Pass-band Modulation

From Dr Mohamed Khairy’s slides
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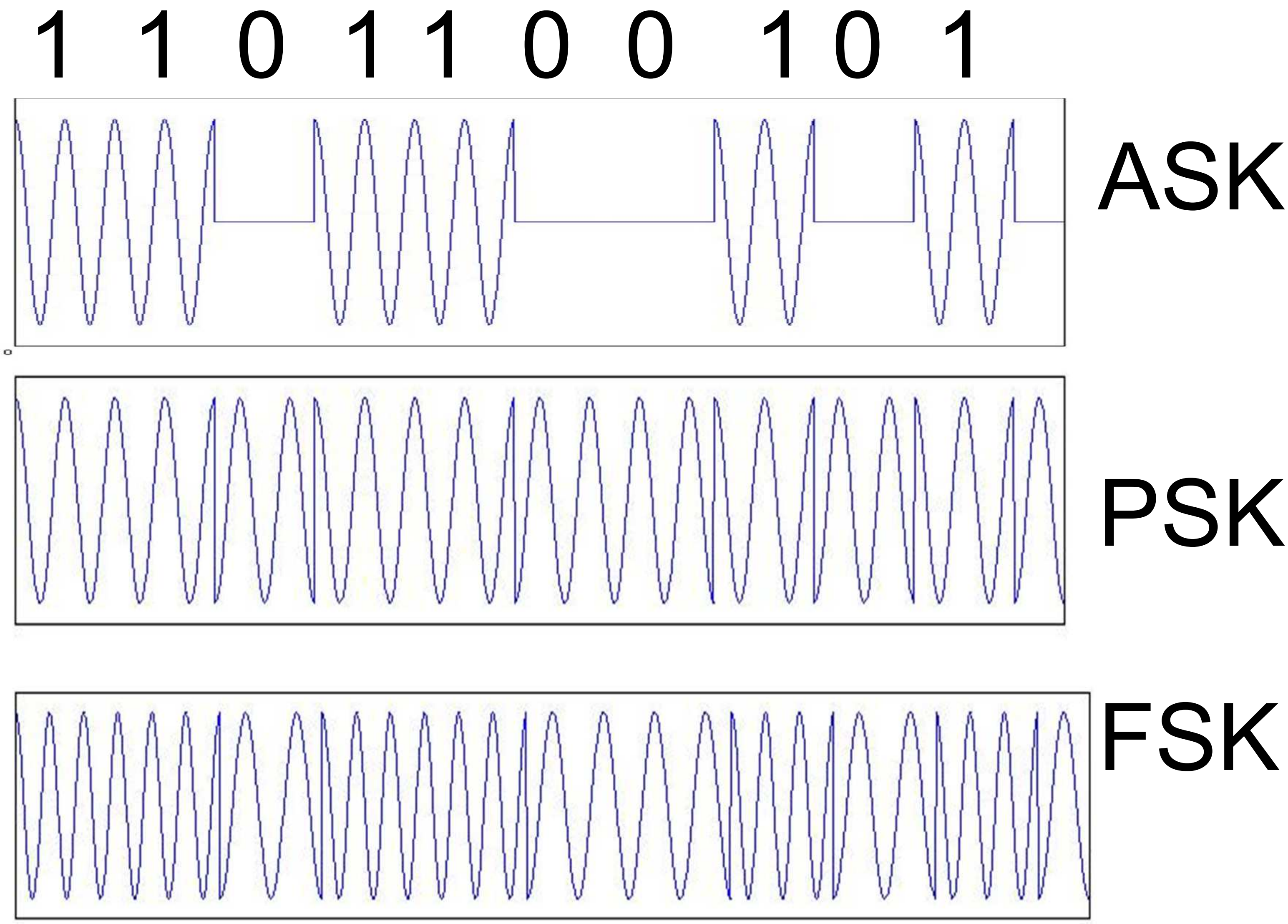
Introduction

- The incoming data stream is modulated onto a carrier
- The communication channel used for passband transmission maybe a microwave radio link, a satellite channel, ...
- The modulation process involves switching the amplitude, frequency, or phase of a sinusoidal carrier in some fashion in accordance with the incoming data
- There are three basic signaling schemes, they are known as Amplitude Shift Keying (ASK), Frequency Shift Keying (FSK), Phase Shift Keying (PSK)


1 1 0 1 1 0 0 1 0 1
ASK
PSK
FSK

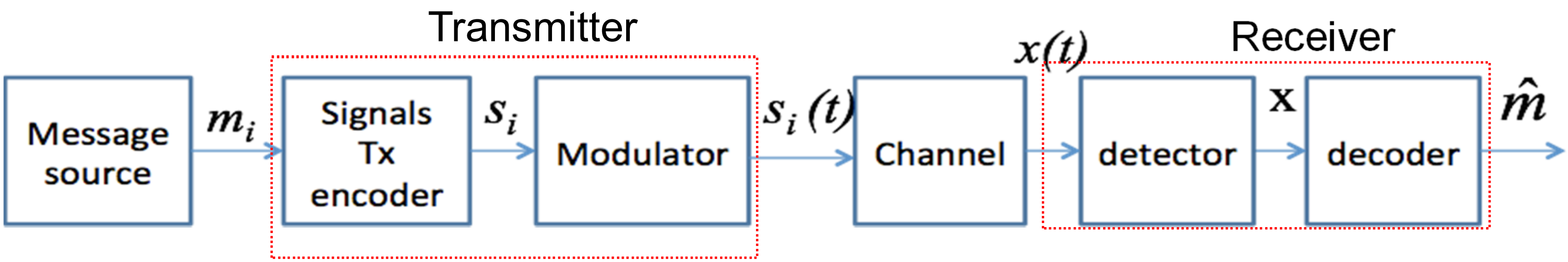
Hierarchy of Digital Modulation Techniques

- Coherent and noncoherent techniques depending on whether the receiver is equipped with a phase-recovery circuit or not
- The phase recovery circuit ensures that the oscillator supplying the locally generated carrier wave in the receiver is synchronized (in both frequency and phase) to the oscillator supplying the carrier wave used to modulate the data stream in the transmitter
- DPSK and M-ary FSK are the commonly used forms of noncoherent systems

Hierarchy of Digital Modulation Techniques

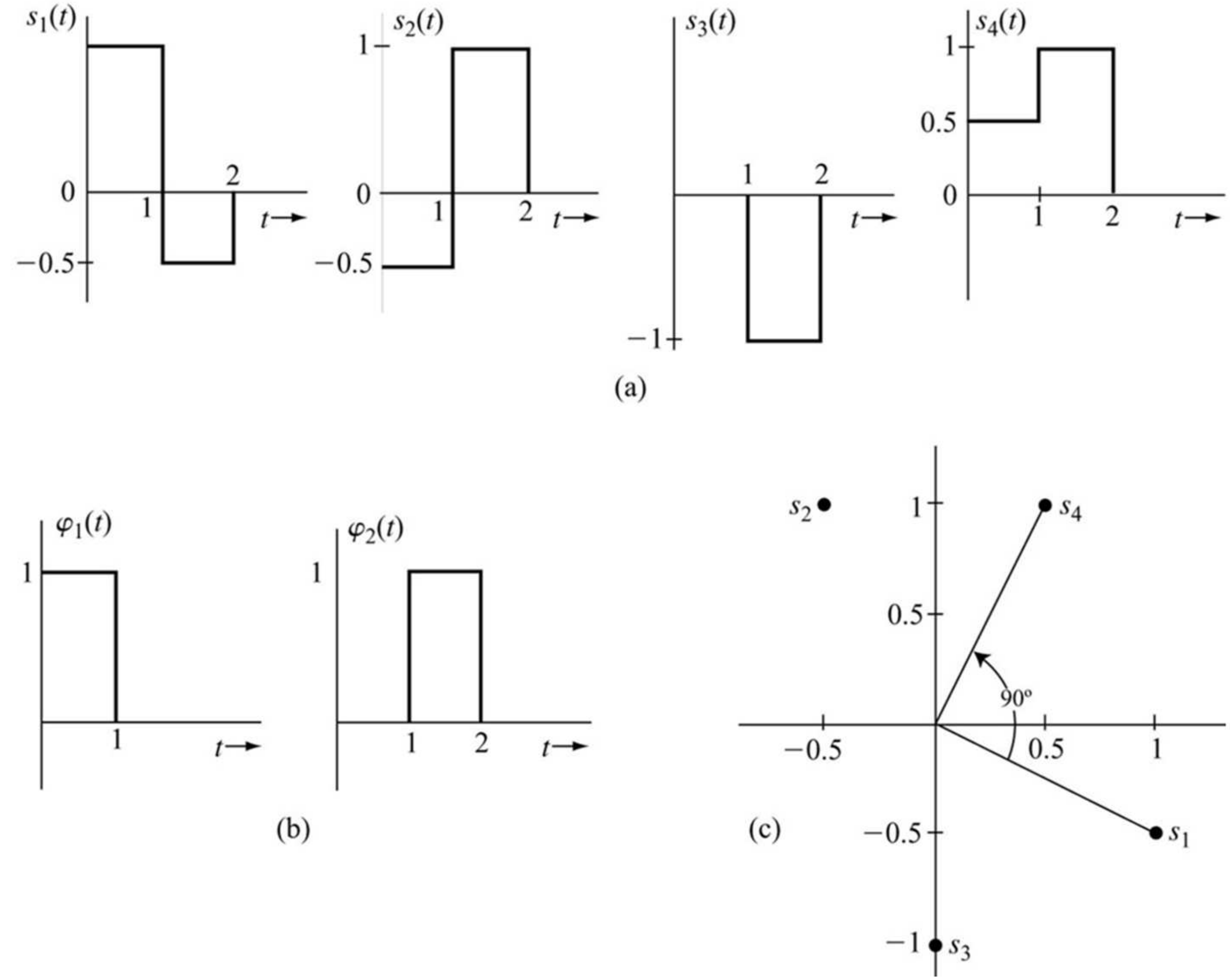
- For M-ary signaling schemes, one of the M waveforms is transmitted every $T=nT_b$, $M=2^n$, T_b is the bit duration
- M-ary signaling schemes reduce the BW by a factor of $n = \log_2 M$ over binary PSK for the same bit rate
- The price paid is increased power to achieve the same BER as BPSK
- Define bandwidth efficiency $\rho=R_b/B$

Passband Transmission Model



- Channel is linear with wide enough BW to accommodate Tx of $s_i(t)$ with no distortion
- Channel noise is AWGN $w(t)$ with zero mean and PSD=$N_0/2$
- The receiver reverses the operations performed by the transmitter
- The receiver minimizes the effect of channel noise on the estimate of the symbol m_i

Signal Space Representation



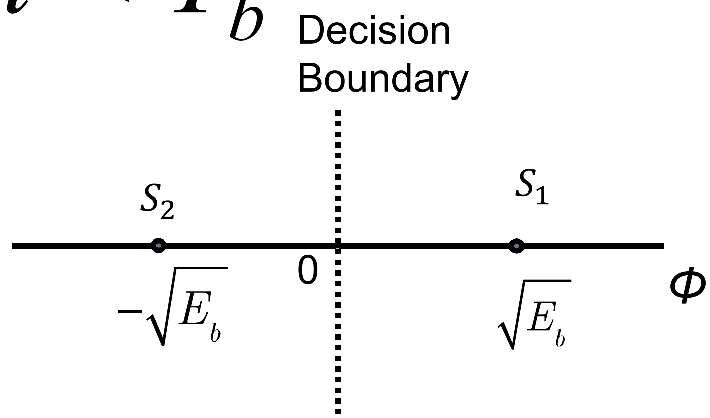
Binary Phase Shift Keying (BPSK)

The signal set can be represented by one basis function:

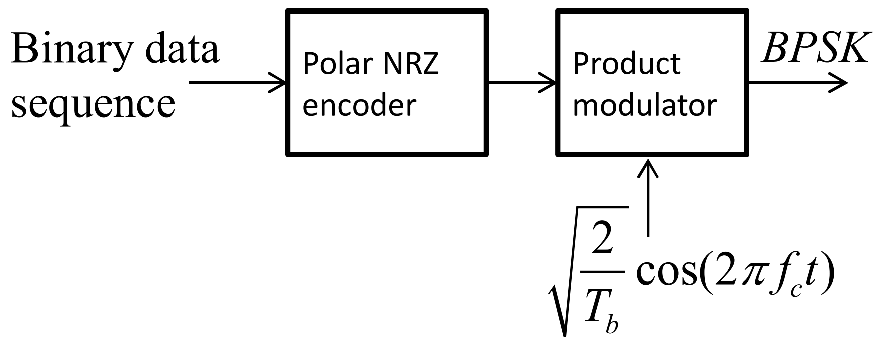
$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t),\ 0 \le t < T_b$$

$$s_1(t) = \sqrt{E_b}\,\phi_1(t)$$

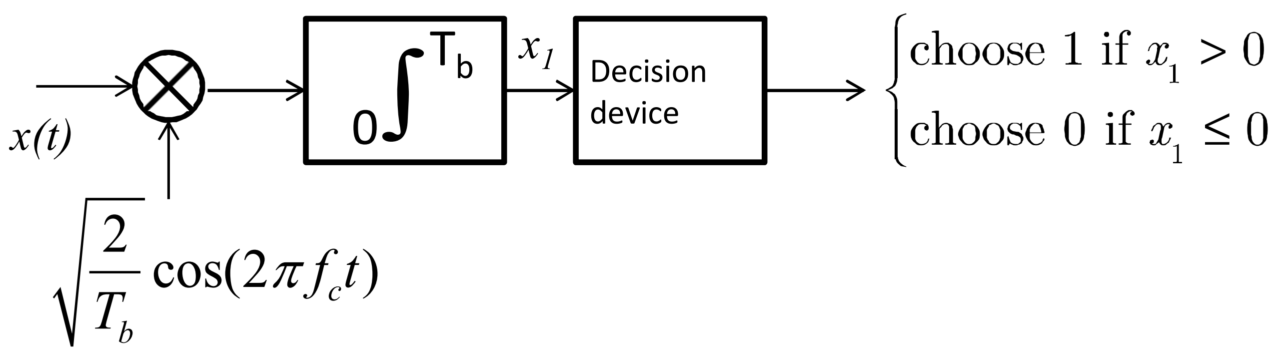
$$s_2(t) = -\sqrt{E_b}\,\phi_1(t)$$



Binary Phase Shift Keying (BPSK)



Transmitter



Receiver

Probability of error for BPSK

$$x(t) = s_i(t) + w(t)$$

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)\,dt = \int_0^{T_b} (s_i(t) + w(t))\phi_1(t)\,dt$$

$$f(x_1 \mid 1) = \mathscr{N}\left(\sqrt{E_b}, N_0 / 2\right)$$

$$f(x_1 \mid 0) = \mathscr{N}\left(-\sqrt{E_b}, N_0 / 2\right)$$

$$P_{10} = \text{prob}(\text{dec } 1 \mid 0 \text{ sent})$$

$$= \int_0^{\infty} f(x_1 \mid 0)\,dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} e^{-\left(x_1 + \sqrt{E_b}\right)^2 / N_0}\,dx_1$$

$$\text{let } \frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}} = z$$

$$P_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-z^2}\, dz = 0.5 \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$\text{where } \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-z^2}\, dz$$

Since the signal space is symmetric, therefore P_{01}, the conditional probability of the receiver deciding in favor of symbol 0, given that 1 was transmitted also has the same value as P_{10}. Thus the average probability of symbol error P_e is

$$P_e = 0.5 \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-y^2}\, dy$$

$$\frac{1}{2} erfc(x) = \int_{x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}\, dy = \int_{x}^{\infty} pdf\ of\ y \sim N(0{,}0.5)\ dy$$

Generally

$\int_{x}^{\infty} pdf\ of\ y \sim N(\mu, \sigma^2)\ dy = \frac{1}{2}\ erfc(\frac{1}{\sqrt{2}} \times \frac{x-\mu}{\sqrt{\sigma^2}})$

Try it yourself by defining a new variable $z = \frac{x-\mu}{\sqrt{\sigma^2}}$

$$erfc(-x) = \frac{2}{\sqrt{\pi}} \int_{-x}^{\infty} e^{-y^2}\, dy$$

$$\frac{1}{2} erfc(-x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{\infty} e^{-y^2}\, dy = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-x} e^{-y^2}\, dy$$

Let $z = -y$

$$\frac{1}{2} erfc(-x) = 1 - \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} e^{-z^2}\, dz = 1 - \frac{1}{2} erfc(x)$$

$$P_{01} = \int_{-\infty}^{0} f(x_1|1)\, dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-\frac{\left(x_1-\sqrt{E_b}\right)^2}{N_0}}\, dx_1$$

Let $z = \frac{x_1-\sqrt{E_b}}{\sqrt{N_0}}$

$$P_{01} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{\sqrt{E_b}}{\sqrt{N_0}}} e^{-z^2}\, dx_1 = 1 - \frac{1}{\sqrt{\pi}} \int_{-\frac{\sqrt{E_b}}{\sqrt{N_0}}}^{\infty} e^{-z^2} dz$$

$P_{01} = 1 - \frac{1}{2} erfc\left(-\frac{\sqrt{E_b}}{\sqrt{N_0}}\right) = \frac{1}{2} erfc\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right) = P_{10}$

Since the signal space is symmetric, therefore P_{01}, the conditional probability of the receiver deciding in favor of symbol 0, given that 1 was transmitted also has the same value as P_{10}.

$$P_{01} = P_{10} = \frac{1}{2}\mathrm{erfc}(\sqrt{\frac{E_b}{N_0}})$$

Thus the average probability of symbol error P_e is

$$P_e = P_0 P_{10} + P_1 P_{01} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Where $P_0 = P_1 = 0.5$

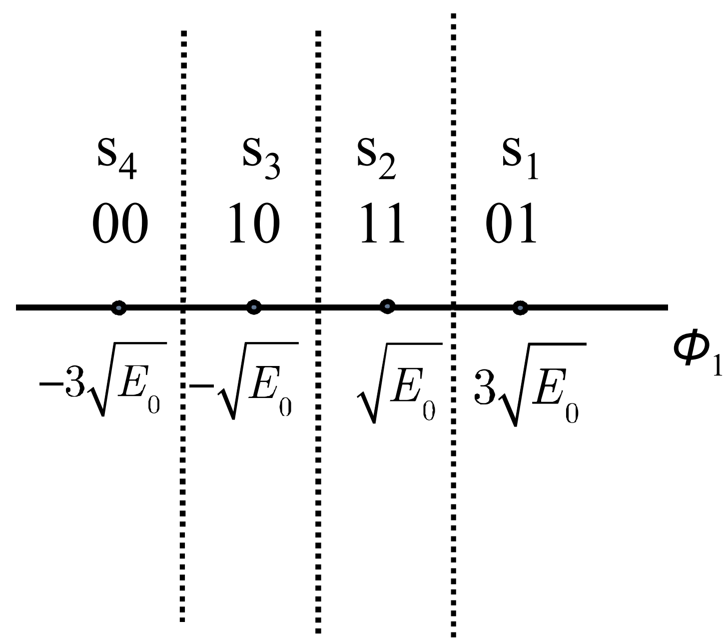
Amplitude Shift Keying (ASK)

$$s_i(t) = \sqrt{\frac{2E_0}{T}}\; a_i \cos\left(2\pi f_c t\right) \quad 0 \le t \le T, i = 1, 2, \ldots, M$$

$$a_i = \pm 1, \pm 3, \pm 5, \ldots$$

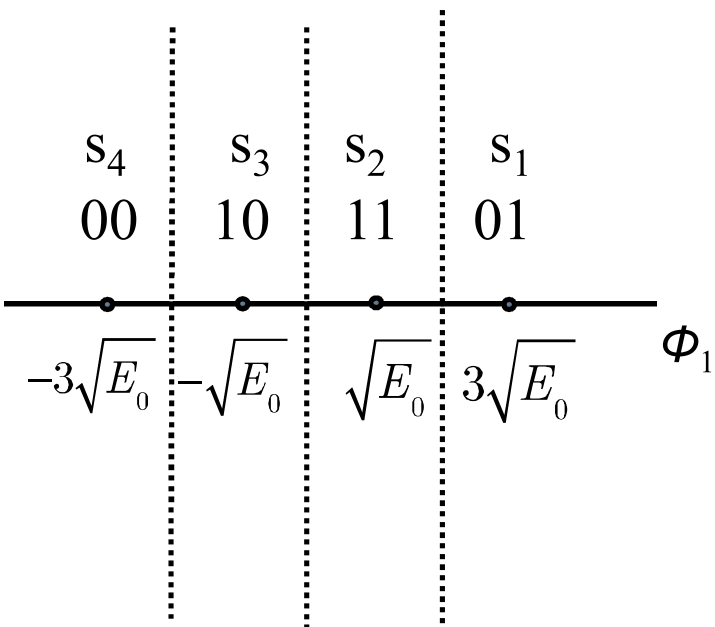
$$\phi_1 = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$$

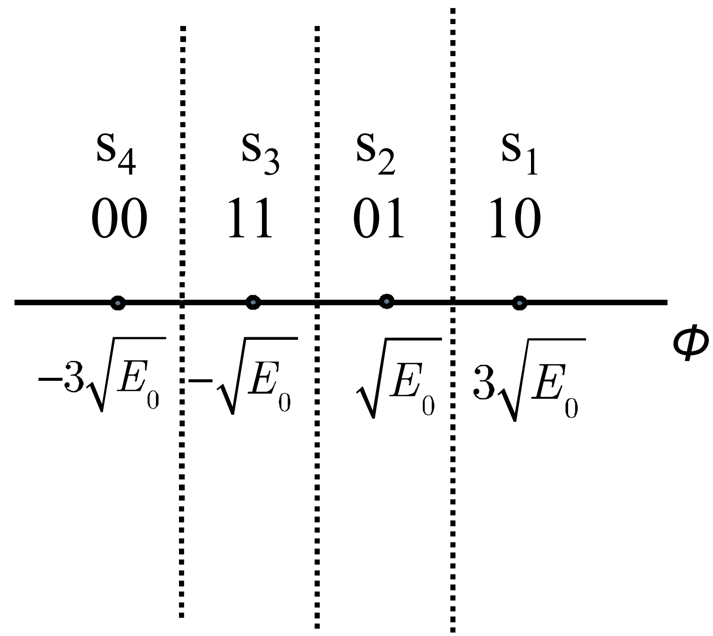
$$x_1 \sim \mathscr{N}\left(a_i \sqrt{E_0}, N_0 / 2\right)$$



Gray encoding

- Which is better?





SER of ASK

$$P_e = \frac{1}{2}P\left(e \mid s_1\right) + \frac{1}{2}P\left(e \mid s_2\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{2\sqrt{E_0}} e^{-\left(x_1 - 3\sqrt{E_0}\right)^2/N_0}\,dx_1 + \frac{1}{2}\frac{2}{\sqrt{\pi N_0}}\int_{2\sqrt{E_0}}^{\infty} e^{-\left(x_1 - \sqrt{E_0}\right)^2/N_0}\,dx_1$$

$$= \frac{3}{4}\operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right)$$

$$E_b = \frac{E_0 + 9E_0}{4} = 2.5E_0$$

$$P_e = \frac{3}{4}\operatorname{erfc}\left(\sqrt{\frac{E_b}{2.5N_0}}\right)$$

Quadri-Phase Shift Keying (QPSK)

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + (2i-1)\frac{\pi}{4}\right) \quad 0 \le t \le T, i = 1,2,3,4$$

Where E is the energy per symbol

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left((2i-1)\frac{\pi}{4}\right)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin\left((2i-1)\frac{\pi}{4}\right)\sin(2\pi f_c t)$$

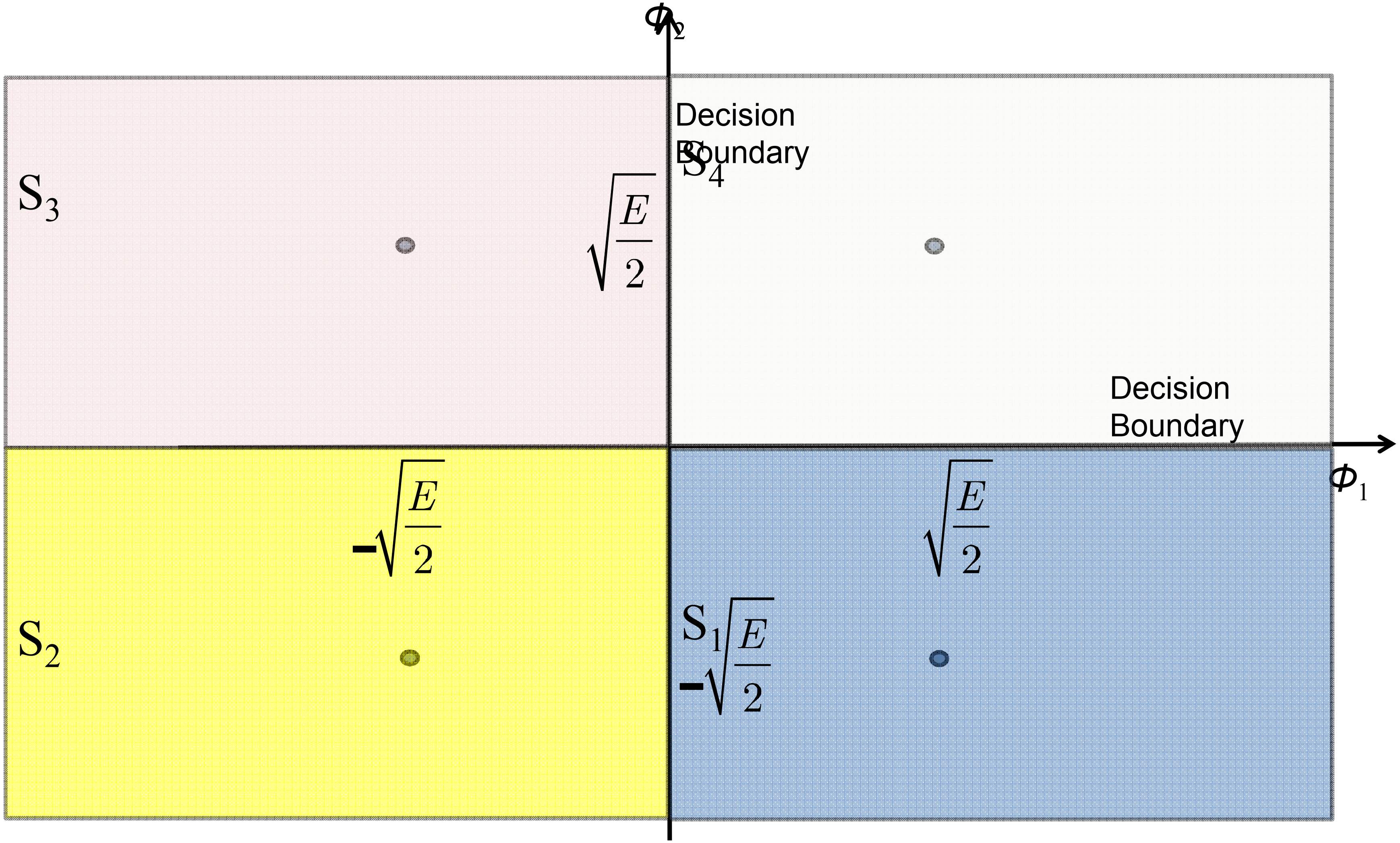
$$\phi_1 = \sqrt{\frac{2}{T}}\cos(2\pi f_c t), \qquad \phi_2 = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$s_i(t) = \sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right)\phi_1 - \sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right)\phi_2$$

$$s_1 = \sqrt{\frac{E}{2}}\phi_1 - \sqrt{\frac{E}{2}}\phi_2, \qquad s_2 = -\sqrt{\frac{E}{2}}\phi_1 - \sqrt{\frac{E}{2}}\phi_2$$

$$s_3 = -\sqrt{\frac{E}{2}}\phi_1 + \sqrt{\frac{E}{2}}\phi_2, \qquad s_4 = \sqrt{\frac{E}{2}}\phi_1 + \sqrt{\frac{E}{2}}\phi_2$$

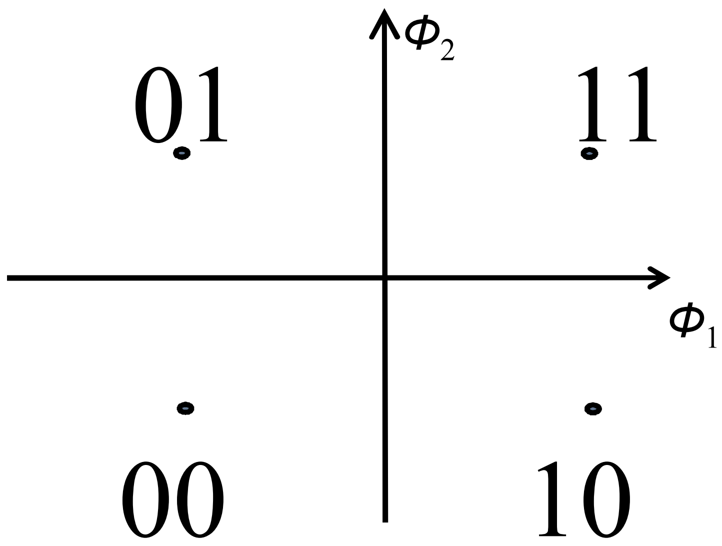
Signal space of QPSK

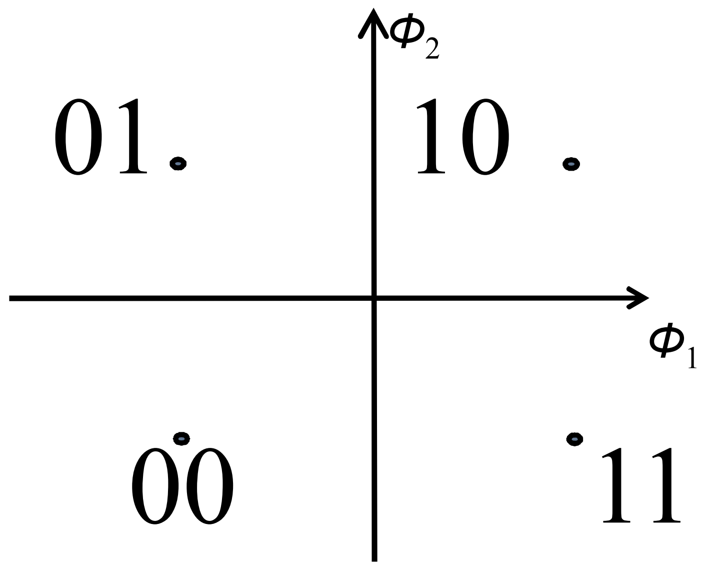


Gray Encoding

How would you assign bits to symbols such that BER is minimized for the same SER?

Which assignment is better?





Error probability of QPSK

$$x(t) = s_i(t) + w(t)$$

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)\,dt = \int_0^{T_b} (s_i(t) + w(t))\phi_1(t)\,dt = \pm\sqrt{\frac{E}{2}} + w_1$$

$$x_2 = \int_0^{T_b} x(t)\phi_2(t)\,dt = \int_0^{T_b} (s_i(t) + w(t))\phi_2(t)\,dt = \pm\sqrt{\frac{E}{2}} + w_2$$

x_1 and x_2 are independent Gaussian random variables with means

$\pm\sqrt{\frac{E}{2}}$ and $\sigma^2 = \frac{N_0}{2}$

- Coherent QPSK is equivalent to 2 coherent BPSK systems working in parallel & using 2 carriers that are in phase quadrature. x_1 and x_2 can be viewed as the individual O/Ps of the 2 coherent BPSK systems, but note that the signal energy is $E/2$

$$P_e^{'} = 0.5 \ \mathrm{erfc}\left(\sqrt{\frac{E/2}{N_0}}\right)$$

Note that $E_b=E/2$

$$P_e^{'} = 0.5 \ \mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Probability of correctly receiving a symbol= $(1-P`_e)^2$
Probability of symbol error $P_e=2P`_e-P`_e{}^2$