"A Relativist's Toolkit" by Eric Poisson

3.13 problems, page 115

1. Consider hypersurface $T = \text{const.}$ in Schwarzschild spacetime, where

$$T = t + 4M\left[\sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\left(\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1}\right)\right]$$

(a) Calculate n_α and find parametric equations that describe the hypersurface.

$$\rightarrow \quad n_\alpha = \frac{\varepsilon T_{,\alpha}}{|g^{\mu\nu} T_{,\mu} T_{,\nu}|^{\frac{1}{2}}}$$

For ε, not clear yet whether hypersurface spacelike or timelike. Since $T = T(t, r)$, it seems neither. Is it then null? Can a hypersurface be neither timelike, spacelike nor null?

Anyway, $\Phi_{,\alpha} = T_{,\alpha} = \frac{\partial T}{\partial x^\alpha}$

$$T_{,t} = \frac{\partial T}{\partial t} = 1$$

$$T_{,r} = \frac{\partial T}{\partial r} = \frac{-M\sqrt{\frac{2r}{M}}}{2M-r} = \frac{\sqrt{\frac{2rM^2}{M}}}{r-2M}$$

$$T_{,r} = \frac{\sqrt{2rM}}{r-2M}$$

$$T_{,\theta} = T_{,\phi} = 0$$

☆ According to Poisson, $T=$ constant defines <u>spacelike</u> hypersurface $\rightarrow$ $\varepsilon = -1$.

Defining $\phi(r) = 1 - \frac{2M}{r}$,

$$ds^2 = -\phi(r)\,dt^2 + \frac{1}{\phi(r)}\,dr^2 + r^2 d\Omega_2^2$$

$\rightarrow$ $g_{tt} = -\phi(r)$, $g_{rr} = \frac{1}{\phi(r)}$

$g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin\theta$

$$\Rightarrow \quad g_{\mu\nu} T_{,\mu} T_{,\nu} = g_{tt} T_{,t} T_{,t} + g_{rr} T_{,r} T_{,r} + g_{\theta\theta} \cancelto{0}{T_{,\theta} T_{,\theta}} + g_{\phi\phi} \cancelto{0}{T_{,\phi} T_{,\phi}}$$

$$= g_{tt} T_{,t} T_{,t} + g_{rr} T_{,r} T_{,r}$$

$$g^{\mu\nu} T_{,\mu} T_{,\nu} = -\phi(r)(1) + \frac{1}{\phi(r)} \left(\frac{\sqrt{2rM}}{r-2M} \right)$$

$$= -\left(1 - \frac{2M}{r}\right) + \frac{1}{1 - \frac{2M}{r}} \frac{\sqrt{2rM}}{r-2M}$$

$$= \left(\frac{2M}{r} - 1\right) + \frac{1}{\left(\frac{r-2M}{r}\right)} \frac{\sqrt{2rM}}{r-2M}$$

$$= \frac{2M-r}{r} + \frac{r}{r-2M} \frac{\sqrt{2rM}}{r-2M}$$

$$= \frac{-(r-2M)}{r} + \frac{r\sqrt{2rM}}{(r-2M)^2}$$

$$g^{\mu\nu} T_{,\mu} T_{,\nu} = \frac{-(r-2M)^3 + r^2\sqrt{2rM}}{r(r-2M)^2}$$

$$\left| g^{\mu\nu} T_{,\mu} T_{,\nu} \right|^{\frac{1}{2}} = \sqrt{\left| \frac{-(r-2M)^3 + r^2\sqrt{2rM}}{r(r-2M)^2} \right|}$$

$$= \frac{1}{r-2M} \sqrt{\left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|}$$

$$n_r = \frac{\varepsilon T_{,r}}{\left| g_{\mu\nu} T_{,\mu} T_{,\nu} \right|^{\frac{1}{2}}} = \frac{-\left(\frac{\sqrt{2rM}}{r-2M} \right)}{\frac{1}{r-2M} \left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}}}$$

$$= \frac{-\sqrt{2rM}}{\left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}}}$$

$$n_r = \frac{-\sqrt{2M}}{\left| r^2\sqrt{2rM} - (r-2M)^3 \right|}$$

$$n_t = \frac{\varepsilon T_{,t}}{\left| g^{\mu\nu} T_{,\mu} T_{,\nu} \right|^{\frac{1}{2}}} = \frac{-1}{\frac{1}{r-2M} \left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}}}$$

$$= -\frac{1}{\frac{1}{r-2M} \left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}}}$$

$$= -(r-2M) \left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}}$$

$$n_t = (2M-r) \left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}}$$

$n_\theta = 0, \; n_\phi = 0$

To find parametric equations, we use the fact that these describe tangents along the hypersurface Σ, i.e.

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$n_\alpha e_a^\alpha = 0$, where $e_a^\alpha = \dfrac{\partial x^\alpha}{\partial y^a}$, and $x^\alpha(y^a)$ is the parametric equation we want

$$n_\alpha e^\alpha_a = n_r e^r_a + n_t e^t_a$$

$$= n_r \frac{\partial r}{\partial y^a} + n_t \frac{\partial t}{\partial y^a} = 0$$

$$-\frac{\sqrt{2M}}{\left| r^2\sqrt{2rM} - (r-2M)^3 \right|} \frac{\partial r}{\partial y^a} + (2M-r)\left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}} \frac{\partial t}{\partial y^a} = 0$$

$$\frac{\sqrt{2M}}{\left| r^2\sqrt{2rM} - (r-2M)^3 \right|^{\frac{1}{2}}} \frac{\partial r}{\partial y^a} = (2M-r)\left| \frac{r^2\sqrt{2rM} - (r-2M)^3}{r} \right|^{\frac{1}{2}} \frac{\partial t}{\partial y^a}$$

$$S_{GHY} = -\frac{1}{8\pi}\left[-32\pi^2 i M(2r-3M)\right]$$

$$S_{GHY} = +4\pi M i\,(2r-3M)$$

$$S_0 = \frac{1}{8\pi}\left[-32\pi^2 M i\left(1-\frac{2M}{r}\right)2r\right]$$

$$= -4\pi M i\,(2r-4M)$$

$$S_{GHY} + S_0 = 4\pi M i\left[(\cancel{2r}-3M)-(\cancel{2r}-4M)\right]$$

$$= 4\pi M^2 i$$

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