Renormalization Group Flows

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These are much condensed versions of **David Skinner's lecture notes**.

1 Set-up

Couldn't word it any better than Skinner himself.

"Even a humble glass of pure water consists of countless H_2O molecules, which are made from atoms that involve many electrons perpetually executing complicated orbits around a dense nucleus, the nucleus itself is a seething mass of protons and neutrons glued together by pion exchange, these hadrons are made from the complicated and still poorly understood quarks and gluons which themselves maybe all we can make out of tiny vibrations of some string, or modes of a theory yet undreamed of. How then is it possible to understand anything about water without first solving all the deep mysteries of Quantum Gravity?

In classical physics the explanation is really an aspect of the Principle of Least Action: if it costs a great deal of energy to excite a degree of freedom of some system, either by raising it up its potential or by allowing it to whizz around rapidly in space–time, then the least action configuration will be when that degree of freedom is in its ground state. The corresponding field will be constant and at a minimum of the potential. This constant is the zero mode of the field, and plays the role of a Lagrange multiplier for the remaining low–energy degrees of freedom. You used Lagrange multipliers in mechanics to confine wooden beads to steel hoops. This is a good description at low energies, but my sledgehammer can excite degrees of freedom in the hoop that your Lagrange multiplier doesn't reach.

We must re-examine this question in QFT because we're no longer constrained to sit at an extremum of the action. The danger is already apparent in perturbation theory, for even in a process where all external momenta are small, momentum conservation at each vertex still allows for very high momenta to circulate around the loop and the value of these loop integrals would seem to depend on all the details of the high–energy theory. The Renormalization Group (RG), via the concept of *universality*, will emerge as our quantum understanding of why it is possible to understand physics at all."

2 Integrating out degrees of freedom

First, impose UV-cutoff Λ_0

Consider the most general action under this cutoff:

$$S_{\Lambda_0}[\varphi] = \int d^dx \left[\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \sum_i \Lambda_0^{d-d_i} g_{i0}\mathcal{O}_i(x)\right] \tag{1}$$

Let's inspect the integrand:

1. "kinetic term": $\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi \sim \frac{1}{2}(\nabla\varphi)^2 \Longrightarrow \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2$
2. "coupling term": $\sum_i \Lambda_0^{d-d_i} g_{i0}\mathcal{O}_i(x)$
   (a) dimensionless **coupling constants**: g_{i0}
   (b) $\Lambda_0^{d-d_i}$ not necessary but included to ensure g_{i0} dimensionless
   (c) $\mathcal{O}_i(x)$ are arbitrary local operators of dimension $d_i > 0$; each $\mathcal{O}_i$ can be a Lorentz-invariant monomial involving some power n_i of fields and their derivatives, e.g. $\mathcal{O}_i \sim (\partial\varphi)^{r_i}\varphi^{s_i}$ where $r_i + s_i = n_i$

Partition function:

$$\mathcal{Z}_{\Lambda_0}(g_{i0}) = \int_{C^\infty(M)_{\leq\Lambda_0}} \mathcal{D}\varphi e^{-S_{\Lambda_0}[\varphi]/\hbar} \tag{2}$$

Split the general field (operator) $\varphi(x)$ into low- and high-energy modes

$$\varphi(x) = \int_{|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x}\tilde{\varphi}(p)$$

$$= \int_{|p|\leq\Lambda} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x}\tilde{\varphi}(p) + \int_{\Lambda\leq|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x}\tilde{\varphi}(p)$$

$$=: \phi(x) + \chi(x) \tag{3}$$

Then we can write the action as

$$S_{\Lambda_0}[\varphi] = S_{\Lambda_0}[\phi + \chi] = S^0[\phi] + S^0[\chi] + S^{int}_{\Lambda_0}[\phi, \chi] \tag{4}$$

Where S^0 gives the kinetic term of each field, respectively, and $S^{int}_{\Lambda_0}$ gives the *effective interaction.*

$$S^{int}_{\Lambda_0} := \int d^d x \left[\sum_i \Lambda_0^{d-d_i} g_{i0}\mathcal{O}_i(x)\right] \tag{5}$$

We also find that splitting φ gives $\mathcal{D}\varphi = \mathcal{D}\phi\mathcal{D}\chi$
Putting (3) and (4) together, we can prove

$$\mathcal{Z}_\Lambda(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0}) \tag{6}$$

Where

$$\mathcal{Z}_\Lambda(g_i(\Lambda)) = \int_{C^\infty(M)\leq\Lambda} \mathcal{D}\varphi e^{-S^{eff}_\Lambda[\varphi]/\hbar} \tag{7}$$

and $S^{eff}_\Lambda[\phi]$ is the *effective action at scale* Λ

$$S^{eff}_\Lambda[\phi] := -\hbar \log\left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0)}} \mathcal{D}\chi \exp(-S_{\Lambda_0}[\phi + \chi]/\hbar\right] \tag{8}$$

Likewise we can define the *effective interaction at scale* Λ

$$S^{int}_\Lambda[\phi] = -\log\left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0)}} \mathcal{D}\chi \exp(-S^0[\chi] - S^{int}_{\Lambda_0}[\phi, \chi])\right] \tag{9}$$

To recap, these are the *renormalization group equations*

1. For effective action at scale Λ:

$$S^{eff}_\Lambda[\phi] := -\hbar \log\left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0)}} \mathcal{D}\chi \exp(-S_{\Lambda_0}[\phi + \chi]/\hbar\right] \tag{10}$$

2. For effective interaction at scale Λ:

$$S^{int}_\Lambda[\phi] = -\log\left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0)}} \mathcal{D}\chi \exp(-S^0[\chi] - S^{int}_{\Lambda_0}[\phi, \chi])\right] \tag{11}$$

Importantly[1], the partition function obtained from the effective action at scale Λ is *exactly the same* as the original partition function we started with, which depended explicitly on the cutoff scale.

$$\mathcal{Z}_\Lambda(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0}) \tag{12}$$

As the scale is lowered infinitesimally, the equivalence (12) becomes the differential equation

$$\Lambda\frac{d\mathcal{Z}_\Lambda(g)}{d\Lambda} = \left(\Lambda\frac{\partial}{\partial\Lambda}|_{g_i} + \Lambda\frac{\partial g_i(\Lambda)}{\partial\Lambda}\frac{\partial}{\partial g_i}|_\Lambda\right)\mathcal{Z}_\Lambda(g) = 0 \tag{13}$$

[1]although I haven't been able to prove this for myself yet

This is known as the *renormalization group equation for the partition function*; this is an example of a *Callan-Symanzik equation.*

So we have found that our coupling constants indeed depend on the scale we choose. More precisely, "as we change the scale by integrating out [presumably higher-energy] modes, the coupling constants in the effective action S_Λ^{eff} vary to account for the change in the degrees of freedom over which we take the path integral,...

Trying to physicalize this

Consider his bead-on-hoop example. If we're looking at it from the macro (classical) scale, we integrate out the quark-gluon interactions. We don't care about them, and the partition function that actually gives us useful (classical) information about the system is the one that can be derived from the *effective* action.

...I think?

so that the partition function is in fact independent of the scale at which we define our theory, provided this scale is below our initial cut-off Λ_0."