CE Sheet 1

- Laplace transforms mentioned in the tutorial beyond those in the lecture

$$\cos(at) \longleftrightarrow \frac{s}{s^2+a^2}$$

$$\sin(at) \longleftrightarrow \frac{a}{s^2+a^2}$$

Can you prove me with $e^{-at} \longleftrightarrow \frac{1}{s+a}$? (bonus)

⇒ After getting the inverse laplace transform using the table remember to multiply everything by $u(t)$ (because we (and thus the table) assume that the signal has $f(t) = u(t)f(t)$)

- Some signals have no laplace transform (the integral doesn't converge)
- Similarily, some signals in s have no inverse laplace transform (can't find a signal where the laplace transform would yield it)

⇒ Laplace transform of $\frac{d^n x}{dt^n}$

- let $n=4$) $\mathcal{L}\left\{\frac{d^4 x}{dt^4}\right\} = s^4 X(s) - (s^3 X(s) + s^2 \dot{X}(s) + s\ddot{X}(s) + \dddot{X}(s))$

• That is, after writing the first term $S^n X(S)$

Subtract $\sum_{i=0}^{n-1} S^{(n-1)-i} \cdot \left.\frac{d^i x}{dt^i}\right|_{t=0}$

• Do not mix up S and 5

• We use $u(t)$ to represent input (like $x(t)$)

It will be assumed that we know how to do Partial fractions decomposition.

Possible Cases

• $\frac{\text{———}}{\text{———}}$ ← smaller degree ← than this
else, long div.

Distinct linear factors

$$\frac{\cdots}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

• Cover up method works here.

Distinct irreducible quadratic factors

$$\frac{\cdots}{(x^2+bx+c)(x-a)} = \frac{A}{x-a} + \frac{Bx+C}{(x^2+bx+c)}$$

repeating linear factors

$$\frac{\cdots}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$$

repeating irreducible quadratic factors

$$\frac{\cdots}{(x^2+bx+c)^2} = \frac{A}{x^2+bx+c} + \frac{B}{(x^2+bx+c)^2}$$

• Else, generally for m unknown constants do m substitutions → Solve the resulting system

• Smart substitutions help (e.g. multiply eqn by $(x-a)$ then set $x=a$ for a smaller eqn)

1) Solve the following differential equations

* like done in CE lecture 1

1. Take laplace transform 2. Algebra 3. Partial fractions 4. Inverse laplace (through table) (& × u(t))

a) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \theta(t)$ ← unit step

$y(0)$	-1
$\dot{y}(0)$	2

1)
$$S^2Y(S) - (-S+2) + 3(SY(S)+1) + 2Y(S) = \frac{1}{S}$$

2) $$Y(S)(S^2+3S+2) + (S+1) = \frac{1}{S}$$

$$Y(S) = \frac{S^{-1}-1-S}{S^2+3S+2} = \frac{1-S-S^2}{S(S^2+3S+2)}$$

$$= \frac{1-S-S^2}{S(S+2)(S+1)}$$

3) $$Y(S) = \frac{1+S-S^2}{S(S+2)(S+1)} = \frac{0.5}{S} + \frac{-0.5}{S+2} + \frac{-1}{S+1}$$

• Cover denominator in LHS then plug to get numerator.

4) $$y(t) = (0.5 - 0.5e^{-2t} - e^{-t})\,u(t)$$

b)

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) + 3x(t)$$

$y(0) \quad 1$ and $x(t) = e^{-4t}$

$\dot{y}(0) \quad 0$

» Plugging with the input yields the differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = -4e^{-4t} + 3e^{-4t} = -e^{-4t}$$

1) $S^2 Y(S) - (S) + 3(SY(S) - 1) + 2Y(S) = \frac{-1}{S+4}$

2) $Y(S)(S^2 + 3S + 2) - (S+3) = \frac{-1}{S+4}$

$$Y(S) = \left((S+3) - \frac{1}{S+4}\right) \cdot \frac{1}{S^2+3S+2}$$

$$= \frac{(S+3)(S+4) - 1}{(S+4)(S^2+3S+2)}$$

$$= \frac{(S+3)(S+4) - 1}{(S+4)(S+1)(S+2)}$$

3) $Y(S) = \frac{5/3}{S+1} + \frac{-1/2}{S+2} + \frac{-1/6}{S+4}$

4) $y(t) = u(t)\left(\frac{5}{3}e^{-t} - \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-4t}\right)$

2) Linearity

Recall that if it violates either homogeneity or superposition then its not linear.

$\Rightarrow$ must satisfy both for linearity.

a) • Let $u_1(t) \longrightarrow y_1(t)$ and $u_2(t) \longrightarrow y_2(t)$ and

Let $u_3(t) = a u_1(t) + b u_2(t)$

$\Rightarrow$ The system is linear iff

$$y_3(t) = a y_1(t) + b y_2(t)$$

} You should be aware of this

• The system is $y(t) = \alpha u(t)$

Thus, $y_3(t) = \alpha (a u_1(t) + b u_2(t))$

$= a(\alpha u_1(t)) + b(\alpha u_2(t))$

$= a y_1(t) + b y_2(t)$

b) The system clearly violates homogeneity

• let $u_1(t) \rightarrow y_1(t)$

then a system $y(t) = u^3(t)$ has

$$\underbrace{a u_1(t)}_{u_3(t)} \longrightarrow \underbrace{a^3 u^3(t)}_{y_3(t)} \neq \underbrace{a u^3(t)}_{a y_1(t)}$$

c)

$$y(t) = e^{u(t)}$$

Again, violates homogeneity from first sight. (multiplying $u(t)$ by a doesn't result in $y(t)$ being multiplied a)

$$\underbrace{a u_1(t)}_{u_3(t)} \rightarrow \underbrace{e^{a u_1(t)}}_{y_3(t)} \neq \underbrace{a e^{u_1(t)}}_{a y_1(t)}$$

d) $\frac{d^2 y(t)}{dt^2} + a \frac{dy(t)}{dt} + y(t) = u(t)$, $y(0) = \dot{y}(0) = 0$

Let $u_1(t) \rightarrow y_1(t)$

$$\frac{d^2 y_1(t)}{dt^2} + a \frac{d y_1(t)}{dt} + y_1(t) = u_1(t), \; y_1(0) = \dot{y}_1(0) = 0$$

Let $u_2(t) \longrightarrow y_2(t)$

$$\frac{d^2 y_2(t)}{dt^2} + a \frac{d y_2(t)}{dt} + y_2(t) = u_2(t), \; y_2(0) = \dot{y}_2(0) = 0$$

- Let also $u_3(t) = \alpha u_1(t) + B u_2(t)$

• Multiply the 1St differential eqn. by α the Second by B then add

$$\left(\alpha \frac{d^2 y_1(t)}{dt^2} + B \frac{d^2 y_2(t)}{dt^2}\right)$$

$$+ a\left(\alpha \frac{d y_1(t)}{dt} + B \frac{d y_2(t)}{dt}\right)$$

$$+\left(\alpha y_1(t) + B y_2(t)\right) = \alpha u_1(t) + B u_2(t)$$

• $y_1(0) = y_2(0) = 0$

• $\dot{y}_2(0) = \dot{y}_1(0) = 0$

• Let $y_3(t) = \alpha y_1(t) + B y_2(t)$

Then its true that

$$\frac{d^2 y_3(t)}{dt^2} + a \frac{d y_3(t)}{dt} + y_3(t) = u_3(t)$$

• $y_3(0) = \dot{y}_3(0) = 0$

• This differential equation implies that

$u_3(t) \rightarrow y_3(t)$

and indeed,

$u_3(t) = \alpha u_1(t) + B u_2(t)$

$y_3(t) = \alpha y_1(t) + B y_2(t)$ $\square$

• The Previous method was confirmed by TA (another way was used in the tutorial)

Proving Linearity of Convolution

• will be needed.

⇒ let $h(t)$ be some signal.

⇒ We want to prove that for any signal $x(t)$ convolving with $h(t)$ is a linear operation.

By defenition

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Homogeneity:

• let $x_1(t) \rightarrow y_1(t)$

let $x_3(t) = a x_1(t)$ then

$$y_3(t) = \int_{-\infty}^{\infty} (a x_1(\tau)) h(t-\tau)\, d\tau$$

$$= a \int_{-\infty}^{\infty} x_1(\tau) h(t-\tau)\, d\tau$$

$$= a\, y_1(t)$$

Additivity:

- Let $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$
- Let $x_3(t) = x_1(t) + x_2(t)$ then

$$y_3(t) = \int_{-\infty}^{\infty} (x_1(t) + x_2(t))h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x_1(t)h(t-\tau)d\tau + \int_{-\infty}^{\infty} x_2(t)h(t-\tau)d\tau$$

$$= y_1(t) + y_2(t)$$

- Satisfies both homogeneity & additivity and its thereby linear

⇒ Now back to our Problem

$$\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt} + y(t) = u(t), \quad y(0) = \dot{y}(0) = 0$$

Laplace Transform for the transfer function

$$S^2Y(S) + aSY(S) + Y(S) = X(S)$$

$$Y(S)(S^2 + aS + 1) = X(S)$$

$$Y(S) = \frac{1}{S^2 + aS + 1} X(S)$$

• Regardless to whether the quadratic's roots are real or not, we have that

$$K(s)\left\{\frac{1}{s^2+as+1}=\frac{1}{(s+\beta_1)(s+\beta_2)}=\frac{\alpha_1}{s+\beta_1}+\frac{\alpha_2}{s+\beta_2}\right.$$

$$\mathcal{L}^{-1}\{K(s)\}=\alpha_1 e^{-\beta_1 t}+\alpha_2 e^{-\beta_2 t}$$

⇒ Since $Y(s)=K(s)\,X(s)$ we have that

$$Y(s)=X(s)\,K(s)\rightarrow y(t)=x(t)*K(t)$$

which was proven to be linear earlier.

• This isn't exactly how it was solved in the tutorial. Perhaps, a longer route was taken by proving linearity of convolution for the specific $h(t)=\alpha e^{-\beta t}$

⇒ Regardless, just keep in mind that you might need to prove linearity of operators if required (even if its as simple as multiplication)

Q3) Find the transfer function of the following

a) $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) + 3x(t)$

⇒ Recall that the transfer function is evaluated at 0 initial conditions

$$S^2 Y(S) + 3SY(S) + 2Y(S) = SX(S) + 3X(S)$$

$$Y(S)(S^2 + 3S + 2) = X(S)(S+3)$$

$$\frac{Y(S)}{X(S)} = \frac{S+3}{S^2+3S+2} = \frac{S+3}{(S+1)(S+2)}$$

b) $\dot{y}(t) + y(t) = x(t-T)$ • Assume $T \geq 0$

$$SY(S) + Y(S) = X(S).e^{-ST}$$

$$Y(S)(S+1) = X(S).e^{-ST}$$

$$\frac{Y(S)}{X(S)} = \frac{e^{-ST}}{S+1}$$

My thanks !!