AdS$_3$/CFT$_2$ Holographic Entanglement Entropy

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Hartman Chapter 21: Holographic Entanglement Entropy

$$S_A = \frac{\text{area}(\gamma_A)}{4G_N} \tag{1}$$

1 Exercise: 2-dimensional CFT in vacuum

Consider a 2D CFT in vacuum. Let A be a region of L_A

$$x \in [-\frac{L_A}{2}, \frac{L_A}{2}] \tag{2}$$

The dual geometry is empty AdS$_3$, whose metric is given by

$$ds^2 = \frac{L_A^2}{z^2}(-dt^2 + dz^2 + dx^2) \tag{3}$$

Where z is the radial direction and dx^2 can be thought of as the attached 1-sphere/1-hyperboloid.

The codimension-2 (i.e. spacelike) hypersurface has metric

$$\partial_A = ds^2 = \frac{L_A^2}{z^2}(dz^2 + dx^2) \tag{4}$$

This is anchored to the AdS boundary, $z = 0$.

γ_A also has this same metric, but it is extremal. This means it is the *geodesic* connecting the two endpoints of A: $P_1 = (z_1, x_1) = (0, -\frac{L_A}{2})$ and $P_2 = (z_2, x_2) = (0, \frac{L_A}{2})$

1.1 Finding the geodesic

We could have also done this with the geodesic equation and SymPy/Mathematica in literally 10 minutes. But this is more intuitve

$$ds^2 = \frac{L_A^2}{z^2}(dz^2 + dx^2)$$
$$\frac{ds^2}{dz^2} = \frac{L_A^2}{z^2}(1 + \frac{dx^2}{dz^2})$$
$$ds = \frac{L_A}{z}\sqrt{1 + \left(\frac{dx}{dz}\right)^2}\, dz$$

Then the length of the interval is[1]

$$\text{area}(\gamma_A) = \int ds = 2\int_{\epsilon}^{z_{max}} \frac{L_A}{z}\sqrt{1+(x'(z))^2}dz = 2\int \mathcal{L}\left[x'(z), x(z), z\right]dz \tag{5}$$

No higher-order dependence, so we can plug into the usual Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dz}\left(\frac{\partial \mathcal{L}}{\partial x'}\right)$$

Clearly, $\frac{\partial \mathcal{L}}{\partial x} = 0$, so

$$\frac{\partial \mathcal{L}}{\partial x'} = const. = R$$

$$R = \frac{L_A}{z}\left(\frac{x'}{\sqrt{1+(x')^2}}\right)$$

With a bit of arithmetic[2]...

$$\implies x' = \frac{Rz}{\sqrt{L_A^2 - R^2 z^2}} \tag{6}$$

How to find R? Let's check the limits of $z \in [\epsilon, z_{max}]$.
At large z, $x' \sim z$. This means the maximum value of z is at $x \to \infty$:

$$\lim_{z \to z_{max}} x' = \infty \tag{7}$$

We see that $x \to \infty$ as the denominator goes to zero, i.e. $\lim_{z\to z_{max}} L_A^2 - R^2 z^2 = 0$

$$\implies R = \frac{L_A}{z_{max}} \tag{8}$$

Which gives

$$x' = \frac{z}{z_{max}\sqrt{1 - \frac{z^2}{z_{max}^2}}}$$

$$x' = \frac{dx}{dz} = \frac{z}{\sqrt{z_{max}^2 - z^2}} \tag{9}$$

Which gives

$$x = \int dz \frac{z}{\sqrt{z_{max}^2 - z^2}}$$

$$= \int dz \frac{\frac{z}{z_{max}}}{\sqrt{1 - \frac{z^2}{z_{max}^2}}} \tag{10}$$

Substitute $\frac{z}{z_{max}} = sin(\lambda) \implies dz = z_{max}cos(\lambda)d\lambda$

$$x = \int z_{max}cos(\lambda)d\lambda \frac{sin(\lambda)}{\sqrt{1 - sin^2(\lambda)}}$$

$$= \int z_{max}sin(\lambda)d\lambda$$

$$= z_{max}cos(\lambda) \tag{11}$$

So this gives a parametrized solution for our geodesic:

$$z = z_{max}sin(\lambda) \tag{12}$$

[1] the factor of 2 is because the integral really goes $z : \epsilon \to z_{max} \to \epsilon$, but we instead assume the integrand is symmetric on either side of z_{max} and only integrate $z : \epsilon \to z_{max}$

[2] and absorbing the $\pm$ into the constant R

$$x = z_{max} cos(\lambda) \tag{13}$$

This tell us that our geodesic is a semicircle[3] of radius z_{max}!

From the x-bounds of our boundary (2) it is clear that this radius is also $L_A/2$. Therefore,

$$z_{max} = \frac{L_A}{2} \tag{14}$$

Finally, we find

$$z = \frac{L_A}{2} sin(\lambda) \tag{15}$$

$$x = \frac{L_A}{2} cos(\lambda) \tag{16}$$

1.2 Computing the path length

$$\begin{aligned} \text{area}(\gamma_A) &= 2 \int_{\epsilon}^{z_{max}} \frac{L_A}{z} \sqrt{1 + (x'(z))^2} dz \\ &= 2 \int_{\epsilon}^{L_A/2} \frac{L_A}{z} \sqrt{1 + (x'(z))^2} dz \end{aligned}$$

Substituting for $x'(z)$...

$$\text{area}(\gamma_A) = 2 \int_{\epsilon}^{L_A/2} \frac{L_A}{z} \sqrt{\frac{(L_A/2)^2}{(L_A/2)^2 - z^2}} dz \tag{17}$$

Then we substitute (15)...

$$\text{area}(\gamma_A) = 2L_A \int_{\lambda_0}^{\lambda_f} \frac{1}{sin(\lambda)} d\lambda \tag{18}$$

where $\lambda_0 = sin^{-1}(\frac{2\epsilon}{L_A})$ and $\lambda_f = sin^{-1}(1)$

$$\begin{aligned} \text{area}(\gamma_A) &= 2L_A \ln\left(tan(\frac{\lambda}{2})\right)|_{\lambda_0}^{\lambda_f} \\ &= 2L_A \ln\left(tan(\frac{\lambda_f}{2})\right) - 2L_A \ln\left(tan(\frac{\lambda_0}{2})\right) \\ &= 2L_A \ln\left(tan(\frac{sin^{-1}(1)}{2})\right) - 2L_A \ln\left(tan(\frac{sin^{-1}(2\epsilon/L_A)}{2})\right) \\ &= 2L_A \ln\left(tan(\frac{\pi}{4})\right) - 2L_A \ln\left(tan(\frac{sin^{-1}(2\epsilon/L_A)}{2})\right) \\ &= -2L_A \ln\left(tan(\frac{sin^{-1}(2\epsilon/L_A)}{2})\right) \\ &\approx -2L_A \ln\left(tan(\frac{\epsilon}{L_A})\right) \\ &\approx -2L_A \ln\left(\frac{\epsilon}{L_A}\right) \end{aligned}$$

At last,

$$\text{area}(\gamma_A) \approx 2L_A \ln\left(\frac{L_A}{\epsilon}\right) \tag{19}$$

[3]Recall that we chose to keep only the positive definition of x' in Eqn. (9)

1.3 Computing the HEE

Just substitute (19) into (1).

$$S_A = \frac{L_A}{2G_N} \ln\left(\frac{L_A}{\epsilon}\right) \tag{20}$$

With $c = \frac{3L_A}{2G_N}$ being the central charge for a 2D CFT, we get the familiar result

$$S_A = \frac{c}{3} \ln\left(\frac{L_A}{\epsilon}\right) \tag{21}$$

2 Exercise: d-dimensional CFT