**Chapter 13**

# Experiments with Random Factors

**Solutions**

**13.1S.** An experiment was performed to investigate the capability of a measurement system. Ten parts were randomly selected, and two randomly selected operators measured each part three times. The tests were made in random order, and the data are shown in Table P13.1.

**Table P13.1**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | Operator 1 | | | Operator 2 | | |
| Measurements | | | Measurements | | |
| 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 50 | 49 | 50 | 50 | 48 | 51 |
| 2 | 52 | 52 | 51 | 51 | 51 | 51 |
| 3 | 53 | 50 | 50 | 54 | 52 | 51 |
| 4 | 49 | 51 | 50 | 48 | 50 | 51 |
| 5 | 48 | 49 | 48 | 48 | 49 | 48 |
| 6 | 52 | 50 | 50 | 52 | 50 | 50 |
| 7 | 51 | 51 | 51 | 51 | 50 | 50 |
| 8 | 52 | 50 | 49 | 53 | 48 | 50 |
| 9 | 50 | 51 | 50 | 51 | 48 | 49 |
| 10 | 47 | 46 | 49 | 46 | 47 | 48 |

(a) Analyze the data from this experiment.

Minitab Output

**ANOVA: Measurement versus Part, Operator**

Factor Type Levels Values

Part random 10 1 2 3 4 5 6 7

8 9 10

Operator random 2 1 2

Analysis of Variance for Measurem

Source DF SS MS F P

Part 9 99.017 11.002 18.28 0.000

Operator 1 0.417 0.417 0.69 0.427

Part\*Operator 9 5.417 0.602 0.40 0.927

Error 40 60.000 1.500

Total 59 164.850

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Part 1.73333 3 (4) + 3(3) + 6(1)

2 Operator -0.00617 3 (4) + 3(3) + 30(2)

3 Part\*Operator -0.29938 4 (4) + 3(3)

4 Error 1.50000 (4)

(b) Find point estimates of the variance components using the analysis of variance method.

 

 

 

 

All estimates agree with the *Minitab* output.

**13.2S.** An article by Hoof and Berman (“Statistical Analysis of Power Module Thermal Test Equipment Performance”, *IEEE Transactions on Components, Hybrids, and Manufacturing Technology* Vol. 11, pp. 516-520, 1988) describes an experiment conducted to investigate the capability of measurements on thermal impedance (Cº/W x 100) on a power module for an induction motor starter. There are 10 parts, three operators, and three replicates. The data are shown in Table P13.2.

**Table P13.2**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Part Number | Inspector 1 | | | Inspector 2 | | | Inspector 3 | | |
| Test 1 | Test 2 | Test 3 | Test 1 | Test 2 | Test 3 | Test 1 | Test 2 | Test 3 |
| 1 | 37 | 38 | 37 | 41 | 41 | 40 | 41 | 42 | 41 |
| 2 | 42 | 41 | 43 | 42 | 42 | 42 | 43 | 42 | 43 |
| 3 | 30 | 31 | 31 | 31 | 31 | 31 | 29 | 30 | 28 |
| 4 | 42 | 43 | 42 | 43 | 43 | 43 | 42 | 42 | 42 |
| 5 | 28 | 30 | 29 | 29 | 30 | 29 | 31 | 29 | 29 |
| 6 | 42 | 42 | 43 | 45 | 45 | 45 | 44 | 46 | 45 |
| 7 | 25 | 26 | 27 | 28 | 28 | 30 | 29 | 27 | 27 |
| 8 | 40 | 40 | 40 | 43 | 42 | 42 | 43 | 43 | 41 |
| 9 | 25 | 25 | 25 | 27 | 29 | 28 | 26 | 26 | 26 |
| 10 | 35 | 34 | 34 | 35 | 35 | 34 | 35 | 34 | 35 |

(a) Analyze the data from this experiment, assuming both parts and operators are random effects.

Minitab Output

**ANOVA: Impedance versus Inspector, Part**

Factor Type Levels Values

Inspecto random 3 1 2 3

Part random 10 1 2 3 4 5 6 7

8 9 10

Analysis of Variance for Impedanc

Source DF SS MS F P

Inspecto 2 39.27 19.63 7.28 0.005

Part 9 3935.96 437.33 162.27 0.000

Inspecto\*Part 18 48.51 2.70 5.27 0.000

Error 60 30.67 0.51

Total 89 4054.40

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Inspecto 0.5646 3 (4) + 3(3) + 30(1)

2 Part 48.2926 3 (4) + 3(3) + 9(2)

3 Inspecto\*Part 0.7280 4 (4) + 3(3)

4 Error 0.5111 (4)

(b) Estimate the variance components using the analysis of variance method.

 

 

 

 

All estimates agree with the *Minitab* output.

**13.3.** Reconsider the data in Problem 5.11. Suppose that both factors, machines and operators, are chosen at random.

(a) Analyze the data from this experiment.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  | Machine |  |
| Operator | 1 | 2 | 3 | 4 |
| 1 | 109 | 110 | 108 | 110 |
|  | 110 | 115 | 109 | 108 |
|  |  |  |  |  |
| 2 | 110 | 110 | 111 | 114 |
|  | 112 | 111 | 109 | 112 |
|  |  |  |  |  |
| 3 | 116 | 112 | 114 | 120 |
|  | 114 | 115 | 119 | 117 |

The following *Minitab* output contains the analysis of variance and the variance component estimates:

Minitab Output

**ANOVA: Strength versus Operator, Machine**

Factor Type Levels Values

Operator random 3 1 2 3

Machine random 4 1 2 3 4

Analysis of Variance for Strength

Source DF SS MS F P

Operator 2 160.333 80.167 10.77 0.010

Machine 3 12.458 4.153 0.56 0.662

Operator\*Machine 6 44.667 7.444 1.96 0.151

Error 12 45.500 3.792

Total 23 262.958

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Operator 9.0903 3 (4) + 2(3) + 8(1)

2 Machine -0.5486 3 (4) + 2(3) + 6(2)

3 Operator\*Machine 1.8264 4 (4) + 2(3)

4 Error 3.7917 (4)

(b) Find point estimates of the variance components using the analysis of variance method.

 

 

 

 

These results agree with the *Minitab* variance component analysis.

**13.4.** Suppose that in Problem 5.15 the furnace positions were randomly selected, resulting in a mixed model experiment. Reanalyze the data from this experiment under this new assumption. Estimate the appropriate model components.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Temperature (°C) |  |
| Position | 800 | 825 | 850 |
|  | 570 | 1063 | 565 |
| 1 | 565 | 1080 | 510 |
|  | 583 | 1043 | 590 |
|  |  |  |  |
|  | 528 | 988 | 526 |
| 2 | 547 | 1026 | 538 |
|  | 521 | 1004 | 532 |

The following analysis assumes a restricted model:

Minitab Output

**ANOVA: Density versus Position, Temperature**

Factor Type Levels Values

Position random 2 1 2

Temperat fixed 3 800 825 850

Analysis of Variance for Density

Source DF SS MS F P

Position 1 7160 7160 16.00 0.002

Temperat 2 945342 472671 1155.52 0.001

Position\*Temperat 2 818 409 0.91 0.427

Error 12 5371 448

Total 17 958691

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Position 745.83 4 (4) + 9(1)

2 Temperat 3 (4) + 3(3) + 6Q[2]

3 Position\*Temperat -12.83 4 (4) + 3(3)

4 Error 447.56 (4)

 

 

 

These results agree with the *Minitab* output.

**13.5S.** Reanalyze the measurement systems experiment in Problem 13.1, assuming that operators are a fixed factor. Estimate the appropriate model components.

**Table P13.1**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | Operator 1 | | | Operator 2 | | |
| Measurements | | | Measurements | | |
| 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 50 | 49 | 50 | 50 | 48 | 51 |
| 2 | 52 | 52 | 51 | 51 | 51 | 51 |
| 3 | 53 | 50 | 50 | 54 | 52 | 51 |
| 4 | 49 | 51 | 50 | 48 | 50 | 51 |
| 5 | 48 | 49 | 48 | 48 | 49 | 48 |
| 6 | 52 | 50 | 50 | 52 | 50 | 50 |
| 7 | 51 | 51 | 51 | 51 | 50 | 50 |
| 8 | 52 | 50 | 49 | 53 | 48 | 50 |
| 9 | 50 | 51 | 50 | 51 | 48 | 49 |
| 10 | 47 | 46 | 49 | 46 | 47 | 48 |

The following analysis assumes a restricted model:

Minitab Output

**ANOVA: Measurement versus Part, Operator**

Factor Type Levels Values

Part random 10 1 2 3 4 5 6 7

8 9 10

Operator fixed 2 1 2

Analysis of Variance for Measurem

Source DF SS MS F P

Part 9 99.017 11.002 7.33 0.000

Operator 1 0.417 0.417 0.69 0.427

Part\*Operator 9 5.417 0.602 0.40 0.927

Error 40 60.000 1.500

Total 59 164.850

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Part 1.5836 4 (4) + 6(1)

2 Operator 3 (4) + 3(3) + 30Q[2]

3 Part\*Operator -0.2994 4 (4) + 3(3)

4 Error 1.5000 (4)

 

 

 

These results agree with the *Minitab* output.

**13.6.** Reanalyze the measurement system experiment in Problem 13.2S, assuming that operators are a fixed factor. Estimate the appropriate model components.

**Table P13.2**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Part Number | Inspector 1 | | | Inspector 2 | | | Inspector 3 | | |
| Test 1 | Test 2 | Test 3 | Test 1 | Test 2 | Test 3 | Test 1 | Test 2 | Test 3 |
| 1 | 37 | 38 | 37 | 41 | 41 | 40 | 41 | 42 | 41 |
| 2 | 42 | 41 | 43 | 42 | 42 | 42 | 43 | 42 | 43 |
| 3 | 30 | 31 | 31 | 31 | 31 | 31 | 29 | 30 | 28 |
| 4 | 42 | 43 | 42 | 43 | 43 | 43 | 42 | 42 | 42 |
| 5 | 28 | 30 | 29 | 29 | 30 | 29 | 31 | 29 | 29 |
| 6 | 42 | 42 | 43 | 45 | 45 | 45 | 44 | 46 | 45 |
| 7 | 25 | 26 | 27 | 28 | 28 | 30 | 29 | 27 | 27 |
| 8 | 40 | 40 | 40 | 43 | 42 | 42 | 43 | 43 | 41 |
| 9 | 25 | 25 | 25 | 27 | 29 | 28 | 26 | 26 | 26 |
| 10 | 35 | 34 | 34 | 35 | 35 | 34 | 35 | 34 | 35 |

Minitab Output

**ANOVA: Impedance versus Inspector, Part**

Factor Type Levels Values

Inspecto fixed 3 1 2 3

Part random 10 1 2 3 4 5 6 7

8 9 10

Analysis of Variance for Impedanc

Source DF SS MS F P

Inspecto 2 39.27 19.63 7.28 0.005

Part 9 3935.96 437.33 855.64 0.000

Inspecto\*Part 18 48.51 2.70 5.27 0.000

Error 60 30.67 0.51

Total 89 4054.40

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Inspecto 3 (4) + 3(3) + 30Q[1]

2 Part 48.5353 4 (4) + 9(2)

3 Inspecto\*Part 0.7280 4 (4) + 3(3)

4 Error 0.5111 (4)

 

 

 

These results agree with the *Minitab* output.

**13.7S.** In problem 5.11, suppose that there are only four machines of interest, but the operators were selected at random.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Operator | Machine | | | |
|  | 1 | 2 | 3 | 4 |
| 1 | 109 | 110 | 108 | 110 |
|  | 110 | 115 | 109 | 108 |
|  |  |  |  |  |
| 2 | 110 | 110 | 111 | 114 |
|  | 112 | 111 | 109 | 112 |
|  |  |  |  |  |
| 3 | 116 | 112 | 114 | 120 |
|  | 114 | 115 | 119 | 117 |

(a) What type of model is appropriate?

A mixed model is appropriate.

(b) Perform the analysis and estimate the model components.

The following analysis assumes a restricted model:

Minitab Output

**ANOVA: Strength versus Operator, Machine**

Factor Type Levels Values

Operator random 3 1 2 3

Machine fixed 4 1 2 3 4

Analysis of Variance for Strength

Source DF SS MS F P

Operator 2 160.333 80.167 21.14 0.000

Machine 3 12.458 4.153 0.56 0.662

Operator\*Machine 6 44.667 7.444 1.96 0.151

Error 12 45.500 3.792

Total 23 262.958

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Operator 9.547 4 (4) + 8(1)

2 Machine 3 (4) + 2(3) + 6Q[2]

3 Operator\*Machine 1.826 4 (4) + 2(3)

4 Error 3.792 (4)

 

 

 

These results agree with the *Minitab* output.

**13.8.** Rework Problem 13.4 using the REML method.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Temperature (°C) |  |
| Position | 800 | 825 | 850 |
|  | 570 | 1063 | 565 |
| 1 | 565 | 1080 | 510 |
|  | 583 | 1043 | 590 |
|  |  |  |  |
|  | 528 | 988 | 526 |
| 2 | 547 | 1026 | 538 |
|  | 521 | 1004 | 532 |

The JMP REML output is shown below. The variance components are similar to those calculated in Problem 13.4.

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.993347 |
| RSquare Adj | 0.99246 |
| Root Mean Square Error | 21.15551 |
| Mean of Response | 709.9444 |
| Observations (or Sum Wgts) | 18 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 709.94444 | 19.94444 | 1 | 35.60 | 0.0179\* |
| Temperature[800] |  | -157.6111 | 6.741707 | 2 | -23.38 | 0.0018\* |
| Temperature[825] |  | 324.05556 | 6.741707 | 2 | 48.07 | 0.0004\* |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Position | 1.6760179 | 750.11111 | 1126.0118 | -1456.832 | 2957.0538 | 63.309 |
| Position\*Temperature | -0.028674 | -12.83333 | 149.33585 | -305.5262 | 279.85956 | -1.083 |
| Residual |  | 447.55556 | 182.71379 | 230.13858 | 1219.556 | 37.774 |
| Total |  | 1184.8333 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Position** | **Position\*Temperature** | **Residual** |
| --- | --- | --- | --- |
| Position | 1267902.7 | -6197.276 | -1.412e-9 |
| Position\*Temperature | -6197.276 | 22301.197 | -11128.11 |
| Residual | -1.412e-9 | -11128.11 | 33384.329 |

**Fixed Effect Tests**

| **Source** | **Nparm** | **DF** | **DFDen** | **F Ratio** | **Prob > F** |  |
| --- | --- | --- | --- | --- | --- | --- |
| Temperature | 2 | 2 | 2 | 1155.518 | 0.0009\* |  |

**13.9.** Rework Problem 13.5 using the REML method.

**Table P13.1**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | Operator 1 | | | Operator 2 | | |
| Measurements | | | Measurements | | |
| 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 50 | 49 | 50 | 50 | 48 | 51 |
| 2 | 52 | 52 | 51 | 51 | 51 | 51 |
| 3 | 53 | 50 | 50 | 54 | 52 | 51 |
| 4 | 49 | 51 | 50 | 48 | 50 | 51 |
| 5 | 48 | 49 | 48 | 48 | 49 | 48 |
| 6 | 52 | 50 | 50 | 52 | 50 | 50 |
| 7 | 51 | 51 | 51 | 51 | 50 | 50 |
| 8 | 52 | 50 | 49 | 53 | 48 | 50 |
| 9 | 50 | 51 | 50 | 51 | 48 | 49 |
| 10 | 47 | 46 | 49 | 46 | 47 | 48 |

The JMP REML output is shown below. The variance components are similar to those calculated in Problem 13.5.

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.420766 |
| RSquare Adj | 0.410779 |
| Root Mean Square Error | 1.224745 |
| Mean of Response | 49.95 |
| Observations (or Sum Wgts) | 60 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 49.95 | 0.42821 | 9 | 116.65 | <.0001\* |
| Operator[Operator 1] |  | 0.0833333 | 0.100154 | 9 | 0.83 | 0.4269 |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | 1.1555556 | 1.7333333 | 0.8656795 | 0.0366326 | 3.430034 | 59.078 |
| Part Number\*Operator | -0.199588 | -0.299383 | 0.1464372 | -0.586394 | -0.012371 | -10.204 |
| Residual |  | 1.5 | 0.3354102 | 1.0110933 | 2.4556912 | 51.126 |
| Total |  | 2.9339506 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Part Number** | **Part Number\*Operator** | **Residual** |
| --- | --- | --- | --- |
| Part Number | 0.749401 | -0.004472 | -1.43e-14 |
| Part Number\*Operator | -0.004472 | 0.0214438 | -0.0375 |
| Residual | -1.43e-14 | -0.0375 | 0.1125 |

**Fixed Effect Tests**

| **Source** | **Nparm** | **DF** | **DFDen** | **F Ratio** | **Prob > F** |  |
| --- | --- | --- | --- | --- | --- | --- |
| Operator | 1 | 1 | 9 | 0.6923 | 0.4269 |  |

**13.10.** Rework Problem 13.6 using the REML method.

**Table P13.2**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Part Number | Inspector 1 | | | Inspector 2 | | | Inspector 3 | | |
| Test 1 | Test 2 | Test 3 | Test 1 | Test 2 | Test 3 | Test 1 | Test 2 | Test 3 |
| 1 | 37 | 38 | 37 | 41 | 41 | 40 | 41 | 42 | 41 |
| 2 | 42 | 41 | 43 | 42 | 42 | 42 | 43 | 42 | 43 |
| 3 | 30 | 31 | 31 | 31 | 31 | 31 | 29 | 30 | 28 |
| 4 | 42 | 43 | 42 | 43 | 43 | 43 | 42 | 42 | 42 |
| 5 | 28 | 30 | 29 | 29 | 30 | 29 | 31 | 29 | 29 |
| 6 | 42 | 42 | 43 | 45 | 45 | 45 | 44 | 46 | 45 |
| 7 | 25 | 26 | 27 | 28 | 28 | 30 | 29 | 27 | 27 |
| 8 | 40 | 40 | 40 | 43 | 42 | 42 | 43 | 43 | 41 |
| 9 | 25 | 25 | 25 | 27 | 29 | 28 | 26 | 26 | 26 |
| 10 | 35 | 34 | 34 | 35 | 35 | 34 | 35 | 34 | 35 |

The JMP REML output is shown below. The variance components are similar to those calculated in Problem 13.6.

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.992005 |
| RSquare Adj | 0.991821 |
| Root Mean Square Error | 0.71492 |
| Mean of Response | 35.8 |
| Observations (or Sum Wgts) | 90 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 35.8 | 2.20436 | 9 | 16.24 | <.0001\* |
| Inspector[Inspector 1] |  | -0.9 | 0.244725 | 18 | -3.68 | 0.0017\* |
| Inspector[Inspector 2] |  | 0.6666667 | 0.244725 | 18 | 2.72 | 0.0139\* |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | 94.485507 | 48.292593 | 22.906727 | 3.3962334 | 93.188952 | 97.498 |
| Part Number\*Inspector | 1.4243156 | 0.7279835 | 0.3010625 | 0.1379119 | 1.3180552 | 1.470 |
| Residual |  | 0.5111111 | 0.0933157 | 0.3681575 | 0.757543 | 1.032 |
| Total |  | 49.531687 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Part Number** | **Part Number\*Inspector** | **Residual** |
| --- | --- | --- | --- |
| Part Number | 524.71813 | -0.02989 | -2.76e-13 |
| Part Number\*Inspector | -0.02989 | 0.0906386 | -0.002903 |
| Residual | -2.76e-13 | -0.002903 | 0.0087078 |

**Fixed Effect Tests**

| **Source** | **Nparm** | **DF** | **DFDen** | **F Ratio** | **Prob > F** |  |
| --- | --- | --- | --- | --- | --- | --- |
| Inspector | 2 | 2 | 18 | 7.2849 | 0.0048\* |  |

**13.11.** Rework Problem 13.7 using the REML method.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Operator | Machine | | | |
|  | 1 | 2 | 3 | 4 |
| 1 | 109 | 110 | 108 | 110 |
|  | 110 | 115 | 109 | 108 |
|  |  |  |  |  |
| 2 | 110 | 110 | 111 | 114 |
|  | 112 | 111 | 109 | 112 |
|  |  |  |  |  |
| 3 | 116 | 112 | 114 | 120 |
|  | 114 | 115 | 119 | 117 |

The JMP REML output is shown below. The variance components are similar to those calculated in Problem 13.7.

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.78154 |
| RSquare Adj | 0.748771 |
| Root Mean Square Error | 1.94722 |
| Mean of Response | 112.2917 |
| Observations (or Sum Wgts) | 24 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 112.29167 | 1.827643 | 2 | 61.44 | 0.0003\* |
| Machine[1] |  | -0.458333 | 0.964653 | 6 | -0.48 | 0.6515 |
| Machine[2] |  | -0.125 | 0.964653 | 6 | -0.13 | 0.9011 |
| Machine[3] |  | -0.625 | 0.964653 | 6 | -0.65 | 0.5410 |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Operator | 2.3974359 | 9.0902778 | 10.035225 | -10.5784 | 28.758958 | 61.804 |
| Operator\*Machine | 0.481685 | 1.8263889 | 2.2841505 | -2.650464 | 6.3032416 | 12.417 |
| Residual |  | 3.7916667 | 1.5479414 | 1.9497217 | 10.332013 | 25.779 |
| Total |  | 14.708333 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Operator** | **Operator\*Machine** | **Residual** |
| --- | --- | --- | --- |
| Operator | 100.70575 | -1.154578 | 1.686e-12 |
| Operator\*Machine | -1.154578 | 5.2173434 | -1.198061 |
| Residual | 1.686e-12 | -1.198061 | 2.3961227 |

**Fixed Effect Tests**

| **Source** | **Nparm** | **DF** | **DFDen** | **F Ratio** | **Prob > F** |  |
| --- | --- | --- | --- | --- | --- | --- |
| Machine | 3 | 3 | 6 | 0.5578 | 0.6619 |  |

**13.12.** Consider the three-factor factorial design in Example 13.4. Propose appropriate test statistics for all main effects and interactions. Repeat for the case where *A* and *B* are fixed and *C* is random.

If all three factors are random there are no exact tests on main effects. We could use the following:



If *A* and *B* are fixed and *C* is random, the expected mean squares are (assuming the restricted form of the model):

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | F | F | R | R |  |
|  | *a* | *b* | *c* | *n* |  |
| Factor | *i* | *j* | *k* | *l* | E(*MS*) |
|  | 0 | *b* | *c* | *n* |  |
|  | *a* | 0 | *c* | *n* |  |
|  | *a* | *b* | 1 | *n* |  |
|  | 0 | 0 | *c* | *n* |  |
|  | 0 | *b* | 1 | *n* |  |
|  | *a* | 0 | 1 | *n* |  |
|  | 0 | 0 | 1 | *n* |  |
|  | 1 | 1 | 1 | 1 |  |

These are exact tests for all effects.

**13.13.** Consider the experiment in Example 13.5. Analyze the data for the case where *A*, *B*, and *C* are random.

Minitab Output

**ANOVA: Drop versus Temp, Operator, Gauge**

Factor Type Levels Values

Temp random 3 60 75 90

Operator random 4 1 2 3 4

Gauge random 3 1 2 3

Analysis of Variance for Drop

Source DF SS MS F P

Temp 2 1023.36 511.68 2.30 0.171 x

Operator 3 423.82 141.27 0.63 0.616 x

Gauge 2 7.19 3.60 0.06 0.938 x

Temp\*Operator 6 1211.97 202.00 14.59 0.000

Temp\*Gauge 4 137.89 34.47 2.49 0.099

Operator\*Gauge 6 209.47 34.91 2.52 0.081

Temp\*Operator\*Gauge 12 166.11 13.84 0.65 0.788

Error 36 770.50 21.40

Total 71 3950.32

x Not an exact F-test.

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Temp 12.044 \* (8) + 2(7) + 8(5) + 6(4) + 24(1)

2 Operator -4.544 \* (8) + 2(7) + 6(6) + 6(4) + 18(2)

3 Gauge -2.164 \* (8) + 2(7) + 6(6) + 8(5) + 24(3)

4 Temp\*Operator 31.359 7 (8) + 2(7) + 6(4)

5 Temp\*Gauge 2.579 7 (8) + 2(7) + 8(5)

6 Operator\*Gauge 3.512 7 (8) + 2(7) + 6(6)

7 Temp\*Operator\*Gauge -3.780 8 (8) + 2(7)

8 Error 21.403 (8)

\* Synthesized Test.

Error Terms for Synthesized Tests

Source Error DF Error MS Synthesis of Error MS

1 Temp 6.97 222.63 (4) + (5) - (7)

2 Operator 7.09 223.06 (4) + (6) - (7)

3 Gauge 5.98 55.54 (5) + (6) - (7)

Since all three factors are random there are no exact tests on main effects. *Minitab* uses an approximate *F* test for the these factors.

**13.14.** Derive the expected mean squares shown in Table 13.10.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | F | R | R | R |  |
|  | *a* | *b* | *c* | *n* |  |
| Factor | *i* | *j* | *k* | *l* | E(*MS*) |
|  | 0 | *b* | *c* | *n* |  |
|  | *a* | 1 | *c* | *n* |  |
|  | *a* | *b* | 1 | *n* |  |
|  | 0 | 1 | *c* | *n* |  |
|  | 0 | *b* | 1 | *n* |  |
|  | *a* | 1 | 1 | *n* |  |
|  | 0 | 1 | 1 | *n* |  |
|  | 1 | 1 | 1 | 1 |  |

**13.15S.** In Problem 5.21, assume that the three operators were selected at random. Analyze the data under these conditions and draw conclusions. Estimate the variance components.

Minitab Output

**ANOVA: Score versus Cycle Time, Operator, Temperature**

Factor Type Levels Values

Cycle Ti fixed 3 40 50 60

Operator random 3 1 2 3

Temperat fixed 2 300 350

Analysis of Variance for Score

Source DF SS MS F P

Cycle Ti 2 436.000 218.000 2.45 0.202

Operator 2 261.333 130.667 39.86 0.000

Temperat 1 50.074 50.074 8.89 0.096

Cycle Ti\*Operator 4 355.667 88.917 27.13 0.000

Cycle Ti\*Temperat 2 78.815 39.407 3.41 0.137

Operator\*Temperat 2 11.259 5.630 1.72 0.194

Cycle Ti\*Operator\*Temperat 4 46.185 11.546 3.52 0.016

Error 36 118.000 3.278

Total 53 1357.333

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Cycle Ti 4 (8) + 6(4) + 18Q[1]

2 Operator 7.0772 8 (8) + 18(2)

3 Temperat 6 (8) + 9(6) + 27Q[3]

4 Cycle Ti\*Operator 14.2731 8 (8) + 6(4)

5 Cycle Ti\*Temperat 7 (8) + 3(7) + 9Q[5]

6 Operator\*Temperat 0.2613 8 (8) + 9(6)

7 Cycle Ti\*Operator\*Temperat 2.7562 8 (8) + 3(7)

8 Error 3.2778 (8)

The following calculations agree with the *Minitab* results:

 

 

 

 

 

**13.16.** Consider the three-factor factorial model



Assuming that all the factors are random, develop the analysis of variance table, including the expected mean squares. Propose appropriate test statistics for all effects.

|  |  |  |
| --- | --- | --- |
| Source | DF | E(MS) |
| *A* | *a*-1 |  |
| *B* | *b*-1 |  |
| *C* | *c*-1 |  |
| *AB* | (*a*-1)(*b*-1) |  |
| *BC* | (*b*-1)(*c*-1) |  |
| Error (*AC + ABC*) | *b*(*a*-1)(*c*-1) |  |
| Total | *abc*-1 |  |

There are exact tests for all effects except *B*. To test *B*, use the statistic 

**13.17.** In the two-factor mixed model analysis of variance, show that  for *i*≠*i'*.

Since (constant) we have , which implies that 

**13.18.** Show that the method of analysis of variance always produces unbiased point estimates of the variance component in any random or mixed model.

Let **g** be the vector of mean squares from the analysis of variance, chosen so that *E*(**g**) does not contain any fixed effects. Let  be the vector of variance components such that , where **A** is a matrix of constants. Now in the analysis of variance method of variance component estimation, we equate observed and expected mean squares, i.e.



Since  always exists then,



Thus is an unbiased estimator of . This and other properties of the analysis of variance method are discussed by Searle (1971a).

**13.19.** Analyze the data in Problem 13.1, assuming that the operators are fixed, using both the unrestricted and restricted forms of the mixed models. Compare the results obtained from the two models.

The restricted model is as follows:

Minitab Output

**ANOVA: Measurement versus Part, Operator**

Factor Type Levels Values

Part random 10 1 2 3 4 5 6 7

8 9 10

Operator fixed 2 1 2

Analysis of Variance for Measurem

Source DF SS MS F P

Part 9 99.017 11.002 7.33 0.000

Operator 1 0.417 0.417 0.69 0.427

Part\*Operator 9 5.417 0.602 0.40 0.927

Error 40 60.000 1.500

Total 59 164.850

Source Variance Error Expected Mean Square for Each Term

component term (using restricted model)

1 Part 1.5836 4 (4) + 6(1)

2 Operator 3 (4) + 3(3) + 30Q[2]

3 Part\*Operator -0.2994 4 (4) + 3(3)

4 Error 1.5000 (4)

The second approach is the unrestricted mixed model.

Minitab Output

##### ANOVA: Measurement versus Part, Operator

Factor Type Levels Values

Part random 10 1 2 3 4 5 6 7

8 9 10

Operator fixed 2 1 2

Analysis of Variance for Measurem

Source DF SS MS F P

Part 9 99.017 11.002 18.28 0.000

Operator 1 0.417 0.417 0.69 0.427

Part\*Operator 9 5.417 0.602 0.40 0.927

Error 40 60.000 1.500

Total 59 164.850

Source Variance Error Expected Mean Square for Each Term

component term (using unrestricted model)

1 Part 1.7333 3 (4) + 3(3) + 6(1)

2 Operator 3 (4) + 3(3) + Q[2]

3 Part\*Operator -0.2994 4 (4) + 3(3)

4 Error 1.5000 (4)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Source | Sum of Squares | DF | Mean Square | E(MS) | *F*-test | *F* |
| *A* | 99.016667 | *a*-1=9 | 11.00185 |  |  | 18.28 |
| *B* | 0.416667 | *b*-1=1 | 0.416667 |  |  | 0.692 |
| *AB* | 5.416667 | (*a*-1)(*b*-1)=9 | 0.60185 |  |  | 0.401 |
| Error | 60.000000 | 40 | 1.50000 |  |  |  |
| Total | 164.85000 | *nabc*-1=59 |  |  |  |  |

In the unrestricted model, the *F*-test for *A* is different. The *F*-test for *A* in the unrestricted model should generally be more conservative, since *MS*AB will generally be larger than *MS*E. However, this is not the case with this particular experiment.

**13.20.** Consider the two-factor mixed model. Show that the standard error of the fixed factor mean (e.g. *A*) is .

The standard error is often used in Duncan’s Multiple Range test. Duncan’s Multiple Range Test requires the variance of the difference in two means, say



where rows are fixed and columns are random. Now, assuming all model parameters to be independent, we have the following:



and



Since  estimates , we would use



as the standard error to test the difference. However, the table of ranges for Duncan’s Multiple Range test already includes the constant 2.

**13.21.** Consider the variance components in the random model from Problem 13.1.

(a) Find an exact 95 percent confidence interval on σ2.







(b) Find approximate 95 percent confidence intervals on the other variance components using the Satterthwaite method.

 and  are negative, and the Satterthwaithe method does not apply. The confidence interval on   
 is

 









**13.22.** Use the experiment described in Problem 5.11 and assume that both factors are random. Find an exact 95 percent confidence interval on *σ*2. Construct approximate 95 percent confidence interval on the other variance components using the Satterthwaite method.

 







Satterthwaite Method:

 





 and  were estimated by extrapolating 1.27869 between degrees of freedom of one and two in *Microsoft Excel.* More precise methods can be used as well.





 , this variance component does not have a confidence interval using Satterthwaite’s Method.

 







 and  were estimated by extrapolating 1.64108 between degrees of freedom of one and two in *Microsoft Excel.* More precise methods can be used as well.



**13.23.** Consider the three-factor experiment in Problem 5.21 and assume that operators were selected at random. Find an approximate 95 percent confidence interval on the operator variance component.

 





 and  were estimated by extrapolating 1.90085 between degrees of freedom of one and two in *Microsoft Excel.* More precise methods can be used as well.





**13.24S.** Rework Problem 13.19 using the modified large-sample approach described in Section 13.6.2. Compare the two sets of confidence intervals obtained and discuss.

 







**13.25.** Rework Problem 13.21 using the modified large-sample method described in Section 13.6.2. Compare this confidence interval with the one obtained previously and discuss.

 







**13.26.** Consider the experiment described in Problem 5.11. Estimate the variance component using the REML method. Compare the CIs to the approximate CIs found in Problem 13.22.

The JMP REML analysis below was performed with both factors, Operator and Machine, as random.

The CIs for the error variance are similar to those found in Problem 13.22. The upper CIs for the other variance components are much larger than those estimated in the JMP REML output below.

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.742044 |
| RSquare Adj | 0.742044 |
| Root Mean Square Error | 1.94722 |
| Mean of Response | 112.2917 |
| Observations (or Sum Wgts) | 24 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 112.29167 | 1.789728 | 1.831 | 62.74 | 0.0005\* |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Operator | 2.3974359 | 9.0902778 | 10.035225 | -10.5784 | 28.758958 | 64.198 |
| Machine | -0.144689 | -0.548611 | 0.9124188 | -2.336919 | 1.239697 | -3.874 |
| Operator\*Machine | 0.481685 | 1.8263889 | 2.2841505 | -2.650464 | 6.3032416 | 12.898 |
| Residual |  | 3.7916667 | 1.5479414 | 1.9497217 | 10.332013 | 26.778 |
| Total |  | 14.159722 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Operator** | **Machine** | **Operator\*Machine** | **Residual** |
| --- | --- | --- | --- | --- |
| Operator | 100.70575 | 0.3848594 | -1.154578 | 2.151e-12 |
| Machine | 0.3848594 | 0.8325081 | -1.539438 | -1.17e-13 |
| Operator\*Machine | -1.154578 | -1.539438 | 5.2173434 | -1.198061 |
| Residual | 2.151e-12 | -1.17e-13 | -1.198061 | 2.3961227 |

**13.27S.** Consider the experiment described in Problem 13.1. Analyze the data using REML. Compare the CIs to those obtained in Problem 13.21.

The JMP REML analysis below was performed with both factors, Part Number and Operator, as random.

The CIs for the Operator and Part Number Operator interaction were not calculated in Problem 13.21 due to negative estimates for the corresponding variance components. The error variance estimates and CIs found in the JMP REML output below are the same as those calculated in Problem 13.21. The upper CI for the Part Number variance estimated with the Satterthwaite method in Problem 13.21 is approximately twice the value estimated in the JMP REML analysis.

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.388009 |
| RSquare Adj | 0.388009 |
| Root Mean Square Error | 1.224745 |
| Mean of Response | 49.95 |
| Observations (or Sum Wgts) | 60 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 49.95 | 0.424591 | 8.563 | 117.64 | <.0001\* |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | 1.1555556 | 1.7333333 | 0.8656795 | 0.0366326 | 3.430034 | 59.203 |
| Operator | -0.004115 | -0.006173 | 0.0218 | -0.0489 | 0.0365544 | -0.211 |
| Part Number\*Operator | -0.199588 | -0.299383 | 0.1464372 | -0.586394 | -0.012371 | -10.226 |
| Residual |  | 1.5 | 0.3354102 | 1.0110933 | 2.4556912 | 51.233 |
| Total |  | 2.9277778 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Part Number** | **Operator** | **Part Number\*Operator** | **Residual** |
| --- | --- | --- | --- | --- |
| Part Number | 0.749401 | 0.0004472 | -0.004472 | 9.256e-14 |
| Operator | 0.0004472 | 0.0004752 | -0.000894 | -3.32e-16 |
| Part Number\*Operator | -0.004472 | -0.000894 | 0.0214438 | -0.0375 |
| Residual | 9.256e-14 | -3.32e-16 | -0.0375 | 0.1125 |

**13.28.** Rework Problem 13.24 using REML. Compare all sets of CIs for the variance components.

JMP uses the unrestricted approach for estimating variance components with mixed models. The JMP REML output is shown below.

The lower confidence interval on  (Operator) comparisons between the Satterthwaite method in Problem 13.21, the modified large sample approach in Problem 13.30, and the JMP REML output below are:

Satterthwaite 

Modified 

REML 

JMP Output

|  |  |
| --- | --- |
| RSquare | 0.388009 |
| RSquare Adj | 0.388009 |
| Root Mean Square Error | 1.224745 |
| Mean of Response | 49.95 |
| Observations (or Sum Wgts) | 60 |

**Parameter Estimates**

| **Term** |  | **Estimate** | **Std Error** | **DFDen** | **t Ratio** | **Prob>|t|** |
| --- | --- | --- | --- | --- | --- | --- |
| Intercept |  | 49.95 | 0.424591 | 8.563 | 117.64 | <.0001\* |

**REML Variance Component Estimates**

| **Random Effect** | **Var Ratio** | **Var Component** | **Std Error** | **95% Lower** | **95% Upper** | **Pct of Total** |
| --- | --- | --- | --- | --- | --- | --- |
| Part Number | 1.1555556 | 1.7333333 | 0.8656795 | 0.0366326 | 3.430034 | 59.203 |
| Operator | -0.004115 | -0.006173 | 0.0218 | -0.0489 | 0.0365544 | -0.211 |
| Part Number\*Operator | -0.199588 | -0.299383 | 0.1464372 | -0.586394 | -0.012371 | -10.226 |
| Residual |  | 1.5 | 0.3354102 | 1.0110933 | 2.4556912 | 51.233 |
| Total |  | 2.9277778 |  |  |  | 100.000 |

**Covariance Matrix of Variance Component Estimates**

| **Random Effect** | **Part Number** | **Operator** | **Part Number\*Operator** | **Residual** |
| --- | --- | --- | --- | --- |
| Part Number | 0.749401 | 0.0004472 | -0.004472 | 9.256e-14 |
| Operator | 0.0004472 | 0.0004752 | -0.000894 | -3.32e-16 |
| Part Number\*Operator | -0.004472 | -0.000894 | 0.0214438 | -0.0375 |
| Residual | 9.256e-14 | -3.32e-16 | -0.0375 | 0.1125 |

**13.29.** The levels of a random factor are sampled from a large population of possible levels.

**True** False

**13.30.** The REML method is preferred as a technique for estimating variance components because it can also provide the standard error of the variance component leading to a confidence interval.

**True** False

**13.31.** Both the ANOVA method and REML will produce the same point estimate of a variance component if the design is balanced.

**True** False

**13.32.** Random factors are always categorical.

True **False**

**13.33.** In a mixed model the statistical tests on fixed factors are always exactly the same as they are in the fixed effects model.

True **False**

**13.34.** A negative point estimate of a variance component implies that the variance of that fixed effect is zero.

True **False**

**13.35.** In a two-factor factorial experiment with at least one random factor

(a) the statistical tests on main effects are identical to the corresponding main effects tests in the fixed-effects case.

**(b) the statistical tests on main effects are different from the corresponding main effects tests in the fixed-effects case.**

(c) no test on the two-factor interaction may be made.

(d) none of the above (a-c) are true

(e) all of the above (a-c) are true

**13.36.** In a two-factor factorial experiment with at least one random factor

**(a) the statistical test on interaction effect is identical to the corresponding interaction effect test in the fixed-effects case.**

(b) the statistical test on interaction effect is different from the corresponding interaction effect test in the fixed-effects case.

(c) no test on the two factor interaction may be made.

(d) none of the above (a-c) are true

(e) all of the above (a-c) are true