CMP205: Computer Graphics

Lecture 3: Transformations II

Ahmed S. Kaseb
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Slides by: Dr. Mohamed Alaa El-Dien Aly

Agenda

- Transformations Vs Coordinate Change
- Arbitrary 3D Rotations
- Transforming Normal Vectors
- Coordinate Transformation
- Windowing Transforms

Acknowledgments: Some slides adapted from Steve Marschner and Fredo Durand.

Transformation Vs Coordinate Change

We can view the same rotation matrix in two ways:

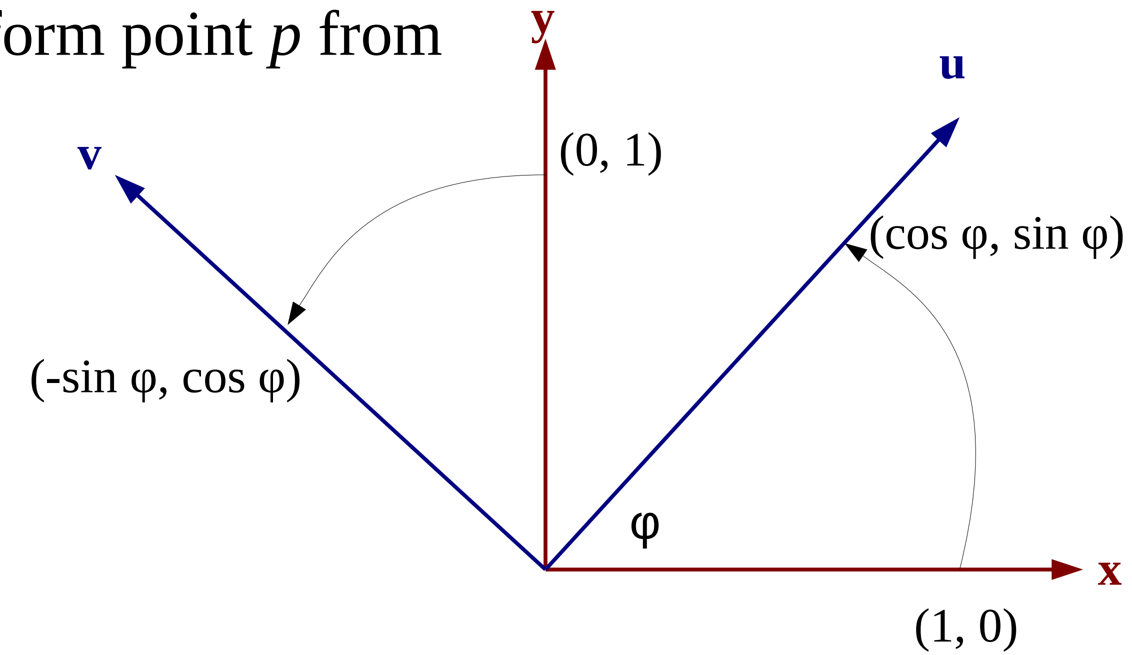
1)As a transformation matrix to transform point p to point p' in the same frame

$$R=\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

2)As a coordinate change to transform point p from frame uv to frame xy

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

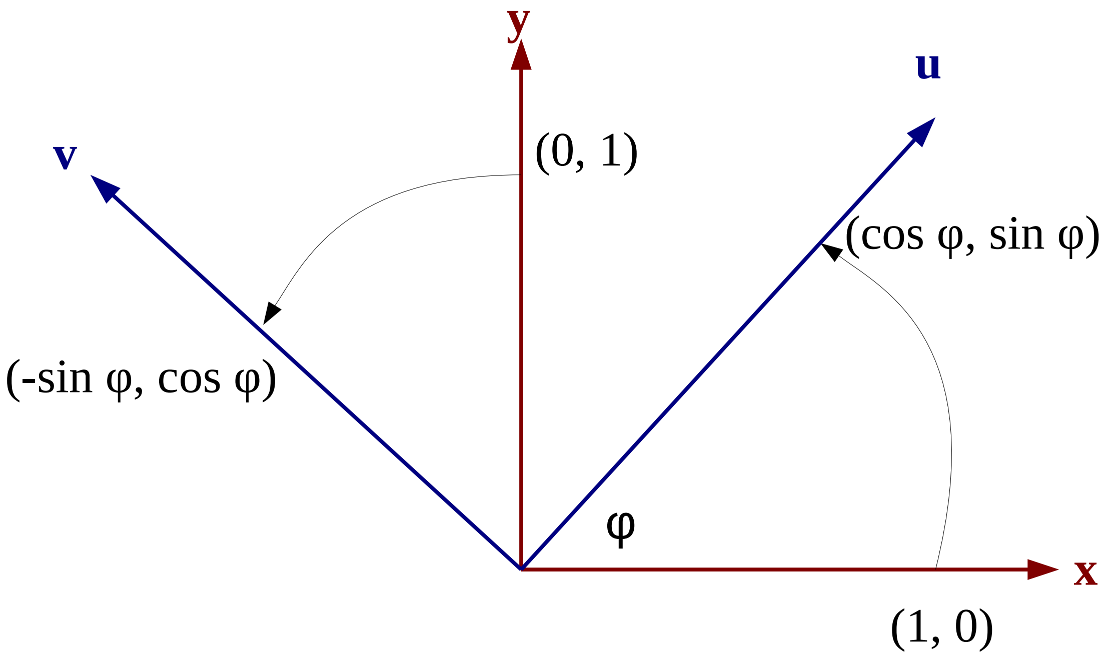
$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$



Transformation Vs Coordinate Change

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Transformation:
$p' = R\,p$



Coordinate Change:
${}^{xy}p = R\,{}^{uv}p$

R transforms points in xy coordinates OR transforms uv coordinates to xy coordinates

What about R^T?

Arbitrary Rotation

A 3x3 unitary matrix can represent arbitrary rotation around any axis

$$R = \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix} = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}$$

$$R\,R^T = I$$

$$Ru = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x$$

R takes (or rotates) uvw to xyz

$$R^T x = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix} = u$$

R^T takes (or rotates) xyz to uvw

Arbitrary Rotation

- To rotate about an arbitrary axis *a* that passes through the origin with an angle φ:
  - Create axes *uvw* s.t. *w* coincides with *a*
  - Change *xyz*-frame to *uvw*-frame using *R* (Recall that *R* rotates *uvw* to *xyz*)
  - Perform the rotation in *uvw* around *w*-axis (vector *a*)
  - Change back to *xyz*-frame using R^T

$$\begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix}$$

Now, how do we know *uvw*?

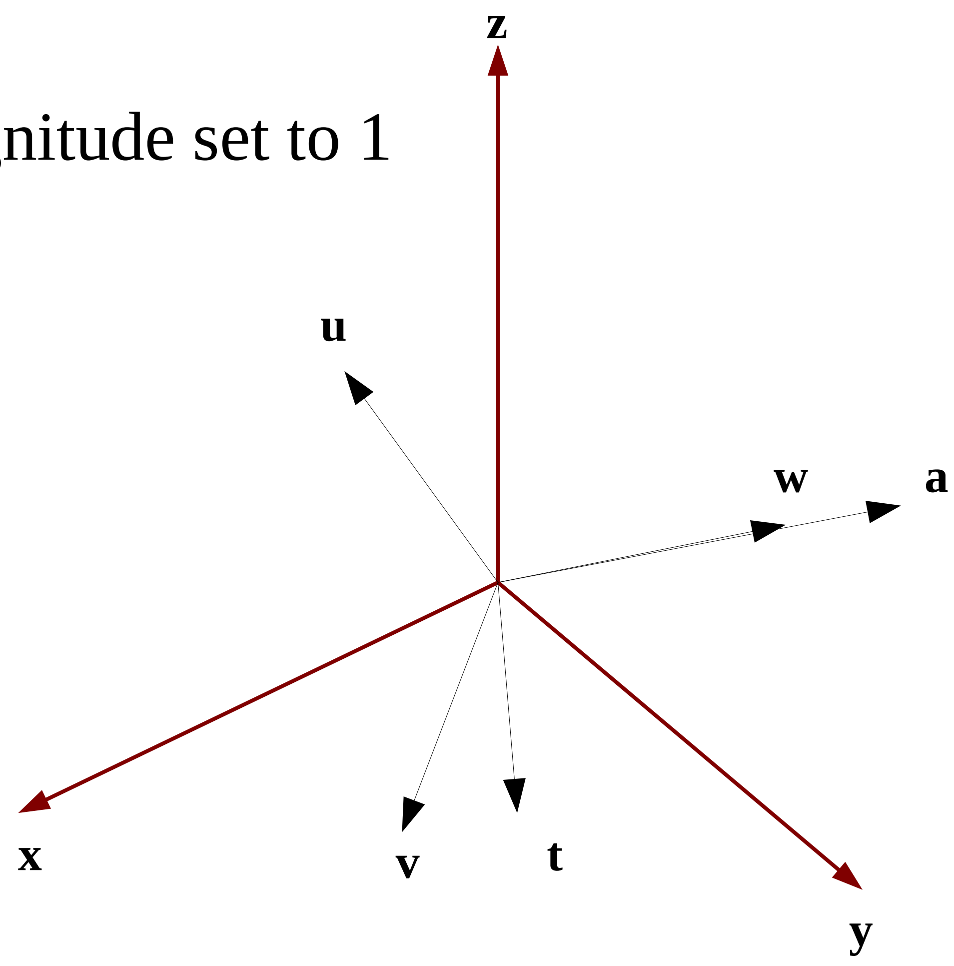
Arbitrary Rotation

$$w = \frac{a}{\|a\|}$$

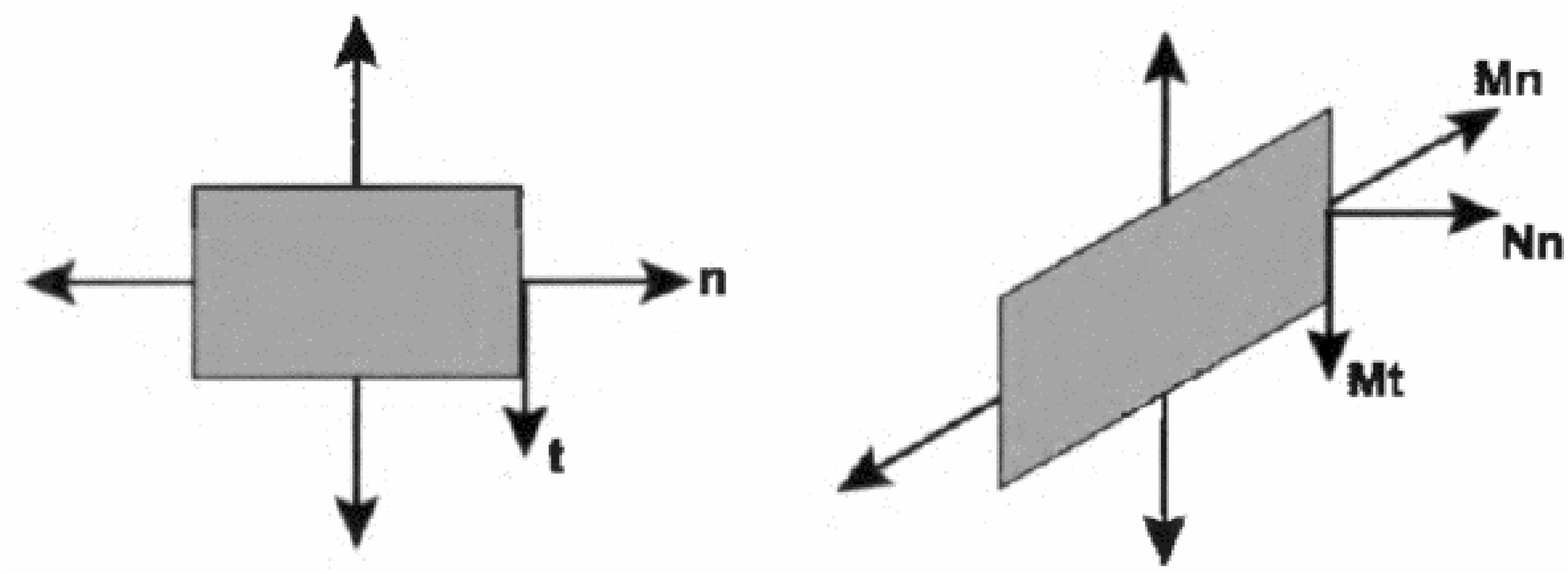
$t = w'$ i.e. w with lowest magnitude set to 1

$$u = \frac{t \times w}{\|t \times w\|}$$

$$v = w \times u$$



Transforming Normal Vectors



Mn is not normal to the surface!

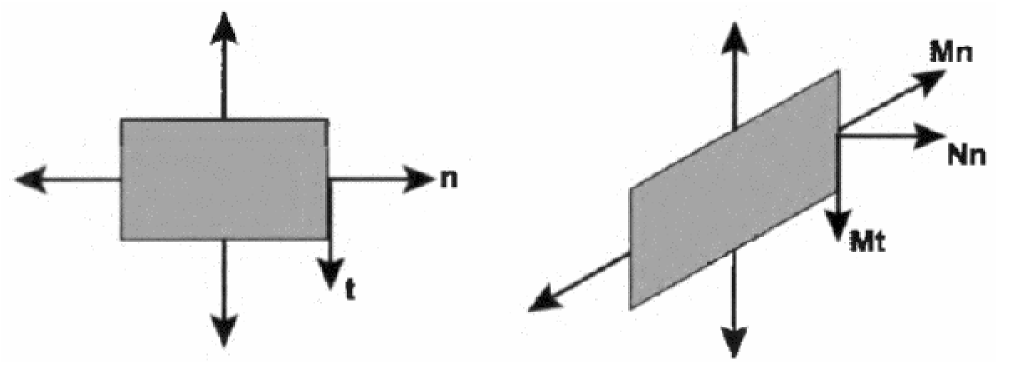
What is N?

Transforming Normal Vectors

Derivation

$n' = N\,n$ and $t' = M\,t$

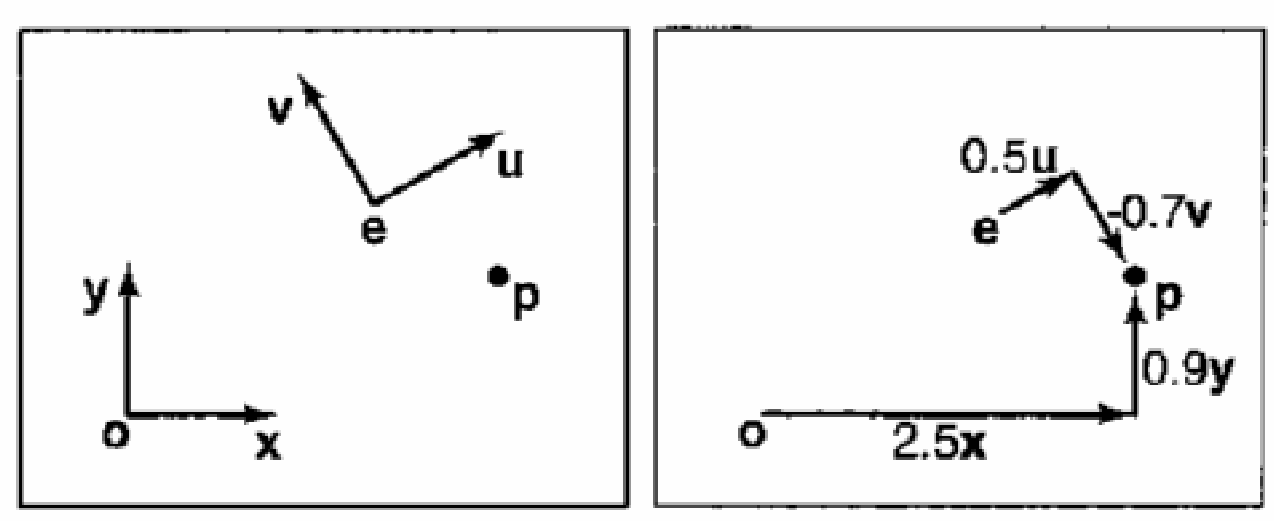
$$n^T t = 0$$
$$n^T M^{-1} M\, t = 0$$
$$\left(n^T M^{-1}\right)\left(M\, t\right) = 0$$
$$\left(\left(M^{-1}\right)^T n\right)^T \left(M\, t\right) = 0$$
$$\left(n'\right)^T t' = 0$$



$$N = \left(M^{-1}\right)^T$$

Coordinate Transformations

$$\boldsymbol{p} = \boldsymbol{o} + x_p \boldsymbol{x} + y_p \boldsymbol{y}$$
$$\boldsymbol{p} = \boldsymbol{e} + u_p \boldsymbol{u} + v_p \boldsymbol{v}$$

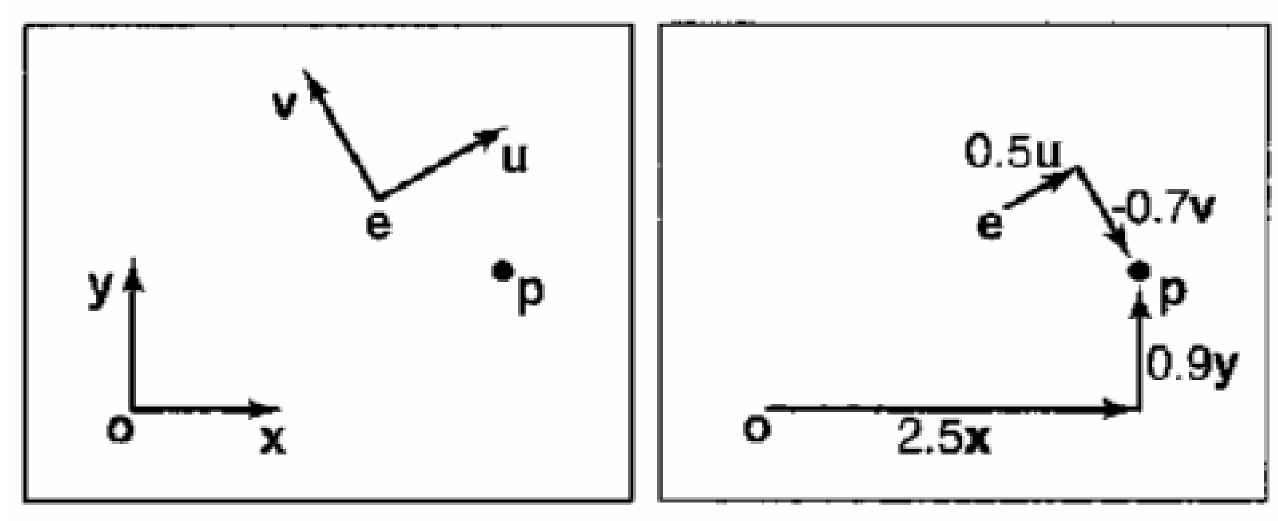


How to find (x_p, y_p) from (u_p, v_p) and vice versa?

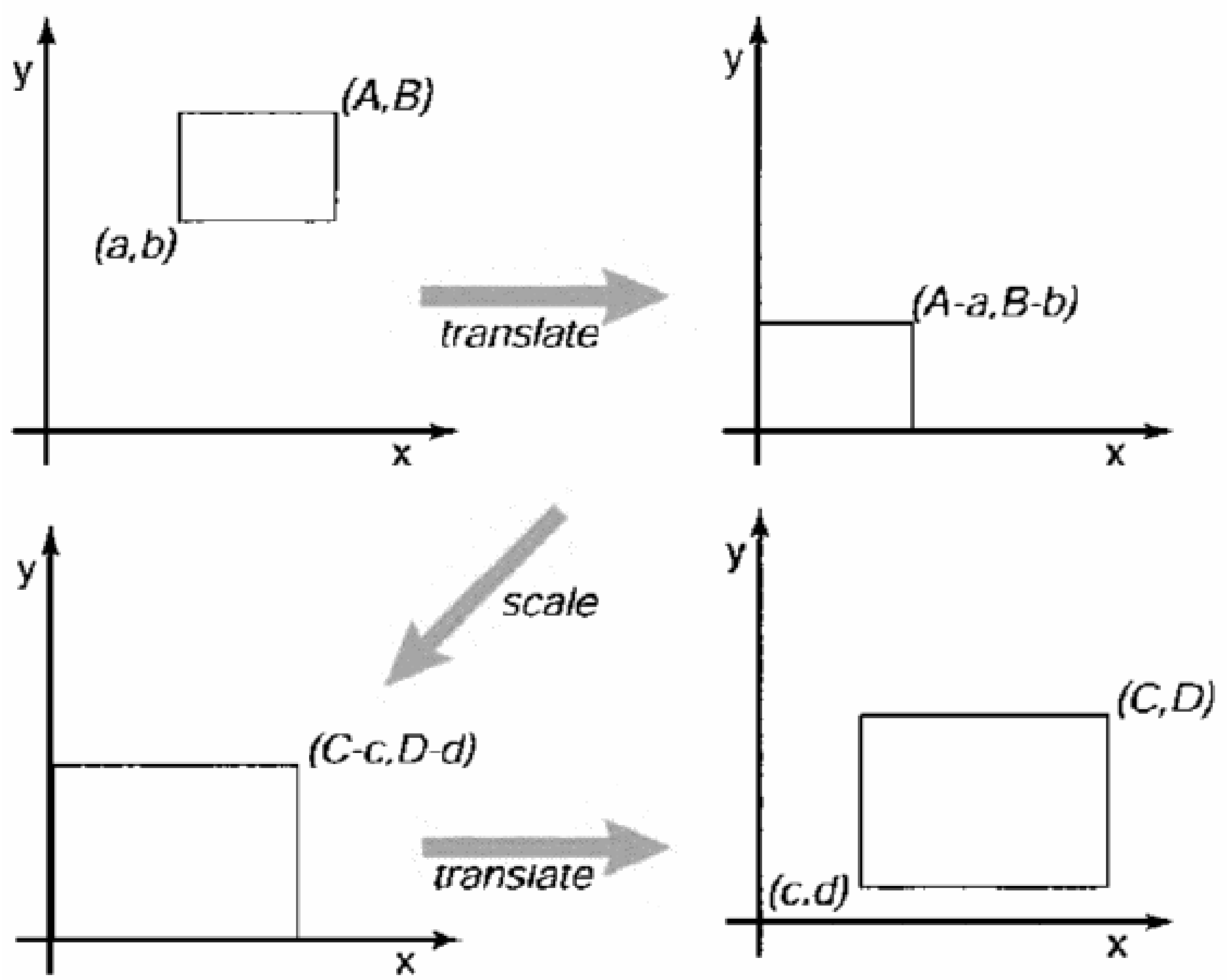
Coordinate Transformations

$$\begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_e \\ 0 & 1 & y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & 0 \\ x_v & y_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_e \\ 0 & 1 & -y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix}$$

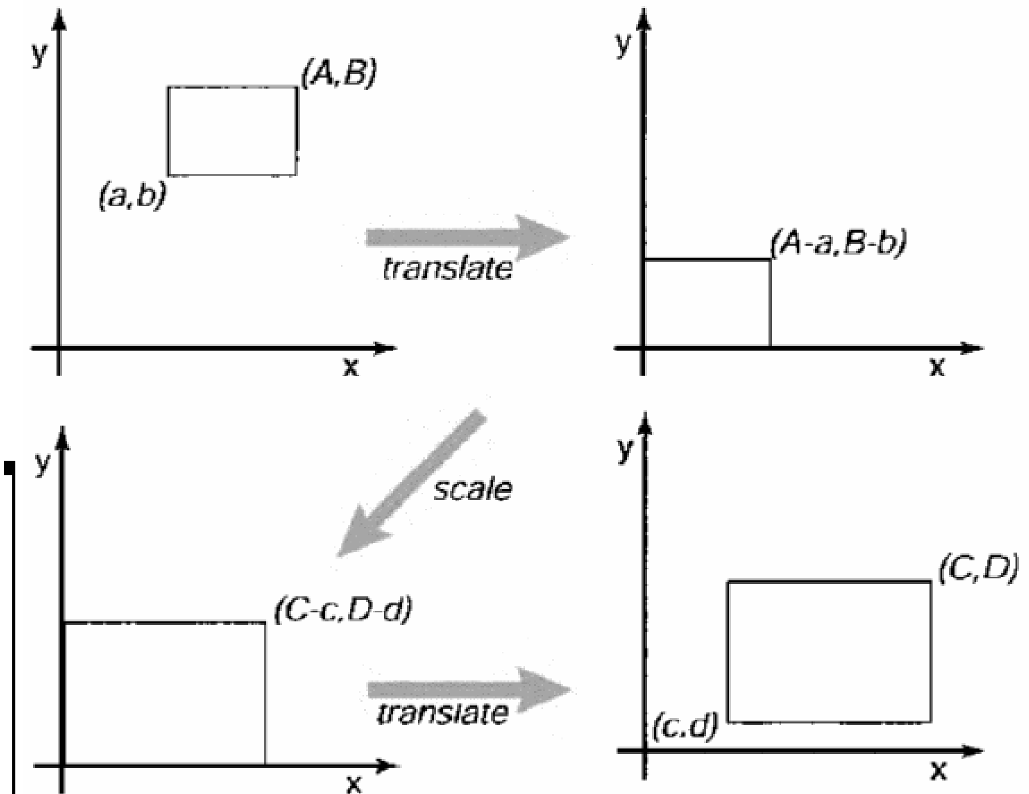


Windowing Transforms



Windowing Transforms

$$\text{translate}(-a,-b)=\begin{bmatrix}1 & 0 & -a\\ 0 & 1 & -b\\ 0 & 0 & 1\end{bmatrix}$$



$$\text{scale}\left(\frac{C-c}{A-a},\frac{D-d}{B-b}\right)=\begin{bmatrix}\frac{C-c}{A-a} & 0 & 0\\ 0 & \frac{D-d}{B-b} & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\text{translate}(c,d)=\begin{bmatrix}1 & 0 & c\\ 0 & 1 & d\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}\frac{C-c}{A-a} & 0 & \frac{cA-Ca}{A-a}\\ 0 & \frac{D-d}{B-b} & \frac{dB-Db}{B-b}\\ 0 & 0 & 1\end{bmatrix}$$

Recap

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