DC Sheet 6 Sol.

Gam-Schmidt Orthogalization

• Given M Signals $g_1(t), g_2(t), \ldots, g_M(t)$

For $i = 1, 2, \ldots, M$

if $i > 1$ $\quad g_{ij} = \int_0^T g_i(t)\,\phi_j(t)\,dt$ for $j = 1, 2, \ldots, i-1$

$$\tilde{\phi}_i(t) = g_i(t) - \underbrace{\sum_{j=1}^{i-1} g_{ij}\,\phi_j(t)}_{\text{if } i>1}$$

$$\phi_i(t) = \frac{\tilde{\phi}_i(t)}{\sqrt{E_{\tilde{\phi}_i}}}$$

Now

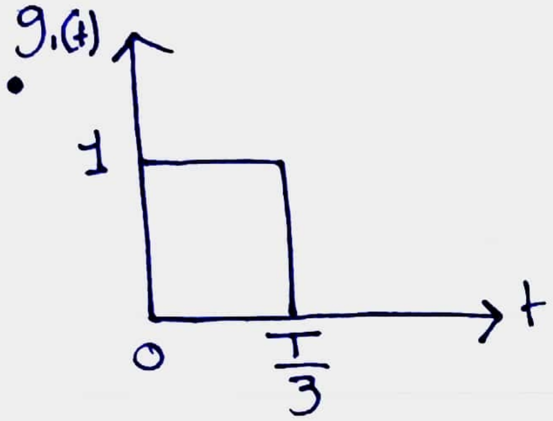
$$g_i = \sum_{j=1}^{i-1} g_{ij}\,\phi_j + \sqrt{E_{\tilde{\phi}_i}}\,\phi_i$$

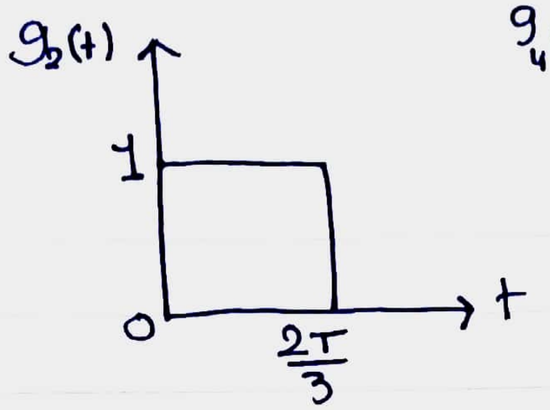
is the Corresponding Point in the Constellation.

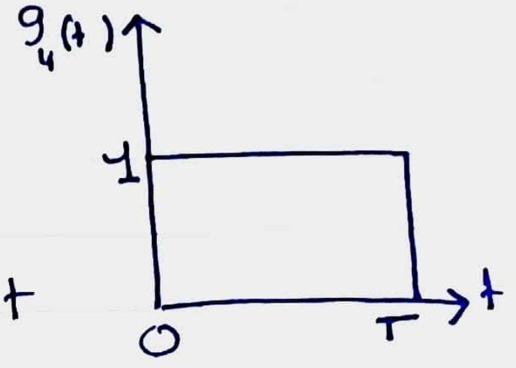
* if $\tilde{\phi}_i = 0$ then exclude it (the g_i at hand is described by other ϕs)

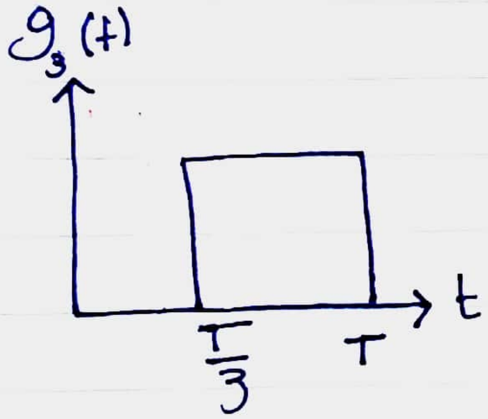
Such order is up to us. • Note that the basis found is generally not unique (depends on order of $g_i(t)$s)

Problem 1)

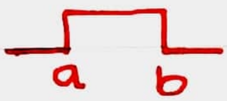








• Clearly, $g_4(t) = g_1(t) + g_3(t)$
So we should only apply Gram-Schmidt on $g_1(t), g_2(t), g_3(t)$

• let $rec(a,b) =$  $\updownarrow 1$

→ has energy $b-a$

$i=1$

→ $\tilde{\phi}_1(t) = g_1(t) = rec(0,\frac{T}{3})$ $\div \sqrt{E_{\tilde{\phi}_1}}$

$$\phi_1(t) = \sqrt{\frac{3}{T}}\, rec(0,\frac{T}{3})$$

• $g_1 = \sqrt{\frac{T}{3}}\,\phi_1$

$i=2$

→ $$g_{21} = \int_0^T g_2(t)\phi_1(t)\,dt$$

$$= \int_0^T rec(0,\frac{2T}{3})\, rec(0,\frac{T}{3})\sqrt{\frac{3}{T}}\,dt$$

$$= \sqrt{\frac{3}{T}}\underbrace{\int_0^T rec(0,\frac{T}{3})\,dt}_{\frac{T}{3}} = \sqrt{\frac{T}{3}}$$

$$\tilde{\phi}_2(t) = g_2(t) - g_{21}\,\phi_1(t)$$

$$= rec(0, \tfrac{2T}{3}) - \sqrt{\tfrac{T}{3}} \cdot \sqrt{\tfrac{3}{T}}\; rec(0, T/3)$$

$$= rec(\tfrac{T}{3}, \tfrac{2T}{3}) \quad \div \sqrt{E_{\tilde{\phi}_2}}$$

$$\phi_2(t) = \sqrt{\tfrac{3}{T}}\; rec(\tfrac{T}{3}, \tfrac{2T}{3})$$

- $g_2 = g_{21}\,\phi_1 + \tilde{\phi}_2 = g_{21}\,\phi_1 + \sqrt{E_{\tilde{\phi}_2}}\;\phi_2$

$$= \sqrt{\tfrac{T}{3}}\,\phi_1 + \sqrt{\tfrac{T}{3}}\,\phi_2$$

$i = 3$

$$\rightarrow \quad g_{31} = \int_0^T g_3(t)\,\phi_1(t)\,dt$$

$$= \int_0^T rec(\tfrac{T}{3}, T)\sqrt{\tfrac{3}{T}}\; rec(0, \tfrac{T}{3})\,dt = 0$$

$$g_{32} = \int_0^T g_3(t)\,\phi_2(t)\,dt$$

$$= \int_0^T rec(\tfrac{T}{3}, T).\sqrt{\tfrac{3}{T}}\; rec(\tfrac{T}{3}, \tfrac{2T}{3})\,dt$$

$$= \sqrt{\tfrac{3}{T}} \int_0^T rec(\tfrac{T}{3}, \tfrac{2T}{3})\,dt = \sqrt{\tfrac{3}{T}} \cdot \tfrac{T}{3} = \sqrt{\tfrac{T}{3}}$$

$$\tilde{\phi}_3(t) = g_3(t) - g_{31}\phi_1(t) - g_{32}\phi_2(t)$$
$$= rec\left(\frac{T}{3}, T\right) - \sqrt{\frac{T}{3}}\, rec\left(\frac{T}{3}, \frac{2T}{3}\right)\sqrt{\frac{3}{T}}$$
$$= rec\left(\frac{2T}{3}, T\right)$$

$$\phi_3(t) = rec\left(\frac{2T}{3}, T\right) . \sqrt{\frac{3}{T}}$$

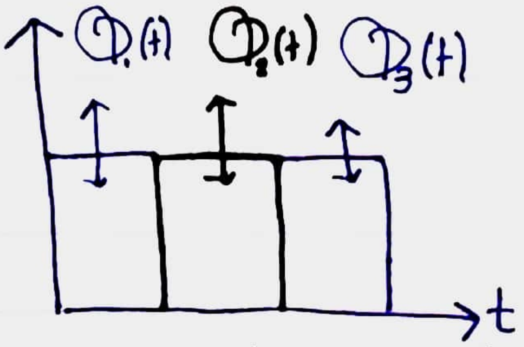
- $g_3 = g_{31}\phi_1 + g_{32}\phi_2 + \sqrt{E_{\tilde{\phi}_3}}\,\phi_3$

$$= \sqrt{\frac{T}{3}}\,\phi_2 + \sqrt{\frac{T}{3}}\,\phi_3$$

$\rightarrow$ $i = M$ so we stop.

we have

$$\phi_1(t) = \sqrt{\frac{3}{T}}\, rec\left(0, \frac{T}{3}\right)$$
$$\phi_2(t) = \sqrt{\frac{3}{T}}\, rec\left(\frac{T}{3}, \frac{2T}{3}\right)$$
$$\phi_3(t) = \sqrt{\frac{3}{T}}\, rec\left(\frac{2T}{3}, T\right)$$

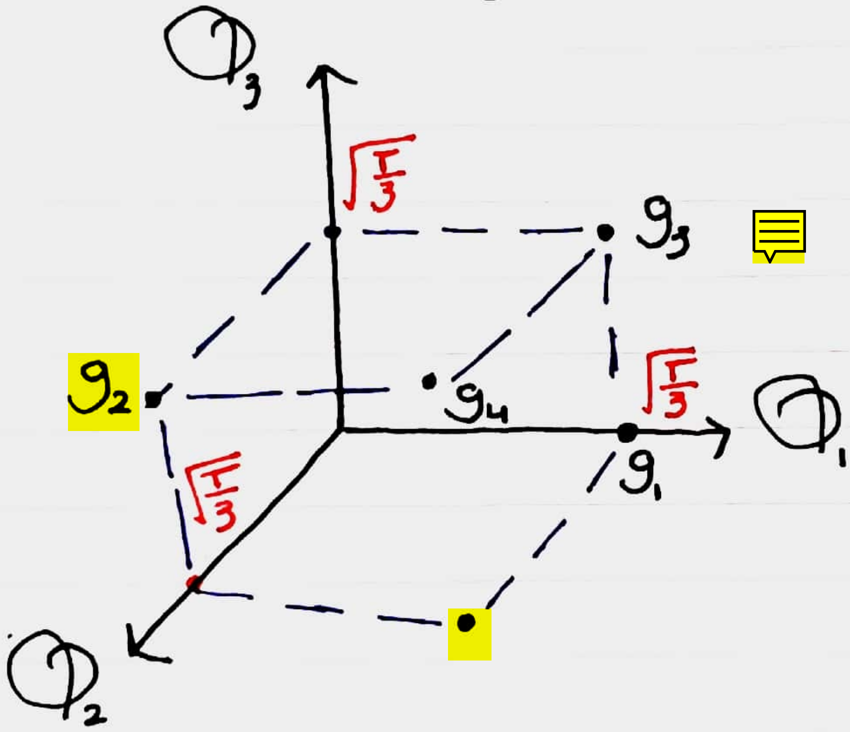


- Can visually confirm being able to generate any of the pulses by scaling ϕ_1, ϕ_2, ϕ_3 then adding

• $g_1 = \sqrt{\frac{T}{3}}\,\phi_1$ • $g_2 = \sqrt{\frac{T}{3}}\,\phi_1 + \sqrt{\frac{T}{3}}\,\phi_2$

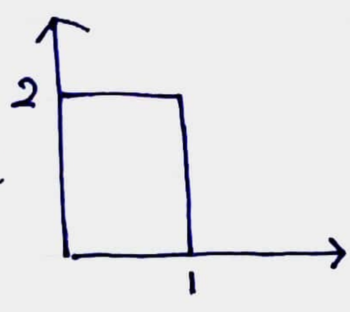
• $g_3 = \sqrt{\frac{T}{3}}\,\phi_2 + \sqrt{\frac{T}{3}}\,\phi_3$ • $g_4 = g_1 + g_3$
$=$

Signal Space Diagram



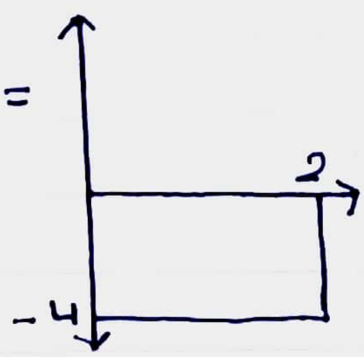
Problem 2)

$g_1(t) =$

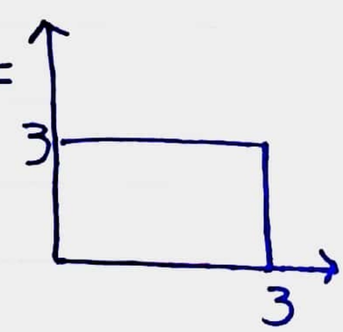


• Start with smaller Pulse.

$g_2(t) =$



$g_3(t) =$



$i = 1$

$\underbrace{\tilde{\phi}_1(t) = 2\,rec(0,1)}_{g_1(t)}$, $E_{\tilde{\phi}_1} = 4$

$\phi_1(t) = rec(0,1)$ • $g_1 = 2\phi_1$

$i=2$

$$g_{21} = \int_0^T g_2(t)\,\phi_1(t)\,dt$$

$$= \int_0^3 -4\,rec(0,2)\,rec(0,1)\,dt = -4$$

$$\tilde{\phi}_2(t) = g_2(t) - g_{21}\phi_1(t)$$

$$= -4\,rec(0,2) + 4\,rec(0,1) = -4\,rec(1,2)$$

$$\phi_2(t) = -rec(1,2) \quad \leftarrow \div \sqrt{E_{\tilde{\phi}}} = \sqrt{16}$$

*

- $g_2 = g_{21}\phi_1 + \sqrt{E_{\tilde{\phi}_2}}\,\phi_2 = -4\phi_1 + 4\phi_2$

$i=3$

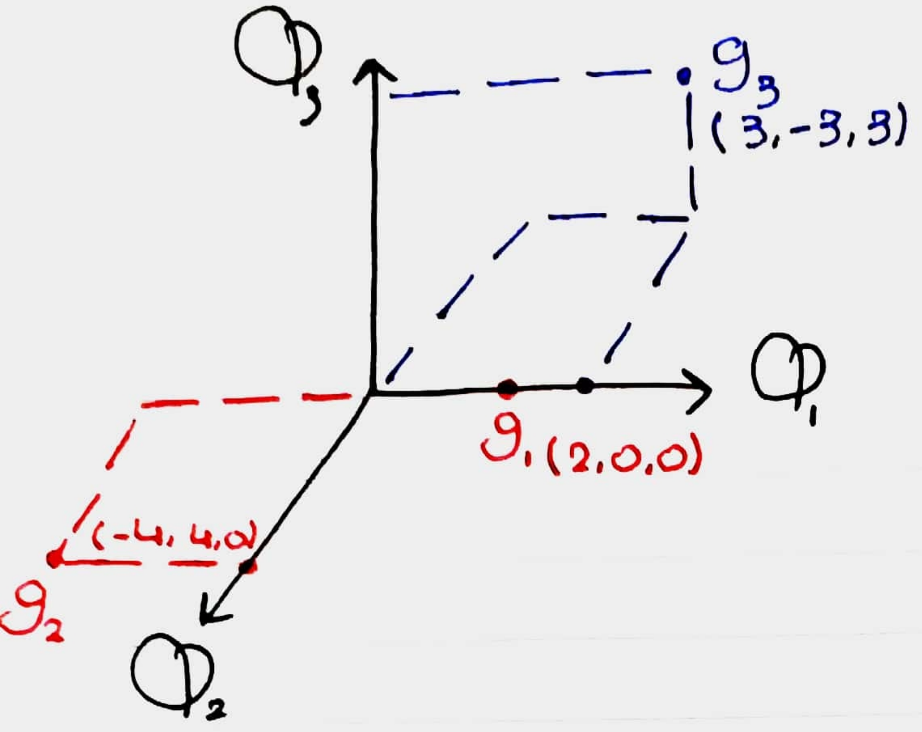
$$g_{31} = \int_0^3 3\,rec(0,3)\,rec(0,1)\,dt = 3$$

$$g_{32} = \int_0^3 3\,rec(0,3)\cdot -rec(1,2)\,dt = -3$$

$$\tilde{\phi}_3(t) = 3\,rec(0,3) - 3\,rec(0,1) - 3\,rec(1,2)$$

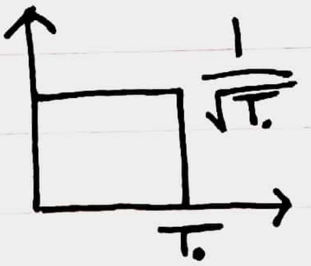
$$= 3\,rec(2,3) \longrightarrow \phi_3(t) = rec(2,3)$$

- $g_3 = 3\phi_1 - 3\phi_2 + 3\phi_3$

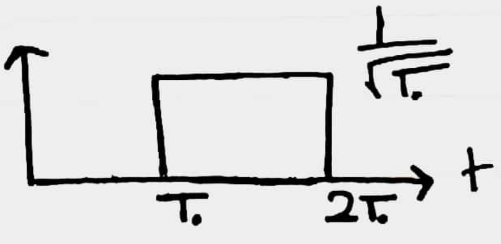


Problem 3)

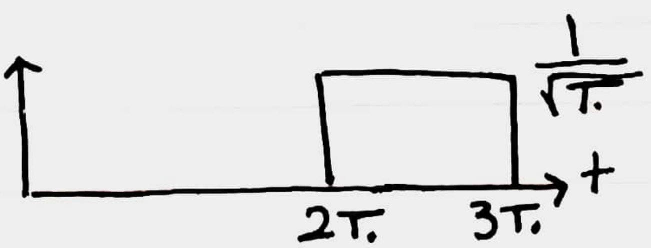
$\phi_1(t) = P(t) =$



$\phi_2(t) = P(t-T_0) =$

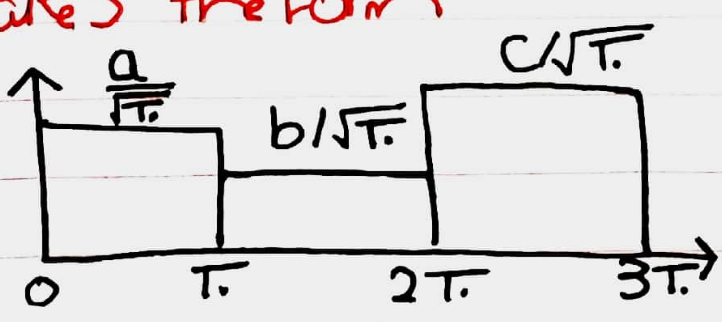


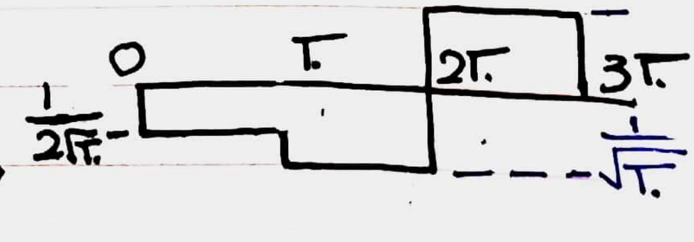
$\phi_3(t) = P(t-2T_0) =$



- Any Signal in the Space has Components (a,b,c) and takes the form

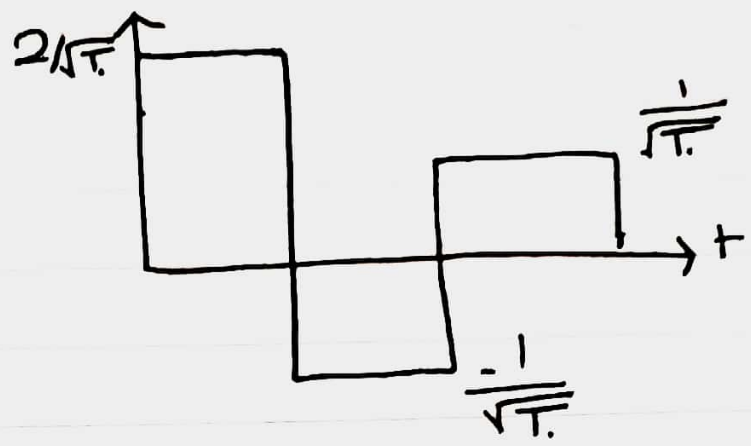
e.g. if $g_1 = (-\frac{1}{2}, -1, 1)$



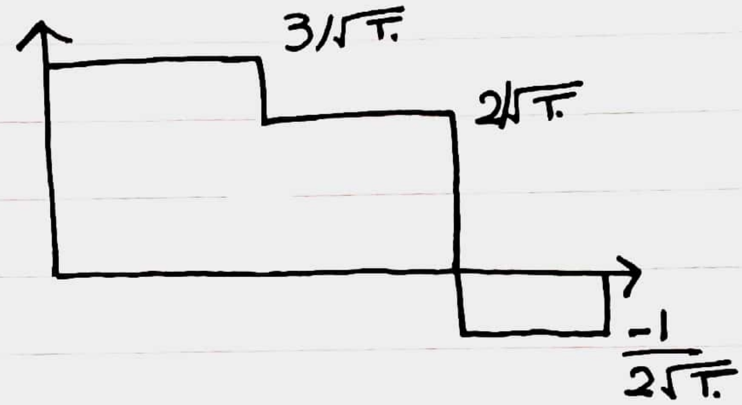


$(1,1,0)$

$(2,-1,1)$



$(3, 2, \frac{-1}{2})$



Problem 4)

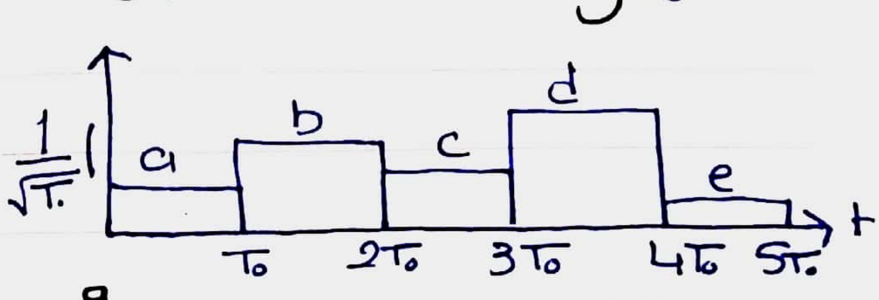
$\varphi_1(t) = P(t)$

$\varphi_2(t) = P(t - T_0)$

...

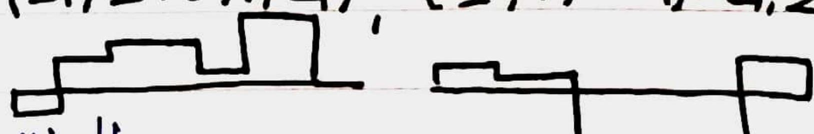
$\varphi_5(t) = P(t - 4T_0)$

• So any signal $g_i(t)$ can be represented by $g_i = (a, b, c, d, e)$



g_1

• given are 4 signals $(-1, 2, 3, 1, 4)$, $(2, 1, -4, -4, 2)$, ...



→ Recall, $g_i^T g_k = \int_0^T g_i(t)\, g_k(t)\, dt$

• $\|g_i\| = \sqrt{E_{g_i}}$ (e.g. $E_{g_1} = 1^2 + 2^2 + 3^2 + 1^2 + 4^2 = 31$) • $g_i^T g_k = 0$ Only for $i = 3, k = 4$