

# Digital Communications (ELC 325b)

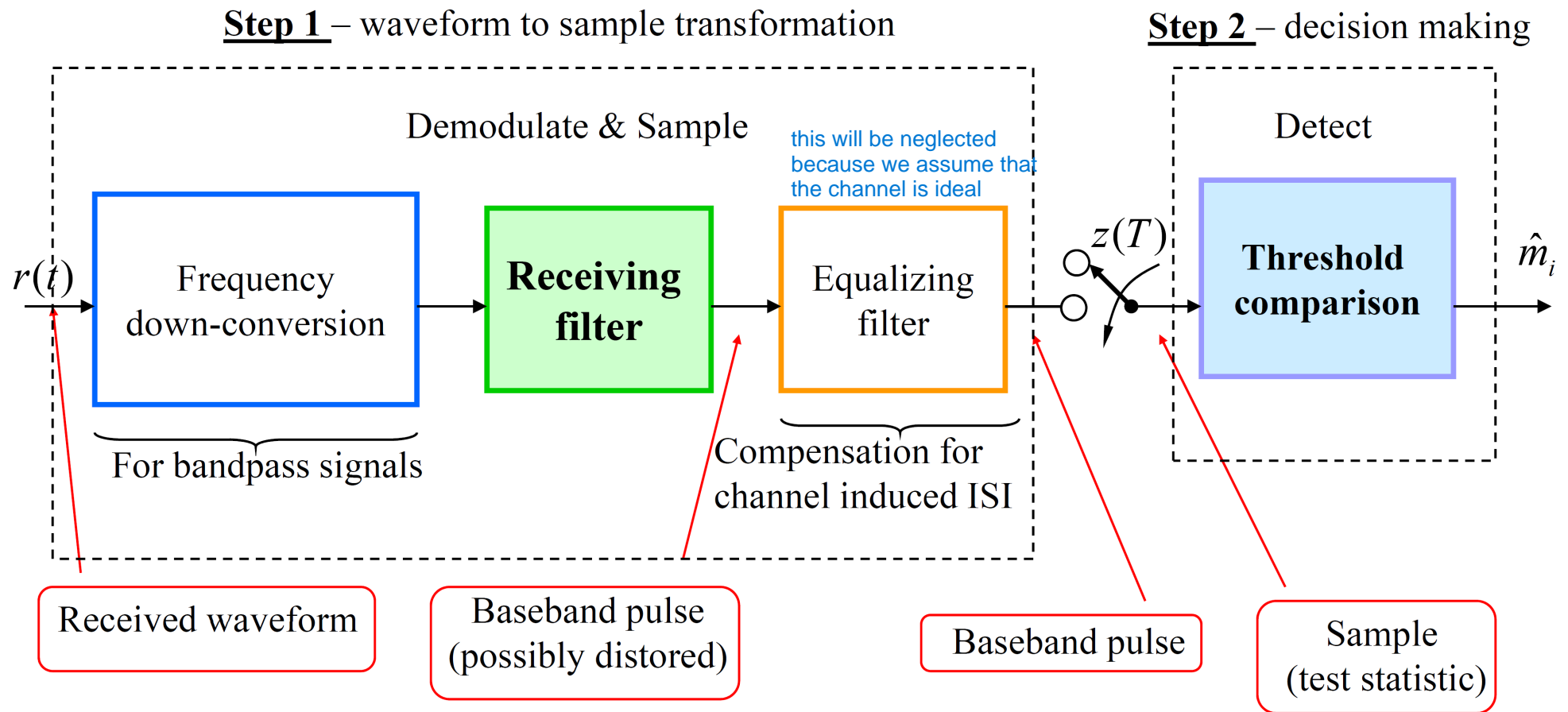
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- 1 Base-Band Transmission and Structure of Optimum Receiver
  - Introduction on Base-Band Transmission
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    - Matched Filter
    - Correlator Receiver
    - Error Rate Calculations
  - Inter-Symbol Interference
  - Design of Optimum ISI-Free Communication System

# Structure of Receiver



# Base-Band Pulse Transmission

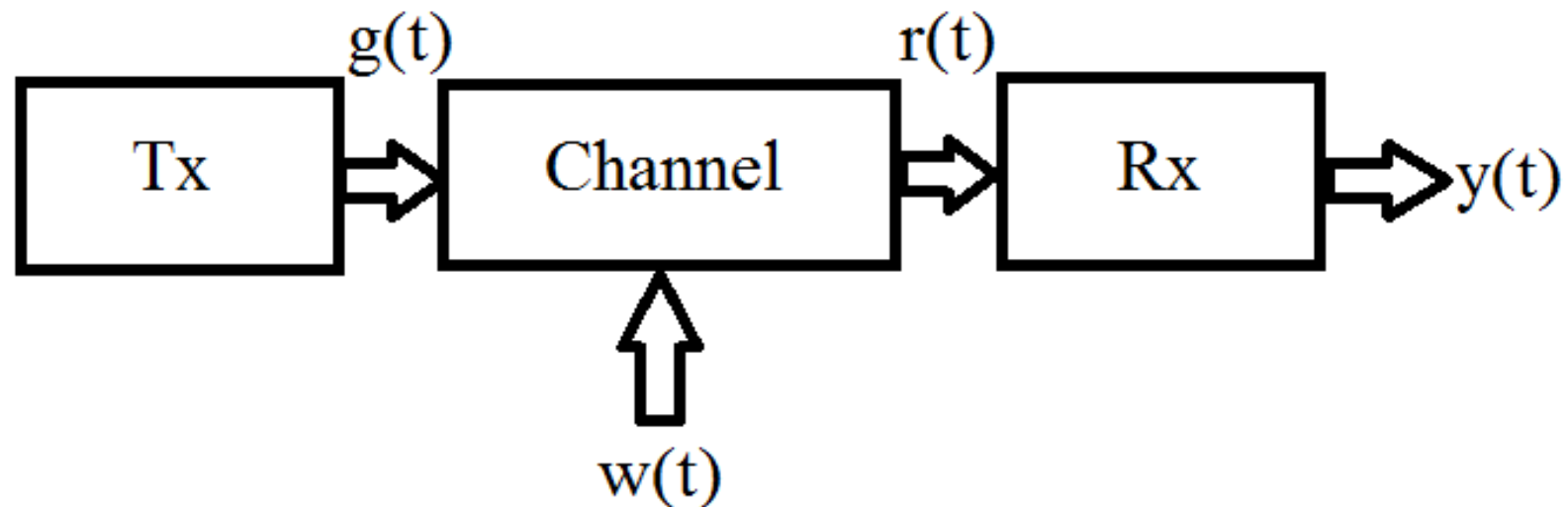
el hadaf mn el lec de enna ne3ml design ll reciever.  
w n3ml maximization ll SNR

## Characteristics

- Digital data have a broad spectrum with low frequency content
- Base-Band transmission of digital data requires the use of low-pass channel with large bandwidth
- Generally, channels are not ideal and are rather dispersive
- Transmission over non-ideal channels causes that the received pulse is affected by adjacent pulses causing inter-symbol interference

# Design of Optimum Receiver in AWGN Channel

ms2lt optimization



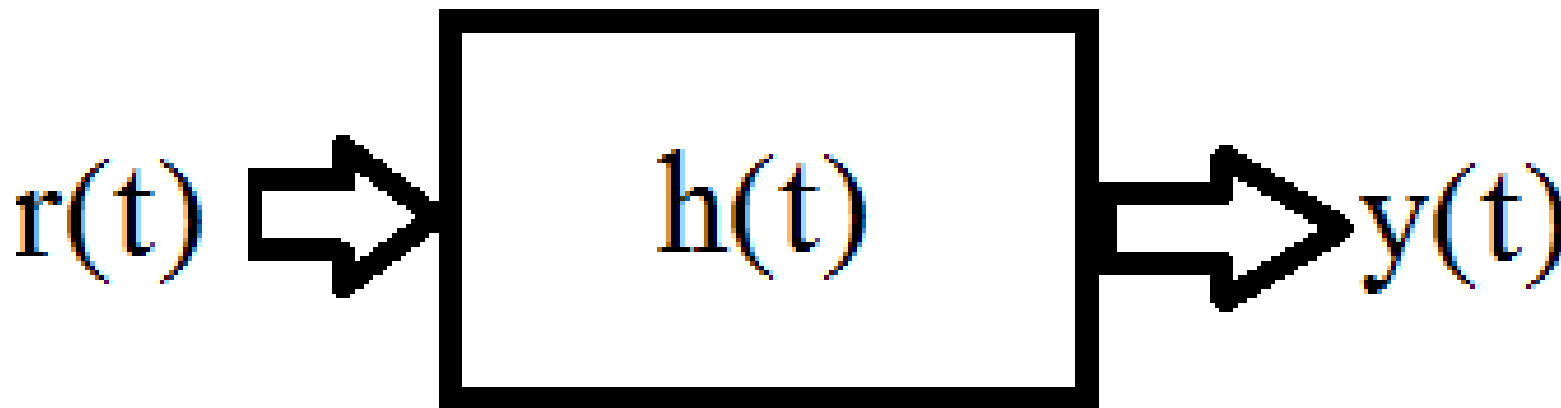
- The receiver receives a pulse signal,  $r(t)$ , of known waveform,  $g(t)$ , immersed in AWGN,  $w(t)$
- The receiver should be able to detect the pulse shape ( $g(t)$  or  $-g(t)$ ) irrespective the noise added from the channel
- For now, we assume the channel is **not bandlimited**, i.e. the pulse shape is not distorted, but may be scaled
- It is assumed the receiver is a filter  $h(t)$

# Design of Optimum Receiver in AWGN Channel

## ✓ Optimality Criteria

The optimality of the receiver design can be based on:

- 1 **Bit Error Rate (BER)** = Probability of errors in the received bits  
An optimum receiver (filter), minimizes the BER
- 2 **Signal-to-Noise Ratio (SNR)** = Signal power to noise power  
An optimum receiver (filter), maximizes the SNR



# Matched Filter

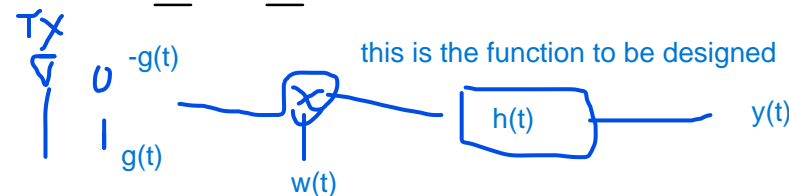
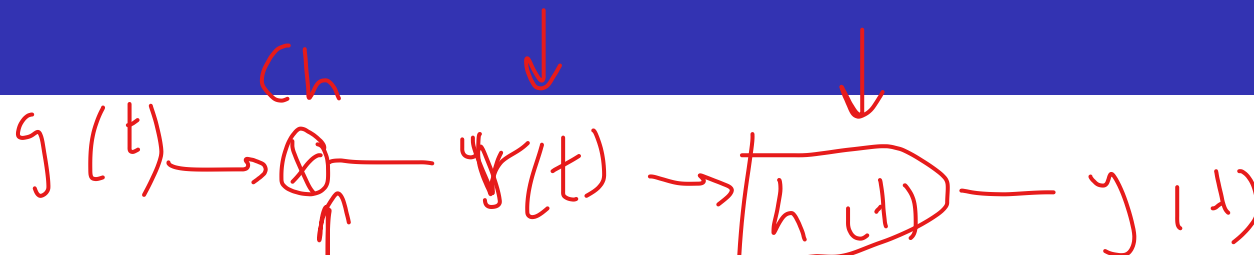
## 1 The Filter Input

$$r(t) = g(t) + w(t), \quad 0 \leq t \leq T$$

## 2 The Filter Output

$$\begin{aligned} y(t) &= r(t) * h(t) \\ &= g(t) * h(t) + w(t) * h(t) \\ &= g_o(t) + n(t) \end{aligned}$$

$$G(f)H(f)$$



ehna 3auzen n3ml max II SNR, 34an a3ml threshold mazbot, lw akbur mno a2ol eny ba3t 1 gher kda ba3t 0 and so on.

It is required that the receiver causes the instantaneous power of the output signal  $g_o(t)$  measured at  $t = T$  as large as possible compared to the average power of the output noise  $n(t)$ .

That is equivalent to **maximizing the peak pulse SNR**

$$G \cdot H$$

$$\eta = \frac{|g_o(T)|^2}{\mathcal{E}\{|n(t)|^2\}}$$

hwa da el rkam el ana 3auzo a3mlo maximization

el goz2 el fo2 msh expected, l2n hwa aslun deterministic fna aslun 3arf kemto.

# Matched Filter

- The output signal



$$g_o(t) = \mathcal{F}^{-1}\{G(f)H(f)\}$$

$$= \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi ft} df$$

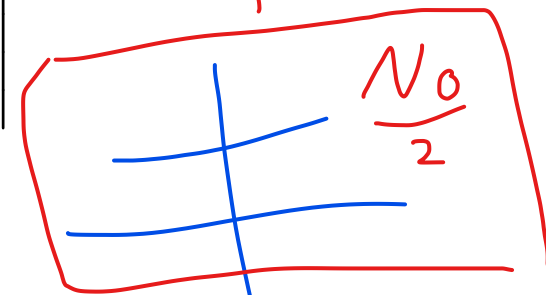
$$|g_o(t)|^2 = \left| \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi ft} df \right|^2$$

$$|g_o(\underline{T})|^2 = \left| \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT} df \right|^2$$

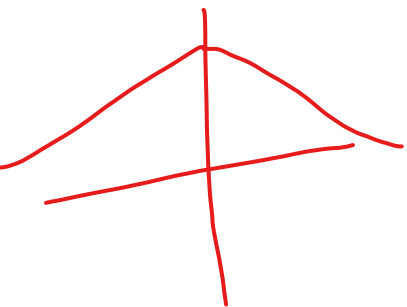
AWGN

$G_f = W(f)$

enta bt7sb el average 34an el noise de random, fa bt7sb el power mn el average bta3ha.



- The output noise



$$S_N(f) = |H(f)|^2 S_W(f) = |H(f)|^2 \frac{N_0}{2}$$

$$\mathcal{E}\{|n(t)|^2\} = \int_{-\infty}^{\infty} S_N(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df$$

$\frac{N_0}{2}$



# Matched Filter

BER  
SNR

$$\eta = \frac{\left| \int_{-\infty}^{\infty} G(f) H(f) e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

$$P = \int_{-\infty}^{\infty} |G(f)|^2 df$$

## Cauchy-Schwarz Inequality

$$\left| \int_{-\infty}^{\infty} \phi_1(x) \phi_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx$$

Equality hold iff  $\phi_1(x) = k \phi_2^*(x)$

$$\left| \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi fT} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |G(f)|^2 df$$

lw 3auz tll3 a7sn 7aga momkena, eshtghl 3nd el equality

# Matched Filter

bndrb el kema el soghyr fe kema soghyra 34an nesbt el error 3la el small signal btb2a akbur, fa da by5lene el mfrod m3tmdsh bshkl kber 7aga bnbsa kbera htll3 ghlr, w nfs el klam 3la el large signal bndrbha fe kema kbera 34an ta2ser el noise 3leha 8alebn msh hayeb2a kber,

$$\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df \quad \text{the energy of the signal}$$

$$\eta_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df = \frac{2}{N_0} \int_{-\infty}^{\infty} |f(t)|^2 dt \quad \text{v.g. } (t-1)$$

The maximum SNR is achieved when  $H(f) = kG^*(f)e^{-j2\pi fT}$

## Matched Filter

34an ne3ml el matched filter  
bn3ml reverse  
w ne3ml shift ymen b T

$$H_{\text{opt}}(f) = kG^*(f)e^{-j2\pi fT} \quad \text{this is the solution for the optimal filter}$$

hadfo enk t2ll effect el noise 3la ad ma t2dr.

$$h_{\text{opt}}(t) = \frac{kg(T-t)}{E}$$

Note:  
enta bt3ml reverse 34an tgeb shkl el matched filter  
w b3d keda bt3ml reverse 34an t3rf te3ml convolution

$$\eta_{\max} = \frac{E}{N_0/2}$$

ehna fl awl bn3ml el desin bta3 el matched filter  
b3d kda bnkhdu 34an nedrbo fl signal 34an ne2dr ntl3 el output

# Matched Filtes

## Properties of Matched Filter

- 1 The impulse response  $h_{opt}(t)$  is uniquely defined by the waveform of the pulse signal  $g(t)$ , the time delay  $T$  and a scaling factor  $k$ .
- 2 The peak pulse SNR of the MF depends only on the ratio of the signal energy to the PSD of the white noise at the filter input.

$$\begin{aligned} G_o(f) &= H_{opt}(f)G(f) = k |G(f)|^2 e^{-j2\pi fT} \\ g_o(T) &= \int_{-\infty}^{\infty} G_o(f) e^{j2\pi fT} df \\ &= k \int_{-\infty}^{\infty} |G(f)|^2 df \\ &= k \int_{-\infty}^{\infty} |g(t)|^2 dt = kE \\ \mathcal{E}\{|n(t)|^2\} &= \frac{N_0}{2} k^2 E \quad \Rightarrow \quad \eta_{\max} = \frac{E}{N_0/2} \end{aligned}$$

# Correlator Receiver

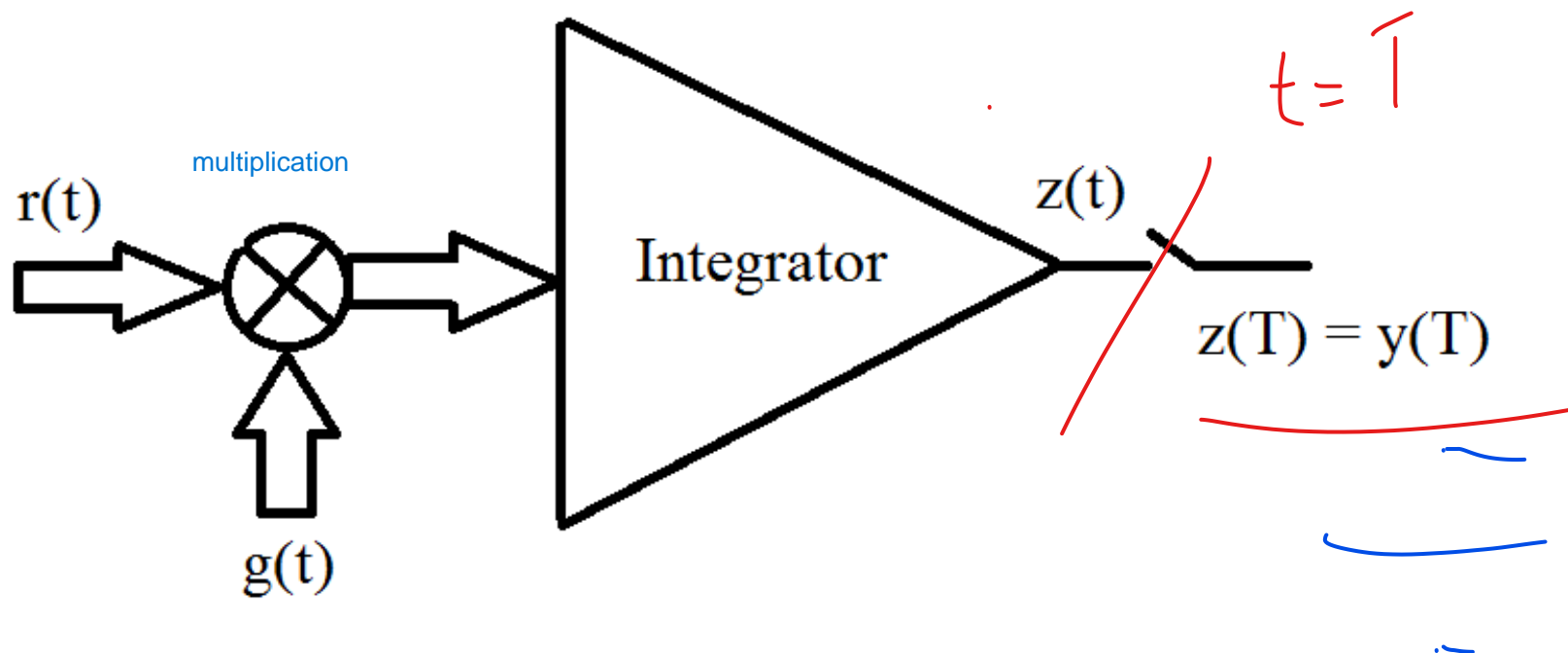
eny a3ml convolution lel matched filter m3 el signal el gyaly 34an ashel el noise.

$$h(t) = g(T - t)$$

$$y(t) = r(t) * h(t)$$

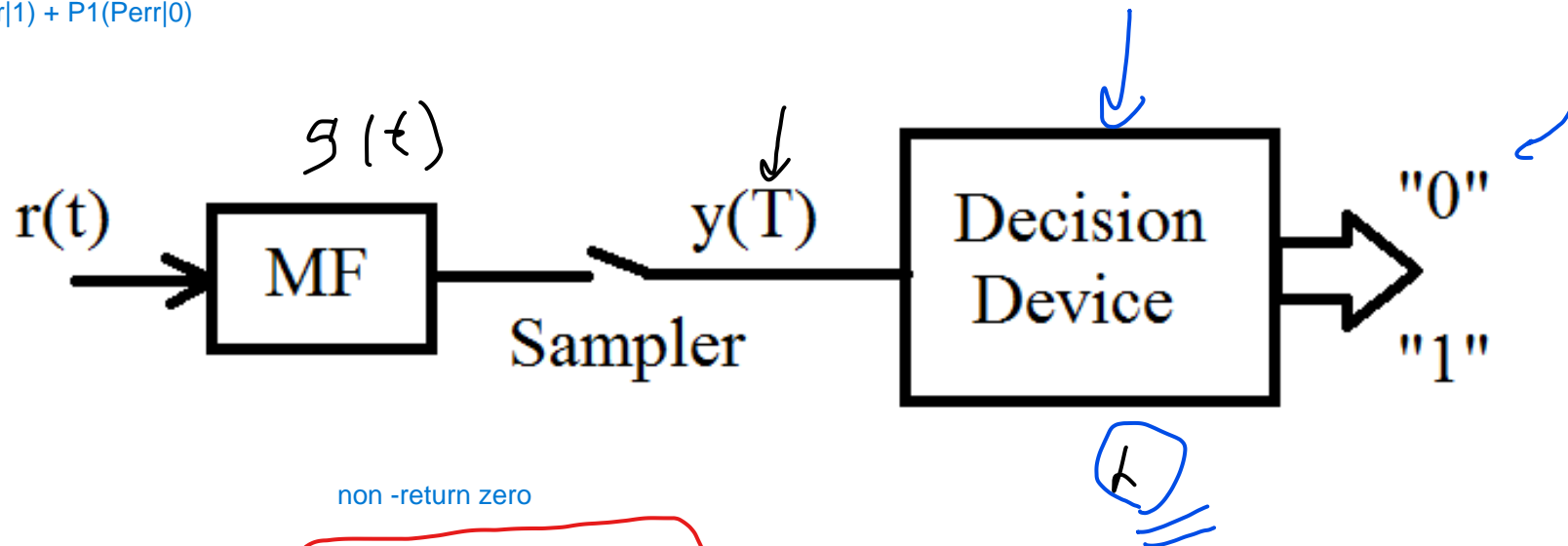
$$= \int_{-\infty}^{\infty} r(\tau) h(t - \tau) d\tau$$

$$y(T) = \int_{-\infty}^{\infty} r(\tau) h(T - \tau) d\tau$$
$$= \int_{-\infty}^{\infty} \underline{r(\tau)} \underline{g(\tau)} d\tau \quad \checkmark$$



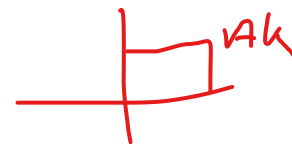
# Error Rate Calculations

$$P_e = P_0P(\text{err}|1) + P_1P(\text{err}|0)$$



Considering a Polar NRZ signaling,

$$r(t) = \begin{cases} +A + w(t), & \text{for bit '1', } 0 \leq t \leq T_b \\ -A + w(t), & \text{for bit '0', } 0 \leq t \leq T_b \end{cases}$$



The receiver is required to make a decision for each signaling interval

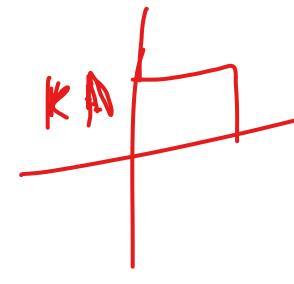
**Note:** For this signaling, the MF is matched to a rectangular pulse ( $A, T_b$ )

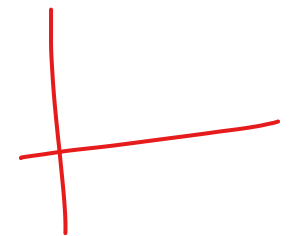
The filter output is sampled at the end of each signaling interval

The sample values are compared to a preset threshold  $\lambda$  to make a decision

# Error Rate Calculations

$$\begin{aligned}
 y(T_b) &= \int_{-\infty}^{\infty} r(\tau)g(\tau)d\tau \\
 &= \int_0^{T_b} \underline{kAr(t)}dt \\
 &= \int_0^{T_b} \frac{1}{T_b} r(t)dt
 \end{aligned}$$

$g(t) = KA$ 


$\boxed{kAT_b = 1}$ 


Then,

$$y = y(T_b) = \pm A + n(t), \quad n(t) = \frac{1}{T_b} \int_0^{T_b} w(t)d\tau$$

**Note:**

$\underline{n(t)}$  is Gaussian distributed, with zero mean and variance  $\sigma^2 = \frac{1}{T_b} N_0/2$

$y(T_b)$  is Gaussian distributed, with  $\pm A$  mean and variance  $\sigma^2 = \frac{1}{T_b} N_0/2$

# Error Rate Calculations

The conditional PDF of the sampled output signal is expressed as

$$p(y|'0') = \frac{1}{\sqrt{\pi N_0/T_b}} \exp \left[ -\frac{(y + A)^2}{N_0/T_b} \right] \quad (1)$$

$$p(y|'1') = \frac{1}{\sqrt{\pi N_0/T_b}} \exp \left[ -\frac{(y - A)^2}{N_0/T_b} \right] \quad (2)$$

Assume bit '0' was transmitted, a decision is considered erroneous if the receiver decides that bit '1' was transmitted. The receiver makes such decision if  $y > \lambda$ . The probability of such decision is

$$P(e|'0') = P\{y > \lambda|'0'\} = \int_{\lambda}^{\infty} p(y|'0') dy$$

Similarly,

$$P(e|'1') = P\{y < \lambda|'1'\} = \int_{-\infty}^{\lambda} p(y|'1') dy$$

34an bnbd2 el evaluation mn 3nd  
el A, w b3de

# Error Rate Calculations

## Probability of Error if '0' was Transmitted

$$\begin{aligned} P(e|'0') &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp \left[ -\frac{(y+A)^2}{N_0/T_b} \right] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{\lambda+A}{\sqrt{N_0/T_b}}}^{\infty} \exp [-z^2] dz \quad \Leftarrow \left[ z = \frac{y+A}{\sqrt{N_0/T_b}} \right] \\ &= \frac{1}{2} \operatorname{erfc} \left( \frac{\lambda+A}{\sqrt{N_0/T_b}} \right) \end{aligned}$$

Note:

$$\begin{aligned} \operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \\ \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \end{aligned}$$

0  
1



# Error Rate Calculations

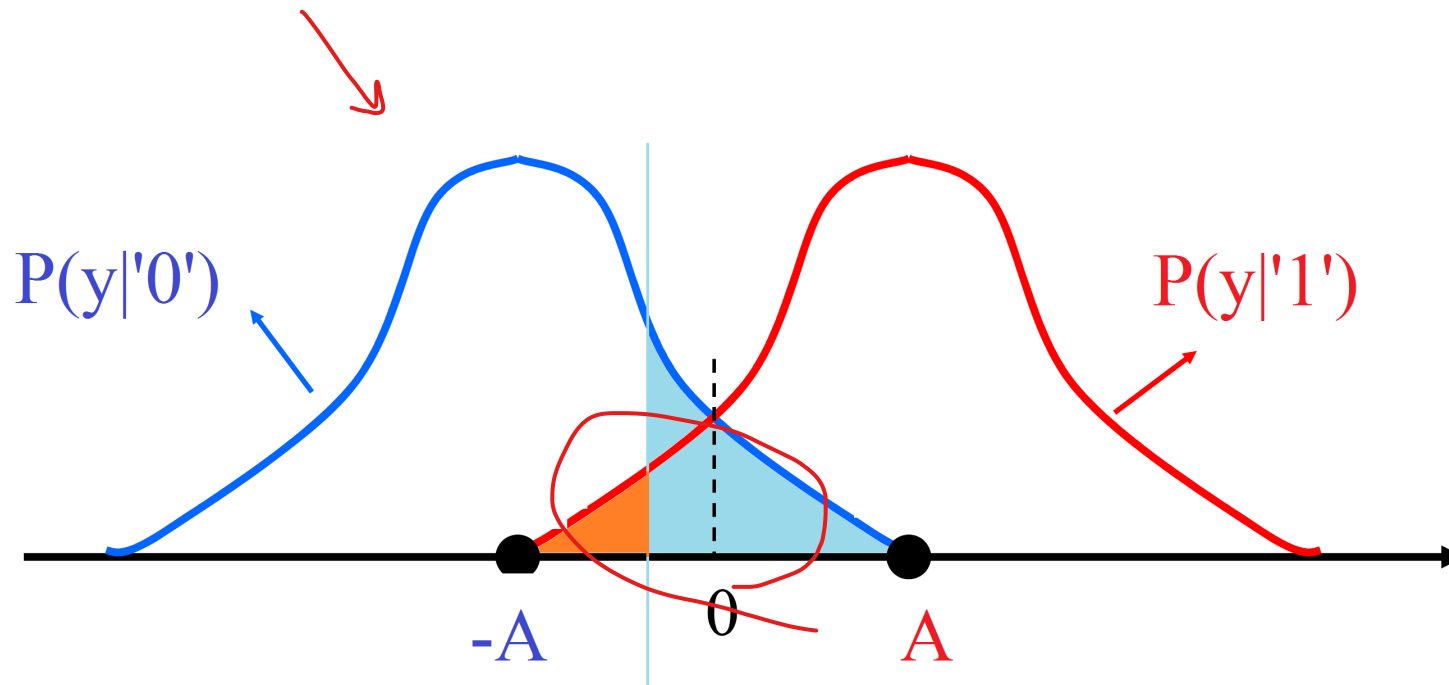
## Probability of Error if '1' was Transmitted

$$\begin{aligned} P(e|'1') &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{-\infty}^{\lambda} \exp \left[ -\frac{(y-A)^2}{N_0/T_b} \right] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{-\lambda+A}{\sqrt{N_0/T_b}}}^{\infty} \exp [-z^2] dz \quad \Leftarrow \left[ z = -\frac{y-A}{\sqrt{N_0/T_b}} \right] \\ &= \frac{1}{2} \operatorname{erfc} \left( \frac{-\lambda+A}{\sqrt{N_0/T_b}} \right) \end{aligned}$$

Note:

$$\begin{aligned} \operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \\ Q(x) &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \end{aligned}$$

# Error Rate Calculations



# Error Rate Calculations

## Average Error Probability

$$P(e) = P(e|'0')P('0') + P(e|'1')P('1') = f(\lambda)$$

In order to minimize the average error probability,  $\lambda$  should be optimally chosen. This is achieved for

$$\lambda_{opt} = \frac{N_0}{4AT_b} \ln \left( \frac{P('0')}{P('1')} \right)$$

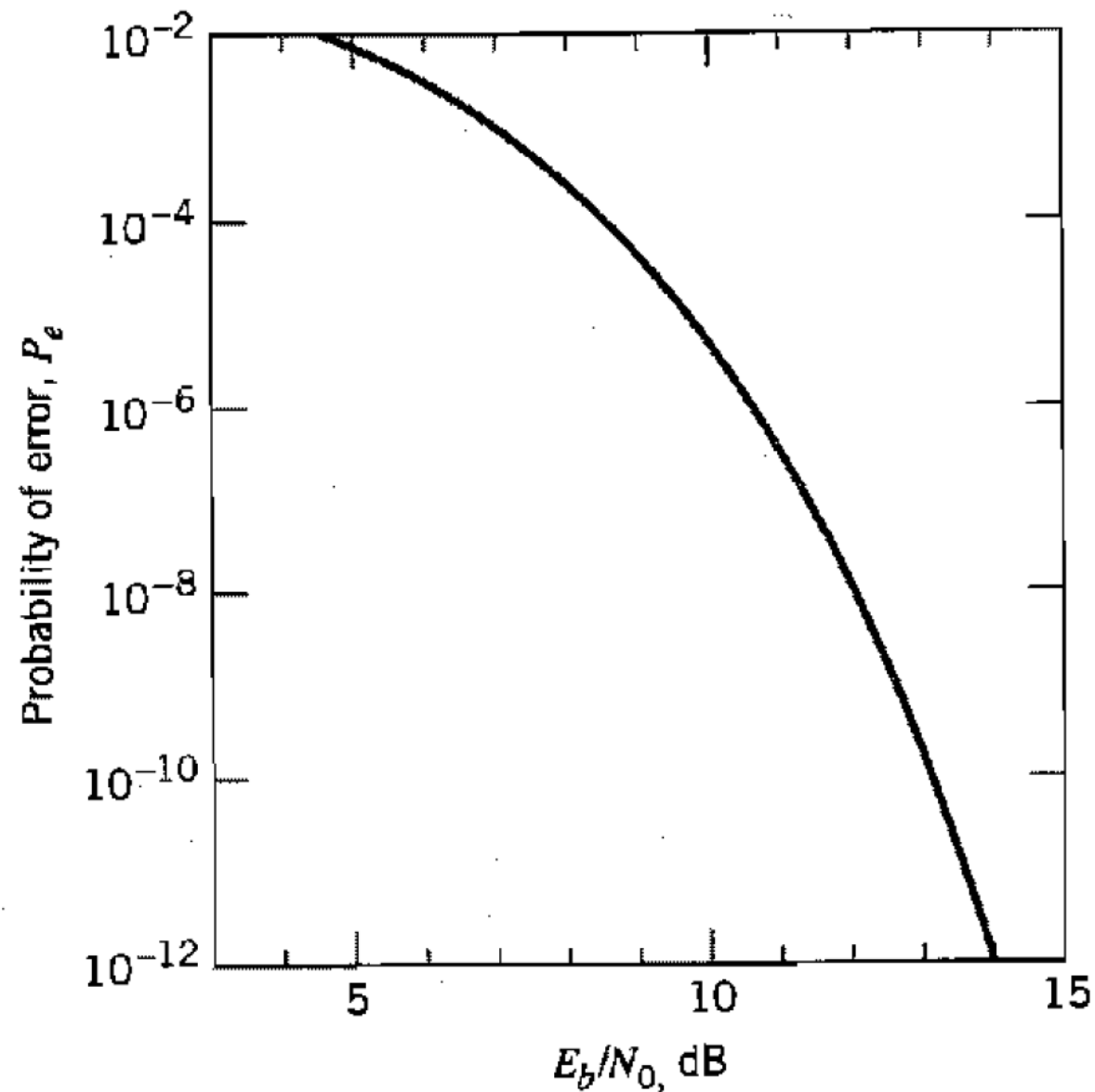
mantkya gedan, da l2n enta lw el probability bta3t enk tb3t 0 zyha zy el probabiltiy enk tb3t 1, sa3tha enta msh htfdl 7d 3n 7d, fa 5las khud el lambda 0, lakn lw el P(1) kan hwa akbur da m3nah en forst enk tb2t 1 akbur mn forst enk ttl3 0, sa3tha el lambda btro7 lel -ve 34an ana el mfrod a3ml avoid le eny a3ml decode lel 1 ghalat l2no byegy aktur. bs da brdu by5ly nesbt eny a3ml wrong evaluation lel 0 akbur, bs still e7tmalyt en el 0 tegy aslun olayela.

## Special Case

If  $P('0') = P('1') = 0.5$ , then  $\lambda_{opt} = 0$ . In this case

$$P(e) = P(e|'0') = P(e|'1') = \frac{1}{2} \operatorname{erfc} \left( \frac{A}{\sqrt{N_0/T_b}} \right) = \frac{1}{2} \operatorname{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right)$$

# Error Rate Calculations



# Error Rate Calculations

## Notes

- 1 Binary Symmetric Channel

$$P(e|'0') = P(e|'1')$$

- 2 The average error probability decreases rapidly as  $E_b/N_0$  increases
- 3 If  $P('0') \gg P('1')$ ,  $\lambda_{opt} \approx \infty$  in order to reduce  $P(e|'0')$
- 4 If  $P('0') \ll P('1')$ ,  $\lambda_{opt} \approx -\infty$  in order to reduce  $P(e|'1')$

# References



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# Thank You

Questions ?