CMP205: Computer Graphics

Lecture 2: Transformations I

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Agenda

- 2D Transformations
- 3D Transformations
- 2D & 3D Translation

Acknowledgments: Some slides adapted from Steve Marschner and Fredo Durand.

2D Transformations

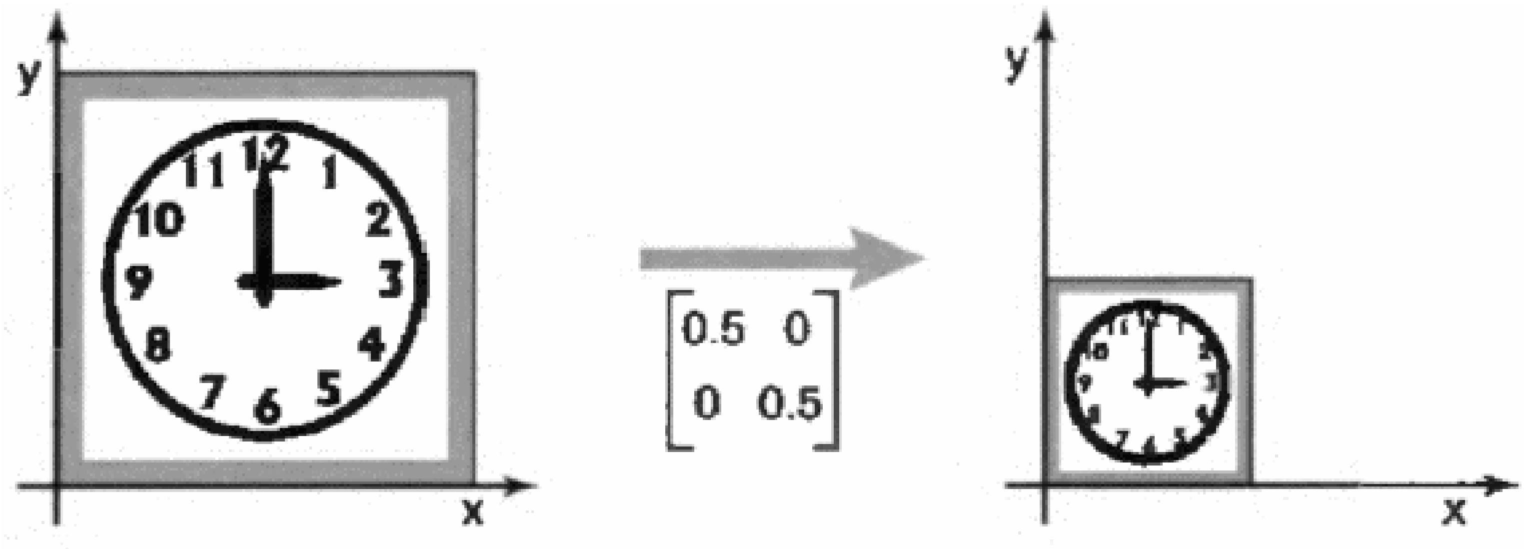
- Scale
- Shear
- Rotation
- Reflection

Look at *linear* transformations in the form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

Scale

Just “scales” all the points by multiplying them with a scale factor



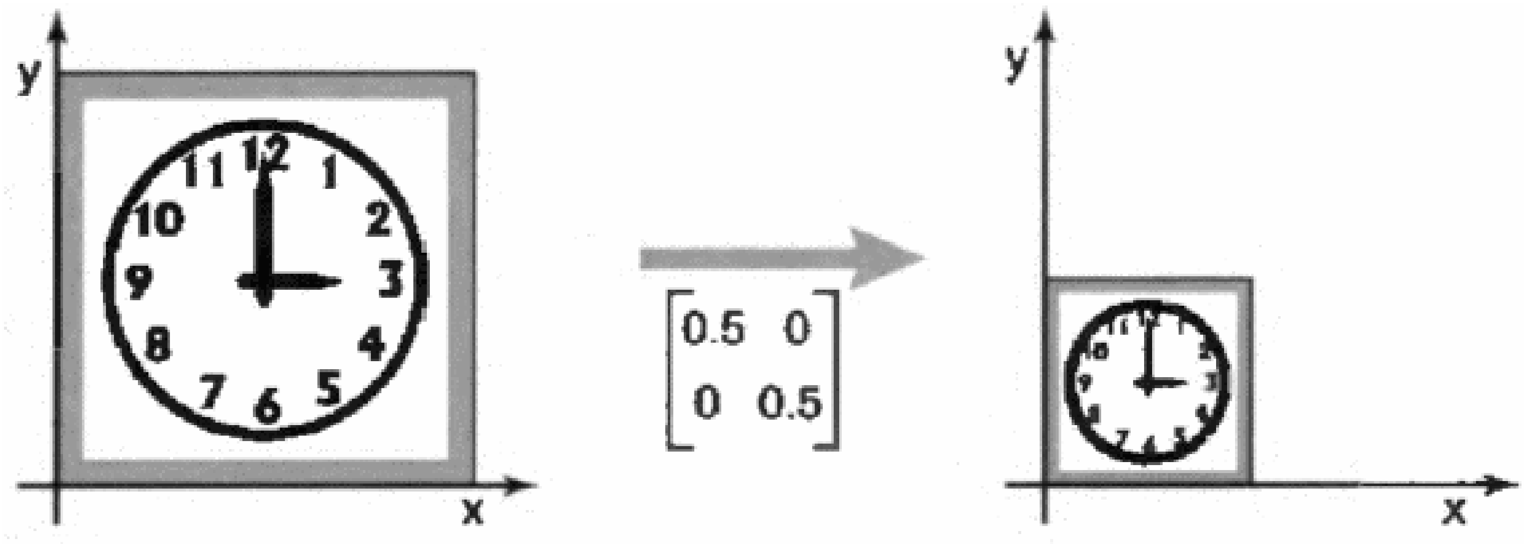
Scale

We can define it by the matrix: $\mathrm{Scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

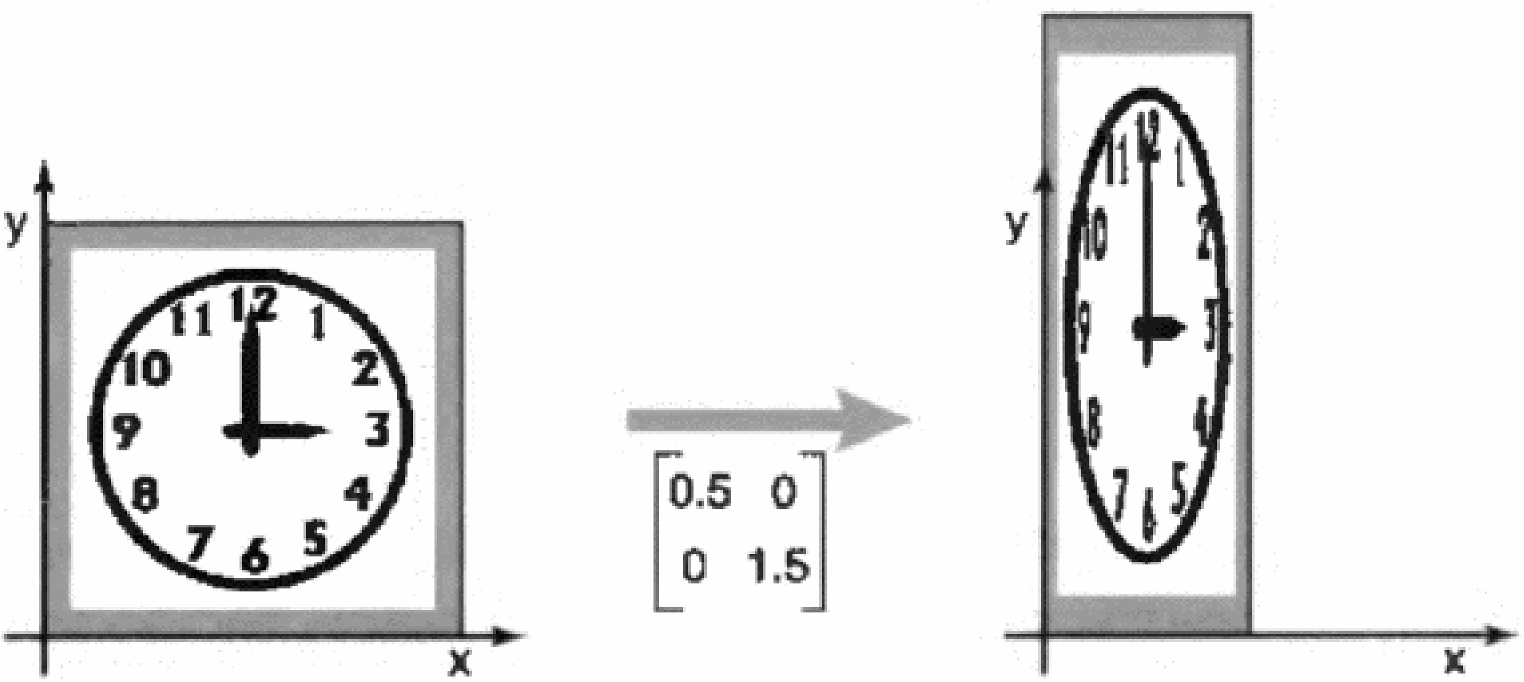
Scale

- Uniform Scale: equal factors in *x* and *y* directions



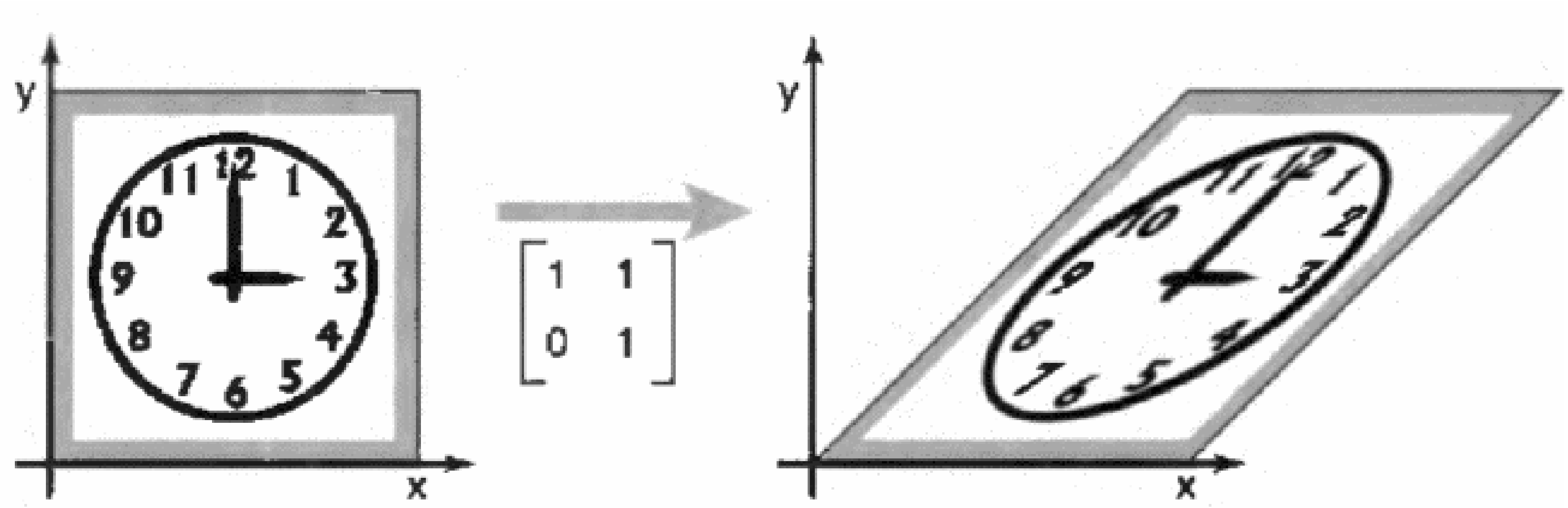
Scale

- Nonuniform Scale: different factors in *x* and *y* directions



Shearing

Shears the points by stretching along one direction



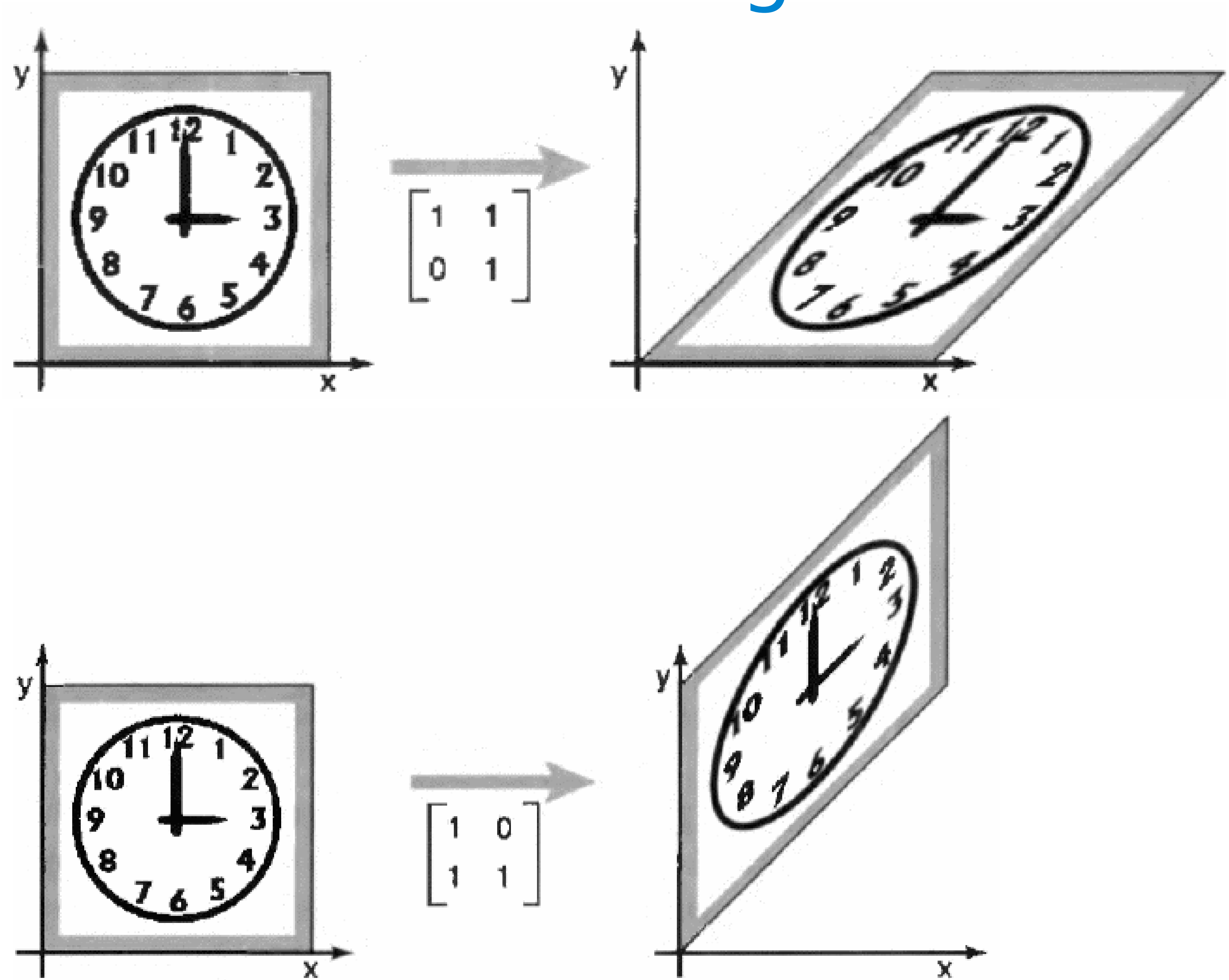
Shearing

$$\text{shear-x}(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + sy \\ y \end{bmatrix}$$

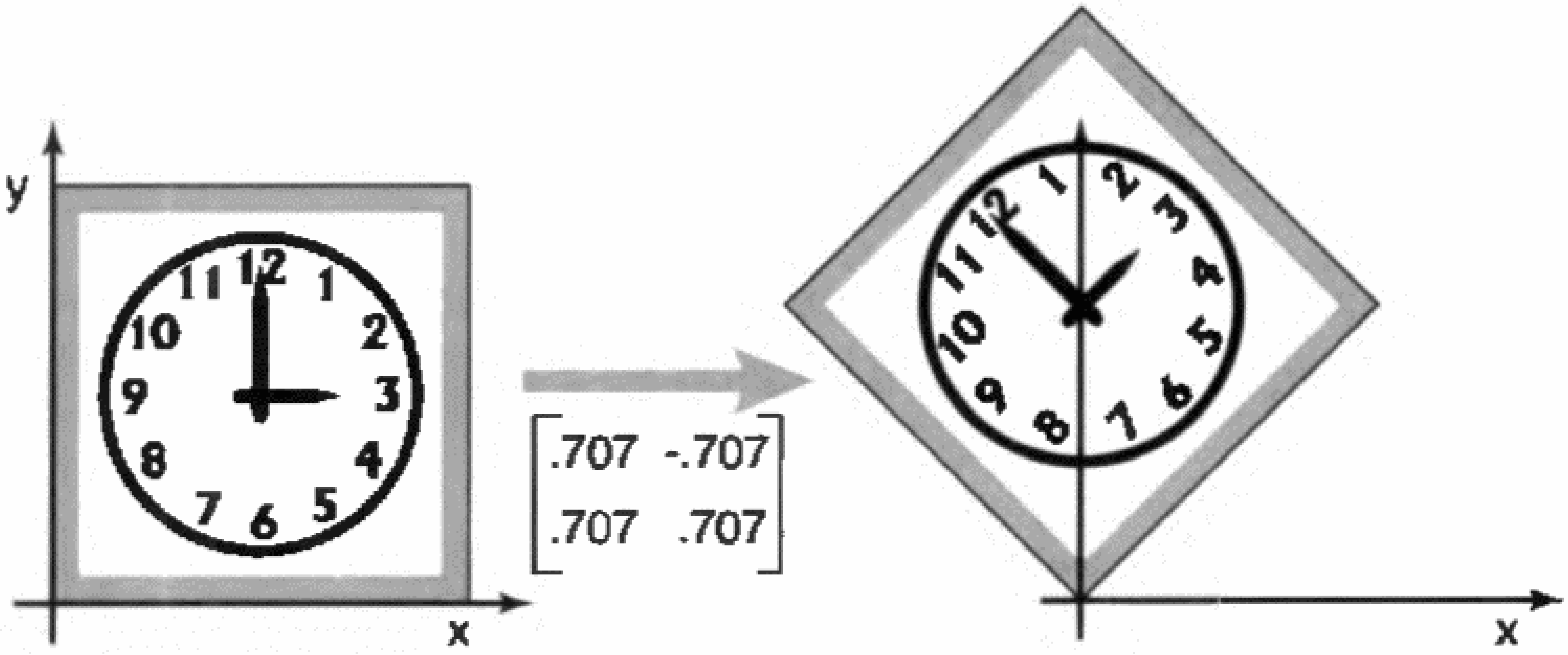
$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

Shearing



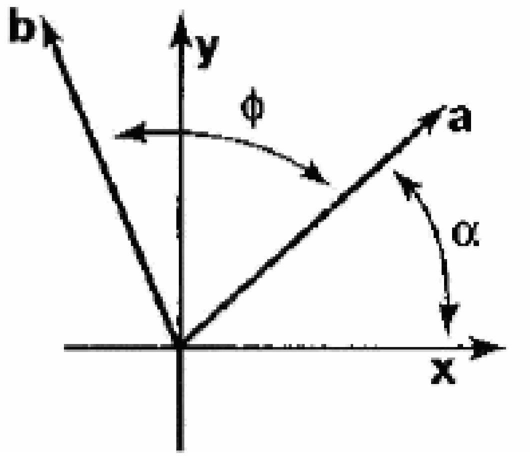
Rotation

Rotate all the points around the origin



Rotation

$$x_a = r\cos\alpha \qquad x_b = r\cos(\alpha+\phi)$$
$$y_a = r\sin\alpha \qquad y_b = r\sin(\alpha+\phi)$$



and we know that:

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$
$$\sin(\alpha+\phi) = \cos\alpha\sin\phi + \sin\alpha\cos\phi$$

$$x_b = x_a\cos\phi - y_a\sin\phi$$
$$y_b = x_a\sin\phi + y_a\cos\phi$$

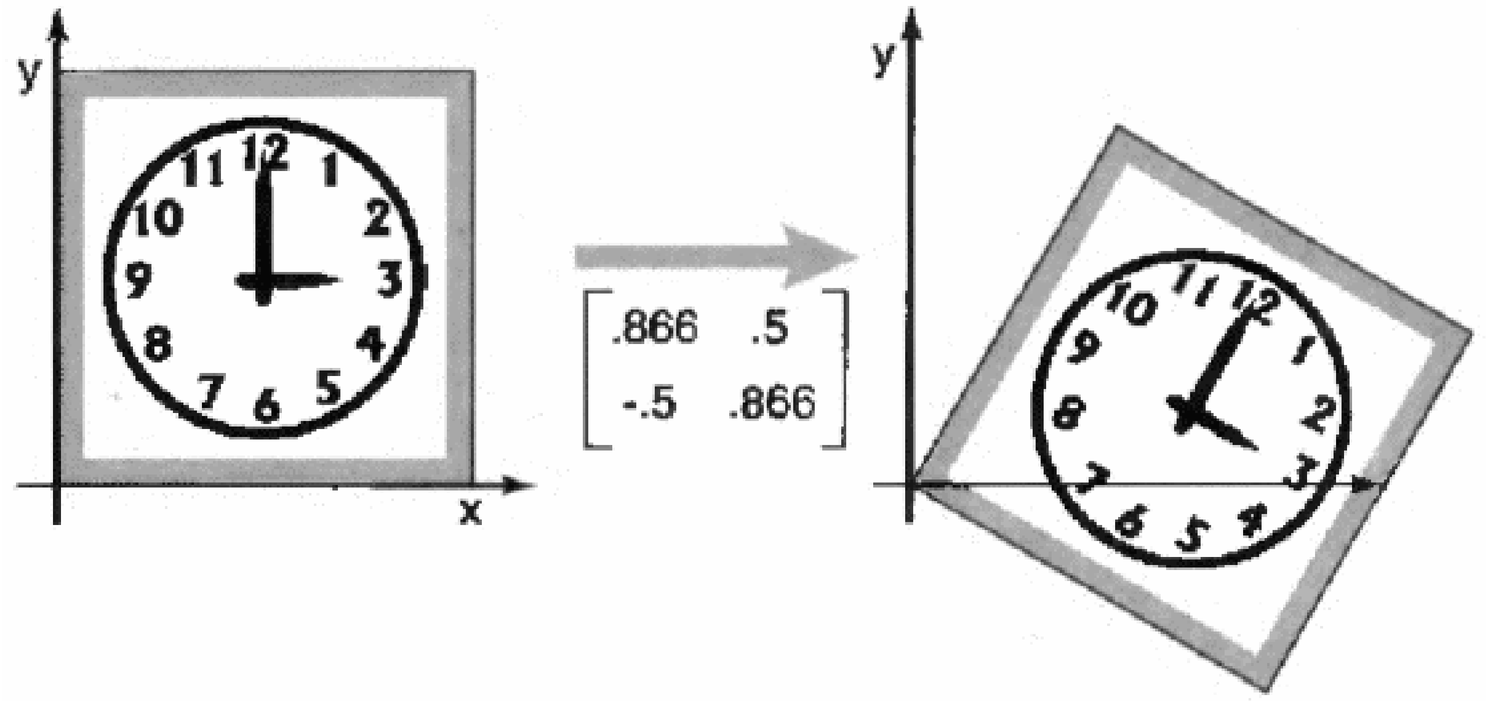
Rotation

$$\mathrm{rotate}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

What's the important property of this matrix?

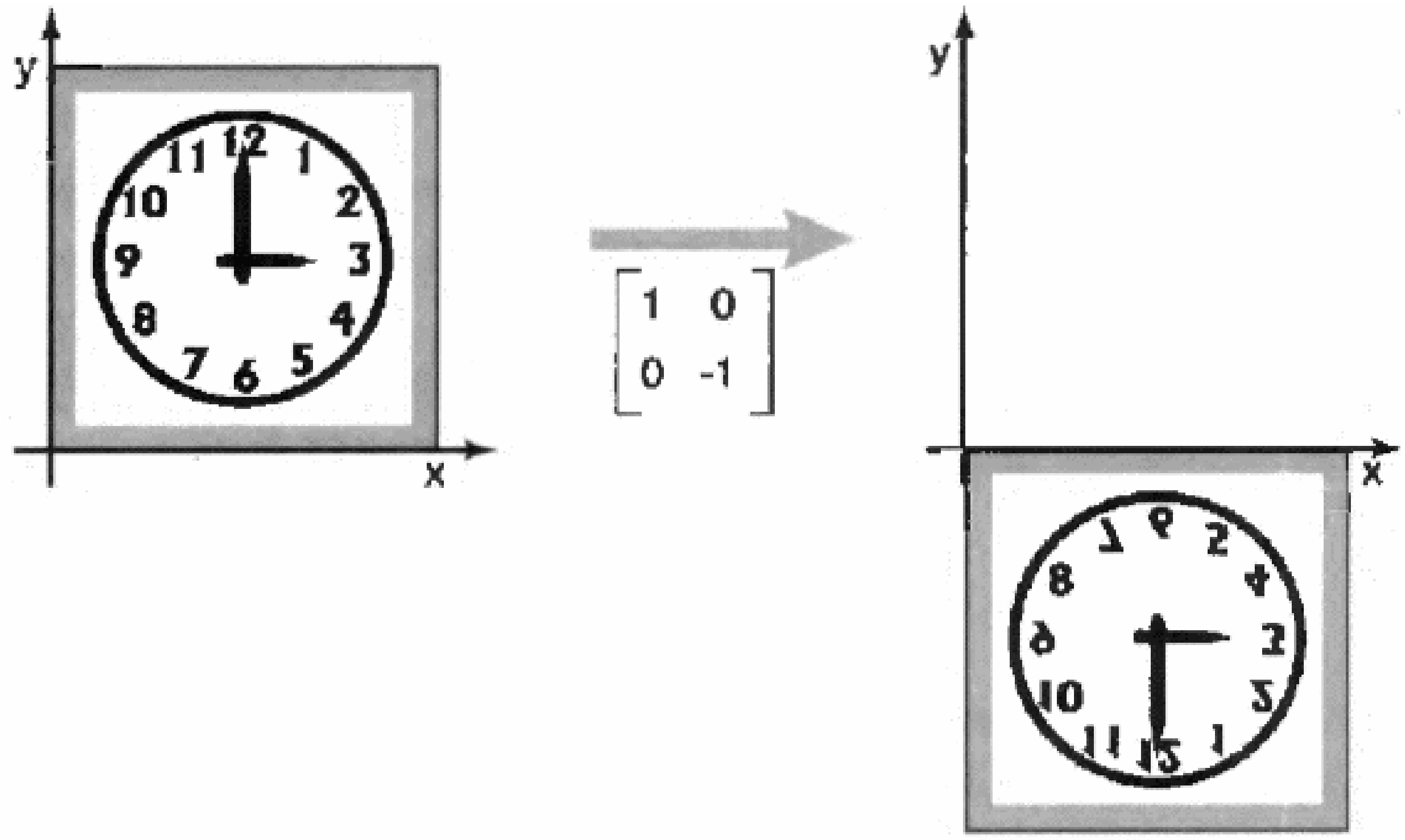
It's orthonormal i.e. $R R^T = I$

Rotation



Reflection

Reflects points around some axis



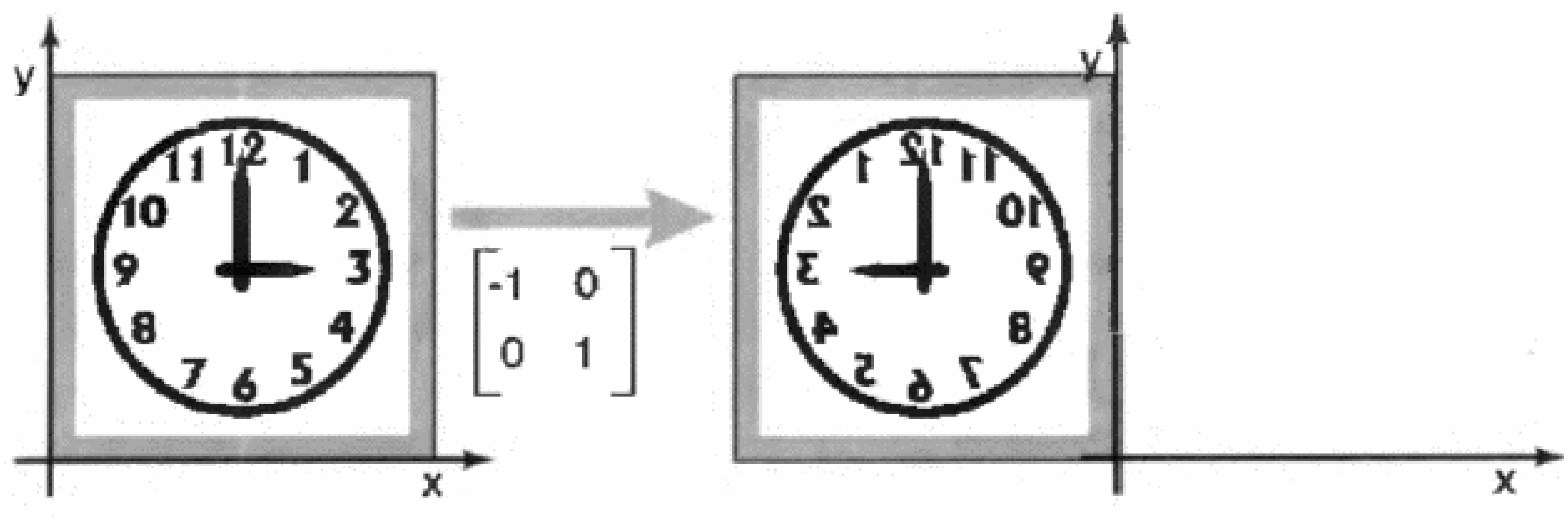
Reflection

$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

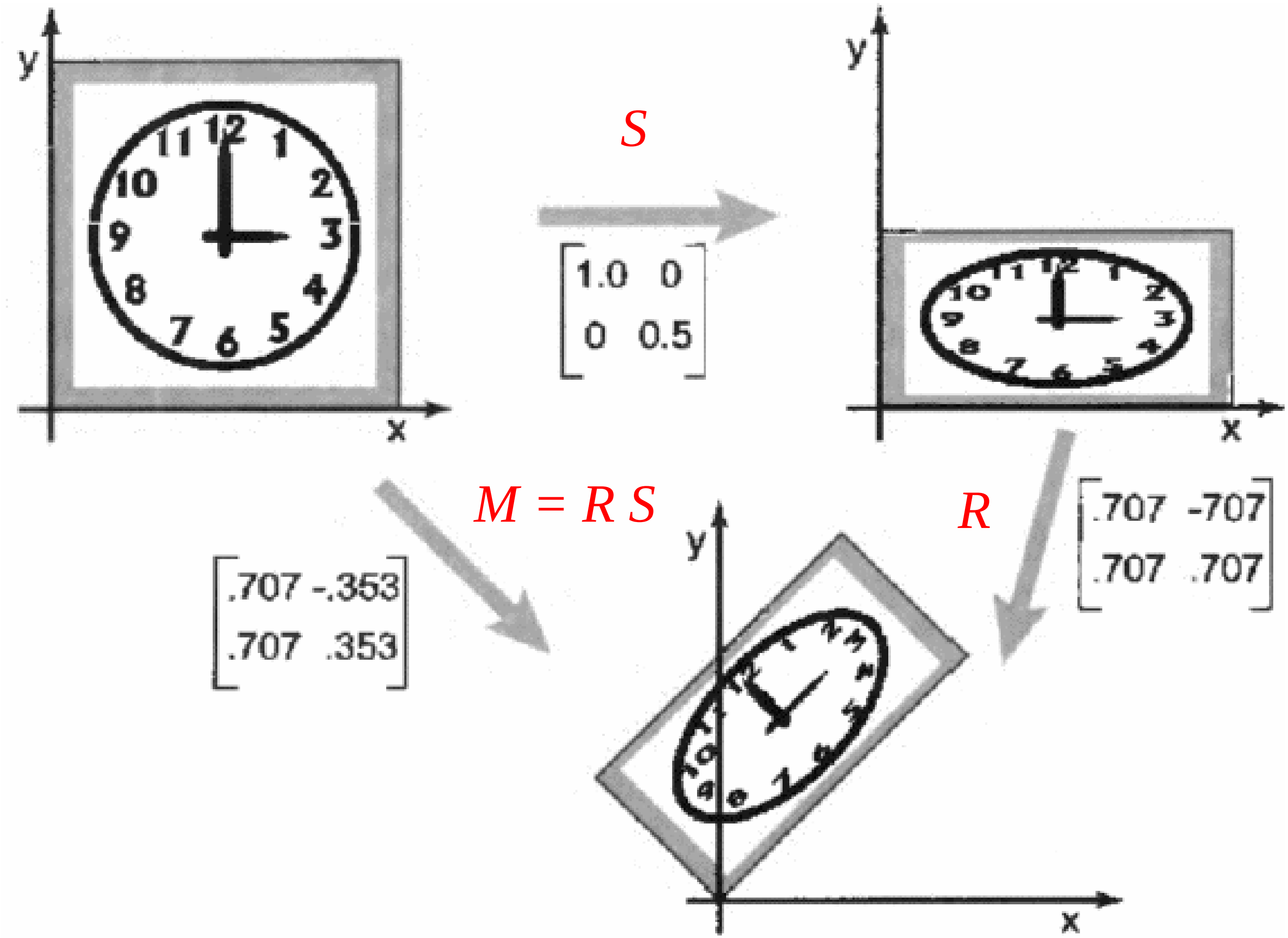
$$\text{reflect-y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection



Composition of 2D Transforms

Two (or more) transformation matrices can be combined in one matrix



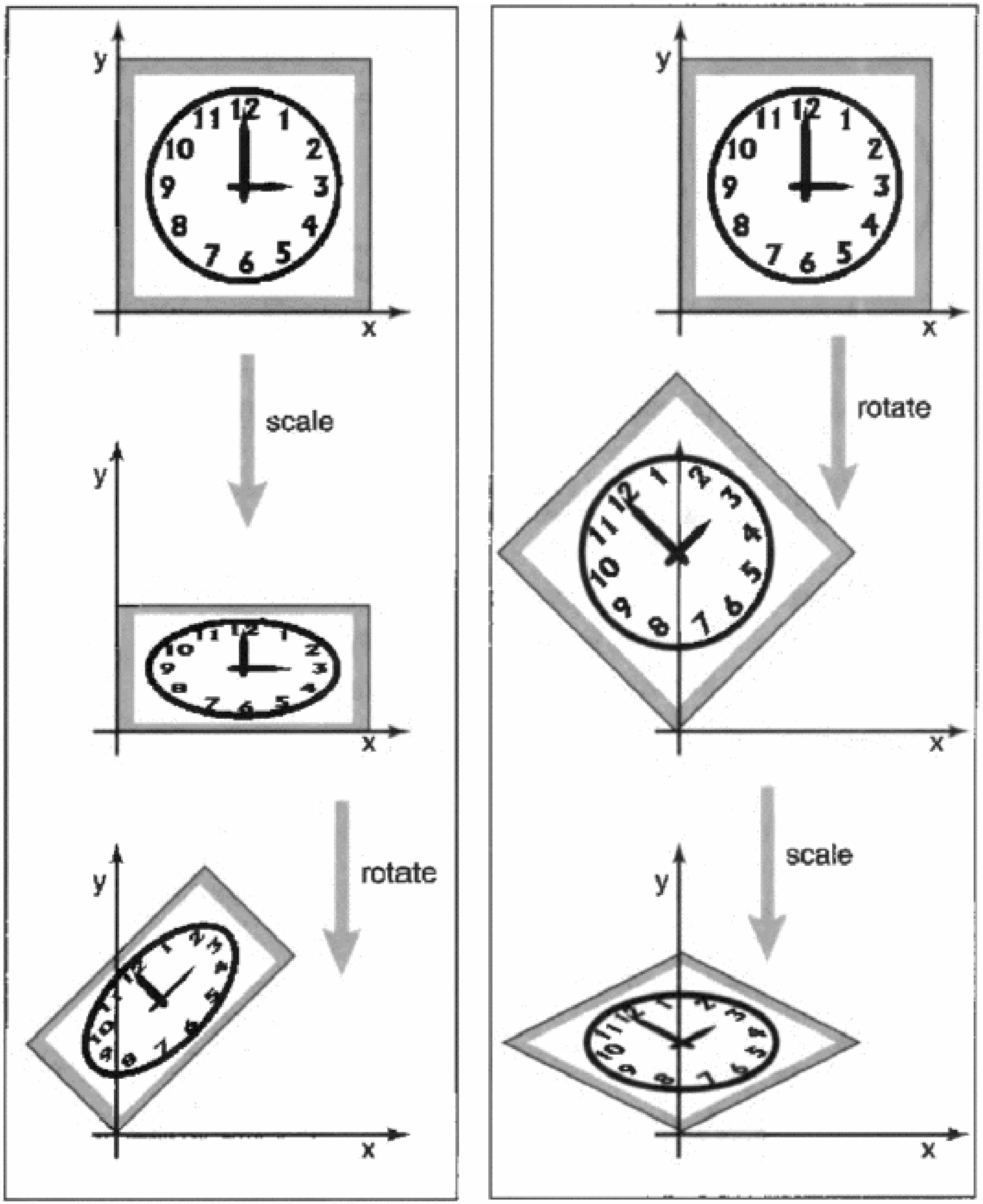
$$v_2 = S\, v_1$$
$$v_3 = R\, v_2$$
$$\downarrow$$
$$v_3 = (R\,S)\, v_1$$
$$v_3 = M\, v_1$$

where:
$$M = R\,S$$

Composition of 2D Transforms

Beware that the order of the transformations matters!

$$RS \neq SR$$



3D Scaling

Here we have scaling in three dimensions instead of two!

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

3D Rotation

Here we have three possible *standard* axes to rotate around:

$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$\text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

3D Shear

Instead of just *one* factor for *y*, we have *two* factors for *y* and *z*:

$$\text{shear-x}(d_y, d_z) = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2D Translations

2D Transformations

$$p' = M_{2\times 2}\, p$$

$$x' = m_{11}x + m_{12}y$$
$$y' = m_{21}x + m_{22}y$$

Translation

$$x' = x + t_x$$
$$y' = y + t_y$$

How can we represent *Translation* as Matrix Multiplication?

2D Translations

Solution: add z=1 to 2D points

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

What about vectors?

$z = 0$!

Homogeneous Coordinates

Convert 2D points into 3D points

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \tilde{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Transformations with 3x3 matrix

$$\tilde{v}' = M \tilde{v}$$

2D Transformations

Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Can represent any combination by a 3x3 matrix

2D Transformations

Rotation/Scale/Shear + Translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{2\times 2} & t_{2\times 1} \\ 0^T & 1 \end{bmatrix}$$

Rotation part + translation part

3D Translations

Homogeneous Coordinates

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

3D Transformations

Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Can represent any combination by a 4x4 matrix

3D Transformations

Rotation/Scale/Shear + Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{21} & a_{22} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{21} & a_{22} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{3\times 3} & t_{3\times 1} \\ 0^T & 1 \end{bmatrix}$$

Rotation part + translation part

Transformation Inverse

$$M \rightarrow M^{-1}$$

$$\mathit{Rotation}\ R \rightarrow R^{T}$$

$$\mathit{translation}(\boldsymbol{t}) \rightarrow \mathit{translation}(-\boldsymbol{t})$$

$$\mathit{scale}(s_x, s_y, s_z) \rightarrow \mathit{scale}(1/s_x, 1/s_y, 1/s_z)$$

$$M_1 M_2 \ldots M_n \rightarrow M_n^{-1} \ldots M_2^{-1} M_1^{-1}$$