CE Sheet 5

1. System has

$$G(S) = \frac{10^4}{S^2(S+50)} \qquad H(S) = 1$$

- Given 2 Constraints
- No Steady State requirements (no need for $\int$)

→ Consider a controller with two unknowns

Let

$$G_c(S) = K_p(1 + T_d S)$$

Now

$$G(S) = \frac{10^4 K_p(1 + T_d S)}{S^2(S+50)} \qquad H(S) = 1$$

$$D(S) = 10^4 K_p + 10^4 T_d K_p S + S^3 + 50S^2$$

$$= S^3 + 50S^2 + (10^4 K_p T_d)S + 10^4 K_p$$

Comparing with the general 3rd order System

$$S^3 + S^2(2\zeta\omega_n + \alpha) + S(2\zeta\omega_n\alpha + \omega_n^2) + \alpha\omega_n^2$$

We get

$$50 = 2\zeta\omega_n + \alpha \quad ①$$

$$10^4 K_p T_d = 2\zeta\omega_n\alpha + \omega_n^2 \quad ②$$

$$10^4 K_p = \alpha\omega_n^2 \quad ③$$

From the constraints we as well have

$$M_p = 0.1 \longrightarrow \zeta = \frac{-\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}} = 0.59$$

$$t_s = 1 \longrightarrow t_s = \frac{4}{\zeta\omega_n} \longrightarrow \omega_n = \frac{4}{\zeta t_s} = 6.78 \text{ rad/s}$$

Now from ①

$$\alpha = 50 - 2 \times 0.59 \times 6.78 = 42$$

. indeed, $\alpha > 5\zeta\omega_n = 20$

From ③

$$K_p = \frac{\alpha\omega_n^2}{10^4} = \frac{42 \times 6.78^2}{10^4} = 0.193$$

From ② $$T_d = \frac{2 \times 0.59 \times 6.78 \times 42 + 6.78^2}{10^4 \times 0.193} = 0.198$$

The required PD Controller is hence

$$G_c(S) = 0.193(1 + 0.198 S)$$

→ Notice that if we set $T_d = 0$ (P controller) then the System of equations lacks a Solution (as $2\zeta w_n \alpha + w_n^2 \neq 0$)

• $\zeta, w_n, \alpha > 0$ and the System is thereby Stable

• We'll skip this step later but do it.

2. $G(S) = \frac{2}{S(S+2)(S+8)}$ $\quad H(S) = 1$

a) $K_p = \lim_{S \to 0} GH(S) = \infty$

$$K_v = \lim_{S \to 0} S.GH(S) = \frac{2}{2 \times 8} = \frac{1}{8}$$

$$K_a = \lim_{S \to 0} S^2.GH(S) = 0$$

$$E(S) = \frac{R(S)}{1 + GH(S)} = \frac{1/S^2}{1 + \frac{2}{S(S+2)(S+8)}}$$

$$= \frac{S(S+2)(S+8)}{(S(S+2)(S+8) + 2)S^2}$$

$S^2 + 10S + 16$

$$E(S) = \frac{(S+2)(S+8)}{S^3+10S^2+16S+2} \cdot \frac{1}{S}$$

$$= \frac{(S+2)(S+8)}{S(S+8.04)(S+0.1365)(S+1.822)}$$

$$= \frac{0.0576}{S+1.822} + \frac{-8.06}{S+0.1365} + \frac{-6.1146\times10^{-4}}{S+8.04}$$

$$+ \frac{8.001}{S}$$

• coverup method x4

Thus,

$$e(t) = (0.0576\, e^{-1.822t} - 8.06\, e^{-0.1365} - 6.11\times10^{-4}\, e^{-8.04}$$
$$+ 8.001)\, u(t)$$

b) Want dominant Poles to be at

$$S = -1 \pm j2\sqrt{3} = -Z\omega_n \pm j\omega_n\sqrt{1-Z^2}$$

$$Z\omega_n = 1 \rightarrow \omega_n = \frac{1}{Z}$$

$$\omega_n\sqrt{1-Z^2} = 2\sqrt{3} \rightarrow \frac{\sqrt{1-Z^2}}{Z} = 2\sqrt{3}$$

$1-Z^2 = (2\sqrt{3})^2 Z^2 \rightarrow 1 = 13Z^2 \rightarrow Z = \sqrt{\frac{1}{13}}$

• $Z>0$ for Stability (other → refused)

So $\omega_n = \sqrt{13}$.

→ Given 2 Constraints, no Steady State requirements.

• Choose a PD controller (has 2 unknowns)

$G_c(S) = K_p + K_d S$

• $K_d = K_p T_d$

By introducing the Controller

$G(S) = \dfrac{2(K_p + K_d S)}{S(S+2)(S+8)}$, $H(S) = 1$

$D(S) = \underbrace{S(S+2)(S+8)}_{S(S^2+10S+16)} + \underbrace{2(K_p + K_d S)}_{S(2K_d) + 2K_p}$

$= S^3 + 10S^2 + (16 + 2K_d)S + 2K_p$

System is

$\omega_n = \sqrt{13}$, $Z = 1/\sqrt{13}$

① $10 = 2Z\omega_n + \alpha \rightarrow 10 = 2 + \alpha \rightarrow \alpha = 8 > 5Z\omega_n$

② $16 + 2K_d = 2Z\omega_n\alpha + \omega_n^2 = 2\alpha + 13 \rightarrow K_d = 6.5$

③ $2K_p = \alpha\omega_n^2 \rightarrow K_p = 52$

The needed Controller is here

$$G_c(S) = 52 + 6.5S$$

3.

Initially

$$G(S) = \frac{10N}{S(S+1)(S+10)} \underset{N=20}{=} \frac{200}{S(S+1)(S+10)}$$

Need $G_c(S)$ Such

$M_p = 0 \longrightarrow Z \geq 1$ take $Z = 1$

$t_s = 2.5 \longrightarrow t_s = \frac{4}{Z\omega_n} \rightarrow \omega_n = 1.6$

. No Steady State requirements

. Two Constraints

→ Use a PD Controller $G_c(S) = K_P + K_d S$

Now

$$G(s) = \frac{200(K_P + K_d S)}{S(S+1)(S+10)}$$

$$D(S) = S\underbrace{(S+1)(S+10)}_{S^2+11S+10} + 200(K_P + K_d S)$$

$$S^3 + 11S^2 + S(10 + 200K_d) + 200K_p$$

which by comparing gives the system

$$11 = 2\zeta\omega_n + \alpha$$

$$10 + 200K_d = 2\zeta\omega_n\alpha + \omega_n^2$$

$$200K_p = \alpha\omega_n^2$$

$$\alpha = 11 - 2\times1\times1.6 = 7.8 \overset{?}{>} 5\overbrace{\zeta\omega_n}^{8}$$

$$K_d = 0.0876$$

$$K_p = \frac{7.8\times1.6^2}{200} = 0.1$$

So the needed controller is

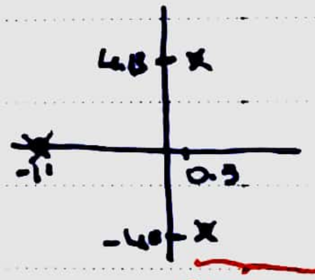
$$G_c(S) = 0.1 + 0.0876S$$

$$TF(S) = \frac{200\,G_c(S)}{S(S+1)(S+10) + 200\,G_c(S)}$$

- System's steady state response before adding the controller ($G_c(S) = 1$)

$$\Rightarrow D(S) = S(S+1)(S+10) + 200$$
$$= S^3 + 11S^2 + 10S + 200$$

Poles



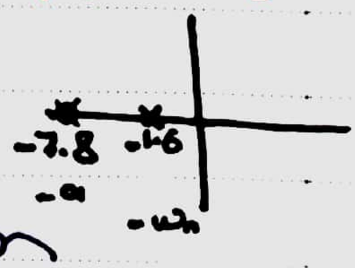
2 Poles in RHP

Since $11>0,\ 10>0,\ 200>0,\ 1>0$
and $11\times10 < 200$

. The system is unstable (has no steady state)

- After adding $G_c(S) = 0.1 + 0.0876S$

$\Rightarrow Z = 1,\ \omega_n = 1.6,\ \alpha = 7.8$ ($Z, \omega_n, \alpha > 0$ $\therefore$ Stable System)



. Since no input is specified study steady state response (C_{ss}) for step, ramp, acceleration inputs.

$$C_{ss} = \lim_{S\to0} S.C(S) = \lim_{S\to0} S.R(S).TF(S)$$

• when $r(t) = M \longleftrightarrow R(S) = \frac{M}{S}$

$$C_{ss} = \lim_{S \to 0} M \cdot \frac{200(0.1 + 0.0876S)}{S(S+1)(S+10) + 200(0.1+0.0876S)}$$

$$= M \cdot \frac{200 \times 0.1}{200 \times 0.1} = 1$$

• when $r(t) = Mt \longleftrightarrow R(S) = \frac{M}{S^2}$

$$C_{ss} = \lim_{S \to 0} M \cdot \frac{1}{S} \cdot \frac{200(0.1+0.0876S)}{S(S+1)(S+10) + 200(0.1+0.0876S)}$$

$$= \infty$$

• when $r(t) = \frac{M}{2} t^2 \longleftrightarrow R(S) = \frac{M}{S^3}$

$$C_{ss} = \lim_{S \to 0} M \cdot \frac{1}{S^2} \cdot \frac{200(0.1+0.0876S)}{S(S+1)(S+10) + 200(0.1+0.0876S)}$$

$$= \infty$$

Before adding the Controller	after adding the Controller
• unstable System (Output Could go to ∞ regardless of input)	• System is stable with Steady State response = M for unit Step Mu(t) input.
• No Steady State/meaningless	

4.

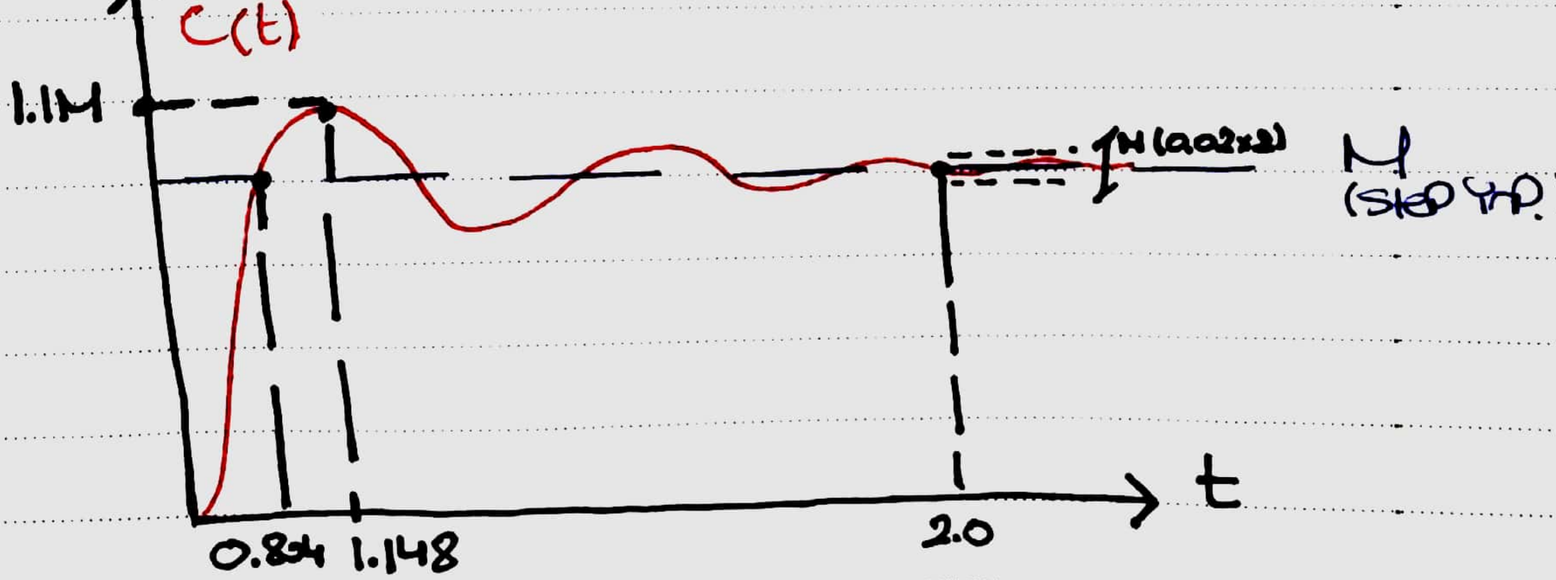
$$M_{p_{Max}} = 0.1 \longrightarrow Z_{min} = \frac{-\ln(0.1)}{\sqrt{\pi^2 + (\ln(0.1))^2}} = 0.59$$

$$T_{s_{max}} = 2 \longrightarrow \omega_{n_{min}} = \frac{4}{2Z} = 3.389$$

» In case we're able to satisfy these transient requirements (we will, you can do this step last) then the resulting response has:

- $M_p = 0.1$, $t_p = \dfrac{\pi}{\omega_n\sqrt{1-Z^2}} = 1.148$
- $T_s = 2$
- $T_{rise} = \dfrac{\pi - \cos^{-1} Z}{\omega_n\sqrt{1-Z^2}} = 0.8047$

Plot:



Consider using a PD controller

$$G_c(S) = K_p + K_d S$$

→ Now by considering the system for R(S) only. $(D(S) = 0)$

[illegible]$(S) = \dfrac{10(K_p + K_d S)}{S\underbrace{(S+5)(S+10)}_{S^2+15S+50}}$, $H(S) = 1$

$$D(S) = S^3 + 15S^2 + S(50 + 10K_d) + 10K_p$$

by comparing we have

$$15 = 2\zeta\omega_n + \alpha$$

$$50 + 10K_d = 2\zeta\omega_n\alpha + \omega_n^2$$

$$10K_p = \alpha\omega_n^2$$

→ $\alpha = 15 - 2 \times 0.59 \times 3.389 = 11 > 5\zeta\omega_n$

→ $K_d = 0.547$

→ $K_p = 12.633$

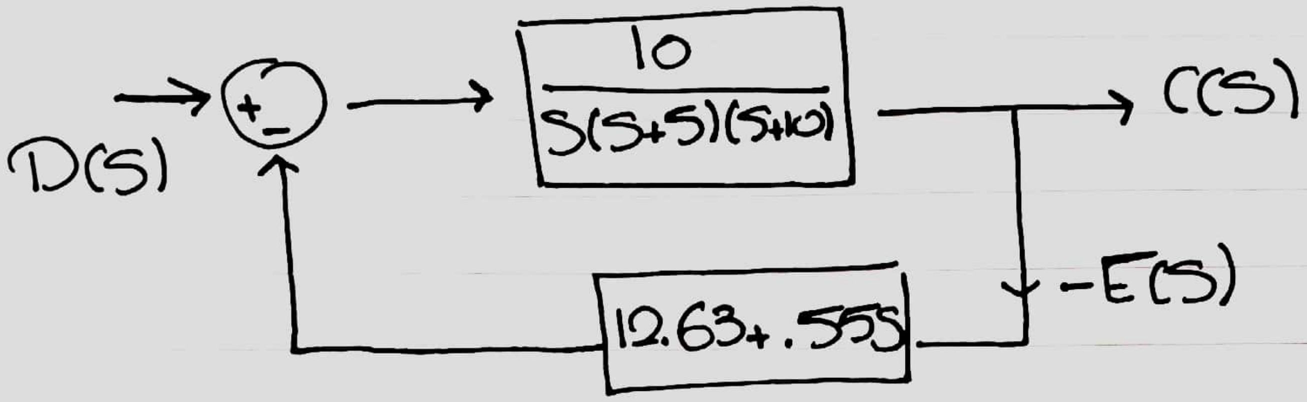
$$G_c(S) = 12.633 + 0.547S$$

Steady State Output due to unit Step disturbance

//R(S)=0

$$D(S) = \frac{\varphi}{S}$$

like in lecture



$$\frac{C(S)}{D(S)} = \frac{10}{10(12.63+0.55S)+S(S+5)(S+10)}$$

TF(S) due to D(S) ←

$$\lim_{t\to\infty} c(t)\big|_{D(t)} = \lim_{S\to 0} S.C(S)$$

$$= \lim_{S\to 0} S.\frac{\varphi}{S}.\frac{10}{10(12.63+0.55S)+S(S+5)(S+10)}$$

$$= \varphi.\frac{10}{10(12.63)} = 0.079\varphi$$

• Meanwhile that due to input $(D(S)=0)$

$$G(S) = \frac{10(12.63+0.55S)}{S(S+5)(S+10)}, \quad H(S)=1$$

$$\frac{C(s)}{R(s)} = \frac{10(12.63 + 0.55s)}{10(12.63+0.55s) + s(s+5)(s+10)}$$

• let $R(s) = \frac{M}{s}$ then

$$\lim_{s\to 0} s.C(s) = \frac{M.\overset{10}{12.63}}{10\times 12.63} = M$$

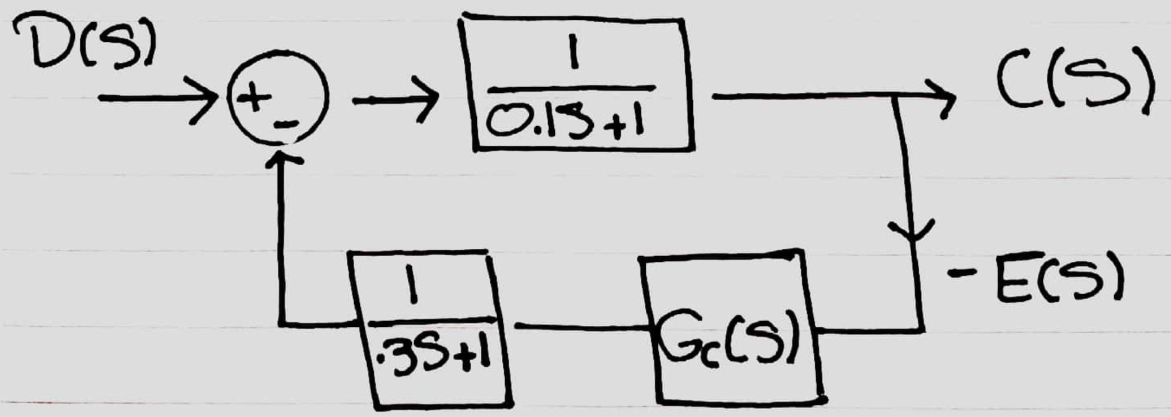
$$\lim_{t\to\infty} c(t) = M + 0.079Q$$

M: Due to step input

$0.079Q$: Due to step disturbance

5.

Steady state error due to D(s) should be zero



$$G(s) = \frac{1}{0.1s+1}, \quad H(s) = \frac{K_p + T_d s + \overset{1/s}{T_i/s}}{0.3s+1}$$

$$\frac{C(S)}{D(S)} = \frac{0.3S+1}{(K_p + T_d S + \hat{T_i}/S) + (0.3S+1)(0.1S+1)}$$

$$\lim_{S\to 0} S.E(S) = \lim_{S\to 0} -S.C(S)$$

$$= \lim_{S\to 0} -S.\frac{Q}{S}.$$

$$= \lim_{S\to 0} \frac{-S(0.3S+1)}{(K_p S + T_d S^2 + \hat{T_i}) + S\underbrace{(0.3S+1)(0.1S+1)}_{0.03S^2+0.4S+1}}$$

$= 0$ (regardless of the choice of const.s

→ Also needed

$M_p \leq 0.05 \rightarrow Z \geq 0.69$

$t_s \leq 0.2 \rightarrow \omega_n \geq \frac{4}{Z \times 0.2}$

take $Z = 0.69$ so $\omega_n = 29$

Char. equation:

$$0.03S^3 + 0.4S^2 + T_d S^2 + S + K_P S + \hat{T}_i = 0$$

$\div 0.03$

$$S^3 + 33.3(0.4 + T_d)S^2 + 33.3(1 + K_P)S +$$
$$+ 33.3\hat{T}_i = 0$$

by Comparison to

$$S^3 + (2Z\omega_n + \alpha)S^2 + (2Z\omega_n\alpha + \omega_n^2)S + \alpha\omega_n^2 = 0$$

$\Rightarrow$ Have $T_d, K_P, \hat{T}_i, \alpha, \omega_n, Z$ unknowns
but Just 5 equations
. use $\alpha = 5Z\omega_n = 5 \times 0.69 \times 29 \simeq 100$

$$33.3(0.4 + T_d) = 2Z\omega_n + \alpha \rightarrow T_d = 3.804$$

$$33.3(1 + K_P) = 2Z\omega_n\alpha + \omega_n^2 \rightarrow K_P = 144.4$$

$$\alpha\omega_n\ 33.3\hat{T}_i = \alpha\omega_n^2 \rightarrow T_i = 3.959 \times 10^{-4}$$

Choosing a Controller

P

- $K_p \propto \frac{1}{e_{ss}}$

(lower e_{ss} but $\neq 0$)

PI

- eliminate e_{ss} (Steady State)

* Type ↑ by 1

PD

- 2 Transient requirements ($\zeta, \omega_n, M_p, t_s, t_p, t_r$)

PID

- Both transient and steady state requirements