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Webinar 1 – MFDS

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How to find Rank of matrix – Using Determinants

Find the rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Method 1 : Using the determinants
Determinant of the given matrix = 1(-9+8) -1(6 -12) -1(-4 +9)
= -1 + 6 − 5 = 0
Hence Rank ≠ 3. It must be less than 3
Consider a minor of order 2

$\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$ = -3- 2 = -5 ≠ 0. Hence there is at least a minor of order 2 which is not zero. So **rank of A = 2**

How to find Rank of matrix – Using Row Reduction

Find the rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

This is echelon form. Since number of non zero rows is 2 so rank (A) = 2

Finding the rank using Determinants

$$\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$$

3 X 3 Matrix
Case 1: |A| = 0 → Rank ≠ 3
Case 2: |A| ≠ 0 → Rank = 3

When it is case 1

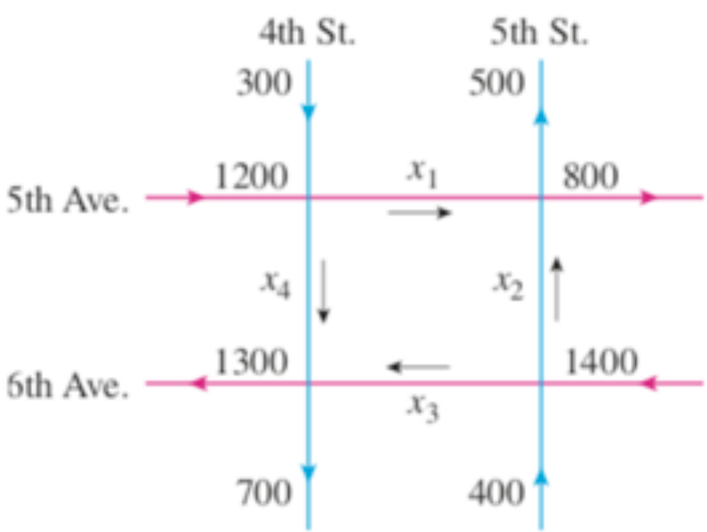
2 X 2 Matrix
Case 1: |A| = 0 → Rank ≠ 2
Case 2: |A| ≠ 0 → Rank = 2

When it is case 1

1 X 1 Matrix
Case 1: |A| = 0 → Rank ≠ 1 →Rank = 0
Case 2: |A| ≠ 0 → Rank = 1

Application Example – Traffic control

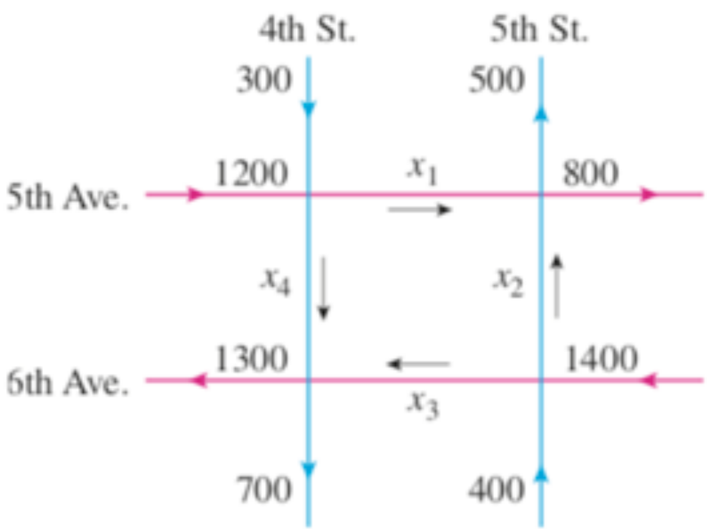
- Figure shows the flow of downtown traffic in a certain city during the rush hours on a typical week day. The arrows indicate the direction of traffic flow on each one – way road, and the average number of vehicles per hour entering and leaving each intersection appears beside the road. 5th Avenue and 6th Avenue can each handle up to 2000 vehicles per hour without causing congestion, whereas the maximum capacity of both 4th Street and 5th Street is 1000 vehicles per hour. The flow of traffic is controlled by traffic lights installed at each of the four intersections.



Q1 : Write a general expression involving the rates of flow – x_1, x_2 ,x_3 ,x_4 and suggest two possible flow patterns that will ensure no traffic congestion

Q2: Suppose the part of 4th street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must be therefore reduced to at most 300 vehicles per hour. Find two possible flow patterns that will result in a smooth flow of traffic

Traffic control



System of linear equations:

$x_1 + x_4 = 1500$
$x_1 + x_2 = 1300$
$x_2 + x_3 = 1800$
$x_3 + x_4 = 2000$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1300 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right] \xrightarrow{\text{Row operations}}$$

$$\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$x_1 = 1500 - t$
$x_2 = -200 + t$
$x_3 = 2000 - t$
$x_4 = t$

For a meaningful solution we must have $200 \leq t \leq 1000$ as x_1 , x_2, x_3, x_4 must be non negative

Rank Nullity Theorem

The null space of a real m x n matrix A is defined to be set of all real solutions to the associated homogeneous linear system Ax = 0

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The dimension of null space is referred as nullity of A, denoted as nullity(A)
To find nullity (A) we need to determine a basis for nullspace (A)

(Rank-Nullity Theorem)

For any $m \times n$ matrix A,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Matrix Representation, Rank and Nullity of Linear Transformation T : $R^2 \rightarrow R^3$

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}.$$

(a) Find the matrix representation of T (with respect to the standard basis for $\mathbb{R}^2$).

(b) Determine the rank and nullity of T.

The matrix representation A of the linear transformation is given by

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)],$$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Standard basis vectors for R^2

Matrix Representation, Rank and Nullity of Linear Transformation T : $R^2 \rightarrow R^3$

To determine $T(\mathbf{e}_1)$, we first express $\mathbf{e}_1$ as a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as follows.

Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ is $3 \cdot 3 - 4 \cdot 2 = 1 \neq 0$ and thus it is invertible.

Hence we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Hence $a = 3$ and $b = -2$.
It yields that

$$\mathbf{e}_1 = 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

It follows that

$$\begin{aligned} T(\mathbf{e}_1) &= T\left(3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \\ &= 3T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) - 2T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \qquad \text{by linearity of } T \\ &= 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 7 \end{bmatrix}. \end{aligned}$$

Matrix Representation, Rank and Nullity of Linear Transformation T : $R^2 \rightarrow R^3$

Similarly, we compute $T(\mathbf{e}_2)$ as follows.

Let

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 3 \\ 2 \end{bmatrix} + d \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

This can be written as

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, we obtain

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix},$$

and $c = -4, d = 3$.

Hence

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Matrix Representation, Rank and Nullity of Linear Transformation T : $R^2 \rightarrow R^3$

$$T(\mathbf{e}_2) = T\left(-4\begin{bmatrix}3\\2\end{bmatrix} + 3\begin{bmatrix}4\\3\end{bmatrix}\right)$$

$$= -4T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) + 3T\left(\begin{bmatrix}4\\3\end{bmatrix}\right) \qquad \text{by linearity of } T$$

$$= -4\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}0\\-5\\1\end{bmatrix} = \begin{bmatrix}-4\\-23\\-9\end{bmatrix}.$$

Therefore the matrix representation A of T is

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix}.$$

Matrix Representation, Rank and Nullity of Linear Transformation T : $R^2 \rightarrow R^3$

$$A = \begin{bmatrix} 3 & -4 \\ 16 & -23 \\ 7 & -9 \end{bmatrix}.$$

Let us first determine the rank of A.

We have

$$A = \begin{bmatrix} 3 & -4 \\ 16 & -23 \\ 7 & -9 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 3 & -4 \\ 16 & -23 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 \\ 16 & -23 \\ 3 & -4 \end{bmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-16R_1} \begin{bmatrix} 1 & -1 \\ 0 & -7 \\ 0 & -1 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & -1 \\ 0 & -7 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_2+7R_3]{R_1+R_3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, the reduced row echelon form matrix of A has two nonzero rows.

So the rank of A is 2.

By the rank-nullity theorem, we know that

$$(\text{rank of } A) + (\text{nullity of } A) = 2.$$

As the rank of A is 2, we see that the nullity of A is 0.

Rank Nullity Theorem verified