CR Sheet 1

1.1)

- A Person receives a Positive on a First-Stage test for a Serious rare disease. ①

- test reports are False Positive with a Probability of 0.05

  i.e., $P(test = +ve \mid truth = -ve) = 0.05$

- For Simplicity, assume no False negatives

  i.e., $P(test = -ve \mid truth = +ve) = 0$

- Find the Probability of the Person ①, actually Suffering from the disease

  i.e., $P(truth = +ve \mid test = +ve)$

- Add to your Knowledge that, one in every 50000 in the Population Suffer from disease.

  i.e., $P(truth = +ve) = \frac{1}{50000}$

⇒ Sol.

$$P(truth = +ve \mid test = +ve)$$

• Bayes Rule

$$= \frac{P(test = +ve \mid truth = +ve)\,P(truth = +ve)}{P(test = +ve \mid truth = +ve)\,P(truth = +ve) + P(test = +ve \mid truth = -ve)\,P(truth = -ve)}$$

$$= \frac{(1-0)(1/50000)}{(1-0)(1/50000) + (0.05)(1-1/50000)} = 3.99 \times 10^{-4}$$

- Although the Probability is Pretty Small, Knowing that the test is +ve has increased the Probablity by a factor of 20. $(4\times10^{-4}/2\times10^{-5})$

1.2)

- A robot is equipped with an unreliable Person detector that outputs "Person" or "No Person".

- If there's a Person in front of the door, it indicates "Person" with a Probability of 0.7

i.e., $P(\text{Output}=\text{"Per"} \mid \text{actual}=\text{Per}) = 0.7$

- If there's no Person in front of the door the detector indicates "Person" with Probability 0.2

i.e., $P(\text{Output}=\text{"Per"} \mid \text{actual}=\neg\text{Per}) = 0.2$

- Prior belief before observing detector is 0.5

i.e., $P(\text{actual}=\text{Per}) = 0.5$

→ Find Posterior Prob. of Person being in front of door when "no Person" is detected.

- $$P(\text{Per} \mid \neg\text{"Per"}) = \frac{P(\neg\text{"Per"} \mid \text{Per})\,P(\text{Per})}{P(\neg\text{"Per"} \mid \text{Per})\,P(\text{Per}) + P(\neg\text{"Per"} \mid \neg\text{Per})\,P(\neg\text{Per})}$$

$$= \frac{(1-0.7)(0.5)}{(1-0.7)(0.5)+(1-0.2)(0.5)}$$

$$= 0.27273$$

Recall,

$$P(Y \mid X) = \frac{P(X \mid Y)\,P(Y)}{\sum_y P(X \mid y)\,P(y)}$$

1.3)

- Consider a 2D State $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ← Position in m ← Velocity in m/s

- distance between two time steps t and t+1 is 0.1 seconds

→ describe a matrix A that maps from x_t to x_{t+1} in the noiseless case $X_{t+1} = AX_t$

⇒ Sol.

- Equations of Motion

$$V = V_0 + a(t - t_0)$$
$$S = V_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 + S_0$$

- By assuming $a = 0$ (next Problem it won't) and taking t_0 to be the last time step:

$$V_t = V_{t-1}$$

$$S_t = S_{t-1} + V_{t-1}(0.1)$$

- recall, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

hence,

$$\begin{pmatrix} S \\ V \end{pmatrix}_t = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} S \\ V \end{pmatrix}_{t-1}$$

- In this Problem's notation:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{t+1} = \underbrace{\begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_t$$

1.4) • Consider control action U_t that acts as a const. acceleration (m/s^2)

• How should the matrix B look like ($X_{t+1} = AX_t + BU_t$)

→ Follow up on previous problem

⇒ Sol.

• Equations of Motion

$$V = V_0 + a(t-t_0)$$

$$S = S_0 + V_0(t-t_0) + \frac{1}{2}a(t-t_0)^2$$

→ V, S are part of the state and a is part of the action.

• Rewriting for convenience

$$V_t = 0 \times S_{t-1} + 1 \times V_{t-1} + (0.1) \times a$$

$$S_t = 1 \times S_{t-1} + (0.1) \times V_{t-1} + (0.05)a$$

$$\begin{pmatrix} S \\ V \end{pmatrix}_t = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S \\ V \end{pmatrix}_{t-1} + \begin{pmatrix} 0.05 \\ 0.1 \end{pmatrix} (a)$$

hence,

$$X_{t+1} = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix} X_t + \begin{pmatrix} 0.05 \\ 0.1 \end{pmatrix} U_t$$

← describes how the action (acc. at any second) affects comp. of the state

• As a check, recall if X is $(n \times 1)$ and U is $(l \times 1)$ then A is $(n \times n)$ and B is $(n \times l)$.

1.5) $\cdot\ X_t = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- You can only measure velocity
  i.e., Z_t is a $K \times 1$ vector where $K=1$

  $Z_t = (Z_2)$

- Find the matrix C that maps state to measurment
  i.e., C in $Z_t = C X_t$ $\qquad\qquad$ $\cdot$ C is $(K \times n)$

$\Rightarrow$ Sol.

- Clearly, if the sensor measures velocity then it must be that $Z_2 = x_2$

  i.e., $Z_2 = 0 \times x_1 + 1 \times x_2 = (0 \quad 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

  $$Z_t = \underbrace{(0 \quad 1)}_{C} X_t$$

- Of course in reality, noise δ_t is added to the measurment.

1.6)

Kalman Filter:

- Prediction

  $\bar{\mu}_t = A_t \mu_{t-1} + B u_t$

  $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

- Correction

  $S_t = C_t \bar{\Sigma}_t C_t^T + Q_t$

  $K_t = \bar{\Sigma}_t C_t^T S_t^{-1}$

  $\mu_t = \bar{\mu}_t + K_t (Z_t - C_t \bar{\mu}_t)$

  $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

• Start with

$$X_0 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ and } \Sigma_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

→ Motion has noise Covariance $R = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.04 \end{pmatrix}$

• Find Kalman filter Prediction for $t=1$ $(\equiv t = 0.1s)$
when $u_1 = 3\ m/s^2$

⇒ Sol.

• We know ~~for certainty~~ * that initially $X_0 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and hence that's the best value for μ_0

* or estimation

• By Plugging in Prediction eqn.s

$$\bar{\mu}_t = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0.05 \\ 0.1 \end{pmatrix}(3)$$

$$= \begin{pmatrix} 3 - 0.1 + 0.05 \times 3 \\ -1 + 0.1 \times 3 \end{pmatrix} = \begin{pmatrix} 2.915 \\ -0.7 \end{pmatrix}$$

$$\bar{\Sigma}_t = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0.1 & 1 \end{pmatrix} + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.04 \end{pmatrix}$$

• $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} \dots \end{pmatrix} = \begin{pmatrix} \dots \end{pmatrix}\begin{matrix} \times a \\ \times b \end{matrix}$

$$= \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0.1 & 1 \end{pmatrix} + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.04 \end{pmatrix}$$

$$= \begin{pmatrix} 4.01 & 0.1 \\ 0.1 & 1 \end{pmatrix} + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.04 \end{pmatrix}$$

$$= \begin{pmatrix} 4.11 & 0.1 \\ 0.1 & 1.04 \end{pmatrix}$$

• Symmetric as expected

- Notice that we were at $x_1 = 3$ and moving back at $x_2 = 1$ m/s, now that one time step has passed, given that acceleration is pushing forward at 3 m/s^2 we think we are at $x_1 = 2.914$ and that $x_2 = 0.7$ m/s backward (Slower due to opposite acceleration $-1 + 0.1 \times 3 = -0.7$ by physics, even x_1 can be confirmed by physics)

- There is an obvious increase in uncertainty, looking at Σ

1.7) • We make a position measurment of $z_1 = 2$m with $\sigma = 0.1$
- Find mean and covariance of corrected state

$\Rightarrow$ Sol
- In this case, for $Z_t = (Z_1)_t = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_t$ it must be that $C = (1 \quad 0)$
  - $\rightarrow$ we previously assumed the opposite (only velocity measurment, $C = (0 \quad 1)$)

- Q_t is $K \times K$ (where Z_t is $K \times 1$)
  - $\rightarrow$ i.e., $Q_t = (0.1^2)$

  • Cov[i,i] is variance

- Thereby, by plugging in correction equations:

$$S_t = (1 \quad 0) \begin{pmatrix} 4.11 & 0.1 \\ 0.1 & 1.04 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0.01)$$

• $W^T A W = \sum_{i,j} w_i w_j A_{ij}$

$$= 4.11 \times 1 \times 1 + 0.01 = 4.12$$

$$K_t = \begin{pmatrix} 4.11 & 0.1 \\ 0.1 & 1.04 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{4.12} = \begin{pmatrix} 4.11 \\ 0.1 \end{pmatrix} \cdot \frac{1}{4.12}$$

$$= \begin{pmatrix} 0.998 \\ 0.024 \end{pmatrix}$$

• Will keep 3 digits after dot.

$$M_t = \begin{pmatrix} 2.915 \\ -0.7 \end{pmatrix} + \begin{pmatrix} 0.998 \\ 0.024 \end{pmatrix} \overbrace{\left((2) - (1 \;\; 0) \begin{pmatrix} 2.915 \\ -0.7 \end{pmatrix} \right)}^{-0.915}$$

$$= \begin{pmatrix} 2.915 - 0.915 \times 0.998 \\ -0.7 - 0.024 \times 0.915 \end{pmatrix} = \begin{pmatrix} 2.002 \\ -0.722 \end{pmatrix}$$

$$\Sigma_t = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.998 \\ 0.024 \end{pmatrix} (1 \;\; 0) \right) \begin{pmatrix} 4.11 & 0.1 \\ 0.1 & 1.04 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.998 & 0 \\ 0.024 & 0 \end{pmatrix} \right) \begin{pmatrix} 4.11 & 0.1 \\ 0.1 & 1.04 \end{pmatrix}$$

$$= \begin{pmatrix} 0.002 & 0 \\ -0.024 & 1 \end{pmatrix} \begin{pmatrix} 4.11 & 0.1 \\ 0.1 & 1.04 \end{pmatrix}$$

$$= \begin{pmatrix} 0.008 & 0 \\ 0.001 & 1.038 \end{pmatrix}$$

* Notice that because the measurm. had a different opinion about the position and had low variance, a large correction for position was made (2.915 → 2.002)
- Meanwhile, Kalman gain for velocity was low (small correction) because our position measurment isn't so helpful for that, (backward velocity must've been able to more strongly resist acc. if we actually ended up at 2)

- much of this can be done via calculator
- digits considered while rounding will affect the answer.

explaining the increase 10.71 → 10.7221