Sheet 1

$\boxed{1}$ Counter example

$4 \mid (1+3)$ but $4 \nmid 1 \wedge 4 \nmid 3$

$\boxed{2}$ if $3 \mid a$, proof is done

if $3 \nmid a \rightarrow a = 3q + r \ \forall\, r \in \{1, 2\}$

Case I: $\boxed{r = 1}$

$a = 3q + 1$

$a + 2 = 3q + 3$

$a + 2 = 3(q+1)$

$3 \mid (a+2)$

Case II: $\boxed{r = 2}$

$a = 3q + 2$

$a + 4 = 3q + 6$

$a + 4 = 3(q+2)$

$3 \mid (a+4)$.

$\boxed{3}$ let $a = 2K+1$ & $b = 2q+1$

from Binomial theorem

1 4 6 4 1

$$a^4 = (2K+1)^4 = (2K)^4 + 4(2K)^3 + 6(2K)^2 + 4(2K) + 1$$

$$a^4 = 16K^4 + 32K^3 + 24K^2 + 8K + 1$$

$$b^4 = 16q^4 + 32q^3 + 24q^2 + 8q + 1$$

$$\therefore a^4 + b^4 - 2 = 16(K^4 + q^4 + 2K^3 + 2q^3) + 8(K(3K+1) + q(3q+1)).$$

$$16 \mid 16(K^4 + q^4 + 2K^3 + 2q^3) \quad (1)$$

for $8(K(3K+1) + q(3q+1))$

if K & q are odd,

then $(3K+1)$ & $(3q+1)$ are even

$3K+1 = 2m$ & $3q+1 = 2n$

$$\therefore 8(K(3K+1) + q(3q+1)) = 16(Km + qn)$$

$\therefore 16 \mid 16(Km+qn)$, hence $16 \mid 8(K(3K+1)+q(3q+1))$ (i) if K & q are even,

then $K=2m$ & $q=2n$

$$8(K(3K+1)+q(3q+1)) = 16(m(6m+1)+n(6n+1))$$

$$\therefore 16 \mid 16(m(6m+1)+n(6n+1))$$

$$\therefore 16 \mid 8(K(3K+1)+q(3q+1)) \quad \text{(ii)}$$

W.L.O.G. if K is even & q is odd

$K=2m$ & $3q+1=2n$

$$8(K(3K+1)+q(3q+1)) = 16(m(6m+1)+(2n-1)n)$$

$$\therefore 16 \mid 16(m(6m+1)+n(2n-1))$$

$$\therefore 16 \mid 8(K(3K+1)+q(3q+1)) \quad \text{(iii)}$$

from (i), (ii) & (iii)

$$16 \mid 8(K(3K+1)+q(3q+1)) \quad (2)$$

from (1) & (2)

$$16 \mid (a^4+b^4-2).$$

(4) Counter example.

$9 \mid (3 * 15)$ but $9 \nmid 3$ & $9 \nmid 15$.

(5) Case 1: n is odd

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1$

since $4k^2 \equiv 0 \pmod 4$

$4k \equiv 0 \pmod 4$

$1 \equiv 1 \pmod 4$

then, $n^2 \equiv 1 \pmod 4$

Case 2: n is even

$n = 2k$

$n^2 = 4k^2$

since $4k^2 \equiv 0 \pmod 4$

then $n^2 \equiv 0 \pmod 4$

from Case 1 & 2

$n^2 \equiv 0$ or $1 \pmod 4$.

(b) let $n = 2K+1$ where $K \geq 0$

$$n^2 = 4K^2 + 4K + 1$$

$$n^2 = 4(K(K+1)) + 1$$

Case 1: K is odd

$\therefore$ $K+1$ is even

$K+1 = 2m$

$$n^2 = 8((2m-1)m) + 1$$

$$8((2m-1)m) \equiv 0 \pmod 8$$

$$1 \equiv 1 \pmod 8$$

$\therefore$ $n^2 \equiv 1 \pmod 8$

Case 2: K is even

$\therefore$ $K = 2m$

$$n^2 = 8(m(2m+1)) + 1$$

since $8(m(2m+1)) \equiv 0 \pmod 8$

$$1 \equiv 1 \pmod 8$$

$$n^2 \equiv 1 \pmod 8$$

from Case 1 & 2 $n^2 \equiv 1 \pmod 8$.

(7) $n|m \rightarrow m = cn$ where $c \in \mathbb{Z}^+$

$a \equiv b \pmod{m} \rightarrow m \mid (a-b)$

$a - b = km$

$a - b = kcn$

$\therefore n \mid (a-b)$

hence, $a \equiv b \pmod{n}$.