1. select the features.
2. feature selection and extraction
3. select the classifier to be applied.

Pattern Classification

05. Density Estimation

after this lecture, we can build a bayes classifier.
8aleban da el lab el gay.

AbdElMoniem Bayoumi, PhD

Fall 2021

Recap: Gaussian Densities

- Assume a multi dimensional Gaussian density for each $P(\underline{X}|C_i)$

- Features may be independent (or conditionally independent), i.e., independent Gaussians

- Features may be dependent in other cases

Recap: Applying Bayes Rule

- One way on how to apply Bayes rule in practical situations:
  - Obtain the training set $\underline{X}(1), \underline{X}(2) \cdots \underline{X}(M)$
  - Assume a multi-dimensional Gaussian density for each class, i.e., $P(\underline{X}|C_i)$
  - To obtain the form of each density we need $\underline{\mu}_i$ and Σ_i for each class i → estimate from training set
  - Estimate the a priori probabilities $P(C_i)$ from the training set, i.e., according to the frequencies of each class
  - Using the obtained estimates, plug in Bayes rule to obtain the classification rule

Density Estimation

- In Bayes rule, the probability densities have to be estimated

- One way is to assume that they are multivariate Gaussian and estimate μ & Σ of these distributions

- Estimate the densities from data

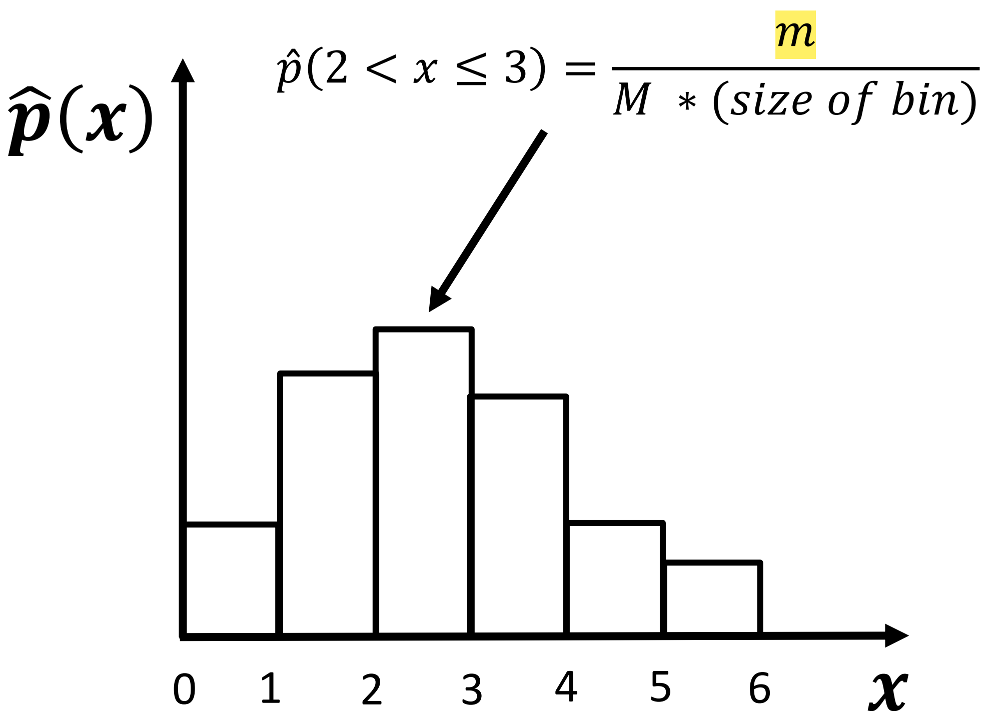
Histogram Analysis

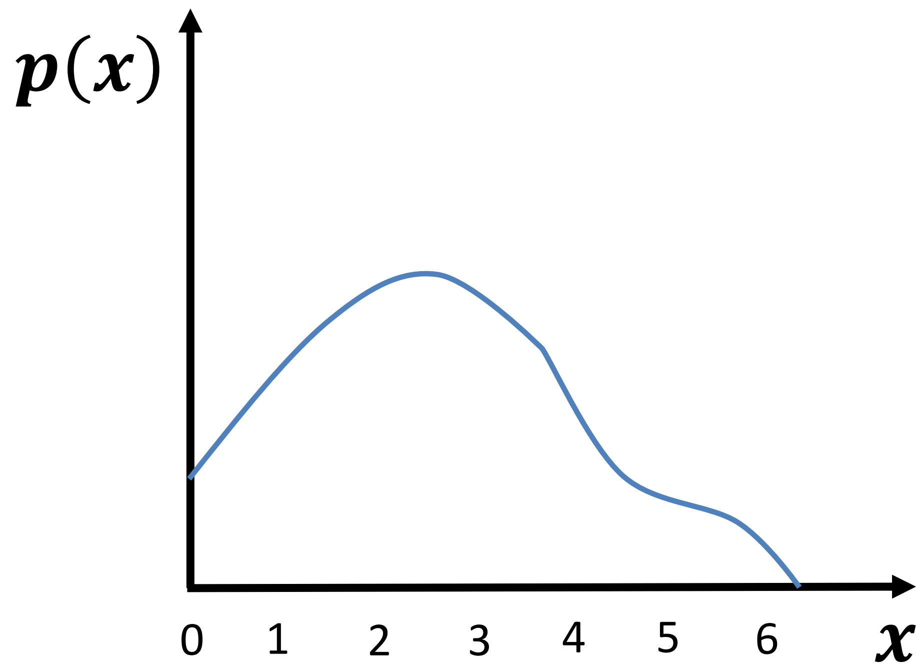
$$\hat{p}(x) = \frac{m}{M \ * (size\ of\ bin)}$$

- m is the number of data points falling within a given range, i.e., histogram bin
- M is the total number of points (that belongs to the same class)
- Size of bin: size of the histogram bin

Histogram Analysis

- Consider 1-D example:
  - m is number of data points within the given range, e.g., $2 < x \leq 3$





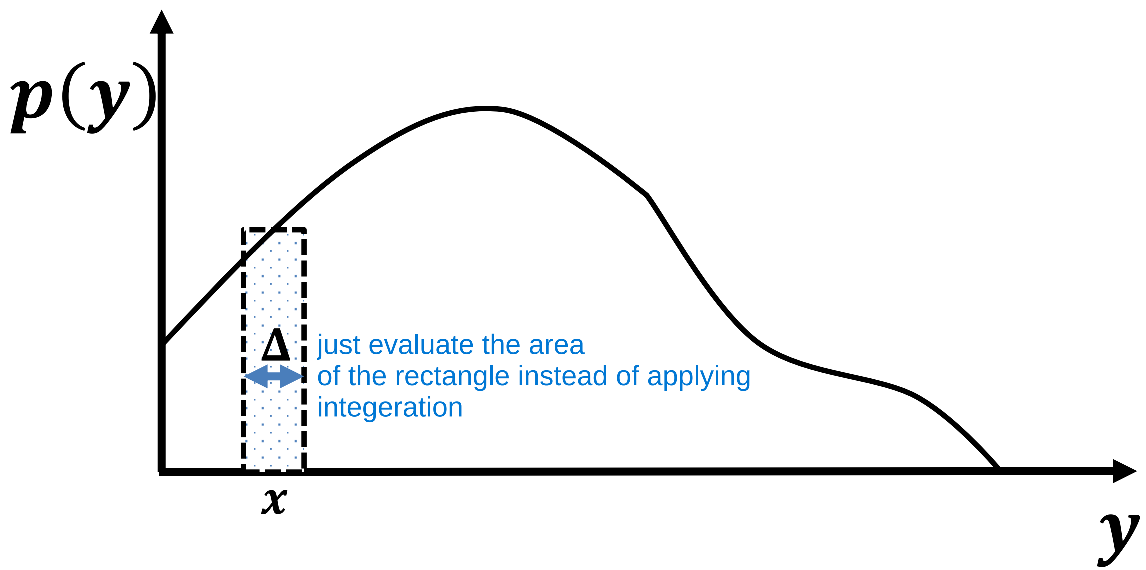
Data was originally generated from this density

Histogram Analysis

proof for the equation

$$\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} p(x)dx \approx \boldsymbol{\Delta}\,.\,p(x)$$

- Probability (generated point $\epsilon \left[x - \frac{\Delta}{2}, x + \frac{\Delta}{2}\right]) \approx \boldsymbol{\Delta}\,.\,p(x) \equiv z$



Bin size $\equiv\ \Delta$

Histogram Analysis

$$\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} p(x)dx \approx \mathbf{\Delta}\,.\,p(x)$$

- Probability (generated point $\epsilon \left[\boldsymbol{X} - \frac{\Delta}{2}\,,\ \ \boldsymbol{X} + \frac{\Delta}{2}\right)) \approx \mathbf{\Delta}\,.\,p(x) \equiv z$

- Assume we draw a number M of points according to p(x) → binomial distribution

- Binomial distribution with probability z for number of points falling in BIN

Histogram Analysis

$$P(k\ points\ falling\ in\ BIN\ out\ of\ M\ points)$$
$$= \binom{M}{k} z^k (1-z)^{M-k}$$

$$E(\#\ points\ in\ BIN) = M.z$$
$$= M.p(x).\Delta$$

- Example: flip a coin 10 times

$$P(8\ Heads) = \binom{10}{8} p^8 (1-p)^{10-8}$$
$$E(\#\ Heads) = p.M = 0.5 * 10 = 5$$
$$p \equiv probability\ of\ head$$

Histogram Analysis

$$E(\#\ points\ in\ BIN) = M.z$$
$$= M.p(x).\Delta$$

- If k points fall in the histogram range then assuming:

$$k \approx M.p(x).\Delta$$

- Then, estimate of p(x) is:

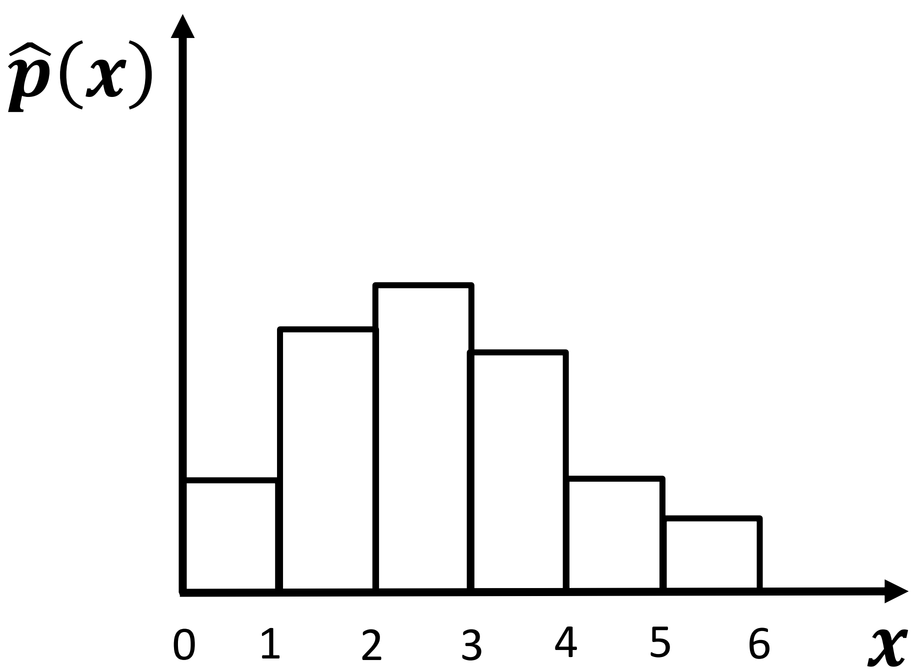
$$p(x) = \frac{k}{M\Delta}$$

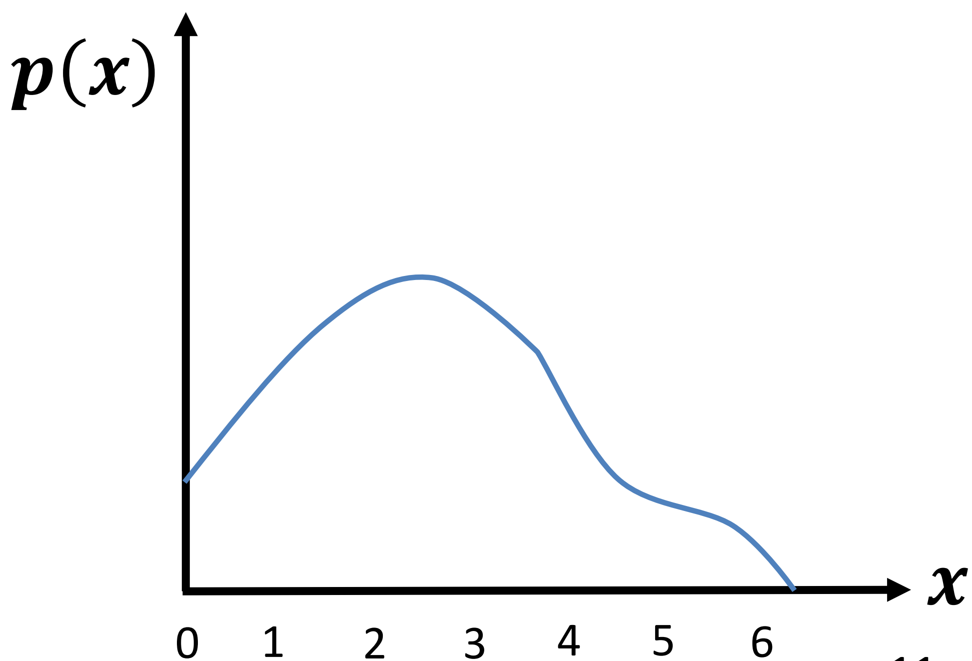
Recall:

$$\hat{p}(x) = \frac{m}{M\ *\ (size\ of\ bin)}$$

Histogram Analysis

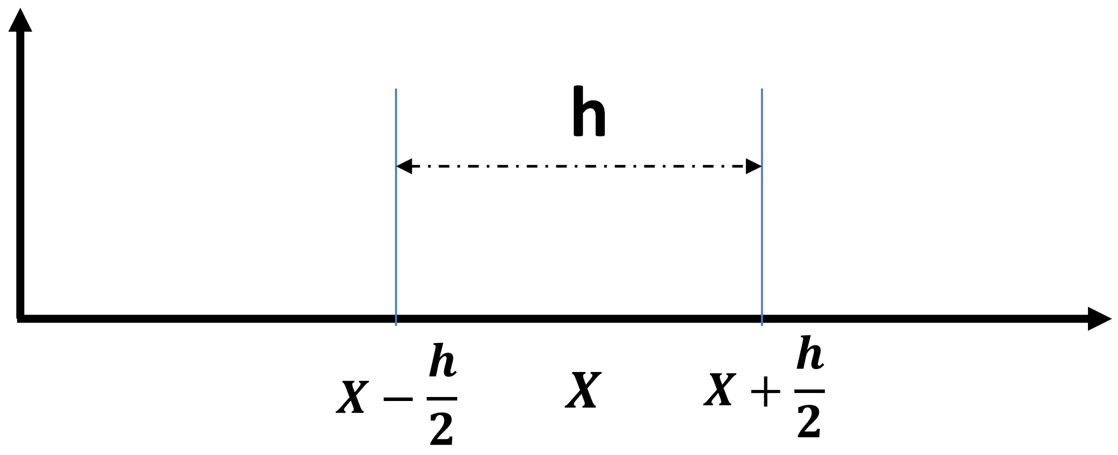
- Weak method of estimation
- Discontinuity of these density estimates, even though the true densities are assumed to be smooth





Naïve Estimator

- Instead of partitioning X, i.e., feature space, into a number of prespecified ranges, we perform a similar range analysis for every X



$$\hat{P}(X) = \frac{\#points\ falling\ in\ \left[\boldsymbol{X}-\frac{\boldsymbol{h}}{\boldsymbol{2}},\ \ \boldsymbol{X}+\frac{\boldsymbol{h}}{\boldsymbol{2}}\right)}{Mh}$$

Naïve Estimator

- Drawbacks:
  - Discontinuity of the density estimates
  - All data points are weighted equally regardless of their distance to the estimation point, i.e, X

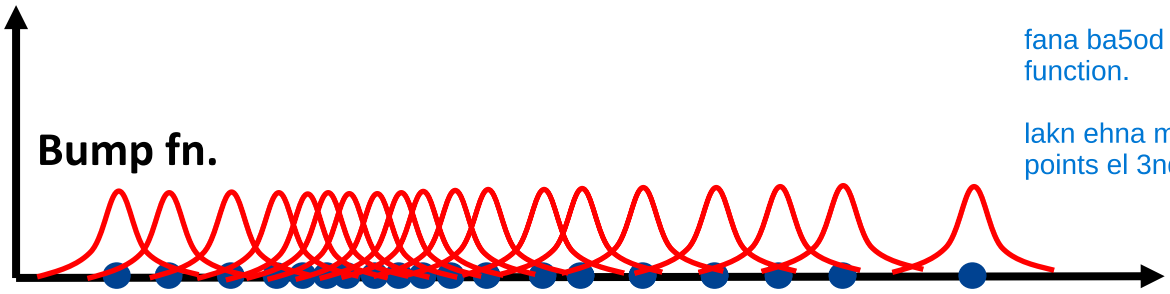
Kernel Density Estimator

- a.k.a. Parzen Window Density Estimator
- Choose a bump function

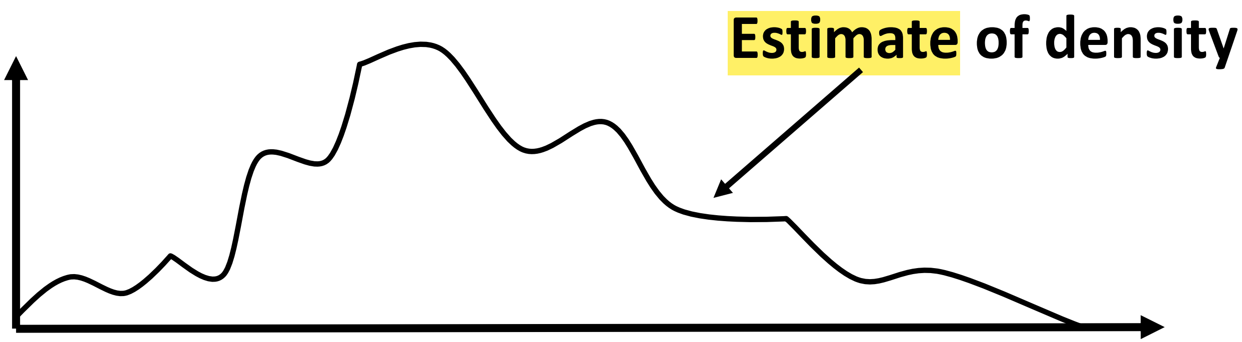
kol point mn hena aknha one class

fana ba5od kol class w aro7 a3ml 3leh el bump function.

lakn ehna msh bn3ml el bump funciton 3la kol el points el 3ndna fl dataset, la kol class.



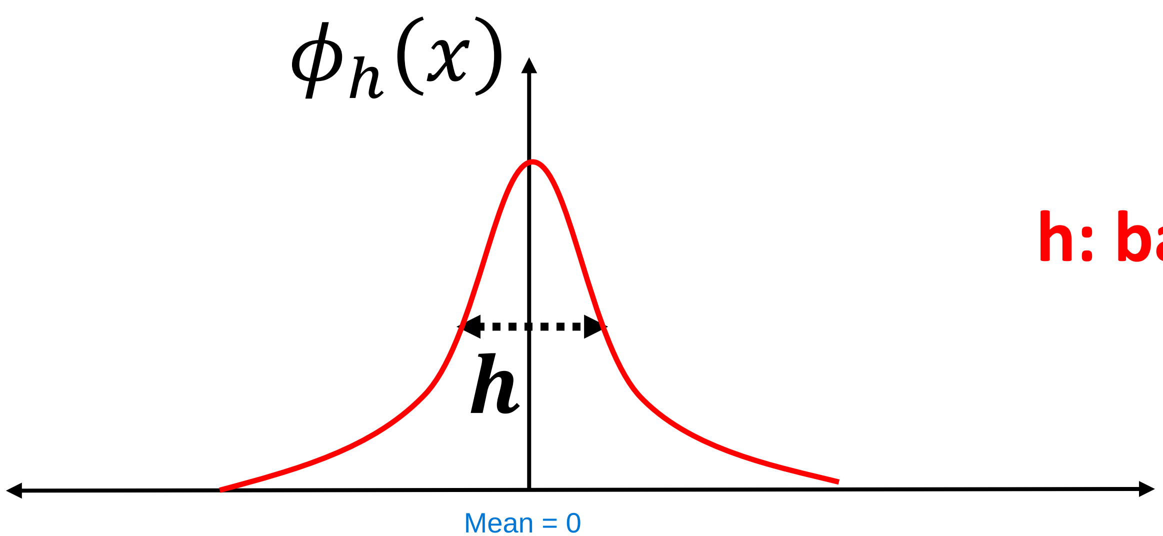
- Summation of bump functions:



Kernel Density Estimator

- Choose bump function as Gaussian with standard deviation (bandwidth) h:

$$\phi_h(x) = \frac{e^{\frac{-x^2}{2h^2}}}{\sqrt{2\pi}h}$$



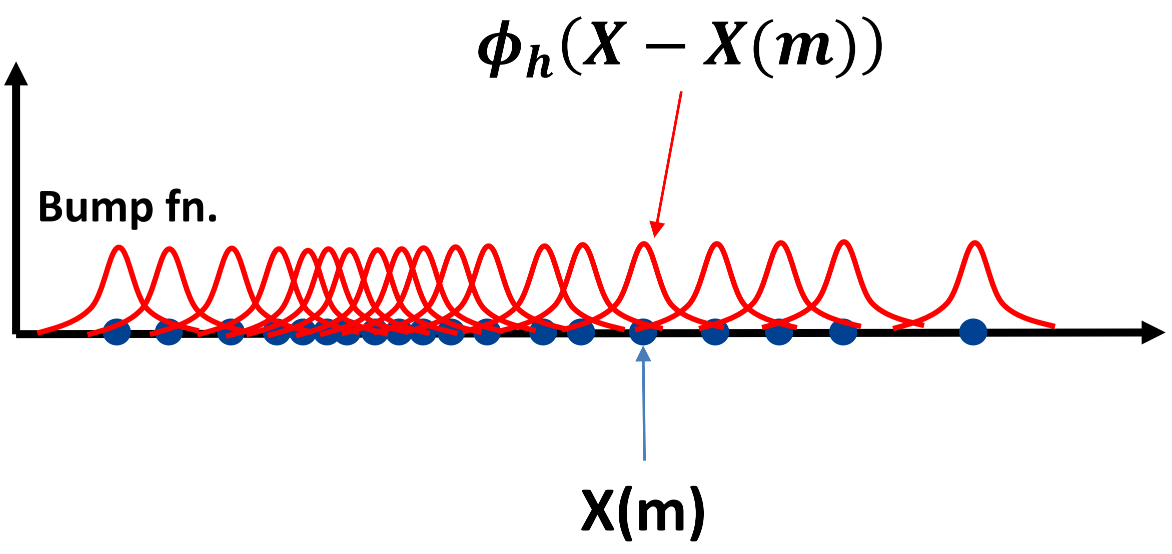
h: bandwidth

Kernel Density Estimator

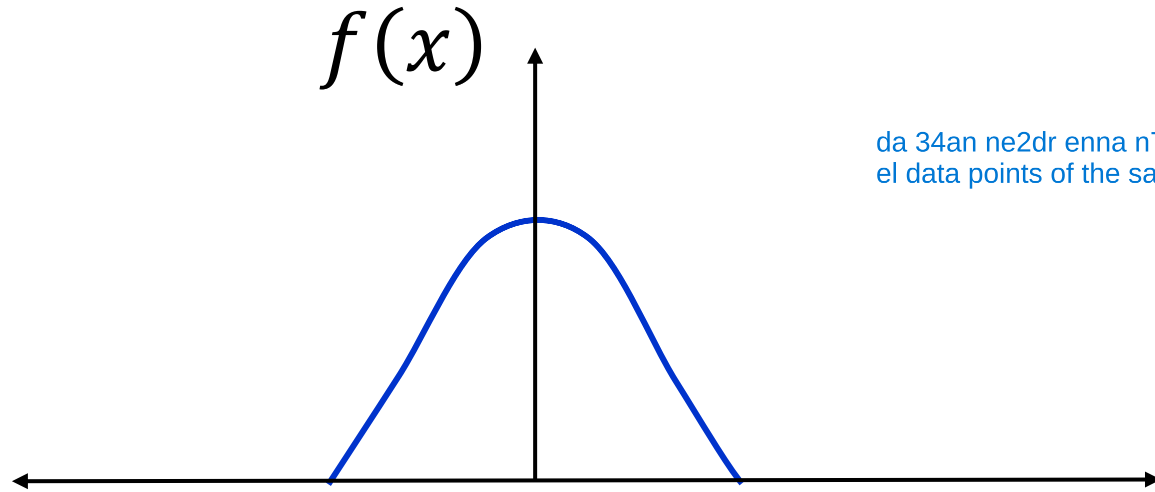
- Choose bump function as Gaussian with standard deviation (bandwidth) h:

$$\phi_h(x) = \frac{e^{\frac{-x^2}{2h^2}}}{\sqrt{2\pi}h}$$

- X(m) are the points generated from the density P(X) that we want to estimate

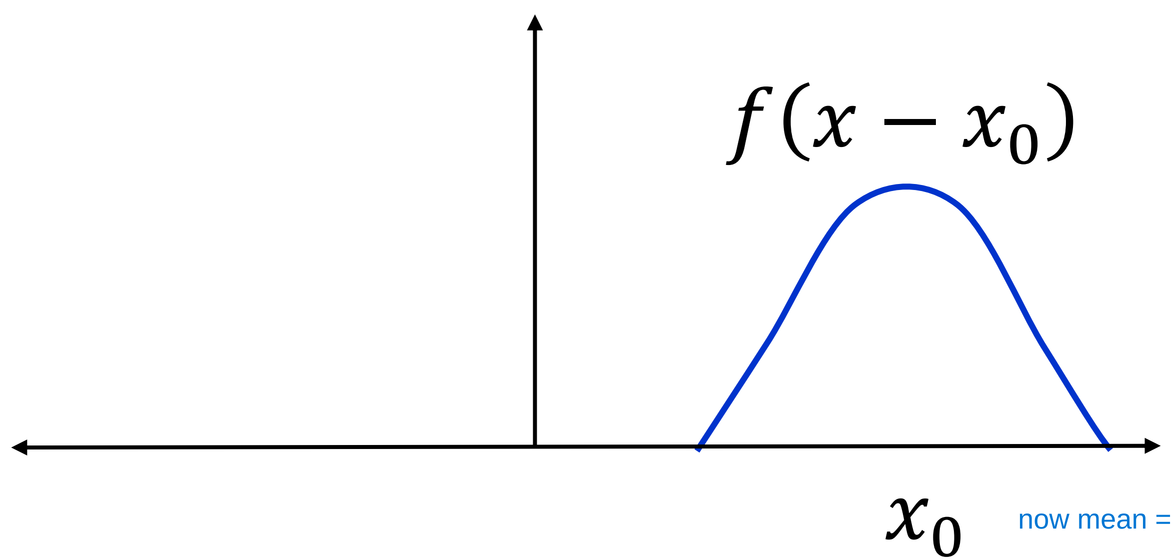


Kernel Density Estimator



da 34an ne2dr enna n7ot el bump function 3la kol el data points of the same class.

signals



now mean = Xo

Kernel Density Estimator

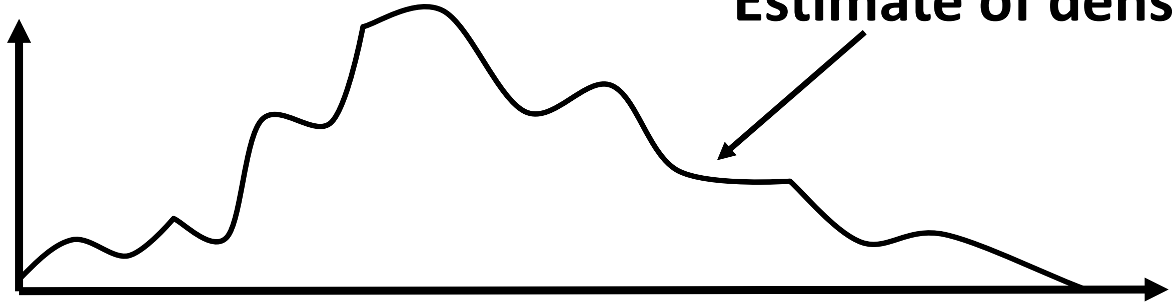
- Summation of bump functions:

$$\widehat{P}(X) = \frac{1}{M} \sum_{m=1}^{M} \phi_h(X - X(m))$$

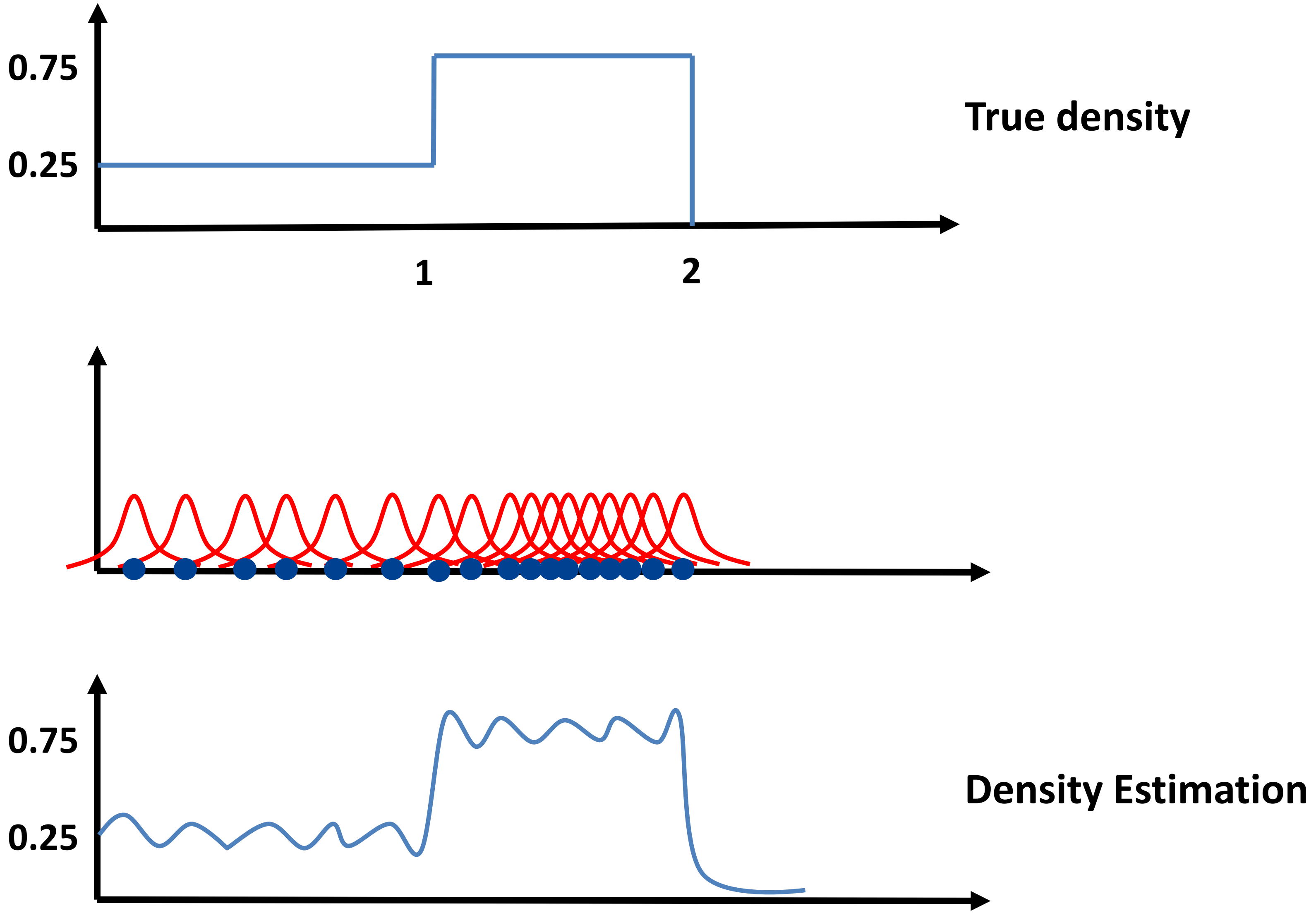
P hat mean P estimated
w estimated 34an ehna bngebha
mn el data points w de btb2a 3bara
3n samples msh el 7a2e2a kolaha

Summation over # of generated points

Estimate of density



Kernel Density Estimator



kol 7aga momkena el muhem teb2a fahem enta bt3ml a

characteristics for any bump function

Kernel Density Estimator

- $\boldsymbol{\phi_h}$ does not have to be Gaussian

but it is prefered

this is the generic form

$$\boldsymbol{\phi_h = \frac{1}{h} g(\frac{X}{h})}$$

where $\boldsymbol{g}(\cdot)$ is any suitable bump function that integrates to 1:

$$\boldsymbol{\int_{-\infty}^{\infty} g(x)dx = 1}$$

density function

symmetric around the mean
(y axis)
w da 34an mt3mlsh favoring bump 3n el tany.
fa tb2a fair.

e.g.

$$\mathrm{g(x)} = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \quad \rightarrow \quad \phi_h(x) = \frac{e^{\frac{-x^2}{2h^2}}}{\sqrt{2\pi}h}$$

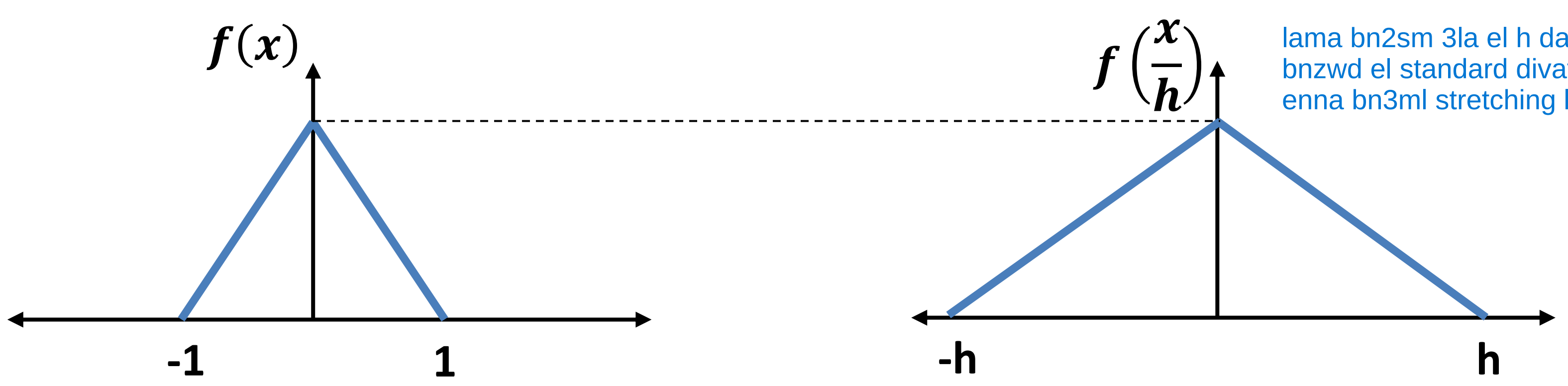
Kernel Density Estimator

- ϕ_h does not have to be Gaussian

$$\phi_h = \frac{1}{h} g\left(\frac{X}{h}\right)$$

where $g(\cdot)$ is any suitable bump function that integrates to 1:

$$\int_{-\infty}^{\infty} g(x)\,dx = 1$$



lama bn2sm 3la el h da m3nah enna bnzwd el standard divation fa da m3nah enna bn3ml stretching lel signal.

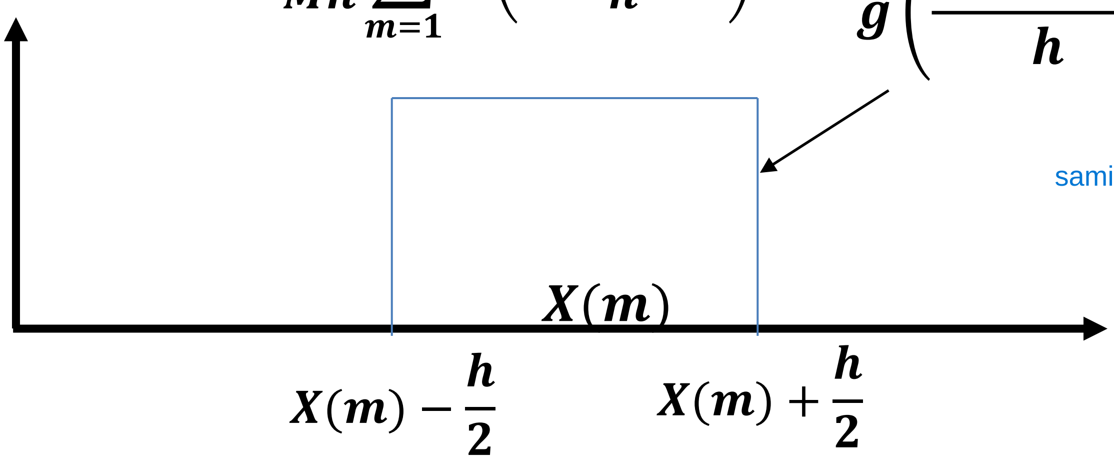
Kernel Density Estimator

- Naïve estimator is equivalent to a Parzen window estimator with:

$$g(x) = \begin{cases} 1, & -\frac{1}{2} \leq x < \frac{1}{2} \\ 0, & otherwise \end{cases}$$

- In this case:

$$\hat{P}(X) = \frac{1}{Mh} \sum_{m=1}^{M} g\left(\frac{X - X(m)}{h}\right)$$



Kernel Density Estimator

X(m) da 3bara 3n mkan el class el bn3ml 3ndo el shift.
aknk fe el signals w 3auz t3ml shift lel signal ymen fa 34an
n3ml keda bn2ol el shifted version = f(x-xo)
xo hena baa hya el X(m)

1-D form:

$$\hat{P}(X) = \frac{1}{M}\sum_{m=1}^{M}\phi_h(X - X(m))$$

$$= \frac{1}{Mh}\sum_{m=1}^{M} g\left(\frac{X - X(m)}{h}\right)$$

look at slide 21

$$\int_{-\infty}^{\infty} g(x)dx = 1$$

$$\int_{-\infty}^{\infty} \hat{P}(x)dx = 1$$

Kernel Density Estimator

Multi-dimension Form:

$$\hat{\boldsymbol{P}}(\underline{\boldsymbol{X}}) = \frac{\mathbf{1}}{\boldsymbol{M}\boldsymbol{h}^{\boldsymbol{N}}} \sum_{\boldsymbol{m}=\mathbf{1}}^{\boldsymbol{M}} \boldsymbol{g}\left(\frac{\underline{\boldsymbol{X}} - \underline{\boldsymbol{X}}(\boldsymbol{m})}{\boldsymbol{h}}\right)$$

$$\int_{-\infty}^{\infty} \boldsymbol{g}(\underline{\boldsymbol{X}})\, \boldsymbol{d}\underline{\boldsymbol{X}} = \mathbf{1}$$

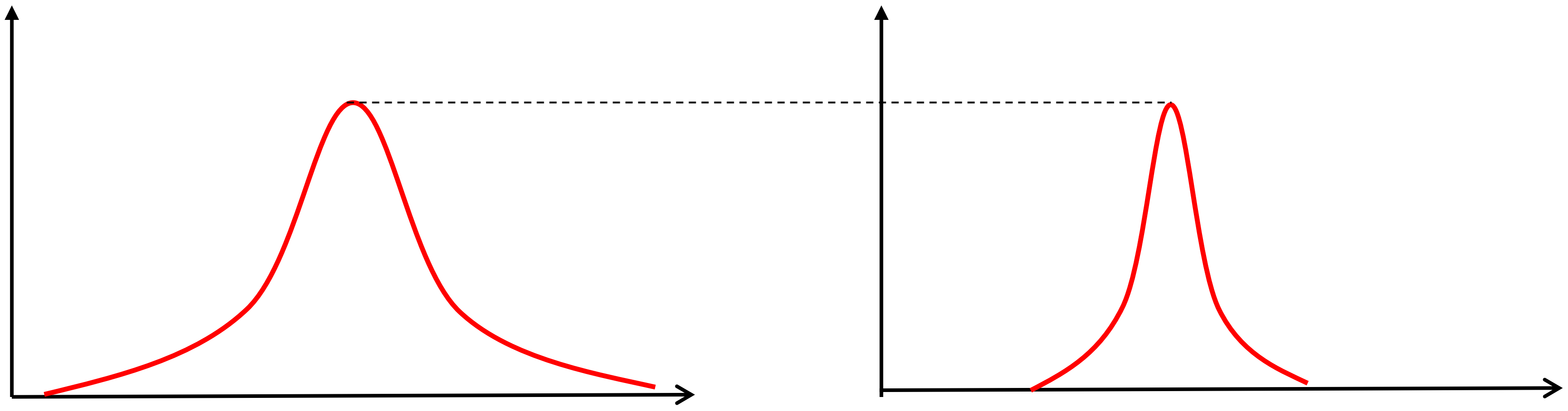
For example: multi-dimension independent Gaussian density:

$$g(\underline{X}) = \frac{e^{-\sum_{i=1}^{N} \frac{x_i^2}{2}}}{(2\pi)^{N/2}}$$

How to choose h?

eh faydet el h?
kol ma tzwd el h kol ma btgeb details aktur w vice versa baa.

bumby 34an enta btgm3 el signal 3la b3d, ersmhom gmb b3d htfhm, ana fhmtha fl mo7dra.

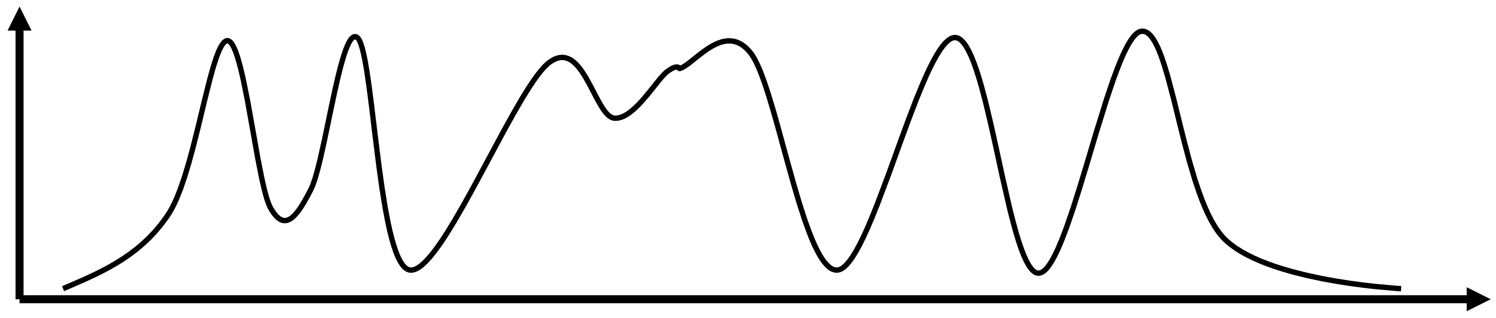


Large h

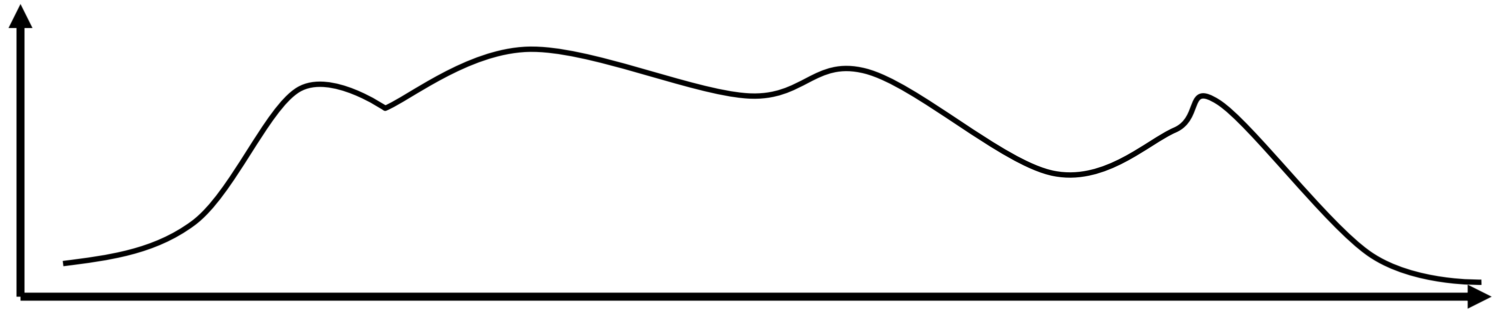
Small h

How to choose h?

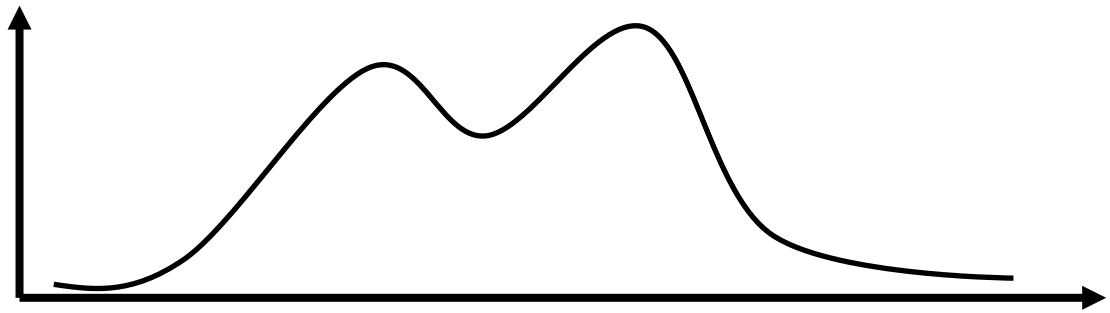
- Too small h → bumpy estimate or non-smooth



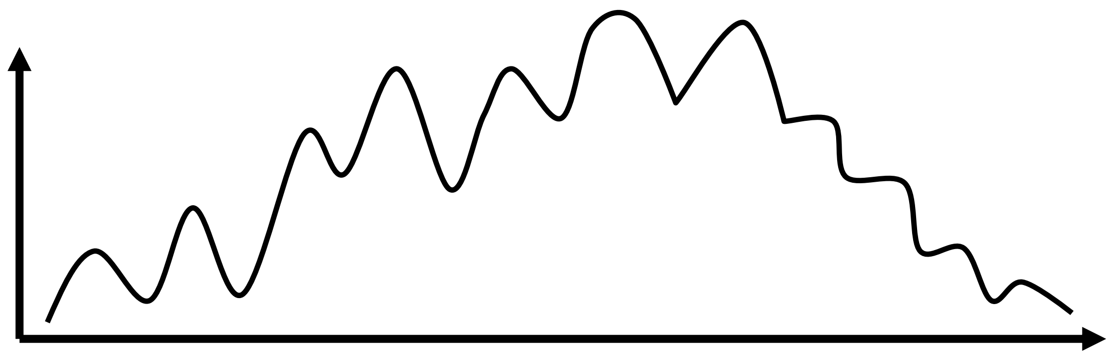
- Too large h → the estimate could be too smooth that essential details of the density will be lost or smoothed out



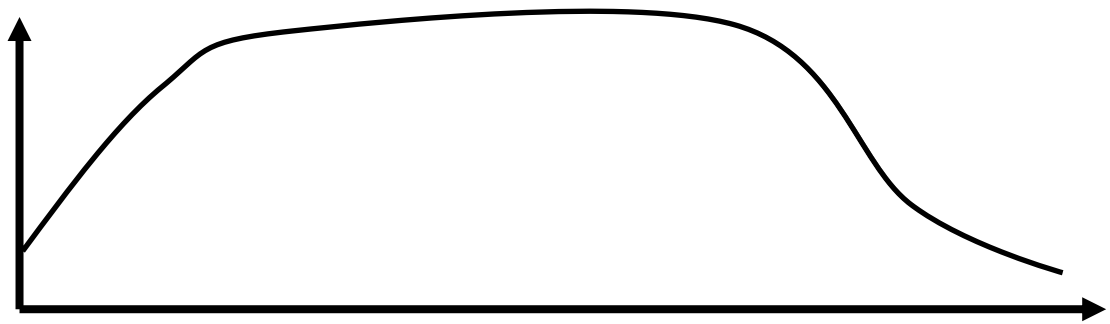
How to choose h?



True Density



Too small h



Too large h

34an nsahel el denya,
1. bnftrd enha independet.
2. bngeb el average bta3 kol dimension
3. w bngeb el optimal h w nw7do lehom kolohom.

Optimal h

- The optimal H (diagonal bandwidth matrix) can be approximated as :

variance

$$\boldsymbol{H_i = \sigma_i \left[\frac{4}{(N+2)M}\right]^{\frac{1}{N+4}}}$$

kol ma el Hi yekbr da m3nah enny akbur el width bta3y, 34an da m3nah en el data scattered aslun

normal reference rule

\# of points of the same class

where

$$\sigma_i = \sqrt{[\Sigma_X]_{i,i}}$$

gaya mn el covariance matrix

- Σ_X is the estimated covariance matrix, i.e.,

$$\Sigma_X = \frac{1}{M}\sum_{m=1}^{M}(\underline{X}(m) - \hat{\mu})(\underline{X}(m) - \hat{\mu})^T$$

$$\boldsymbol{h_{opt} = \frac{1}{N}\sum_{i=1}^{N} H_i}$$

$[\Sigma_X]_{i,i} \equiv i^{th}$ diagonal element of Σ_X

$N \equiv$ dimensions

number of diminsions

For multi-variate normal kernel & diagonal bandwidth matrix

Bowman, A.W., and Azzalini, A. (1997), Applied Smoothing Techniques for Data Analysis, London: Oxford University Press [page 32]. shoof el proof fl makan da fel reference

Acknowledgment

- These slides have been created relying on lecture notes of Prof. Dr. Amir Atiya