Sheet 4

$\boxed{\text{PCA}}$

$\boxed{1}$ $$\hat{\mu} = \frac{1}{M} \sum_{m=1}^{M} \underline{X}(m)$$

$$\underline{Y} = \underline{X} - \hat{\mu}$$

$$\hat{\Sigma} = \frac{1}{M} \sum_{m=1}^{M} \left[(\underline{X}(m) - \hat{\mu}) (\underline{X}(m) - \hat{\mu})^T \right]$$

$$\hat{\Sigma} = \frac{1}{M} \sum_{m=1}^{M} \underline{Y}(m) \, \underline{Y}(m)^T$$

$\boxed{2}$ Compute eigen vectors $\underline{u}_i$ and eigen values $\underline{\lambda}_i$

$\boxed{3}$ Choose L new features u_i corresponding to big eigen values

$\boxed{4}$ perform transformation of data

$$\underline{Z} = \begin{bmatrix} \underline{u}_1^T \\ u_2^T \\ \vdots \\ u_L^T \end{bmatrix} \underline{Y}$$

Problem 1

Recall from lecture

$$U = \begin{bmatrix} | & | & | & \\ u_1 & u_2 & u_3 & \cdots \\ | & | & | & \end{bmatrix} \qquad \Omega = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \end{bmatrix}$$

$$B = \Sigma$$

$$\boxed{\Sigma U = U \Omega}$$

$$\Sigma \underline{u}_i = \underline{u}_i \lambda_i$$

$(\Sigma - \lambda_i I) \underline{u}_i = 0$ $\rightarrow$ eigen vector is non-zero vector

So, $|\Sigma - \lambda_i I| = 0$

Given $\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda_i & 1 \\ 1 & 3-\lambda_i \end{vmatrix} = 0$$

$(2-\lambda_i)(3-\lambda_i) - 1 = 0$

solve the eqn.

$\boxed{\lambda_1 = 3.618}$ $\boxed{\lambda_2 = 1.382}$

eigen Values

To get eigen vectors

$\Sigma u_i = u_i \lambda_i$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3.618\,X \\ 3.618\,Y \end{bmatrix}$$

$2X + Y = 3.618\,X \longrightarrow (1)$

$X + 3Y = 3.618\,Y \longrightarrow (2)$

solve to get X & Y $\quad X = 0.526$ & $Y = 0.851$

$$\underline{u}_1 = \begin{bmatrix} 0.526 \\ 0.851 \end{bmatrix}$$

The same $\Sigma \underline{u}_2 = \underline{u}_2 \lambda_2 \Rightarrow u_2 = \begin{bmatrix} -0.851 \\ 0.526 \end{bmatrix}$

To Check that your answer is correct

$u_1^T u_2 = 0 \rightarrow \underline{u}_1 \cdot \underline{u}_2 = 0 \quad \underline{u}_1 \perp u_2$ orthogonal

* Transform the data

Choose u_1 because λ_1 is bigger

$$\underset{1\times1}{Z} = \underset{1\times2}{u_1^T}\ \underset{2\times1}{Y}$$

Problem 2

$$\Sigma U = U \Omega$$

$$\boxed{\Sigma = U \Omega U^T}$$

$$\Sigma = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ - & u_3 & - \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$