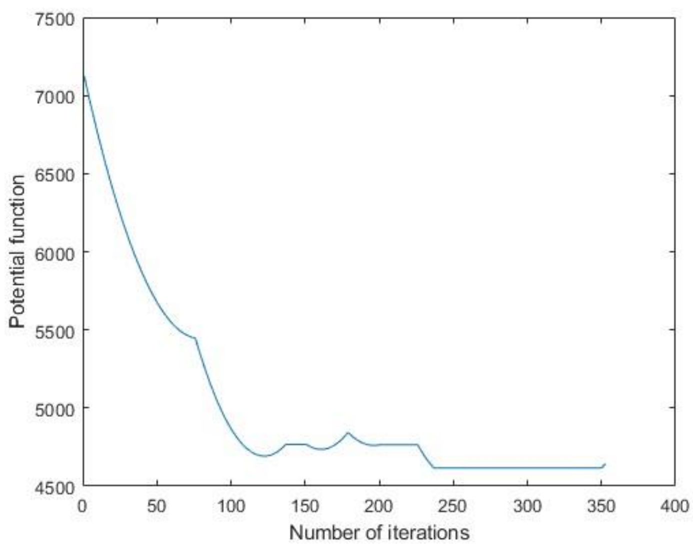


Figure 7: Convergence for M=150

Pathing:

SAD = 39 players with a cost of **30.85min** for each player

SD = 80 players with a cost of **31min** for each player

SABD = 0 players

SBD = 31 players with a cost of **30.875min** for each player

Detailed Pathing:

SA = 39

AD = 39

SD = 80

AB = 0

SB = 31

BD = 31

- **For M = 150**

All starting players choose SAD.

Meaning, all players go through path SA, then AD.

Results:

Iterations = 522

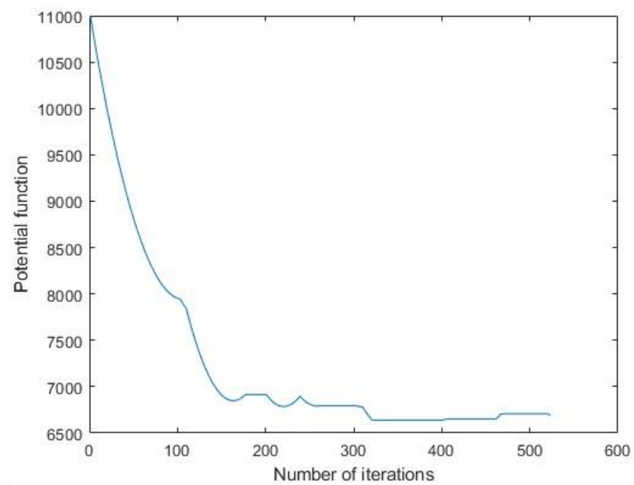


Figure 8: Convergence for M=150

Pathing:

SAD = **56 players** with a cost of **33.4m** for each player

SD = **92 players** with a cost of **33.4m** for each player

SABD = 0 players

SBD = **52 players** with a cost of **33.5m** for each player

Detailed Pathing:

SA = 56

AD = 56

SD = 92

AB = 0

SB = 52

BD = 52

2) (Simultaneous update) Consider M= 200. In the BRD algorithm, the players update their strategies sequentially. In other words, in each iteration, only one player updates his strategy. Repeat the same experiment, but this time, let players update their strategies simultaneously (i.e., every player updates his action (if there exists a better strategy for him) at each iteration). How are the results different from the previously discussed sequential case? Does it converge now? (3 point)

To view this code, check my GitHub repo at https://github.com/abdallah-benelmadjat/Best-response-dynamics-algorithm- and select alternate.m

Note: All the players in my code start in SAD.

- **How are the results different from the previously discussed sequential case? Does it converge now?**

The potential function is increasing then decreasing indefinitely. This **oscillating** behavior is indicative of lack of convergence, and it merely represents the chaotic nature of all players picking the same path together at the same time.

The results are terrible, and they don't show any convergence towards an equilibrium.

All players choose the same pathing at the same time when they choose simultaneously.

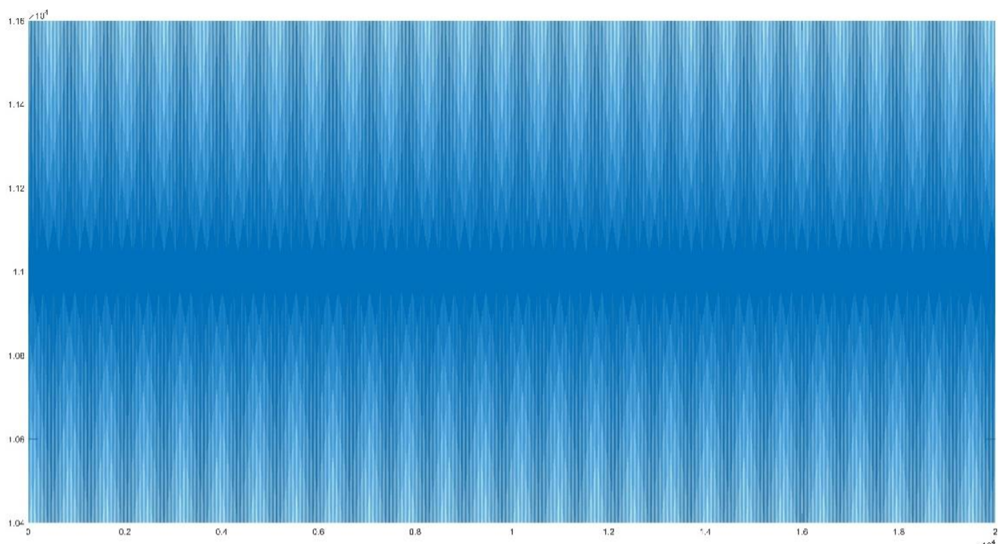


Figure 9: Sign of lack of convergence from oscillating

This obviously does not converge, but to give a more detailed analysis, how are the results different really?

They are different in that all the players have the same cost at the same time, at all times (when they begin in the same path together). They are different because all players also choose the same path. **Thus, all players act technically as a single player.**

Without a loop breaker or a limit, the loop will never break, because **there would never be an equilibrium**.

Here is a much closer look at what is happening with the cost of all players.