Computing the Einstein-Hilbert Action of a Euclidean Schwarzschild Black Hole

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1 Acknowledgements

Haoming Liu was very helpful in getting me started on actually evaluating the integral. I also found the **original GHY paper**, which used a very different method but served as a useful sanity check.

2 Set-up

The starting point is the Einstein-Hilbert action for a gravitational field,

$$S = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{g}R, \tag{1}$$

with the Gibbons-Hawking-York (GHY) boundary-term added in order to ensure stationary action without imposing any boundary conditions on the metric.

$$+\frac{1}{8\pi}\int_{\partial\mathcal{M}} \sqrt{|h|}K, \tag{2}$$

and a counter boundary-term to regulate a divergence of the GHY term at $r_0 \to \infty$

$$+\frac{1}{8\pi}\int_{\partial\mathcal{M}} \sqrt{|h|}K_0 \tag{3}$$

We consider the Euclidean Schwarzschild black hole, which is the same as the normal (Lorentzian) Schwarzschild black hole, but with $t \to -i\tau$.

The metric then looks like

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_2^2 \\ &= \phi(r)d\tau^2 + \frac{dr^2}{\phi(r)} + r^2 d\Omega_2^2 \end{aligned} \tag{4}$$

Where

$$\phi(r) = 1 - \frac{2M}{r} \tag{5}$$

Thoughts: Thermal equilibrium as a prerequisite for Euclidean time?

Time can be periodic only if no *net, long-term* changes are happening (i.e. thermal equilibrium). If the same thing happens every day, who's to say you're not just repeating the *same* day over and over?

For the Maldacena Information Loss paradox: can the black hole be in thermal equilibrium *while* it's absorbing a new particle? It seems like it's experiencing some *net, everlasting* change in its total energy/thermodynamics. Information is seemingly lost because, if you assume the black hole is in constant thermal equilibrium for all time, that implicitly requires no net change due to the newly-introduced particle, i.e. the particle "disappears".

Periodic time seems valid at the level of closed, quantum systems, but at the macro-scale it contradicts the second law of thermodynamics: entropy must increase, and net, everlasting changes have to happen at some point. Closed systems don't stay closed.

But what if we consider a system with radius $r = \infty$? Wouldn't this system be eternally closed?

3 Bulk term

We say that the bulk term, (1), vanishes because our spacetime is vacuum i.e. $R = 0$.

Thoughts: Ignoring the Bulk Term and AdS/CFT

Well, technically *almost* all of our spacetime is in vaccum. But what about the singularity?!

Surely there's some relation between the ignored bulk term and all the boundary terms we've added in order to regulate divergences, which seems like a statement of AdS $\longleftrightarrow$ CFT.

Digression and speculation:

1. It seems valid that the divergent term at $r \to 0$ should affect the boundary at $r \to \infty$, by virtue of being infinite.
    - Technically the black hole would have to be eternal in order to impose boundary conditions on $r = \infty$. In order to be specific about time conditions, think *retardation and radiation* of EM waves.
2. Ignoring this divergent origin term then creates a "virtual" divergent boundary term. Mathematically speaking, this divergent boundary term appears because we've ignored its divergent counter-term at the origin.
3. In order to regulate these divergences then, we can just check that the *interaction* between $r = 0$ and $r = \infty$ is finite/normalizable/physical. We can learn about the two boundaries, $r = 0$ and $r = \infty$, by operating on the d+1 *bulk* between them.
4. If things are "conformal", there's some level of *predictability* about how local behavior is correlated with boundary behavior. If there were some stochastic, highly nonlinear divergence of behavior at different scales, we would need a much more complicated (information-heavy) theory.
5. Overall this also seems similar to a Green's function.

4 Gibbons-Hawking-York boundary term

As our boundary, we choose a timelike hypersurface at some fixed radius $r = r_0$, whose metric is given by:

$$ds^2_{bdry} = \phi(r_0)d\tau^2 + r_0^2 d\Omega_2^2 \tag{6}$$

This metric basically describes a spherical shell of fixed radius $r = r_0$, with constantly-varying time separation across the entire surface. Using this hypersurface as our metric means we've chosen to foliate spacetime into such concentric[1] timelike hypersurfaces, one at each value of $r_0 \in [0, \infty]$.

Thoughts: Timelike hypersurface at the singularity and discretization of spacetime

What happens at the $r \to 0$ timelike hypersurface? Well, there's zero spatial or angular separation between events on this hypersurface. Squeezing a finite set of spacetime events (e.g. the large yet finite mass/informational-history of a black hole) into this infinitesimally small spatial distribution requires an infinite extension of the time dimension. But that doesn't mean the set of spacetime events on this hypersurface is suddenly anything but finite.

It also seems to me that these spacetime events (i.e. the "information") are the actual physical/fundamental quantities[a]. It's not obvious that the infinite time-separation between any two members of the information-set is necessarily a "divergence" in the physical sense, because despite the infinite *separation* you never actually get an infinite *density*.

Is the notion of "discretization" important here? In second quantization, even a finite density and finite integration limits can create divergences if you allow arbitrarily high frequencies - i.e. arbitrarily small discretization of spacetime. In set theory, for the finite interval $1 \to 2$, the set $[1, 2] \in \mathbb{N}$ has finite cardinality while the set $[1, 2] \in \mathbb{R}$ has infinite cardinality.

In our example, the timelike hypersurface at $r \to 0$ gives infinite separation between members of a finite set, which seems inverse to the set theory example. Instead of finite intervals (say, $1 \to 1.5$) between members of a set of infinite cardinality ($[1, 2] \in \mathbb{R}$), we have infinite intervals between members of a finite set.

Is there a discretization limit on the metric? At what point is it no longer differentiable? And what happens as you approach this limit?

[a] I could be wrong, who knows

5 Enough talk, let's fight

We want

$$S_{GHY} = -\frac{1}{8\pi} \int_{\partial \mathcal{M}} \sqrt{|h|} K \tag{7}$$

We know

$$n = -\sqrt{\phi(r)} \frac{\partial}{\partial r} \tag{8}$$

$$h_{\alpha\beta} = \begin{bmatrix} \phi(r_0) & 0 & 0 \\ 0 & r_0^2 & 0 \\ 0 & 0 & r_0^2 sin^2(\theta) \end{bmatrix} \implies \sqrt{|h|} = r_0^2 sin(\theta) \sqrt{\phi(r_0)} \tag{9}$$

$$\begin{aligned} K = \nabla_\alpha n^\alpha &= \partial_\alpha n^\alpha + \Gamma^\alpha_{\alpha\lambda} n^\lambda \\ &= \partial_r n^r + \Gamma^\alpha_{\alpha r} n^r \\ &= \partial_r n^r + (\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) n^r \\ &= \partial_r n^r + \left(\frac{M}{r^2 \phi(r)} - \frac{M}{r^2 \phi(r)} + 0 + 0 \right) n^r \\ &= \partial_r n^r = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n^r)|_{r=r_0} \end{aligned} \tag{10}$$

[1] The fact that they are concentric may seem intuitive, but mathematically speaking it makes some assumptions about the symmetries of your spacetime, etc. See Carroll Ch. 9

$$= -\frac{1}{r_0}\left(2\sqrt{\phi(r_0)} + \frac{M}{r_0\sqrt{\phi(r_0)}}\right) \tag{11}$$

Then

$$\begin{aligned}
I &= \int_{\partial\mathcal{M}} \sqrt{|h|}K \\
&= \int_0^\beta \int_0^\pi \int_0^{2\pi} (r_0^2 sin(\theta)\sqrt{\phi(r_0)}) \left(-\frac{1}{r_0}\left(2\sqrt{\phi(r_0)} + \frac{M}{r_0\sqrt{\phi(r_0)}}\right)\right) d\tau sin(\theta) d\theta d\phi \\
&= -2\pi\beta \int_0^\pi (r_0 sin(\theta)\sqrt{\phi(r_0)}) \left(2\sqrt{\phi(r_0)} + \frac{M}{r_0\sqrt{\phi(r_0)}}\right) sin(\theta) d\theta \\
&= -2\pi\beta(r_0\sqrt{\phi(r_0)}) \left(2\sqrt{\phi(r_0)} + \frac{M}{r_0\sqrt{\phi(r_0)}}\right) \int_0^\pi sin^2(\theta) d\theta \\
&= -2\pi\beta[2r_0\phi(r_0) + M](\frac{\pi}{2}) \\
&= -2\pi^2\beta r_0\phi(r_0) - \pi^2\beta M \\
&= -2\pi^2\beta r_0(1 - \frac{2M}{r_0}) - \pi^2\beta M \\
&= -2\pi^2\beta r_0 + 3\pi^2\beta M
\end{aligned} \tag{12}$$

I'm off by a factor relative to Hartman's expression. Let's see how much

$$I_{me} = -2\pi^2\beta r_0 + 3\pi^2\beta M \tag{13}$$

$$I_H = \beta(8\pi r_0 - 12\pi M) = C I_{me} \tag{14}$$

$$\begin{aligned}
C = \frac{I_H}{I_{me}} &= \frac{\beta(8\pi r_0 - 12\pi M)}{-2\pi^2\beta r_0 + 3\pi^2\beta M} \\
&= \frac{8r_0 - 12M}{-2\pi r_0 + 3\pi M} \\
&= \frac{8(r_0 - \frac{3}{2}M)}{-2\pi(r_0 - \frac{3}{2}M)} \\
&= -\frac{1}{4\pi}
\end{aligned} \tag{15}$$

$$I_{me} = -4\pi I_H \tag{16}$$

Fortunately, we also have the **original GHY paper**. Let's check their expression.

$$I_{GH} = -32\pi^2 iM(2r_0 - 3M) = C_2 I_{me} \tag{17}$$

$$\begin{aligned}
C_2 = \frac{I_{GH}}{I_{me}} &= \frac{-32\pi^2 iM(2r_0 - 3M)}{-2\pi^2\beta r_0 + 3\pi^2\beta M} \\
&= \frac{32iM(2r_0 - 3M)}{2\beta r_0 - 3\beta M} \\
&= \frac{32iM(2r_0 - 3M)}{\beta(2r_0 - 3M)} \\
&= \frac{32iM}{\beta} = \frac{32iM}{8\pi M}
\end{aligned} \tag{18}$$

$$= \frac{4i}{\pi} \tag{19}$$

$$I_{me} = -\frac{i\pi}{4} I_{GH} = -4\pi I_H \tag{20}$$

Uhhh... let's maybe put a pin in this. Moving on: the boundary counter-term at $g \to$ Minkowski

6 Boundary Counter Term

To get rid of that divergent r_0 dependence at $r_0 \to \infty$, we have to *subtract* what is basically the GHY term but this time, with a boundary/hypersurface that is embedded in flat space instead of Schwarzschild. Here we go.

Since this is still a timelike hypersurface, but it's now embedded in flat spacetime:

$$n = \frac{\partial}{\partial r} \tag{21}$$

$$h_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_0^2 & 0 \\ 0 & 0 & r_0^2 sin^2(\theta) \end{bmatrix} \implies \sqrt{|h|} = r_0^2 sin(\theta) \tag{22}$$

Employing the following relation,

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu) \tag{23}$$

K becomes

$$K = \nabla_\alpha n^\alpha = \frac{1}{\sqrt{|h|}} \frac{\partial}{\partial r} (\sqrt{|h|} n^r)$$

$$= \frac{1}{r^2 sin(\theta)} \frac{\partial}{\partial r} (r^2 sin(\theta))|_{r=r_0}$$

$$K = K_0 = \frac{2}{r_0} \tag{24}$$

Then

$$I_0 = \int_{\partial \mathcal{M}} \sqrt{|h|} K_0$$

$$= \int_0^\beta \int_0^\pi \int_0^{2\pi} r_0^2 sin(\theta) \left(\frac{2}{r_0} \right) d\tau sin(\theta) d\theta d\phi$$

$$= 2\pi\beta(2r_0) \int_0^\pi sin^2(\theta) d\theta \tag{25}$$

$$= 4\pi\beta r_0 (\frac{\pi}{2}) \tag{26}$$

$$I_0 = 2\pi^2 \beta r_0 \tag{27}$$

7 Expression for Action

Finally,

$$S = S_{EH} + S_{GHY} + S_0$$

$$= -\frac{1}{16\pi} \int_{\mathcal{M}} \sqrt{g} R + \frac{1}{8\pi} \int_{\partial \mathcal{M}} \sqrt{|h|} K + \frac{1}{8\pi} \int_{\partial \mathcal{M}} \sqrt{|h|} K_0$$

$$= 0 + \frac{1}{8\pi}(-2\pi^2 \beta r_0 + 3\pi^2 \beta M) + \frac{1}{8\pi} 2\pi^2 \beta r_0$$

$$= \frac{3\pi\beta M}{8} = \frac{3\pi(8\pi M)M}{8} = 3\pi^2 M^2 \tag{28}$$

Bruh