Sec 5 Info theory

1] $P(A,B) = P(A|B)\,P(B)$

2] $P(A) = \sum_{b_i \in B} P(A,b_i) = \sum_{b_i \in B} P(A|b_i)\,P(b_i)$

3] if independant $\rightarrow$ $P(A|B) = P(A)$

$\rightarrow$ $P(A,B) = P(A)\,P(B)$

4] $I = \log_2 1/P$ in bits

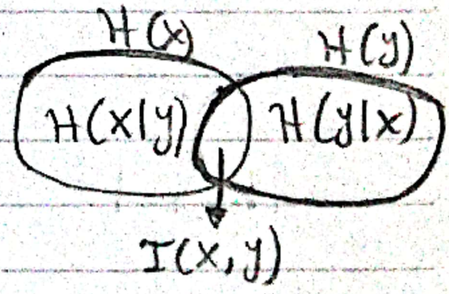
5] Entropy $H(X) = \sum P(x_i)\,\underbrace{I(x_i)}_{\log_2(1/P(x_i))}$

$0 \leq H(x) \leq \log_2(\underbrace{n}_{n:\ \#\text{symb}})$

6] $H(x,y) = \sum\sum P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)} = H(x) + H(y|x) = H(y) + H(x|y)$

$H(x|y) = \sum\sum \overbrace{P(x_i, y_j)}^{P(x|y)\,P(y)} \log_2 \frac{1}{P(x_i | y_j)}$

$I(x,y) = H(Y) - H(y|x) = H(x) - H(x|y)$



$0 \leq H(x|y) \leq H(x)$ same for $H(y|x)$

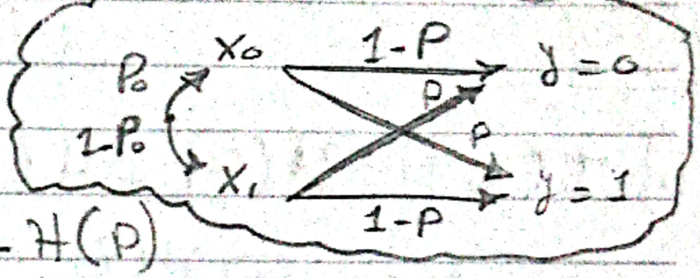
7] $I(x,y) = H(x) + H(y) - H(x,y)$

$= \sum\sum \underbrace{P(x_i, y_j)}_{P(y_i|x_i)\,P(x_i)} \log_2 \underbrace{\frac{P(x_i, y_j)}{P(x_i)\,P(y_j)}}_{\frac{P(y_i|x_i)}{\sum_{x_i} P(y_j|x_i)\,P(x_i)}}$

8] $C = Max[I(x,y)]$

Q:1 $P_0 \rightarrow x_0 = 0$

$\underbrace{1-P_0}_{P_1} \rightarrow x_1 = 1 \implies$



a] R.T.P: $I(x,y) = H(Z) - H(P)$

$H(Z) = Z\log_2\left(\frac{1}{Z}\right) + (1-Z)\log_2\left(\frac{1}{1-Z}\right)$

$Z = P_0 P + (1-P_0)(1-P)$

$H(P) = P\log_2(1/P) + (1-P)\log_2\left(\frac{1}{1-P}\right)$

$I(X,Y) = H(Y) - H(Y|X)$

* $H(Y) = \sum P(y_i)\log_2 \frac{1}{P(y_i)}$

$P(y_1) = \sum_{x_i} P(y_1, x_i) = \sum P(y_1|x_i)P(x_i)$

$= P P_0 + (1-P)(1-P_0) = Z$

$P(y_0) = \sum P(y_0|x_i)P(x_i) = (1-P)P_0 + P(1-P_0) = 1-Z$

$\therefore H(Y) = 1-Z\log_2\left(\frac{1}{1-Z}\right) + Z\log_2 1/Z = H(Z)$

* $H(Y|X) = P(x_0)H(Y|x_0) + P(x_1)H(Y|x_1)$

$P_0\left[(1-P)\log_2 \frac{1}{1-P} + P\log_2 1/P\right] + (1-P_0)\left[P\log_2 1/P + (1-P)\log_2\frac{1}{1-P}\right]$

$= \left[P\log_2 1/P + (1-P)\log_2 \frac{1}{1-P}\right]$

$= H(P)$

$\therefore I(X,Y) = H(Y) - H(Y|X) = H(Z) - H(P)$

$\boxed{Q:1}$

b] $I(x,y) = H(z) - H(P)$

$\frac{\partial I(x,y)}{\partial P_0} = \frac{\partial H(z)}{\partial P_0} - zero$ $\qquad$ $H(P)$ is not func of (P_0)

$\therefore$ we want P_0 that maximizes $H(z)$

But, $H(z) = H(Y)$ $\rightarrow$ The value of P_0 to maximize $H(Y)$ is 0.5 [$\because$ Binary [$\frac{1}{n}$] Channel]

c] R.T.P: $C = 1 - H(P)$

$\because C = Max(I(x,y))$

$\therefore C = I(x,y)_{P_0=1/2}$

$= H(z)_{P_0=1/2} - H(P)_{P_0=1/2}$ $\qquad$ | $z_{P_0=1/2} = 1/2$

$= 1 - H(P)$ $\quad$ Q.E.D. $\qquad$ | $H(z)_{z=1/2} = 1$

$\boxed{Q:2}$ $\begin{bmatrix} P(y_1|x_1) & P(y_2|x_2) \\ P(y_1|x_1) & P(y_2|x_2) \end{bmatrix} \Rightarrow \begin{array}{c|cc} & y_1 & y_2 \\ x_1 & 2/3 & 1/3 \\ x_2 & 1/10 & \frac{9}{10} \end{array}$

$P(x_1) = 1/3$, $P(x_2) = 2/3$

Get: $H(x)$, $H(x|y)$, $H(y)$, $H(y|x)$ & $I(x,y)$

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* $H(x) = \sum_{x_i} P(x_i) \log_2 \frac{1}{x_i} = \frac{1}{3} \log 3 + \frac{2}{3} \log 3/2$

$= 0.918$ bits

* $H(y|x) = \sum P(x_i) H(y|x_i) = P(x_1) H(y|x_1) + P(x_2) H(y|x_2)$

$= 1/3 [\frac{2}{3} \log 3/2 + 1/3 \log 3] + \frac{2}{3} [\frac{1}{10} \log_2 10 + \frac{9}{10} \log_2 \frac{10}{9}]$

$= 0.61876$ bits

* $H(y) = \sum_{y_i} P(y_i) \log_2 1/P(y_i)$

$= \frac{13}{45} \log_2 (\frac{45}{13}) + \frac{32}{45} \log_2 (\frac{45}{32}) \Rightarrow$

	y_1	y_2
x_1	$2/9$	$1/9$
x_2	$1/15$	$3/5$
$P(y)$	$\frac{13}{45}$	$\frac{32}{45}$

$= 0.8672$

* $I(x,y) = H(y) - H(y|x) = 0.248$ bits
$= H(x) - H(x|y)$

* $H(x|y) = H(x) - I(x,y) = 0.66948$ bits

OR $= P(y_1) H(x|y_1) + P(y_2) H(x|y_2)$