DC Sheet 2

Problem 1)

- $BW = 1MHz$

$\rightarrow F_s = 1.5 \times (2 \times 1M)\, Hz = 3\, MHz$

// Sampled at rate 50% higher than Nyquist rate.

- $L = 256$

thus, $n = 8$ (quantization bits)

- $\mu = 255$

a) Determine SQNR

- For μ-law, $SQNR \simeq \dfrac{3L^2}{(\ln(1+\mu))^2}$

$$SQNR \simeq \frac{3 \times 256^2}{(\ln(1+255))^2} = 6393.966$$

which in dB is $10\log 6393.966 = 38.06\, dB$

- use 20log for amplitudes (eg. volt) as $P \propto V^2$ (meh)
- $10^{\frac{x}{10}}$ to convert back

b) • Need to improve the SQNR we just computed.

→ must be increased by at least 10 dB ①

→ Without increasing transmission bandwidth ②

→ While sampling at 20% above the Nyquist rate ③

would it be possible?

how?

and at what SQNR?

①

$$SQNR \geqslant 48.06 \, db = 10^{\frac{48.06}{10}}$$

thus,

$$SQNR \geqslant 63973.48$$

$$\frac{3L^2}{(\ln(1+\mu))^2} \geqslant 63973.48$$

thus,

$L \geqslant 809$ and since $L \in \{1, 2, 4, 8, 16, 32, ..\}$

this is only true when

$L \geqslant 1024$

which using $L = 2^n$ yields

$n \geqslant 10$ bits

③ $f_s = 1.2 \times (2 \times 1 \, MHz) = 2.4 \, MHz$

② $B_T = \frac{R_b}{n} = \frac{f_s \times n}{n}$

- the Channel is the Same (n) (can work with R_b)
- lowest B_T given ① and ③ are not violated is then

$$B_{T_{min}} = \frac{1}{n}(f_s \times n_{min}) = \frac{1}{n}(24 \times 10^6)\ Hz$$

Now to See if ② is violated, need to Calculate the Original B_T

$$B_T = \frac{1}{n}(3 \times 10^6 \times 8) = \frac{1}{n}(24 \times 10^6)\ Hz$$

→ Clearly if we go beyond $n = 10$ we violate ②
(among $n \geq 10$ only $n = 10$ Satisfies ①, ②, ③)

- The only (and max) SNR that could be achieved this way is thus

$$SQNR \simeq \frac{3 \times 1024^2}{(\ln(1+255))^2} = 102303.448$$

$$= 50.09\ dB$$

* Suppose both $n = 10, n = 11$ Satisfied ①, ② ③ then we'd go with $n = 11$ to maximize the SQNR

Problem 2)

Sinusoidal Signal of random Phase

$$X(t) = \underbrace{A}_{\text{Const.}} \cos(2\pi \underbrace{f_c}_{\text{Const.}} t + \underbrace{\Theta}_{\Theta \sim U(-\pi,\pi)})$$

$$2\pi \cdot \frac{1}{2\pi} = 1$$

1) Find ACF:

$$R_x(\tau) = E(X(t)X(t-\tau))$$

$$= A^2 E(\cos(2\pi f_c t+\Theta) . \cos(2\pi f_c t + 2\pi f_c \tau + \Theta))$$

$$= \frac{A^2}{2}(E(\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta))$$

- We've just used $\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$
- We'll now use $E(X+Y) = E(X) + E(Y)$
- Then we'll use $E(a) = a\,E(1) = a$

$$= \frac{A^2}{2}(\underbrace{E(\cos(2\pi f_c \tau)}_{\cos(2\pi f_c \tau)} + \underbrace{E(\cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta))}_{\text{Recall}})$$

2

$\cos(2\pi f_c \tau)$
(has no random var.)

Recall
$$E(g(\theta)) = \int_{-\infty}^{\infty} g(\theta) . f(\theta)\, d\theta$$

② $E(\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta))$

$$= \int_{-\pi}^{\pi} \underset{g(\theta)}{\cos}(\underbrace{4\pi f_c t + 2\pi f_c \tau}_{a} + 2\theta) \underbrace{\frac{1}{2\pi}}_{f(\theta)} \, d\theta$$

$$= \int_{-\pi}^{\pi} \cos(a + 2\theta) \frac{1}{2\pi} \, d\theta = 0$$

- $\cos(\theta)$ (and hence $\cos(a+\theta)$) has period 2π in θ
- $\cos(a + 2\theta)$ has period π
- the area under one period of cosine is zero
- we integrated over two periods thus result is zero

Consequently,

$$R_x(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau)$$

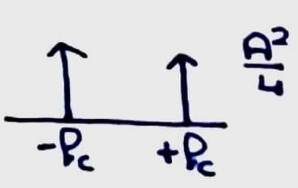
2) Find PSD

$$G_x(f) = \mathcal{F}_\tau\{R_x(\tau)\}$$

$$\cos(2\pi f_c \tau) \longleftrightarrow \frac{1}{2}(\delta(f - f_c) + \delta(f + f_c))$$

Thus,

$$G_x(f) = \frac{A^2}{4}(\delta(f - f_c) + \delta(f + f_c))$$

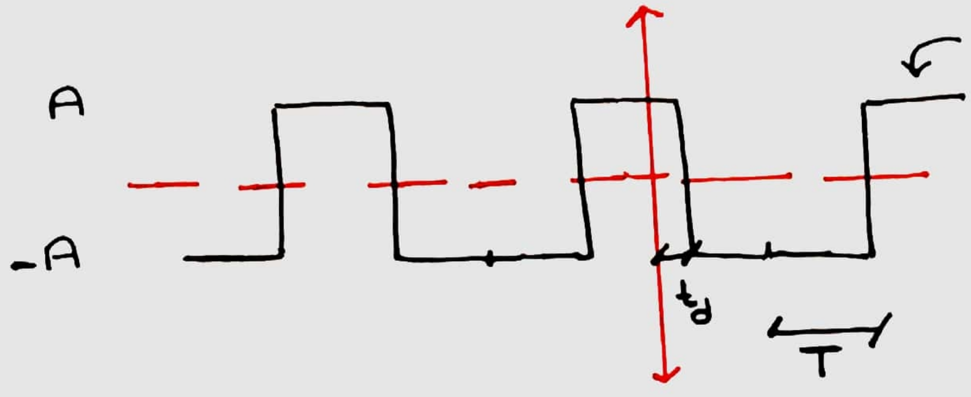


Problem 3)

- $X(t)$ is a random sequence of 0s and 1s each resembled by a Pulse of amplitude $-A$ or A for T seconds
  (0 under $-A$, 1 under A)

- The Starting time of the first complete Pulse for +ve time is a random variable t_d that's equally likely to fall anywhere from 0 to T.

- During any interval $(n-1)T < t - t_d < nT$ whether the bit corresponding to the Pulse is 0 or 1 is decided via a coin toss
  → The result for any interval is independent of all others.

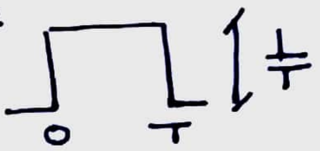


- We will use ~ (hat) for random variables ($t_d = \tilde{t}_d$)

- So the random Process is Pretty much a Square wave where
  → The value of each Pulse is a random variable $\tilde{A}$ where $\tilde{A} = \begin{cases} A & P = 0.5 \\ -A & P = 0.5 \end{cases}$

→ The whole square wave is shifted randomly by $t_d \sim U(0,T)$

$P(t_d) =$

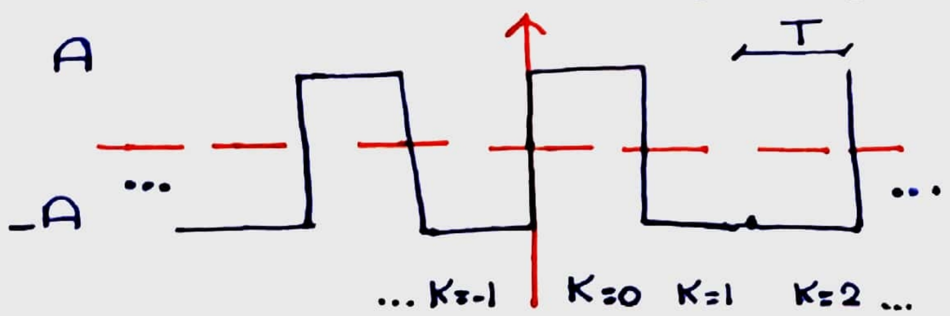


• Now if we let $P(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$ we can write a closed-form for the random process, namely:

$$Y(t) = \sum_{K=-\infty}^{\infty} \hat{A}_K \, P(t-KT)$$

↳ For each K, we get another random variable $\hat{A}_K =$

Y(t) sample function (example realization):



• We clearly aren't considering the random shift yet, thus our random process is

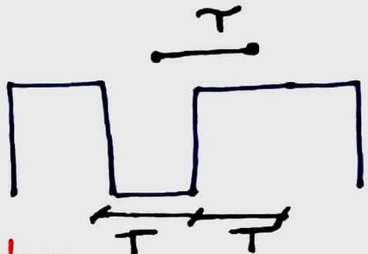
$$X(t) = Y(t-t_d) = \sum_{K=-\infty}^{\infty} \hat{A} \, P(t-\hat{t}_d-KT)$$

* Notice that unlike previous problems we solved, this one involves 2 random variables.

Now need to find the ACF.

$$R_x(\tau) = E(X(t)X(t-\tau))$$

• Suppose $|\tau| \geq T$



→ then clearly the value of the random variable $X(t)$ is independent from that of $X(t+\tau)$ since $|\tau| > T$ insinuates that they belong to different (and thus independent) pulses.

In this case,

$$R_x(\tau) = E(X(t))\,E(X(t-\tau))$$

• Due to independence ①

• Bonus: It is WSS so they must be =

$$E(X(t)) = E\left(\sum_{k=-\infty}^{\infty} \hat{A}_k P(t-\hat{t}_d - kT)\right)$$

$$= \sum_{k=-\infty}^{\infty} E(\hat{A}_k)\,E(P(t-\hat{t}_d-kT)) \quad ①$$

0.5

-A A

$$-A \times 0.5 + A \times 0.5 = 0$$

$$= 0$$

Hence,

$$R_x(\tau) = 0$$

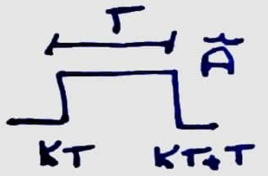
• May also make the conclusion graphically or say that if at t $X(t) = \tilde{A}_K$ then for sure at $t+\tau$ $X(t+\tau) = \tilde{A}_{K'}$ with $K \neq K'$ and the expectation of the values taken at either is 0.

• comment.

Notice)

• $X(t)$ takes $\tilde{A}_K$ through the whole interval given by $KT + \hat{t}_d \leq t < (K+1)T + \hat{t}_d$

(This is given, but to justify we had for $Y(t)$ which after the random shift becomes



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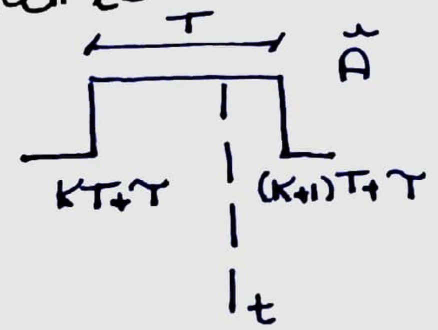
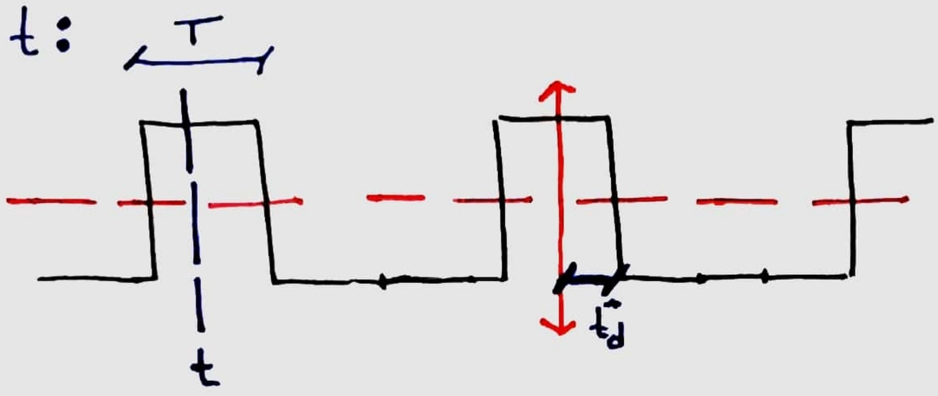


→ Consequently, at any time t the location of the start of the next pulse is a random variable distributed over $(t, t+T]$ uniformly (and implicitly the end of the previous pulse is uniformly distributed over $(t-T, t)$.)

To see this, imagine a random realization & mark a time t:

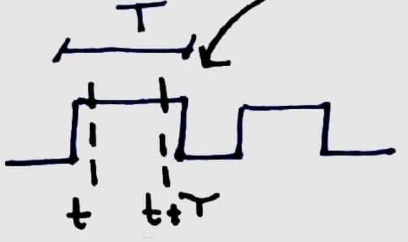


- Now animate in your brain all shift from 0 to T

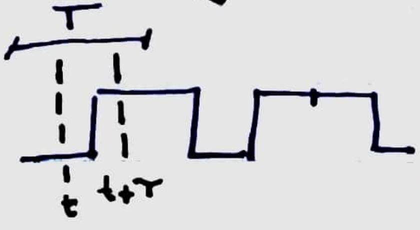
  → There's exactly one shift where the next pulse is 0 units ahead and exactly one shift where it occurs T units ahead. and it all occurs randomly according to $\hat{t}_d$ which is uniformly distributed

* Now let's consider $|\tau| < T$

- Two possible scenarios



$X(t)$ & $X(t+\tau)$ are in the same pulse



they're in different pulses.

→ Because we know that at any time t the next Pulse Starts at any value from 0 to T according to the uniform distribution ⟶ The Probability of $X(t+\tau)$ leaving the Pulse is equal to the Probability of the next Pulse Starting within $[t, t+\tau)$ which is $\frac{|\tau|}{T}$

- Consequently, the Probability of Staying within the Same Pulse is $1-\frac{|\tau|}{T}$

Hence, $X(t)X(t+\tau) = \begin{cases} \hat{A}_k^2 & P = 1-\frac{|\tau|}{T} \\ \hat{A}_k \hat{A}_{k'} \; (k \neq k') & P = \frac{|\tau|}{T} \end{cases}$

- Same Pulse → Same rand. var.

- $E(\hat{A}_k^2) = E(A^2) = A^2$
- $E(\hat{A}_k \hat{A}_{k'}) = 0$

- $A.A = -A.-A = A^2$

Thus,

$$R_x(\tau) = E(X(t)X(t+\tau))$$

$$= A^2\left(1-\frac{|\tau|}{T}\right) + 0\left(\frac{|\tau|}{T}\right)$$

$$= A^2\left(1-\frac{|\tau|}{T}\right)$$

- $|\tau| < T$

To Summarize all we did:

- For $|\tau| \geq T$

→ $X(t)$ and $X(t+\tau)$ are independent (belong to different pulses regardless of shift)

Hence,

$$E(X(t)X(t+\tau)) = E(\hat{A}_k \hat{A}_{k'})_{k \neq k'}$$

$$= \underbrace{E(\hat{A}_k)}_{0} \underbrace{E(\hat{A}_{k'})}_{0} = 0$$

- For $|\tau| < T$

→ $X(t)$ and $X(t+\tau)$ may or may not fall in different pulses

- Probability of falling in different pulses = Probability of the next edge occuring within τ from t which is $\frac{|\tau|}{T}$ as the next edge occurs randomly & uniformly from 0 to T.

Hence,

$$X(t)X(t+\tau) = \begin{cases} \hat{A}_k^2 & P = \left(1 - \frac{|\tau|}{T}\right) \text{ · same pulse} \\ \hat{A}_k \hat{A}_{k'} & P = \frac{|\tau|}{T} \end{cases}$$

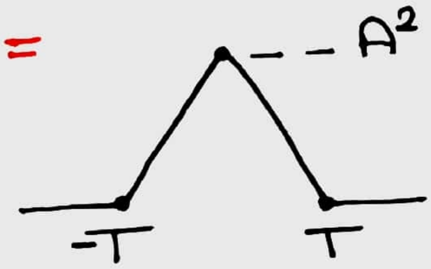
$$E(X(t)X(t+\tau)) = E(\hat{A}_k^2).\left(1 - \frac{|\tau|}{T}\right) + E(\hat{A}_k \hat{A}_{k'}) \frac{|\tau|}{T}$$

$$= A^2\left(1 - \frac{|\tau|}{T}\right)$$

$$R_x(\tau) = \begin{cases} A^2\left(1-\frac{|\tau|}{T}\right) & |\tau|<T \\ 0 & \text{Otherwise} \end{cases}$$

- at $\tau=0 \rightarrow R_x(\tau) = A^2$
- at $\tau=\pm T \rightarrow R_x(\tau)=0$

Hence, $R_x(\tau) =$

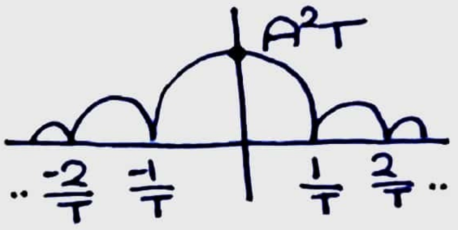


$A^2 \, tri\left(\frac{t}{T}\right)$

$$G_x(f) = \mathcal{F}_T\{R_x(\tau)\}$$

$$= \mathcal{F}_T\left\{A^2 \, tri\left(\frac{t}{T}\right)\right\}$$

$$= A^2 . T . \operatorname{Sinc}^2(fT)$$



Problem 4)

- The time-average (DC Component) of a deterministic Signal is given by

$$X_{DC} = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{x(t)}{T}\, dt$$ //returns a const.

Can write such signal as

$$x(t) = X_{DC} + x_{ac}(t)$$

→ Now its easy to see that $\lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{x_{ac}(t)}{T}\, dt = 0$

(the other component $x_{ac}(t) = x(t) - X_{DC}$ is purely AC)

* It's called a DC Component as

$$X(f) = X_{DC}\,\delta(f) + X_{ac}(f)$$

→ It corresponds to a delta (frequency component) @ $f=0$ (Similar to how DC current has 0 frequency)

Example) $x(t) = \underbrace{2\cos^2(t)}_{\text{has AC \& DC}} = \underbrace{1}_{DC} + \underbrace{\cos(2t)}_{AC}$

• The same applies if it was rather a random process (WSS) $X(t)$ except that

$$\rightarrow X_{DC} = E(X(t))$$

$$\rightarrow X(t) = X_{DC} + X_{ac}(t) \text{ where } E(X_{ac}(t)) = 0$$

Given)

→ Random Process $Y(t)$

→ Consists of a DC Component of $\sqrt{\frac{3}{2}}$ and a Periodic Component of random shift** and a random Component

$$\text{Periodic Component} \rightarrow G(t)$$

$$\text{random Component} \rightarrow X(t)$$

$$\text{Hence, } Y(t) = \underbrace{A}_{DC} + \underbrace{\overbrace{G(t) + X(t)}^{Gx(t)}}_{AC}$$

• where $A = \sqrt{\frac{3}{2}}$

• Recall that, $P_{avg\,Y} = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df$ hence finding R is a good start.

$$R_Y(\tau) = E(Y(t)Y(t+\tau))$$

$$= E((A + Gx(t))(A + Gx(t+\tau)))$$

$$= E[A^2 + AGx(t+\tau) + AGx(t) + Gx(t)Gx(t+\tau)]$$

$$= A^2 + A\,E[Gx(t+\tau)] + AE[Gx(t)] + E[Gx(t)Gx(t+\tau)]$$

Zero expectation

• Since $GX(t)$ is A.C., the second and third terms vanish and we're left with

$$\underbrace{R_Y(\tau)}_{\substack{\text{Power}\\ \text{of } Y(t)\\ \text{if } \tau=0}} = \underbrace{A^2}_{\substack{\text{Power of}\\ \text{DC component}\\ \text{by definition}}} + \underbrace{E[GX(t)GX(t+\tau)]}_{\substack{R_{Gx}(\tau)\\ \text{if we set } \tau=0 \text{ then this}\\ \text{is the AC Power by definition}}}$$

$$R_{Gx}(\tau) = E[(G(t)+X(t))(G(t+\tau)+X(t+\tau))]$$

$$= E[G(t)G(t+\tau)] + E[G(t)X(t+\tau)]$$

$$+ E[X(t)G(t+\tau)] + E[X(t)X(t+\tau)]$$

• The 2nd and 3rd terms are 0 as X and G are independent and thus $E[G(\ )X(\ )] = E(\underbrace{G(\ )}_{AC})E(\underbrace{X(\ )}_{AC}) = 0$

$$R_{Gx}(\tau) = \underbrace{E[G(t)G(t+\tau)]}_{R_G(\tau)} + \underbrace{E[X(t)X(t+\tau)]}_{R_x(\tau)}$$

Thus,

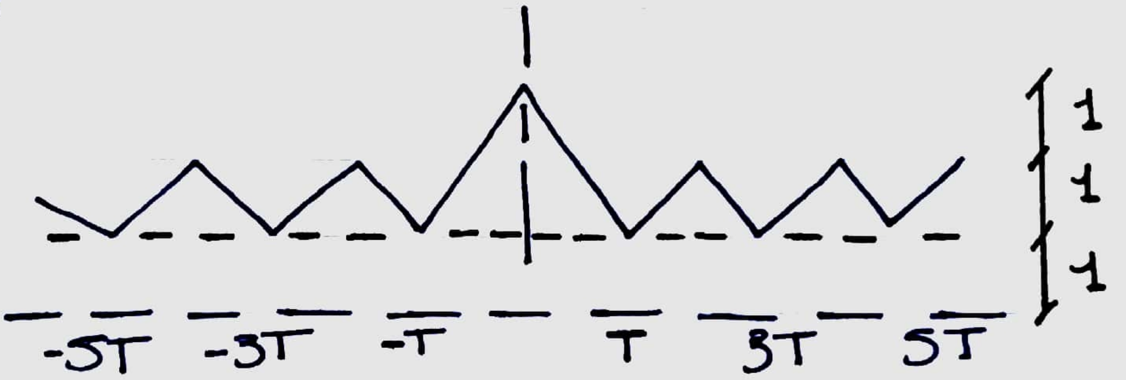
$$R_Y(\tau) = A^2 + R_G(\tau) + R_x(\tau)$$

$$\underbrace{R_Y(0)}_{\text{Tot. avg. Power}} = \underbrace{A^2}_{\text{DC Power}} + \overbrace{\underbrace{R_G(0)}_{\cdot\text{Avg. Power of } G} + \underbrace{R_x(0)}_{\cdot\text{Avg. Power of } X}}^{\text{AC Power}}$$

$\Rightarrow$ First equation implies the existence of A^2 and $R_G(\tau)$ and $R_X(\tau)$ such that the sum is $R_Y(\tau)$

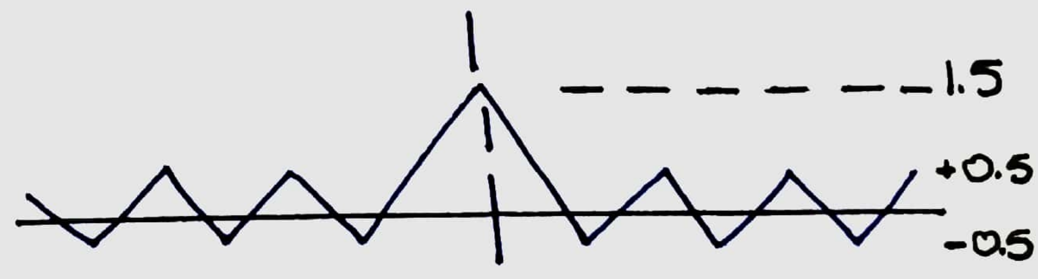
* We have A^2 and $R_Y(\tau)$ can we uniquely identify $R_G(\tau)$ and $R_X(\tau)$ that satisfy the equation?

$R_Y(\tau) =$



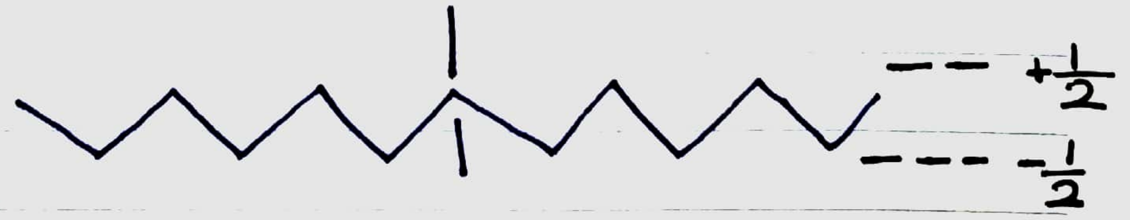
Hence,

$R_Y(\tau) - A^2$
$= R_X(\tau) + R_G(\tau) =$



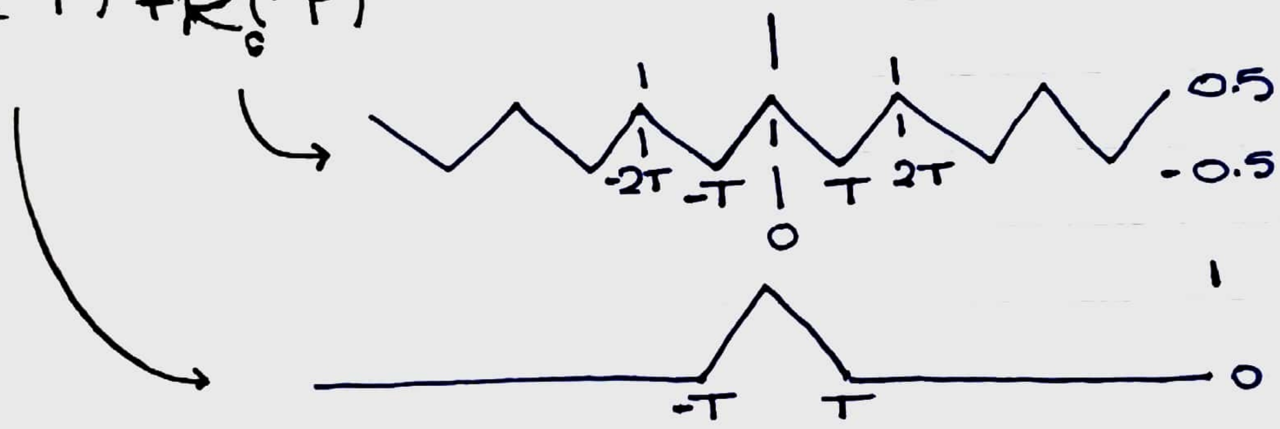
• Recall that the autocorrelation of a periodic random signal is also periodic (and at the same period).
Hence, a visual suggestion is

$\Rightarrow R_G(\tau) =$



which is only different from $R_Y(\tau)$ in that the triangle at the origin ranges (-0.5 → 0.5) rather than (-0.5 → 1.5)

It thus must be the case that $R_X(\tau)$ is a triangle ranging (0 → 1) so that indeed $R_Y(\tau) - A^2 = R_X(\tau) + R_C(\tau)$



• At this point its easy to use $\tau = 0$ to get the power of every component of the signal

$$R_Y(0) = A^2 + R_C(0) + R_X(0)$$

3 (bonus check) 1.5 0.5 1 } from the graphs

• Don't forget units V^2/load

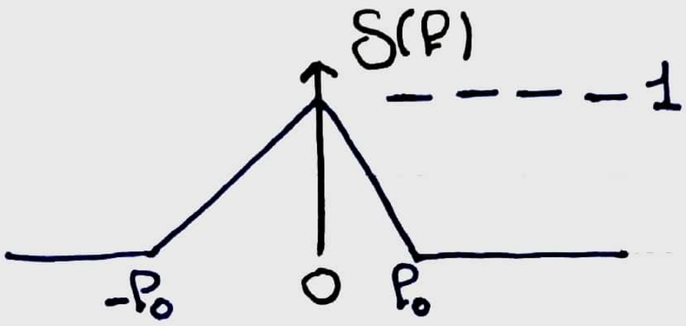
• $R_C(\tau) = tri\left(\frac{t}{T}\right)$

• $R_X(\tau) = \sum_{k=-\infty}^{\infty} \left(tri\left(\frac{t-2kT}{T}\right)\right) - 0.5$

} Perhaps, one can prove that these aren't unique solutions

Problem 5)

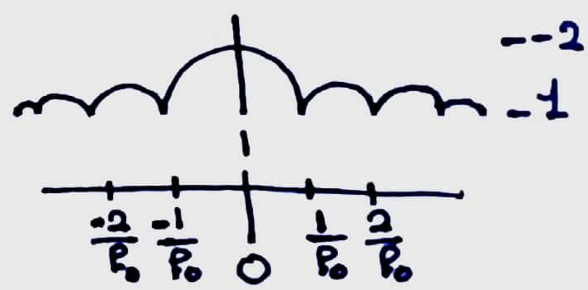
$S_x(f) =$



a) $R_x(\tau) = \mathcal{F}_\tau^{-1}\{S_x(f)\}$

$= \mathcal{F}_\tau^{-1}\{ tri(\frac{t}{f_0}) + \delta(f) \}$

$= f_0 . \text{Sinc}^2(f_0 \tau) + 1$



b) DC Power in $X(t)$:

→ The DC Component is $\underset{(\text{at } f=0)}{}$ $\delta(f) \longleftrightarrow \underbrace{1}_{R_{DC}(\tau)}$

• Don't confuse with the DC value.

• Hence $P_{avg_{DC}} = R_{DC}(0) = 1$ // also area under $\delta(f)$

→ Can alternatively claim that $\lim_{T\to\infty} \frac{f_0}{T} \int_{-T/2}^{T/2} \text{Sinc}^2(f_0 . \tau) d\tau$

$= 0$

(Can't be written as const. + Signal)

• Dies as $T \to \infty$

c) AC Power in $X(t)$:

→ The AC Component is $tri\left(\frac{t}{f_0}\right) \leftrightarrow \underbrace{f_0 . Sinc^2(f_0 \tau)}_{R_{AC}(\tau)}$

- $P_{\underset{AC}{avg.}} = R_{AC}(0) = f_0 . 1$

also true that $P_{\underset{AC}{avg}} = \int_{-\infty}^{\infty} S_{AC}(f)\, df = \frac{1}{2} \times 2f_0 \times 1 = f_0$

* Could've also done

$$P_{\underset{AC}{avg.}} = P_{AVG} - P_{\underset{DC.}{AVG}} = (1+f_0) - 1 = f_0$$

⇒ Proof that we can add Powers

$$E[(X_{DC} + X_{AC}(t))(X_{DC} + X_{AC}(t+\tau))]$$

$$=$$

$$E[X_{DC}^2] + X_{DC} \underbrace{E[X_{AC}(t)] + X_{DC} E[X_{AC}(t+\tau)]}_{0 \text{ (like the last Problem)}}$$

$$+ E[X_{AC}(t) X_{AC}(t+\tau)]$$

$$= X_{DC}^2 + R_{AC}(\tau) \underset{\tau=0}{=} X_{DC}^2 + R_{AC}(0)$$

- DC Power by def.
- AC Power by def.

rea under $S_x(f)$ is another way to Prove it.

d)

Solution #1)

- Two random variables are uncorrelated if their covariance is zero.

→ The autocovariance due to samples at t and $t+\tau$ should be zero

$$K_x(\tau) = E[X(t)X(t+\tau)] - \mu_{xt}\,\mu_{xt+\tau}$$

$$= R_x(\tau) - \mu_{xt}^2 \qquad \text{(it's WSS)}$$

Recall, $E(X(t)) = \mu_{xt}$ = DC value

. Since $P_{DC} = 1$ then $\mu_{xt} = \sqrt{1} = 1$

- if DC is A volts
- Power is A^2 volts

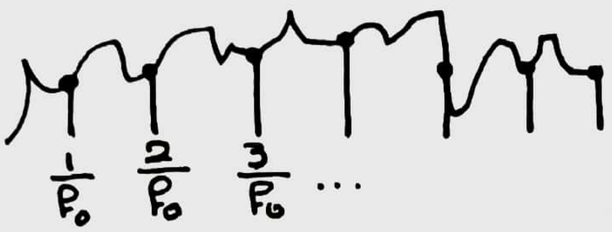
Hence, $0 = R_x(\tau) - 1 \rightarrow R_x(\tau) = 1$

→ which is true for all $\tau = \frac{n}{f_0}$, $n \in \mathbb{Z}$ (except for $n=0$)

* We conclude that whenever $\tau = \frac{n}{f_0}$ the samples will be uncorrelated.

→ Hence the sampling rate needed is also any integer multiple of $\frac{1}{f_0}$ (that isn't 0) $t = \frac{n}{f_0}$, $n \neq 0$

- For instance

$X(t) =$ 

* All Possible Samples are separated by an integer multiple of $\frac{1}{F_0}$

→ Hence its always the case that

$R_x(\tau) = 1$

Solution #2 (Section)

- Take the definition of 'uncorrelated samples' to mean 0 auto correlation
- A vertical shift won't make two uncorrelated samples correlated, hence subtract the DC.
- Now solve $R_x(\tau) = 0$ and arrive at the same sampling rate as above.

Thank you.