Number Theory

Sheet 1 — MTH3251

Basic Concepts in Number Theory

1. Let $a, b, c \in \mathbb{Z}$, prove the following properties

   i If $c \mid a$ and $c \mid b$, then $c \mid a \pm b$

   ii If $a \mid b$, then $a \mid bc$

   iii If $a \mid b$ and $b \mid c$, then $a \mid c$

   iv If $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for $m, n \in \mathbb{Z}$

Done

2. Let a, b, c, $d \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, prove the following congruence identities:

   i If $a \equiv b \pmod{m}$, then $(a + c) \equiv (b + c) \pmod{m}$

   ii If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $(a + c) \equiv (b + d) \pmod{m}$

   iii If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$

3. Use induction to prove that $5 \mid n^5 - n$ for any positive integer n

4. Use induction to prove that $n! > n^2$ for every integer $n \geq 4$, whereas $n! > n^3$ for every integer $n \geq 6$.

5. Prove the Bernoulli inequality: If $1 + a > 0$, then $(1 + a)^n \geq 1 + na$

6. The numbers 1051, 1529, and 2246 have the same remainder r when divided by some integer d. Find d and r.

7. Use the Division Algorithm to prove the following:

   i The square of any integer is either of the form $3k$ or $3k + 1$

   ii The cube of any integer has one of the forms: $9k$, $9k + 1$, or $9k + 8$

   iii The fourth power of any integer is either of the form $5k$ or $5k + 1$

8. For $n \geq 1$, prove that $n(n+1)(2n+1)/6$ is an integer. [Hint: By the Division Algorithm, n has one of the forms $6k, 6k + 1, ..., 6k + 5$; prove the result in each of these six cases.]

9. For $n \geq 1$, prove that the integer $n(7n^2 + 5)$ is of the form $6k$.

10. If n is an odd integer, show that $n^4 + 4n^2 + 11$ is of the form $16k$.

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