Gravitational Path Integrals

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These are much condensed versions of **Grabovsky's lecture notes**. Do keep those handy because I skip some important things. At the moment (April 2023), chapters 3 (*Euclidean Quantum Gravity*, the whole reason I came here in the first place) and 4 (*Entropy and the Replica Trick*) of the lectures notes aren't yet available.

1 Convention

Natural units except for G, and sometimes $\hbar$ but only when stated.

2 The Canonical Path Integral

The starting point is a review of the canonical quantum-mechanical path integral. Recall the Heisenberg equation of motion:

$$\frac{d}{dt}\hat{\mathcal{O}}(t) = i[\hat{H}, \hat{\mathcal{O}}(t)] + \frac{\partial \hat{\mathcal{O}}(t)}{\partial t} \tag{1}$$

If $\frac{\partial \hat{\mathcal{O}}(t)}{\partial t} = 0$, then

$$\frac{d}{dt}\hat{\mathcal{O}}(t) = i[\hat{H}, \hat{\mathcal{O}}(t)] \tag{2}$$

Let's consider the case of the position operator, $\hat{\mathcal{O}}(t) = \hat{p}(t)$. Recall the equal-time commutation between position and momentum, $[\hat{q}(t), \hat{p}(t)] = i$. In order to find the non-equal-time commutator, $[\hat{q}(t), \hat{p}(t')]$, we have to actually solve the equations of motion; in other words, *the different-time commutator depends on* $\hat{H}$.

Thoughts: Equal-time commutation

So, commutation between position and momentum is Hamiltonian-independent at equal times, but Hamiltonian-dependent if there is time-separation. How does this compare with what general relativity has to say on the matter?

Well it's hard to even formulate an analogous notion in GR, since q is as much of a coordinate as t (i.e. there is no $\hat{q}$ in GR, afaik).

However, at a philosophical and not-mathematically-rigorous level, a Hamiltonian-independent commutation relation seems to contradict GR. In GR, the relation between position and momentum is always metric-dependent, which seems to imply that it is always Hamiltonian dependent - since the gravitational potential that gives the metric should show up in the Hamiltonian.

Maybe, in locally flat space, there is no gravitational potential and thus the position-momentum relation is Hamiltonian-independent.

Now, the operator $\hat{q}(t)$ has a complete set of eigenstates $\{|q,t\rangle\}$, which are orthonormal *at equal times*: $\langle q', t|q, t\rangle = \delta(q - q')$. To find the overlap between different q's at different t's, we need the *propagator*,

$$K(q_i, t_f; q_f, tf) \equiv \langle q_f, t_f | q_i, t_i\rangle \tag{3}$$

which surely must depend on the time-evolution of the states, i.e. $\hat{H}$.

2.1 Constructing the propagator

"To construct the path integral that computes the propagator, we will proceed in four steps:

- formally solve equation (2) in the case $\hat{\mathcal{O}}(t) = \hat{q}(t)$, and thereby relate the $\hat{q}$-eigenstates at times t_i and t_f
- slice up the interval $[t_i, t_f]$ into a large number N of infinitesimal time steps, and resolve the identity at each step
- resolve a tricky ordering ambiguity in the Hamiltonian
- take the continuum limit to obtain the path integral"

2.1.1 Step 1: Formally solve the Heisenberg EOM for $\hat{\mathcal{O}}(t) = \hat{q}(t)$

$$\frac{d\hat{q}}{dt} = i[\hat{H}(t), \hat{q}(t)] \Longrightarrow \hat{q}(t_f) = U(t_i, t_f)\hat{q}(t_i)U(t_i, t_f)^{-1} = U(t_i, t_f)\hat{q}(t_i)U(t_f, t_f)$$
$$= \mathcal{P}\exp\left[i\int_{t_i}^{t_f} dt\hat{H}(t)\right]\hat{q}(t_i)\mathcal{P}\exp\left[i\int_{t_f}^{t_i} dt\hat{H}(t)\right] \tag{4}$$

"Notice that inverting U is tantamount to reversing the path, so as to evolve backwards in time."

"Here $\mathcal{P}$ is the path ordering symbol" $\Longleftarrow$ Explanation via **Stackexchange**: "A related concept is the so-called 'time-ordered product'... Basically, a time-ordered product of several operators means you have to put the operator corresponds to larger t to the left of the operator with respect to smaller t." So I supposed the $\mathcal{P}$ orders our operators based on how far along they are in the path. Operators which act on earlier parts of the path move to the right (?), and those which show up later move to the left. **Try to confirm this interpretation with someone.**

Anyway, we see that the time evolution of a position-space ket is given by

$$|q, t_f\rangle = U(t_i, t_f)|q, t_i\rangle = \mathcal{P}\exp\left[\int_{t_i}^{t_f} dt\hat{H}(t)\right]|q, t_i\rangle \tag{5}$$

And we can check that $|q, t_f\rangle$ is indeed an eigenket of $\hat{q}(t_f)$

$$\hat{q}(t_f)|q, t_f\rangle = U(t_i, t_f)\hat{q}(t_i)U(t_i, t_f)^{-1}U(t_i, t_f)|q, t_i\rangle$$
$$= U(t_i, t_f)\hat{q}(t_i)|q, t_i\rangle$$
$$= qU(t_i, t_f)|q, t_i\rangle$$
$$\hat{q}(t_f)|q, t_f\rangle = q|q, t_f\rangle \tag{6}$$

2.1.2 Step 2: Slice up the interval $[t_i, t_f]$ into a large number N of infinitesimal time steps

$$K(q_i, t_i; q_f, t_f) \equiv \langle q_f, t_f|q_i, t_i\rangle = \langle q_f, t_i|U(t_i, t_f)^{-1}|q_i, t_i\rangle \tag{7}$$

Note that we've chosen to time-evolve the bra instead of the ket.

$$K(q_i, t_i; q_f, t_f) = \langle q_f, t_f|q_i, t_i\rangle = \langle q_f, t_i|U(t_f, t_i)|q_i, t_i\rangle = \langle q_f, t_i|\mathcal{P}\exp\left[\int_{t_f}^{t_i} dt\hat{H}(t)\right]|q_i, t_i\rangle \tag{8}$$

"The *path-ordered exponential* is defined as the following product:

$$K(q_i, t_i; q_f, t_f) = \langle q_f, t_i|U(t_f, t_i)|q_i, t_i\rangle = \lim_{\Delta t\to 0} = \langle q_f, t_i|e^{-i\hat{H}(t_i)\Delta t}...e^{-i\hat{H}(t_f)\Delta t}|q_i, t_i\rangle \tag{9}$$

The operator $U(t_f, t_i)$ appearing here is a string of infinitesimal time evolution operators proceeding from t_f to t_i in steps of size Δt. Thus the rightmost $\hat{H}$ is evaluated at t_f , the one before is evaluated at $t_f\Delta t$, and so on until the leftmost $\hat{H}$ is evaluated at t_i. The minus signs above are responsible for undoing the backwards evolution of each step.

Next, we denote $q_N \equiv q_f$ and $q_0 \equiv q_i$ and insert $N-1$ position-space resolutions of the identity between each exponential factor, where $N = \frac{|t_f - t_i|}{\Delta t}$ is the number of time steps:

$$K(q_i, t_i; q_f, t_f) = \lim_{N\to\infty} \int dq_N ... dq_{N-1} \langle q_N, t| e^{-i\hat{H}(t)\Delta t} |q_{N-1}, t\rangle \langle q_{N-1}, t| ... |q_1, t\rangle \langle q_1, t| e^{-i\hat{H}(t)\Delta t} |q_0, t\rangle \tag{10}$$

"

So it seems the only identity we don't insert is $\int dq_0 |q_0, t\rangle \langle q_0, t|$

Let's evaluate the last one of these factors

$$K_1 = \langle q_1, t| e^{-i\hat{H}(t_f)} |q_0, t\rangle \tag{11}$$

Using the Dyson series expansion and working to first order in Δt,

$$K_1 = \langle q_1, t| \left(1 - i\hat{H}(t_f)\Delta t + O(\Delta t^2)\right) |q_0, t\rangle \tag{12}$$

This is apparently difficult to calculate. Insert momentum-space identity:

$$K_1 = \int dp_1 \langle q_1, t|p_1, t\rangle \langle p_1, t| \left(1 - i\hat{H}(t_f)\Delta t + O(\Delta t^2)\right) |q_0, t\rangle \tag{13}$$

Importantly, at this point we assume that $\hat{H}$ *does not explicitly depend on time.* $\hat{H}(t) = \hat{H}(t_f) = \hat{H}$. According to author, "this restriction can be loosened with some work".

2.1.3 Step 3: Resolve a tricky ordering ambiguity in the Hamiltonian

"The problem with definining $\hat{H} = \hat{H}(\hat{q}(t), \hat{p}(t))$ as a 'function' of the $\hat{q}$ and $\hat{p}$ operators is that there are ordering ambiguities due to noncommutativity." $\Longleftarrow$ this is exactly the issue you came across with the $\hat{G}$ propagator in Masha's reading.

TODO: Do more research on the following, regarding *Weyl ordering.*

"We will choose to define $\hat{H}$ so that all $\hat{p}$'s are moved to the left and all $\hat{q}$'s are moved to the right. This convention is called Weyl ordering, and we define the Weyl symbol $H_W(q, p)$ to be the following possibly complex-valued c-number function:

$$H_W(q, p) = \frac{\langle p, t| \hat{H} |q, t\rangle}{\langle p, t|q, t\rangle} \tag{14}$$

"

No idea what just happened here.

Anyway, somehow this results in

$$K_1 = \int dp_1 \langle q_1, t|p_1, t\rangle \langle p_1, t|q_0, t\rangle \left(1 - iH_W(q_1, p_1)\Delta t + O(\Delta t^2)\right) \tag{15}$$

So we also apparently pulled the Hamiltonian operator out from the second braket. Moving on,

$$K_1 = \int dp_1 \frac{1}{\sqrt{2\pi}} e^{ip_1 q_1} \frac{1}{\sqrt{2\pi}} e^{-ip_1 q_0} \left(1 - iH_W(q_1, p_1)\Delta t + O(\Delta t^2)\right) \tag{16}$$

Inserting the approximation $e^{-iH_W(q_1,p_1)\Delta t} \approx 1 + (-iH_W(q_1, p_1)\Delta t)$[1], this gives

$$K_1 = \frac{1}{2\pi} \int dp_1 e^{ip_1 q_1} e^{-ip_1 q_0} e^{-iH_W(q_1,p_1)\Delta t} \tag{17}$$

Defining $\dot{q}_1 = \frac{q_1 - q_0}{\Delta t}$, we get

$$K_1 = \frac{1}{2\pi} \int dp_1 \exp\left(i[p_1 \dot{q}_1 - H_W(q_1, p_1)\Delta t]\right) \tag{18}$$

"We recognize the term in brackets as the contribution to the (canonical) action on a small interval $[t_0, t_1]$ of length Δt"

[1] and apparently also the approximations $e^{1-iH_W(q_1,p_1)\Delta t + O(\delta t^2)} \approx e^{-iH_W(q_1,p_1)\Delta t}$ and $1 + (-iH_W(q_1, p_1)\Delta t) \approx -iH_W(q_1, p_1)\Delta t$

2.1.4 Step 4: Take the continuum limit to obtain the path integral

So we've got the expression for just one of those infinitesimal K's. Plugging in for each one, we find

$$K_{discrete} = \int dq_1...dq_{N-1}K_1...K_N \tag{19}$$

Take the continuum limit to find the real thing

$$K(q_i,t_i;q_f,t_f) = \lim_{N\to\infty}\int dq_1...dq_{N-1}K_1...K_N$$

$$= \lim_{N\to\infty}\left(\frac{1}{2\pi}\right)^N\int dq_1...dq_{N-1}dp_1...dp_N \exp\left(i\sum_{k=1}^{N}[p_k\dot{q}_k - H_W(q_k,p_k)]\Delta t\right)$$

$$\equiv \int_{q(t_i)=q_i}^{q(t_f)=q_f}\mathcal{D}q\mathcal{D}p\exp\left(\int_{t_i}^{t_f}dt[p\dot{q}-H_W(q,p)]\right)$$

$$K(q_i,t_i;q_f,t_f) = \int_{q(t_i)=q_i}^{q(t_f)=q_f}\mathcal{D}q\mathcal{D}pe^{iS_L[q,p]} \tag{20}$$

[2]

"The infinite-dimensional measure $\mathcal{D}q\mathcal{D}p$ on the set of phase space trajectories is a formal expression, and is not rigorously defined. It instructs us to sum over "all" paths taken by a particle between positions q_i and q_f during the time interval $[t_i,t_f]$. The contribution of a given path to the propagator is weighted by the *Lorentzian action.*

$$S_L[q,p] = \int_{t_i}^{t_f}dt(p\dot{q}-H_W(q,p)) \tag{21}$$

"

2.2 Comments on and extensions of the propagator

2.2.1 The measure factor

If S_L is quadratic in p, then the path integral over p is a Gaussian and can be computed exactly. This isn't always the case, but if you are able to perform an exact integral over p, you'd be left with a "measure factor"[3], say $\mu[q]$. To be precise, you could write

$$K(q_i,t_i;q_f,t_f) = \int_{q(t_i)=q_i}^{q(t_f)=q_f}\mathcal{D}q\mu[q]e^{iS_L[q,\dot{q}]} \tag{22}$$

Where $\dot{q} = \frac{\partial H_W}{\partial p}$. In the semiclassical limit we can often ignore this "measure factor" i.e. set $\mu = 1$, because "it's nontrivial dependence on q is always of order $\hbar$". $\Longleftarrow$ is that *always* always, or just when S_L is quadratic in p?

2.2.2 Boundary conditions

Our definition of S_L has the correct form to yield a well-posed variational problem because the initial and final positions are fixed. It seems what the author is implying is that, in general, a "well-posed" variational problem requires fixed boundary values (which kind of makes sense intuitively).

[2] Uhhhh, that coefficient in the second line looks like it's tending to zero there... I suppose it's fair since the integrand is infinite dimensional, might contribute some counter-term. Mathematically speaking though, it's still suspicious because we'd still be taking some finite number to the power of infinity. If each of the integrands contributes *exactly* 2π, then sure it's a finite term. But if each of the integrands contributes less or more than 2π, then the whole thing should go to either 0 or ∞.

[3] What this is exactly, I don't know, and they don't really explain

2.2.3 General states

Consider two arbitrary states, $|\psi\rangle$ and $|\psi'\rangle$. The path integral expression for their overlap is written

$$\begin{aligned}\langle\psi'|\psi\rangle &= \int dq_i dq_f \, \langle\psi'|q_f,t_f\rangle \langle q_f,t_f|q_i,t_i\rangle \langle q_i,t_i|\psi\rangle \\ &= \int dq_i dq_f \psi'(q_f,t_f)^* \psi(q_i,t_i) \langle q_f,t_f|q_i,t_i\rangle \end{aligned} \tag{23}$$

Now insert $\langle q_f,t_f|q_i,t_i\rangle = K(q_i,t_i;q_f,t_f) = \int_{q_i}^{q_f} \mathcal{D}q\mathcal{D}p e^{iS_L[q,p]}$

$$\langle\psi'|\psi\rangle = \int dq_i dq_f \psi'(q_f,t_f)^* \psi(q_i,t_i) \left(\int_{q_i}^{q_f} \mathcal{D}q\mathcal{D}p e^{iS_L} \right)$$

[4] "Here we absorb the two position integrals dq and dq' into the path integral measure $\mathcal{D}q$. This unfortunate ambiguity in the notation is often left for readers to resolve."

$$\langle\psi'|\psi\rangle = \int \mathcal{D}q\mathcal{D}p \psi'(q_f)^* \psi(q_i) e^{iS_L}$$

Okay, I'm switching up my studying style. Writing all this down is just inefficient. From here on, I just read. How about just questions though?

3 Questions

- Matrix elements of operators = correlators? How are those $\mathcal{O}_j$ matrix *elements* instead of matrices/operators? S1.2.4
- Weyl symbol? S1.2.4
- His explanation of OTOCs is actually very helpful. S1.2.5
- Loops and traces (S1.2.6) is *very* important.
- Having trouble understanding Figure 1. Maybe ask Gustavo to explain, since he's so good with the geometric figure stuff. I just don't get why the path has to have those switchbacks. Wouldn't it make more sense for the dotted-lines to be ordered according to actual time, and then have the path zigzag across?. Why are their dotted-lines ordered according to the path instead of actual time?
- "The p integral evaluates to a q-independent number, so the result is a covariant (Lagrangian), rather than a canonical (Hamiltonian), Path-Integral over positions" $\Longleftarrow$ excuse me?

4 Topics to look up

- Lorentzian Action
- Measure factor

5 Very big question

How can you enforce locality in the continuum limit? Zeno's paradox. Think UV modes across the boundary vs. IR modes between the bulks

[4] I believe it's supposed to be $S_L[q,p]$, right? Maybe he's just doing it this way for conciseness.