A Finite Entanglement Entropy and the c-theorem

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Based on the **paper by Casini and Huerta**.

1 Set-up

Consider the vacuum state $|0\rangle$ in some region of space denoted by V. The reduced density matrix is obtained by tracing over all degrees of freedom that are localized on a region of space outside of V, i.e. all observables that are inaccessible to an observer in V.

$$\rho_V = tr_V \, |0\rangle \langle 0| \tag{1}$$

Then the entanglement entropy[1] is the von Neumann entropy of ρ_V

$$S(V) = -tr(\rho_V \log \rho_V) \tag{2}$$

This is typically divergent in the UV. Recap from Hartman: this is what gives the ∼area-law scaling for entanglement entropy; the UV modes contribute the leading term to the entanglement entropy[2] and the number of UV modes is proportional to the area.

However: "...it is possible to define a finite quantity $F(A, B)$ for two given different subsets A and B which measures the degree of entanglement between their respective degrees of freedom. We show that the function $F(A, B)$ is severely constrained by the Poincare symmetry[3,4] and the mathematical properties of the entropy. In particular, for one component sets[5] in two dimensional conformal field theories its general form is completely determined. Moreover, it allows to prove an alternative entropic version of the c-theorem for 1 + 1 dimensional QFT. We propose this well defined quantity as the meaningful entanglement entropy and comment on possible applications in QFT and the black hole evaporation problem."

We begin with a bit of set-theory recap:

Given two non-intersecting sets A and B, we have that

1. $$\mathcal{H}_{A\cup B} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{3}$$

2. $$\rho_A = tr_{\mathcal{H}_B} \rho_{A\cup B} \tag{4}$$

Also, recall the strong subadditive inequality (SSA), which holds for *any* two sets A and B:

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B) \tag{5}$$

Moreover, since the vacuum state is a pure state,

$$S(A) = S(-A) \tag{6}$$

[1] Apparently entanglement entropy is a big deal in the condensed matter DMRG method

[2] Mathematically speaking, this statement has been proven rigorously. Intuitively speaking, I rationalize it as being due to the higher energy of these modes; another way of thinking of it is that smaller wavelengths allow for a greater information-density.

[3] Recall: the Poincare group includes rotations, translations, and boosts

[4] It seems this entire formalism only applies for Minkowski space

[5] This restriction seems quite important. I suppose they mean sets composed of a single spacetime event? Or maybe we're only talking about spatial sets here, and the single component is a single spatial element, i.e. one 'vertex' of the lattice

Read the following closely to understand: "In Minkowski space the independent degrees of freedom (i.e. disjoint Hilbert spaces) should be assigned to subsets of general Cauchy surfaces and an entanglement entropy must correspond to any of these subsets."

"Moreover, the causal structure and unitarity imply that this entropy must be the same for different spatial sets having the same causal domain of dependence[6]. It has been shown that the combination of the SSA property, the positivity of the entropy, and the Poincare invariance of the vacuum state in Minkowski space strongly constrain the function S in any dimensions. The result is that if S is finite for at least one arbitrary set then it has to be finite and proportional to the boundary area for any other set[7], plus a constant term"

Anyway, we all know that S is divergent in the UV. "Let A and B be two *spatial* sets in Minkowski space which belong to the same Cauchy surface[8]." Introduce the symmetric function $F(A, B)$,

$$F(A, B) = S(A) + S(B) - S(A \cap B) - S(A \cup B) \tag{7}$$

$F(A, B)$ has a few cool qualities:

1. $F(A, B)$ is finite and free from divergences due to the following cancellation:

   $$area(A) + area(B) = area(A \cup B) + area(A \cap B) \tag{8}$$

   We take this to mean that the UV divergences - which are proportional to the area and are supposedly the only source of divergence - cancel out.

2. $F(A, B)$ is positive by the SSA inequality.

3. $F(A, B) = F(-A, -B)$

4. Perhaps most importantly, $F(A, B)$ is monotonically increasing. For non-intersecting sets A and B, with $B \subseteq C$ and $A \cap C = 0$, we find that

   $$F(A, B) \leq F(A, C) \tag{9}$$

   . This property makes $F(A, B)$ a good measure of the degree of entanglement in QFT[9].

For two non-intersecting sets A and B, $F(A, B)$ can be written as

$$F(A, B) = Tr(\rho_{A\cup B} \log \rho_{A\cup B}) - Tr(\rho_{A\cup B} \log(\rho_A \otimes \rho_B)) = S(\rho_{A\cup B}|\rho_A \otimes \rho_B) \tag{10}$$

Where $S(\rho_1|\rho_2) = Tr(\rho_1(\log(\rho_1) - log(\rho_2))$ is known as the relative entropy for two states ρ_1 and ρ_2 in the same Hilbert space. In the context of QI, the particular relative entropy in Eqn. (10) is the *mutual information* between ρ_A and ρ_B in the composite quantum system.

"$F(A, B)$ for intersecting sets can be defined in terms of the relative entropy for non-intersecting sets with the use of additional auxiliary sets".

"The origin for the divergence of the standard entanglement entropy can be traced to the impossibility of expressing the Hilbert space H as a tensor product of $H_V \otimes H_{-V}$ in the relativistic case"[10] $\Longleftarrow$ **big if true**. "... This partition is unambiguous if the system is defined on a lattice, with the local Hilbert spaces being generated by the local degrees of freedom (for example the spin-like operators in a given region)."

[6]The causal diamond thing

[7]Any other set on the same Cauchy surface, or any other set in general? Assuming the former.

[8]A Cauchy surface is basically a spacelike hypersurface of the spacetime

[9]Not necessarily the degrees of freedom; we'll get to that in a bit

[10]I thought we could simply represent causally-disconnected spacetime intervals as disjoint Hilbert spaces? Maybe they mean specifically the GR case? Moreover, while it seems easy to represent causally-disconnected sets (disjoint Hilbert spaces) or full-causally-connected sets (the same Hilbert space, since they must have the same entropy), it's not obvious how to represent partially-causally-connected sets (though the notion of a reduced density matrix comes to mind).

1.1 Let's dissect this last paragraph more closely...

Through footnotes, that is.

"However, in a relativistic QFT the axiomatic investigations[11] have shown that, although there is a well defined notion of local algebras of operators in a volume V (type III Von Neumann algebras[12] in general), there is no Lorentz invariant partition of the total Hilbert space into a tensor product $H_V \otimes H_{-V}$. One heuristic reason for this can be seen considering the theory of a free scalar field ϕ. In the classical theory the phase space attached to a bounded region V is given by initial data $\phi(x)$ and $\dot{\phi}(x)$ vanishing outside V[13]. In the quantum case however, only positive energy solutions of the wave equation appear in the annihilation part of the operator $\phi(x)$[14]. In momentum space the time derivative[15] is then $\epsilon_p = -i\sqrt{p^2+m^2}$,[16]. Now, in real space, the operator given by ϵ_p is antilocal[17,18], which means that it transforms any function vanishing outside of an open set V into a function which does not vanish identically in any open subset outside of V[19]. This implies that one can not have $\langle 0|\, \phi(x)\, |\psi\rangle$ and $\langle 0|\, \dot{\phi}(x)\, |\psi\rangle$ vanishing simultaneously in the same region of space[20], where $|\psi\rangle$ is any one particle state. In general the moral is that in QFT there is no covariant meaning for the localization of states[21], and what can be unambiguously localized are the field operator algebras[22]. In this sense it is remarkable that even if a notion of entropy for a state in a given algebra may not exist (type III algebras do not contain a trace[23]), the relative entropy of two states for the same Von Neumann algebra has a well defined mathematical meaning[24]."

2 Entanglement entropy in 1+1 dimensions

Through some strenuous computation, we can show that

$$F(A,B) = G(A) + G(B) - G(A \cap B) - G(A \cup B) \tag{11}$$

where G, unlike S, is finite.

"Since the meaningful quantity is F, the function $G(A)$ is defined up to an arbitrary term proportional to the number m_A of connected component[25] in A. Thus, for the one-component sets there is a 'gauge' symmetry associated to an additive constant"

Positivity of F implies SSA for G, and we can also prove $G(A) = G(-A)$.

The crucial difference between G and S is that G can be negative.

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[11] Axiomatic field theory

[12] Might want to dig into this

[13] Assuming they mean $\phi(x)$ in V is the initial data and $\dot{\phi}(x)$ vanishing outside of V is the (Neumann) boundary condition

[14] It *seems* like this is related to the practice (in the SHO/second-quantization formalism) of writing the Hamiltonian/position/momentum operators as a product/sum of the creation and annihilation operators. Is it possible to decompose *any* operator into a product or sum of a and $a^\dagger$?

[15] Time derivative of what? $\phi(x)$? $\phi(p)$, since we're in momentum space?

[16] No clue how they got this or what they actually differentiated

[17] Develop a more mathematical definition, analogous to how 'local' fields must always commute at spatially-separated locations

[18] Does this follow from the previous sentence?

[19] Develop this mathematically

[20] This kind of makes sense from the previous sentence, but still requires a bit more meditation

[21] Develop this mathematically

[22] Develop this mathematically

[23] Follow up on this

[24] Maybe an example will make this clearer

[25] m_A is analogous to the area of A

2.1 Entanglement entropy in conformal field theory

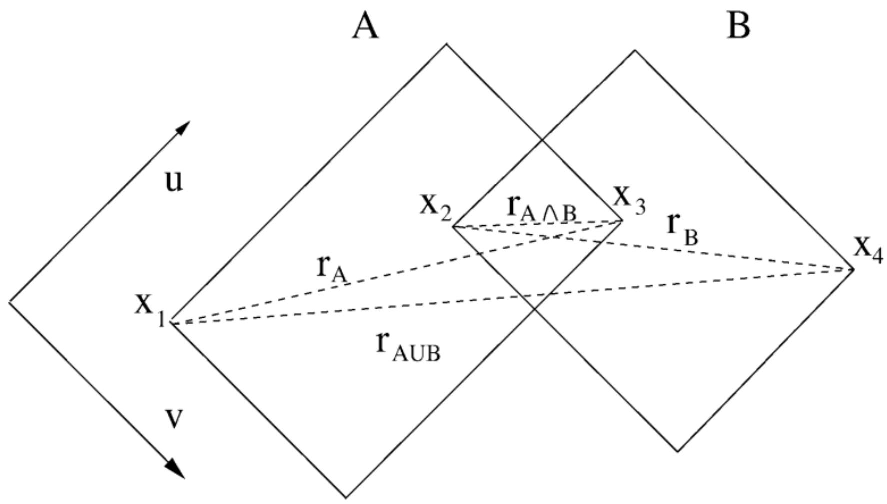

Figure 1: Two intersecting one component sets A and B whose straight Cauchy surfaces have sizes r_A and r_B, respectively. The diamonds formed by the intersection and union (followed by causal completion[a]) of A and B have sizes $r_{A\cap B}$ and $r_{A\cup B}$. This configuration of sets is uniquely determined by the position of the points x_1, x_2, x_3 and x_4 forming the spatial corners of the diamonds.

[a] What is causal completion?

"For the special case of a conformal field theory (CFT) and when the sets A and B are one component and intersecting, Eqn. (11) allows us to obtain the general form of $F(A,B)$. We see from Fig. (1) that only four points determine the position of the diamonds corresponding to A and B. Since $F(A,B)$ must be invariant under global conformal transformations, general results ensure that it must be a function of the cross ratios[26]."

$$F(A,B) = F(\eta_\mu, \eta_\nu) \tag{12}$$

Where

$$\eta_\mu = \frac{u_{23}u_{14}}{u_{13}u_{24}} = \frac{(u_2-u_3)(u_1-u_4)}{(u_1-u_3)(u_2-u_4)} \tag{13}$$

$$\eta_\nu = \frac{v_{23}v_{14}}{v_{13}v_{24}} = \frac{(v_2-v_3)(v_1-v_4)}{(v_1-v_3)(v_2-v_4)} \tag{14}$$

And, for example, the null coordinates of x_1 are (u_1, v_2). Now, this is honestly quite a big leap. Why these ratios in particular?

"On the other hand, G can be written as a function of the length of the (causal) diamond base[27] $r_{ij} = \sqrt{(x_i-x_j)^2} = \sqrt{u_{ij}v_{ij}}$. Using this and Eqs. (11) and (12), we get the general form for G and F

$$G(A) = k\,\log\,(r_A) + \beta \tag{15}$$

$$F(A,B) = k\,\log\,\left(\frac{r_A r_B}{r_{A\cap B} r_{A\cup B}}\right) \tag{16}$$

"

Maybe we don't need to really understand all this in too much depth? I'm just trying to understand the geometric argument here. Maybe Hartman's lecture notes are better.

Let's come back to this

[26]This kind of makes sense. Since the theory is conformal, the *absolute* sizes of the sets don't matter, just the relative sizes i.e. the ratios of sizes

[27]So these are the causal diamonds we're seeing in Fig. (1)

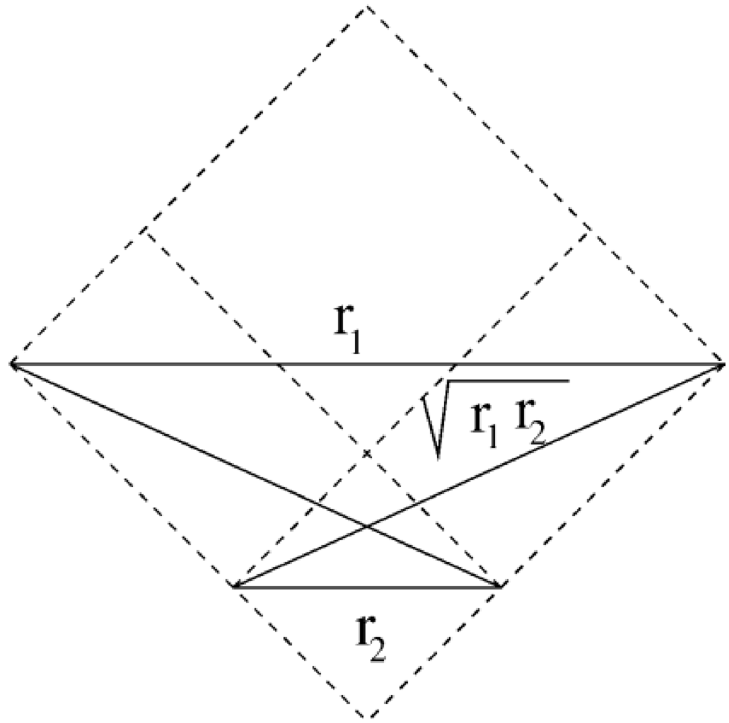

Figure 2: Configuration of two intersecting sets of size $\sqrt{r_1 r_2}$ from which Eq. (17) can be obtained by means of the SSA inequality

$$G(\sqrt{r_1, r_2}) \geq \frac{1}{2}(G(r_1) + G(r_2)) \tag{17}$$