Cairo University Faculty of Engineering		3rd Year Comp. MTH3251- Fall 2022 Number theory - Sheet 5

(1) Prove the following.

(a) $\tau(n)$ is an odd integer if and only if n is a perfect square.
(b) $\sigma(n)$ is an odd integer if and only if n is a perfect square or twice a perfect square.
[*Hint:* If p is an odd prime, then $1 + p + p^2 + \cdots + p^k$ is odd only when k is even.]

(2) If $n > 1$ is a composite number, then $\sigma(n) > n + \sqrt{n}$.
[*Hint:* Let $d \mid n$, where $1 < d < n$, so $1 < n/d < n$. If $d \leq \sqrt{n}$, then $n/d \geq \sqrt{n}$.]

(3) (a) Find the form of all positive integers n satisfying $\tau(n) = 10$. What is the smallest positive integer for which this is true?
(b) Show that there are no positive integers n satisfying $\sigma(n) = 10$.
[*Hint:* Note that for $n > 1$, $\sigma(n) > n$.]

(4) For $k \geq 2$, show each of the following:
(a) $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$.
(b) If $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$.
(c) If $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies $\sigma(n) = 2n + 2$.

(5) For a fixed integer k, show that the function f defined by $f(n) = n^k$ is multiplicative.

(6) Let $\omega(n)$ denote the number of distinct prime divisors of $n > 1$, with $\omega(1) = 0$. For instance, $\omega(360) = \omega(2^3 \cdot 3^2 \cdot 5) = 3$. Show that $2^{\omega(n)}$ is a multiplicative function.

(7) Given $n \geq 1$, let $\sigma_s(n)$ denote the sum of the sth powers of the positive divisors of n; that is,

$$\sigma_s(n) = \sum_{d \mid n} d^s$$

Verify the following:
(a) $\sigma_0 = \tau$ and $\sigma_1 = \sigma$.
(b) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of n, then

$$\sigma_s(n) = \left(\frac{p_1^{s(k_1+1)} - 1}{p_1^s - 1}\right)\left(\frac{p_2^{s(k_2+1)} - 1}{p_2^s - 1}\right) \cdots \left(\frac{p_r^{s(k_r+1)} - 1}{p_r^s - 1}\right)$$

(8) Show that if $\gcd(a, n) = \gcd(a - 1, n) = 1$, then

$$1 + a + a^2 + \cdots + a^{\phi(n)-1} \equiv 0 \pmod{n}$$

[*Hint:* Recall that $a^{\phi(n)} - 1 = (a - 1)(a^{\phi(n)-1} + \cdots + a^2 + a + 1)$.]

(9) If m and n are relatively prime positive integers, prove that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \pmod{mn}$$

(10) Find the units digit of 3^{100} by means of Euler's theorem.

(11) If $\gcd(a, n) = 1$, show that the linear congruence $ax \equiv b \pmod{n}$ has the solution $x \equiv ba^{\phi(n)-1} \pmod{n}$.

(12) For any integer a, show that a and a^{4n+1} have the same last digit.

(13) For any prime p, establish each of the assertions below:

(a) $\tau(p!) = 2\tau((p-1)!)$.

(b) $\sigma(p!) = (p+1)\sigma((p-1)!)$.

(c) $\phi(p!) = (p-1)\phi((p-1)!)$.