Entanglement Entropy and the Renormalization Group

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May 2023

Based on **David Skinner's lecture notes**. See *Renormalization Group Flows* for a more mathematically-complete version of these notes.

1 Integrating out degrees of freedom

So we have the renormalization group equations, which basically tell us that correlation functions of low-energy observables and the partition function are independent of one's choice of scale Λ'[1]. The only scale that the partition function explicitly depends on is the UV cutoff scale Λ_0

We find that the coupling constants, as well as the (normalization) constant in front of the kinetic term, *do* depend on one's choice of scale.

We can use the renormalization group equations to go from a theory at a higher-energy-scale (where by "theory" we mean the action integral, integrated over all possible fields in your theory), to a theory at a lower-energy-scale. These group equations include (but aren't necessarily limited to) the group equations for the effective action, the effective interaction, and the partition function.

The RG equation for the partition function is basically a differential equation that determines how the the coupling constants change as one varies the scale. In words, this equations states: "As we change the scale by integrating out modes, the couplings in the effective action S_Λ^{eff} vary to account for the change in the degrees of freedom over which we take the path integral, so that the partition function is in fact independent of the scale at which we define our theory, provided this scale is below our initial cut–off Λ_0."

The RG equation for the effective action involves integrating out degrees of freedom that are higher-energy. This means our lower-energy theories necessarily have a smaller number of degrees-of-freedom, i.e. a smaller number of dimensions. This is what Hartman is talking about at the very top of the chapter: RG "flows" from higher-dimensional CFTs to lower-dimensional CFTs.

On the topic of CFTs. Each coupling constant has a *beta function*, basically telling us how the coupling varies w.r.t. the scale Λ. The *anomalous dimension* is basically the beta function of the kinetic term. The beta function (anomalous dimension) for any coupling (field operator kinetic term) depends on all the other couplings (other field operators). They later state (in 5.2) that the anomalous dimension actually depends on the other couplings.

We also learn that the couplings and wavefunction normalizations change such that *correlators* are unaltered by change of scale. (Equation 5.22).

Re-read section 5.1.2 "Anomalous dimensions", especially to understand the meaning of "anomalous dimension". Kind of skipped a lot in this section.

[1] I *believe* this is at least partly because we enforce unitarity

2 Renormalization Group Flow

"Critical point": choice of coupling constants such that the beta functions vanish, i.e. the coupling constants no longer depend on the scale. Since the anomalous dimension is a function of these couplings, it's also scale-independent at the critical point.

"...power law behavior of correlation functions is characteristic of scale-invariant theories."

Critical theories are special. The metric appears in the action, we find that the variation in the partition function due to a variation of the metric is proportional to the expectation value of the stress-energy tensor. If the metric transformation is just a scale transformation then, scale invariance at a critical point implies that the expectation of the (trace of) the stress-energy tensor is zero.

"In fact, all known examples of Lorentz–invariant, unitary QFTs that are scale invariant are actually invariant under the larger group of conformal transformations and it's believed that all critical points of RG flows are CFTs" $\Longleftarrow$ very important.

Apparently, by definition the beta functions vanish when the coupling constants are stricly real-valued. $\Longleftarrow$ follow up on that one.

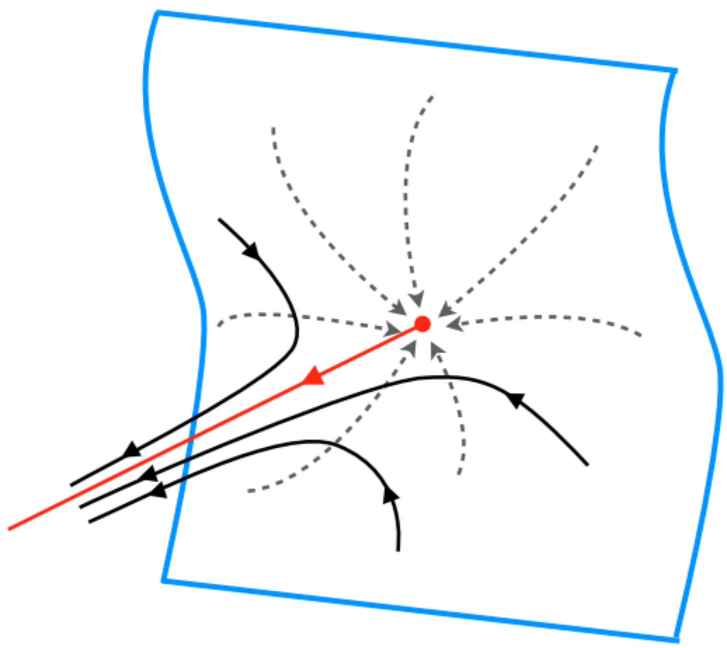

Figure 1: Theories on the critical surface flow (dashed lines) to a critical point in the IR. Turning on relevant operators drives the theory away from the critical surface (solid lines), with flow lines focussing on the (red) trajectory emanating from the critical point.

"Now consider starting near a critical point and turning on the coupling to any operator with $\Delta_i > d$. According to (5.33) this coupling becomes smaller as the scale Λ is lowered, or as we probe the theory in the IR." $\Longleftarrow$ Does this mean that there are no couplings in the IR? Everything is a free field? Wait this only applies to operators with $\Delta_i > d \Longleftarrow$ we call such an operator "irrelevant"; turning it on just makes us flow back to the original critical point.

The "critical surface" $\mathcal{C}$ is defined as the set of coordinates (at least in the local vicinity of g_i^*) of irrelevant operators, such that the flow from any point on this surface (i.e. any combination of irrelevant operators) takes us back to the critical point g_i^*.

Relevant operators have $\Delta_i < d$ and grow as the scale is lowered. "If our action contains vertices with relevant couplings then RG flow will drive us away from the critical surface $\mathcal{C}$ as we head into the IR. Starting precisely

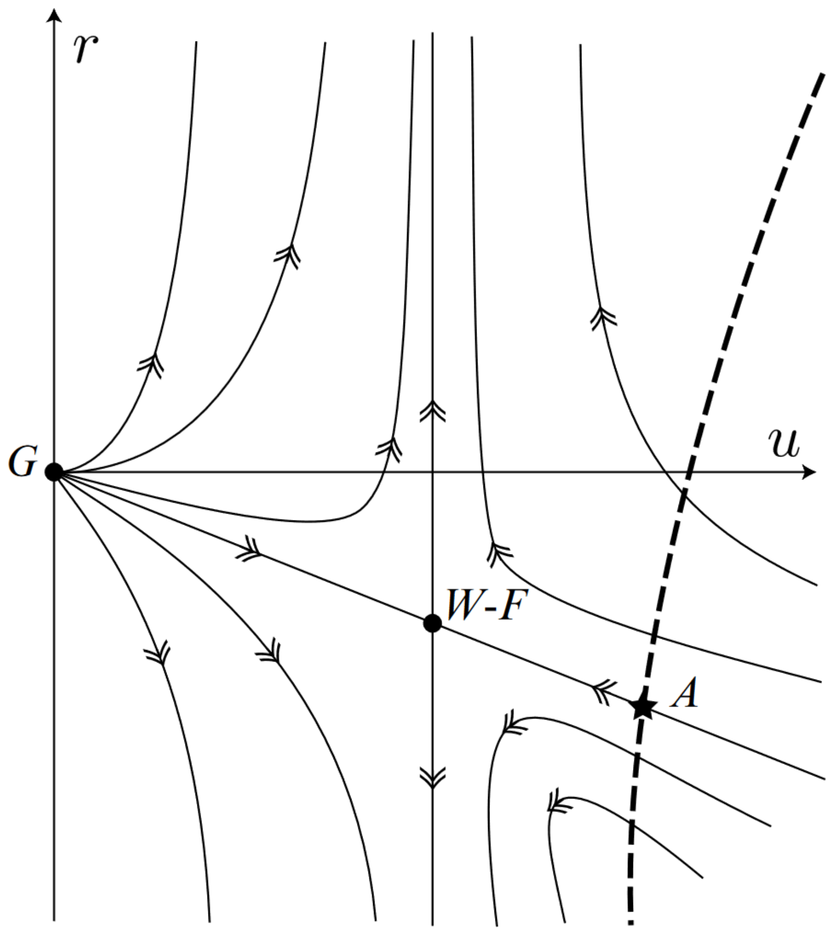

Figure 2: The dashed line shows the path of a single theory as one varies the coupling constants. In some condensed matter systems, one does this by varying the temperature.

from a critical point and turning on a relevant operator generates what is known as a *renormalized trajectory*: the RG flow emanating from the critical point. As we probe the theory at lower and lower scales we evolve along the renormalized trajectory either forever or until we eventually meet another critical point g_i^{**}."

Typically the critical surface has finite codimension.

Marginal operators have vanishing eigenvalues, and therefore neither increase nor decrease under RG flow. However, quantum corrections can make them either *marginally relevant* or *marginally irrelevant* under a long-enough RG flow. "Provided the non–zero eigenvalues of these operators are sufficiently small, the size of such nearly marginal couplings can be unchanged for long periods of RG flow — although ultimately they will either be irrelevant or relevant. Such operators play an important role phenomenologically, as we will see."

Then the greatest paragraph I have ever read, ever:

"A generic QFT will have an action that involves all types of operators and so lies somewhere off the critical surface. Under RG evolution, all the many irrelevant operators are quickly suppressed, while the relevant ones grow just as before. The flow lines of a generic theory thus strongly focus onto the renormalized trajectory, and so in the IR a generic QFT will closely resemble a theory emanating from the critical point, where only relevant operators have been turned on. The fact that many different high energy theories will flow to look the same in the IR is known as *universality.* It assures us that the properties of the theory in the IR are determined not by the infinite set of couplings g_i, but only by the couplings to a few relevant operators. We say that theories whose RG flows are all focused onto the same trajectory emanating from a given critical point are in the same universality class. Theories in a given universality class could look very different microscopically, but will all end up looking the same at large distances. In particular, deep in the IR, these theories will all flow to the second critical point g_i^{**}. **This is the reason you can do physics!** To study a problem at a given energy scale you don't first need to worry about what the degrees of freedom at much higher energies are doing. They are, quite literally, irrelevant"

2.1 The Continuum Limit

But what about dependence on Λ_0? Well, suppose we start on some critical surface, within the domain of attraction of some g_i^*. As we take $\Lambda_0 \to \infty$, all the irrelevant operators are suppressed arbitrarily by some power of $\frac{\Lambda}{\Lambda_0}$. Provided S_{Λ_0} lies on some critical surface, the limit

$\lim_{\Lambda_0\to\infty} \left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0)}} \mathcal{D}\chi \, \exp(-S_{\Lambda_0}[\phi+\chi]/\hbar)\right] = \lim_{\Lambda_0\to\infty} \exp(-S_{\Lambda_0}/\hbar)$ exists[2] and "the resulting effective theory will be a CFT, independent of Λ. Since $\mathcal{C}$ (the critical surface) has only finite codimension, we only have to tune finitely many coefficients (those of all the relevant operators) in order to ensure that $g_{i0} \in \mathcal{C}$." $\Longleftarrow$ Does this mean that all CFTs are independent of the UV-cutoff? I suppose that only makes sense.

2.2 What about non-CFTs?

e.g. Yang-Mills, QCD $\Longleftarrow$ both have relevant and marginally relevant operators turned on[3].

Consider a theory which passes *close* to $\mathcal{C}$. It's RG flow points first toward the nearest g_i^*, and then redirects and points along the renormalized trajectory. Consider $\Lambda = \mu$, the energy scale at which this non-CFT passes closest to g_i^*. Since the RG flow is determined by initial conditions, μ depends only on our initial theory: $\mu = \Lambda_0 f(g_{i0})$.

"To obtain a theory with relevant or marginal (marginally relevant) operators, we tune the initial couplings g_{i0} so that μ remains finite as we take $\Lambda_0 \to \infty$. If $\mathrm{codim}(\mathcal{C}) = r$ then this is one condition on r parameters — the coefficients of the relevant operators in the initial action. The theory we end up with thus depends on $(r-1)$ parameters, together with the scale μ."

He goes on to make this mathematically rigorous, exactly how we finetune these coupling constants using so-called "counterterms" S_{CT} that depend on the field as well as the UV cutoff. **Make S_{CT} explicit, instead of modifying g_{i0}, so that we can work perturbatively.**

However, in doing so we have ignored marginally irrelevant operators. These will change as $\Lambda_0 \to \infty$ in a way that is not prescribed by our fine-tuning of S_{CT}, i.e. they will be relevant in the low-energy theory but will vanish

[2] Provided you take the limit *after* computing the path integral

[3] by "turned on", one means that the operator has a non-zero coupling

at some finite energy. Their presence indicates our theory cannot be valid up to arbitrarily high energies... ***There must be an energy scale at which new physics comes into play.*** $\Longleftarrow$ I'm not crying, you're crying

Examples:

1. In the case of pion–nucleon scattering, this scale is $\sim$ 217 MeV and indicates the presence of quarks, gluons and the whole structure of QCD.
2. For radioactive β-decay, the scale is $\sim$ 250 GeV and indicates the electroweak theory,
3. For gravity the scale is $\sim$ 1019 GeV, where probably the whole notion of QFT itself must give way.
4. Perhaps most interesting of all are marginally irrelevant operators, like the quartic coupling (Φ^4) of the Higgs in the Standard Model.

"Strictly speaking, just like irrelevant operators, marginally irrelevant operators are arbitrarily suppressed as the cut–off is removed. However, they typically decay only logarithmically as Λ_0 is raised, rather than as a power law. Such operators thus afford us a tiny glimpse of new physics at exponentially high energy scales, far beyond the range of current accelerators."

3 Calculating RG evolution

Let's come back to this. It's a fantastic topic but we do need to finish this chapter in Hartman. Stay tuned for more *Casini and Huerta*.