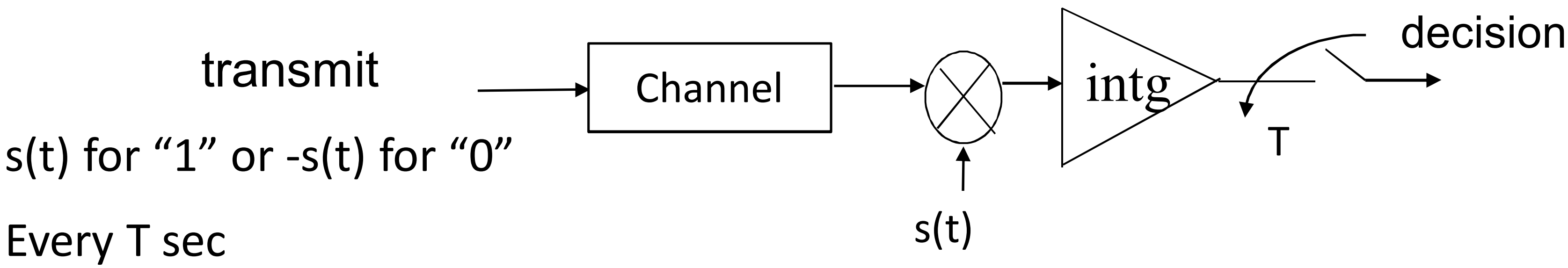
Signal Space Analysis

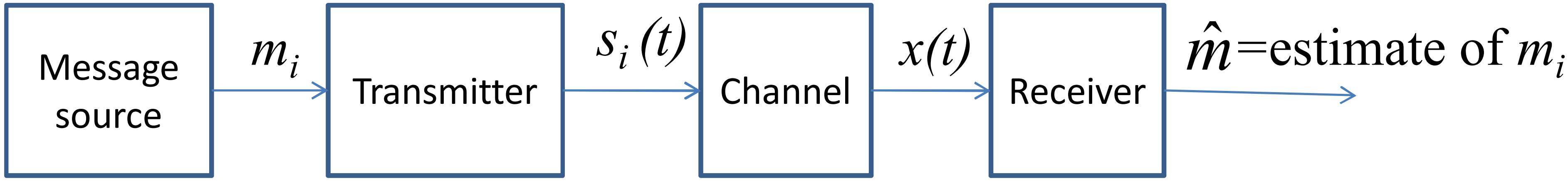
Motivation

- In case of binary transmission, the optimum system is as follows



- What would the system look like if different signals are transmitted for the “1” and the “0” or if the signal set has more than 2 signals?

Block diagram of a generic comm. system



- The source emits 1 message every T seconds, with the symbols belonging to an alphabet of M symbols denoted by $m_1, m_2, \ldots, m_M$
- The *apriori* probabilities of $m_1, m_2, \ldots, m_M$ specify the message source output
- The transmitter takes the message source output and codes it into distinct signal $s_i(t)$ suitable for transmission over the channel

Channel Model

- The channel is assumed linear with a wide enough BW to accommodate transmission of $s_i(t)$ without distortion
- The noise, $w(t)$, is a sample function of a *zero-mean white Gaussian RP*
- The received signal $x(t)$ is given by

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

Receiver Objective

- The receiver has the task of observing the received signal $x(t)$ for a duration T seconds and making a best estimate of the transmitted message m_i.
- Since the random noise is present, the receiver will occasionally make errors.
- The requirement is therefore to design the receiver so as to minimize the average probability of symbol error given by

$$P_e = \sum_{i=1}^{M} p_i P(\hat{m} \neq m_i \mid m_i)$$

Geometric Representation of Signals

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

A set of M signals $s_i(t)$ are represented as a linear combination of N orthonormal basis functions, $M \geq N$

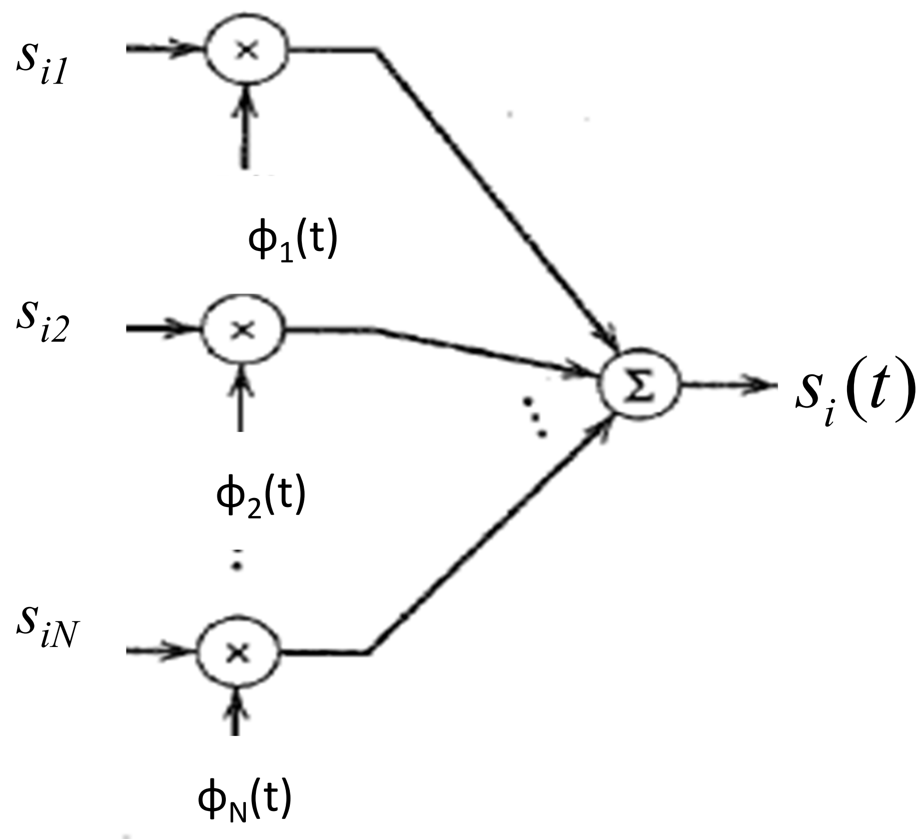
$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \qquad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases}$$

The orthonormal basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are orthonormal, i.e.

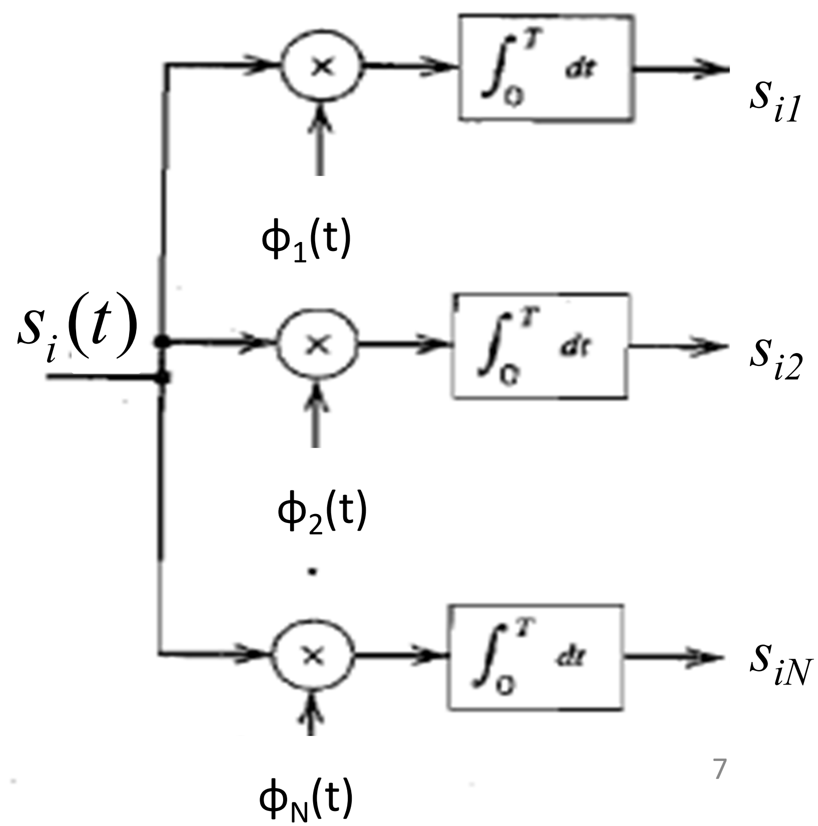
$$\int_0^T \phi_i(t)\phi_j(t)dt = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- The set of coefficients $\left\{s_{ij}\right\}_{j=1}^{N}$ may be viewed as an N-dimensional vector denoted by $\mathbf{s}_i$
- $\mathbf{s}_i$ bears a one-to-one relationship with $s_i(t)$

Synthesizer for generating $s_i(t)$



Analyzer for generating signal vector



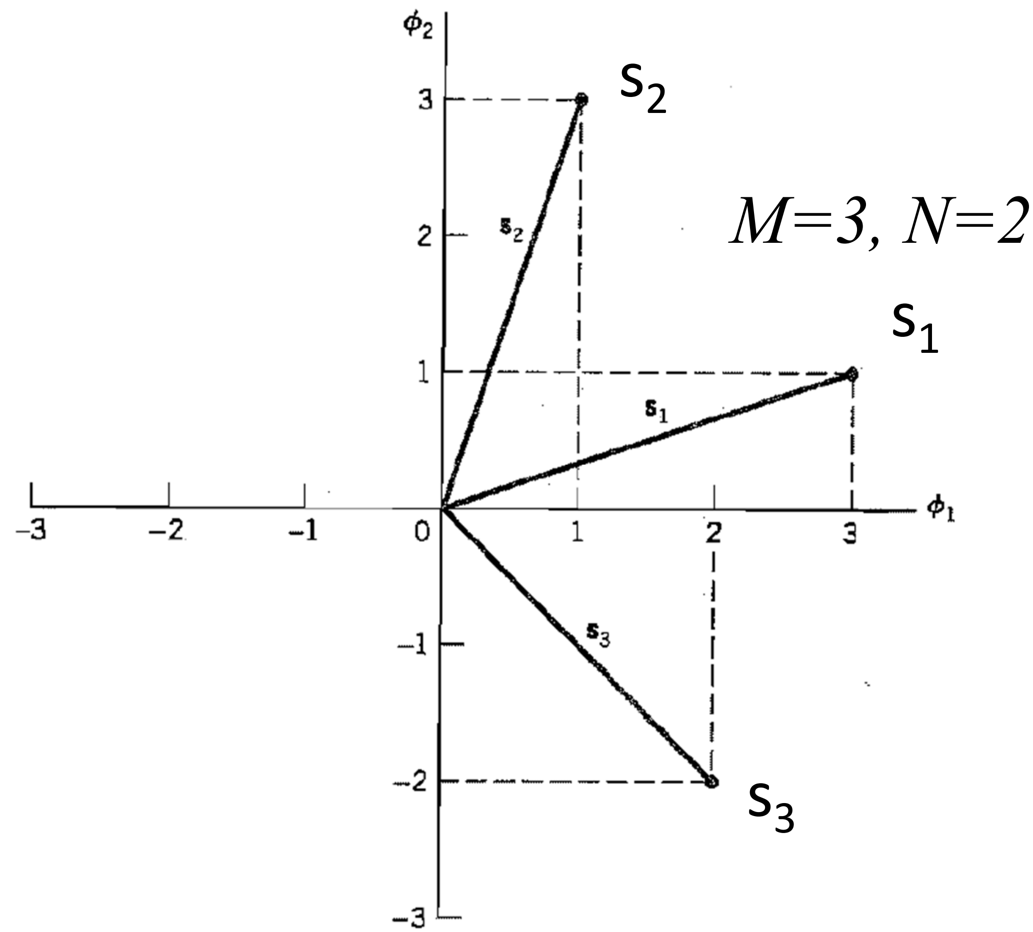
- We may state that each signal in the set $\{s_i(t)\}$ is completely determined by the vector of its coefficient (signal vector)

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \ldots M$$

- We may visualize the set of signal vectors $\{\mathbf{s}_i \mid i = 1, 2, \ldots, M\}$ as defining a corresponding set of M-points in an N-dimensional Euclidean space called the Signal Space
- The signal space analysis helps in performing noise analysis for digital communication systems

Signal space

The idea of visualizing a set of signals geometrically, provides the mathematical basis for performing noise analysis of digital Communication systems in a conceptually satisfying manner.



Energy and correlation

The energy of a signal can be calculated as follows:

$$E_i = \int_0^T s_i^2(t)dt = \sum_{j=1}^{N} s_{ij}^2 = \left\| \mathbf{s}_i \right\|^2 = \mathbf{s}_i^T \mathbf{s}_i$$

The correlation between signals can be calculated as follows:

$$\int_0^T s_i(t) s_k(t) dt = \mathbf{s}_i^T \mathbf{s}_k$$

Gaussian Random Process (GRP)

- Assume $X(t)$ is a Gaussian RP

  Then $Y = \int_0^T g(t)X(t)dt$ is a Gaussian Random Variable

- If $X(t)$ (a GRP) is applied to a stable linear filter, then the RP $Y(t)$ developed at the O/P of the filter is also Gaussian

$$Y(t) = \int_0^T h(t-\tau)X(\tau)d\tau \qquad 0 \le t \le \infty$$

Gaussian Random Process (GRP)

- The set of RV $x(t_1), x(t_2), \ldots, x(t_n)$ obtained by sampling a GRP $X(t)$ at times $t_1, t_2, \ldots, t_n$ are jointly Gaussian for any n
- If $x(t_1), x(t_2), \ldots, x(t_n)$ are uncorrelated, then they are independent

Conversion of the continuous AWGN channel into a vector channel

- Suppose that the I/P to the bank of N correlators in not $s_i(t)$ but $x(t)$

$$x(t) = s_i(t) + w(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

where $w(t)$ is a sample function of an AWGN RP of zero mean and PSD=$No/2$.

- The O/P of correlator j, x_j, is given by

$$x_j = \int_0^T x(t)\phi_j(t)dt = s_{ij} + w_j, \qquad j = 1, 2, \ldots, N$$

where

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \qquad\qquad w_j = \int_0^T w(t)\phi_j(t)dt$$

- x_j is a Gaussian process, we will find its mean and variance

$$E(X_j) = E(s_{ij} + W_j) = s_{ij}$$

$$\sigma_{x_j}^2 = E\left(W_j^2\right) = E\left(\int_0^T\int_0^T W(t)\phi_j(t)W(u)\phi_j(u)dtdu\right)$$

$$= \int_0^T\int_0^T \phi_j(t)\phi_j(u)R_w(t,u)dtdu$$

- For AWGN

$$R_w(t,u) = \frac{N_0}{2}\delta(t-u)$$

$$\sigma_{x_j}^2 = \int_0^T \int_0^T \phi_j(t)\phi_j(u)\frac{N_0}{2}\delta(t-u)\,dt\,du$$

$$= \frac{N_0}{2}\int_0^T \phi_j^2(t)\,dt = \frac{N_0}{2}, \quad \text{for all } j$$

- Also $\operatorname{cov}(X_j X_k) = E\left\{\left(X_j - \mu_{x_j}\right)\left(X_k - \mu_{x_k}\right)\right\} = 0, \quad j \neq k$

$\therefore X_j$ are independent,

$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$ are elements of independent GRV with mean value s_{ij} and $\sigma^2 = N_0/2$

PDF of the correlators O/P

$$f(\mathbf{x} \mid m_i) = \prod_{j=1}^{N} f(x_j \mid m_i), \qquad i = 1, 2, \ldots, M$$

$$f(x_j \mid m_i) = \frac{1}{\sqrt{\pi N_0}} e^{\frac{-1}{N_0}\left(x_j - s_{ij}\right)^2}, \quad \begin{matrix} j = 1, 2, \ldots, N \\ i = 1, 2, \ldots, M \end{matrix}$$

$$f(\mathbf{x} \mid m_i) = \left(\pi N_0\right)^{-N/2} e^{\frac{-1}{N_0}\sum_{j=1}^{N}\left(x_j - s_{ij}\right)^2}, \qquad i = 1, 2, \ldots, M$$

Correlator O/P Properties

- The correlator O/Ps determined by the received signal *x(t)* are the only data that are useful for the decision-making process and hence represent the sufficient statistic for the problem

- Insofar as signal detection in AWGN is considered, only the projections of the noise onto the basis functions of the signal set $\{s_i(t)\}_{i=1}^{M}$ affects the sufficient statistics of the detection problem, the remainder of the noise is irrelevant

Gram-Schmidt Orthogonalization

- How to find the bases functions $\phi(t)$ of a set of signals $s(t)$:

1. $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$ $\Rightarrow$ $s_1(t) = \sqrt{E_1}\phi_1(t) = s_{11}\phi_1(t)$

2 $g_2(t) = s_2(t) - s_{21}\phi_1(t)$ where $s_{21} = \int_0^T s_2(t)\phi_1(t)dt$

3. $\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$ And so on