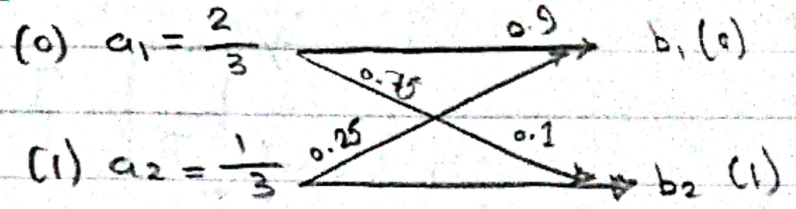
[Final 2015 $\xrightarrow{B}$ Q2
Final 2009 → Q6
Final 2008 Ext → Q 1b]

Channel



$$
\begin{array}{c|cc}
 & b_1 & b_2 \\
\hline
a_1 & P(b_1|a_1) & P(b_2|a_1) \\
a_2 & P(b_1|a_2) & P(b_2|a_2)
\end{array}
$$

$\Downarrow$

$$
\begin{array}{ccc}
 & b_1 & b_2 \\
\begin{matrix} a_1 \\ a_2 \end{matrix} & \left[\begin{matrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{matrix}\right] & \begin{matrix} =1 \\ =1 \end{matrix}
\end{array}
$$

Channel Matrix

Output Prob:

$$P(b_1) = P(b_1|a_1)P(a_1) + P(b_1|a_2)P(a_2)$$
$$= 0.9 * \frac{2}{3} + 0.25 * \frac{1}{3} = \frac{41}{60}$$

$$P(b_2) = P(b_2|a_1)P(a_1) + P(b_2|a_2)P(a_2)$$
$$= 0.1 * \frac{2}{3} + 0.75 * \frac{1}{3} = \frac{19}{60}$$

[$P(b_1) + P(b_2) = \frac{41}{60} + \frac{19}{60} = 1$ ✓✓]

Joint Prob:

$$P(a_1, b_1) = P(b_1|a_1)P(a_1) = 0.9 * \frac{2}{3} = 3/5$$
$$P(a_1, b_2) = P(b_2|a_1)P(a_1) = 0.1 * \frac{2}{3} = 1/15$$
$$P(a_2, b_1) = P(b_1|a_2)P(a_2) = 0.25 * 1/3 = 1/12$$
$$P(a_2, b_2) = P(b_2|a_2)P(a_2) = 0.75 * 1/3 = 1/4$$

Conditional Input Prob:

$$P(a_1|b_1) = \frac{P(a_1, b_1)}{P(b_1)} = \frac{3/5}{41/60} = \frac{36}{41}$$

$$P(a_1|b_2) = \frac{P(a_1, b_2)}{P(b_2)} = \frac{1/15}{19/60} = \frac{4}{19}$$

$$P(a_2|b_1) = \frac{P(a_2, b_1)}{P(b_1)} = \frac{1/12}{41/60} = \frac{5}{41}$$

$$P(a_2|b_2) = \frac{P(a_2, b_2)}{P(b_2)} = \frac{1/4}{19/60} = \frac{15}{19}$$

Mutual Info:

$$I(A,B) = H(A) - H(A|B) = H(B) - H(B|A)$$

$$H(B) = -\sum_{1}^{2} P(b_J) \log_2 b_J = -\left[P(b_1)\log_2 P(b_1) + P(b_2)\log_2 P(b_2)\right]$$

$$= -\left[\frac{41}{60} \times \log_2 \frac{41}{60} + \frac{19}{60} \times \log_2 \frac{19}{60}\right]$$

$$= 0.90071$$

$$H(B|A) = -\sum\sum P(b_J, a_i)\log_2 P(b_J|a_i) = -[P(b_1,a_1)\log_2(b_1|a_1) +$$

$$P(b_2,a_1)\log_2(b_2|a_1) +$$

$$P(b_1,a_2)\log_2(b_1,a_2) +$$

$$P(b_2,a_2)\log_2(b_2,a_2)]$$

$$= -\left[\frac{3}{5}\log_2(0.9) + \frac{1}{15}\log_2(0.1) + \frac{1}{12}\log_2(0.25) + \frac{1}{4}\log_2(0.75)\right]$$

$$= 0.58308$$

$$\therefore I(A,B) = H(B) - H(B|A)$$

$$= 0.90071 - 0.58308$$

$$= 0.31762$$

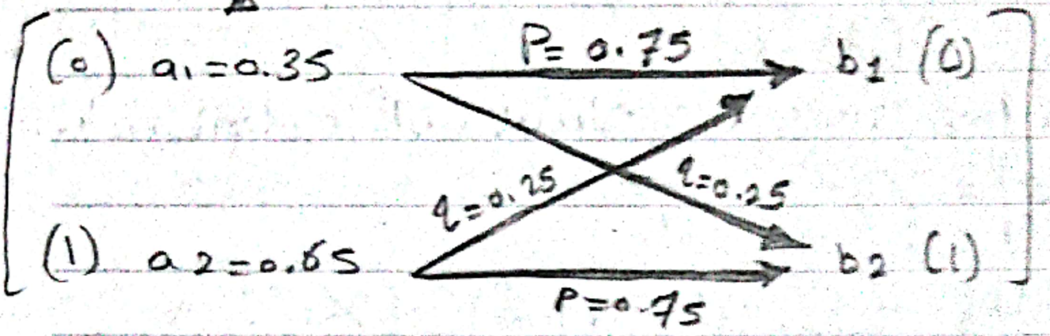
calc H(A|B)

Method 1 $\rightarrow H(A|B) = -\sum\sum P(a_i, b_J)\log_2(a_i|b_J)$

Method 2 $\rightarrow H(A|B) = H(A) - I(A,B)$

[final 2014 → Q6 (2)]
[final 2016 → Q5 (2)]

Model



Channel transition Matrix

$$\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

∴ Binary symmetric ∴ $\underbrace{P_{11} = P_{22}}_{\text{correct reception } (P)}$, $P_{12} = P_{21}$

- الصح زي بعض
- الغلط زي بعض!

src entropy:

$H(A) = -\sum P(a_i) \log_2 P(a_i) = -[0.35 \log_2 0.35 + 0.65 \log_2 0.65]$

$\boxed{H(A) = 0.934}$

Channel Capacity

∴ Binary symmetric $\longrightarrow C = 1 + P \log_2 P + \underbrace{(1-P)}_{q} \log_2 \underbrace{(1-P)}_{q}$

∴ $C = 1 + P \log_2 P + q \log_2 q$

$C = 1 + 0.75 \log_2 0.75 + 0.25 \log_2 0.25$

$\boxed{C = 0.18872}$

Proof 1: Calculate or Derive a formula for the max entropy $H(x)_{max}$:

* Entropy reaches its maximum value when uncertainty is at its max value. ($\because H(x) \propto$ uncertainty)
* That only happens when all the symboles have the same Probability i.e $P_1 = P_2 = --- = P_n = \frac{1}{n}$ @

($\because$ uncer. $\propto \frac{1}{P(S)}$)

$\therefore \quad H(x) = \sum_{i=1}^{n} P_i \log_2 \frac{1}{P_i}$

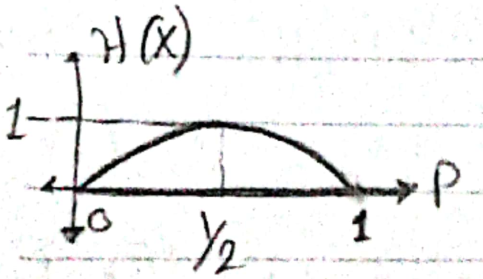
$\therefore \quad H_{max}(x) = \sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{1/n}$

$H_{max}(x) = n \times \frac{1}{n} \log_2 n$

$H_{max}(x) = \log_2 n$ bits/symbol Q.E.D

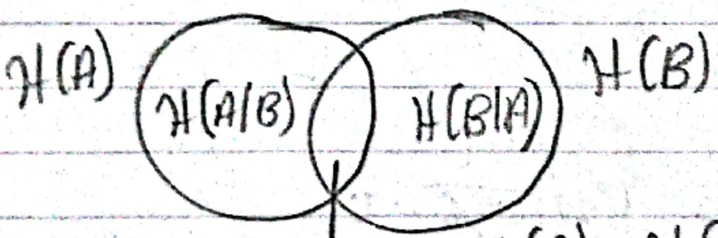
Bonus:

* For a binary src $\rightarrow x_1 = P$, $\rightarrow x_2 = 1-P$ $\Big]$ $H_{max}(x) = \log_2 2 = 1$ bit/symbol



Proof 2: Derive a formula for the Mutual Information:

* By Shannon's 1st theorem, we need an avg of $H(A)$ bits to specify one input symbol (a_i).
* On average $H(A|B)$ bits are lost in the channel!
* Therefore, observation of a single output symbol provides us with $[\underbrace{I(A;B)}_{\text{mutual info}} = \underbrace{H(A)}_{\text{info sent}} - \underbrace{H(A|B)}_{\text{info lost}}]$ bits of info.



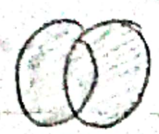
$$\underbrace{H(A) - H(A|B)}_{I(A;B)} = \underbrace{H(B) - H(B|A)}_{I(B;A)}$$

where $I(A;B) = I(B;A) \geq 0$

— Bonus:

* Relation bet. Mutual info and Joint entropy

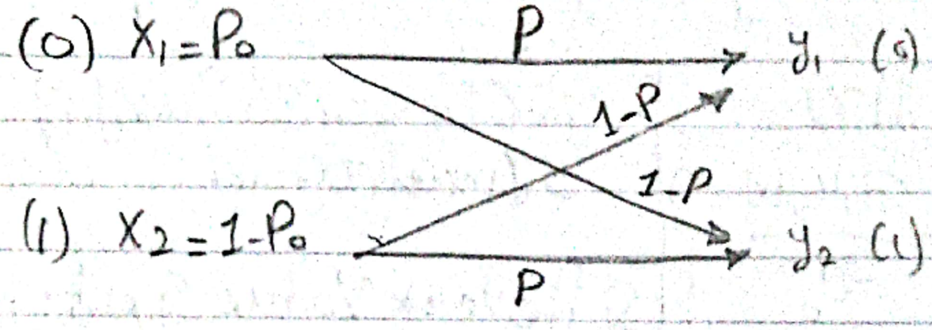
$$H(A,B) = H(A) + H(B|A)$$
$$= H(B) + H(A|B)$$
$$= H(A) + H(B) - I(A;B)$$



* $I(A;B)$
  - $= 0$ if, $H(A|B) = H(A)$ i.e. all info lost
  - $= H(A)$ if, $H(A|B) =$ zero i.e. No info lost

<u>Proof 3</u>: Prove that the Capacity of binary symmetric Channels is

$$\to C = 1 + P\log_2 P + (1-P)\log_2(1-P)$$



* We know that

$$C = Max[I(X;Y)]$$

$$* \; I(X;Y) = H(Y) - H(Y|X)$$

$$* \; H(Y) = \Sigma P(y_i)\log_2 \frac{1}{P(y_i)}$$

$$\to P(y_1) = \Sigma P(y_1|x_i)P(x_i) = \overbrace{PP_0 + (1-P)(1-P_0)}^{\text{let } Z =}$$

$$\to P(y_2) = 1 - P(y_1) = 1 - Z$$

$$H(Y) = Z\log_2 1/Z + (1-Z)\log_2\frac{1}{1-Z}$$

$$* \; H(Y|X) = \Sigma P(x_i)H(Y|x_i) = \Sigma\Sigma P(x_i)P(y_j|x_i)\log_2(y_j|x_i)$$

$$= P_0\left[(1-P)\log_2\frac{1}{1-P} + P\log_2\frac{1}{P}\right] + (1-P_0)\left[P\log_2 1/P + (1-P)\log_2\frac{1}{1-P}\right]$$

$$= P\log_2 1/P + (1-P)\log_2\frac{1}{1-P}$$

where, $H(Y|X) = \Sigma\Sigma P(y_j, x_i)\log_2(y_j|x_i)$

$$= \Sigma\Sigma P(y_j|x_i)P(x_i)\log_2(y_j|x_i)$$

follow
Proof 3

* to Calc $Max[I(x;y)]$, get value of P_0 to maximize it!

$$I(x;y) = H(y) - H(y|x)$$

$$\frac{\partial I(x;y)}{\partial P_0} = \frac{\partial H(y)}{\partial P_0} - \underbrace{\frac{\partial H(y|x)}{\partial P_0}}_{0}$$

at $\frac{\partial H(y)}{\partial P_0} = \text{zero} \rightarrow P_0 = \frac{1}{2}$

∴ The channel is binary & symmetric

∴ the max value is at $P_0 = 1 - P_0 = \frac{1}{2}$ i.e. $\frac{1}{n}$

اسهل واحد منهم !

* at $P_0 = \frac{1}{2}$ → $H(y) = 1$ ∴ $Z_{P_0 = \frac{1}{2}} = \frac{1}{2}$

→ $H(y|x) = P\log_2 \frac{1}{P} + (1-P)\log_2 \frac{1}{1-P}$

$= -[P\log_2 P + (1-P)\log_2 (1-P)]$ Not func of P_0

$$\therefore C = Max[I(x;y)]$$

$$= 1 - H(y|x)$$

$$= 1 + P\log_2 P + (1-P)\log_2(1-P)$$

Q.E.D