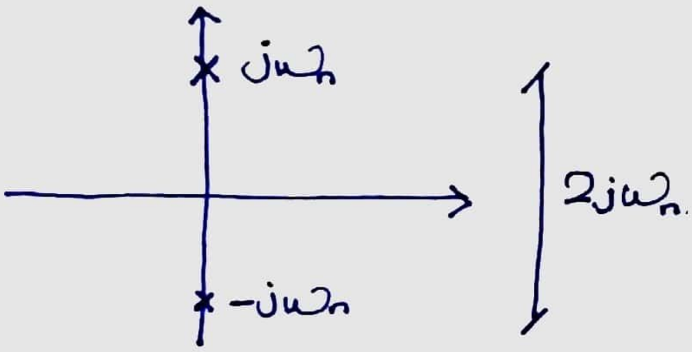
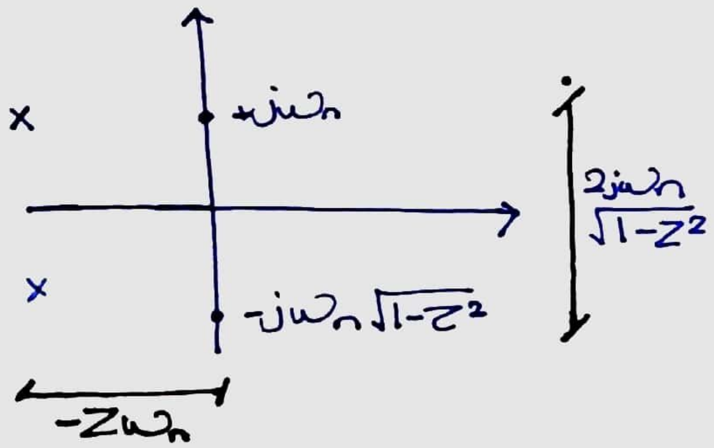
CE Sheet 4

- An animated overview of the damping ratio and poles

- At $Z=0$

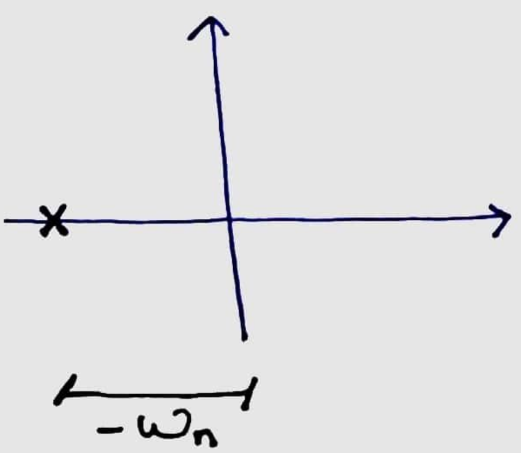


- As Z increases ($0<Z<1$)



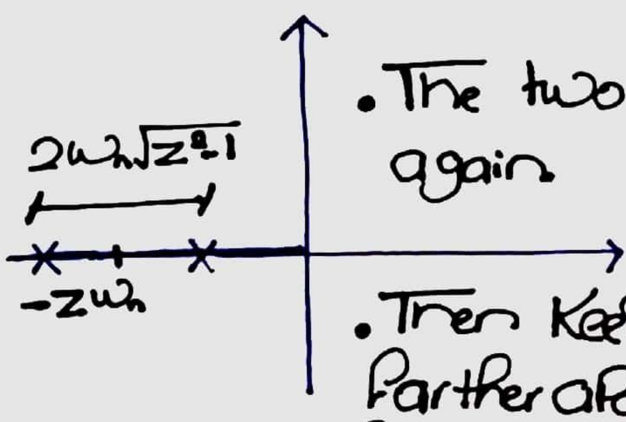
- The two poles get vertically closer to each other and farther from the imaginary axis

- At $Z=1$



- The vertical distance becomes zero (they merge)

- As Z further increases ($Z>1$)



- The two poles split again
- Then keep getting farther apart horizontally forever.

• *wipe tears*

- Notice that after the ~~break up~~ (after $Z = 1$) the Pole closer to the origin controls the ~~relationship~~ System's Performance.

→ Reason being that the other Pole (big α in $e^{-\alpha t}$) will die out very quickly and we'll be left with the other one (small α in $e^{-\alpha t}$).

  - Have shown that reaching Steady State requires contribution from both (their constants add to steady state value)

    → Hence the one closer to the origin acts as a bottleneck

  - In such case, $t_s = \frac{4}{Z\omega_n}$ won't work for ← Derived for $0<Z<1$

    Settle time (it'll depend on the real part of the closer Pole.)

System Specifications

relative (÷) to M

Maximum Relative Overshoot

$$M_p = e^{-Z\pi/\sqrt{1-Z^2}}$$

- Thus, $Z = -\frac{\tilde{M}}{\sqrt{\pi^2+\tilde{M}_p^2}}$ ($\sim \ln M_p$)
- $t_p = \pi/\omega_d$

reach Steady State with tolerance a%

Settle time

$$t_s = \begin{cases} 4/Z\omega_n & (2\%) \\ 3/Z\omega_n & (5\%) \end{cases}$$

- generally solve $\frac{a}{100} = e^{-Z\omega_n t}$

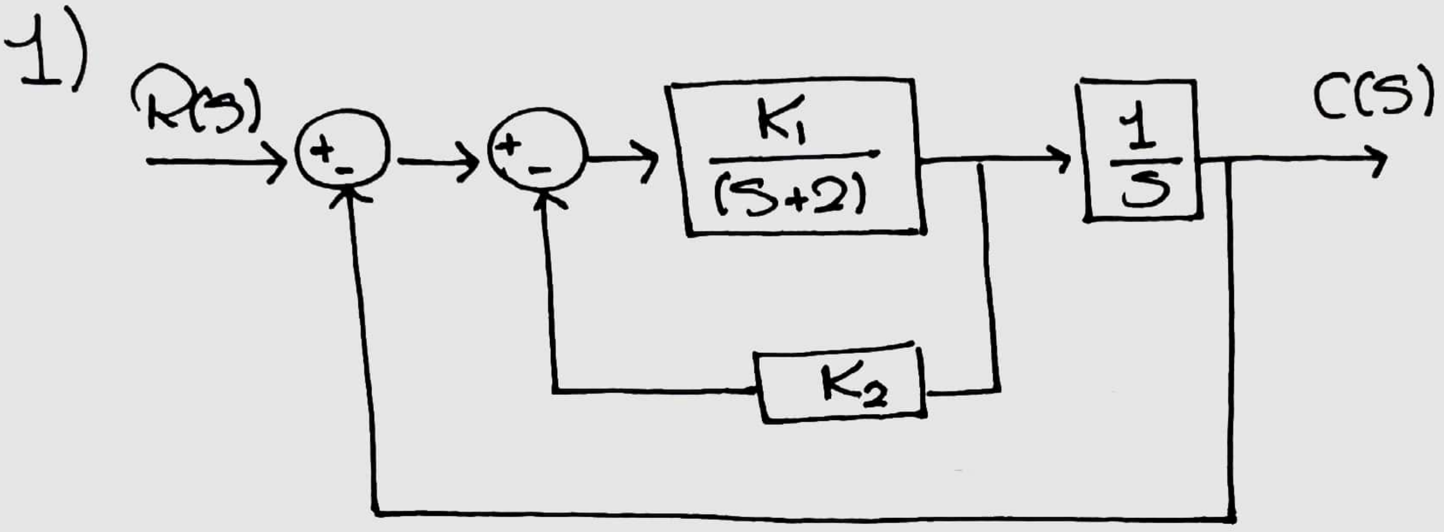
1st time to hit M

Rise time

$$t_r = \frac{\pi - \cos^{-1} Z}{\omega_d}$$

- $\omega_d = \omega_n\sqrt{1-Z^2}$

1)



- Need to find values for K_1, K_2
- Such that

$\rightarrow M_p = 0.25$

$\rightarrow t_p = 2$

- whenever we studied transient response → it was due to unit step

Thus, $Z = \dfrac{-\ln(0.25)}{\sqrt{\pi^2 + (\ln(0.25))^2}} = 0.4$

$t_p = \dfrac{\pi}{w_n\sqrt{1-Z^2}} \Rightarrow w_n = \dfrac{\pi}{t_p\sqrt{1-Z^2}} = 1.71$

- must be the Z and w_n for this system

→ Our Objective is to find K_1, K_2 that make this so.

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- Start by finding the characteristic equation of the given system ①

→ Then compare it to the standard form characteristic equation at the right choice of Z, ω_n

2nd Order: $S^2 + 2Z\omega_n S + \omega_n^2$

3rd Order: $(S+\alpha)(S^2+2Z\omega_n S+\omega_n^2)$

- The transfer function of the system at hand is

inner system) $G(S) = \frac{K_1}{S+2}$, $H(S) = \frac{K_2}{1}$

$$TF_{inner}(S) = \frac{K_1}{K_1 K_2 + (S+2)}$$

whole system) $G(S) = \frac{K_1}{K_1K_2+(S+2)} \cdot \frac{1}{S}$ $H(S)=1$

$$TF(S) = \frac{K_1}{S(K_1K_2+S+2) + K_1}$$

This is what we really need ①

- Simplifies to

$$S^2 + S(2+K_1K_2) + K_1$$

Standard form is

$$S^2 + S(2Z\omega_n) + \omega_n^2$$

Notice that need both to have an S^2 coefficient of 1 before comparing. (Divide the 1st otherwise)

Hence,

$$K_1 = \omega_n^2$$

$$2 + K_1K_2 = 2Z\omega_n \rightarrow K_2 = \frac{2Z\omega_n - 2}{K_1}$$

Crunching the numbers ($\omega_n = 1.71$, $Z = 0.4$) we get

$$K_1 = 2.9241 \qquad K_2 = -0.216$$

2) Inner System has

$$G(S) = \frac{10}{S(S+1)(S+10)} \qquad H(S) = K_tS$$

- Thus, $TF_{inner}(S) = \dfrac{10}{10K_tS + S(S+1)(S+10)}$

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Thus, For whole System

$$G(S) = K \cdot \frac{10}{10K_t S + S(S+1)(S+10)} \qquad H(S) = 1$$

- The characteristic equation $1 + GH(S) = 0$ is thus

$$10K_t S + \underbrace{S(S+1)(S+10)}_{S(S^2+11S+10)} + 10K = 0$$

- Could've also started by writing TF

$$S^3 + 11S^2 + S(10 + 10K_t) + 10K = 0$$

- The Problem asks to find K, K_t such that

→ $K_v = 1$ (i.e. $\lim_{S \to 0} S^1 GH(S) = 1$)

→ $Z = 0.707$ for the dominant Poles

Recall that when dealing with high order systems (e.g. 3rd like this one) we assume that

3rd Pole → $\alpha \geq 5 Z \omega_n$ (So that the dominant Poles are the conjugate Poles relating to Z, ω_n ⟶ Can analyze the System through known methods)

- Now before comparing the characteristic equation to $(S+\alpha)(S^2+2Z\omega_n S+\omega_n^2)$ ②
Let's see what $K_v=1$ gives

$$\lim_{S\to 0} S.GH(S) = \lim_{S\to 0} \frac{S.\,10K}{10K_t S + S(S+1)(S+10)}$$

$$= \lim_{S\to 0} \frac{10K}{10K_t+(0+1)(0+10)} = 1$$

- Thus we get $K = K_t + 1$
- We already have $Z = 0.707$

- The system's characteristic equation

$$S^3 + 11S^2 + 10(K_t+1)S + 10K = 0$$

- The standard 3rd order system

by expanding ② . can memorize

$$S^3 + S^2(\alpha+2Z\omega_n) + (2\alpha Z\omega_n + \omega_n^2)S + \alpha\omega_n^2$$

By comparing coefficients, we also get

$$\alpha + 2Z\omega_n = 11$$
$$2\alpha Z\omega_n + \omega_n^2 = 10(K_t+1)$$
$$\alpha\omega_n^2 = 10K$$

- Now we have a System of 5 unknowns ($Z, \omega_n, \alpha, K, K_t$) and 5 equations (or 4 and 4 after Plugging for Z)

→ Although we Seek K, K_t we can't get them without α, ω_n, Z first (that's why we need to Solve the whole System)

$$\left.\begin{aligned} &K = K_t + 1 \\ &\alpha + 1.414\omega_n = 11 \\ &1.414\alpha\omega_n + \omega_n^2 = 10(K_t+1) \\ &\alpha\omega_n^2 = 10K \end{aligned}\right\} \text{after } Z = 0.707$$

$$\left.\begin{aligned} &\alpha + 1.414\omega_n = 11 \\ &1.414\alpha\omega_n + \omega_n^2 = 10K \\ &\alpha\omega_n^2 = 10K \end{aligned}\right\} \text{after } K = K_t + 1$$

$$\left.\begin{aligned} &\alpha + 1.414\omega_n = 11 \\ &1.414\alpha\omega_n + \omega_n^2 = \alpha\omega_n^2 \end{aligned}\right\} \text{after } 10K = \alpha\omega_n^2$$

$$\rightarrow 1.414\alpha\omega_n = \omega_n^2(\alpha - 1)$$

- cancel out ω_n ($\omega_n = 0$)
- refuse → $K=0$: no system

after Simplify and $\alpha = 11 - 1.414\omega_n$

$$1.414\underbrace{(11 - 1.414\omega_n)}_{\alpha} = \omega_n\underbrace{(10 - 1.414\omega_n)}_{\alpha - 1}$$

Solve for ω_n (then backpropagate for all others)

$$15.554 - 2\omega_n = 10\omega_n - 1.414\omega_n^2$$

$$1.414\omega_n^2 - 12\omega_n + 15.554 = 0$$

$$\omega_n = 6.89 \quad \text{or} \quad \omega_n = 1.5965$$

Since $\alpha = 11 - 1.414\omega_n$ then

$$\alpha = 1.2575 \quad \text{or} \quad \alpha = 8.7425$$

Since $10K = \alpha\omega_n^2$ then

→ Before proceeding with the rest of the unknowns check $\alpha \geq 5Z\omega_n$ (whenever have α, Z, ω_n)

- $1.2575 \overset{?}{\geq} 5 \times 0.707 \times 6.89$

- $8.7425 \overset{?}{\geq} 5 \times 0.707 \times 1.5965$ yes!

→ Hence, $\omega_n = 6.89$ is no longer a solution

- Only $\omega_n = 1.5965$ qualifies → $\alpha = 8.7425$

→ $K = \dfrac{\alpha\omega_n^2}{10} = 2.23$ → $K_t = K - 1 = 1.23$

- System is stable as expected $\alpha, \omega_n^2, Z\omega_n > 0$ • we did FVT

Conclusions from last two Problems

→ We're given a system with characteristic equation $s^2 + bs + c = 0$ or $s^3 + as^2 + bs + c = 0$ where a, b, c are functions of some parameters of the system that we wish to find such that a set of constraints is satisfied.

- For n parameters to be found given m constraints
  → 2nd order case:

$$\#\text{unknowns} = \underbrace{n}_{\text{Parameters}} + \underbrace{2}_{z,\,\omega_n}$$

$$\#\text{equations} = \underbrace{m}_{\text{constraints}} + \underbrace{2}_{\text{by Comparing coefficients (b and c)}}$$

- The equation to compare to is

$$s^2 + (2z\omega_n)s + \omega_n^2$$

→ 3rd Order Case:

$$\#\text{unknowns} = n + \underbrace{3}_{z,\,\omega_n,\,\alpha}$$

$$\#\text{equations} = m + \underbrace{3}_{\text{by comparing a, b, c}}$$

- The equation to compare to is

$$s^3 + (2z\omega_n + \alpha)s^2 + (2\alpha z\omega_n + \omega_n^2)s + \alpha\omega_n^2$$

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- In the 3rd order case it must always hold that $\alpha \geq 5\zeta\omega_n$

→ Must check that it holds after finding a solution (and refuse any solution that violates it)

→ Furthermore, if $n > m$ such $n = m - 1$ then its legit to assume $\alpha = 5\zeta\omega_n$ and the system no longer has ∞ solutions

3)

Inner System) $G(s) = \dfrac{100}{(1+0.1s)(1+0.5s)}$

$$H(s) = K_t$$

Thus, $TF_{inner}(s) = \dfrac{100}{100K_t + (1+0.1s)(1+0.5s)}$

Whole System)

$$G(s) = \frac{K}{20s} \cdot \frac{100}{100K_t + (1+0.1s)(1+0.5s)}$$

$H(s) = 1$

→ Thus, the characteristic equation is

$$20s\,(100K_t + \underbrace{(1+0.1s)(1+0.5s)}_{0.05s^2+0.6s+1}) + 100K = 0$$

By Simplifying,

$$s^3 + s^2(12) + s(1+100K_t).20 + 100K = 0$$

Constraints:

$$M_p = \frac{4.3}{100} \rightarrow Z = \frac{-\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}} = 0.7076$$

$$t_r = 2 \rightarrow t_r = \frac{\pi - \cos^{-1} Z}{\omega_d} \rightarrow \omega_n = \frac{\pi - \cos^{-1} Z}{t_r\sqrt{1-Z^2}}$$

rad

$$= 1.668 \text{ rad/s}$$

The Standard Form

$$s^3 + s^2(2Z\omega_n + \alpha) + s(2\alpha Z\omega_n + \omega_n^2) + \alpha\omega_n^2$$

Thus the Corresponding System:

$$\left.\begin{aligned} &Z = 0.7076 \\ &\omega_n = 1.668 \text{ rad/s} \\ &12 = 2Z\omega_n + \alpha \\ &20(1+100K_t) = 2\alpha Z\omega_n + \omega_n^2 \\ &100K = \alpha\omega_n^2 \end{aligned}\right\} \begin{gathered}5 \\ \text{equations} \\ \text{and 5} \\ \text{unknowns}\end{gathered}$$

• By using the first and second equations in the third.

• $Z = 0.7576$
$\omega_n = 1.668$ rad/s

$\alpha = 12 - 2Z\omega_n = 9.64$

Check $\alpha \geq 5Z\omega_n \rightarrow$ holds indeed.

• Now $K = \dfrac{\alpha \omega_n^2}{100} = 0.268$

$K_t = ((2\alpha Z\omega_n + \omega_n^2)/20 - 1)/100$

$= 0.00277$

* The Steady State error

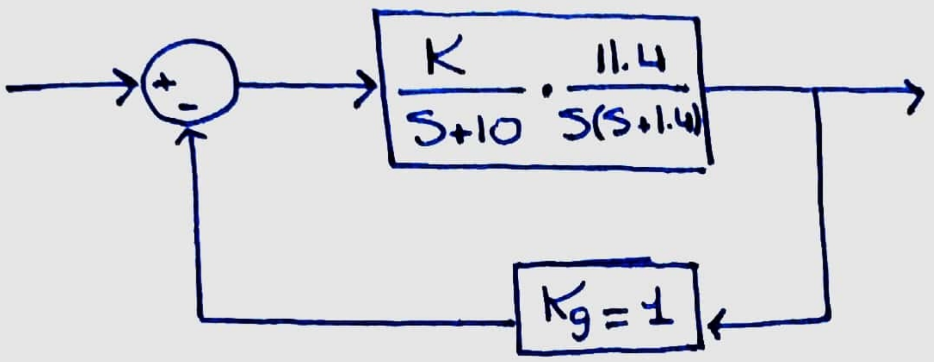
• $\alpha, \omega_n, Z\omega_n > 0$ and the system is thus stable as expected.

$$GH(S) = \frac{100K}{20S(100K_t + (1+0.1S)(1+0.5S)}$$

• It's a type one system; thus, $e_{ss} = \dfrac{M}{K_v}$ M 1 as its unit ramp

$K_v = \lim\limits_{S \to 0} S.GH(S) = \dfrac{5K}{100K_t + 1} = 1.0493 \rightarrow e_{ss} = 0.9523$

Q4)



$$G(S) = \frac{11.4K}{S(S+1.4)(S+10)} \qquad H(S) = \frac{K_g}{1} = 1$$

. Characteristic equation

$$S(S+1.4)(S+10) + 11.4K = 0$$

$(S^2 + 11.4S + 14)$

$$S^3 + 11.4S^2 + 14S + 11.4K = 0$$

. Constraints

→ Fast response with $M_p \leq 9.5\%$

Thus, $Z \geq \frac{-\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}} = 0.6$

. Take $Z = 0.6$ ("Fast" does not insinuate taking any specific value > 0.6 it would still be correct if we do and t_{settle} will go down but it can make $\alpha \geq Z\omega_n.5$ harder) the Standard 3rd order System:

$$S^3 + S^2(2Z\omega_n + \alpha) + S(2Z\omega_n\alpha + \omega_n^2) + \alpha\omega_n^2 = 0$$

- By Comparing coefficients:

$$2Z\omega_n + \alpha = 11.4$$

$$2Z\omega_n\alpha + \omega_n^2 = 14$$

$$\alpha\omega_n^2 = 11.4K$$

also have

$$Z = 0.6$$

* 4 equations and 4 unknowns

→ First eqn: $\alpha = 11.4 - 1.2\omega_n$

.Plug in 2nd eqn:

$$1.2\omega_n(11.4 - 1.2\omega_n) + \omega_n^2 = 14$$

$$13.68\omega_n - 1.2^2\omega_n^2 + \omega_n^2 - 14 = 0$$

$$-0.44\omega_n^2 + 13.68\omega_n - 14 = 0$$

$$\omega_n = 30.3 \quad \text{or} \quad \omega_n = 1.059$$

So

$\alpha = -24.96$ or $\alpha = 10.1292$

unstable $\alpha \geq 5 \times Z\omega_n$

Accepted

Now $K = \dfrac{\alpha \omega_n^2}{11.4} = 0.94095$

- t_r, t_p, t_s are functions of the pole locations (Z, ω_n)

$Z = 0.6$ and $\omega_n = 1.059$

- $t_p = \dfrac{\pi}{\omega_n \sqrt{1 - Z^2}} = 3.7\,s$

- $t_r = \dfrac{\pi - \cos^{-1} Z}{\omega_n \sqrt{1 - Z^2}} = 2.6\,s$

- $t_s = \dfrac{4}{Z\omega_n} = 6.295\,s$

b)

$$G(S) = \frac{11.4K}{S(S+1.4)(S+10)} \qquad H(S) = 1$$

$\cdot\, G(S) = GH(S)$

$$K_p = \lim_{S\to 0} S^0 . G(S) = \infty$$

$$K_v = \lim_{S\to 0} S^1 . G(S) = \frac{11.4K}{1.4\times 10} = 0.81K$$

$$K_a = \lim_{S\to 0} S^2 G(S) = 0$$

because its 'unit'

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0.81K}$$

because its 'ramp'

• The highest value of K (while system is stable) would clearly minimize e_{ss}

→ The characteristic equation is

$$S^3 + 11.4S^2 + 14S + 11.4K = 0$$

Stability Conditions (3rd order system)

→ $1, 11.4, 14, 11.4K > 0 \rightarrow K > 0$

→ $11.4\times 14 - 1\times 11.4K > 0 \rightarrow K < 14$ $(K_{max} = 14)$

Thus, $e_{ss_{min}} = (0.81\times 14)^{-1} = 0.088$