Recall conformal transformation preserves angles

(1) $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ s.t. $g_{\alpha\beta}(\sigma) \to \Omega^2 g_{\alpha\beta}(\sigma)$

A CFT is a field theory which is invariant under such (conformal) x-forms.

We will consider the case of a CFT in a fixed (not dynamical) background metric.

↳ Meaning (1) is a global symmetry with associated conserved (Noether) currents.

Not worried about particles, S-matrices.

"We will be concerned with correlation functions and the behavior of different operators under conformal x-forms".

Start with 2D Minkowski coordinates

$\rightarrow (\sigma^0, \sigma^1)$

Now get Euclidean worldsheets

$\longrightarrow (\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$

Now form complex coordinates from Euclidean coordinates

$\rightarrow \quad z = \sigma^1 + i\sigma^2, \quad \bar{z} = \sigma^1 - i\sigma^2$

$\quad = \sigma^1 - \sigma^0 \qquad\qquad = \sigma^1 + \sigma^0$

$(z, \bar{z})$ are "Euclidean analogue of light cone/null coordinates". Length always equal to zero/all intervals are null.

Recall: Holomorphic is like analytic.

Every holomorphic function is analytic, (according to GPT-4)

Holomorphic Fns "left-moving" and anti-holomorphic "right-moving"

The holomorphic derivatives are:

$$\partial_z = \partial = \frac{1}{2}(\partial_1 - i\partial_2)$$

$$\partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$

- $\partial z = \bar{\partial}\bar{z} = 1$
- $\partial \bar{z} = \bar{\partial} z = 0$

We will usually work in Flat, Euclidean space: $ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz\,d\bar{z}$

$$\Rightarrow \quad g_{zz} = g_{\bar{z}\bar{z}} = 0$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$$

The measure factor is:

$$dz\,d\bar{z} = 2d\sigma^1 d\sigma^2$$

Define delta-Fn: $\int d^2z\, \delta(z,\bar{z}) = 1$

Also : $\int d^2\sigma\, \delta(\sigma) = 1$

meaning $\delta(\sigma) = 2\,\delta(z,\bar{z})$ (I think. Might be $\frac{1}{2}$).

Contravariant vectors: $V^z = V^1 + iV^2$

$$V^{\bar{z}} = V^1 - iV^2$$

Covariant: $V_z = \frac{1}{2}(V^1 - iV^2)$

$$V_{\bar{z}} = \frac{1}{2}(V^1 + iV^2)$$

In flat Euclidean space, complex coordinates, conformal x-forms are any holomorphic change of coordinates:

$$z \to z' = f(z), \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z})$$

Then $ds^2 = dz\,d\bar{z} \to \left|\frac{df}{dz}\right|^2 dz\,d\bar{z}$

- $\exists$ Infinite number of conformal x-forms
  - $\rightarrow$ special to 2D CFTs

    In higher dimensions, space of conformal x-forms is finite-dimensional group.

☆ For a $\mathbb{R}^{p,q}$ theory, the conformal group is $SO(p+1, q+1)$ when $p+q>2$. ☆

Some simple + important examples:

- $z \to z + a$; translation
- $z \to \zeta z$
  - if $|\zeta| = 1$, rotation
  - if $\zeta \neq 1$, $\zeta \in \mathbb{R}$, then scale x-form aka dilatation

- If we treat $z, \bar{z}$ as independent variables (i.e. $\bar{z}$ is <u>not</u> defined as $\bar{z} = z^*$), we extend the worldsheet from $\mathbb{R}^2$ + $\mathbb{C}^2$.
  - → Just recall that at the end of the day, we're sitting on the real slice $\mathbb{R}^2 \subset \mathbb{C}^2$ defined by $\bar{z} = z^*$.

4.1.1 : Stress-Energy Tensor

Aka energy-momentum tensor

* "Defined in the usual way as the matrix of conserved currents which arise from translational invariance"

→ $\delta\sigma^\alpha = \varepsilon^\alpha$ (i.e. $\sigma^\alpha \to \sigma^\alpha + \varepsilon^\alpha$)

→ $\delta S = \int d^2\sigma \, J^\alpha \partial_\alpha \varepsilon$, for some function of the fields J^α

- δS must be zero when ε constant; this is the definition of a symmetry.

- $\delta S = 0$ also when EOMs obeyed, for all variations of $\varepsilon(\sigma)$. So when EOMs obeyed:

$\partial_\alpha J^\alpha = 0$, J^α is the conserved current.

Let's see this for translational invariance:

- Promote ε to a fn. of coordinates/worldsheet variables: $\varepsilon^\alpha = \varepsilon^\alpha(\sigma)$

- Then translation given by $\delta\sigma^\alpha = \varepsilon^\alpha(\sigma)$.
  - Under a dynamical metric, this is a diffeomorphism (recall, we're assuming theory is invariant under x-form), so the change: $\delta g_{\alpha\beta} = \partial_\alpha \varepsilon_\beta + \partial_\beta \varepsilon_\alpha$ leaves theory unchanged.
- So in a fixed metric, change in action is opposite to what we get if we just change the metric (since both changes together give an invariant action).

Thus: $\delta S = \int d^2\sigma \, J^\alpha \partial_\alpha \varepsilon$

should technically be $\frac{\delta S}{\delta g_{\alpha\beta}}$

$$= -\int d^2\sigma \frac{\partial S}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} = -2\int d^2\sigma \frac{\partial S}{\partial g_{\alpha\beta}} \partial_\alpha \varepsilon_\beta$$

The conserved current is $J_\alpha = -2 \frac{\partial S}{\partial g_{\alpha\beta}}$

Add a normalization (standard in string theory, not necessarily others) to get the stress-energy tensor:

S as in the density, the one inside the integral of δS

$$\boxed{\bigstar \quad T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}} \qquad (4.4)}$$

Recall the "usual definition": "The matrix of conserved currents which arise from translational invariance".

In flat space: $\partial^\alpha T_{\alpha\beta} = 0$

In curved space: $\nabla^\alpha T_{\alpha\beta} = 0$ (covariantly conserved)

* Stress-Energy Tensor is Traceless in a CFT:

Consider scale x-form (special case of conformal x-form)

$$\delta g_{\alpha\beta} = \varepsilon g_{\alpha\beta}$$

$$\downarrow$$

$$\delta S = \int d^2 z \, \frac{\partial S}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta}$$

Partial w.r.t. contravariant is covariant

Partial w.r.t. covariant is contravariant

$$\delta S = \int d^2 z \left(- \frac{\sqrt{g}}{4\pi} T^{\alpha\beta} \right) \left(\varepsilon g_{\alpha\beta} \right)$$

$$= - \frac{1}{4\pi} \int d^2 z \sqrt{g} \, \varepsilon \, T^{\alpha}_{\ \alpha}$$

CFT invariant under such x-forms meaning

$\delta S = 0 \implies T^{\alpha}_{\ \alpha} = 0$, $T_{\alpha\beta}$ traceless

Many theories have traceless $T_{\alpha\beta}$ at classical level, but this often fails at quantum level.

"Technically the difficulty arises due to the need to introduce a scale when regulating the theories"

☆ "We will be interested in 2D theories that succeed in preserving conformal symmetry at quantum level".

☆ "Looking ahead: Even when the conformal invariance survives in a 2D quantum theory, the vanishing trace $T^{\alpha}_{\alpha} = 0$ will only turn out to hold in flat space. We will derive this result in 4.4.2".

In complex coordinates

$$T^{\alpha}_{\ \alpha} = 0 \rightarrow T_{z\bar{z}} = 0$$

"conservation eqn." $\partial_\alpha T^{\alpha\beta} = 0$

$$\partial T^{zz} = 0$$

$$\updownarrow$$

$$\bar{\partial} T_{zz} = 0$$

$$\partial T_{\bar{z}\bar{z}} = 0$$

$$\updownarrow$$

$$\bar{\partial} T^{\bar{z}\bar{z}} = 0$$

In other words, $T_{zz} = T_{zz}(z)$ is holomorphic
$T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})$ is anti-holomorphic

Simplified notation moving forward:

$$T_{zz}(z) \equiv T(z), \quad T_{\bar{z}\bar{z}}(\bar{z}) \equiv \bar{T}(\bar{z})$$

skimming 4.1.2 : Noether currents, invariance of T under independent x-forms of $z, \bar{z}$

* First two lines of 4.6 important for learning how

$$\delta g^{\alpha\beta} = \partial^\alpha \delta\sigma^\beta$$

4.1.2: Noether Currents

$T_{\alpha\beta}$ provides Noether currents for translation

How about other conformal x-forms?

Consider $z' = z + \varepsilon(z)$, $\bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z})$

$\varepsilon =$ constant. $\rightarrow$ translation

$\varepsilon \sim z \longrightarrow$ rotation, dilatation

First promote $\varepsilon(z) \rightarrow \varepsilon(z, \bar{z})$ (so it depends on the coordinates. Not just scales with the coordinate it x-forms).

$$\Rightarrow \quad \delta S = -\int d^2z \sqrt{g}\, \frac{\partial S}{\partial g^{\alpha\beta}}\, \delta g^{\alpha\beta}$$

$$\frac{\partial S}{\partial g^{\alpha\beta}} = -\frac{1}{4\pi} T_{\alpha\beta}$$

$$\delta g^{\alpha\beta} = \partial^\alpha \varepsilon^\beta + \partial^\beta \varepsilon^\alpha$$

$$= 2\partial^\alpha \varepsilon^\beta \quad \text{because } g^{\alpha\beta} \text{ symmetric}$$

$$= 2\partial^\alpha \delta\sigma^\beta$$

$$= \frac{1}{2\pi}\int d^2z\, T_{\alpha\beta}\left(\partial^\alpha \delta\sigma^\beta\right)$$

Important takeaway from 4.1.2:

Noether currents of conformal x-forms are proportional to the stress-energy tensor

* For $\delta z = \varepsilon(z)$, $\delta \bar{z} = 0$

$\hookrightarrow$ $J^z = 0$, $J^{\bar{z}} = T_{zz}(z)\varepsilon(z) \equiv T(z)\varepsilon(z)$

For $\delta z = 0$, $\delta \bar{z} = \bar{\varepsilon}(\bar{z})$

$\hookrightarrow$ $\bar{J}^z = \bar{T}(\bar{z})\bar{\varepsilon}(z)$, $\bar{J}^{\bar{z}} = 0$

4.1.3 Example: Free Scalar Field

$$X = X(\sigma), \quad S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X\, \partial^\alpha X$$

(No minus sign because Euclidean signature)

1. Check if conformal.

consider rescaling: $\sigma^\alpha \to \lambda\sigma^\alpha$.

Viewed as an active x-form, coordinates remain fixed but value of field X at σ gets moved to point $\lambda\sigma$. Meaning:

$$X(\sigma) \to X(\lambda^{-1}\sigma)$$

$$\frac{\partial X(\sigma)}{\partial\sigma^\alpha} \to \frac{\partial X(\lambda^{-1}\sigma)}{\partial\sigma^\alpha} = \frac{\partial(\lambda^{-1}\sigma^\alpha)}{\partial\sigma^\alpha}\frac{\partial X(\lambda^{-1}\sigma)}{\partial(\lambda^{-1}\sigma^\alpha)}$$

$$= \frac{1}{\lambda}\frac{\partial X(\tilde{\sigma})}{\partial\tilde{\sigma}} \quad \text{where } \tilde{\sigma} = \lambda^{-1}\sigma$$

$$\frac{\partial X(\sigma)}{\partial\sigma^\alpha} \longrightarrow \frac{1}{\lambda}\frac{\partial X(\tilde{\sigma})}{\partial\tilde{\sigma}}$$

Also $d^2 z = d^2(\lambda \lambda^{-1} z) = d^2(\lambda \tilde{z}) = \lambda^2 d^2 \tilde{z}$

$$\Rightarrow \quad S = \frac{1}{4\pi\alpha'} \int d^2 z \, \partial_\alpha X \, \partial^\alpha X$$

$$\longrightarrow \quad \tilde{S} = \frac{1}{4\pi\alpha'} \int d^2 \tilde{z} \, \partial_{\tilde{\alpha}} X \, \partial^{\tilde{\alpha}} X$$

Note that any polynomial interaction term for X would thus break conformal invariance.

Since $z_0 \in (-\infty, \infty)$ OR $(0, \infty)$ And $z_1 \in (-\infty, \infty)$, then the limits of integration are the same. This means $S = \tilde{S}$, the action is invariant under the generalized conformal x-form $\Rightarrow$ theory is conformal!!!

Probably the more rigorous way to do this is explicit evaluation of δS. But we're on a tight schedule.

2. $T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\partial S}{\partial g^{\alpha\beta}}$, $S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \partial^\alpha X$

☆ $\sqrt{g}$ always implicit

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \, \partial_\alpha X \partial^\alpha X$$

$$\text{"}S\text{"} = \frac{1}{4\pi\alpha'} \sqrt{g} \, \partial_\alpha X \partial^\alpha X$$

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\partial \text{"}S\text{"}}{\partial g^{\alpha\beta}}$$

$$\sqrt{g} = \sqrt{g_{\alpha\beta} g^{\alpha\beta}} = \sqrt{\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 6 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \sqrt{1} = 1$$

$$T_{\alpha\beta} = -4\pi \frac{\partial \text{"}S\text{"}}{\partial g^{\alpha\beta}} = -4\pi \frac{\partial}{\partial g^{\alpha\beta}} \left(\frac{1}{4\pi\alpha'} \sqrt{g} \, \partial_\alpha X \partial^\alpha X \right)$$

$$= -\frac{1}{\alpha'} \frac{\partial}{\partial g^{\alpha\beta}} \left(\sqrt{g} \, \partial_\alpha X \partial^\alpha X \right)$$

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\frac{\partial}{\partial g^{\alpha\beta}}\left(\sqrt{g^{\mu\nu}g_{\mu\nu}}\;\partial_\alpha X\, g^{\alpha\rho}\,\partial_\rho X\right)$$

$$= -\frac{1}{\alpha'}\partial_\alpha X\,\partial_\rho X\,\frac{\partial}{\partial g^{\alpha\beta}}\left(g^{\alpha\rho}\sqrt{g^{\mu\nu}g_{\mu\nu}}\right)$$

$$= -\frac{1}{\alpha'}\partial_\alpha X\,\partial_\rho X\left(\frac{\partial g^{\alpha\rho}}{\partial g^{\alpha\beta}}\sqrt{g^{\mu\nu}g_{\mu\nu}} + g^{\alpha\rho}\frac{\partial}{\partial g^{\alpha\beta}}\sqrt{g^{\mu\nu}g_{\mu\nu}}\right)$$

★ Carroll 4.69

$$= -\frac{1}{\alpha'}\partial_\alpha X\,\partial_\rho X\left[\delta^\rho_\beta\sqrt{g} + g^{\alpha\rho}\left(-\frac{1}{2}\sqrt{g}\,g_{\alpha\beta}\right)\right]$$

$$= -\frac{1}{\alpha'}\partial_\alpha X\,\partial_\rho X\left[\delta^\rho_\beta - \frac{1}{2}g^{\alpha\rho}g_{\alpha\beta}\right]$$

$$= -\frac{1}{\alpha'}\partial_\alpha X\,\partial_\rho X\,\delta^\rho_\beta + \frac{1}{2}\frac{1}{\alpha'}\partial_\alpha X\,\partial_\rho X\, g^{\alpha\rho}g_{\alpha\beta}$$

$$= -\frac{1}{\alpha'}\partial_\alpha X\,\partial_\beta X + \frac{1}{2}\frac{1}{\alpha'}\partial_\alpha X\,\partial^\alpha X\,\underbrace{g_{\alpha\beta}}_{=\delta_{\alpha\beta}\text{ in flat Euclidean space}}$$

$$\downarrow$$

★ 4.9: $$T_{\alpha\beta} = -\frac{1}{\alpha'}\left[\partial_\alpha X\,\partial_\beta X - \frac{1}{2}(\partial X)^2\delta_{\alpha\beta}\right]$$

3. Check $T^{\alpha}_{\alpha} = 0$

$$T^{\alpha}_{\alpha} = -\frac{1}{\alpha'}\left(\partial_{\alpha} X \partial^{\alpha} X - \frac{1}{2}\delta^{\alpha}_{\alpha}(\partial X)^2\right)$$

$$= -\frac{1}{\alpha'}\Big[\partial_0 X \partial^0 X - \frac{1}{2}\delta^0_0 (\partial X)^2$$
$$+ \partial_1 X \partial^1 X - \frac{1}{2}\delta^1_1 (\underset{\sim}{\partial X})^2\Big] \checkmark$$

Meaning non-covariant gradient, NOT $\partial = \partial_z$.
i.e partial gradient

(NOT $\partial X = \partial_z X$. Annoying, ik)

4. Check $T_{z\bar{z}} = 0$

★ $T_{\alpha\beta} = -\frac{1}{\alpha'}\left[\partial_\alpha X \partial_\beta X - \frac{1}{2}(\partial X)^2 \delta_{\alpha\beta}\right]$ is a tensor statement, meaning either LHS or RHS transform as tensors. Therefore the statement is true in any coordinate system (so long as you're consistent in employing the correct coordinate x-forms / Jacobians). Although partial derivatives are not tensors necessarily, in flat space they are equivalent to the covariant derivative, which is a tensor.

★ Important to be clear on choice of coordinates though. As stated, T4.9, we have:

$\alpha \in [\sigma_1, \sigma_2]$, $\beta \in [\sigma_1, \sigma_2] \Rightarrow \alpha' \in [z, \bar{z}]$, $\beta' \in [z, \bar{z}]$

We know this from $S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X \partial^\alpha X$ (not numbered, top of T on section 4.1.3), used to define $T_{\alpha\beta}$ as given, because the integral measure is $d^2\sigma$.

$$T_{\alpha'\beta'} = \Lambda^{\alpha}_{\alpha'} \Lambda^{\beta}_{\beta'} T_{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} T_{\alpha\beta} \quad \star$$

$$T_{z\bar{z}} = \frac{\partial b^1}{\partial z}\frac{\partial b^1}{\partial \bar{z}} T_{b_1 b_1} + \frac{\partial b^1}{\partial z}\frac{\partial b^2}{\partial \bar{z}} T_{b_1 b_2} + \frac{\partial b^2}{\partial z}\frac{\partial b^1}{\partial \bar{z}} T_{b_2 b_1} + \frac{\partial b^2}{\partial z}\frac{\partial b^2}{\partial \bar{z}} T_{b_2 b_2}$$

i. $\frac{\partial b^1}{\partial z}\frac{\partial b^1}{\partial \bar{z}} = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$\frac{\partial b^2}{\partial z}\frac{\partial b^2}{\partial \bar{z}} = \frac{1}{2i}\left(-\frac{1}{2i}\right) = \frac{1}{4}$

$$\Rightarrow \frac{\partial b^1}{\partial z}\frac{\partial b^1}{\partial \bar{z}} T_{b_1 b_1} + \frac{\partial b^2}{\partial z}\frac{\partial b^2}{\partial \bar{z}} T_{b_2 b_2} = \frac{1}{4} T^{\alpha}{}_{\alpha} = 0$$

because $T_{\alpha\beta}$ traceless

$$\Rightarrow T_{z\bar{z}} = \frac{\partial b^1}{\partial z}\frac{\partial b^2}{\partial \bar{z}} T_{b_1 b_2} + \frac{\partial b^2}{\partial z}\frac{\partial b^1}{\partial \bar{z}} T_{b_2 b_1}$$

ii. $T_{b_1 b_2} = -\frac{1}{\alpha'}\left(\partial_{b_1} X \partial_{b_2} X - \frac{1}{2}\cancel{\delta_{b_1 b_2}}(\partial X)^2\right)$

$T_{b_1 b_2} = -\frac{1}{\alpha'}\left(\frac{\partial X}{\partial b_1}\frac{\partial X}{\partial b_2}\right), \quad T_{b_2 b_1} = -\frac{1}{\alpha'}\left(\frac{\partial X}{\partial b_2}\frac{\partial X}{\partial b_1}\right)$

Nobody has explicitly said that the stress-energy tensor is symmetric, yet. Just so happens to be symmetric here.

$$\Rightarrow T_{z\bar{z}} = \left(\frac{\partial b_1}{\partial z}\frac{\partial b_2}{\partial \bar{z}} + \frac{\partial b_2}{\partial z}\frac{\partial b_1}{\partial \bar{z}}\right) T_{b_1 b_2}$$

iii. $b_1 = \frac{1}{2}(z + \bar{z})$, $b_2 = \frac{1}{2i}(z - \bar{z})$

$$\Rightarrow \quad T_{z\bar{z}} = \left[\frac{1}{2}\left(-\frac{1}{2i}\right) + \frac{1}{2i}\left(\frac{1}{2}\right)\right] T_{b_1 b_2}$$

$$T_{z\bar{z}} = 0 \checkmark$$

5. $$T_{\alpha'\beta'} = \frac{\partial \sigma_1}{\partial x^{\alpha'}}\frac{\partial \sigma_1}{\partial x^{\beta'}} T_{\sigma_1\sigma_1} + \frac{\partial \sigma_1}{\partial x^{\alpha'}}\frac{\partial \sigma_2}{\partial x^{\beta'}} T_{\sigma_1\sigma_2} + \frac{\partial \sigma_2}{\partial x^{\alpha'}}\frac{\partial \sigma_1}{\partial x^{\beta'}} T_{b_2 b_1} + \frac{\partial \sigma_2}{\partial x^{\alpha'}}\frac{\partial \sigma_2}{\partial x^{\beta'}} T_{b_2 b_2}$$

a. Confirm $T = T_{zz} = -\frac{1}{\alpha'}\partial X \partial X$

$$T_{zz} = \frac{\partial \sigma_1}{\partial z}\frac{\partial \sigma_1}{\partial z} T_{\sigma_1\sigma_1} + \frac{\partial \sigma_1}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{\sigma_1\sigma_2} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_1}{\partial z} T_{b_2 b_1} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{b_2 b_2}$$

$$T_{zz} = \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right] + \frac{\partial \sigma_1}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{\sigma_1\sigma_2} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_1}{\partial z} T_{b_2 b_1} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{b_2 b_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right] + \frac{\partial \sigma_1}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{\sigma_1\sigma_2} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_1}{\partial z} T_{b_2 b_1} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{b_2 b_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right] + \left(\frac{1}{2}\right)\left(\frac{1}{2i}\right) T_{\sigma_1\sigma_2} + \left(\frac{1}{2i}\right)\left(\frac{1}{2}\right) T_{b_2 b_1} + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{b_2 b_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right] + \frac{1}{4i}\left(T_{\sigma_1\sigma_2} + T_{b_2 b_1}\right) + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{b_2 b_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X\, \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right]$$

$$+ \frac{1}{4i}\left(T_{\sigma_1\sigma_2} + T_{\sigma_2\sigma_1}\right) + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{\sigma_2\sigma_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X\, \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right]$$

$$+ \frac{1}{4i}\left[-\frac{1}{\alpha'}\left(\frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_2}\right) - \frac{1}{\alpha'}\left(\frac{\partial X}{\partial \sigma_2}\frac{\partial X}{\partial \sigma_1}\right)\right] + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{\sigma_2\sigma_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X\, \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right]$$

$$- \frac{1}{2i\alpha'}\left(\frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_2}\right) + \frac{\partial \sigma_2}{\partial z}\frac{\partial \sigma_2}{\partial z} T_{\sigma_2\sigma_2}$$

$$= \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_1} X\, \partial_{\sigma_1} X - \frac{1}{2}(\partial X)^2\right)\right]$$

$$- \frac{1}{2i\alpha'}\left(\frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_2}\right) - \frac{1}{4}\left[-\frac{1}{\alpha'}\left(\partial_{\sigma_2} X\, \partial_{\sigma_2} X - \frac{1}{2}(\partial X)^2\right)\right]$$

$$= \frac{1}{4\alpha'}\left[-\left(\partial_{\sigma_1} X\, \partial_{\sigma_1} X - \frac{1}{2}(\cancel{\partial X})^2\right) + \left(\partial_{\sigma_2} X\, \partial_{\sigma_2} X - \frac{1}{2}(\cancel{\partial X})^2\right)\right]$$

$$- \frac{1}{2i\alpha'}\left(\frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_2}\right)$$

$$= \frac{1}{4\alpha'}\left(\partial_{\sigma_2} X\, \partial_{\sigma_2} X - \partial_{\sigma_1} X\, \partial_{\sigma_1} X\right) - \frac{1}{2i\alpha'}\left(\frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_2}\right)$$

$$= \frac{1}{4\alpha'}\left(\frac{\partial X}{\partial \sigma_2}\frac{\partial X}{\partial \sigma_2} - \frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_1}\right) - \frac{1}{2i\alpha'}\left(\frac{\partial X}{\partial \sigma_1}\frac{\partial X}{\partial \sigma_2}\right)$$

$$= \frac{1}{4\alpha'}\left[\left(\frac{\partial X}{\partial z}\frac{\partial z}{\partial \sigma_2}\right)^2 - \left(\frac{\partial X}{\partial z}\frac{\partial z}{\partial_1}\right)^2\right] - \frac{1}{2i\alpha'}\left(\frac{\partial x}{\partial \sigma_1}\frac{\partial x}{\partial \sigma_2}\right)$$

$$= \frac{1}{4\alpha'}\left[\left(\frac{\partial X}{\partial z}\frac{\partial z}{\partial \sigma_2}\right)^2 - \left(\frac{\partial X}{\partial z}\frac{\partial z}{\partial_1}\right)^2\right] - \frac{1}{2i\alpha'}\left(\frac{\partial x}{\partial \sigma_1}\frac{\partial x}{\partial \sigma_2}\right)$$

$$= \frac{1}{4\alpha'}\left[\left(\frac{\partial X}{\partial z}\frac{\partial z}{\partial \sigma_2}\right)^2 - \left(\frac{\partial X}{\partial z}\frac{\partial z}{\partial_1}\right)^2\right] - \frac{1}{2i\alpha'}\left[\left(\frac{\partial x}{\partial z}\right)^2\frac{\partial z}{\partial \sigma_1}\frac{\partial z}{\partial \sigma_2}\right]$$

$$= \frac{1}{4\alpha'}\left(\frac{\partial x}{\partial z}\right)^2\left[\left(\frac{\partial z}{\partial \sigma_2}\right)^2 - \left(\frac{\partial z}{\partial \sigma_1}\right)^2\right]$$

$$- \frac{1}{2i\alpha}\left(\frac{\partial x}{\partial z}\right)^2\left(\frac{\partial z}{\partial \sigma_2}\frac{\partial z}{\partial \sigma_1}\right)$$

$$= \frac{1}{4\alpha'}\left(\frac{\partial x}{\partial z}\right)^2\left(i^2 - 1^2\right) - \frac{1}{2i\alpha}\left(\frac{\partial x}{\partial z}\right)^2\left(i \cdot 1\right)$$

$$= -\frac{2}{4\alpha'}\left(\frac{\partial x}{\partial z}\right)^2 - \frac{1}{2\alpha}\left(\frac{\partial x}{\partial z}\right)^2$$

$$= -\frac{1}{\alpha'}\partial X \partial X \quad \checkmark$$

Skimmig 6. Derive EOM $\partial\bar{\partial}X = 0$

7. Derive general soln. for EOM:

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$$

8. Why T holomorphic, $\bar{T}$ anti-holomorphic when evaluated on soln. from #7.

4.2.1: Operator Product Expansion

- Terminology: "Field" refers to both primary and derivative/"descendant" fields

  e.g. $\phi, \partial^n \phi, e^{i\phi}$

  ↳ "In CFT the term 'field' refers to any <u>local</u> expression that we can write down". As opposed to QFT, where it only refers to primary fields.

"The OPE is a statement about what happens as local operators approach each other. The idea is that two local operators inserted at nearby points can be closely approximated by a string of operators at one of these points."

★ 4.10:

$$\mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w}) = \sum_k \underbrace{C_{ij}^{k}(z-w,\bar{z}-\bar{w})}\,\mathcal{O}_k(w,\bar{w})$$

set of fns which, because of translation invariance, depend only on the separation between the two operators.

★ 4.10 holds as an operator insertion inside time-ordered functions. "

$$\langle \mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w})\ldots\rangle = \sum_k C_{ij}^{k}(z-w,\bar{z}-\bar{w})\,\langle \mathcal{O}_k(w,\bar{w})\ldots\rangle$$

The ... is not always explicitly written, usually implicit.

- Correlation fns always assumed to be time-ordered (or path-ordered?)

→ "This means that, as far as the OPE is concerned, everything commutes since the ordering of the operators is determined inside the correlation fn. anyway".

This seems like a oxymoron. Ask Gustavo.

- The other operator insertions are arbitray except they should be at a large distance w.r.t. $|z-w|$.
- In a CFT the OPE is an exact statement, has "radius of convergence" equal to distance to nearest other insertion.
- OPEs have singular behavior at $z \to w$. This is what Hartman means

by "hitting other operator insertions (inside correlation functions)". This singular behavior will be the only thing we care about. ☆ "IT WILL TURN OUT TO CONTAIN THE SAME INFORMATION AS COMMUTATION RELATIONS, AS WELL AS TELLING US HOW OPERATORS TRANSFORM UNDER SYMMETRIES"

$$\Updownarrow$$

SINGULAR BEHAVIOR AT OPERATOR COLLISIONS DETERMINES THE ALGEBRA OF OBSERVABLES

☆ "Indeed, in many equations we will simply write the singular terms in the OPE and denote the non-singular terms as '+...'."

4.2.2. Ward Identities

- Set of operator equations
- Like Noether's theorem but quantized, for QFTs.

"We start by considering a general theory with a symmetry. Later we will restrict to conformal symmetries".

Recall path integral $Z = \int \underbrace{D\phi}_{} \, e^{-S[\phi]}$

QFT rigorous defn. of fields. Primary fields

A symmetry of the quantum theory is s.t. the infinitesimal x-form: $\phi' = \phi + \varepsilon \delta \phi$ leaves both the action and the measure invariant: $S[\phi'] = S[\phi]$, $D\phi' = D\phi$

"To first order, change has to be prop. to ε":

$$\phi \to \phi + \varepsilon \delta\phi$$

$$Z \to \int D\phi' \exp(-S[\phi'])$$

By definition of $S[\phi']$

$$= \int D\phi \exp\left(-S[\phi] - \frac{1}{2\pi}\int J^\alpha \partial_\alpha \varepsilon\right)$$

$$= \int D\phi \, e^{-S[\phi]}\left(1 - \frac{1}{2\pi}\int J^\alpha \partial_\alpha \varepsilon\right)$$

Taylor expand e^x, drop quad order and higher

★ Not clear why this must be equal to $\int D\phi e^{-S[\phi]}$ for finite number of fields. Or even maybe <u>some</u> cardinality of infinite number of ~~a~~ fields (e.g. for a countably infinite set, you should be able to distinguish between ϕ and ϕ'). ← Ahhh, but each one still weighted appropriately by $e^{-S[\phi]}$ or $e^{-S[\phi']}$, ★ respectfully. I think that answers my question.

★ Also, we stated if $\varepsilon\delta\phi$ is symmetry, then action and measure invariant $S[\phi'] = S[\phi]$, $D\phi' = D\phi$

$$Z \to \int D\phi \, e^{-S[\phi]}\left(1 - \frac{1}{2\pi}\int J^{\alpha}\partial_{\alpha}\varepsilon\right)$$

$$\star = \int D\phi \, e^{-S[\phi]}$$

$$\Rightarrow \int D\phi \, e^{-S[\phi]}\left(\int J^{\alpha}\partial_{\alpha}\varepsilon\right) = 0$$

"This must hold for all ε..."

$$\Longrightarrow \langle \partial_{\alpha} J^{\alpha} \rangle = 0$$

"This gives the quantum version of Noether's theorem: the vacuum expectation value of the divergence of the [conserved] current vanishes"

Why specifically the vacuum state?

di Francesco et. al. pg. 45: "Remembering that the correlation function is the vacuum expectation value of a time-ordered product in the operator formalism"

→ Also, recall: In the UV all finite-energy states look the same. Might as well use vacuum since most tractable.

Employ the same trick for time-ordered correlation functions:

$$\langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle = \frac{1}{Z} \int D\phi \, e^{-S[\phi]} \underbrace{\mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n)}$$

These operators can be primary or descendant fields

Under x-form: $\mathcal{O}_i \rightarrow \mathcal{O}_i + \varepsilon \delta \mathcal{O}_i$

Once again promote $\varepsilon \rightarrow \varepsilon(z)$.

$$\langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle \rightarrow \frac{1}{Z} \int D\phi' \, e^{-S[\phi']} \mathcal{O}_1'(z_1) \ldots \mathcal{O}_n'(z_n)$$

$e^{-S[\phi']}$ $$= \frac{1}{Z} \int D\phi' \exp(-S[\phi']) \, \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n)$$

Because we chose $\varepsilon(z)$ to only have support away from the other operator insertions, i.e

$\mathcal{O}_i \rightarrow \mathcal{O}_i + \varepsilon \delta \mathcal{O}_i = \mathcal{O}_i$

$$= \frac{1}{Z} \int D\phi \, \exp\left(-S[\phi] - \frac{1}{2\pi} \int J^\alpha \partial_\alpha \varepsilon\right) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n)$$

$$\overset{\text{1st order}}{\approx} \frac{1}{Z} \int D\phi \, e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int J^\alpha \partial_\alpha \varepsilon\right) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n)$$

Phillipe says integration by parts

still not 100% sure about switching order

$$\Rightarrow \int D\phi\, e^{-S[\phi]} \left(\int \partial_\alpha J^\alpha \varepsilon\right) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) = 0$$

holds for all ε...

$$\Rightarrow \langle \partial_\alpha J^\alpha(z) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle = 0$$

For z not belonging to $\{z_1, \ldots, z_n\}$

What if $z \in \{z_1, \ldots, z_n\}$

☆ See "di Francesco 2.4" notes for derivation of Ward identity.

As stated by Tong, the Ward identity is:

4.11 : $$-\frac{1}{2\pi} \int_\varepsilon \partial_\alpha \langle J^\alpha(z) \mathcal{O}_1(z_1) \ldots \rangle = \langle \delta \mathcal{O}_1(z_1) \ldots \rangle$$

over region of support for ε, i.e where $\varepsilon \neq 0$

As stated by Hartman H.23.36:

$$\partial_\mu \langle J^\mu(y) \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \sum_i \delta(y - x_i) \langle \mathcal{O}_1 \ldots \delta\mathcal{O}_i \ldots \mathcal{O}_n \rangle$$

☆ Ward Identities For Conformal x-Forms

For any vector J^α ...

stokes — 2D, Euclidean signature

$$\int_{\Sigma} \partial_\alpha J^\alpha \overset{\downarrow}{=} \oint_{\partial\Sigma} J_\alpha \hat{n}^\alpha \overset{\downarrow}{=} \oint_{\partial\Sigma} \left(J_1 d\sigma^2 - J_2 d\sigma^1 \right)$$

switch to z coordinates

$$\overset{\downarrow}{=} -i \oint_{\partial\Sigma} \left(J_z dz - \bar{J}_{\bar{z}} d\bar{z} \right)$$

Apply this to LHS of Ward identity (4.11)

$\downarrow$

$$\frac{i}{2\pi} \oint_{\partial\Sigma} dz \langle J_z(z,\bar{z}) \mathcal{O}_1(\sigma_1) \ldots \rangle - \frac{i}{2\pi} \oint_{\partial\Sigma} d\bar{z} \langle \bar{J}_{\bar{z}}(z,\bar{z}) \mathcal{O}_1 \ldots \rangle$$

$$= \langle \delta \mathcal{O}_1(\sigma_1) \ldots \rangle$$

For conformal x-forms, we know J is holomorphic and $\bar{J}$ anti-holomorphic (see Tong section 4.1.2). (Recall holomorphic $\sim$ analytic) which means we can use residue theorem for the contour integral

↓

$$\frac{i}{2\pi}\oint_{\partial\xi} dz\, J_z(z)\,\mathcal{O}_i(b_i) = -\mathrm{Res}[J_z \mathcal{O}_i]$$

$$\frac{i}{2\pi}\oint_{\partial\xi} d\bar{z}\, J_{\bar{z}}(\bar{z})\,\mathcal{O}_i(b_i) = -\mathrm{Res}[J_{\bar{z}} \mathcal{O}_i]$$

Employing OPE (Tong 4.10)

$$J_z(z,\bar{z})\,\mathcal{O}_i(b_i) = \sum_k C^k(z - z(b_i), \bar{z} - \bar{z}(b_i))\,\mathcal{O}_k(b_i)$$

$$\text{OR} = \sum_k C^k\left(b_i - \frac{z+\bar{z}}{2}, b_i - \frac{z-\bar{z}}{2i}\right)\mathcal{O}_k(z,\bar{z})$$

↓ For $\delta z = \varepsilon(z)$

$$\delta\mathcal{O}_i(b_i) = -\mathrm{Res}[J_z(z)\,\mathcal{O}_i(b_i)] = -\mathrm{Res}[\varepsilon(z)\,T(z)\,\mathcal{O}_i(b_i)]$$

Not sure how this equal to $J_z\mathcal{O}_i$.

Possibly: $\delta\mathcal{O} = \mathcal{O}' - \mathcal{O}$

In general: if we know OPE between an operator and the stress-tensors $T(z), \bar{T}(\bar{z})$, We know how the operator transforms under conformal symmetry.

Shouldn't this word be x-form?

Alternatively, if we know how it transforms then we know at least part of its OPE with T and $\bar{T}$.

To understand why

$$J(z)\,\mathcal{O}_1(w,\bar{w}) = -\mathrm{Res}\left[J_z \mathcal{O}_1\right] = \ldots + \frac{\mathrm{Res}\left[J_z \mathcal{O}_1(w,\bar{w})\right]}{z-w} + \ldots$$

Re-read residue theorem in Boas.

Possibly something to do with principal term in Laurent expansion.

☆ Maybe ask Gustavo:

The residue is the principal term in the Laurent expansion

☆ Important definition:

" An operator $\mathcal{O}$ is said to have <u>weight</u> $(h, \tilde{h})$ if, under $\delta z = \varepsilon z$ and $\delta \bar{z} = \bar{\varepsilon} \bar{z}$, $\mathcal{O}$ transforms as

$$\delta \mathcal{O} = -\varepsilon(h\mathcal{O} + z\partial_z \mathcal{O}) - \bar{\varepsilon}(\tilde{h}\mathcal{O} + \bar{z}\partial_{\bar{z}}\mathcal{O})$$

☆ Both h and $\tilde{h}$ are real numbers.

△ In a unitary CFT, $h, \tilde{h} \geq 0$ for all operators.

☆ Recall <u>spin</u> is the eigenvalue under rotation: $s = h - \tilde{h}$

☆ Recall scaling dimension is $\Delta = h + \tilde{h}$

☆ Recall rotations are implemented (generated?) by the angular momentum operator:

$$L = -i(\sigma^1 \partial_{\sigma_2} - \sigma^2 \partial_{\sigma_1})$$

* And scalings are generated by the dilation operator:

$$D = x^\alpha \partial_\alpha = z\partial_z + \bar{z}\partial_{\bar{z}}$$

Read up on scaling dimension

Compare transformation law,

4.16: $\delta\mathcal{O} = -\varepsilon(h\mathcal{O} + z\partial_z\mathcal{O}) - \bar{\varepsilon}(\tilde{h}\mathcal{O} + \bar{z}\partial_{\bar{z}}\mathcal{O})$

With ward identity,

(4.12)

$$\delta\mathcal{O}_1(b_1) = -\mathrm{Res}[J_z(z)\mathcal{O}_1(b_1)] = -\mathrm{Res}[\varepsilon(z)T(z)\mathcal{O}_1(b_1)]$$

Keeping in mind the Noether current due to rotation + scaling $\delta z = \varepsilon z$

4.7: $J^z = 0$, $J^{\bar{z}} = T_{zz}(z)\varepsilon(z) \equiv T(z)\varepsilon(z)$

(From $\delta\bar{z} = \bar{\varepsilon}\bar{z}$ we get $\bar{J}^{\bar{z}} = 0$, $\bar{J}^z = \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z})$)

The residue of the JO OPE will determine the $\frac{1}{z^2}$ terms in the TO OPE. From transformation law, we know the OPE between T and and operator $\mathcal{O}$ with weight $(h, \tilde{h})$ must take the form:

Very important for Hartman 4.4.2: Weyl Anomaly

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \underbrace{\frac{\partial\mathcal{O}(w,\bar{w})}{z-w}}$$

Torg Bottom of pg. 75

Residue only due to the principal term right?

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \ldots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}}$$

A primary operator is one whose OPE with T and $\bar{T}$ has no higher singularities than $(z-w)^{-2}$ or $(\bar{z}-\bar{w})^{-2}$, respectively.

For a primary operator,

$$T(z)\mathcal{O}(w,\bar{w}) = h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \text{non-singular}$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \text{non-singular}$$

Since we know all the singularities in the OPE of a primary operator, we can reconstruct the x-form of the operator under all conformal x-forms:

Focusing on just $\delta z = \varepsilon(z)$...

$$\underbrace{\delta\mathcal{O}(w,\bar{w})} = -\operatorname{Res}\left[\varepsilon(z)T(z)\mathcal{O}(w,\bar{w})\right]$$

still not sure relation between $T\mathcal{O}$ and $\delta\mathcal{O}$

H says $\delta\mathcal{O} \equiv \mathcal{O}' - \mathcal{O}$

But not sure how to derive this expression

★ Singularities in OPE determine how it x-forms.

$$= -\operatorname{Res}\left[\varepsilon(z)\left(h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots\right)\right]$$

Tong 4.2.2 bottom of pg 73

For smooth, conformal x-forms, $\varepsilon(z)$ has no singularities at $z=w$, so it can be Taylor expanded about $z=w$:

$$\varepsilon(z)=\varepsilon(\omega)+\varepsilon'(\omega)(z-\omega)+\frac{\varepsilon''(\omega)}{2}(z-\omega)^2+\dots$$

Plugging this into the residue expression on the RHS of $\delta O(w,\bar{\omega})$...

$$\delta O(w,\bar{w})=-h\,\varepsilon'(\omega)O(\omega,\bar{w})-\varepsilon(\omega)\partial O(\omega,\bar{\omega})$$

Since no singularities higher than $(z-\omega)^{-2}$, the Taylor expansion ensures no singularities at all besides these 2 (since it has powers of $(z-\omega)^n$ for $n\in[0,\infty]$)

For anti-holomorphic x-form $\delta\bar{z}=\bar{\varepsilon}(\bar{z})$:

My guess

$$\delta O(w,\bar{w})=-\tilde{h}\,\bar{\varepsilon}'(\bar{\omega})O(\omega,\bar{\omega})-\bar{\varepsilon}(\bar{\omega})\bar{\partial}\,O(\omega,\bar{\omega})$$

That's for an infinitesimal x-form. For finite x-form $z \to z'(z)$ and $\bar{z} \to \bar{z}'(\bar{z})$, just integrate up.

☆ In general, the transformation of a primary operator [under <u>conformal</u> x-forms] is given by:

$$\mathcal{O}(z,\bar{z}) \to \underline{\mathcal{O}'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\tilde{h}} \mathcal{O}(z,\bar{z})}$$

<u>T 4.18</u>

One of our main interests for a CFT is finding the spectrum of weights $(h,\tilde{h})$ of primary fields. This will be equivalent to computing the particle mass spectrum in a QFT. In stat. mech., the weights of primary operators are the <u>critical exponents</u>.

Don't know what those are

4.3 Example: Free Scalar Field

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha X\, \partial^\alpha X$$

Classical EOM is $\partial^2 X = 0$ (in terms of σ).

Top of pg. 83: Why must TT OPE take that form based on the weight of TT? ← Tong bottom of pg. 75
These (my) notes, pg. 41

☆ T is not primary unless $c=0$
⇒ In general this is never the case

☆ Mapping between Euclidean cylinder and Euclidean plane is <u>conformal</u>

☆ $\frac{E}{2\pi} = -\frac{1}{12}$, Ground state energy of free-scalar field (on a cylinder)

↑

Funny, that's equal to $\sum_{n=0}^{\infty} n$!

That's exactly what Tong uses to derive the same result in section 2.2.2!

4.2.2: The Weyl anomaly

"The purpose of this section is to show that:

$$\langle T^{\alpha}{}_{\alpha} \rangle = -\frac{c}{12} R \qquad (4.35)$$

which obviously vanishes in flat space. So, in curved space, stress-energy not necessarily traceless.

Gravitational anomaly: require $c = \tilde{c}$, so theory can be consistent in fixed, curved backgrounds

What about in a dynamical metric?

Why must $c = \tilde{c}$ for "theories to be consistent", and why specifically in a fixed background?

4.4.3: c is also responsible for the density of high-energy states.

Euclidean torus (with compactified time) tells us that the high- and low-energy partition function are related.

→ "Modular invariance": $Z[\frac{4\pi^2}{\beta}] = Z[\beta]$

☆ "In a CFT, the entropy of high-energy states is given by:

Cardy's Formula: $S(E) \sim \sqrt{cE}$

And the high-energy density scales as

$$S(E) \to 2\pi \sqrt{\frac{c}{6}\left(ER - \frac{c}{24}\right)} \qquad (4.44)$$

✯ Familiar topics Tong has touched upon:

- Central charge counts degrees of freedom, but now from an entirely new approach. Pg. 83. "This is true in a deep sense!"
- C-theorem

"We expect information is lost as we flow from a UV theory to the IR. The c-theorem makes this intuition precise"

Never thought of that: could the c-theorem be related to the information paradox.

↳ Like I said before, it's all about degrees-of-freedom. The inside of a black hole has so few, it's like a theory with very small c_0 (a long way down the

RG Flow).

Must be

↳ Related to area-law scaling.

"It may be that the final theory is gapped - meaning that everything is massless"

← That's a good one. Ask Gustavo maybe.