[2] Derive an expression for the channel capacity & mutual info of a binary symmetric channel.

<u>where:</u> $P_0 = P(X = x_0)$ $P_1 = P(X = x_1) = 1 - P_0$

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[1] lets start by explaining some terms:

- Binary means that we have only 2 inputs x_0 x_1 & 2 outputs y_0 y_1
- Symmetric means → $P(y_0 | x_0) = P(y_1 | y_1) = P$
  $P(y_1 | x_0) = P(y_0 | x_1) = 1 - P$

Now we can build our TMx safely → $TMx = \begin{array}{c c} & \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \end{array} & \begin{pmatrix} P & 1-P \\ 1-P & P \end{pmatrix} \end{array}$

[2] Now lets define the Mutual Information

$$I(X;Y) = H(Y) - H(Y|X)$$

$\because H(Y) = -\sum_i P(y_i) \log_2 P(y_i)$

$\because H(Y|X) = -\sum_i (P(y_i, x_i) \log_2 (P(y_i | x_i))$

$\therefore$ to be able to evaluate $I(X;Y)$ we need <u>JM</u>

$\therefore$ to be able to evaluate the Joint Matrix to get $H(Y)$

[3] Now lets evaluate the Joint Matrix to get $H(Y)$

$JM = \begin{pmatrix} P P_0 & (1-P) P_0 \\ (1-P) P_1 & P P_1 \end{pmatrix}$ where P_0 & P_1 are defined at the problem Header☺

$\therefore \cancel{H(Y) = -(P P_0 \log_2 (P P_0) +}$

[4] Now lets evaluate $P(y_i)$

$P(y_0) = P P_0 + (1-P) P_1$ let it $= Z$ | $P(y_1) = (1-P) P_0 + P P_1$ let it $= 1 - Z$

5) Now we can easily compute $H(Y)$

$$\because H(Y) = -\sum_i P(y_i) \log_2 (P(y_i))$$

$$\therefore H(Y) = -\left[Z \log_2 (Z) + (1-Z) \log_2 (1-Z) \right]$$

6) Now lets evaluate $H(Y|X)$

$$\because H(Y|X) = -\sum P(x,y) \log_2 (P(y|x))$$

$$= -\Big(P P_0 \log_2 (P) + (1-P) P_0 \log_2 (1-P) + P P_1 \log (P) + (1-P) P_1 \log_2 (1-P) \Big)$$

// By taking common factors $\hookrightarrow$ $P \lg P$, $(1-P) \log (1-P)$

$$= -\left(P \lg P \, (P_0 + P_1) + (1-P) \lg (1-P) \, (P_0 + P_1) \right)$$

$\because P_0 + P_1 = 1$

$$\therefore H(Y|X) = \left(P \lg P + (1-P) \lg (1-P) \right)$$

let it $= H(P)$ to avoid long writing only

7) Now we can evaluate Mutual Information

$$I(X;Y) = H(Z) + H(P)$$

8) lets find channel capacity $\rightarrow$ $C = \max (I(X;Y))$

this occurs when $\boxed{Z = \frac{1}{2}}$ By applying partial drivitives on P_0

Now lets drive the formula for $\underline{C}$

$Z = 1/2$

$$P P_0 + (1-P) P_1 = Z = \tfrac{1}{2} \quad \therefore P_1 = (1 - P_0)$$

$$\therefore P P_0 + (1-P)(1-P_0) = \frac{1}{2}$$

$$\therefore P P_0 + 1 - P - P_0 + P_0 P = 1/2$$

$$\therefore P_0 (2P - 1) + 1 - P = 1/2$$

$$\therefore P_0 \cancel{(2P - 1)} = P - \frac{1}{2} = \frac{\cancel{2P - 1}}{2}$$

$$\boxed{\therefore P_0 = \frac{1}{2}} \qquad \boxed{\therefore P_1 = \frac{1}{2}}$$

Hence the channel capacity can be defined

$$\boxed{C = 1 - H(P)}$$

Done

Because : $\left(\frac{1}{2} \log_2 (0.5) + \frac{1}{2} \log_2 (0.5)\right) = \boxed{1}$

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ary for the Proofs: easy

Derive an expression for maximum $H(S)$.

ets start by defining that entropy is a quantization for the uncertainaty, so it will be maximized if all the inputs have the same Probability $\rightarrow P(X) = \frac{1}{\ell}$

$$\therefore H(S) = -\sum_{0}^{\ell-1} P(X) \log_2 P(X) = -\sum_{0}^{\ell-1} \frac{1}{\ell} \log_2 \left(\frac{1}{\ell}\right)$$

$$\therefore H(S) = -\left(\frac{1}{\ell}\right)(\ell)\left(\log(P(X))\right) = -\log(P(X)) = I(S)$$

which is the Average Amount of Information content.

[2] Drive expression for the channel capacity & mutual information of a binary symmetric channel where:

ne Mutual Information Equation:-

From Shanon's 1st theorm, we need atleast $H(A)$ bits to convey one symbol of $\underline{A}$

2] On average we lose $H(A|B)$ bits of information due to the channel noise ☹

3] thus, on average, the observation of single output can be evaluated as:

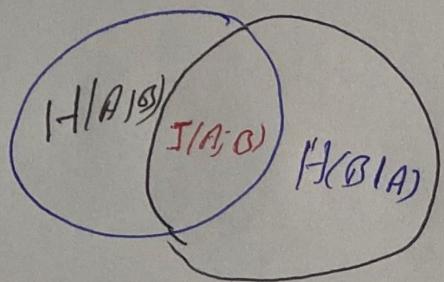
$$\boxed{I(A;B) = H(A) - H(A|B)}$$

4] Also we can conclude more properties

a) $I(A;B) > 0$

b) $I(A;B) \leq H(A)$

c) $I(A;B) = I(B;A)$



Summary of IS

1) Information ($I(S)$):-

$I(S) = -\log_2(P(S))$ } this represents the Information content of each message.

2) Entropy ($E(S)$):

$$E(S) = E(I(S)) = \sum_{i=0}^{n-1} P(i) \log\left(\frac{1}{P_i}\right) = -\sum_{i=0}^{n-1} P(i) \log(P_i)$$

- It quantifies the randomness of the random variable. بنا عن معلومات قد إيه كدا
- on average

3) Max Probability of Entropy:-

this occurs when all the values possible for the random variable s has the same probability $\rightarrow \frac{1}{K}$

Proof:

let $P(i) = \frac{1}{K}$ | $\therefore H(S) = -\sum_{0}^{K-1} P(i) \log_2(P(i))$ $\frac{1}{K}$

$$\therefore H(S) = -\sum_{0}^{K-1} \frac{1}{K} \log_2\left(\frac{1}{K}\right) = -\frac{1}{K}(K) \log(P(i))$$

$\because$ summation from 0 to $K-1$, $\therefore$ their sum $= K$

$$\therefore H(S) = -\log(P(i)) = I(S)$$

which is the maximum possible value for the entropy & H is equal to the Information content.

Note: Entropy represents the minimum # of bits.

Information Channel:

1- Transition Matrix : this Matrix defines the behaviour of the channel.
(TMX)

ie: If we have 2 Inputs & 2 outputs → a_i → input → b_i → output

$$\therefore TMx = \begin{array}{cc} \begin{array}{cc} b_1 & b_2 \end{array} & \\ \begin{pmatrix} P(b_1|a_1) & P(b_2|a_1) \\ P(b_1|a_2) & P(b_2|a_2) \end{pmatrix} & \begin{array}{c} a_1 \\ a_2 \end{array} \end{array}$$

marked in red are the correct outputs

they're probabilities of correctness

2. Joint Matrix (JM):-

هنا بنحسب احتمالية حدوث b_i مع a_i مع بعض

their joint

JM for the same upper Matrix:

$$JM = \begin{array}{cc} \begin{array}{cc} b_1 & b_2 \end{array} & \\ \begin{pmatrix} P(b_1|a_1)\,P(a_1) & P(b_2|a_1)\,P(a_1) \\ P(b_1|a_2)\,P(a_2) & P(b_2|a_2)\,P(a_2) \end{pmatrix} & \begin{array}{c} a_1 \\ a_2 \end{array} \end{array}$$

في هنا بقى معلومة مهمة ← إن مجموع العامود بتاعه column تتعبر عن ال total probability بتاعة حدوث ال output ⇒ $P(b_i)$

ie: $P(b_1) = P(b_1|a_1) \cdot P(a_1) + P(b_1|a_2)\,P(a_2)$

$P(b_2) = P(b_2|a_1) \cdot P(a_1) + P(b_2|a_2)\,P(a_2)$

كده هنا lec1 تخلص

Steps of solving Problems:-

[1] get TMx

[2] get JM

[3] calculate $P(b_i)$ from JM

الحمد لله وكفى والصلاة والسلام
على الحبيب المصطفى ♡

rive A rent Entropies:-

bin

1] Source Entropy: $H(A) = -\sum P(a_i)\log_2(P(a_i))$

: the Average amount of uncertainity @ source

2] Destination Entropy: $H(B) = -\sum_j P(b_j)\log_2(P(b_j))$

• the Average amount of uncertainity @ destination.

3] Joint Entropy: $H((A,B)) = -\sum_{ij} P(a_i,b_j)\log_2(P(a_i,b_j))$

Best case is when $A = B$ $\therefore H(A,B) = H(A)$

Average amount of Information due to simultaneously observing A & B

What If they were Independent?

$\therefore H(A,B) = H(A) + H(B)$ → worst possible scenario

4] Conditional Entropy:-

$H(B|A) = -\sum P(x,y)\log_2(P(x|y))$

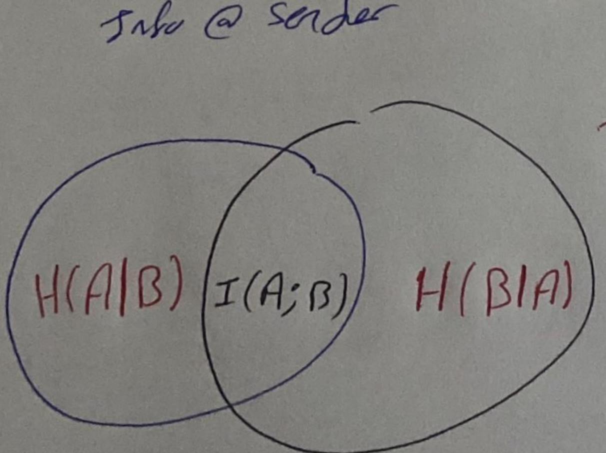
also $H(B|A) = H(B,A) - H(A)$

Mutual Information:-

$I(A;B) = H(A) - H(A|B) =$ المعلومات اللي بتحصل عليها .

$H(A)$: Info @ sender

$H(A|B)$: # of Bits you lose



$I(A;B) = H(A) + H(B) - H(A,B)$!!