Digital Communications (ELC 325b)

© Samy S. Soliman

Electronics and Electrical Communication Engineering Department
Cairo University, Egypt

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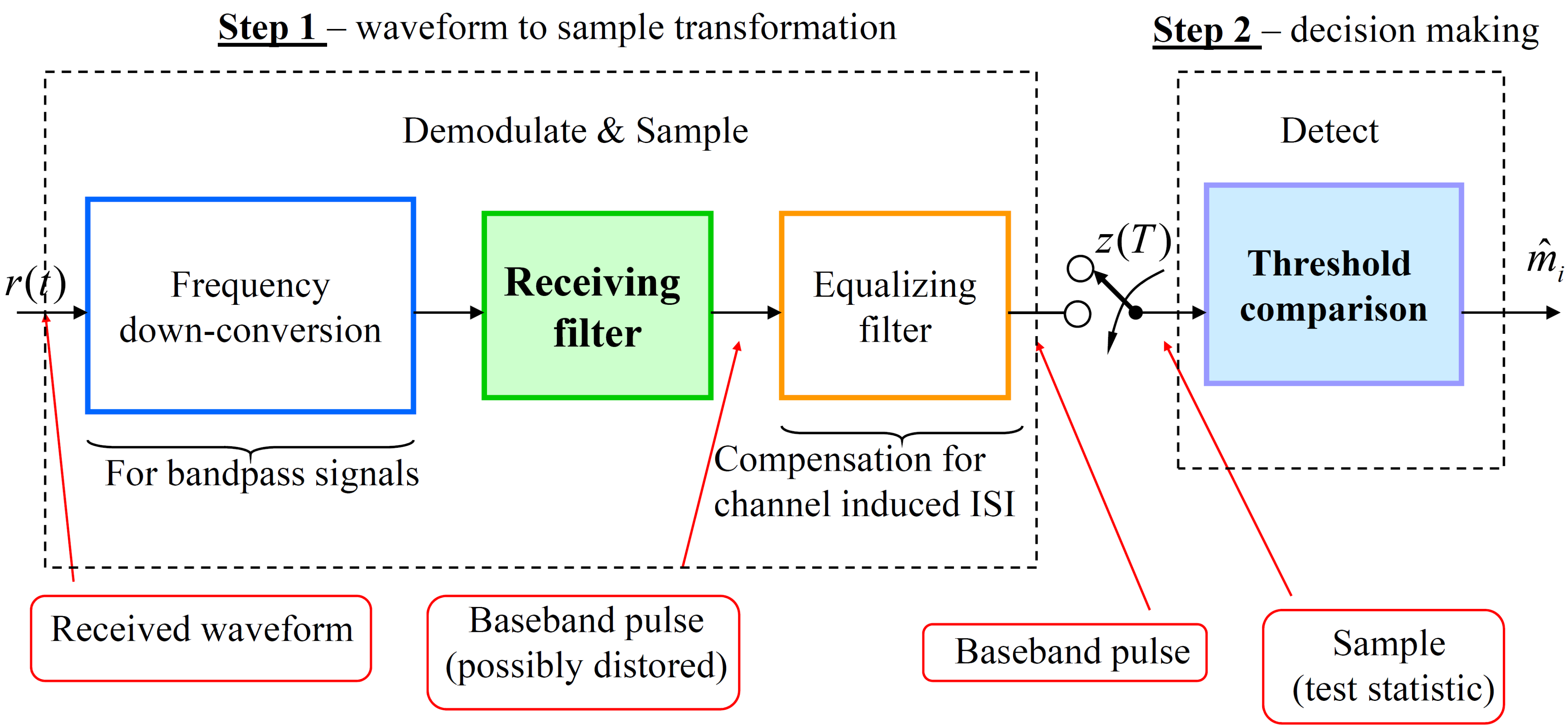
Outline

Structure of Receiver

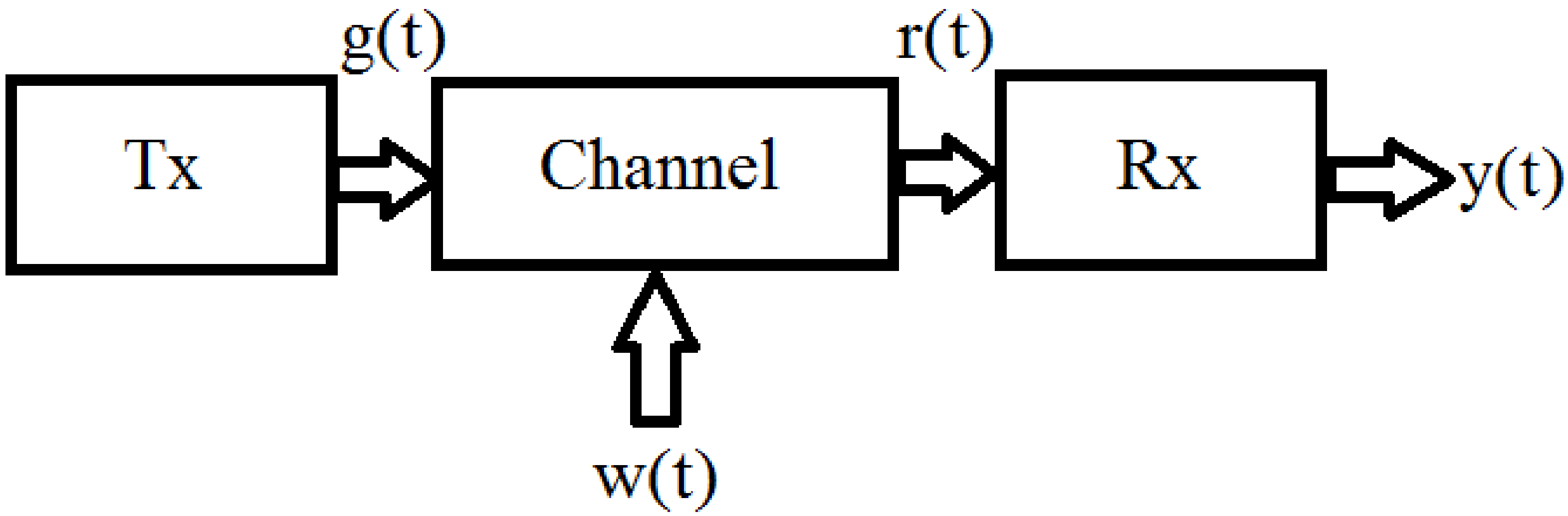


Base-Band Pulse Transmission

Characteristics

- Digital data have a broad spectrum with low frequency content
- Base-Band transmission of digital data requires the use of low-pass channel with large bandwidth
- Generally, channels are not ideal and are rather dispersive
- Transmission over non-ideal channels causes that the received pulse is affected by adjacent pulses causing inter-symbol interference

Design of Optimum Receiver in AWGN Channel

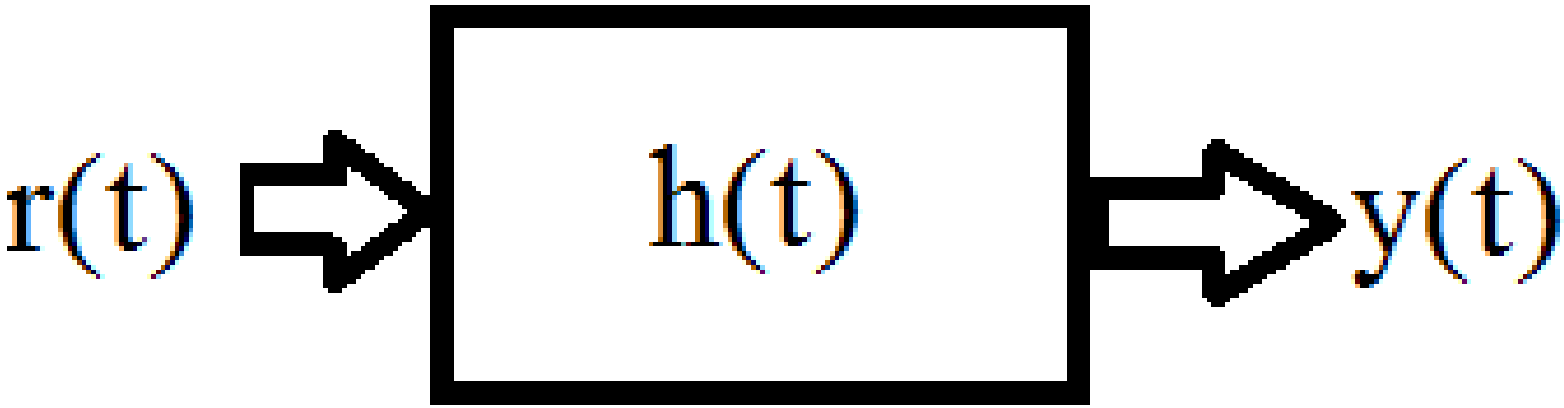


- The receiver receivers a pulse signal, $r(t)$, of known waveform, $g(t)$, immersed in AWGN, $w(t)$
- The receiver should be able to detect the pulse shape ($g(t)$ or $-g(t)$) irrespective the noise added from the channel
- For now, we assume the channel is **not bandlimited**, i.e. the pulse shape is not distorted, but may be scaled
- It is assumed the receiver is a filter $h(t)$

Optimality Criteria

The optimality of the receiver design can be based on:

1. **Bit Error Rate (BER)** = Probability of errors in the received bits
   An optimum receiver (filter), minimizes the BER
2. **Signal-to-Noise Ratio (SNR)** = Signal power to noise power
   An optimum receiver (filter), maximizes the SNR



Matched Filter

1. The Filter Input

$$r(t) = g(t) + w(t), \qquad 0 \leq t \leq T$$

2. The Filter Output

$$\begin{aligned} y(t) &= r(t) * h(t) \\ &= g(t) * h(t) + w(t) * h(t) \\ &= g_o(t) + n(t) \end{aligned}$$

It is required that the receiver causes the instantaneous power of the output signal $g_o(t)$ measured at $t = T$ as large as possible compared to the average power of the output noise $n(t)$.
That is equivalent to **maximizing the peak pulse SNR**

$$\eta = \frac{|g_o(T)|^2}{\mathcal{E}\{|n(t)|^2\}}$$

Matched Filter

- The output signal

$$
\begin{aligned}
g_o(t) &= \mathcal{F}^{-1}\{G(f)H(f)\} \\
&= \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi ft}df \\
|g_o(t)|^2 &= \left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi ft}df\right|^2 \\
|g_o(T)|^2 &= \left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df\right|^2
\end{aligned}
$$

- The output noise

$$
\begin{aligned}
S_N(f) &= |H(f)|^2 S_W(f) = |H(f)|^2\frac{N_0}{2} \\
\mathcal{E}\{|n(t)|^2\} &= \int_{-\infty}^{\infty} S_N(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df
\end{aligned}
$$

Matched Filter

$$\eta = \frac{\left| \int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

Cauchy-Schwarz Inequality

$$\left| \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 \, dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 \, dx$$

Equality hold iff $\phi_1(x) = k\phi_2^*(x)$

$$\left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 \, df \int_{-\infty}^{\infty} |G(f)|^2 \, df$$

Matched Filter

$$\begin{aligned}\eta &\leq \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2\,dx\\ \eta_{\max} &= \frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2\,df = \frac{2}{N_0}\int_{-\infty}^{\infty}|f(t)|^2\,dt\end{aligned}$$

The maximum SNR is achieved when $\boxed{H(f) = kG^*(f)e^{-j2\pi fT}}$

Matched Filter

$$\begin{aligned}H_{opt}(f) &= kG^*(f)e^{-j2\pi fT}\\ h_{opt}(t) &= kg(T-t)\\ \eta_{\max} &= \frac{E}{N_0/2}\end{aligned}$$

Matched Filtes

Properties of Matched Filter

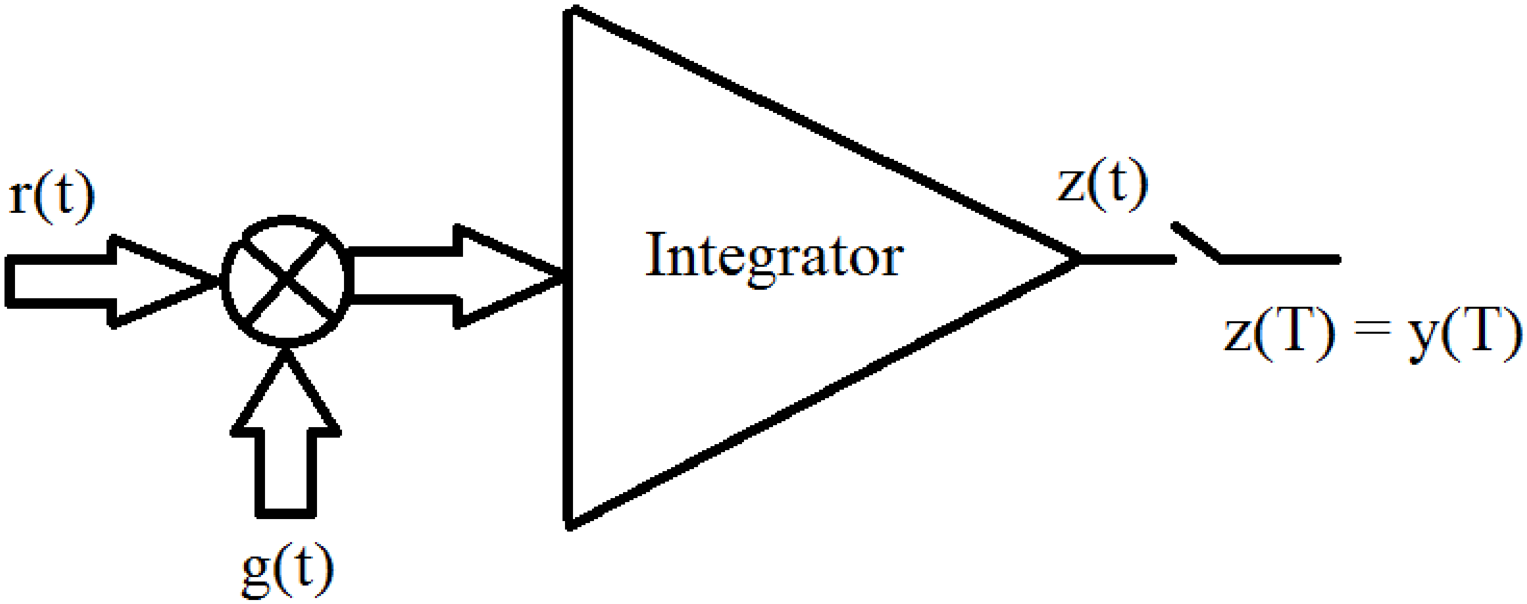
1. The impulse response $h_{opt}(t)$ is uniquely defined by the waveform of the pulse signal $g(t)$, the time delay T and a scaling factor k.
2. The peak pulse SNR of the MF depends only on the ratio of the signal energy to the PSD of the white noise at the filter input.

$$
\begin{aligned}
G_o(f) &= H_{opt}(f)G(f) = k\left|G(f)\right|^2 e^{-j2\pi fT} \\
g_o(T) &= \int_{-\infty}^{\infty} G_o(f)e^{j2\pi fT}\,df \\
&= k\int_{-\infty}^{\infty} \left|G(f)\right|^2 df \\
&= k\int_{-\infty}^{\infty} \left|g(t)\right|^2 dt = kE \\
\mathcal{E}\{|n(t)|^2\} &= \frac{N_0}{2}k^2E \qquad\qquad \Rightarrow \eta_{max} = \frac{E}{N_0/2}
\end{aligned}
$$

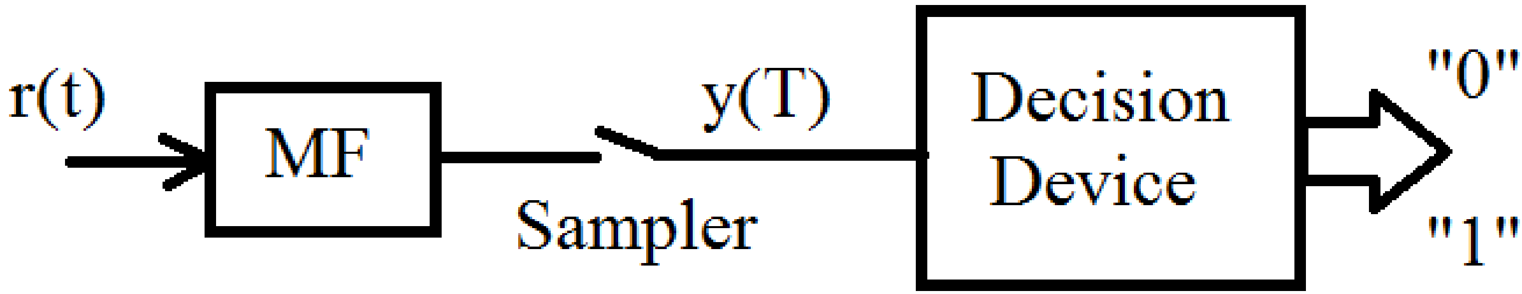
Correlator Reciever

$$
\begin{aligned}
h(t) &= g(T-t) \\
y(t) &= r(t) * h(t) \\
&= \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
y(T) &= \int_{-\infty}^{\infty} r(\tau)h(T-\tau)d\tau \\
&= \int_{-\infty}^{\infty} r(\tau)g(\tau)d\tau
\end{aligned}
$$



Error Rate Calculations



Considering a Polar NRZ signaling,

$$r(t) = \begin{cases} +A + w(t), & \text{for bit '1'}, \quad 0 \leq t \leq T_b \\ -A + w(t), & \text{for bit '0'}, \quad 0 \leq t \leq T_b \end{cases}$$

The receiver is required to make a decision for each signaling interval

Note: For this signaling, the MF is matched to a rectangular pulse (A, T_b)

The filter output is sampled at the end of each signaling interval

The sample values are compared to a preset threshold λ to make a decision

$$
\begin{aligned}
y(T_b) &= \int_{-\infty}^{\infty} r(\tau)g(\tau)d\tau \\
&= \int_0^{T_b} kAr(t)d\,\mathrm{t} \\
&= \int_0^{T_b} \frac{1}{T_b} r(t)d\mathrm{t} \qquad\qquad kAT_b = 1
\end{aligned}
$$

Then,

$$
y = y(T_b) = \pm A + n(t), \qquad n(t) = \frac{1}{T_b}\int_0^{T_b} w(t)d\tau
$$

Note:

$n(t)$ is Gaussian distributed, with zero mean and variance $\sigma^2 = \frac{1}{T_b}N_0/2$

$y(T_b)$ is Gaussian distributed, with $\pm A$ mean and variance $\sigma^2 = \frac{1}{T_b}N_0/2$

Error Rate Calculations

The conditional PDF of the sampled output signal is expressed as

$$p(y/'0') = \frac{1}{\sqrt{\pi N_0/T_b}} \exp\left[-\frac{(y+A)^2}{N_0/T_b}\right] \quad (1)$$

$$p(y/'1') = \frac{1}{\sqrt{\pi N_0/T_b}} \exp\left[-\frac{(y-A)^2}{N_0/T_b}\right] \quad (2)$$

Assume bit '0' was transmitted, a decision is considered erroneous if the receiver decides that bit '1' was transmitted. The receiver makes such decision if $y > \lambda$. The probability of such decision is

$$P(e|'0') = P\{y > \lambda|'0'\} = \int_{\lambda}^{\infty} p(y/'0')dy$$

Similarly,

$$P(e|'1') = P\{y < \lambda|'1'\} = \int_{-\infty}^{\lambda} p(y/'1')dy$$

Error Rate Calculations

Probability of Error if '0' was Transmitted

$$
\begin{aligned}
P(e|'0') &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp\left[-\frac{(y+A)^2}{N_0/T_b}\right] dy \\
&= \frac{1}{\sqrt{\pi}} \int_{\frac{\lambda+A}{\sqrt{N_0/T_b}}}^{\infty} \exp\left[-z^2\right] dz \qquad \Leftarrow \left[z = \frac{y+A}{\sqrt{N_0/T_b}}\right] \\
&= \frac{1}{2} erfc\left(\frac{\lambda+A}{\sqrt{N_0/T_b}}\right)
\end{aligned}
$$

Note:

$$
\begin{aligned}
erfc(x) &= \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt \\
erf(x) &= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt
\end{aligned}
$$

Error Rate Calculations

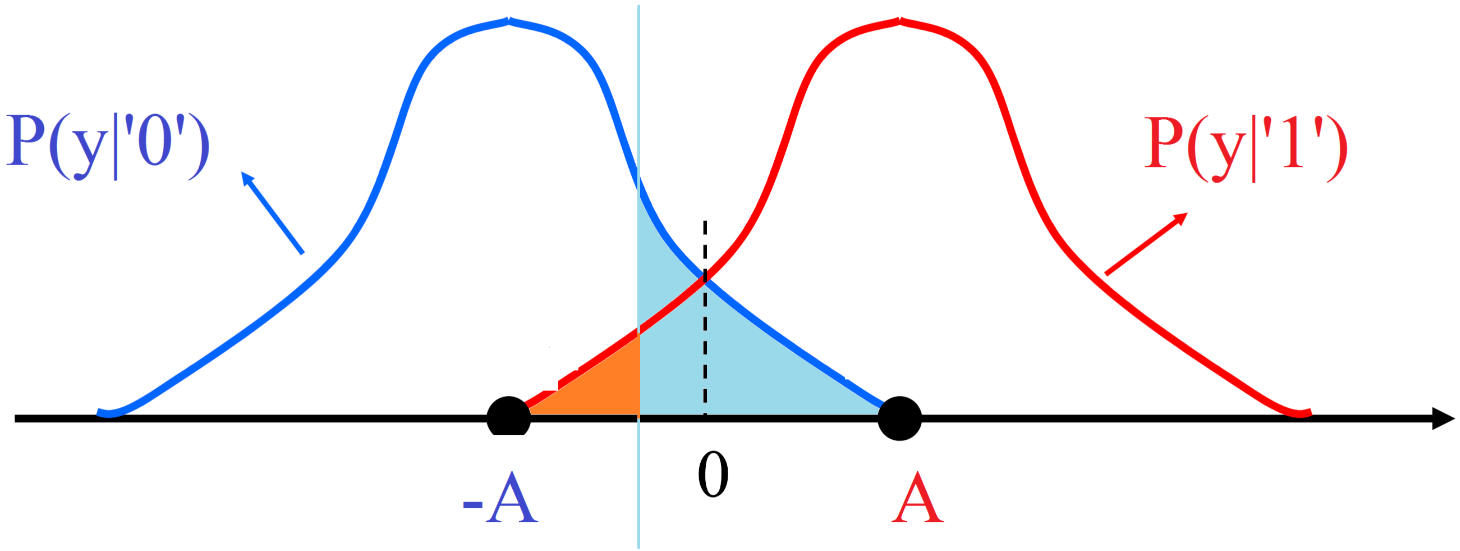
Probability of Error if '1' was Transmitted

$$
\begin{aligned}
P(e|'1') &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{-\infty}^{\lambda} \exp\left[-\frac{(y-A)^2}{N_0/T_b}\right] dy \\
&= \frac{1}{\sqrt{\pi}} \int_{\frac{-\lambda+A}{\sqrt{N_0/T_b}}}^{\infty} \exp\left[-z^2\right] dz \qquad \Leftarrow \left[z = -\frac{y-A}{\sqrt{N_0/T_b}}\right] \\
&= \frac{1}{2} erfc\left(\frac{-\lambda+A}{\sqrt{N_0/T_b}}\right)
\end{aligned}
$$

Note:

$$
\begin{aligned}
erfc(x) &= \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt \\
Q(x) &= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt
\end{aligned}
$$

Error Rate Calculations



Error Rate Calculations

Average Error Probability

$$P(e) = P(e|'0')P('0') + P(e|'1')P('1') = f(\lambda)$$

In order to minimize the average error probability, λ should be optimally chosen. This is achieved for

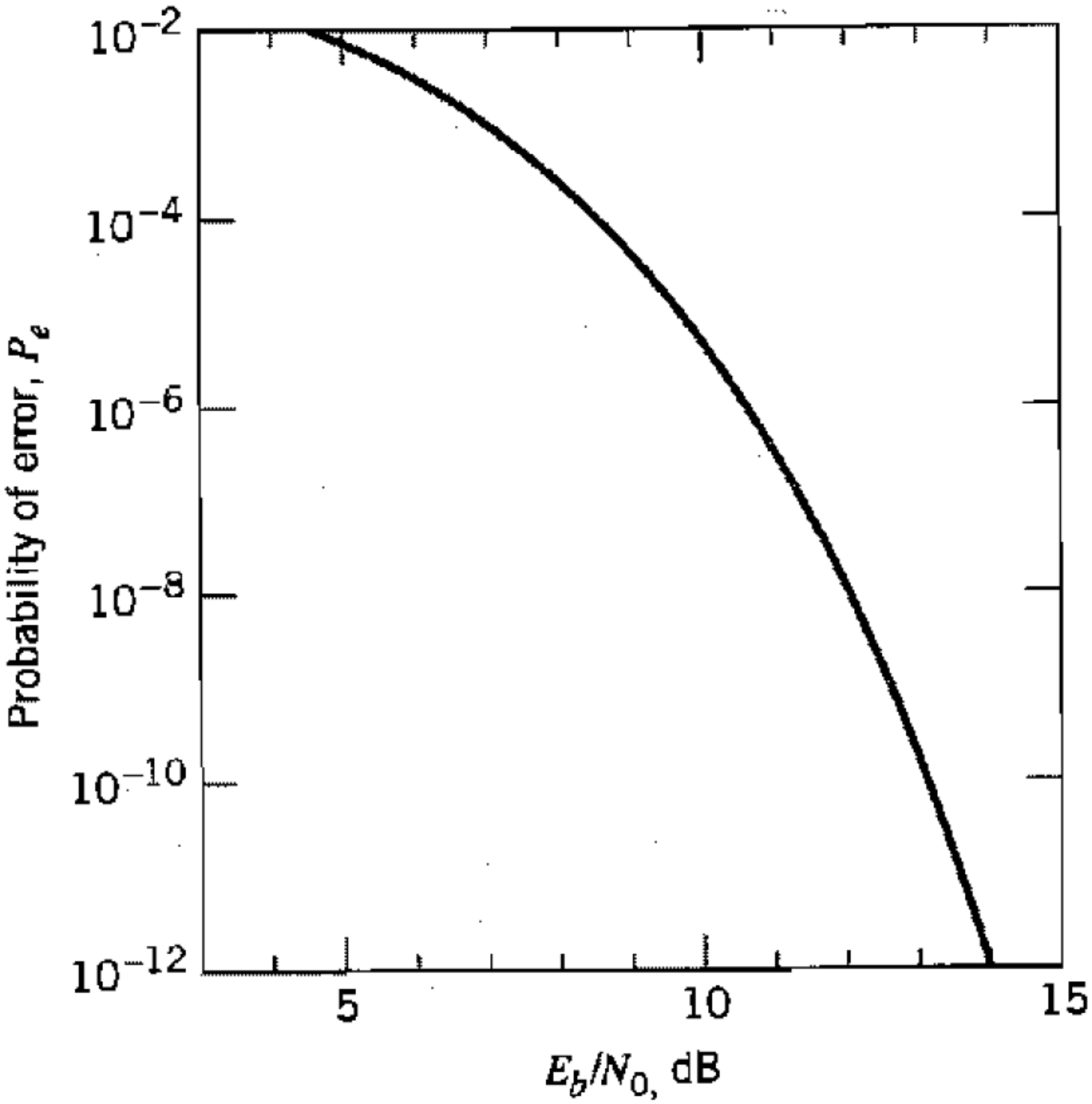
$$\lambda_{opt} = \frac{N_0}{4AT_b} \ln\left(\frac{P('0')}{P('1')}\right)$$

Special Case

If $P('0') = P('1') = 0.5$, then $\lambda_{opt} = 0$. In this case

$$P(e) = P(e|'0') = P(e|'1') = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0/T_b}}\right) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Error Rate Calculations



Error Rate Calculations

Notes

1. Binary Symmetric Channel

$$P(e|'0') = P(e|'1')$$

2. The average error probability decreases rapidly as E_b/N_0 increases
3. If $P('0') \gg P('1')$, $\lambda_{opt} \approx \infty$ in order to reduce $P(e|'0')$
4. If $P('0') \ll P('1')$, $\lambda_{opt} \approx -\infty$ in order to reduce $P(e|'1')$

References

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Thank You

Questions ?