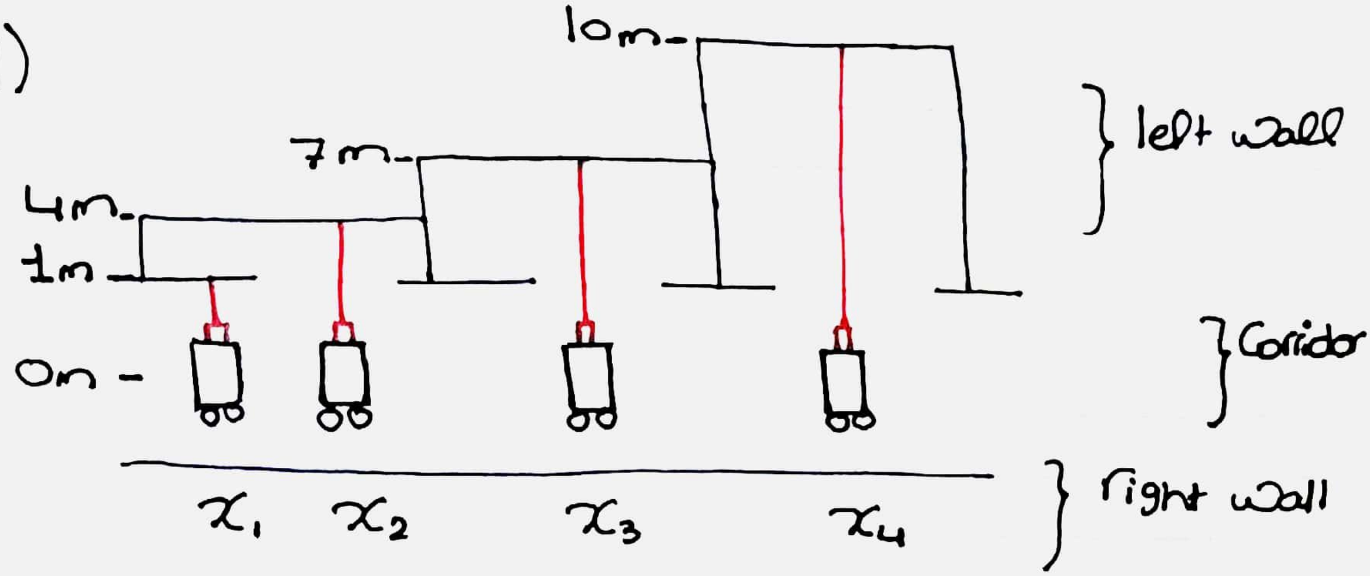
CR Sheet 2

2.1)



1. Robot moves along corridor & has accurate map

2. At locations x_1, x_2, x_3, x_4 it takes measurment Z_k using laser beam to find distance to left wall.

3. The measured distances are

Z_1	Z_2	Z_3	Z_4
2.5m	8m	6m	9m

and its known that they are corrupted with additive Gaussian noise $\sim N(\mu = 1m, \sigma = 2m)$

4. The mapping between Z_k and x_i is unknown

we don't know what x_i Caused reading Z_k

* We know that

$$Z_k = h(x) + \delta$$

noise

actual reading

reading Corresponding to State ignoring noise

- Known for x_1, x_2, x_3, x_4 from map

a) • For each of the four measurments, determine the most likely robot pose

1

• Assume an equally distributed prior & assert that in the end $\sum_i P(x_i \mid z) = 1$

5

$\Rightarrow$ Sol.

For each reading z_K
See which location x_i
is most probable given the reading z_K

i.e., we want $P(x_i \mid z_K)$ for $K = 1, 2, 3, 4$ and $i = 1, 2, 3, 4$

• Suppose the reading z_1 was due to x_2, then
$Z_1 = h(x_2) + \delta = 7 + \delta \sim N(\mu = 7+1, \sigma = 2)$
which corresponds to the probability

$$P(z_1 \mid x_2) = \frac{1}{\sqrt{2\pi \cdot 2}} \cdot e^{-\frac{(z_1 - 8)^2}{8}}$$

Can plug with our r

$\hookrightarrow$ Prob. of z_1 given that it was due to x_2.

$\leftarrow$ Can plug our z_1 reading

• Can as well find it for x_1, x_3, x_4

• Then by Bayes: $P(x_i \mid z_1) = \dfrac{P(z_1 \mid x_i) P(x_i)}{P(z_1)}$

$\rightarrow P(x_i) = \frac{1}{4}$ due to 1

$\longrightarrow$ After computing for all x_i, predict the location to be the one that maximizes it.

• Generaling the Computation, we have:

$$P(Z_K | x_i) = \frac{1}{2\sqrt{2\pi}} \cdot e^{\frac{-(Z_K-(h(x_i)+1))^2}{8}}$$

$$P(x_i | Z_K) = \frac{P(Z_K | x_i)\, P(x_i)}{\sum_{j=1}^{4} P(Z_K | x_j) P(x_j)}$$

• $P(x_i) = \frac{1}{4}$

• Can cancel out $\frac{1}{2\sqrt{2\pi}}$

$$= \frac{e^{-(Z_K - h(x_i) - 1)^2/8}}{\sum_{j=1}^{4} e^{-(Z_K - h(x_j) - 1)^2/8}} \quad (1)$$

	$h(x_1)=1$ x_1	$h(x_2)=4$ x_2	$h(x_3)=7$ x_3	$h(x_4)=10$ x_4
$Z_1=2.5$	0.96	0.45	0.0228	0.0001
$Z_2=8$	0.011	0.32	1	0.32
$Z_3=6$	0.135	0.882	0.606	0.044
$Z_4=9$	0.0022	0.135	0.882	0.606

• By Plugging for the numerator

• Dividing by the Sum of the row is eq. to dividing by (1)

• Circled is the best Pred. for each Z

Converting to Prob. (normalizing) we get

	x_1	x_2	x_3	x_4	
$z_1=2.5$	0.67	0.314	0.016	0.0000698	• Predict x_1
$z_2=8$	0.0067	0.194	0.606	0.194	• Predict x_3
$z_3=6$	0.0809	0.529	0.363	0.026	• Predict x_2
$z_4=9$	0.0013	0.0832	0.543	0.372	• Predict x_3

- Notice that it makes sense that the robot is less confident about its decision for z_3 and z_4 (highest Prob. not much higher than others) because the measured z_3 and z_4 lie between x_2, x_3 and x_3, x_4 respectively and each of these pairs involve two locations really close to each other (3 units difference when $\sigma=2$)

b) Robot believes that taking measurements at positions x_2 and x_3 is in general two times as likely as doing so at x_1 and x_4

→ Use the new prior info to calculate the Probs

$$P(x_i \mid z_k) = \frac{P(z_k \mid x_i)P(x_i)}{\sum_{j=1}^{4} P(z_k \mid x_j)P(x_j)}$$

- We know

→ $P(x_1)+P(x_2)+P(x_3)+P(x_4)=1$

→ $P(x_3)=P(x_2)=2P(x_1)=2P(x_4)$

Thus, $P(x_2)=P(x_3)=\frac{1}{3}$ and $P(x_1)=P(x_4)=\frac{1}{6}$

$$P(x_i \mid z_k) \propto \frac{1}{6} e^{-(z_k - h(x_i) -1)^2/8} \qquad i=1,4$$

$$P(x_i \mid z_k) \propto \underbrace{\frac{1}{3} e^{-(z_k - h(x_i) -1)^2/8}}_{\text{numerator of last eqn.}} \qquad i=2,3$$

- This implies multiplying the 1st & last col. in the table 2 pages ago by $\frac{1}{6}$, the 2nd & 3rd col. by $\frac{1}{3}$ then renormalizing to get table like in last page

c) - Laser scanner reports a faulty measurment of $Z=30$ in 10% of all cases, no matter the actual distance

→ Means that given any x_i, there is 90% chance Z_K will be caused by it (i.e., will come from the dist. $\frac{1}{\sqrt{2\pi}\,2} e^{-(Z_K - h(x_i) -1)^2/8}$) and 10% chance it will by 30 (i.e., will come from the dist.* that's 0 everywhere but 1 at 30)

- Consequently,

$$P(Z_K \mid x_i) = 0.9\left(\frac{1}{\sqrt{2\pi}\,2} \cdot e^{-(Z_K - h(x_i) -1)^2/8}\right) + 0.1(\underbrace{1.\,I(Z_K=30)}_{1 \text{ if } Z_K=30 \text{ else zero}})$$

→ Since $\tau_K = 30$ never occurs for $\tau_1, \tau_2, \tau_3, \tau_4$ we have that

$$P(\tau_K | x_i) = 0.9 \frac{1}{\sqrt{2\pi} 2} e^{-(\tau_K - h(x_i) - 1)^2/8} \quad , K = 1,2,3,4$$

- 0.9 will cancel out from the denominator & numerator while computing $P(x_i | \tau_K)$

→ hence, no probabibilities in a or b will change.

* Notice that this would've not been the case if
  - $Z_K = 30$ occured in our measurment (it would increase its prob. given x_i)
  - The faulty measurment doesn't necessarily result in $Z=30$ (e.g. follows some dist.) ← add instead of 1
  - Perhaps, will see more of this when we get to sensor models.

2.2)

Extended Kalman Filter $(\mu_{t-1}, \Sigma_{t-1})$

Prediction:

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$$

Correction:

$$\hat{z}_t = h(\bar{\mu}_t)$$

$$S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$$

$$K_t = \bar{\Sigma}_t H_t^T S_t$$

$$\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$$

$$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$$

* Will continue solving in word.

Component	What it represents	Why its needed
$\boldsymbol{\mu}_{t-1}$	1. The mean of the robot's belief about the state in the last time step 2. Because the distribution its Gaussian, it's also the best candidate for the estimated state in that time step.	1. To predict the state (belief mean) in the current time step after executing an action $\bar{\mu}_t$ 2. Can also be used for correction μ_t when there's no action
$\boldsymbol{\Sigma}_{t-1}$	1. The covariance of the robot's belief about the state in the last time step	1. To predict the state (belief mean) in the current time step after executing an action $\bar{\Sigma}_t$ 2. Can also be used for correction $\boldsymbol{\Sigma}_t$ when there's no action
$\bar{\boldsymbol{\mu}}_t$	The mean of the predicted belief (also corresponds to predicted state)	Gives an estimate of the belief's mean after an action is executed. E.g., needed so the robot can know where it is after executing action.
$\bar{\boldsymbol{\Sigma}}_t$	The covariance of the predicted belief	Gives an estimate of the belief's covariance after an action is executed. E.g., needed so the uncertainty in the robot's prediction of the state can be quantified.
$\boldsymbol{g}(\boldsymbol{u}_t, \boldsymbol{x}_{t-1})$	The ideal, generally nonlinear mapping from previous state and action to next state. It represents the action/motion model.	1. For the robot to predict the mean of the belief after executing an action $\bar{\boldsymbol{\mu}}_t = \boldsymbol{g}\left(\boldsymbol{u}_t, \boldsymbol{\mu}_{t-1_{t-1}}\right)$ 2. Also needed for G_t
$\boldsymbol{G}_t$	The Jacobian matrix of $g(u_t, x_{t-1})$ with the respect to the state x_{t-1} and evaluated at $x_{t-1} = \mu_{t-1}$	Provides a first-order approximation of g(.) which justifies the predicted mean computation as well as the predicted covariance where its directly involved $\bar{\boldsymbol{\Sigma}}_t = \boldsymbol{G}_t \boldsymbol{\Sigma}_{t-1} \boldsymbol{G}_t^T + \boldsymbol{R}_t$

$\boldsymbol{R_t}$	The covariance matrix of the zero-mean additive Gaussian noise that affects the action model	Needed to perform the covariance prediction (as above) which model's the robot's uncertainty after doing an action
$\boldsymbol{\hat{z}_t}$	The mean of the predicted measurement, given the noisy sensor model	Needed for comparison with the actual measurement later to decide, based on the amount of uncertainty in that prediction, the amount of correction needed to compute μ_t
$\boldsymbol{S_t}$	The covariance of the predicted measurment, given the noisy sensor model. It represents the uncertainty in the predicted measurment	Needed to compute the Kalman Gain K_t
$\boldsymbol{H_t}$	The Jacobian matrix of h(x_t) with the respect to the state x_t and evaluated at $x_t = \bar{\mu}_t$	Provides a first-order approximation of h(.) which justifies the predicted measurment mean computation as well as the predicted measurment covariance where its directly involved $S_t = \boldsymbol{H_t}\bar{\Sigma}_t\boldsymbol{H_t^T} + \boldsymbol{Q_t}$
$\boldsymbol{Q_t}$	The covariance matrix of the zero-mean additive Gaussian noise that affects the sensor model	Needed to perform the covariance prediction (as above) which model's the uncertainty in the predicted measurment
$\boldsymbol{K_t}$	The Kalman Gain. Represents the uncertainty of the robot's prediction of the measurment assuming no noise relative to the uncertainty in the measurment assuming noise.	Decides how much of the difference between the actual measurment and the predicted one should be propagated as a correction.
$\boldsymbol{\mu_t}$	The mean of the corrected belief	Estimate the current state and used to estimate next state belief
$\boldsymbol{\Sigma_t}$	The covariance of the corrected belief	Know the uncertainty in the current state estimation and used to estimate next state belief

2.3)

EKF	UKF
Commonalities	
Both generalize the Kalman filter to handle nonlinear action/sensor models where the world is no longer Guaranteed to be Gaussian.	
Both can hence be perceived as Bayes filters that approximate the world as Gaussian	
Neither is optimal, they are just approximations	
Both can diverge if the non-linearities are extreme enough	
Both have the same computational complexity (highly efficient)	
Differences	
In practice, faster with a constant factor	
Accurate to only 1st order terms in the Taylor expansion of the nonlinearity	Better approximation; accurate up to the 2nd order terms
Linearizes the non-linearity then transforms the previous Gaussian through it to get the new Gaussian	Choose sigma points and transforms them through the non-linear Gaussian then uses weights to construct the new Gaussian
Need to be able to compute the Jacobians of the nonlinearities	It's a derivative-free filter. Can handle cases where the Jacobians are hard to compute or can't be computed (indifferentiability).
	Involves tuning hyperparameters
Can be perceived as a Kalman filter if we let A=G and C=H while using the linearized equations/non-linearities for mean updates.	