Induced Dirac-Schrödinger operators on S^1-semi-free quotients

Disputation der Dissertation von

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zur Erlangung des akademischen Grades Dr. rer. nat.

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Outline

Semi-free action? An example

The $\sigma_{S^1}(M)$ signature
- Definition and properties
- Proof of Lott's formula for $\sigma_{S^1}(M)$

The signature operator on a manifold with a conical singular stratum
- Definition and local description
- Index computation (Witt case)

The induced operator Dirac-Schrödinger operator
- Brüning & Heintze Construction
- Definition of $\mathscr{D}$
- Example revisited
- Local description of the potential
- Index computation (Witt case)
- Index computation (non-Witt case)?

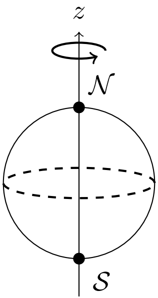
$$y = \frac{45}{1 + e^{-40(x-0.1)}} + 2$$

$$x = -\frac{1}{40} \log\left(\frac{45}{y-2} - 1\right) + 0.1$$

Example: What is a semi-free action?

Let S^1 act on $M = S^2 \subset \mathbb{R}^3$ by rotations around the z-axis.

- $M^{S^1} := \{\mathcal{N}, \mathcal{S}\}$ fixed point set.
- On the principal orbit $M_0 := S^2 - M^{S^1}$, the action is free.

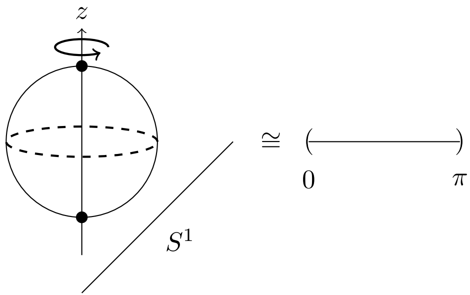


For a semi-free action the isotropy groups $S^1_x := \{g \in S^1 \mid gx = x\}$ must be either $\{1\}$ or S^1 for all $x \in S^2$.

Example: A S^1-semi-free quotient

As the action on M_0 is free, the quotient space $M_0/S^1 = (0, \pi) =: I$ is a smooth manifold.

$M_0 := S^2 - M^{S^1}$



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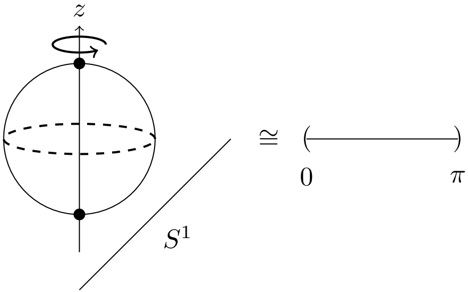

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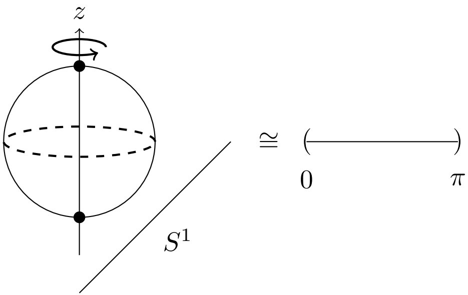

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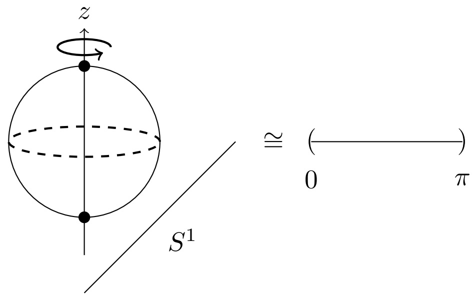

- Equip S^2 with the round metric $g^{TS^2} = d\theta^2 + \sin^2\theta d\phi^2$.
- The quotient metric $g^{TI} := d\theta^2$ is incomplete.
- The Hodge - de Rham operator

$$D_I := \begin{pmatrix} 0 & -\partial_\theta \\ \partial_\theta & 0 \end{pmatrix}$$

defined on $\Omega_c(I)$, is not essentially self-adjoint in $L^2(\wedge^* I)$.

Lott's S^1-equivariant signature

- (M, g^{TM}): $4k+1$-dimensional closed oriented Riemannian manifold on which S^1 acts by orientation preserving isometries.
- M^{S^1}: fixed point set and M_0 the principal orbit.
- V: generating vector field of the action.
- $\alpha := V^\flat/\|V\|^2 \in \Omega^1(M_0)$ satisfies $\alpha(V) = 1$.

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$$\Omega_{\mathrm{bas},c}(M_0) := \{\omega \in \Omega_c(M_0) \mid L_V \omega = 0 \text{ and } \iota_V \omega = 0\}.$$

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Define $\sigma_{S^1}(M)$ to be the signature of the symmetric quadratic form,

$$\begin{aligned} H^{2k}_{\mathrm{bas},c}(M_0) \times H^{2k}_{\mathrm{bas},c}(M_0) &\longrightarrow \mathbb{R} \\ ([\omega], [\omega']) &\longmapsto \int_M \alpha \wedge \omega \wedge \omega'. \end{aligned}$$

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- It does not depend on the Riemannian metric.
- It is invariant under S^1-homotopy equivalences.

$\sigma_{S^1}(M)$ formula for semi-free actions

Theorem (Lott, 00')

Suppose S^1 acts effectively and semi-freely on M, then

$$\sigma_{S^1}(M) = \int_{M_0/S^1} L(T(M_0/S^1), g^{T(M_0/S^1)}) + \eta(M^{S^1}).$$

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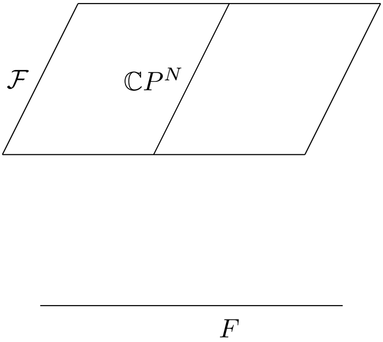
If the codimension of M^{S^1} in M is divisible by four we call M/S^1 a Witt space and

- $\eta(M^{S^1}) = 0$.
- $L(T(M_0/S^1), g^{T(M_0/S^1)})$ represents the homology L-class of M/S^1.
- $\sigma_{S^1}(M)$ equals the intersection homology signature of M/S^1.

Strategy of Lott's proof

- $F \subset M^{S^1}$: connected closed $4k - 2N - 1$ dimensional manifold.
- Let $NF \longrightarrow F$ be the normal bundle of F.
- SNF/S^1 is the total space of a Riemannian $\mathbb{C}P^N$-fiber bundle $\mathcal{F}$.
- Model M/S^1 as the mapping cylinder $C(\pi_{\mathcal{F}} : \mathcal{F} \longrightarrow F)$.
- For $r > 0$ small enough $\sigma_{S^1}(M) = \sigma(M/S^1 - N_r(F))$.
- Study the limit of the APS signature theorem as $r \longrightarrow 0$.
  - Use Dai's formula for the adiabatic limit of the eta invariant.
  - Prove that the form $\widetilde{\eta}$ and Dai's tau invariant $\tau_{\mathcal{F}}$ vanish.



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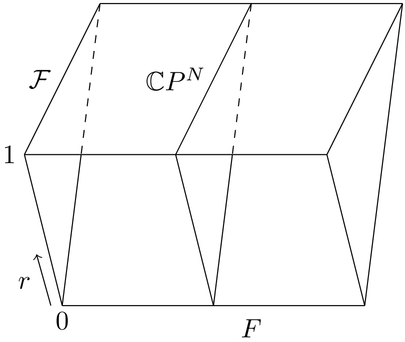

$$dr^2 \oplus g^{T_H\mathcal{F}} \oplus r^2 g^{T_V\mathcal{F}}$$

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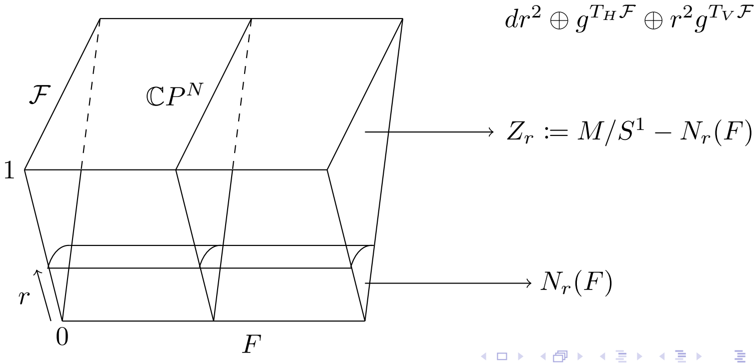

Does there exist an operator whose index is $\sigma_{S^1}(M)$?

Natural Candidate: the signature operator on M_0/S^1

- Consider the de Rham complex:

$$\begin{tikzcd} \cdots \arrow[r] & \Omega_c^{j-1}(M_0/S^1) \arrow[r, "d"] & \Omega_c^{j}(M_0/S^1) \arrow[r, "d"] \arrow[l, bend left, "d^\dagger"] & \Omega_c^{j+1}(M_0/S^1) \arrow[r] \arrow[l, bend left, "d^\dagger"] & \cdots \end{tikzcd}$$

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- $\mathrm{ind}(D^+) =?$ where $D^+ : \Omega_c^+(M_0/S^1) \longrightarrow \Omega_c^-(M_0/S^1)$.

Local description of the signature operator

Close to the fixed point set:

$$D = \gamma\left(\frac{\partial}{\partial r} + \bigstar \otimes A(r)\right) \Longrightarrow D^{+} = \frac{\partial}{\partial r} + A(r),$$

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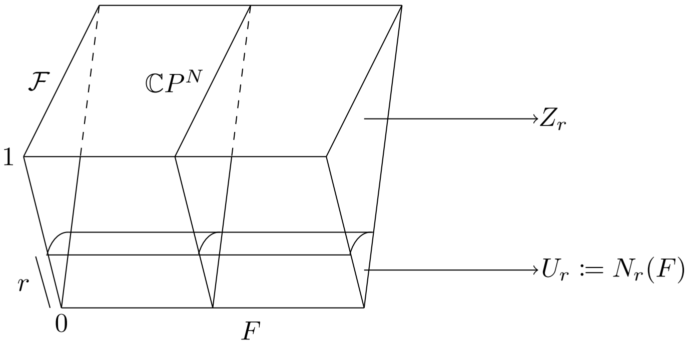
  - If N odd (Witt case) this is the case.
  - If N even (non-Witt case) $\Longrightarrow$ need boundary conditions.

Index formula for the Witt case (Brüning 09')



Use the Dirac-Systems formalism (Ballmann, Brüning & Carron, 08').

- In particular the gluing index formula:

$$\mathrm{ind}(D^+) = \mathrm{ind}\left(D^+_{Z_r, Q_<(A(r))(H)}\right) + \mathrm{ind}\left(D^+_{U_r, Q_\geq(A(r))(H)}\right).$$

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3. Finally,
   $$\lim_{r \to 0^+} \operatorname{ind}\left(D^+_{Z_r, Q_<(A_0(r))(H)}\right) + \frac{1}{2}\dim(\ker A_0(r)) = \sigma_{S^1}(M)$$

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Theorem (Brüning & Heintze, 79')

$$\begin{tikzcd}
L^2(X,E)^G \arrow[r, "P|_{Dom(P)\cap L^2(X,E)^G}"] \arrow[d, "\Phi"'] & L^2(X,E)^G \arrow[d, "\Phi"] \\
L^2(X_0/G,F,h) \arrow[r, "T"] & L^2(X_0/G,F,h)
\end{tikzcd}$$

Which self-adjoint operator to consider?

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We conjugate by multiplication by $U := h^{-1/2} = \|V\|^{-1/2}$,

$$U^{-1}TU = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\hat{c}(\bar{\kappa}) & \varepsilon(\bar{\varphi}_0 \wedge) \\ \varepsilon(\bar{\varphi}_0 \wedge)^\dagger & -\frac{1}{2}\hat{c}(\bar{\kappa}) \end{pmatrix}$$

where $\hat{c}(\bar{\kappa}) := \bar{\kappa} \wedge + \iota_{\bar{\kappa}^\sharp}$.

An induced operator Dirac-Schrödinger operator

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\Omega_c^{\mathrm{ev}}(M_0)^{S^1} \arrow[r, "B := -c(\chi)d_M + d_M^\dagger c(\chi)"] & \Omega_c^{\mathrm{ev}}(M_0)^{S^1} \\
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has a structure of a first order regular singular operator.
- The spectrum of the cone coefficient satisfies

$$\operatorname{spec}\begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \cap (-1/2, 1/2) = \emptyset.$$

Thus, $\mathscr{D}$ is essentially self-adjoint.

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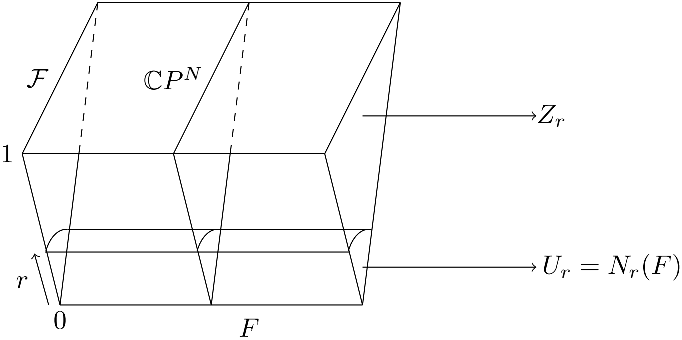
- The spectrum of the cone coefficient restricted to vertical harmonic forms is

$$2j - N \pm \frac{1}{2} \notin \left(-\frac{1}{2}, \frac{1}{2}\right),$$

thus we see $\mathscr{D}$ is indeed essentially self-adjoint.

Index formula for the Witt case



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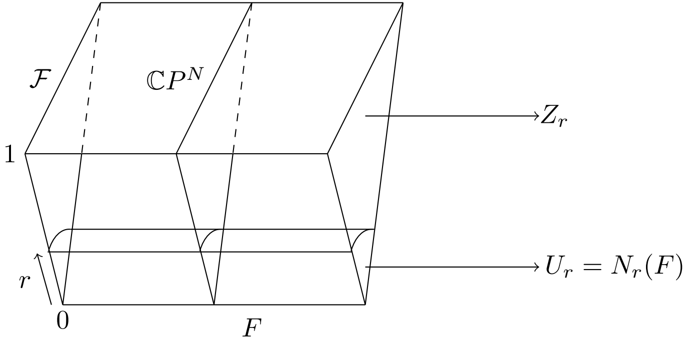

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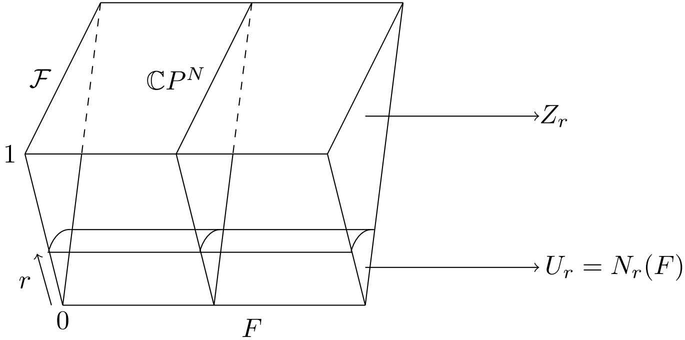

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- From a variation of Brüning's method we can prove

$$\lim_{r \to 0^+} \operatorname{ind}\left(\mathscr{D}^+_{Z_t, Q_<(A(r))(H)}\right) = \sigma_{S^1}(M).$$

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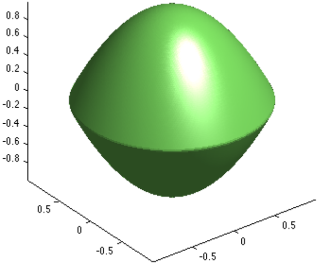
- Study the case of the spin Dirac operator (Studied in the Witt case by Albin and Gell-Redman).



Thank you!
Vielen Dank!
Gracias!
Gràcies!