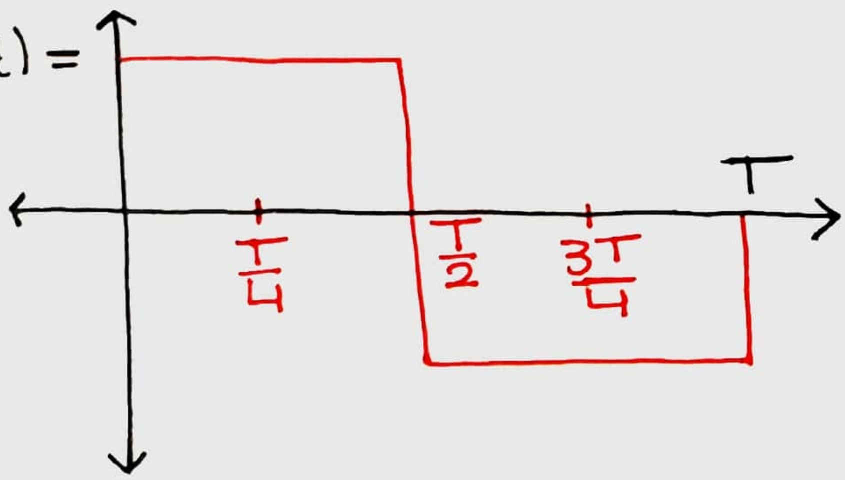
DC Sheet 3

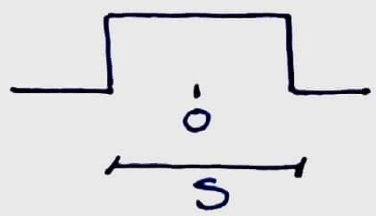
Problem 1)

$s(t) =$



$$s(t) = \frac{A}{2}\,\mathrm{rect}\left(\frac{t-0.25T}{0.5T}\right) - \frac{A}{2}\,\mathrm{rect}\left(\frac{t-0.75T}{0.5T}\right)$$

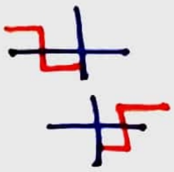
where $\mathrm{rect}\left(\frac{t}{s}\right) =$



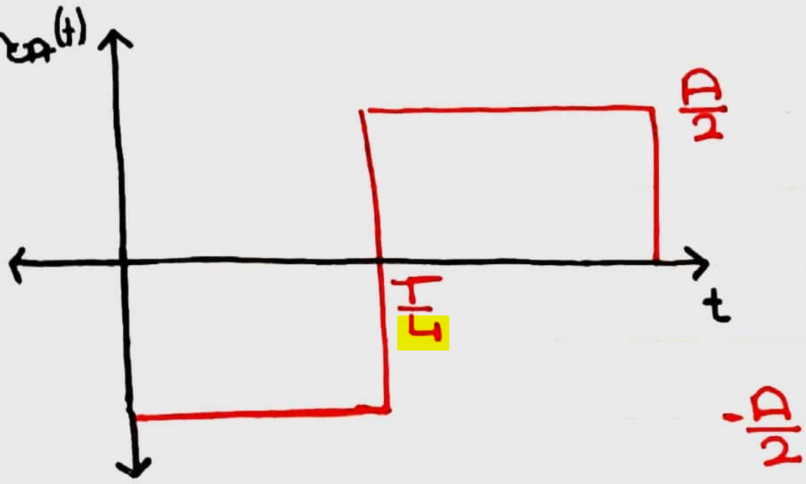
1) Impulse response of matched filter

$h_{opt} = s(T-t)$

- Shift $s(t)$ by T $\quad s(t+T)$
- Reflect $\quad s(T-t)$



$h_{opt}(t)$



- Clearly,

$$h_{opt}(t) = -s(t) = \frac{A}{2}\left(\mathrm{rect}\left(\frac{t-0.75T}{0.5T}\right) - \mathrm{rect}\left(\frac{t-0.25T}{0.5T}\right)\right)$$

• We could've also instead directly plugged with $T-t$ in place of t (and perhaps used the fact that rect is even.)

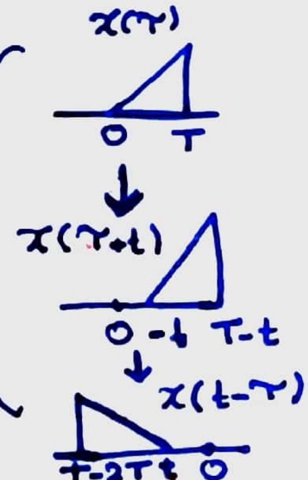
b) Already did that in a)

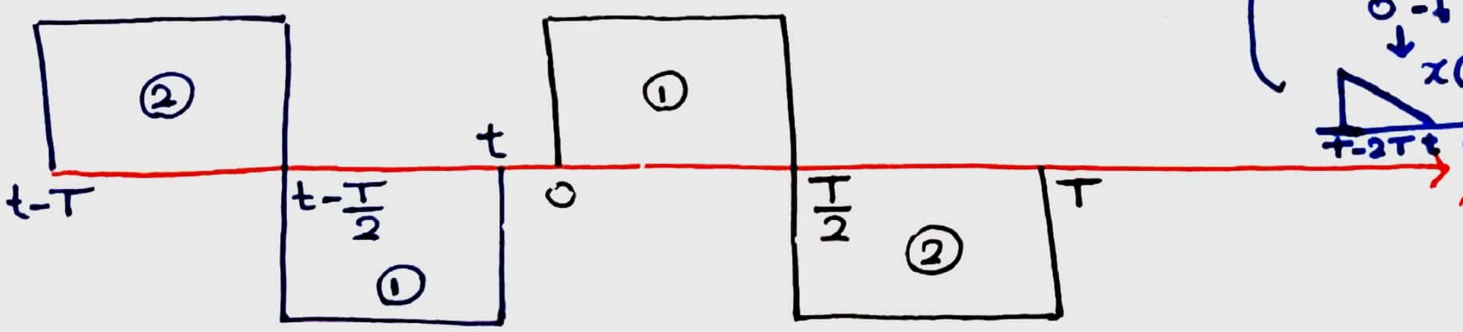
c) Let $s_o(t)$ be the matched filter's output due to $s(t)$

$$s_o(t) = s(t) * h(t) = -s(t) * s(t)$$

$$= \int_{-\infty}^{\infty} s(\tau) h(t-\tau)\, d\tau$$

Recall





Overlap cases)

→ ① with ① → $t \in [0, \frac{T}{2})$

→ ① with ② → $t \in [\frac{T}{2}, T)$ also implies

$t - \frac{T}{2} \in [0, \frac{T}{2})$ (② Partial in with ①)

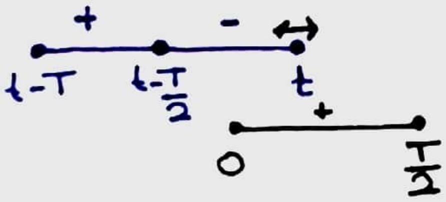
• Partial Out.

→ ② with ① → $t - T \in [0, \frac{T}{2})$ $(t \in [T, 3T/2))$

→ ② with ② → $t - T \in [\frac{T}{2}, T)$ $(t \in [\frac{3T}{2}, T))$

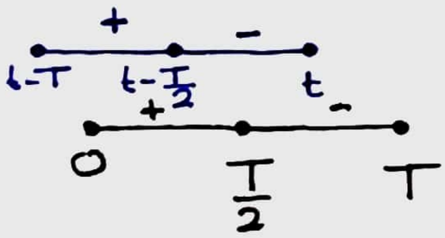
⇒ Due matching sizes, no "full overlap" intervals

- $t \in [0, \frac{T}{2})$



$$S_o(t) = \int_0^t \frac{A}{2} \cdot \frac{-A}{2}\, d\tau = -\frac{A^2}{4} t$$

- $t \in [\frac{T}{2}, T)$



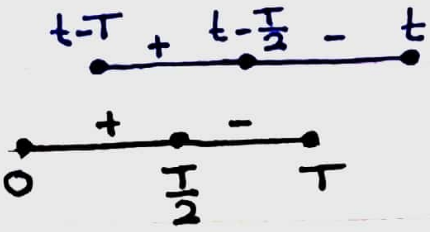
$$\left.\begin{aligned} S_o(t) = &\int_0^{t-\frac{T}{2}} \frac{A}{2} \cdot \frac{A}{2}\, d\tau \\ &+ \int_{t-\frac{T}{2}}^{\frac{T}{2}} \frac{A}{2} \cdot -\frac{A}{2}\, d\tau \\ &+ \int_{\frac{T}{2}}^{t} -\frac{A}{2} \cdot -\frac{A}{2}\, d\tau \end{aligned}\right\}$$

or directly write ± width×h

$$= \frac{A^2}{4}\left((t-\frac{T}{2}) - (T-t) + (t-\frac{T}{2}) \right)$$

$$= \frac{A^2}{4}(3t - 2T)$$

- $t \in [T, \frac{3T}{2})$ $\quad (t-T \in [0, T/2))$

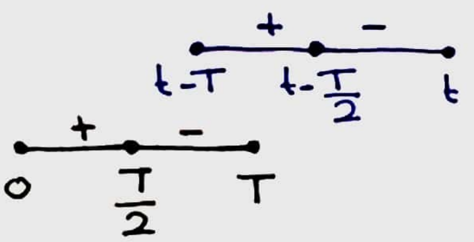


$$\begin{aligned} S_o(t) = &\int_{t-T}^{T/2} A^2/4\, d\tau \\ &+ \int_{T/2}^{t-T/2} -A^2/4\, d\tau \\ &+ \int_{t-\frac{T}{2}}^{T} \frac{A^2}{4}\, d\tau \end{aligned}$$

$$S_o(t) = \frac{A^2}{4}\Big((0.5T-(t-T)) - ((t-0.5T)-0.5T) + (0.5T-(t-0.5T))\Big)$$

$$= \frac{A^2}{4}(\; 4T - 3t)$$

- $t \in [\frac{3T}{2}, 2T)$ $\qquad (t-T \in [\frac{T}{2}, T))$



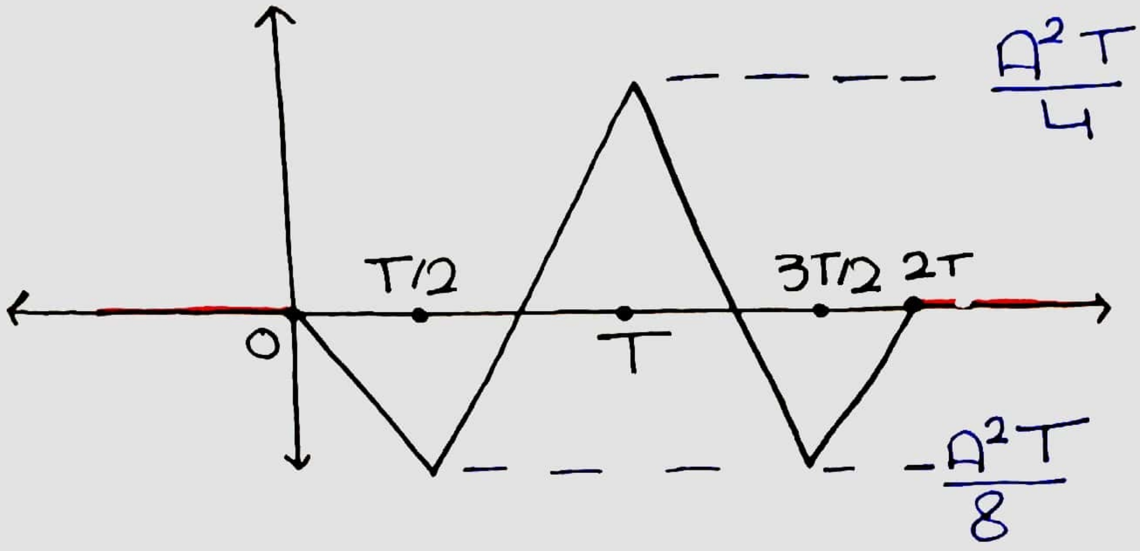
$$S_o(t) = \frac{-A^2}{4}(T-(t-T))$$

$$= -\frac{A^2}{4}(2T-t)$$

$$S_o(t) = \begin{cases} \frac{-A^2}{4}t & 0 \leqslant t < T/2 \\ \frac{A^2}{4}(3t-2T) & T/2 \leqslant t < T \\ \frac{A^2}{4}(\;4T-3t) & T \leqslant t < 3T/2 \\ \frac{-A^2}{4}(2T-t) & 3T/2 \leqslant t < 2T \\ 0 & \text{otherwise} \end{cases}$$

- Clearly 4 line segments over 4 intervals
  $\rightarrow$ result must be continuous, evaluate $S_o(t)$ at $\frac{T}{2}, T, \frac{3T}{2}$ and no δ connect points

$$\frac{A^2}{4}\left(-\frac{1}{2}T\right) \quad \frac{A^2}{4}(T) \quad -\frac{A^2}{4}\left(\frac{T}{2}\right)$$



Other Options besides Convolution

$$S_o(t) = -S(t) * S(t) \leftrightarrow S_o(f) = -(S(f))^2$$

$$S(t) = \frac{A}{2}\left(\mathrm{rect}\left(\frac{t-0.25T}{0.5T}\right) - \mathrm{rect}\left(\frac{t-0.75T}{0.5T}\right)\right)$$

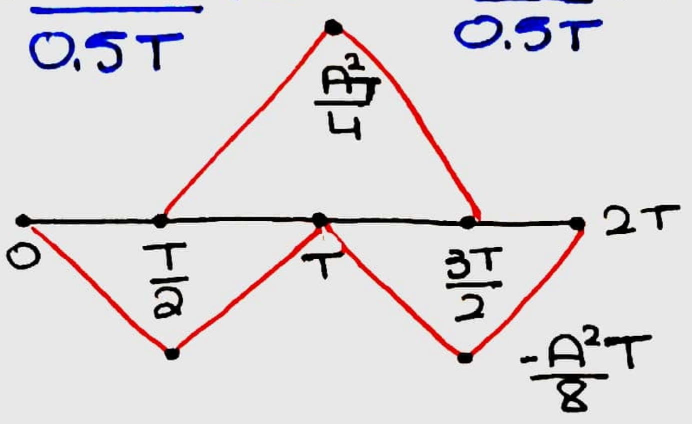
$$S(f) = \frac{A}{2}\left(0.5T.\,\mathrm{Sinc}(0.5Tf).\,e^{-j2\pi f(0.25T)} - 0.5T.\,\mathrm{Sinc}(0.5Tf).\,e^{-j2\pi f(0.75T)}\right)$$

$$S_o(f) = -\left(\frac{A}{2}.0.5T.\,\mathrm{Sinc}(0.5Tf)\right)^2\left(e^{-j2\pi f\frac{T}{4}} - e^{-j2\pi f\frac{3T}{4}}\right)^2$$

$$= \underbrace{-\frac{A^2}{16}T}_{-\frac{A^2}{8}.0.5T}.\,\mathrm{Sinc}^2(0.5Tf)\left(e^{-j2\pi f.\frac{T}{2}} - 2e^{-j2\pi fT} + e^{-j2\pi f.\frac{3T}{2}}\right)$$

$$S_o(t) = -\frac{A^2}{8}T\left(tri\left(\frac{t-T/2}{0.5T}\right) - 2\,tri\left(\frac{t-T}{0.5T}\right) + tri\left(\frac{t-3/2T}{0.5T}\right)\right)$$

The overlapping lines should be found & added up.



4) Max value is clearly $\frac{A^2}{4}T$

→ In the lecture we showed that under the optimum impulse response (matched to the input signal $g(t)$), we have

$$\to |g_o(T)|^2 = (KE)^2$$

Thus, $|g_o(T)| = E_g$ (here $K=1$)

Recall, $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$ (area under the curve)

- Besides of the value at T of the output resembling the original signal's energy, it's also the unique global maximum.

That is, we can always find the output's peak by computing the area under $|g(t)|^2$ ($|S(t)|^2$ here.)

• & possibly $\times K$

- Note that we assumed $K=1$ for the matching filter (since its optimal regardless of it). However, if another K was used then the output (& Peak) would get scaled by that.

Problem 2)

$$S(t) = \frac{A}{T}\cos(2\pi f_0 t), \quad 0 \leq t \leq T, \quad f_0 = \frac{1}{T}$$

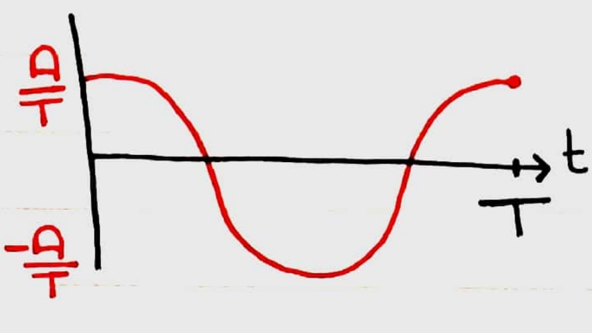
1) $S(T-t) = h_{opt}(t) = \frac{A}{T}\cos(2\pi f_0 (T-t))$

$0 \leq T-t \leq T$

$0 \geq t-T \geq -T \quad \rightarrow +T$

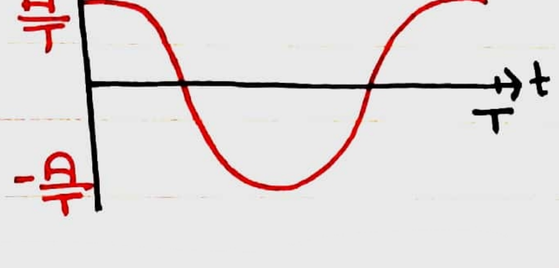
$0 \leq t \leq T$

$S(t) =$



$h_{opt}(t) = S(T-t)$

- Shift left by T
- Reflect



Indeed, $h_{opt}(t) = S(t)$

as $\cos(2\pi f_0 T - 2\pi f_0 t)$

$= \cos(2\pi - 2\pi f_0 t)$ Periodic

$= \cos(+2\pi f_0 t)$ even

2) Already drawn in 1

$$h_{opt}(t)$$

3) $S_o(t) = S(t) * S(t)$

$$= \int_{-\infty}^{\infty} S(\tau)\, S(t-\tau)\, d\tau$$

• No need to perform convolution since we're only interested in $t=T$

$$S_o(T) = \int_{-\infty}^{\infty} S(\tau)\, S(T-\tau)\, d\tau$$

But have shown already that $S(T-t) = S(t)$

$$= \int_{-\infty}^{\infty} (S(\tau))^2\, d\tau$$

• Can use t instead

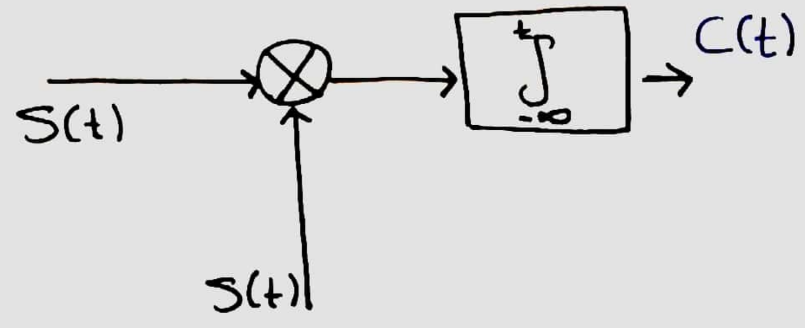
$$= \int_{0}^{T} \frac{A^2}{T^2} \cos^2(2\pi f_0 \tau)\, d\tau$$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$= \left(\frac{A}{T}\right)^2 \int_{0}^{T} (\underbrace{1}_{\text{gives } T} + \underbrace{\cos(4\pi f_0 \tau)}_{\text{integrating over 2 periods}})\, d\tau \cdot \frac{1}{2}$$

$$= \frac{A^2}{2T}$$

4)



$$c(t) = \int_{-\infty}^{t} (s(t))^2 dt \xrightarrow{@t=T} c(T) = \int_{0}^{T} (s(t))^2 dt$$

→ s(t) is 0 until t=0

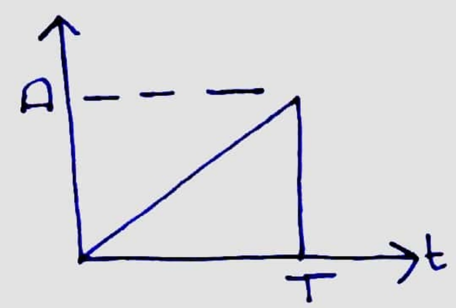
$$= \frac{A^2}{2T}$$

• The integral is the same & thus so is the output.

→ In fact,
the correlator is what's used to implement the matching filter. (lecture)

Problem 3)

$s(t) =$



$s(T-t) =$

• Shift ←
• Reflect.

intercept

$\frac{\Delta y}{\Delta x} = m$

1) $h_{opt}(t) = s(T-t) = \begin{cases} A - \frac{A}{T} t & 0 \leq t \leq T \\ 0 & \text{o.w.} \end{cases}$

2) Done already.

3) The output is maximized at $t=T$ and the value there corresponds to the input signal's energy.

4) It will be the area under $|s(t)|^2$

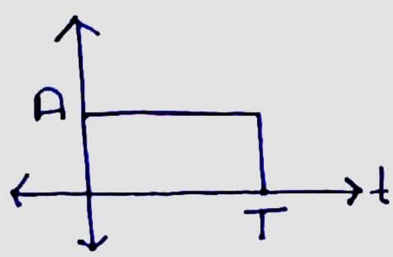
can be justified as at $t=T$ we get the largest overlap (Convolution)

$$s_o(T) = E_{s(t)} = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

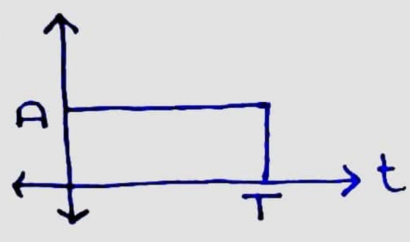
$$= \int_0^T \frac{A^2 \cdot t^2}{T^2} dt = A^2 \frac{T}{3}$$

* Can also arrive at this starting from $\int_{-\infty}^{\infty} s(\tau) \underbrace{h(t-\tau)|_{t=T}}_{s(\tau)} d\tau$

Problem 4)

$s(t) =$ 

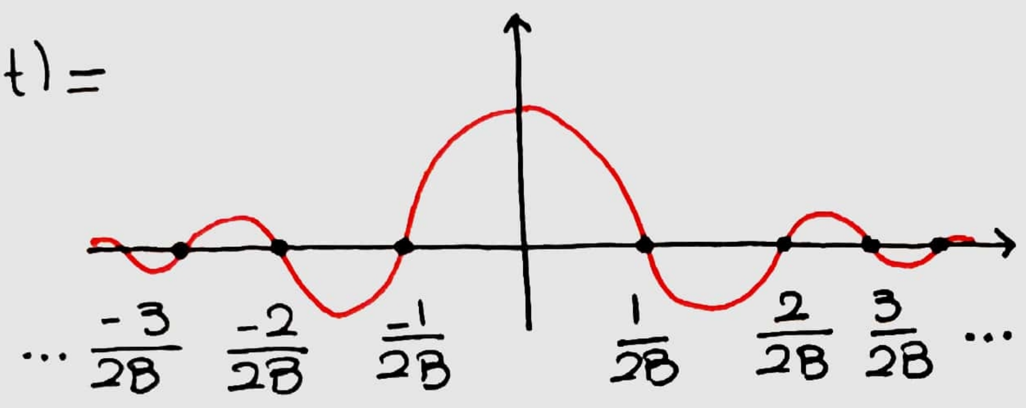
$h_{opt}(t) = s(T-t)$

$=$ 

Now instead of $h_{opt}(t)$, we have

$$H(f) = \text{rect}\left(\frac{f}{2B}\right) \longleftrightarrow h(t) = 2B.\text{Sinc}(2Bt)$$

$h(t) =$



• Sinc has zeros whenever $2Bt$ is integer $\neq 0$.

* We know that hopt naturally maximizes the Peak Pulse SNR

⇒ We succeed in approximating the matching filter using an LPF if choose B such that the PSNR is maximized.

$$PSNR = \frac{|S_o(T)|^2}{E[|n(t)|^2]}$$

← instantaneous power of the output at T

← Average power of output noise

$$\bullet\ S_o(t) = S(t) * h(t)$$

$$= \int_{-\infty}^{\infty} S(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} S(t-\tau)\, h(\tau)\, d\tau$$

← can now use the fact that $S(T-\tau) = S(\tau)$

$$S_o(T) = \int_{-\infty}^{\infty} S(\tau)\, h(\tau)\, d\tau$$

$$S_o(T) = \int_0^T A \cdot 2B\,\mathrm{Sinc}(2B\tau)\, d\tau$$

↳ s(t) is nonzero only over this

$$= 2AB \int_0^T \mathrm{Sinc}(2B\tau)\, d\tau$$

• Can't be expressed in terms of common standard functions.

Use fact

$$\mathrm{Sinc}(x) \simeq \cos\left(\frac{\pi}{2}x\right)$$

← very good for $\frac{-1}{2B} < t < \frac{1}{2B}$

$$= 2AB \int_0^T \cos(\pi B\tau)\, d\tau$$

$$= 2AB \left.\frac{\sin(\pi B\tau)}{\pi B}\right|_0^T = \frac{2A}{\pi} \cdot \sin(\pi B T)$$

$$|S_o(t)|^2 = \left(\frac{2A}{\pi}\right)^2 \cdot \sin^2(\pi B T)$$

$$E[|n(t)|^2] = \int_{-\infty}^{\infty} S_n(f)\, df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 \underbrace{S_w(f)}_{\frac{N_0}{2}}\, df$$ #LTI System (like in Rec.)

$$= \frac{N_0}{2} \cdot \int_{-B}^{B} 1^2\, df = N_0 B$$

$$\overbrace{\text{PSNR}}^{\eta} = \left(\frac{2A}{\pi}\right)^2 \cdot \frac{\sin^2(\pi BT)}{N_0 B}$$

- At this point, we have an expression for the PSNR in terms of B $\longrightarrow$ time to maximize it

$\Rightarrow$ Set $\dfrac{\partial \eta}{\partial B} = 0$

- maximizing η $\rightarrow$ maximizing $\ln \eta$

$$\ln(\eta) = 2\ln\left(\frac{2A}{\pi}\right) + 2\ln\sin(\pi BT) - \ln(N_0 B)$$

$$\frac{\partial \ln(\eta)}{\partial B} = 0 + 2 \cdot \frac{\pi T \cdot \cos(\pi BT)}{\sin(\pi BT)} - \frac{N_0}{N_0 B} = 0$$

flip

$$\frac{1}{2\pi T} \cdot \tan(\pi BT) = B$$

$$\tan(\pi BT) = 2\pi BT$$

Let $x = \pi BT$ now it boils down to solving

$$\tan(x) = 2x$$

has no closed form solution, Wolfram gives

$x = 0, \pm 1.166, \pm 4.604, \pm 14.1, \ldots$

↳ global max (not just π)

• $x = 1.166$

$\rightarrow B = \dfrac{1.166}{\pi T}$

$$PSNR = \frac{4A^2}{\pi N_0} \cdot \frac{\sin^2(\pi BT)}{\pi B} \cdot \frac{T}{T}$$

or just write it in terms of x

$$= \frac{4A^2 T}{\pi N_0} \cdot \frac{\sin^2 x}{x}$$

$x = 1.166$

$$PSNR\Big|_{Max} = PSNR\Big|_{x=1.166} = \frac{4A^2 T}{\pi N_0} \cdot 0.7246$$

2) In case of the matched filter, we've shown in the lecture:

$$PSNR\Big|_{Max} = PSNR\Big|_{@h_{opt}} = \frac{E_{s(t)}}{N_0/2}$$

• $E_{s(t)} = \int_{-\infty}^{\infty} |s(t)|^2 dt = A^2 T$

Thus,

- $PSNR_{\max\, h(t)} = \frac{4A^2T}{\pi N_0} \cdot 0.7246$

$$= 0.92259 \frac{A^2T}{N_0}$$

- $PSNR_{\max\, h_{opt}(t)} = 2\frac{A^2T}{N_0}$

- Hence, Clearly $\frac{PSNR_{h(t)}}{PSNR_{h_{opt}(t)}} = \frac{0.92259}{2}$

it only has 46% of the truly optimal SNR.

$\Rightarrow$ In dB, that is $10\log\left(\frac{0.92259}{2}\right) = -3.36\,dB$

Problem 5)

• Just like the previous problem but $H(f) = \frac{1}{1+\frac{jf}{f_0}}$

(instead of $H(f) = \text{rect}\left(\frac{f}{2B}\right)$ & we

should optimize over f_0)

$\Rightarrow$ Again, $PSNR_{\substack{h_{opt} \\ max}} = \frac{2A^2T}{N_0}$

$$H(f) = \frac{f_0(2\pi)}{2\pi f_0 + j2\pi f}$$

since $e^{-at}u(t) \longleftrightarrow \frac{1}{j2\pi f + \alpha}$

• its in the sheet but can be derived from laplace. $(s = j\omega,\ \omega = 2\pi f)$

then $h(t) = 2\pi f_0 e^{-2\pi f_0 t} u(t)$

• $S_0(T) = \int_{-\infty}^{\infty} \underbrace{S(T-\tau)}_{S(\tau)} h(\tau)\, d\tau$

$$= \int_0^T 2\pi f_0 e^{-2\pi f_0 \tau} \cdot A\, d\tau$$

$$= \Big|_T^0 e^{-2\pi f_0 \tau} A = A\left(1 - e^{-2\pi f_0 T}\right)$$

$\cdot\ |S_o(T)|^2 = A^2(1-e^{-2\pi f_o T})^2$

$\cdot\ E[|n(t)|^2] = \int_{-\infty}^{\infty} |H(f)|^2 \frac{N_o}{2}\, df$

$= \frac{N_o}{2} \int_{-\infty}^{\infty} |h(t)|^2 dt$

$= \frac{N_o}{2} . (2\pi f_o)^2 . \int_{0}^{\infty} e^{-2\pi f_o t . 2} dt$

$= \frac{N_o}{2} . (2\pi f_o)^2 . \frac{1}{2\pi f_o (2)} \underbrace{\Big|_{\infty}^{0} e^{-4\pi f_o t}}_{1}$

$= \frac{N_o}{8\pi f_o} . 4\pi^2 f_o^2 = \frac{N_o \pi}{2} . f_o$

Thus, $\overbrace{PSNR}^{n}_{h(t)} = \frac{A^2(1-e^{-2\pi f_o T})^2}{N_o \pi f_o / 2}$

$\cdot$ To maximize:

$\frac{\partial \ln(n)}{\partial f_o} = 0 = 0 + \frac{2.(2\pi\ T).e^{-2\pi f_o T}}{1-e^{-2\pi f_o T}} - \frac{1}{f_o}$

• Take $\frac{1}{P_o}$ to the Other side & flip fractions

$$P_o = \frac{1 - e^{-2\pi P_o T}}{4\pi T e^{-2\pi P_o T}}$$

$$2(2\pi P_o T)e^{-2\pi P_o T} = 1 - e^{-2\pi P_o T}$$

• let $x = 2\pi P_o T$

$$2xe^{-x} = 1 - e^{-x}$$

$$e^{-x}(2x+1) = 1 \longrightarrow 2x+1 = e^{x}$$

• Numerically yields $x = 1.25643$

$\Rightarrow 2\pi P_o T = 1.25643$ implies $P_o \simeq \frac{0.2}{T}$

• We had

$$PSNR = \frac{A^2(1-e^{-2\pi P_o T})^2 . 2}{N_o . \pi P_o} \times \frac{2T}{2T}$$

• Now Can Sub. $x = 2\pi P_o T$

$$= \frac{2A^2T}{N_o} . \frac{2(1-e^{-x})^2}{x}$$

$$PSNR_{\max_{h(t)}} = \frac{2A^2T}{N_0} \cdot \frac{2(1-e^{-x})^2}{x}\Bigg|_{x=1.25643}$$

$$= \frac{2A^2T}{N_0} \cdot 0.81452$$

80% of the truly opt. SNR!

Thus, $$\frac{PSNR_{h(t)}}{PSNR_{h_{oA}(t)}} = 0.81452 = -0.89\,dB$$

(or if we just use $0.8 \rightarrow -0.969\,dB$)

Thank you

#Don't Lose Focus!