# N=2 Supersymmetric SYK Supercharge Anticommutation

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Source: **Supersymmetric SYK Models**

I believe they made a typo in equation (5.2), and wrote commutators instead of anticommutators. Here's what is given in the paper:

$$[Q, \psi^a] = 0 \tag{1}$$

$$[Q, \bar{\psi}_a] = \bar{b^i} \equiv i \sum_{1 \leq j < k \leq N} C_{ijk} \psi^j \psi^k \tag{2}$$

Here's what is actually the case:

$$\{Q, \psi^a\} = 0 \tag{3}$$

$$\{Q, \bar{\psi}_a\} = \bar{b^i} \equiv i \sum_{1 \leq j < k \leq N} C_{ijk} \psi^j \psi^k \tag{4}$$

## 1 Starting point

Definition of supercharge, Q:

$$Q = i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^i \psi^j \psi^k \tag{5}$$

Anticommutation relations between fermions:

$$\{\psi^i, \bar{\psi}_j\} = \delta^i_j \tag{6}$$

$$\{\psi^i, \psi^j\} = 0 \tag{7}$$

$$\{\bar{\psi}_i, \bar{\psi}_j\} = 0 \tag{8}$$

## 2 Proof: $\{Q, \psi^a\} = 0$

$$\{Q, \psi^a\} = Q\psi^a + \psi^a Q$$
$$= i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^i \psi^j \psi^k \psi^a + i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^a \psi^i \psi^j \psi^k$$

Plugging in the anticommutation relation (7)...

$$= -i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^a \psi^i \psi^j \psi^k + i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^a \psi^i \psi^j \psi^k$$
$$= 0 \tag{9}$$

## 3 Proof: $\{Q, \bar{\psi}_a\} = \bar{b}_i \equiv i \sum_{1\le j<k\le N} C_{ijk}\psi^j\psi^k$

Buckle in, this is a wild one.

$$\begin{aligned} \{Q, \bar{\psi}_a\} &= Q\bar{\psi}_a + \bar{\psi}_a Q \\ &= i \sum_{1\le i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k \end{aligned} \tag{10}$$

### 3.1 Second term

Let's consider just the second term on the RHS of eqn (10).

$$A \equiv i \sum_{1\le i<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k$$

We can split the sum up into the following:

1. $1 \le a < i$
2. $1 \le a = i$
3. $i < a < j$
4. $i < a = j$
5. $j < a < k$
6. $j < a = k$
7. $k < a \le N$

$$A = i \sum_{1\le a<i<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i=a<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<a<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k$$

$$+ i \sum_{1\le i<j=a<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<a<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<k=a\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<k<a\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k$$

Group terms together by whether or not any of $(i, j, k)$ is equal to $a$

$$A = i \sum_{1\le a<i<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<a<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<a<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<k<a\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k$$

$$+ i \sum_{1\le i=a<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j=a<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<k=a\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k$$

Anticommute wherever $a \notin (i, j, k)$...

$$= -i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+ i \sum_{1\le i=a<j<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j=a<k\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k + i \sum_{1\le i<j<k=a\le N} C_{ijk}\bar{\psi}_a\psi^i\psi^j\psi^k$$

Plug in wherever $a \in (i,j,k)$

$$= -i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+i \sum_{1\le a<j<k\le N} C_{ajk}\bar{\psi}_a\psi^a\psi^j\psi^k + i \sum_{1\le i<a<k\le N} C_{iak}\bar{\psi}_a\psi^i\psi^a\psi^k + i \sum_{1\le i<j<a\le N} C_{ija}\bar{\psi}_a\psi^i\psi^j\psi^a$$

In the 6th and 7th terms, anticommute to bring $\psi^a$ next to $\bar{\psi}_a$

$$= -i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+i \sum_{1\le a<j<k\le N} C_{ajk}\bar{\psi}_a\psi^a\psi^j\psi^k - i \sum_{1\le i<a<k\le N} C_{iak}\bar{\psi}_a\psi^a\psi^i\psi^k + i \sum_{1\le i<j<a\le N} C_{ija}\bar{\psi}_a\psi^a\psi^i\psi^j$$

Relabel indeces in the last three terms

$$= -i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+i \sum_{1\le a<j<k\le N} C_{ajk}\bar{\psi}_a\psi^a\psi^j\psi^k - i \sum_{1\le j<a<k\le N} C_{jak}\bar{\psi}_a\psi^a\psi^j\psi^k + i \sum_{1\le j<k<a\le N} C_{jka}\bar{\psi}_a\psi^a\psi^j\psi^k \tag{11}$$

Use antisymmetry of $C_{ijk}$ in last three terms

$$= -i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a - i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+i \sum_{1\le a<j<k\le N} C_{ajk}\bar{\psi}_a\psi^a\psi^j\psi^k + i \sum_{1\le j<a<k\le N} C_{ajk}\bar{\psi}_a\psi^a\psi^j\psi^k + i \sum_{1\le j<k<a\le N} C_{ajk}\bar{\psi}_a\psi^a\psi^j\psi^k$$

### 3.2 First term

Now let's inspect the first term on the RHS of eqn (10)

$$B \equiv i \sum_{1\le i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

Expand the sum in the same way as we did $A$

$$B = i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i=a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+i \sum_{1\le i<j=a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<j<k=a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

Group by whether $a \in (i,j,k)$

$$B = i \sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i \sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

$$+i\sum_{1\le i=a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j=a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<k=a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$

Plug in wherever $a \in (i,j,k)$

$$B = i\sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$
$$+i\sum_{1\le a<j<k\le N} C_{ajk}\psi^a\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<a<k\le N} C_{iak}\psi^i\psi^a\psi^k\bar{\psi}_a + i\sum_{1\le i<j<a\le N} C_{ija}\psi^i\psi^j\psi^a\bar{\psi}_a$$

Anticommute in last three terms to bring $\psi^a$ next to $\bar{\psi}_a$

$$B = i\sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$
$$+i\sum_{1\le a<j<k\le N} C_{ajk}\psi^a\bar{\psi}_a\psi^j\psi^k - i\sum_{1\le i<a<k\le N} C_{iak}\psi^i\psi^a\bar{\psi}_a\psi^k + i\sum_{1\le i<j<a\le N} C_{ija}\psi^i\psi^j\psi^a\bar{\psi}_a$$

Anticommute again in last three terms to bring $\psi^a\bar{\psi}_a$ to the front

$$B = i\sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$
$$+i\sum_{1\le a<j<k\le N} C_{ajk}\psi^a\bar{\psi}_a\psi^j\psi^k - i\sum_{1\le i<a<k\le N} C_{iak}\psi^a\bar{\psi}_a\psi^i\psi^k + i\sum_{1\le i<j<a\le N} C_{ija}\psi^a\bar{\psi}_a\psi^i\psi^j$$

Relabel indeces in last three terms

$$B = i\sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$
$$+i\sum_{1\le a<j<k\le N} C_{ajk}\psi^a\bar{\psi}_a\psi^j\psi^k - i\sum_{1\le j<a<k\le N} C_{jak}\psi^a\bar{\psi}_a\psi^j\psi^k + i\sum_{1\le j<k<a\le N} C_{jka}\psi^a\bar{\psi}_a\psi^j\psi^k$$

Use antisymmetry of $C_{ijk}$ in last three terms

$$B = i\sum_{1\le a<i<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<a<j<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<a<k\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a + i\sum_{1\le i<j<k<a\le N} C_{ijk}\psi^i\psi^j\psi^k\bar{\psi}_a$$
$$+i\sum_{1\le a<j<k\le N} C_{ajk}\psi^a\bar{\psi}_a\psi^j\psi^k + i\sum_{1\le j<a<k\le N} C_{ajk}\psi^a\bar{\psi}_a\psi^j\psi^k + i\sum_{1\le j<k<a\le N} C_{ajk}\psi^a\bar{\psi}_a\psi^j\psi^k \tag{12}$$

### 3.3 Sum of first and second terms

Recall $\{Q, \bar{\psi_a}\} \equiv A + B$.

By inspection of the definitions of $A$ and $B$ - equations (11) and (12), respectively - we see that the first four terms of each one will cancel out.

This gives us

$$\{Q, \bar{\psi_a}\} \equiv A + B$$

$$= i \sum_{1 \le a < j < k \le N} C_{ajk} \bar{\psi_a} \psi^a \psi^j \psi^k + i \sum_{1 \le j < a < k \le N} C_{ajk} \bar{\psi_a} \psi^a \psi^j \psi^k + i \sum_{1 \le j < k < a \le N} C_{ajk} \bar{\psi_a} \psi^a \psi^j \psi^k$$

$$+ i \sum_{1 \le a < j < k \le N} C_{ajk} \psi^a \bar{\psi_a} \psi^j \psi^k + i \sum_{1 \le j < a < k \le N} C_{ajk} \psi^a \bar{\psi_a} \psi^j \psi^k + i \sum_{1 \le j < k < a \le N} C_{ajk} \psi^a \bar{\psi_a} \psi^j \psi^k$$

$$= i \sum_{1 \le a < j < k \le N} C_{ajk} \{\bar{\psi_a}, \psi^a\} \psi^j \psi^k + i \sum_{1 \le j < a < k \le N} C_{ajk} \{\bar{\psi_a}, \psi^a\} \psi^j \psi^k + i \sum_{1 \le j < k < a \le N} C_{ajk} \{\bar{\psi_a}, \psi^a\} \psi^j \psi^k$$

$$= i \sum_{1 \le a < j < k \le N} C_{ajk} \psi^j \psi^k + i \sum_{1 \le j < a < k \le N} C_{ajk} \psi^j \psi^k + i \sum_{1 \le j < k < a \le N} C_{ajk} \psi^j \psi^k$$

$$= i \sum_{1 \le j < k \le N, j \ne a, k \ne a} C_{ajk} \psi^j \psi^k$$

$$= i \sum_{1 \le j < k \le N} C_{ajk} \psi^j \psi^k - i \sum_{1 \le j < k \le N} C_{ajk} \psi^j \psi^k |_{j=a} - i \sum_{1 \le j < k \le N} C_{ajk} \psi^j \psi^k |_{k=a}$$

$$\{Q, \bar{\psi_a}\} = i \sum_{1 \le j < k \le N} C_{ajk} \psi^j \psi^k - i \sum_{1 \le a < k \le N} C_{aak} \psi^a \psi^k - i \sum_{1 \le j < a \le N} C_{aja} \psi^j \psi^a$$

Which gives us, by antisymmetry of $C_{ijk}$

$$\{Q, \bar{\psi_a}\} = i \sum_{1 \le j < k \le N} C_{ajk} \psi^j \psi^k \tag{13}$$

Mwahahaha