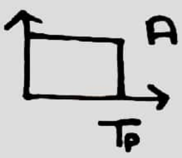
DC Sheet 4

Q01

- Derive the Probability of error for a Polar NRZ Signaling System.

$$g(t) = \begin{cases} +A & 0 \le t \le T_b & \text{"1"} \\ -A & 0 \le t \le T_b & \text{"0"} \end{cases}$$

where the matched filter is matched to 

* Already done in the lecture
* Primary Objective was to know what distribution does the random var. $y(T_b)$ follow.

$\Rightarrow$ It's a random variable as

$$y(T_b) = g(t) * h(t)\Big|_{t=T_b} + \underbrace{w(t) * h(t)\Big|_{t=T_b}}_{\text{Noise}}$$

$$= g_o(T_b) \quad + n(T_b)$$

it's a function of $w(t)$ which at anytime is a random variable $\sim N(\mu = 0, \sigma^2)$

- We used the fact that the PSD of $w(t)$ is flat to get the distribution of $n(T_p)$ which lead to the distributions of $y(T_p) \mid$ "0" and $y(T_p) \mid$ "1" (whether $g(T_p) = \pm A$ for $0 \le t \le T_p$)

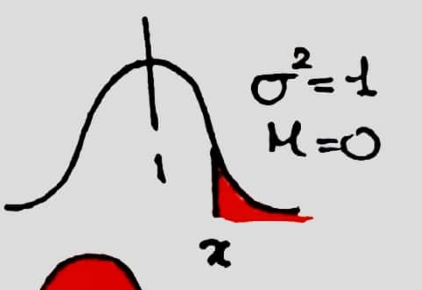
$$\rightarrow P_r(\text{error}) = P_r(\text{error} \mid \text{'0'}) P_r(\text{'0'}) \quad // \text{error} = y > \lambda$$

$$+ P_r(\text{error} \mid \text{'1'}) P_r(\text{'1'}) \quad // \text{error} = y < \lambda$$

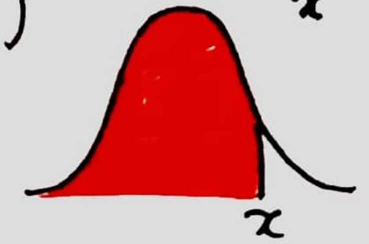
and we only then need to derive the optimal threshold λ to compute it. (was written directly in the lecture).

Recall,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} \, dt$$



$$Q(-x) = 1 - Q(x)$$

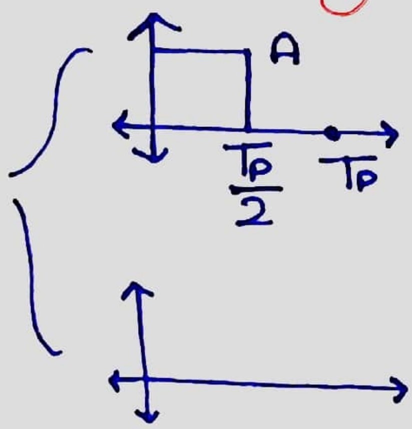


- So once we've standardized the random variable, we can use it directly.

Q 1)

- Binary PCM System
- Uses RZ Signaling to transmit '0' & '1'

$g(t) =$



. to send "1" $// g(t)_1$

. to send "0" $// g(t)_0$

- AWGN Channel ($M = 0$ and $S_w(f) = \frac{N_0}{2}$)
- $P("0") = P("1")$ (any pulse is equally likely to take ⎍ or —)
- Suppose we match a filter to 

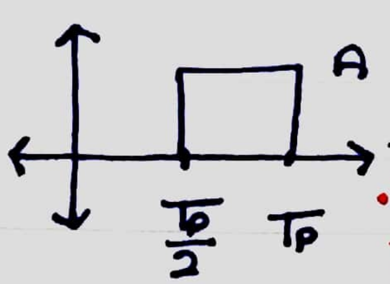
(use a matched filter)

⇒ What will be the Probability of error?

- Need the distribution of $y(T_p) = g(T_p) + n(T_p)$

→ Matched filter:

$h(t) = Kg(T-t),$ ↳ set as 1 $=$



. By shifting ← T then reflecting →

$h(t) = A$ for $0.5T_P \leq t \leq T_P$, 0 elsewhere

• Now Can Start finding $g_o(T_P)$, $n(T_P)$:

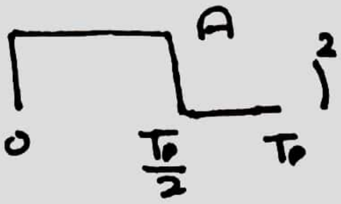
$$g_o(T_P) = \begin{cases} g_o(T_P)_1 & \text{• "1" was sent} \\ g_o(T_P)_0 & \text{• "0" was sent} \end{cases}$$

$$g_o(T_P)_1 = g(t)_1 * h(t) \Big|_{t=T_P}$$

$$= \int_{-\infty}^{\infty} g(\tau)_1 \, \underset{\swarrow \text{Plugged}}{h(T_P - \tau)} \, d\tau \qquad \text{• } h(t) = g(T-t) \rightarrow h(T-t) = g(t)$$

$$= \int_{-\infty}^{\infty} g(\tau)_1 \, g(\tau)_1 \, d\tau = E_{g_1}$$

= Area(



) $= \dfrac{A^2 T_P}{2}$

• Could've used last line directly but note it only holds for the matched symbol. ↳ was derived as a reminder

$$g_o(T_P)_0 = g_o(T_P)_0 = \underbrace{g(t)_0}_{\hookrightarrow 0} * h(t) \Big|_{T_P} = 0$$

$$g_o(T_P) = \begin{cases} \dfrac{A^2 T_P}{2} & \text{"1" Sent} \\ 0 & \text{"0" Sent} \end{cases}$$

Thus, $y(T_p) = \begin{cases} \frac{A^2 T_p}{2} + n(T_p) & \text{"1"} \\ n(T_p) & \text{"0"} \end{cases}$

• $n(T_p) = h(t) * w(t) \big|_{T_p}$

$$= \int_{-\infty}^{\infty} w(\tau)\, h(T_p - \tau)\, d\tau$$

• $h(T_p - \tau) = g(\tau)$.

$$= \int_{-\infty}^{\infty} w(\tau)\, g(\tau)\, d\tau$$

$$= \int_{0}^{T_p/2} A\, w(\tau)\, d\tau = A \int_{0}^{T_p/2} w(\tau)\, d\tau$$

$w(\tau) \sim N(\mu = 0, \sigma^2)$

• $n(T_p)$ is the continuous sum of many Gaussian random variables and is thus itself gaussian.

→ Need its mean & variance

$$\rightarrow E[n(T_p)] = A \int_{0}^{T_p/2} \overbrace{E[w(\tau)]}^{\to 0}\, d\tau = 0$$

• like swapping S.S. (legit)

$$\rightarrow \text{Var}[n(T_p)] = E[n^2(T_p)] - \overbrace{E^2(n(T_p))}^{\to 0}$$

$$= E[n(T_p) . n(T_p)]$$

$$= R_n(0) = \int_{-\infty}^{\infty} S_n(f)\, df$$

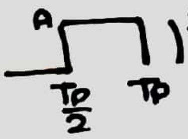
• Write $R_n(\tau)$, $\tau = 0$ and recall E is same for all times WSS.

$$= \int_{-\infty}^{\infty} S_w(f)\, |H(f)|^2\, df$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df = \frac{N_0}{2} \int_{-\infty}^{\infty} |h(t)|^2 \, dt$$ ①

• Energy Theorem (DC Note)

$\downarrow$

Area(()^2)

• Better than what we did in lec. ①

$$= \frac{N_0}{2} \cdot \frac{A^2 T_p}{2}$$

$$= \frac{N_0 A^2 T_p}{4}$$

Thus, $n(T_p) \sim N\left(\mu = 0, \sigma^2 = \frac{N_0 A^2 T_p}{4}\right)$

Consequently,

$$y(T_p) \,|\, \text{"1"} \sim N\left(\mu = \frac{A^2 T_p}{2}, \sigma^2 = \frac{N_0 A^2 T_p}{4}\right)$$

$// \; y(T_p) = \frac{A^2 T_p}{2} + n(T_p)$ $\quad E(a+x) = E(a) + E(x) = a + E(x)$ $\quad Var(a+x) = Var(x)$

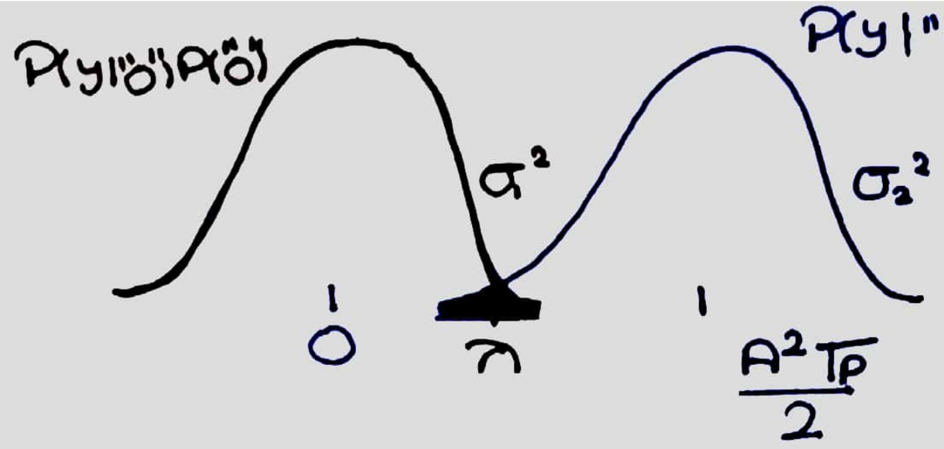
$$y(T_p) \,|\, \text{"0"} \sim N\left(\mu = 0, \sigma^2 = \frac{N_0 A^2 T_p}{4}\right)$$

and

$$P(y \,|\, \text{"1"}) = \frac{2}{\sqrt{2\pi N_0 T_p} \, A} \cdot e^{-\frac{(y - A^2 T_p/2)^2 \cdot 2}{N_0 A^2 T_p}}$$

$$P(y \,|\, \text{"0"}) = \frac{2}{\sqrt{2\pi N_0 T_p} \, A} \cdot e^{-\frac{(y)^2 \cdot 2}{N_0 A^2 T_p}}$$

Just for show really.



- $\sigma_1^2 = \sigma_2^2 = \frac{N_0 A^2 T_p}{4}$
- $P(\text{"0"}) = P(\text{"1"}) = \frac{1}{2}$

- The optimal threshold λ by symmetry is thus clearly $\lambda = \frac{A^2 T_p}{4}$

$$P_r(\text{error}) = P_r(y > \lambda \mid \text{"0"}) P_r(\text{"0"}) + P_r(y < \lambda \mid \text{"1"}) P_r(\text{"1"})$$

$$= \frac{1}{2}\left(P_r(y > \lambda \mid \text{"0"}) + P_r(y < \lambda \mid \text{"1"})\right)$$

- equal by symmetry as well

$$= P_r(y > \lambda \mid \text{"0"})$$

- $y \mid \text{"0"} \sim N\left(0, \sigma^2 = \frac{N_0 A^2 T_p}{4}\right)$
- $Z = \frac{y - \mu}{\sigma} \sim N(0, \sigma^2 = 1)$

$$= P_r\left(Z > \frac{\lambda - 0}{(N_0 A^2 T_p / 4)^{0.5}}\right) = P_r\left(Z > \frac{2\lambda}{(N_0 A^2 T_p)^{0.5}}\right)$$

$$= Q\left(\frac{2\lambda}{A\sqrt{N_0 T_p}}\right)$$

Now since $\lambda = \frac{A^2 T_p}{4}$

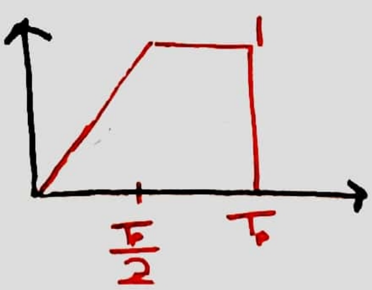
$$P(error) = Q\left(\frac{2}{A\sqrt{N_0 T_p}} \cdot \frac{A^2 T_p}{4}\right)$$

$$= Q\left(\frac{A}{2}\sqrt{\frac{T_p}{N_0}}\right)$$

• more than the Polar NRZ case. (Factor of 2)

Q2)

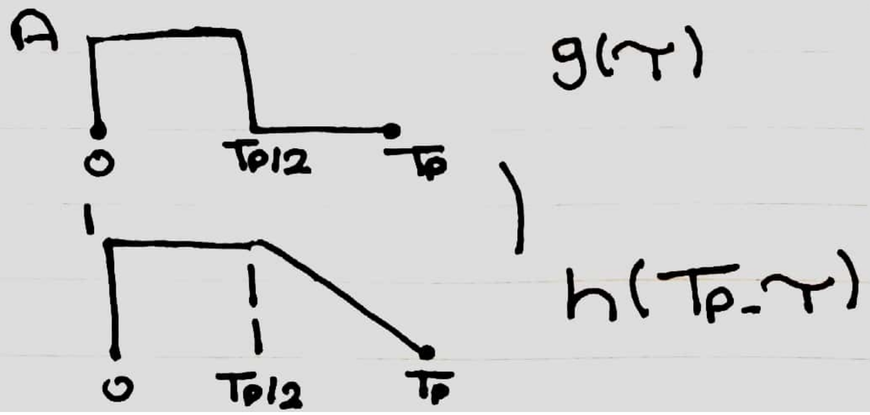
• Consider if instead of the matching filter

$h(t) =$ 

was used.

$$g_o(T_p)_1 = h(t) * g(t)_1 \big|_{t=T_p}$$

$$= \int_{-\infty}^{\infty} g(\tau)_1 \, h(T_p-\tau)\, d\tau$$

= Area (× 

)

$$g_o(T_p)_1 = \frac{A\,T_p}{2}$$

$$g_o(T_p)_0 = h(t) * g(t)_0 \big|_{t=T_p} = 0$$

$$g_o(T_p) = \begin{cases} A\,T_p/2 & \text{"1" sent} \\ 0 & \text{"0" sent} \end{cases}$$

Gaussian   Deterministic   Gaussian

$$\cdot\; n(T_p) = h(t) * w(t) \big|_{t=T_p}$$

$$= \int_{-\infty}^{\infty} w(\tau)\, h(T_p-\tau)\, d\tau$$

$$\sim \mathcal{N}(\mu=0, \sigma^2)$$

$$\rightarrow E[n(T_p)] = \int_{-\infty}^{\infty} E[w(\tau)\, h(T_p-\tau)\, d\tau]$$

$$= \int_{-\infty}^{\infty} \overbrace{E[w(\tau)]}^{0}\, E[h(T_p-\tau)]\, d\tau$$

$$= 0$$

$$\rightarrow Var[n(T_p)] = E[n^2(T_p)] - \overbrace{E^2[n(T_p)]}^{0}$$

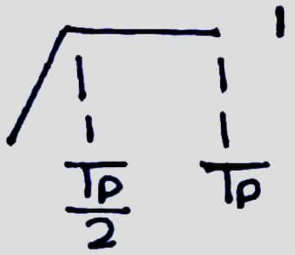
$$= E[n^2(T_p)] = R_n(0)$$

$$\mathrm{Var}[n(T_p)] = \int_{-\infty}^{\infty} S_n(f)\,df$$

$$= \int_{-\infty}^{\infty} S_w(f)\,|H(f)|^2\,df$$

$\hookrightarrow N_0/2$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |h(t)|^2\,dt$$

$h(t) =$ 

$$= \frac{N_0}{2} \left(\int_0^{T_p/2} \left(\frac{2t}{T_p}\right)^2 dt + 1\left(T_p - \frac{T_p}{2}\right)\right)$$

$\int t^2 = \frac{t^3}{3}$

$$= \frac{N_0}{2} \left(\frac{4}{T_p^2} \left(\frac{T_p^3}{3.8}\right) + \frac{T_p}{2} \right)$$

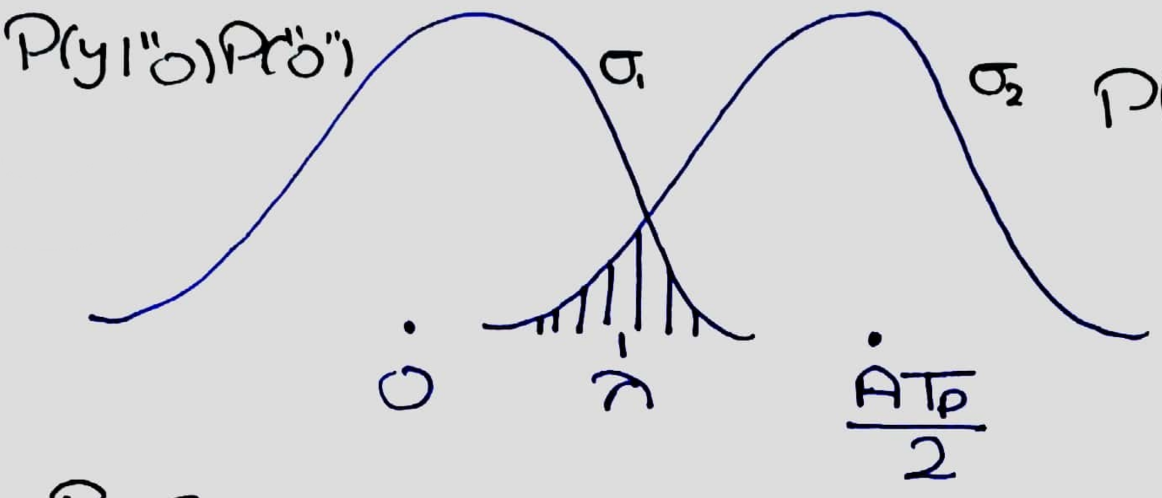
$$= \frac{N_0}{2} \left(\frac{4T_p}{24} + \frac{T_p}{2} \right) = \frac{N_0 T_p}{3}$$

$$n(T_p) \sim N\left(\mu = 0,\ \sigma^2 = \frac{N_0 T_p}{3}\right)$$

Consequently

$$y(T_p) \,|\, \text{"1"} \sim N\left(\mu = \frac{A T_p}{2},\ \sigma^2 = \frac{N_0 T_p}{3}\right)$$

$$y(T_p) \,|\, \text{"0"} \sim N\left(\mu = 0,\ \sigma^2 = \frac{N_0 T_p}{3}\right)$$



- $\sigma_1^2 = \sigma_2^2 = \frac{N_0 T_P}{3}$ ①
- $P(\text{"1"}) = P(\text{"0"}) = \frac{1}{2}$

By Symmetry,

①

$$P(\text{error}) = P(y > \lambda \mid \text{"0"}) \quad \text{with } \lambda = \frac{AT_P}{4}$$

$\downarrow$

$\frac{1}{2}\left(P(y>\lambda|0) + P(y<\lambda|1)\right)$

$$y|\text{"0"} \sim N\left(\mu = 0,\ \sigma = \sqrt{\frac{N_0 T_P}{3}}\right)$$

Let $Z = \frac{y-\mu}{\sigma}$ then $Z \sim N(0,1)$

$$= P\left(Z > \frac{\lambda - 0}{\sqrt{\frac{N_0 T_P}{3}}}\right) = Q\left(\frac{\lambda}{\sqrt{N_0 T_P/3}}\right)$$

$$= Q\left(\frac{A T_P \sqrt{3}}{\sqrt{N_0 T_P}\, 4}\right)$$

$$= Q\left(\frac{\sqrt{3}A}{4} \cdot \sqrt{\frac{T_P}{N_0}}\right)$$

- $\frac{\sqrt{3}}{4} < \frac{1}{2}$ ← unipolar NRZ

& thus error ↑

Q3.

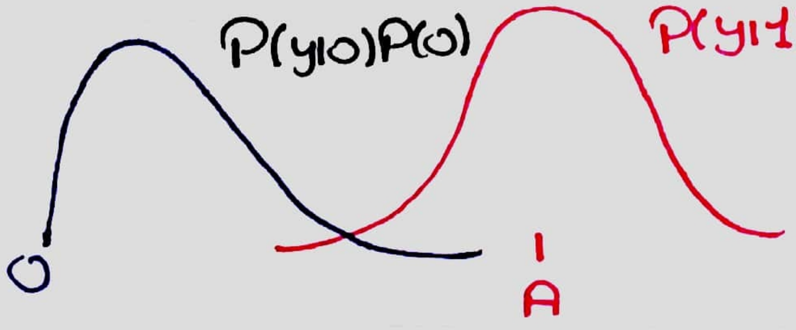
- $P(y|0) = \frac{y}{N_\circ} e^{-y^2/2N_\circ}$ # Rayleigh Distribution

- $P(y|1) = \frac{1}{\sqrt{2\pi N_\circ}} e^{-\frac{(y-A)^2}{2N_\circ}}$, $\lambda > 0$ and $A >> 0$ # Normal Distribution

- Assume P("0") = P("1")

→ Find optimum λ that minimizes Probability of error

→ Find the expression of the Probability of error



$$P(\text{error}) = \underset{\frac{1}{2}}{P(0)}\, \underset{(e|0)}{P(y>\lambda|0)} + \underset{\frac{1}{2}}{P(1)}\, \underset{(e|1)}{P(y<\lambda|1)}$$

• $P(\text{error} \mid 0) = P(y > \lambda \mid 0)$

$$= \int_{\lambda}^{\infty} \frac{y}{N_0} e^{-y^2/2N_0}\, dy$$

let $z = \frac{-y^2}{2N_0}$

then $dz = -\frac{y}{N_0} dy$

$$= \int_{\frac{-\lambda^2}{2N_0}}^{-\infty} -e^{z}\, dz = \left. e^{z} \right|_{-\infty}^{\frac{-\lambda^2}{2N_0}}$$

$$= e^{-\lambda^2/2N_0}$$

• $P(\text{error} \mid 1) = P(y < \lambda \mid 1)$

$y \sim N(\mu = A, \sigma^2 = N_0)$

let $z = \frac{y - \mu}{\sigma}$ then $z \sim (\mu = 0, \sigma = 1)$

$$= P\left(z < \frac{\lambda - A}{\sqrt{N_0}} \mid 1\right)$$

$$= 1 - P\left(z > \frac{\lambda - A}{\sqrt{N_0}} \mid 1\right)$$

$$= Q\left(\frac{A - \lambda}{\sqrt{N_0}}\right)$$

Thus,

$$P(\text{error}) = \frac{1}{2}\left(e^{-\lambda^2/2N_0} + Q\left(\frac{A-\lambda}{\sqrt{N_0}}\right)\right)$$
$\downarrow P_e$

• To find optimum λ (minimize error)

Set $\dfrac{\partial P_e}{\partial \lambda} = 0$

$$\frac{\partial}{\partial \lambda}\frac{1}{2}\left(e^{-\lambda^2/2N_0} + Q\left(\frac{A-\lambda}{\sqrt{N_0}}\right)\right) = 0$$

$$\rightarrow \frac{\partial e^{-\lambda^2/2N_0}}{\partial \lambda} = -\frac{\partial}{\partial \lambda} Q\left(\frac{A-\lambda}{\sqrt{N_0}}\right)$$

$$-\frac{2\lambda}{2N_0} \cdot e^{-\lambda^2/2N_0} = -Q'\left(\frac{A-\lambda}{\sqrt{N_0}}\right) \cdot \frac{-1}{\sqrt{N_0}}$$

$\rightarrow$ Since $Q(x) = \int_x^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt = 1 - \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$

$\rightarrow$ Since by the fundamental theorem of calculus

$$\frac{\partial}{\partial x}\int_a^x f(t)\,dt = f(x)$$

* Then $Q'(x) = \dfrac{-e^{-t^2/2}}{\sqrt{2\pi}}$

So now

$$+\frac{\lambda}{N_0} e^{-\lambda^2/2N_0} = + e^{-\frac{(A-\lambda)^2}{2N_0}} \cdot \frac{1}{\sqrt{2\pi N_0}}$$

$\rightarrow$ Plugged $\varphi(x)$

$\rightarrow$ Notice that if we started with setting

$$P(y|1)P(1) = P(y|0)P(0)$$

then we'd reach this equation rightaway
(the general case however is to set $\frac{\partial P_{error}}{\partial \lambda} = 0$)

By dividing both sides by $e^{-\lambda^2/2N_0}$ we get

$$\frac{\lambda}{N_0} = \frac{e^{-(\cancel{\lambda^2}-2A\lambda+A^2)/2N_0}}{\sqrt{2\pi N_0}}$$

$$\frac{\lambda}{N_0} = \frac{e^{-A(A-2\lambda)/2N_0}}{\sqrt{2\pi N_0}}$$

only 1st 2 terms

• Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \simeq 1 + x \, , \, |x| < 1$

$$\frac{\lambda}{N_0} = \frac{1}{\sqrt{2\pi N_0}}\left(1 - A(A - 2\underline{\lambda})/2N_0\right)$$

$$\frac{\lambda}{N_0} - \frac{1}{\sqrt{2\pi N_0}}\,\frac{\lambda A}{N_0} = \frac{1}{\sqrt{2\pi N_0}}\left(1 - \frac{A^2}{2N_0}\right)$$

$$\lambda_{opt} \simeq \frac{\frac{1}{\sqrt{2\pi N_0}}\left(1 - A^2/2N_0\right)}{\frac{1}{N_0} - \frac{1}{\sqrt{2\pi N_0}}\,\frac{A}{N_0}}$$

$$\simeq \frac{\frac{1}{\sqrt{2\pi N_0}}\left(-A^2/2N_0\right)}{\frac{-1}{\sqrt{2\pi N_0}} \cdot \frac{A}{N_0}}$$

as $A \gg 0$
(can ignore other 1

$$\simeq \frac{A}{2}$$

By Substituting

$$P(error) = \frac{1}{2}\left(e^{-A^2/8N_0} + Q\left(\frac{A}{2\sqrt{N_0}}\right)\right)$$