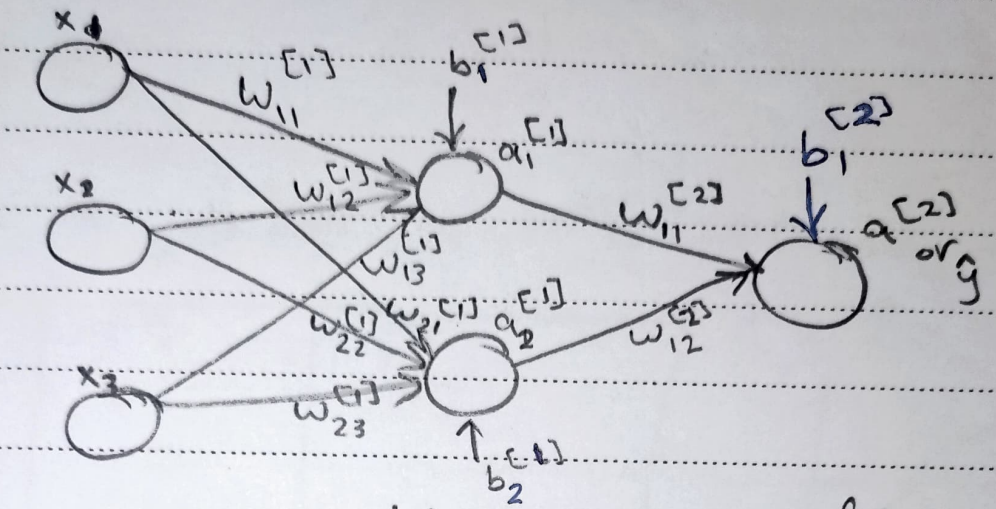
Sheet "5"

Problem ①



input layer — hidden layer — output layer

$a^{[0]} = x$

$b^{[1]} = 0$ $\quad$ $b^{[2]} = 0$

Forward Propagation

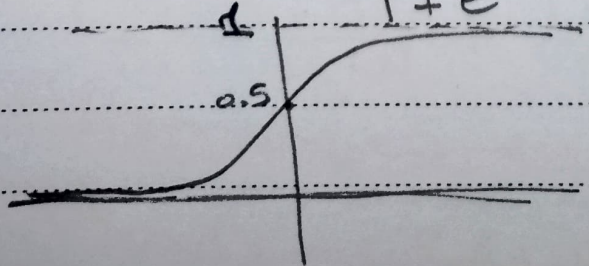
a) $z^{[1]} = W^{[1]} x + b^{[1]}$

$a^{[1]} = \text{sigmoid}(z^{[1]})$

$z^{[2]} = W^{[2]} a^{[1]} + b^{[2]}$

$a^{[2]} = \text{sigmoid}(z^{[2]})$

eqn of sigmoid $\frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$



dimension of

$W^{[1]}_{2\times 3}$

$x = a^{[0]}_{3\times 1}$

$z^{[1]}$ & $a^{[1]}_{2\times 1}$

$W^{[2]}_{1\times 2}$

$a^{[1]}_{2\times 1}$

$z^{[2]}$ & $a^{[2]} = \hat{y}_{1\times 1}$

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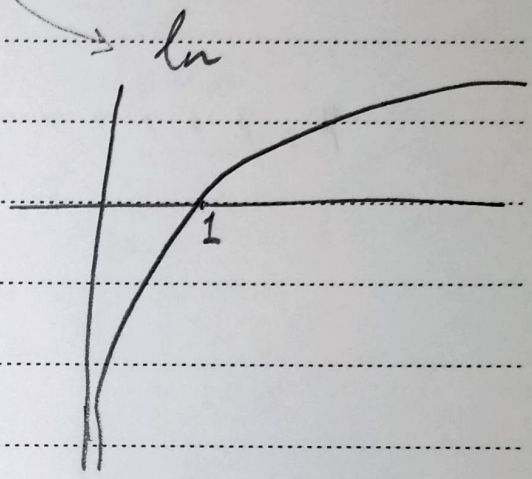
b) Loss (cost) function of binary classification

Cross entropy loss function

$y \rightleftarrows 0, 1$ $\quad \hat{y} \rightleftarrows 0, 1$ $\quad \hat{y} \rightarrow$ output of sigmoid $[0,1]$

$$\mathcal{L} = -\left[y \log \hat{y} + (1-y) \log (1-\hat{y}) \right]$$

y	$\hat{y}$	$\mathcal{L}$
0	0	0 No loss
0	1	big +ve high loss
1	0	big +ve high loss
1	1	0 No loss



c) Back propagation

Required: Minimize loss function w.r.t. W & b

Get:

$$\frac{\partial \mathcal{L}}{\partial W^{[2]}} \;\&\; \frac{\partial \mathcal{L}}{\partial b^{[2]}} \;\&\; \frac{\partial \mathcal{L}}{\partial W^{[1]}} \;\&\; \frac{\partial \mathcal{L}}{\partial b^{[1]}}$$

$$\frac{\partial \mathcal{L}}{\partial W^{[2]}} = \frac{\partial \mathcal{L}}{\partial a^{[2]}} * \frac{\partial a^{[2]}}{\partial z^{[2]}} * \frac{\partial z^{[2]}}{\partial W^{[2]}}$$

($a^{[2]} = \hat{y}$)

$$\frac{\partial \mathcal{L}}{\partial a^{[2]}} = \frac{-y}{a^{[2]}} + \frac{(1-y)}{1-a^{[2]}}$$

$$= \frac{a^{[2]} - y}{a^{[2]}(1-a^{[2]})}$$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$

$$\frac{\partial L}{\partial z^{[2]}} = \frac{\partial L}{\partial a^{[2]}} * \frac{\partial a^{[2]}}{\partial z^{[2]}}$$

$$= \frac{a^{[2]} - y}{a^{[2]}(1-a^{[2]})} * a^{[2]}(1-a^{[2]})$$

$$= a^{[2]} - y$$

$$\star \frac{\partial L}{\partial W^{[2]}}_{1\times 2} = \left(\frac{\partial L}{a^{[2]}} * \frac{\partial a^{[2]}}{\partial z^{[2]}}\right) * \frac{\partial z^{[2]}}{\partial W^{[2]}}$$

$$= (a^{[2]} - y)_{1\times 1} (a^{[1]})^T \quad \text{should be } 1\times 2$$

$$\star \frac{\partial L}{\partial b^{[2]}} = \left(\frac{\partial L}{\partial a^{[2]}} * \frac{\partial a^{[2]}}{\partial z^{[2]}}\right) * \frac{\partial z^{[2]}}{\partial b^{[2]}}$$

$$= (a^{[2]} - y)$$

$$\frac{\partial L}{\partial a^{[1]}}_{2\times 1} = \left(\frac{\partial L}{\partial a^{[2]}} * \frac{\partial a^{[2]}}{\partial z^{[2]}}\right) * \frac{\partial z^{[2]}}{\partial a^{[1]}}$$

$$= \left((a^{[2]} - y)_{1\times 1} W^{[2]}_{1\times 2}\right)^T \rightarrow \text{should be } 2\times 1$$

$$\frac{\partial L}{\partial z^{1}} = \frac{\text{chain of}}{\partial a^{[1]}} * \frac{\partial a^{[1]}}{\partial z^{[1]}}$$

$$= \left((a^{[2]} - y)[W^{[2]}]\right)^T * a^{[1]}(1-a^{[1]})$$

$$\star \frac{\partial L}{\partial W^{[1]}}_{2\times 3} = \frac{\text{chain of}}{\partial z^{(1)}} * \frac{\partial z^{(1)}}{\partial W^{(1)}}$$

$$= \frac{\partial L}{\partial z^{[1]}}_{2\times 1} * (a^{[0]})^T_{1\times 3}$$

$y = \text{sigmoid}(x)$

$$= \frac{1}{1+e^{-x}}$$

$$= (1+e^{-x})^{-1}$$

$$= -(1+e^{-x})^{-2}(-e^{-x})$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{1}{1+e^{-x}} * \frac{e^{-x}+1-1}{1+e^{-x}}$$

$$= y * \left[\frac{1+e^{-x}}{1+e^{-x}} - \frac{1}{1+e^{-x}}\right]$$

$$= y * [1-y]$$

$$\frac{\partial L}{\partial b^{[1]}} = \frac{\partial L}{\partial z^{[1]}} * \frac{\partial z^{[1]}}{\partial b^{[1]}}$$

$$= \frac{\partial L}{\partial z^{[1]}} * 1 = \left((a^{[2]} - y)(w^{[2]})^T\right) * a^{[1]}(1 - a^{[1]})$$

d) $$\underline{x} = \begin{bmatrix} 3 \\ 0.2 \\ -0.1 \end{bmatrix}$$

$$\underline{z}^{[1]} = \underline{W}^{[1]} \underline{x} + b^{[1]}$$

$$= \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.1 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 3 \\ 0.2 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.31 \\ -0.32 \end{bmatrix}$$

$$a^{[1]} = \begin{bmatrix} 0.5769 \\ 0.4207 \end{bmatrix}$$

$$z^{[2]} = \begin{bmatrix} 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 0.5769 \\ 0.4207 \end{bmatrix} = 0.03124$$

$$a^{[2]} = 0.5070 \Rightarrow \hat{y}$$

e) $\hat{y} < 0.6 \Rightarrow$ output: 0

f) Cost: $L = -\left(y \log \hat{y} + (1-y) \log (1-\hat{y})\right)$

$$= -\left(1 \times \ln(1 - 0.5070)\right)$$

$$= 0.70887$$

9) $\frac{\partial L}{\partial W^{[2]}} = (a_i^{[2]} - y)(a_i^{[1]})^T$

$= (0.5078 - 0) * [0.5769 \quad 0.4207]$

$= [0.2929 \quad 0.2136]$ ##

$\frac{\partial L}{\partial b^{[2]}} = a^{[2]} - y = 0.5078$ ##

$\partial L / \partial a^{[1]} = (0.5078 * [0.2 \quad -0.2])^T = \begin{bmatrix} 0.10156 \\ -0.10156 \end{bmatrix}$ element wise

$\frac{\partial L}{\partial z^{[1]}} = \begin{bmatrix} 0.10156 \\ -0.10156 \end{bmatrix} \circledast \begin{bmatrix} 0.5769 \\ 0.4207 \end{bmatrix} \circledast \begin{bmatrix} 1-0.5769 \\ 1-0.4207 \end{bmatrix}$

$= \begin{bmatrix} 0.0247 \\ -0.0247 \end{bmatrix}$

$\frac{\partial L}{\partial W^{[1]}} = \begin{bmatrix} 0.0247 \\ -0.0247 \end{bmatrix} [3 \quad 0.2 \quad -0.1]$

$= \begin{bmatrix} 0.0741 & 4.94 \times 10^{-3} & -2.47 \times 10^{-3} \\ -0.0741 & -4.94 \times 10^{-3} & 2.47 \times 10^{-3} \end{bmatrix}$ ##

$\frac{\partial L}{\partial b^{[1]}} = \begin{bmatrix} 0.02 \\ -0.0247 \end{bmatrix}$

h) Gradient descent

$$W^{[2]} := W^{[2]} - \alpha \frac{\partial L}{\partial W^{[2]}}$$

$$b^{[2]} := b^{[2]} - \alpha \frac{\partial L}{\partial b^{[2]}}$$

$$W^{[1]} := W^{[1]} - \alpha \frac{\partial L}{\partial W^{[1]}}$$

$$b^{[1]} := b^{[1]} - \alpha \frac{\partial L}{\partial b^{[1]}}$$

α is the learning rate $\alpha = 0.05$

$$W^{[2]} := [0.2 \quad -0.2] - 0.05\,[-2.929 \quad 0.2136]$$
$$= [0.1854 \quad 2.136\times10^{-3}]$$

$$b^{[2]} = 0 - 0.05 * 0.5078 = -0.025539$$

$$W^{[1]} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.1 & 0 & 0.2 \end{bmatrix} - 0.05 \begin{bmatrix} 0.0741 & 4.94\times10^{-3} & -2.47\times10^{-3} \\ -0.0741 & -4.94\times10^{-3} & 2.47\times10^{-3} \end{bmatrix}$$

$$= \begin{bmatrix} 0.0963 & 0.199753 & 0.300125 \\ -0.0963 & 2.47\times10^{-4} & 0.1987 \end{bmatrix}$$

$$b^{[1]} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.05 * \begin{bmatrix} 0.0247 \\ -0.0247 \end{bmatrix} = \begin{bmatrix} -1.235\times10^{-3} \\ 1.235\times10^{-3} \end{bmatrix}$$