A comparative analysis of Galerkin's finite element method (GFEM) and Streamline upwind/Petrov-Galerkin (SUPG) methods for convection-diffusion equations

Anar Abdullayev

Department of Mathematics, Middle East Technical University

June 11, 2024

Abstract

In this paper, both Galerkin's finite element method (GFEM) and the streamline upwind/Petrov-Galerkin methods (SUPG) for convection-dominated convection-diffusion equations will be examined. These methods will be introduced separately by discussing their theoretical approaches, followed by numerical experiments to compare them. The experiments will highlight the undesired oscillations in GFEM and the stabilized behavior achieved with SUPG.

1. Problem formulation

Throughout this paper, we consider the following convection-dominated convection-diffusion equation with a nonhomogeneous Dirichlet boundary condition:

$$\begin{aligned} -\varepsilon\Delta\tilde{u} + b\cdot\nabla\tilde{u} &= f \qquad \text{in } \Omega, \\ \tilde{u} &= u_b \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ (resp. $\mathbb{R}^3$) is a polygonal domain (a polyhedral domain), ε is a positive diffusivity constant, $b \in W^{1,\infty}(\Omega)^2$ (resp. $W^{1,\infty}(\Omega)^3$) is the convection coefficient with the incompressibility condition $\nabla \cdot b = 0$, $f \in L^2(\Omega)$ is a source function, and $u_b \in H^{1/2}(\partial\Omega)$ is the Dirichlet boundary condition.

Let $\tilde{u}$ be the solution of the problem (1) such that $\tilde{u} = u + u_p$, where u is the solution of the corresponding homogeneous problem and u_p is a particular solution. The trace of u_p on $\partial\Omega$ is the function u_b, and the function u_p is called a *Dirichlet lift* of u_b in Ω. Then u is the solution of the following homogeneous problem:

$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f + \varepsilon\Delta u_p - b\cdot\nabla u_p \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2}$$

2. The Galerkin Finite Element Method (GFEM)

Let us find a weak (or variational) formulation of the convection-diffusion equation. Define a variational space that incorporates the essential (i.e., Dirichlet) boundary condition.

$$V := H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

Multiply both sides of the equation by a test function $v \in V$ and integrate over the domain Ω.

$$\int_\Omega (\varepsilon(\Delta\tilde{u})v + (b\cdot\nabla\tilde{u})v)\,dx = \int_\Omega fv\,dx.$$

Once the Green's first identity is used, we obtain

$$\varepsilon \int_\Omega \nabla \tilde{u} \cdot \nabla v \, dx - \varepsilon \int_{\partial\Omega} \frac{\partial \tilde{u}}{\partial \vec{n}} v \, ds + \int_\Omega (b \cdot \nabla \tilde{u}) v \, dx = \int_\Omega f v \, dx.$$

Since $v \in H_0^1(\Omega)$, we have

$$\varepsilon \int_\Omega \nabla \tilde{u} \cdot \nabla v \, dx + \int_\Omega (b \cdot \nabla \tilde{u}) v \, dx = \int_\Omega f v \, dx,$$

or equivalently,

$$\varepsilon \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega (b \cdot \nabla u) v \, dx = \int_\Omega f v \, dx - \varepsilon \int_\Omega \nabla u_p \cdot \nabla v \, dx - \int_\Omega (b \cdot \nabla u_p) v \, dx,$$

where $u_p \in H^1(\Omega)$. Then, the weak formulation of the problem (2) is as follows: find $u \in V$ such that

$$a(u, v) = l(v), \quad \forall v \in V, \tag{3}$$

where

$$\begin{aligned} a(u, v) &= \varepsilon \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega (b \cdot \nabla u) v \, dx = \varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u, v), \\ l(v) &= \int_\Omega f v \, dx - \varepsilon \int_\Omega \nabla u_p \cdot \nabla v \, dx - \int_\Omega (b \cdot \nabla u_p) v \, dx = (f, v) - a(u_p, v). \end{aligned}$$

Note that the weak formulation of the problem (1) is given as: find $\tilde{u} \in H^1(\Omega)$ such that $u \in H_0^1(\Omega)$ and

$$\varepsilon(\nabla \tilde{u}, \nabla v) + (b \cdot \nabla \tilde{u}, v) = (f, v), \quad \forall v \in V.$$

Let us now verify that there exists a unique weak solution $u \in H_0^1(\Omega)$.

Theorem 1. *There exists a unique $u \in V$ satisfying* (3).

Proof. The variational space V is a Hilbert space with the inner product

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Now, let us show that $l : V \to \mathbb{R}$ is a continuous linear functional. For any $\alpha, \beta \in \mathbb{R}$ and $v_1, v_2 \in V$, we have

$$\begin{aligned} l(\alpha v_1 + \beta v_2) &= (f, \alpha v_1 + \beta v_2) - a(u_p, \alpha v_1 + \beta v_2) = \\ &= \alpha (f, v_1) + \beta (f, v_2) - \alpha a(u_p, v_1) - \beta a(u_p, v_2) = \\ &= \alpha l(v_1) + \beta l(v_2), \end{aligned}$$

which implies that $l(\cdot)$ is a linear functional. As $l(\cdot)$ is a linear functional, showing its boundedness suffices to establish continuity. Using the Cauchy-Schwarz inequality, the continuity of $a(\cdot, \cdot)$, and the Sobolev embedding theorem, we get the following inequality.

$$\begin{aligned} |l(v)| = \left| \int_\Omega (f, v) - a(u_p, v) \right| &\le \left| \int_\Omega (f, v) \right| + |a(u_p, v)| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|u_p\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\le \left(\|f\|_{L^2(\Omega)} + \|u_p\|_{H^1(\Omega)} \right) \|v\|_{H^1(\Omega)} = M \|v\|_{H^1(\Omega)}, \end{aligned}$$

where $M = \|f\|_{L^2(\Omega)} + \|u_p\|_{H^1(\Omega)} > 0$. Since $l(\cdot)$ is found to be bounded, it is continuous.

Next, we will show that $a : V \times V \to \mathbb{R}$ is a continuous and coercive bilinear form. Notice that for any fixed $w \in V$, $a(w, v)$ is linear for all $v \in V$. Similarly, for any fixed $w \in V$, $a(u, w)$ is linear for all $u \in V$. So, $a(\cdot,\cdot)$ is a bilinear functional on $V \times V$. The bilinear functional $a(\cdot)$ is continuous, as shown below.

$$|a(u,v)| = \left|\varepsilon \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega (b \cdot \nabla u) v \, dx\right| \leq \varepsilon \left|\int_\Omega \nabla u \cdot \nabla v \, dx\right| + \left|\int_\Omega (b \cdot \nabla u) v \, dx\right| \leq$$
$$\leq \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|b \cdot \nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq c_{\text{cont}} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

To show that a bilinear form $a(\cdot,\cdot)$ is coercive on V, we need to find a positive constant c_{coer} such that $a(v,v) \geq c_{\text{coer}} \|v\|_V^2$ for all $v \in V$.

$$a(v,v) = \varepsilon \int_\Omega \nabla v \cdot \nabla v \, dx + \int_\Omega b \cdot (\nabla v) v \, dx = \varepsilon |\nabla v|_{H^1(\Omega)} + \int_\Omega b \cdot (\nabla v) v \, dx.$$

To proceed further, notice that $v(\nabla v) = \frac{1}{2}\nabla(v^2)$ and use integration by parts to obtain the following equality.

$$a(v,v) = \varepsilon |v|^2_{H^1(\Omega)} + \frac{1}{2} \int_\Omega b \cdot \nabla(v^2) \, dx =$$
$$= \varepsilon |v|^2_{H^1(\Omega)} + \frac{1}{2} \int_{\partial\Omega} v^2 (b \cdot \vec{n}) \, ds - \frac{1}{2} \int_\Omega v^2 (\nabla \cdot b) \, dx.$$

Since $v \in H^1_0(\Omega)$ and $\nabla \cdot b = 0$ (incompressibility condition), we have

$$a(v,v) = \varepsilon |v|^2_{H^1(\Omega)}.$$

Once the Poincaré inequality is applied, we obtain

$$a(v,v) = \varepsilon |v|^2_{H^1(\Omega)} = \frac{\varepsilon}{2} |v|^2_{H^1(\Omega)} + \frac{\varepsilon}{2} |v|^2_{H^1(\Omega)} \geq \frac{\varepsilon}{2} |v|^2_{H^1(\Omega)} + \frac{\varepsilon}{2c_{\text{P}}^2} \|v\|^2_{L^2(\Omega)} \geq c_{\text{coer}} \|v\|^2_{H^1(\Omega)}, \tag{4}$$

where $c_{\text{coer}} = \frac{\varepsilon}{2+2c_{\text{P}}^2}$. The inequality (4) shows that $a(\cdot,\cdot)$ is coercive.

Since all conditions of the Lax-Milgram theorem are satisfied, then there exists a unique $u \in V$ satisfying (3). □

Let V_h be a finite-dimensional subspace of V. Then, the Galerkin finite element discretization of (2) is as follows: find $u_h \in V_h$ such that

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$

Then, the desired discrete solution is $\tilde{u}_h = u_h + u_p$, where u_h is the discrete solution of the homogeneous problem (2), and u_p is a particular solution of (1).

3. The Streamline upwind Petrov-Galerkin Method (SUPG)

If convection dominates diffusion (i.e., $\varepsilon \ll |b|$) in (1), then the Galerkin method lacks stability. In this case, the solution of (1) exhibits interior and boundary layers, which are small subregions where the solution's derivatives are extremely large. The widths of these layers are typically much smaller than the mesh size, making them difficult to resolve accurately. Consequently, this results in undesirable spurious (nonphysical) oscillations in the numerical solution (John and Knobloch [4]). To address this, a stabilization term was proposed by Brooks and Hughes [1], known as the SUPG method. In this method, the space V_h is modified to be at least of class H^2. Additionally, the requirement that functions in V_h are continuous across element edges (resp. faces) is dropped to include nonconforming finite element spaces in the variational formulation. Then, the finite-dimensional space V_h satisfies

$$V_h \subset \{v \in L^2(\Omega) : v|_K \in C^\infty(\tilde{K}) \quad \forall K \in \mathcal{T}_h\}.$$

The residual for (2)

$$R_h(u) = -\varepsilon\Delta_h u + b\cdot\nabla_h u - f - \varepsilon\Delta u_p + b\cdot\nabla u_p$$

is well-defined for $u, v \in V_h$. Then, the SUPG method is as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + (R_h(u_h), \tau b\cdot\nabla_h v_h) = (f, v_h) - a(u_{ph}, v_h), \quad \forall v_h \in V_h,$$

where $\tau \in L^\infty(\Omega)$ is a nonnegative *stabilization parameter*. Also, note that the found discrete solution corresponds to the homogeneous problem (2), so the desired discrete solution is $\tilde{u} = u_h + u_p$.

The choice of the parameter τ is not optimal, and there are many proposed stabilization parameters in the literature. In this paper, we will present four of these parameters as given in [4]. The following are the parameters:

$$\tau_1 = \tau_K = \frac{h_K}{2\|b\|_K}\xi_0(Pe) \quad \text{with} \quad \xi_0(\alpha) = \coth\alpha - \frac{1}{\alpha}, \quad Pe = \frac{\|b\|_K h_K}{2\varepsilon},$$

$$\tau_2 = \tau_K = \frac{h}{2\|b\|_K}\xi_1(Pe) \quad \text{with} \quad \xi_1(\alpha) = \max\left\{0, 1 - \frac{1}{\alpha}\right\}, \quad Pe = \frac{\|b\|_K h_K}{2\varepsilon},$$

$$\tau_3 = \tau_K = \frac{h}{2\|b\|_K}\xi_2(Pe) \quad \text{with} \quad \xi_2(\alpha) = \min\left\{1, \frac{\alpha}{3}\right\}, \quad Pe = \frac{\|b\|_K h_K}{2\varepsilon},$$

$$\tau_4 = \tau_K = \frac{\mathrm{diam}(K)}{2\|b\|_K}\xi_0(Pe_K/2) \quad \text{with} \quad \xi_0(\alpha) = \coth\alpha - \frac{1}{\alpha}, \quad Pe_K = \frac{\|b\|_K \mathrm{diam}(K)}{2\varepsilon},$$

where h_K is the element length, $\|b\|_K$ is the Euclidean norm of b, ξ denotes an upwind function, Pe_K is the local Péclet number, and $\mathrm{diam}(K) = \sup\{|x-y| : x, y \in K\}$ is the diameter of the element domains K (for more details, see [3] and [5]).

In the one-dimensional case of (1) with constant data, the SUPG solution with continuous piecewise linear finite elements on a uniform division of Ω is nodally exact if the stabilization parameter is set to τ_1 (Christie et al. [2]). The upwind functions ξ_1 and ξ_2 are referred to as *critical* and *doubly asymptotic* approximations of ξ_0 in [1]. In most cases, ξ_0 yields better results than ξ_1 and ξ_2. However, as these upwind functions approach each other for larger values of the Péclet number, the corresponding discrete solutions are virtually indistinguishable (John and Knobloch [4]).

4. Numerical experiments

This section presents the results of two numerical experiments defined in a two-dimensional domain and discretized by conforming piecewise linear Lagrange finite elements. We are testing four stabilization parameters considered in Section 3. To provide a better impression of the SUPG methods, at the end of each example, GFEM and SUPG solutions of the problem are plotted for a fixed mesh size.

Example 1. Consider the following convection-diffusion equation:

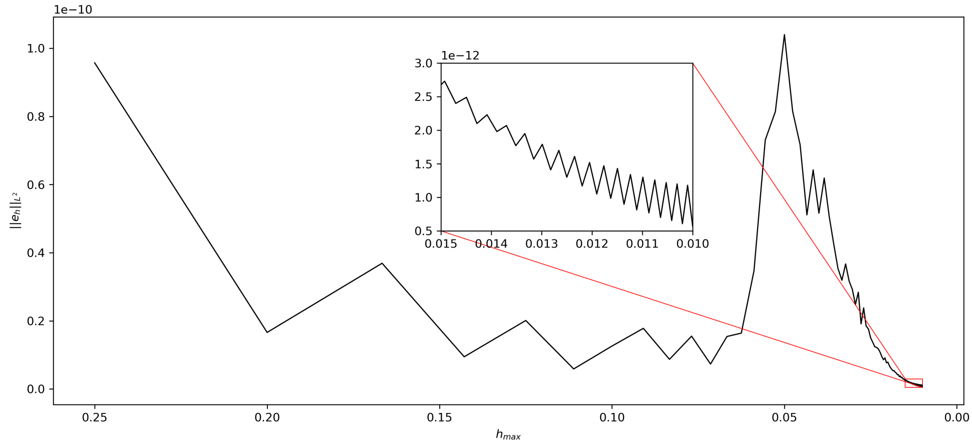
$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f \qquad \text{in } (0,1)\times(0,1), \\ u(x,y) &= 0 \qquad \text{on } [0,1]\times[0,1], \end{aligned} \tag{5}$$

where $\varepsilon = 10^{-8}$, $b = (5,3)^{\mathrm{T}}$ and $f = 2\varepsilon^2\left(y(1-y) + x(1-x)\right) + 5\varepsilon y(1-y)(1-2x) + 3\varepsilon x(1-x)(1-2y)$. The exact solution of the problem is $u(x,y) = \varepsilon xy(1-x)(1-y)$.

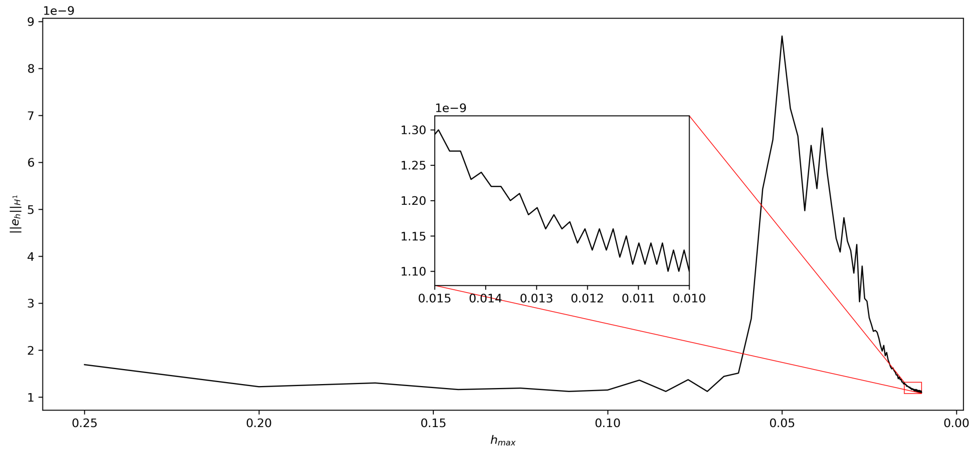
Once we use the Galerkin Finite Element Method (GFEM) to solve the problem, we observe oscillations in the L^2 and H^1 error plots. These oscillations can be clearly seen in Figures 1a and 1b. One can also observe this phenomenon from Table 1 of errors in the norms L^2, H^1, and L^∞ along with their convergence rates.

n	$\|\|e_h\|\|_{L^2}$	L^2 rate	$\|\|e_h\|\|_{H^1}$	H^1 rate	$\|\|e_h\|\|_{L^\infty}$	L^∞ rate
8100	$1.34 \cdot 10^{-12}$	-0.58	$1.15 \cdot 10^{-9}$	-0.03	$6.77 \cdot 10^{-12}$	0.03
8281	$8.14 \cdot 10^{-13}$	0.72	$1.11 \cdot 10^{-9}$	0.04	$6.25 \cdot 10^{-12}$	0.11
8464	$1.3 \cdot 10^{-12}$	-0.68	$1.14 \cdot 10^{-9}$	-0.04	$6.08 \cdot 10^{-12}$	0.04
8649	$7.69 \cdot 10^{-13}$	0.76	$1.11 \cdot 10^{-9}$	0.04	$5.87 \cdot 10^{-12}$	0.05
8836	$1.26 \cdot 10^{-12}$	-0.72	$1.14 \cdot 10^{-9}$	-0.04	$5.62 \cdot 10^{-12}$	0.06
9025	$7 \cdot 10^{-13}$	0.85	$1.11 \cdot 10^{-9}$	0.04	$5.30 \cdot 10^{-12}$	0.08
9216	$1.22 \cdot 10^{-12}$	-0.81	$1.14 \cdot 10^{-9}$	-0.04	$5.16 \cdot 10^{-12}$	0.04
9409	$6.55 \cdot 10^{-13}$	0.90	$1.10 \cdot 10^{-9}$	0.04	$4.97 \cdot 10^{-12}$	0.05
9604	$1.2 \cdot 10^{-12}$	-0.87	$1.13 \cdot 10^{-9}$	-0.04	$4.73 \cdot 10^{-12}$	0.07
9801	$6.08 \cdot 10^{-13}$	0.98	$1.10 \cdot 10^{-9}$	0.04	$4.63 \cdot 10^{-12}$	0.03
10000	$1.18 \cdot 10^{-12}$	-0.96	$1.13 \cdot 10^{-9}$	-0.04	$4.58 \cdot 10^{-12}$	0.02

Table 1: Errors in the L^2, H^1, and L^∞ norms and their convergence rates.



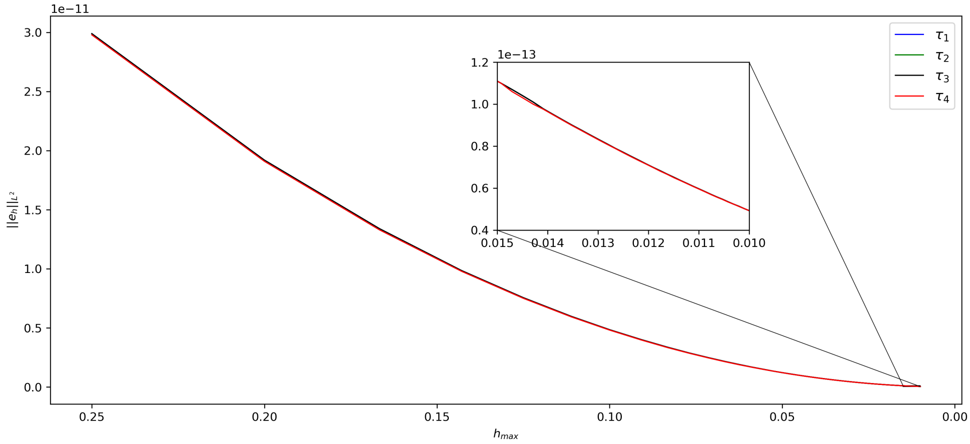
(a) L^2 error plot obtained using GFEM for Example 1.



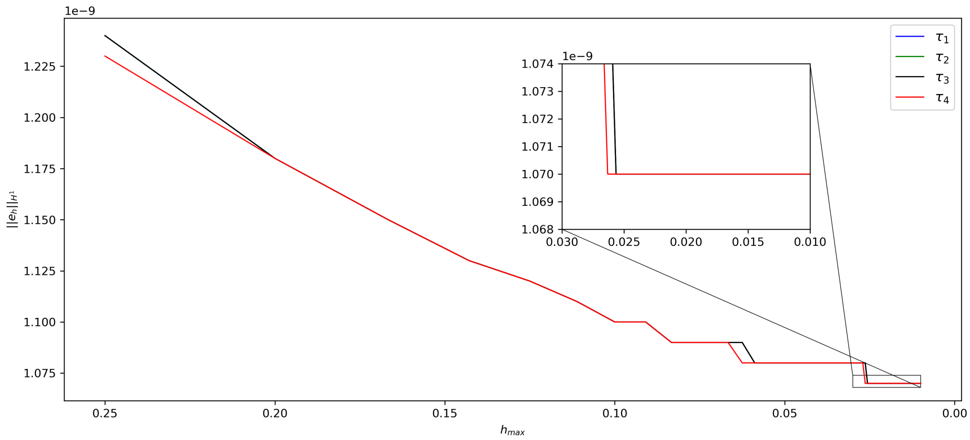
(b) H^1 error plot obtained using GFEM for Example 1.

Figure 1: Error plots for the convection-diffusion equation (5) using GFEM for Example 1.

When the SUPG method is applied with four different stabilization parameters given in Section 3, we obtain Figures 2a and 2b for L^2 and H^1 error plots, respectively. As it can be seen, there are no oscillations in the plots, unlike GFEM. Moreover, notice that data for τ_1, τ_2, and τ_3 are almost the same, and in the plot, only τ_3 can be seen. This is explained in [4] that for larger values of Pe, the results for the upwind functions are very close.



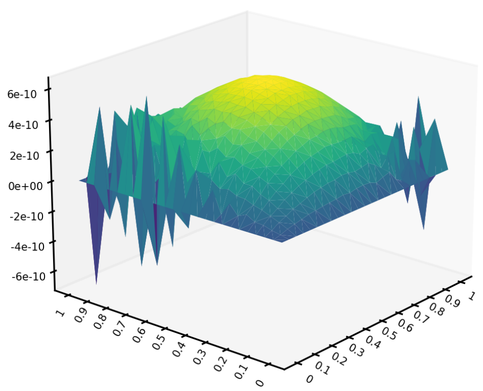
(a) L^2 error plot obtained using SUPG for Example 1.



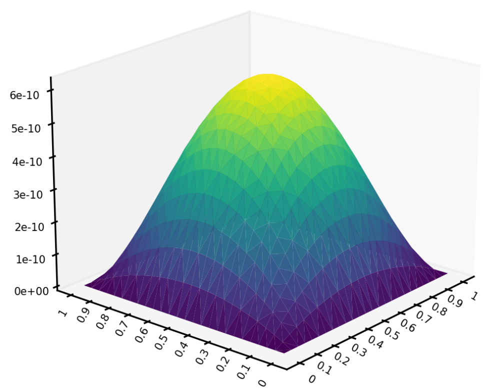
(b) H^1 error plot obtained using GFEM for Example 1.

Figure 2: H^1 error plot obtained using SUPG for Example 1.

In this example, the oscillation that occurred in GFEM can also be observed in Figure 3a. When the SUPG method is applied, there is no distinguishable difference from the exact solution. Note that the GFEM solution does not always show distinguishable oscillation in the discrete solution plot. To demonstrate this, let us examine the second example.



(a) GFEM.



(b) SUPG.

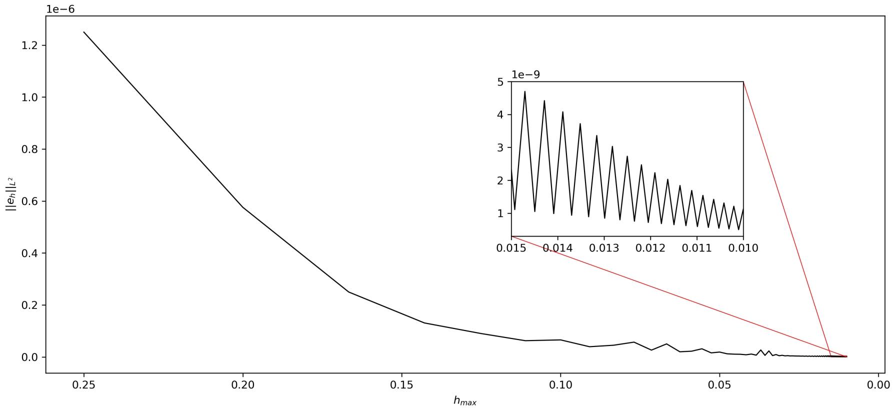
Figure 3: Solutions obtained using GFEM and SUPG with $h_{max} = 0.04$ for Example 1.

Example 2. Consider the following convection-diffusion equation:

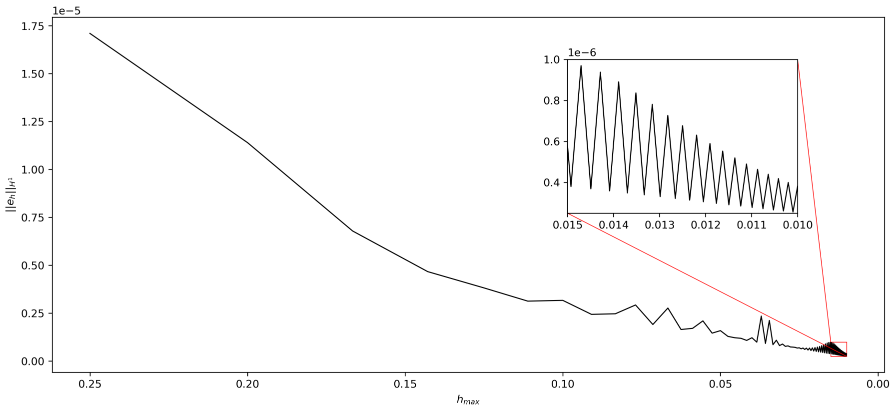
$$\begin{aligned} -\varepsilon\Delta u + b\cdot\nabla u &= f \qquad \text{in } (0,1)\times(0,1), \\ u(x,y) &= 0 \qquad \text{on } [0,1]\times[0,1], \end{aligned} \tag{6}$$

where $\varepsilon = 10^{-6}$, $b = (\cos(\pi/3), -\sin(\pi/3))^{\mathrm{T}}$, and $f = \pi\varepsilon\cos(2\pi x)\sin(3\pi y) + 13\pi^2\varepsilon^2\sin(2\pi x)\sin(3\pi y) - 1.5\sqrt{3}\pi\varepsilon\sin(2\pi x)\cos(3\pi y)$. The exact solution of the problem is $u(x,y) = \varepsilon\sin(2\pi x)\sin(3\pi y)$.

In this example, as well, the same comments as for Example 1 can be made. GFEM plots have oscillations, while SUPG plots do not. Moreover, the stabilization parameters τ_1,τ_2, and τ_3 are almost the same and are indistinguishable in the plots.



(a) L^2 error plot obtained using GFEM for Example 2.



(b) H^1 error plot obtained using GFEM for Example 2.

Figure 4: Error plots for the convection-diffusion equation (6) using GFEM for Example 2.

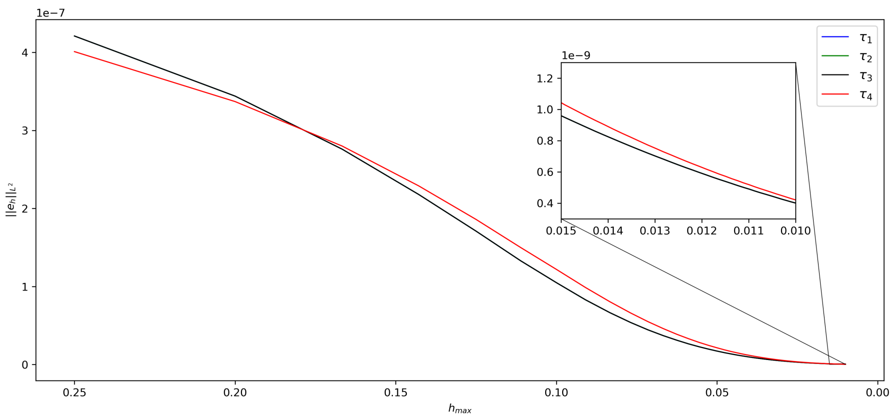


Figure 5: L^2 error plot obtained using SUPG for Example 2.

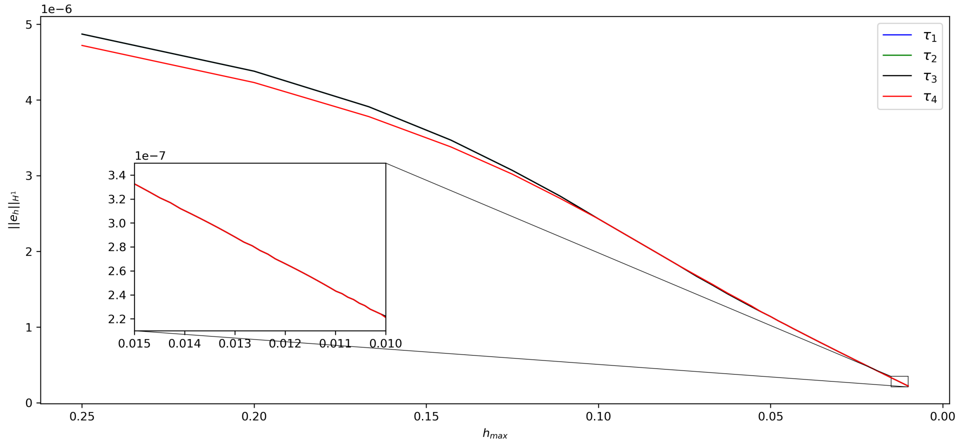
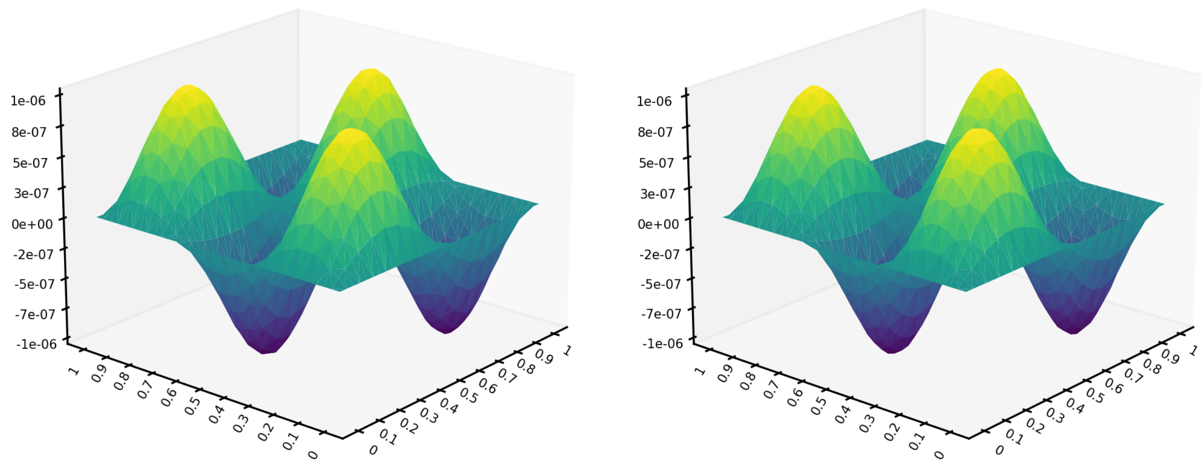


Figure 6: H^1 error plot obtained using SUPG for Example 2.

Once we plot both GFEM and SUPG results, we observe that they are identical, and they match the exact solution. However, when we examine the error plot in different norms (see Figure 5 and 6), it becomes evident that SUPG stabilizes the solution, resulting in smooth error plots.



(a) GFEM. (b) SUPG.

Figure 7: Solutions obtained using GFEM and SUPG with $h_{max} = 0.03125$ for Example 2.

5. Conclusions

The streamline upwind/Petrov-Galerkin (SUPG) method can be easily integrated with preexisting methods for convection-dominated convection-diffusion equations. However, as the best choice of stabilization parameter τ is not known, there will always be room for improvement in the method depending on the given domain and boundary conditions. I am not sure if I am right, but the domain can be scaled accordingly to handle the issue rather than introducing the parameter. I believe this because problems arise when derivatives become significantly larger in certain small regions. These regions can be identified, and then scaling can be carried out so that derivatives remain stable.

References

[1] A.N. Brooks, T.J.R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg. 32 (1982) 199-259.

[2] I. Christie, D.F. Griffiths, A.R. Mitchell, O.C. Zienkiewicz, Finite element methods for second order differential equations with significant first derivatives, Int. J. Numer. Methods Engrg. 10 (1976) 1389-1396.

[3] M. Stynes, L. Tobiska, Necessary L^2-uniform convergence conditions for difference schemes for two-dimensional convection–diffusion problems, Comput. Math. Appl. 29 (1995) 45-53.

[4] V. John, P. Knobloch, On spurious oscillations at layers diminishing (SOLD) methods for convection-diffusion equations: Part I - A review, Comput. Methods Appl. Mech. Engrg. 196 (2007) 2197-2215.

[5] Y.-T. Shih, H.C. Elman, Modified streamline diffusion schemes for convection-diffusion problems, Comput. Methods Appl. Mech. Engrg. 174 (1999) 137-151.