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Abstract. In this paper, we are interested in solving three-dimensional second-order elliptic equations in exterior domains using the inverted finite element method with Dirichlet boundary conditions. We introduce a variational formulation in weighted spaces and demonstrate its well-posedness. The high efficiency of the method is shown by displaying the results of the $3D$ computation.

Keywords Weighted spaces. Inverted finite elements. Exterior domain.

1 Introduction

Consider an elliptic equation of the form

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j}(x) + b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) = f(x) \quad \in \quad \Omega, \tag{1}$$

where $\Omega = \mathbb{R}^3 \backslash \Omega_0$ is the unbounded exterior domain of $\mathbb{R}^3$ with respect to Ω_0, where Ω_0 is a bounded Lipschitz open set (here Ω_0 which can chosen as the exterior region of a ball in a regular tetrahedron centered at the origin). With asymptotic conditions at large distances, that is when $|x| \longrightarrow +\infty$, a_{ij}, b_i and c are variables coefficients. f is a given function and $u(x)$ is the unknown function to be determined.
This equation was studied by Boulmzaoud when $\Omega = \mathbb{R}^3_+$ (see, e.g. [?]), by [?] in the exterior domains of $\mathbb{R}^2$ and in the half space of $\mathbb{R}$ (see, e.g [?]).
In this work, we first give the variational formulation associated with the problem $(??)$ in weighted Sobolev spaces. Then we demonstrate the existence and uniqueness of the solution using the Lax-Milgram theorem. Next, using the Galerkin approach for determining the approximate solution u, where a graduated mesh will also be employed in the unbounded region of the problem, we provide an estimate of the error. Finally, numerical results show the efficiency of the method.

Notations and Preliminaries

we define the basic weight,

$$\langle x \rangle = (1 + |x|^2)^{1/2}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Where $|x| = (x_1^2 + x_2^3 + x_3^2)$ is the distance to origin. $L^2(\Omega)$ designates the usual Lebesgue space of real square integrable functions over Ω, equipped with the norm

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx \right)^{1/2}.$$

*To your mother

$W^{m,p}(\Omega)$ designate the classical Sobolev space, for any integer m

$$W^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega) | \forall \mu \in \mathbb{N}^3 : 0 \leq |\mu| \leq m, D^\mu \in \mathbb{L}^p(\Omega)\},$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \partial x_2^{\mu_2} \partial x_3^{\mu_3}}, \quad |\mu| = (\mu_1 + \mu_2 + \mu_3).$$

$\mathbb{P}_k(K)$ is the space of all polynomials of degree less or equal to k, such that $P(0) = 0$.

Given $m \in \mathbb{N}, \alpha \in \mathbb{R}$, we define $W^{m,p}_\alpha(\Omega)$

Définition 1.1

$$W^{m,p}_\alpha(\Omega) = \{u \in \mathcal{D}'(\Omega) | \forall \mu \in \mathbb{N}^3 : 0 \leq |\mu| \leq m, \langle x \rangle^{\theta - m + |\mu|} D^\mu \in \mathbb{L}^p(\Omega)\},$$

the space of all the functions $u \in L^p(\Omega)$ whose derivatives for $|\mu| \leq m$ satisfy

$$\langle x \rangle^{\alpha - m + |\mu|} D^\mu u \quad \in L^p(\Omega). \tag{2}$$

The space $W^{m,p}_\alpha(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^m_\alpha(\Omega)} = \left(\sum_{|\mu| \leq m} \int_\Omega (1 + |x|^2)^{(\alpha - m + |\mu|)p} |D^\mu u(x)|^p dx \right)^{1/p}.$$

We also consider the space $V^{m,p}_\alpha(\Omega_0)$. This space is topologically and algebraically identical to the space whose functions verify

$$\langle x \rangle^{\theta - m + |\mu|} D^\mu u \quad \in L^2(\Omega_0), \ \forall |\mu| \leq m.$$

Given a four points a_0, a_1, a_2, a_3 of $\mathbb{R}^3$. The infinite simplexes $T(a_0, a_1, a_2, a_3)$ is defined as fellows

$$T(a_0, a_1, a_2, a_3) = \{x = \sum_{i=0}^{3} \lambda_i a_i, \ \lambda_0 \leqslant 0, \ \lambda_i \geqslant 0 \ pour \ i = 1, 2, 3 \ \sum_{i=0}^{3} \lambda_i = 1\},$$

where a_0 denotes the fictitious vertex of $T(a_0, a_1, a_2, a_3)$, while the others its denotes real vertices . We associate to this infinite simplexes the finite simplex $S(a_0, a_1, a_2, a_3)$, such that

$$S(a_0, a_1, a_2, a_3) = \{x = \sum_{i=0}^{3} \lambda_i a_i, \ 0 \leqslant \lambda_i \leqslant 1 \ pour \ i = 1, 2, 3, \ \sum_{i=0}^{3} \lambda_i = 1\}.$$

2 Variational formulation

the objective of this section is to give a weak formulation of the problem, using the properties of weighted Sobolev spaces in the exterior domain.

We search for solution in $W^1_0(\Omega)$ i. e that satisfy

$$\int_\Omega \frac{|u(x)|^2}{|x|^2 + 1} dx < \infty, \ \int_\Omega |\nabla u(x)|^2 < \infty.$$

Lemma 2.1 *A function $u \in W^1_0(\Omega)$ is solution of (??) if*

$$\forall v \in W^1_0(\Omega), \ \mathcal{A}(u, v) = \langle f, v \rangle, \tag{3}$$

where the bilinear form $\mathcal{A}$ define as

$$\mathcal{A}(u, v) = \sum_{i,j=1}^{3} \int_\Omega a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} dx + \sum_{i=1}^{3} \int_\Omega b_i(x) \frac{\partial u(x)}{\partial x_j} v(x) dx + \int_\Omega c(x) u(x) v(x) dx.$$

Assume that

($\mathcal{H}_1$) $a(x) \in L^\infty(\Omega)^{3\times 3}$ and there exists a constant $\eta_0 > 0$ such that

$$\forall \xi = (\xi_{1,\xi_2,\xi_3}) \in \Omega,\ \sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \geqslant \eta_0|\xi|^2 > 0,\ \text{a. e. in } \Omega$$

.

($\mathcal{H}_2$) $\langle x\rangle\ b(x) \in L^\infty(\Omega)^3,\ \langle x\rangle^2\ div\ b(x) \in L^\infty(\Omega)$, and $\langle x\rangle^2\ c \in L^\infty(\Omega)$, that is there exists a constant $\alpha \in \mathbb{R}$, such that

$$|b(x)|^2 + |div\ b(x)| + |c(x)| \leq \frac{\alpha}{1+|x|^2} \text{a. e. in } \Omega$$

.

($\mathcal{H}_3$) $f \in W_0^{-1}$ i. e

$$\int_\Omega (|x|^2+1)|f(x)|^2 dx < \infty.$$

The next proposition concerns well possessedness of the problem considered here

Proposition 2.2 *Suppose that assumptions ($\mathcal{H}_1$), ($\mathcal{H}_2$) and ($\mathcal{H}_3$) are valid, Eq (**??**) satisfying the Dirichlet boundary condition*

$$u = 0 \quad on\ \ \partial\omega, \tag{4}$$

has a unique solution $u \in W_0^1(\Omega)$ and

$$\|u\|_{W_0^1(\Omega)} \leqslant \|f\|_{W_0^{-1}(\Omega)}.$$

For proving proposition we need Hardy's inequality when Ω is an exterior domain

Lemma 2.3 *There exists a constant $\beta > 0$ such that*

$$\forall u \in W_0^1(\Omega), \int_\Omega |\nabla u(x)|^2\, dx \geqslant \beta \int_\Omega \frac{|u(x)|^2}{1+|x|^2} dx.$$

The following assumption is made

($\mathcal{H}_4$) there exists a constant $\beta' < \beta$, such that

$$c(x) - \frac{1}{2} div\ b(x) \geqslant -\frac{\eta_0\beta'}{1+|x|}.$$

proof. Existence and Uniqueness of the solution stem from Lax-Milgram theorem. Moreover, the linear form $\langle f, v\rangle$ and the bilinear form $\mathcal{A}$ are continuous

$$\begin{aligned} |\langle f, v\rangle| &= |\int_\Omega f(x)v(x)dx| \\ &= |\int_\Omega \langle x\rangle f(x)\frac{v(x)}{\langle x\rangle}dx| \\ &\leqslant \tilde{c}\|v\|_{W_0^1(\Omega)}. \end{aligned}$$

$$|\mathcal{A}(u,v)| = |\sum_{i,j=1}^{3}\int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx + \sum_{i=1}^{3}\int_\Omega b_i(x)\frac{\partial u}{\partial x_j}v dx + \int_\Omega c(x)uv dx|. \tag{5}$$

We have

$$|\sum_{i,j=1}^{3}\int_{\Omega}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx| = \sum_{i,j=1}^{3}\|a_{ij}(x)\|_{L^\infty(\Omega)}\int_{\Omega}|\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}|dx$$
$$\lesssim \|\nabla u(x)\|_{L^2(\Omega)}\|\nabla v(x)\|_{L^2(\Omega)}.$$

On the other hand

$$|\sum_{i=1}^{3}\int_{\Omega}b_i(x)\frac{\partial u}{\partial x_j}(x)v(x)dx| = |-\int_{\Omega}\textsf{div}\ b(x)v(x)u(x)dx - \int_{\Omega}b(x)u(x)\nabla v(x)dx|$$
$$\leqslant \int_{\Omega}\langle x\rangle^2|\textsf{div}\ b(x)||\frac{u(x)}{\langle x\rangle}||\frac{v(x)}{\langle x\rangle}|dx + \int_{\Omega}\langle x\rangle|b(x)||\frac{u(x)}{\langle x\rangle}||\nabla v(x)|dx|.$$

Finally

$$|\int_{\Omega}c(x)u(x)v(x)\ dx| \leqslant \int_{\Omega}\langle x\rangle^2|c(x)||\frac{u(x)}{\langle x\rangle}||\frac{v(x)}{\langle x\rangle}|dx.$$

Using $(H_1), (H_2)$ and Hardy inequality we can prove that

$$\mathcal{A}(v,v) \gtrsim \|v\|^2_{W_0^1(\Omega)}. \tag{6}$$

Because

$$\sum_{i,j=1}^{3}\int_{\Omega}a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}dx \geq \eta_0\int_{\Omega}|\nabla v|^2dx$$

on the other hand

$$\sum_{i=1}^{3}\int_{\Omega}b_i(x)\frac{\partial v}{\partial x_j}(x)v(x)dx + \int_{\Omega}c(x)v(x)v(x)\ dx = \int_{\Omega}(c(x)-\frac{1}{2}\textsf{div}\ b(x))|v(x)|^2\ dx$$
$$\geq \frac{\eta_0\beta'}{\beta}\int_{\Omega}|\nabla v(x)|^2dx.$$

thus

$$\mathcal{A}(v,v) \gtrsim \|v\|^2_{W_0^1(\Omega)}. \tag{7}$$

■

3 Discrisation by Inverted finite element method

The Galerkin method for approximating the solution u consists in replacing problem (??) by the finite-dimensional problem
Find $u_h \in W_h$ such that

$$\forall\ v_h \in W_h \quad \mathcal{A}(u_h,v_h) = \langle f, v_h\rangle. \tag{8}$$

Where $W_h \subset W_0^1$.

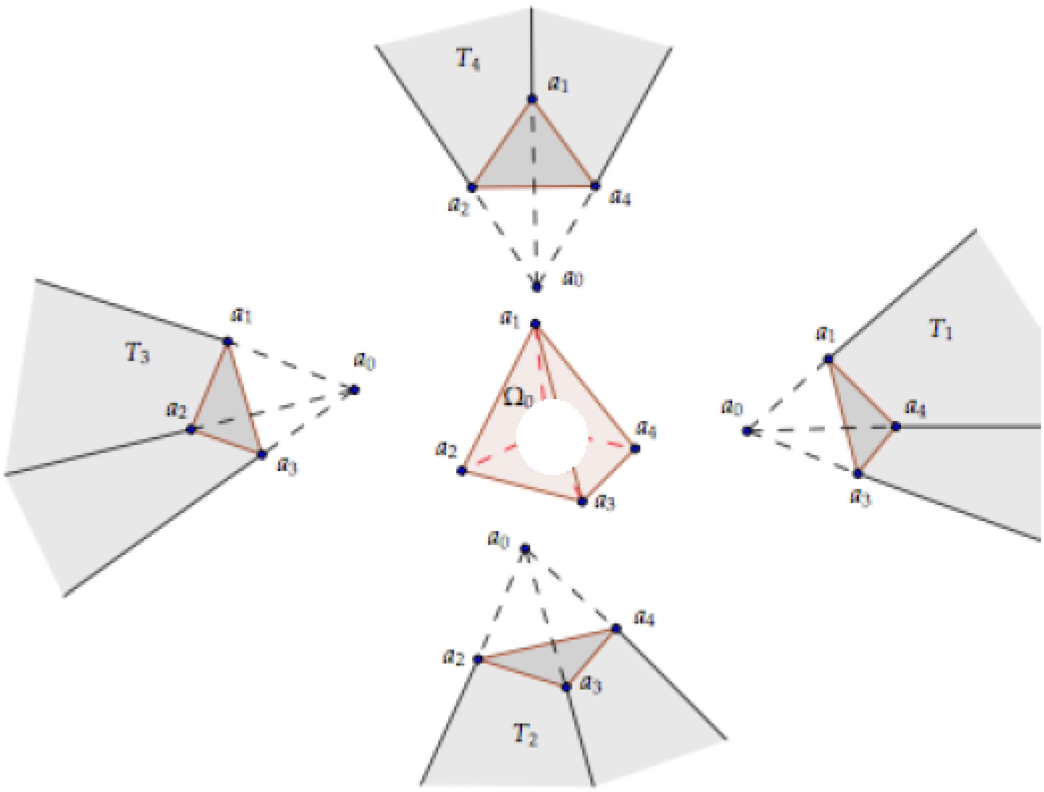

FIGURE 1 – les domaines Ω_0(en blanc), Ω_∞(en gris) et ω(en rouge)

3.1 constructing the space W_h

The first idea for constructing the space W_h is to divide the domain Ω into two sub-domains, such that

$$\bar{\Omega} = \bar{\Omega}_0 \cup \bar{\Omega}_\infty.$$

Ω_∞ :is an unbounded domain which represents the exterior of Ω_0 in $\mathbb{R}^3$ is decomposed into the union of four infinite simplexes T_1, T_2, T_3 and T_4 such that

1. $\bar{\Omega}_\infty = \bigcup_{l=1}^{4} T_l$,
2. $T_1, ..., T_4$ have the same fictitious vertex a_0. Without loss of generality, we suppose that $a_0 = 0$,
3. for any $l, m \leq 4$ with $l \neq m$, the intersection of T_l and T_m is either the empty set or a whole face.

Ω_∞ is transformed into a bounded region Ω_* by an inversion mapping Φ, such that

$$\forall x \in \bar{\Omega}_\infty \cup \bar{\Omega}_* - \{0\};\ \Phi(x) = \frac{x}{r(x)^2}, \tag{9}$$

where

✓ $\Omega_* = \Phi(\Omega_\infty) = int(\cup_{m=1}^{M} S_m) - \{0\}$.

✓ $r(x) = 1$ in the bord $\bar{\Omega}_0 \cap \bar{\Omega}_\infty$ and $r(x) \simeq |x|$ for $x \in \bar{\Omega}_* \cup \bar{\Omega}_\infty - \{0\}$.

the following lemma is useful in the remaining of this paper.

Lemma 3.1 *Let $T(a_0, ..., a_n)$ be an infinite simplex. then*

$$F_T^{-1} \circ \Phi_T = \hat{\Phi} \circ F_T^{-1},$$

$$|det(\nabla \Phi_T)| = \left(\frac{|h_T|^2}{x^t.h_T}\right)^6.$$

Such that, F_T the affine mapping which maps the reference infinite simplex $\hat{T}$ into T. The mapping F_T maps also the reference simplex $\hat{S}$ into the finite simplex S_T associated to T.

Now, we define the polygonal Kelvin transform Λ_γ, as the operator which assigns to each function u defined on Ω_∞ the function $\Lambda_\gamma u$ defined in Ω_* by

$$\forall x \in \Omega_*,\ \ \Lambda_\gamma u(x) = r(x)^{-\gamma} u(\Phi(x)), \tag{10}$$

where γ a real number.
The next proposition describes the image of the weighted space $W_\alpha^{m,p}$ by the operator Λ_γ.

Proposition 3.2 *Let α, γ and p three real numbers with $1 < p < +\infty$. Let $u \in W_\alpha^{m,p}(\Omega_\infty)$. We set*

$$\delta = \gamma - \alpha - \frac{2n}{p},$$

with $m \in \mathbb{N}$, then $u^* \in V_\delta^{m,p}(\Omega_*)$ with $u^* = \Lambda_\gamma u$ and

$$\|u\|_{W_\alpha^{m,p}(T_i)} \backsimeq \|u^*\|_{V_\alpha^{m,p}(S_i)} \tag{11}$$

The proof of this proposition is given in [?].
The last step for constructing the discrete space W_h is the meshing of the domain. we consider two families of mesh
$\{K; K \in T_h^0\}$ is a regular mesh on the bounded sub-domain Ω_0,
$\{K^*; K^* \in \mathcal{T}_h^*\}$ is a $\eta-$graded mesh on the fictitious sub-domain Ω^*.
Then, we defined the space $W_{h,k,\gamma}(\Omega)$ which approaching the space $W_0^1(\Omega)$,

$$W_{h,k,\gamma}(\Omega) = \{u_h \in \mathcal{C}^0(\Omega),\ u_{|K} \in (\mathbb{P}_k)^3,\ \forall K \in \mathcal{T}_h^0,$$

$$u^*_{|K^*} \in (\mathbb{P}_k)^3,\ \forall K^* \in \mathcal{T}_h^*\}.$$

3.2 error estimate

Before starting approximation results, observe that $W_h(\Omega)$ depends the parameter h of the finite element method is supposed tend to zero, depends also the parameters μ and γ are a priori fixed. The choice of this parameters has a serious influence on the quality of approximation. Moreover, we have

Lemma 3.3 *Suppose that $\gamma > \frac{3}{2}$. then the following inclusion holds*

$$W_{h,k,\gamma}(\Omega) \hookrightarrow W(\Omega). \tag{12}$$

In the sequel, we shall suppose that the condition $\gamma > \frac{3}{2}$ and posed $W_h = W_{h,k,\gamma}$, Then the approximate problem associated to (**??**)
Find $u_h \in W_h$ such that

$$\forall\, v_h \in W_h \quad \mathcal{A}(u_h, v_h) = \langle f, v_h \rangle, \tag{13}$$

is well-posed and has a unique solution in W_h.

Théorème 3.4 *Let $u \in W_0^1(\Omega)$ be a solution of* (**??**) *and $u_h \in W_h$ solution of the discrete problem* (**??**). *Suppose that u belongs to $W^{k+1}_{k+\eta}$ for some real $\eta > 0$ and that*

$$\eta - \frac{1}{2} < \gamma < \eta + \frac{1}{2}.$$

Then,

$$\|u - \Pi_h u\|_{W_0^1(\Omega) \lesssim h^{k \min(1,\tau)} \|u\|_{W^{k+1}_{k+\eta}(\Omega)}}, \tag{14}$$

with

$$\tau = \frac{\eta}{k\mu}.$$

proof. Let Π_K be the interpolation operator defined from $\mathcal{C}^0(K)$ into W_h, Π_K is the unique element of $\mathbb{P}_k$ such that

$$\Pi_K v(a) = v(a), \quad for \quad a \in \Sigma_K.$$

Where

$$\Sigma_K = \left\{ x = \sum_{i=0}^{3} \lambda_i a_K^i ; \ \sum_{i=0}^{3} \lambda_i = 1 \ et \ \lambda_i \in \left\{ 0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1 \right\} \right\},$$

We consider the local $\overset{\circ}{\Pi}_K$ interpolation operator $\overset{\circ}{\mathbb{P}}_k$ if $K \in \mathcal{T}_h - \mathcal{T}_h^*(0 \in K)$, $\overset{\circ}{\Pi}_K u$ is the unique element of $\overset{\circ}{\mathbb{P}}_k$ satisfying

$$\overset{\circ}{\Pi}_K v(a) = v(a), \quad \forall a \in \Sigma_K - \{0\},$$

for each $v \in \mathcal{C}^0(K - \{0\})$.
Thus, for any function $v \in \mathcal{C}^0_{loc}(\bar{\Omega})$ we defined the global interpolation

$$v_{h|K} = \Pi_K v_{|K} \quad for \ all \ K \in \mathcal{T}_h, \tag{15}$$

$$v^*_{h|K} = \Pi_K v^*_{|K} \quad for \ all \ K \in \mathcal{T}_h^*, \tag{16}$$

$$v^*_{h|K} = \overset{\circ}{\Pi}_K v^*_{|K} \quad for \ all \ K \in \mathcal{T}_h - \mathcal{T}_h^*. \tag{17}$$

We have

$$\|u - \Pi_h u\|_{W_0^1(\Omega)} \lesssim \|u - \Pi_h u\|_{W^1(\Omega_0)} + \sum_{i=1}^{4} \|u - \Pi_h u\|_{W_0^1(T_i)}.$$

From Céa's Lemma, we have

$$\|u-u_h\|_{W_0^1(\Omega)} \lesssim \inf_{v_h\in V_h(\Omega)} \|u-v_h\|.$$

In the bonded region Ω_0, where finite element method is used. We know that

$$\|u-\Pi_h u\|_{W^1(\Omega_0)} \lesssim h^k \|u\|_{W^{k+1}(\Omega_0)}.$$

Now, we give a proposition to estimate the difference $u-\Pi_h u$ in the unbounded region T_i for each $i \le 4$.

Proposition 3.5 *Let $i \in 1,...,4$, $m \le k$ integer, and δ, θ two real numbers such that*

$$-\frac{5}{2} \le \theta \le -\frac{3}{2}, \tag{18}$$

and

$$V_{k+\theta}^{k+1}(\hat{K}) \hookrightarrow V_{k+\delta}^{m}(\hat{K}). \tag{19}$$

then

$$|u-\Pi_K u|_{V_{k+\delta}^m(K)} \le h^{k\min(1,\tau)} |u|_{V_{k+\theta}^{k+1}(K)} \quad if\ K \in \mathcal{T}_h^*, \tag{20}$$

$$|u-\overset{\circ}{\Pi}_K u|_{V_{k+\delta}^m(K)} \le h^{k\tau} |u|_{V_{k+\theta}^{k+1}(K)} \quad if\ K \in \mathcal{T}_h - \mathcal{T}_h^*, \tag{21}$$

with

$$\tau = \frac{\delta-\theta}{k\mu}. \tag{22}$$

The proof in [?] (see proposition 2).
Posing $\theta=\gamma-\eta-3$, $\Lambda_\gamma \Pi_h u = \Pi_h^* u$ and for each $i \le 4$, we have

$$\begin{aligned}
\sum_{i=1}^{4} \|u-\Pi_h u\|_{W_0^1(T_i)} &\le \sum_{i=1}^{4} \|u^*-\Pi_h^* u\|^2_{V^1_{k+\delta}(S_i)} \\
&\le \sum_{K\in\mathcal{T}_h-\mathcal{T}_h^*} \|u^*-\Pi_h u^*\|^2_{V^1_{k+\delta}(K)} + \sum_{K\in\mathcal{T}_h^*} \|u^*-\Pi_h u^*\|^2_{V^1_{k+\delta}(K)} \\
&\le C h^{k\min(1,\tau)} \sum_{i=1}^{4} |u|_{V_{k+\theta}^{k+1}(S_i)} \\
&\le C h^{k\min(1,\tau)} \sum_{i=1}^{4} \|u\|_{W_{k+\theta}^{k+1}(T_i)}.
\end{aligned}$$

With $\delta=\gamma-3$ and $\tau=\frac{\eta}{k\mu}$ ■

4 Numerical Implementation

Now, We solve a $3D$ test problem numerically to evaluate the validity of inverted finite element method. we're going to show you how to calculate the integral which appears in (??). Assume that $(M_i)_{1\le i\le DOF}$ are the node of the mesh. Consider $(\varphi_i)_{1\le i\le DOF}$ the basic functions of W_h which are defined by
$\varphi_i \in W_h$, $\varphi_i(M_j)=\delta_{ij}$ if $M_j \in K$ for some$K \subset \Omega_0$, $\Lambda_\gamma\varphi_i(M_j)=\delta_{ij}$ if $M_j \in K$ for some $K \subset \Omega_*$.
posing

$$u_h(x) = \sum_{i=1}^{3} u_i\varphi_i,$$

$$u_i = \begin{cases} u(M_i)\ if\ M_i \in K\ ,\ K \subset \Omega_0 \\ \\ \Lambda_\gamma u(M_i)\ if\ M_i \in K\ ,\ K \subset \Omega_* \end{cases}$$

The problem (**??**) is equivalent to finding $U = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that

$$\sum_{j=1}^{DOF} a(\varphi_j, \varphi_i) u_j = b(\varphi_j), 1 \leq i \leq 3.$$

Defined

$$A = (A_{ij})_{1 \leqslant i,j \leqslant DOF} \in \mathbb{R}^{3,3}, \quad A_{ij} = a(\varphi_j, \varphi_i), \tag{23}$$

and

$$B = (B_i)_{1 \leqslant i \leqslant DOF} \in \mathbb{R}^3, \quad B_i = b(\varphi_i), \tag{24}$$

we get the linear system

$$AU = B. \tag{25}$$

The coefficients of A are given by

$$A_{ij} = \int_\Omega [\nabla\varphi_i]^t M \nabla\varphi_j dx + \int_\Omega b(x).\nabla\varphi_i\varphi_j dx + \int_\Omega c(x)\varphi_i\varphi_j dx. \tag{26}$$

The three terms of (**??**) can be written by

$$\int_\Omega [\nabla\varphi_i]^t M \nabla\varphi_j dx = \int_{\Omega_0} [\nabla\varphi_i]^t M \nabla\varphi_j dx + \sum_{m=1}^{4} \int_{T_m} [\nabla\varphi_i]^t M \nabla\varphi_j dx,$$

$$\int_\Omega b(x).\nabla\varphi_i\varphi_j dx = \int_{\Omega_0} b(x).\nabla\varphi_i\varphi_j dx + \sum_{m=1}^{4} \int_{T_m} b(x)\nabla\varphi_i\varphi_j dx,$$

$$\int_\Omega c(x)\varphi_i\varphi_j dx = \int_{\Omega_0} c(x)\varphi_i\varphi_j dx + \sum_{m=1}^{4} \int_{T_m} c(x)\varphi_i\varphi_j dx,$$

with $M = (a_{i,j})_{i,j}$ is a 3-dimensional matrix. All of these formulas start with a Finite Element term in the bounded domain Ω_0, It can be calculated easily as in the classic finite element method.
Now we detail the integrals in unbounded sub-domain $(T_m)_{1 \leq m \leq 4}$. we have

$$\tilde{\varphi}_i(\xi) = \varphi(F_m(\xi)), \quad x = F_m(\xi),$$

$$\int_{T_m} [\nabla\varphi_i]^t M \nabla\varphi_j dx = \sum_{K \subset S_m} \int_{\varphi(K)} [\nabla\varphi_i]^t M \nabla\varphi_j dx.$$

we find

$$\int_{\Phi(K)} [\nabla\varphi_i]^t M \nabla\varphi_j dx = |det\ B_m| \int_{F_m^{-1} \circ \Phi(K)} [\nabla_\xi \tilde{\varphi}_i]^t B_m^{-1} M(F_m(\xi)) B_m^{-t} \nabla_\xi \tilde{\varphi}_j d\xi.$$

B_m the Jacobian matrix of F_m.
Now, we are posing

- ✓ $\hat{\varphi}_l(\hat{x}) = \tilde{\varphi}_l(\hat{\Phi}(\hat{x}))$ for $l = i$ or $l = j$;
- ✓ $\xi = \hat{\Phi}(\hat{x})$;
- ✓ $\hat{x}_*^t = \dfrac{\hat{x}}{r(\hat{x})}$.

we have

$$\int_\Omega [\nabla \varphi_i]^t M \nabla \varphi_j dx = |det\ B_m| \int_{F_m^{-1}(K)} r(\hat{x})^{-2} [\nabla_{\hat{x}} \hat{\varphi}_i]^t A(\hat{x}_*) \nabla_{\hat{x}} \hat{\varphi}_j d\hat{x},$$

where

$$A(x) = (I - 2\hat{x}_*^t.c^t) B_m^{-1}.M(F_m \circ \Phi(\hat{x})) \times B_m^{-t}(I - 2c\hat{x}_*^t), \quad c = (1,1,1)^t,$$

and

$$\hat{\Phi}^{-1} \circ F_m^{-1} \circ \phi = F_m^{-1}.$$

In the same way we get

$$\int_\Omega b(x).\nabla \varphi_i \varphi_j dx = |det\ B_m| \int_{F_m^{-1}(K)} r(\hat{x})^{-4} b(F_m \circ \hat{\Phi}) B_m^{-t}(I - 2c\hat{x}_*^t) \nabla_{\hat{x}} \hat{\varphi}_i(\hat{x}) \hat{\varphi}_j(\hat{x}) d\hat{x}.$$

$$\int_\Omega c(x) \varphi_i \varphi_j dx = |det\ B_m| \int_{F_m^{-1}(K)} r(\hat{x})^{-6} b(F_m \circ \hat{\Phi}) B_m^{-t}(I - 2c\hat{x}_*^t) \hat{\varphi}_i(\hat{x}) \hat{\varphi}_j(\hat{x}) d\hat{x}.$$

Define ω_i as the unique function of $\mathbb{P}_k(F_m^{-1}(K))$ satisfying

- ✓ $\omega(M_i) = 1$ and $\omega_i(\hat{x}) = 0$ if $\hat{x}$ is another node of $F_m^{-1}(K)$;
- ✓ when $k = 1$, $\omega_i(\hat{x}) = a_i + \alpha_i^t \hat{x}$, $a_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}^3$. This function coincides with the barycentric coordinates of

$$F_m^{-1}(K)$$

.

So the right-hand-side integral is written

$$\int_{F_m^{-1}(K)} f(\delta(\hat{x}), \sigma_1, \sigma_2) d\hat{x}, \quad \delta = r(\hat{x}), \quad \sigma_1 = \frac{\hat{x}_1}{\delta},\ \sigma_2 = \frac{\hat{x}_2}{\delta}.$$

We set

$$\delta_K^- = \inf_{x \in F_m^{-1}(K)} r(x), \delta_K^+ = \sup_{x \in F_m^{-1}(K)} r(x),$$

$$\Theta_{K,\delta} = F_m^{-1}(K) \cap \{\hat{x_1} + \hat{x_2} + \hat{x_3} = \delta\}.$$

Thus,

$$\int_{F_m^{-1}(K)} f(\delta(\hat{x}), \sigma_1(\hat{x}), \sigma_2(\hat{x})\ d\hat{x} = \int_{\delta_K^-}^{\delta_K^+} \delta^2 \left(\int_{\Theta_{K,\delta}} f(\delta, \sigma_1, \sigma_2)\ d\sigma_1 d\sigma_2 \right)\ d\delta.$$

Where

$$\int_{\Theta_{K,\delta}} f(\delta, \sigma_1, \sigma_2)\ d\sigma_1 d\sigma_2,$$

are calculated using the Gauss-Labatto quadrature formula.

5 Documentclasses

6 Conclusions

There is no longer LaTeX example which was written by [**?**].

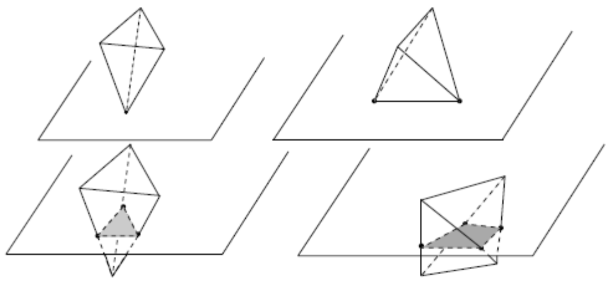

FIGURE 2 – The intersection of a tetrahedron and a plane

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