CE Sheet 1

• Power of a Periodic Signal

$$P = \frac{1}{T}\int_T |x(t)|^2 dt$$

→ if its a Product of Sines/Cosines make it into a Sum of them.

• After squaring a Sine/Cosine use the Cosine double angle rule.

• Integrate

» if the Sine/Cosine being integrated has its Period as a multiple of the original signal's fundam. Period the result is zero since $\int_T \sin(\omega t)\,dt = \int_T \cos(\omega t)\,dt = 0$ and multiplying that by anything still gives zero

Alternatively, use $P = \sum_{k=-\infty}^{\infty} |a_k|^2$ in the frequency domain

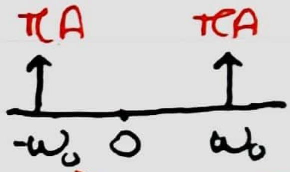
(much easier)

Divide the area of each delta by 2π to get a_k.

* If it's a Sum of Sinusoids Can't get the Power of each independently then Sum (unless they have the same Phase*)

For $x(t) = A\cos(\omega_0 t + \phi)$

$$= \frac{A}{2}\left(e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right)$$



Same if it was sine.

Also applies to Sine (Phase independent)

$$= \frac{A}{2}\left(e^{j\phi}\cdot \underbrace{e^{j\omega_0 t}}_{K=1} + \underbrace{e^{-j\omega_0 t}}_{K=-1}\cdot e^{-j\phi}\right)$$

$$a_1 = \frac{A}{2}e^{j\phi}\ ,\quad a_{-1} = \frac{A}{2}e^{-j\phi}$$

$$P = \sum_{K=-\infty}^{\infty} |a_K|^2 = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

magnitude squared (no regard to Phase)

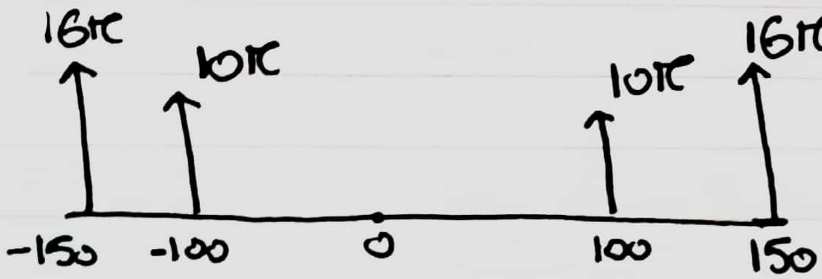
• Proven in the lecture in the classic way.

1) a) $x(t) = 10\cos(100t + \pi/3)$

$$P = \frac{10^2}{2} = 50$$

b)

$$x(t) = 10\cos(100t + \frac{\pi}{3}) + 16\sin(150t + \pi/5)$$

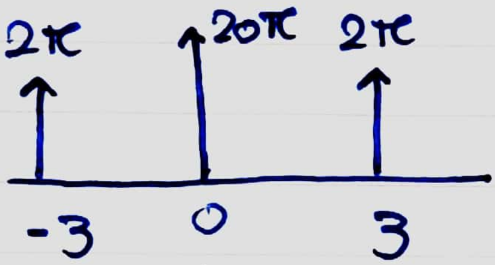


$$P = \sum_{k=-\infty}^{\infty} |a_k|^2 = 2 \times 8^2 + 2 \times 5^2 = 178$$

↳ divide deltas by 2π

c)

$$x_1(t) = 10 + 2\sin 3t$$



$\}$ $x(t) = x_1(t)\cos(10t)$

Shifts $x_1(t)$ to ± 10 & multiplies by half. (Proven in lec.)

↳ $1 \leftrightarrow 2\pi\delta(\omega)$

$\}$ only for visualisation (Previous Part is enough to Compute Power)

$\div 2\pi$

$$P = 4\left(\frac{1}{2}\right)^2 + 2(5)^2$$

$$= 51$$

d) $x(t) = 10\cos 5t \cos 10t$

$$= 5(\cos(15t) + \cos(5t))$$

// $\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$

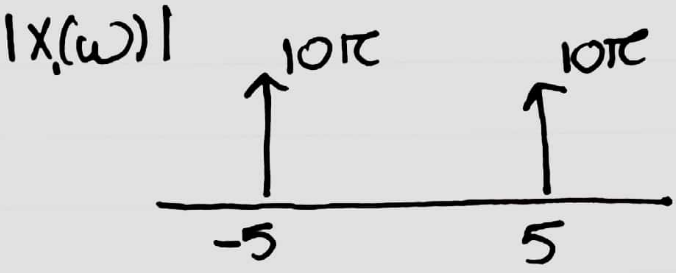
$$P = \frac{5^2}{2} + \frac{5^2}{2} = 25$$

// try doing it in Freq. domain.

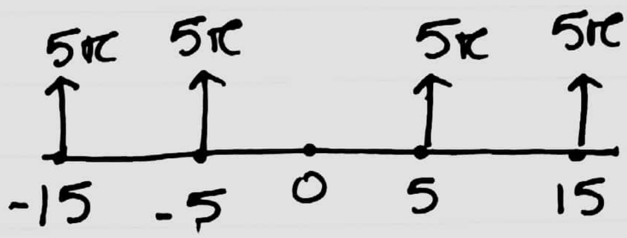
e)

$x_1(t) = 10\sin 5t$

Reminder $X(\omega)$ has $-ve$ and j



$x(t) = x_1(t)\cos(10t)$



Safe to graph in case of overlap.

» Carrier is high freq.

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2 = 4 \times (2.5)^2 = 25$$

Try to solve using $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

f) $x(t) = e^{j\alpha t}\cos\omega_0 t$

↳ Just a Phase $\frac{1}{T}\int_T |x(t).e^{j\alpha t}|^2 dt = \frac{1}{T}\int |x(t)|^2 dt$

$$P = \frac{1^2}{2} = \frac{1}{2}$$

2)

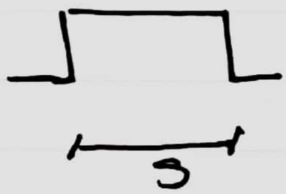
Recall,

$$\text{rect}(t/S) \longleftrightarrow \frac{S}{T_0} \cdot \text{Sinc}\left(k\omega_0 \frac{S}{2}\right)$$

(Periodic with T_0)

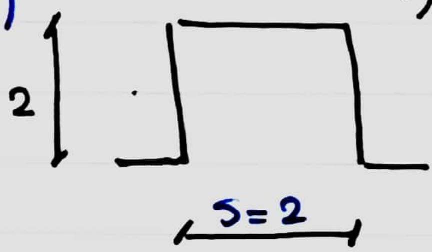
// get $a_0 = \frac{1}{T}\int_T x(t)\,dt$

area



a)

$x_1(t)$



$$\longleftrightarrow 2 \cdot \frac{2}{4}\,\text{Sinc}\left(\frac{k\pi}{2}\right)$$

$T_0 = 4$, $\omega_0 = \frac{\pi}{2}$

Clearly, $x(t) = x_1(t) - 1$

$$a_k = \text{Sinc}\left(\frac{\pi}{2}k\right) \qquad b_k = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

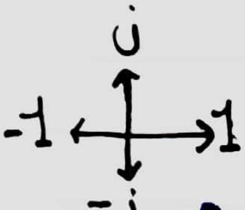
$$a_0 = \frac{1}{4}(2 \times 2) = 1$$

→ linearity of FS

$$x(t) \longleftrightarrow C_k = \begin{cases} 1-1=0 & k=0 \\ \text{Sinc}\left(\frac{\pi}{2}k\right) & k \neq 0 \end{cases}$$

$$\bullet\ \mathrm{sinc}\left(\frac{\pi}{2}k\right) = \frac{e^{j\frac{\pi}{2}\cdot k} - e^{-j\frac{\pi}{2}\cdot k}}{2} \cdot \frac{1}{j\frac{\pi}{2}k}$$

Can be further simplified.



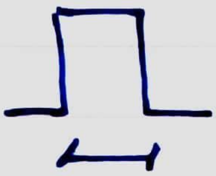
$e^{j\theta} =$)

$$= \frac{j^{k} - j^{-k}}{j\pi k} = \frac{j^{k-1}\left(1-(-1)^{k}\right)}{\pi k}$$

// $j^{-1} = -j$

$$C_k = \begin{cases} 0 & k \text{ is even} \\ \dfrac{2j^{k}}{\pi k} & k \text{ is odd} \end{cases}$$

b)



$\longleftrightarrow \dfrac{2\pi}{10\pi}\,\mathrm{sinc}\left(k\cdot\frac{1}{5}\cdot\pi\right)$

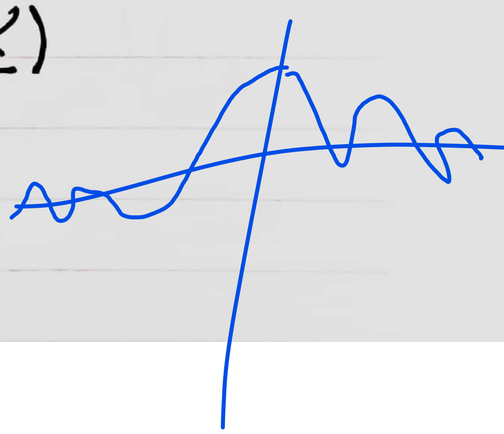
$S = 2\pi$

$T_0 = 10\pi$

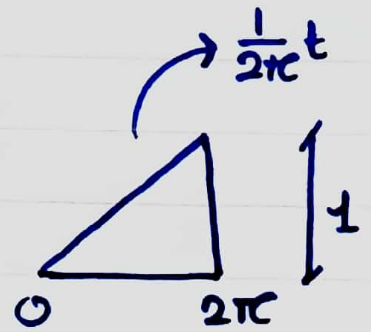
$\omega_0 = \dfrac{1}{5}$

$$a_k = \frac{1}{5}\,\mathrm{sinc}\left(\frac{\pi k}{5}\right)$$

$$a_0 = \frac{1}{10\pi}\cdot 2\pi = \frac{1}{5}$$



c)



$$\omega_0 = \frac{2\pi}{2\pi} = 1$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi} t\,e^{-jk\omega_0 t}\,dt$$

$\omega_0 = 1$

t	e^{-jkt}
1	$e^{-jkt}/(-jk)$
0	$e^{-jkt}/(-jk)^2$

$k \neq 0$

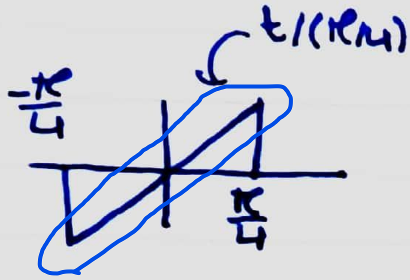
Plug as you write

$$a_k = \left(\frac{2\pi e^{-2\pi jk}}{-jk} + \frac{e^{-2\pi jk}}{-k^2} - \frac{1}{-k^2}\right)\frac{1}{4\pi^2} \qquad e^{j2\pi(-k)} = 1$$

$$= \frac{j2\pi}{k}\cdot\frac{1}{4\pi^2} = \frac{j}{2\pi k} \qquad k \neq 0$$

$$a_0 = \frac{1}{2\pi}\times\frac{2\pi}{2} = \frac{1}{2}$$

d)



$T_0 = \pi \;,\; \omega_0 = 2$

$$a_K = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{t}{\pi/4} \cdot e^{-jK2t}$$

$$\begin{array}{ccc} t & & e^{-jK2t} \\ 1 & \searrow & e^{-jK2t}/(-2jK) \quad K\neq 0 \\ 0 & \searrow & e^{-jK2t}/(-2jK)^2 \end{array}$$

Plug

$$a_K = \frac{4}{\pi^2}\left(\frac{\pi}{4}\frac{e^{-j2K\frac{\pi}{4}}}{-2jK} - \frac{e^{-j2K\frac{\pi}{4}}}{-4K^2} \right.$$
$$\left. - \frac{-\pi}{4}\frac{e^{j2K\pi/4}}{-2jK} - \frac{e^{j2K\pi/4}}{-4K^2} \right)$$

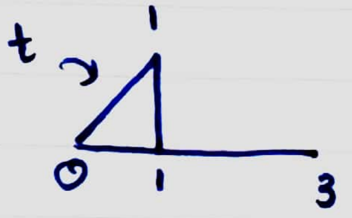
$$= \frac{4}{\pi^2}\left(\frac{\pi}{4}\cdot j\cdot\frac{1}{2K}\left(e^{jK\frac{\pi}{2}} + e^{-jK\frac{\pi}{2}}\right) \right. \quad //\cos$$
$$\left. + \frac{1}{4K^2}\left(e^{-jK\frac{\pi}{2}} - e^{jK\frac{\pi}{2}}\right)\right) \quad //\sin$$

$$= \frac{1}{\pi^2}\left(\frac{\pi j}{k} \cos\left(\frac{k\pi}{2}\right) - \frac{j2}{\pi^2} \sin\left(\frac{k\pi}{2}\right)\right)$$

$k \neq 0$

Wolfram agrees*

e)



$$\left.\right\} \quad T_0 = 3 \qquad \omega_0 = \frac{2\pi}{3}$$

$$a_0 = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$a_k = \frac{1}{3} \int_0^1 t e^{-jk\omega_0 t} \, dt$$

$$\begin{array}{ll} t & e^{-jk\omega_0 t} \\ 1 & e^{-jk\omega_0 t} / (-jk\omega_0) \quad k \neq 0 \\ 0 & e^{-jk\omega_0 t} / (-jk\omega_0)^2 \end{array}$$

$$a_k = \frac{1}{3}\left(\frac{e^{-jk\omega_0}}{-jk\omega_0} - \frac{e^{-jk\omega_0}}{-(k\omega_0)^2} + \frac{1}{-(k\omega_0)^2} \right)$$

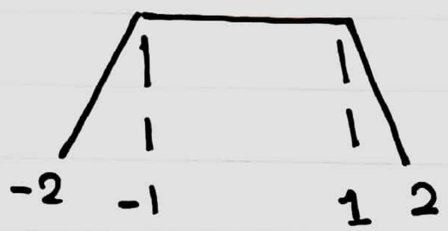
$$= \frac{1}{3}\left(j\frac{e^{-jk\omega_0}}{k\omega_0} + \frac{e^{-jk\omega_0}}{(k\omega_0)^2} - \frac{1}{(k\omega_0)^2} \right) \Bigg\}$$

Can take one or two common factors

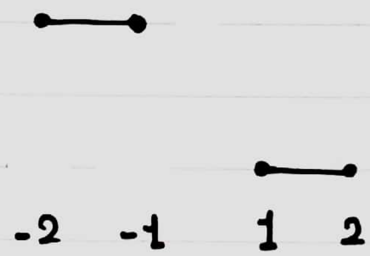
» A Popular technique is to differentiate then apply the integration property.

$$\int_{-\infty}^{t} x(\lambda)\,d\lambda \longleftrightarrow \frac{a_k}{jk\omega_0}$$

E)



$\xrightarrow{\frac{d}{dt}}$



$T = 6,\ \omega_0 = \frac{\pi}{3}$

$$a_k' = \frac{1}{6}\left(\int_{-2}^{-1} e^{-jk\omega_0 t}\,dt - \int_{1}^{2} e^{-jk\omega_0 t}\,dt\right)$$

$$= \frac{1}{6}\ \left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_{-2}^{-1} - \left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_{1}^{2} \qquad k \neq 0$$

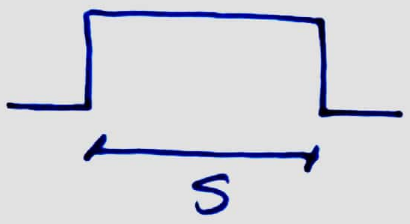
$$= \frac{j}{6k\omega_0}\left(e^{jk\omega_0} - e^{jk2\omega_0} - e^{-jk2\omega_0} + e^{-jk\omega_0}\right)$$

$$= \frac{j}{3k\omega_0}\left(\cos(k\omega_0) - \cos(2k\omega_0)\right)$$

$$a_k = \frac{a_k'}{jk\omega_0} = \frac{13}{(k+e^2)} \cdot \left(\cos(k\omega_0) - \cos(2k\omega_0)\right), \quad k \neq 0$$

$$a_0 = \frac{1}{6}\left(1\times 2 + 1\times 1\times \frac{1}{2} + 1\times 1\times \frac{1}{2}\right) = \frac{1}{2}$$

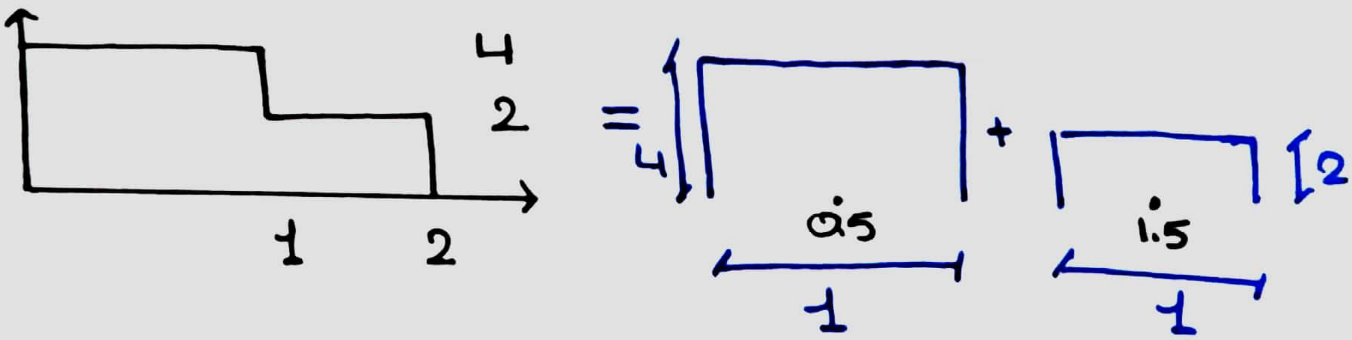
3)



$\longleftrightarrow S \cdot \text{Sinc}\left(\frac{\omega S}{2}\right) \to X_1(\omega)$

rect(t/S)

$x_1(t)$



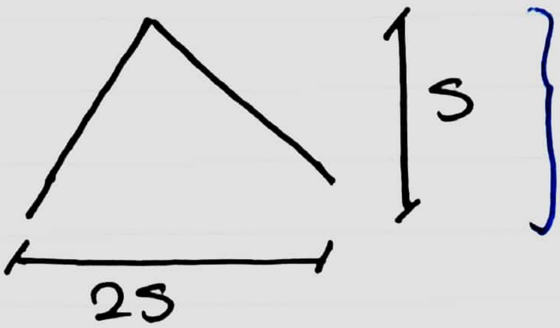
$$x(t) = x_1(t-0.5)\cdot 4 + x_1(t-1.5)\cdot 2$$

Shift Property

$$\mathcal{F}_T\{x(t)\} = 4X_1(\omega)\cdot e^{-0.5j\omega} + 2X_1(\omega)e^{-1.5j\omega}$$

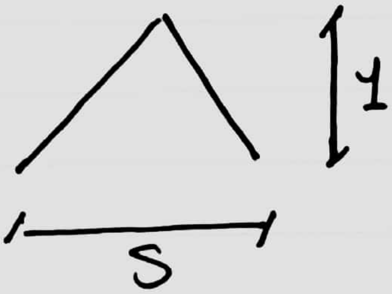
$$= \text{Sinc}(\omega/2)\left(4e^{-\frac{j\omega}{2}} + 2e^{-\frac{3}{2}j\omega}\right)$$

=



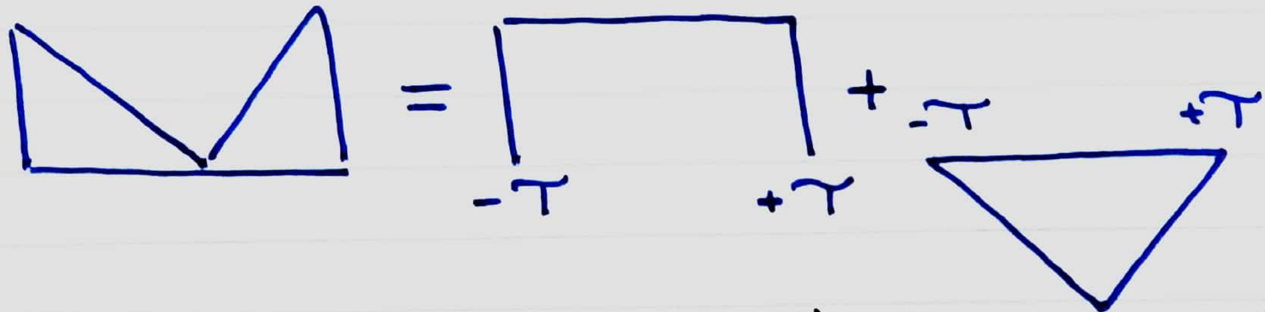
$$X(\omega) = S^2 \cdot \operatorname{sinc}^2\left(\frac{\omega S}{2}\right)$$

Thus,



$$\longleftrightarrow \frac{S}{2} \cdot \operatorname{sinc}^2\left(\frac{\omega S}{4}\right)$$

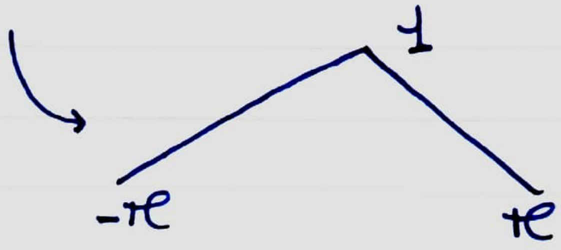
half its width



$$X(\omega) = 2T \cdot \operatorname{sinc}(\omega T) - T \operatorname{sinc}^2\left(\frac{\omega T}{2}\right)$$

4)

a) $x_1(t) = g_1(t)\cos(\omega_0 t)$



$S = 2\pi$

$$\mathcal{F}_T\{x_1(t)\} = \frac{1}{2\pi}\left(G_1(\omega) * \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)\right)$$

$$= \frac{1}{2}\left(G_1(\omega-\omega_0) + G_1(\omega+\omega_0)\right)$$

$$G_1(\omega) = \pi \,\text{sinc}^2\left(\frac{\omega\pi}{2}\right)$$

$$\mathcal{F}_T\{x_1(t)\} = \frac{\pi}{2}\left(\text{sinc}^2\left(\frac{\pi}{2}(\omega-\omega_0)\right) + \text{sinc}^2\left(\frac{\pi}{2}(\omega+\omega_0)\right)\right)$$

b) Same thing but $g_2(t) = g_1(t-2\pi)$ thus

$$G_2(\omega) = G_1(\omega)e^{-j2\pi\omega}$$

$$X_2(\omega) = \frac{\pi}{2}\left(\text{sinc}^2\left(\frac{\pi}{2}(\omega-\omega_0)\right).e^{-j2\pi(\omega-\omega_0)} + \text{sinc}^2\left(\frac{\pi}{2}(\omega+\omega_0)\right).e^{-j2\pi(\omega+\omega_0)}\right)$$

c) $\text{rect}\left(\frac{t}{2\pi}\right) \longleftrightarrow S.\text{Sinc}(\omega S/2)$

$\downarrow g_0(t)$ //S = 2π

$g_3(t) = g_0(t - 2\pi) \longleftrightarrow G_3(\omega) = G_0(\omega)e^{-j2\pi\omega}$

$x_3(t) = g_3(t)\cos(\omega_0 t)$

$X_3(\omega) = \frac{1}{2\pi}(G_3(\omega) * \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$

$= \frac{1}{2}(G_3(\omega-\omega_0) + G_3(\omega+\omega_0))$

$= \frac{1}{2}(2\pi\,\text{Sinc}(\pi(\omega-\omega_0)e^{-j2\pi(\omega-\omega_0)}$

$+$

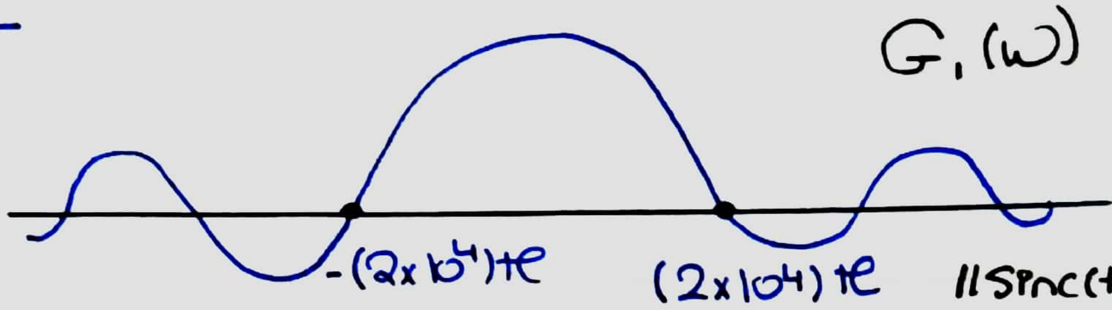
$2\pi\,\text{sinc}(\pi(\omega+\omega_0))e^{j2\pi(\omega+\omega_0)})$

//may or may not come back for sketching (here and in Problem 2)

5) a)

$$g_1(t) \longleftrightarrow 10^4 \cdot \overset{S}{10^{-4}} \cdot \mathrm{sinc}\left(\frac{\omega}{2\times10^4}\right)$$

$G_1(\omega)$



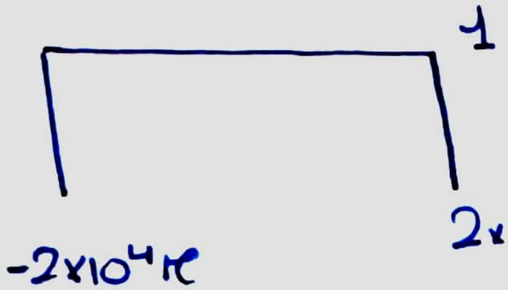
// $\mathrm{sinc}(\pi)=0$

$g_2(t) \longleftrightarrow 1$

$G_2(\omega)$

b)

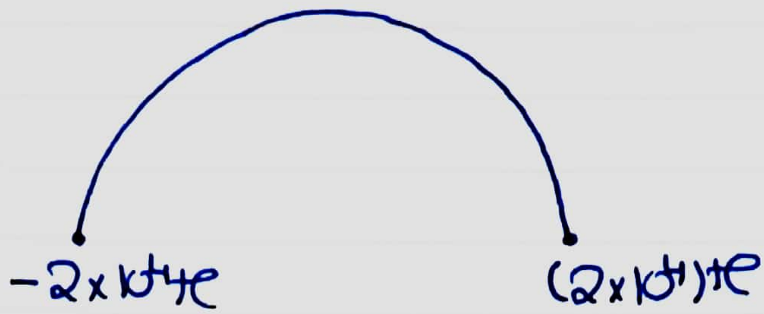
$H_1(\omega)$



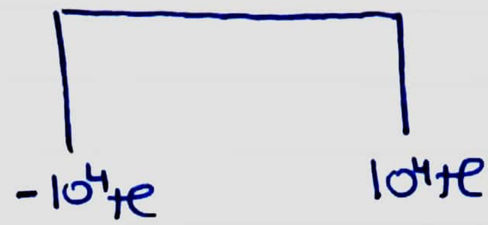
$H_2(\omega)$

c)

$Y_1(\omega) =$



$Y_2(\omega) =$



d)

Signal	Bandwidth
$y_1(t) \leftrightarrow Y_1(\omega)$	$(2\times10^4)\pi/2\pi = 10\,KHz$
$y_2(t) \leftrightarrow Y_2(\omega)$	$(10^4)\pi/2\pi = 5\,KHz$
$y_1(t)\,y_2(t) \leftrightarrow \frac{1}{2\pi}\left(Y_1(\omega) * Y_2(\omega)\right)$	$3\times10^4\pi/2\pi = 15\,KHz$

Bandwidth adds up!

Sum of Start Points of $Y_1(\omega)$, $Y_2(\omega)$

Same for end Points

thank you!! :)