Sheet 306

Q1: Given: $P(R) = 1/5$, $P(B) = 1/5$, $P(G) = 3/5$,

Recall → $P(A,B) = P(A|B) * P(B)$

a) $P(F=0 \mid B=R) = 4/10 = 2/5$

Marginal Prob

b) $P(F=0) = \sum_b P(F=0, B=b) = P(F=0, B=R) + P(F=0, B=B) + P(F=0, B=G)$

$$= \left.\begin{matrix} P(F=0 \mid B=R) * P(B=R) + P(F=0 \mid B=B) * P(B=B) \\ + P(F=0 \mid B=G) * P(B=G) \end{matrix}\right\} \text{Joint Prob}$$

$$= \frac{4}{10} * \frac{1}{5} + \frac{5}{10} * \frac{1}{5} + \frac{3}{10} * \frac{3}{5}$$

$$= \frac{4+5+9}{50} = 18/50 = 9/25$$

- مكانش ينفع نعمل Normalization لل table و نقسم كله على 30 عشان عدد ال boxes غير متساوي !

c) $P(B=R \mid F=0) = \dfrac{P(B=R, F=0)}{P(F=0)} = \dfrac{4/50}{18/50} = 2/9$

$P(B=R, F=0) \to P(F=0 \mid B=R) * P(B=R)$

d) $P(B=G \mid F=A) = \dfrac{P(B=G, F=A)}{P(F=A)} = \dfrac{P(F=A \mid B=G) * P(B=G)}{P(F=A)} = \dfrac{3/10 * 3/5}{\sim}$

$P(F=A) \to \sum P(F=A, B=b)$

$$\frac{3}{10} * \frac{1}{5} + \frac{5}{10} * \frac{1}{5} + \frac{3}{10} * \frac{3}{5} \leftarrow \sum P(B=b) * P(F=A \mid B=b)$$

$$= \frac{9}{3+5+9} = 9/17$$

Notes:

* $P(X=x, Y=y) = P(X=x \mid Y=y) * P(Y=y)$
  $= P(Y=y \mid X=x) * P(X=x)$
* $P(X=x) = \sum_{y_i} P(X=x, Y=y_i)$
* $P(A \mid B) = \dfrac{P(B \mid A)\, P(A)}{P(A)}$

- ممكن تحل دي مباشرة أسهل
- من قولك using Bayes Rule!

$\frac{1}{4}$ Flipped Twice

Q.2 $S = [(h,h), (h,t), (t,h), (t,t)]$

let $A = [(h,h)]$, both are heads

$B = [(h,t),(h,h)]$, First is head

$C = [(h,h),(h,t),(t,h)]$, at least one is head

a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P[(h,h)]}{P[(h,h),(h,t)]} = \frac{1/4}{1/4+1/4} = 1/2$

b) $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{3 \times 1/4} = 1/3$

Q.3 Two coins — Prove that E & F are ind.

$E =$ 1st coin is head $= [(h,h),(h,t)]$, $F =$ 2nd coin is tail $[(h,t),(t,t)]$

** $P(E,F) = P(E)P(F)$

$P(E,F) = P(E \cap F) = P[(h,t)] = 1/4$

$P(E) = 2 * 1/4 = 1/2$

$P(F) = 2 * 1/4 = 1/2$

$P(E)P(F) = 1/2 * 1/2 = 1/4 = P(E,F)$ Q.E.D

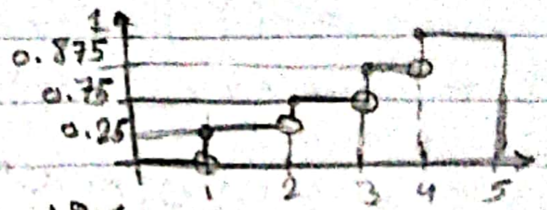
Q.4 $P(1) = 1/4$, $P(2) = 1/2$, $P(3) = 1/8$, $P(4) = 1/8$

a) $\because \sum P(x) = 1/4 + 1/2 + 1/8 + 1/8 = 1$ $\therefore$ it's a valid prob mass func

b) $E[X] = \sum x P(x) = 1 * 1/4 + 2 * 1/2 + 3 * 1/8 + 4 * 1/8 = \frac{17}{8}$

c) $Var(x) = E[X^2] - E[X]^2 = \frac{43}{8} - \left(\frac{17}{8}\right)^2 = \frac{43}{50} \simeq 0.86$ $\sqrt{V} \to \sigma = 0.927$

$\hookrightarrow \sum x^2 P(x) = 1^2 * 1/4 + 2^2 * 1/2 + 3^2 * 1/8 + 4^2 * 1/8 = \frac{43}{8}$

d) 

CDF

e) $E(g) = \sum \ln(x) P(x)$ —— Like b & c

$\therefore Var(ax+b) = a^2 Var(x)$

$\boxed{Q:5}$ $E(X)=1$, $Var(x)=5$

a) $E((2+X)^2) = E(4+4X+X^2) = 4 + 4E(X) + E(X^2) = 14$

but, $Var(x) = E(X^2) - E(X)^2 \rightarrow 5 = E(X^2) - 1 \rightarrow E(X^2) = 6$

b) $Var(4-3X) = (-3)^2 Var(X) = 9 \times 5 = 45$

$\boxed{Q:6}$ Let X be a random vars that represents the points

RND Var $X \rightarrow$ 5.5 , -5

Prob(X) $\rightarrow$ 2 balls same clr , 2 balls diff clr } No replacement!

$P(\text{same}) = P(2\text{balls}=G) + P(2\text{balls}=B) = 2/9 + 2/9 = 4/9$

$P(\text{diff}) = P(1\text{ball}=G, 1\text{ball}=B) = 5/9$

$P(2\text{balls}=G) = \frac{5}{10} \times \frac{4}{9} = 2/9$, $P(2\text{balls}=G) = 2/9$

1st, 2nd

$P(1\text{ball}=G, 1\text{ball}=B) = P(F=G, S=B) + P(F=B, S=G)$

$= 5/10 \times 5/9 + 5/10 \times 5/9$

$= 5/18 + 5/18 = 5/9$

$\therefore X \rightarrow$ 5.5 , -5

$P(X) \rightarrow$ 4/9 , 5/9] $\rightarrow 4/9 + 5/9 = 1$ ✓

$E(X) = \sum x P(x) = 5.5 \times 4/9 + (-5) \times 5/9 = -1/3$

$Var(X) = E(X^2) - E(X)^2 = \underbrace{5.5^2 \times 4/9 + (-5)^2 \times 5/9}_{147/9} - (-1/3)^2 = \frac{146}{9}$

Q:7 $f(x) = \begin{cases} \lambda e^{-x/100}, & x \geq 0 \\ \text{zero}, & x < 0 \end{cases}$

* Compute $\lambda \rightarrow \int_{-\infty}^{\infty} f(x) = 1 \rightarrow \int_{-\infty}^{\infty} \lambda e^{-x/100} dx = 1$

$\int_{0}^{\infty} \lambda e^{-x/100} dx = \lambda \int_{0}^{\infty} e^{-x/100} dx = -100\lambda \int -1/100\, e^{-x/100} dx$

$= -100\lambda * (0-1) = \boxed{100\lambda = 1} \rightarrow \boxed{\lambda = 1/100}$

a) $\int_{50}^{150} f(x) dx = \int_{50}^{100} 1/100\, e^{-x/100} dx = \int_{100}^{50} -1/100\, e^{-x/100} dx$

$= e^{-50/100} - e^{-100/100} = e^{-0.5} - e^{-1}$

b) $\int_{-\infty}^{100} f(x) dx = \int_{-\infty}^{0} f(x) dx + \int_{0}^{100} \frac{1}{100} e^{-x/100} dx = \int_{100}^{0} -1/100\, e^{-x/100} dx$

$= e - e^{-1}$

Q:8 $f(x) = \begin{cases} a + bx^2 & 0 \leq x < 1 \\ \text{zero} & \text{otherwise} \end{cases}$, $E(x) = 35$ & Find a & b

* $\int_{0}^{1} a + bx^2 dx = 1 \rightarrow \left[ax + b\, x^3/3\right]_0^1 = \boxed{a + b/3 = 1}$ ①

* $E(x) = \int_0^1 x f(x) dx = \int_0^1 ax + bx^3 dx = 35 \rightarrow \left[ax^2/2 + \frac{bx^4}{4}\right]_0^1$
$= \boxed{a/2 + b/4 = 35}$ ②

from ① & ② $\rightarrow a = -137,\ b = 414$

Q:9 مكمل الأوراق

From Prob. Last Year!

$\boxed{Q:10}$ X & Y are indep.

- $P(X,Y) = P(X) * P(Y)$
- $P(X|Y) = P(X)$
- $E(XY) = E(X)\,E(Y)$
- $Var(X+Y) = Var(X) + Var(Y) + 2\,\underbrace{Cov(X,Y)}_{0}$
- $Corr(X,Y) = Cov(X,Y)/\sqrt{\sigma_x \sigma_y}$

a) ✗ | e) ✗ → the 2 terms aren't equall!

b) ✗ | F) ✓ → $E(XY) = E(X)E(Y)$

c) ✓ | g) ✗ → the 2 terms aren't Equall!

d) ✓ | h) ✓ → $Var(x+y) - (Var(x) + Var(y)) = $ zero

$\boxed{Q:12}$ PrcN → $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$, at $\mu=0$, $\sigma^2=1$ → $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$E[e^{-ax}] = ?$ Recall: $E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$ → e^{-ax}

$$E[e^{-ax}] = \int_{-\infty}^{\infty} e^{-ax} * \frac{1}{\sqrt{2\pi}} * e^{-x^2/2} dx \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2ax)} dx$$

$$-\tfrac{1}{2}(x^2+2ax+a^2-a^2) = -\tfrac{1}{2}(x+a)^2 + \tfrac{1}{2}a^2$$

$$\rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+a)^2} \cdot e^{\frac{1}{2}a^2} dx = \frac{1}{\sqrt{2\pi}} \cdot e^{a^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\overbrace{x+a}^{z})^2} dx$$

Let $z = x+a$, $dz = dx$ → $e^{a^2/2} * \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dx}_{1}$ Gaussian with $\mu=0$, $\sigma^2=1$

→ $\boxed{E(e^{-ax}) = e^{a^2/2}}$ like Prob Last Year!

$\boxed{Q:13}$

$$E(e^{-x} + e^{-2x} + x^2) = E(e^{-x}) + E(e^{-2x}) + E(x^2)$$
$$= e^{+x/2} + e^{+x} + 1$$

R.T.F!

$* \; \underbrace{Var(x)}_{1} = \underbrace{E(x^2)}_{1} - \underbrace{E(x)^2}_{0}$