Upper-upper variance

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1 Intro

Suppose we want to generate an antisymmetric tensor C in the following way:

1. Create C_0, with non-zero elements only in the upper-upper-triangle (i.e. $i < j < k$). Suppose these upper-upper-triangular elements are randomly selected from a distribution with known variance σ^2_{upper} and mean $\mu_{upper} \equiv 0$

2. Then we add or subtract all transposes of C_0 depending on whether a given transpose is an even or odd permutation of the indeces (i, j, k).

3. C, defined as this antisymmetric sum of tensors, will be antisymmetric.

But what is the variance of C, σ^2_C, in terms of the known variance σ^2_{upper}?

2 Math

By definition, the variance of the distribution from which we select the upper-upper-triangular elements is:

$$\sigma^2_{upper} = \frac{1}{N_{upper}} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} (C^{ijk}_{upper} - \mu_{upper})^2$$

$$\sigma^2_{upper} = \frac{6}{N(N-1)(N-2)} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} (C^{ijk}_{upper})^2 \tag{1}$$

Suppose we want to know the variance of C_0, which has zero elements everywhere except in the upper-upper-triangle, where its elements are drawn from the above distribution. By definition:

$$\sigma^2_0 = \frac{1}{N^3} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (C^{ijk}_0 - \mu_0)^2 \tag{2}$$

Let's compute μ_0 first

$$\begin{aligned}
\mu_0 &= \frac{1}{N^3} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} C^{ijk}_0 \\
&= \frac{1}{N^3} \sum_{upper-upper-triangle} C^{ijk}_0 + \frac{1}{N^3} \sum_{everywhere-else} C^{ijk}_0 \\
&= \frac{1}{N^3} \sum_{upper-upper-triangle} C^{ijk}_{upper} + \frac{1}{N^3} \sum_{everywhere-else} 0 \\
&= \frac{N_{upper}}{N^3} \mu_{upper} + 0 \\
\mu_0 &\equiv 0
\end{aligned} \tag{3}$$

Plugging this into eqn (2):

$$\sigma_0^2 = \frac{1}{N^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(C_0^{ijk})^2$$

$$= \frac{1}{N^3}\sum_{upper-upper-triangle}(C_0^{ijk})^2 + \frac{1}{N^3}\sum_{everywhere-else}(C_0^{ijk})^2$$

$$= \frac{1}{N^3}\sum_{upper-upper-triangle}(C_{upper}^{ijk})^2 + \frac{1}{N^3}\sum_{everywhere-else}0^2$$

$$= \frac{1}{N^3}\sum_{upper-upper-triangle}(C_{upper}^{ijk})^2$$

$$= \frac{1}{N^3}\frac{N_{upper}}{N_{upper}}\sum_{upper-upper-triangle}(C_{upper}^{ijk})^2$$

$$= \frac{N_{upper}}{N^3}\frac{1}{N_{upper}}\sum_{upper-upper-triangle}(C_{upper}^{ijk})^2$$

$$\sigma_0^2 = \frac{N_{upper}}{N^3}\sigma_{upper}^2 \tag{4}$$

Okay, now recall C is an antisymmetric sum of transposes of C_0. To find the variance of C, we will therefore need to use the variance-addition formula:

$$\sigma_{X\pm Y}^2 = \sigma_X^2 + \sigma_Y^2 \pm 2Cov(X,Y) \tag{5}$$

But what is the covariance of C_0 with respect to its transpose?

Well, covariance is defined as:

$$Cov(X,Y) = \frac{1}{N}\sum_i (X_i - \mu_X)(Y_i - \mu_Y) \tag{6}$$

Let's define $C_{0,T}$ as the transpose of C_0 about the first and second indeces:

$$C_{0,T}^{ijk} \equiv C_0^{jik} \tag{7}$$

Plugging C_0 and $C_{0,T}$ into the covariance formula gives:

$$Cov(C_0, C_{O,T}) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(C_0^{ijk} - \mu_0)(C_{0,T}^{ijk} - \mu_0)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_0^{ijk}C_{0,T}^{ijk}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}C_0^{ijk}C_0^{jik}$$

But we know that C_0^{ijk} only has non-zero indeces in the upper-upper-triangle, i.e. where $i < j < k$. Therefore, wherever $C_0^{ijk} \neq 0$, $C_0^{jik} = 0$. In other words, since C_0 is an upper-upper-triangular matrix, it has no non-zero indeces in common with any of its transposes.

All this to say:

$$Cov(C_0, Transpose(C_0)) \equiv 0 \tag{8}$$

for a transpose of C_0 about *any* axes.

Therefore, our variance-addition rule simplifies to that of independent variables:

$$\sigma^2_{X\pm Y} = \sigma^2_X + \sigma^2_Y \tag{9}$$

Since C is a sum of 6 transposes of C_0, it's variance is therefore

$$\sigma^2_C = 6\sigma^2_0$$

Or, in terms of the known variance σ^2_{upper}:

$$\sigma^2_C = 6\frac{N_{upper}}{N^3}\sigma^2_{upper} = \frac{6}{N^3}\frac{N(N-1)(N-2)}{6}\sigma^2_{upper}$$

$$\sigma^2_C = \frac{(N-1)(N-2)}{N^2}\sigma^2_{upper} \tag{10}$$

Or, what we really came here for

$$\sigma^2_{upper} = \frac{N^2}{(N-1)(N-2)}\sigma^2_C \tag{11}$$