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Reduction target structure-based hierarchical attribute reduction for two-category decision-theoretic rough sets

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ABSTRACT

Attribute reduction is an essential subject in rough set theory, but because of quantitative extension, it becomes a problem when considering probabilistic rough set (PRS) approaches. The decision-theoretic rough set (DTRS) has a threshold semantics and decision feature and thus becomes a typical and fundamental PRS. Based on reduction target structures, this paper investigates hierarchical attribute reduction for a two-category DTRS and is divided into five parts. (1) The knowledge-preservation property and reduct are explored by knowledge coarsening. (2) The consistency-preservation principle and reduct are constructed by a consistency mechanism. (3) Region preservation is analyzed, and the separability between consistency preservation and region preservation is concluded; thus, the double-preservation principle and reduct are studied. (4) Structure targets are proposed by knowledge structures, and an attribute reduction is further described and simulated; thus, general reducts are defined to preserve the structure targets or optimal measures. (5) The hierarchical relationships of the relevant four targets and reducts are analyzed, and a decision table example is provided for illustration. This study offers promotion, rationality, structure, hierarchy and generalization, and it establishes a fundamental reduction framework for two-category DTRS. The relevant results also provide some new insights into the attribute reduction problem for PRS.

1. Introduction

Rough set theory (RS theory) can effectively address data analysis problems that involve uncertain, imprecise or incomplete information; Refs. [6,11,35,55,56] report relevant developments in these respects. The classical model (Pawlak model) [28,29] is a qualitative model that does not involve quantitative information. Thus, quantitative and extended models are important to study. Probability is an important tool for describing and measuring uncertainty, and it was further introduced into RS theory to construct the probabilistic rough set (PRS). The PRS approach exhibits outstanding merits, including measurability, generality and flexibility, and has many application models, such as the decision-theoretic rough set (DTRS) [46,50], 0.5-probabilistic rough set [44], variable precision rough set (VPRS) [64], rough membership function [30], parameterized rough set [4,31], Bayesian rough set [39,40] and game-theoretic rough set [1,5]. At the same time, the graded rough set (GRS) [21,48,49] also becomes a fundamentally quantitative and extended model by utilizing an absolute measure,

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which is compared to the relative measure used in PRS. We conducted a comparative study of VPRS and GRS in [59], and we further investigated the double quantification with both relative and absolute measures in [57,58].

PRS models generally require thresholds to divide the universe into application regions; as a result, threshold determination becomes a critical task. However, most thresholds usually lack underlying semantics. Yao et al. [46,50] proposed DTRS by the Bayesian risk-decision and three-way decision semantics and provided a threshold calculation and an explanation. In particular, DTRS can implement the unification of most PRS models, and it also provides a PRS fundamental exploration platform [51]. Much valuable research on DTRS has been performed. Refs. [46,47] investigated three-way decisions; Refs. [15,23,24,34] discussed model development and threshold calculation; Refs. [8,9,13,51,61] studied DTRS attribute reduction; Refs. [12,18,20,53] applied DTRS to the clustering problem; moreover, in contrast to the classical two-category case, Refs. [18,19,22,62] explored multi-category problems.

Attribute reduction is an essential subject in RS theory and underlies many practical applications. A reduct – the core notion of this theory – is a minimum attribute subset that preserves a certain classification property, as provided by entire attributes; thus, reducts hold great significance for optimization and generalization. Different approaches and algorithms have been extensively proposed in recent studies to determine all reducts or a single reduct [14,16,41,42,52,60]. Miao et al. [26] investigated reducts in consistent and inconsistent decision tables of the Pawlak model, and in particular, they analyzed the hierarchy of three types of reducts. With respect to PRS reduction, Refs. [8,9,13,51,61] studied DTRS reduction, Refs. [2,7,25,43,63] studied VPRS reduction and Refs. [40,54] concerned Bayesian rough set reduction. As is well known, a Pawlak reduct preserves the classification positive region (C-POS) due to the monotonicity of C-POS [28,29]. However, PRS usually exhibits C-POS non-monotonicity due to quantitative extension; thus, directly applying a Pawlak reduct would introduce some anomalies [2,25,43,51]. Therefore, quantitative PRS reduction has already transcended qualitative Pawlak reduction, and the relevant construction becomes a problem. For the DTRS reduction issue, Yao and Zhao [51] analyzed the separability of the dependency degree γ and proposed different measures beyond γ (such as confidence, coverage, generality and cost); furthermore, general reducts were explored by considering multiple measures. Moreover, Jia et al. [8,9] proposed a reduct based on minimum cost; Zhao et al. [61] proposed a reduct based on C-POS preservation; and Li et al. [13] proposed a reduct based on the C-POS measure.

DTRS serves as one type of fundamental PRS, and its attribute reduction can well reflect the nature of quantitative reduction. Thus, this paper mainly concentrates on DTRS reduction. In particular, the two-category case is a fundamental form of DTRS and features relatively clear relationships for both set regions and rule consistency; thus, two-category DTRS underlies generalization exploration and can also provide effective degeneration analysis. In view of its relative difficulty, the use of the two-category approach appears to be a rational strategy for conducting a breakthrough study on quantitative attribute reduction. In particular, it should be noted that rough logic is also an important research direction in RS theory. Decision logic [29] is a fundamental approach, and more extensive logic approaches have been reported in [3,10,17,27,36,37]. In fact, decision logic provides an effective theoretical system for attribute reduction and can simplify reduction algorithms, in which consistency/inconsistency plays a core role. Currently, quantitative attribute reduction extensively concerns region targets but rarely considers logic consistency. Determining how to select a rational reduction criterion is undoubtedly the key problem in attribute reduction. For qualitative Pawlak reduction, region preservation and consistency preservation become natural and equivalent reduction targets. For two-category DTRS reduction, we will investigate the rationality, separability and integration of the two reduction targets, establish the double-preservation principle to tackle the reduction criteria problem and further investigate the double-preservation reduct. As a result, the qualitative Pawlak reduct will be promoted and extended to quantitative attribute reduction.

In multi-granule scenarios, granular computing strongly emphasizes structure and hierarchy, and Refs. [32,33,38,45] have discussed many valuable characteristics and results regarding information granules. In fact, RS theory is usually viewed as a concrete model of granular computing, and the concept and method of granular computing can be used to effectively analyze the attribute reduction issue. Attribute reduction is related to knowledge structures, whereas reduction targets provide knowledge structure requirements. Accordingly, a structural study can fully probe the essence of attribute reduction. Moreover, hierarchical reduction targets can lead to hierarchical reducts, and relevant research can deepen our understanding of reduct structure systems. Herein, the above-mentioned separability of the two basic reduction targets provides ample space for implementing structural and hierarchical approaches. Based on reduction target structures, this paper investigates hierarchical attribute reduction for two-category DTRS as follows: (1) The knowledge-preservation property and reduct are explored, and the Pawlak reduct in the knowledge representation system is promoted. (2) The consistency-preservation principle and reduct are constructed by the consistency mechanism, and the Pawlak reduct in the decision table is promoted. (3) Region preservation is analyzed, and the separability between consistency preservation and region preservation is determined; accordingly, the double-preservation principle and reduct are explored, and the Pawlak reduct is extended. (4) The structure targets are proposed based on knowledge structures, and attribute reduction is further described and simulated; accordingly, general reducts are proposed to preserve structure targets or optimal measures. (5) The hierarchical relationships of the relevant four targets and reducts are analyzed, and a decision table example is provided for illustration. In particular, most of the conclusions drawn and results obtained, including the reduction targets and reduct notions, offer some generalization to multi-category and quantitative models, which is also analyzed.

This paper offers the following five innovations: (1) promotion and utilization of the classical Pawlak reduct; (2) rational construction with respect to consistency preservation, region preservation and double preservation; (3) structural reduction construction based on knowledge structures; (4) hierarchical exploration for attribute reduction; and (5) generalization analysis based on two-category DTRS reduction. Through progressive and systemic discussions, this study establishes a fundamental reduction framework for two-category DTRS and also provides new insights into the problems of Pawlak reduction, DTRS reduction and PRS reduction.

The remainder of this paper is organized as follows. Section 2 first reviews the Pawlak model, Pawlak reduct, DTRS model and DTRS reduct. Sections 3–5 describe the principles and reducts applied in this study with respect to knowledge preservation, consistency preservation and double preservation. Section 6 explores the structure targets and general reducts. Section 7 presents the reduct relationships and a sample illustration. Section 8 provides concluding remarks.

2. Preliminaries

Some terms will be used repeatedly throughout this paper. For simplification, the main abbreviations are first introduced in this section. Terms that are replaced by their first letter include the following: Decision → D, Knowledge → K, Consistency → C, Region → R and Preservation → P. Accordingly, the following phrases are clear: D-Table, K-Coarsening, K-Preservation, C-Preservation, R-Preservation and CR-Preservation. Moreover, KP-Reduct, CP-Reduct and CRP-Reduct denote the K-Preservation reduct, C-Preservation reduct and CR-Preservation reduct, respectively. ST indicates structure target in terms such as ST-Preservation and STP-Reduct. KRS indicates knowledge representation system. POS, BND and NEG and C-POS, C-BND and C-NEG denote the set positive, boundary negative regions and the classification positive, boundary and negative regions, respectively.

The next section will provide a review of relevant content associated with the basic models and reducts employed in this paper.

2.1. Pawlak model and Pawlak reduct

Herein, the Pawlak model and Pawlak reduct [28,29] are reviewed.

U is a finite and non-empty universe, $\mathscr{R}$ is a family of equivalence relations over U and $(U,\mathscr{R})$ constitutes a knowledge base. Suppose that $\phi \neq R \subseteq \mathscr{R}, \cap R$ is an equivalence relation, which is denoted as $IND(R)$. The classified structure $U/IND(R)$ serves as knowledge and is sometimes directly referred to as knowledge R. $[x]_R$ denotes the relevant equivalence class, i.e., the basic knowledge granule. Suppose that $X \subseteq U$ and the lower and upper approximations of X are defined by

$$\underline{apr}_R X = \{x|[x]_R \subseteq X\}, \overline{apr}_R X = \{x|[x]_R \cap X \neq \phi\}, \tag{1}$$

$$\mathrm{POS}_R(X) = \underline{apr}_R X, \mathrm{NEG}_R(X) = U - \overline{apr}_R X, \mathrm{BND}_R(X) = \overline{apr}_R X - \underline{apr}_R X, \tag{2}$$

denote POS, NEG and BND, respectively. Given that $P,Q \subseteq \mathscr{R}$, if $IND(P) \subseteq IND(Q)$, then knowledge Q *depends* on knowledge P (while knowledge P *deduces* knowledge Q), which is denoted as $P \Rightarrow Q$, and the knowledge structure about $U/IND(Q)$ is rougher than the knowledge structure about $U/IND(P)$. The *dependency/deducibility* underlies rough reasoning.

Knowledge is usually represented as a value-attribute table, called a knowledge representation system (KRS). KRS is a pair (U, At); U is the universe; At is a finite set of primitive attributes; and each $a \in At$ is a total function $a : U \to V_a$, where V_a is the value set of a. Suppose $\phi \neq A \subseteq C$; then, $IND(A) = \{(x,y) \in U \times U | \forall a \in A, a(x) = a(y)\}$ is the equivalence relation, and $U/IND(A)$ is the knowledge. KRS Pawlak reduct aims to preserve $IND(C)$ to preserve the knowledge. The decision table (D-Table) is a special KRS with a distinguished condition and decision attributes, and it plays an important role in decision making. Let (U, At) be a KRS, and let $C, D \subseteq At$ be the condition and decision attribute subsets, respectively; then, $S = (U, C \cup D)$ constitutes the D-Table. The D-Table is consistent if all of the object pairs that have the same condition values on C also have the same decision values on D; otherwise, the D-Table is inconsistent. The positive region of D on A (i.e., C-POS) is defined by

$$\mathrm{POS}_A(D) = \bigcup_{X \in U/IND(D)} \underline{apr}_{IND(A)} X, \tag{3}$$

whereas $\mathrm{BND}_A(D) = U - \mathrm{POS}_A(D)$ is the classification boundary (i.e., C-BND). If $B' \subseteq B \subseteq C$, then $\mathrm{POS}_{B'}(D) \subseteq \mathrm{POS}_B(D)$; thus, C-POS exhibits monotonicity. In particular, $\gamma_A(D) = \frac{|\mathrm{POS}_A(D)|}{|U|}$ is the classical measure γ for the classification quality, i.e., the dependency degree of D on A.

Definition 2.1 [29]. Suppose that $B \subseteq C$. B is a Pawlak reduct of C if it satisfies the C-POS preservation and set independency, i.e., $\mathrm{POS}_B(D) = \mathrm{POS}_C(D)$, and $\forall b \in B, \mathrm{POS}_{B-\{b\}}(D) \neq \mathrm{POS}_B(D)$. $Core(C) = \{a \in C | \mathrm{POS}_{C-\{a\}}(D) \neq \mathrm{POS}_C(D)\}$ is the core of C.

If $|U/IND(D)| = 2$, then D-Table $S = (U, C \cup D)$ concerns the two-category case. Suppose that $U/IND(D) = \{X, \neg X\}$; then, $\mathrm{POS}_A(D) = \mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X), \mathrm{BND}_A(D) = \mathrm{BND}_A(X), \gamma_A(D) = \frac{|\mathrm{POS}_A(X)|}{|U|} + \frac{|\mathrm{NEG}_A(X)|}{|U|}$. Thus, the set regions and classification regions exhibit clear relationships. Moreover, C-POS preservation is equivalent to region preservation.

Definition 2.2 [29]. Let (C, D) be an algorithm, $a \in C, B \subseteq C$. (1) Suppose that algorithm (C, D) is consistent. If $((C - \{a\}), D)$ is inconsistent, then a is indispensable in (C, D), and the set of all indispensable attributes in (C, D) forms the core of (C, D). B is a reduct of C if (B, D) is consistent, and $\forall b \in B, b$ is independent in (B, D). (2) Suppose that algorithm (C, D) is inconsistent. If

$\mathrm{POS}(C,D) \neq \mathrm{POS}((C-\{a\}),D)$, then a is indispensable in (C,D), and the set of all indispensable attributes in (C,D) forms the core of (C,D). B is a reduct of C if $\mathrm{POS}(B,D) = \mathrm{POS}(C,D)$, and $\forall b \in B, b$ is independent in (B,D).

By Definition 2.2, the decision logic provides the reduct concepts of the consistent/inconsistent algorithm related to the consistent/inconsistent D-Table. Herein, $\mathrm{POS}(C,D)$ is the positive region of the algorithm and reflects the set of all consistent rules in the algorithm. The reduction targets in the decision logic aim to preserve rule consistency in both consistent and inconsistent D-Tables.

2.2. DTRS model and DTRS reduct

Next, the DTRS model is illustrated by the two-category problem [47,50].

There are only two states and three actions (accept, defer and reject). The state set $\Omega = \{X, \neg X\}$ indicates that an element is in X and not in X, and the action set is $\mathscr{A} = \{a_P, a_B, a_N\}$, where a_P, a_B and a_N represent the three actions of deciding that an object is in the sets $\mathrm{POS}(X), \mathrm{BND}(X)$ and $\mathrm{NEG}(X)$, respectively. Moreover, when an object belongs to X, let $\lambda_{PP}, \lambda_{BP}$ and λ_{NP} denote the costs of taking the actions a_P, a_B and a_N, respectively; when an object does not belong to X, then let $\lambda_{PN}, \lambda_{BN}$ and λ_{NN} denote the costs of taking the same three actions, respectively. The loss functions regarding the states X and $\neg X$ can be expressed as a 2×3 matrix, as follows:

	a_P	a_B	a_N
X	λ_{PP}	λ_{BP}	λ_{NP}
$\neg X$	λ_{PN}	λ_{BN}	λ_{NN}

Here, $[x]$ denotes the description of x, and the probabilities for the two complement states are denoted as $P(X|[x]) = \frac{|X \cap [x]|}{|[x]|}$ and $P(\neg X|[x]) = 1 - P(X|[x])$.

The expected costs $\mathscr{R}(a_i|[x])$ of taking individual actions can be expressed as

$$\begin{aligned}
&\mathscr{R}(a_P|[x]) = \lambda_{PP}P(X|[x]) + \lambda_{PN}P(\neg X|[x]),\\
&\mathscr{R}(a_B|[x]) = \lambda_{BP}P(X|[x]) + \lambda_{BN}P(\neg X|[x]),\\
&\mathscr{R}(a_N|[x]) = \lambda_{NP}P(X|[x]) + \lambda_{NN}P(\neg X|[x]).
\end{aligned} \tag{4}$$

The Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) If $\mathscr{R}(a_P|[x]) \leqslant \mathscr{R}(a_B|[x])$ and $\mathscr{R}(a_P|[x]) \leqslant \mathscr{R}(a_N|[x])$, then decide $[x] \subseteq \mathrm{POS}(X)$;
(B) If $\mathscr{R}(a_B|[x]) \leqslant \mathscr{R}(a_P|[x])$ and $\mathscr{R}(a_B|[x]) \leqslant \mathscr{R}(a_N|[x])$, then decide $[x] \subseteq \mathrm{BND}(X)$;
(N) If $\mathscr{R}(a_N|[x]) \leqslant \mathscr{R}(a_P|[x])$ and $\mathscr{R}(a_N|[x]) \leqslant \mathscr{R}(a_B|[x])$, then decide $[x] \subseteq \mathrm{NEG}(X)$.

By the reasonable loss condition, $\lambda_{PP} \leqslant \lambda_{BP} < \lambda_{NP}, \lambda_{NN} \leqslant \lambda_{BN} < \lambda_{PN}$, the minimum-risk decision rules (P)-(B) can be written as

(P1) If $P(X|[x]) \geqslant \alpha$ and $P(X|[x]) \geqslant \gamma$, then decide $[x] \subseteq \mathrm{POS}(X)$;
(B1) If $P(X|[x]) \leqslant \alpha$ and $P(X|[x]) \geqslant \beta$, then decide $[x] \subseteq \mathrm{BND}(X)$;
(N1) If $P(X|[x]) \leqslant \beta$ and $P(X|[x]) \leqslant \gamma$, then decide $[x] \subseteq \mathrm{NEG}(X)$,

where

$$\begin{aligned}
\alpha &= \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})} = \frac{\lambda_{(P-B)N}}{\lambda_{(P-B)N} + \lambda_{(B-P)P}},\\
\beta &= \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})} = \frac{\lambda_{(B-N)N}}{\lambda_{(B-N)N} + \lambda_{(N-B)P}},\\
\gamma &= \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})} = \frac{\lambda_{(P-N)N}}{\lambda_{(P-N)N} + \lambda_{(N-P)P}}.
\end{aligned} \tag{5}$$

Rule (B1) indicates that $\alpha > \beta$, and furthermore, $0 \leqslant \beta < \gamma < \alpha \leq 1$. Thus, the following three-way decisions are obtained:

(P2) If $P(X|[x]) \geqslant \alpha$, then decide $[x] \subseteq \mathrm{POS}(X)$;
(B2) If $\beta < P(X|[x]) < \alpha$, then decide $[x] \subseteq \mathrm{BND}(X)$;
(N2) If $P(X|[x]) \leqslant \beta$, then decide $[x] \subseteq \mathrm{NEG}(X)$.

Hence, the thresholds α, β are determined by the loss functions $\lambda_{..}$ in formula (5), and the three regions are established:

$$\mathrm{POS}^{\alpha,\beta}(X) = \{x|P(X|[x]) \geqslant \alpha\},$$
$$\mathrm{BND}^{\alpha,\beta}(X) = \{x|\beta < P(X|[x]) < \alpha\},$$
$$\mathrm{NEG}^{\alpha,\beta}(X) = \{x|P(X|[x]) \leqslant \beta\}.$$

The two-category case is a basic item for the DTRS model in which only the three-way regions (POS, NEG and BND) can fully describe the region structures. The two-category case can be generalized to the multi-category case and can also be used to conduct relevant degeneration detection. Note that there are two main approaches for the multi-category extension. One approach transforms the multi-category case into multiple two-category cases, whereas the other resorts to performing a Bayesian decision in a high-dimensional space. C-POS is a core concept for multi-category attribute reduction and has two definitions that are related to the two approaches. Here, $S = (U, C \cup D), A \subseteq C, |U/IND(D)| \geqslant 2$.

Definition 2.3 (*Form I*). [13,51]

$$\begin{aligned}
\mathrm{POS}_A^{\alpha,\beta}(D) &= \bigcup_{X \in U/IND(D)} \mathrm{POS}_A^{\alpha,\beta}(X),\\
\mathrm{BND}_A^{\alpha,\beta}(D) &= \bigcup_{X \in U/IND(D)} \mathrm{BND}_A^{\alpha,\beta}(X),\\
\mathrm{NEG}_A^{\alpha,\beta}(D) &= U - \mathrm{POS}_A^{\alpha,\beta}(D) \cup \mathrm{BND}_A^{\alpha,\beta}(D).
\end{aligned} \tag{6}$$

Definition 2.4 (*Form II*). [8,61]

$$\begin{aligned}
\mathrm{POS}_A^{\alpha,\beta}(D) &= \{x|P(D_{max}([x]_A)|[x]_A) \geqslant \alpha\},\\
\mathrm{BND}_A^{\alpha,\beta}(D) &= \{x|\beta < P(D_{max}([x]_A)|[x]_A) < \alpha\},\\
\mathrm{NEG}_A^{\alpha,\beta}(D) &= \{x|P(D_{max}([x]_A)|[x]_A) \leqslant \beta\},
\end{aligned} \tag{7}$$

where $D_{max}([x]_A) = arg\ \max_{X \in U/IND(D)}\{\frac{|[x]_A \cap X|}{|[x]_A|}\}$.

We next conduct degeneration for Forms I and II by the two-category approach.

Proposition 2.5. *For Forms I and II, the corresponding results in the two-category approach are as follows. First,* $\mathrm{NEG}_A^{\alpha,\beta}(D) = \phi$.

(1) *If* $\alpha + \beta = 1$, *then* $\mathrm{POS}_A^{\alpha,\beta}(D) = \mathrm{POS}_A^{\alpha,\beta}(X) \cup \mathrm{NEG}_A^{\alpha,\beta}(X), \mathrm{BND}_A^{\alpha,\beta}(D) = \mathrm{BND}_A^{\alpha,\beta}(X), \mathrm{POS}_A^{\alpha,\beta}(D) \cap \mathrm{BND}_A^{\alpha,\beta}(D) = \phi$,
(2) *If* $\alpha + \beta > 1$, *then* $\mathrm{POS}_A^{\alpha,\beta}(D) \subset \mathrm{POS}_A^{\alpha,\beta}(X) \cup \mathrm{NEG}_A^{\alpha,\beta}(X), \mathrm{BND}_A^{\alpha,\beta}(D) \supset \mathrm{BND}_A^{\alpha,\beta}(X), \mathrm{POS}_A^{\alpha,\beta}(D) \cap \mathrm{BND}_A^{\alpha,\beta}(D) = \phi$,
(3) *Suppose that* $\alpha + \beta < 1$. *For Form I,* $\mathrm{POS}_A^{\alpha,\beta}(D) \supset \mathrm{POS}_A^{\alpha,\beta}(X) \cup \mathrm{NEG}_A^{\alpha,\beta}(X), \mathrm{BND}_A^{\alpha,\beta}(D) \supset \mathrm{BND}_A^{\alpha,\beta}(X), \mathrm{POS}_A^{\alpha,\beta}(D) \cap \mathrm{BND}_A^{\alpha,\beta}(D) \neq \phi$. *For Form II,* $\mathrm{POS}_A^{\alpha,\beta}(D) \supset \mathrm{POS}_A^{\alpha,\beta}(X) \cup \mathrm{NEG}_A^{\alpha,\beta}(X), \mathrm{BND}_A^{\alpha,\beta}(D) \subset \mathrm{BND}_A^{\alpha,\beta}(X), \mathrm{POS}_A^{\alpha,\beta}(D) \cap \mathrm{BND}_A^{\alpha,\beta}(D) = \phi$.

Both forms of the classification regions aim to generalize the two-category results and to multi-category attribute reduction. By degeneration, Proposition 2.5 provides the two-category result on the set regions. C-NEG can be non-empty in the multi-category DTRS model, but compared to the Pawlak model result, C-NEG is still empty in the two-category case. Moreover, form I exhibits an unconventional property in condition $\alpha + \beta < 1$.

Currently, DTRS reduction mainly focuses on multi-category cases. Jia et al. [8] provided a thorough review of the existing methods that can be used to this end, including positive-based reduct [61], nonnegative-based reduct [8] and cost-based reduct [8]. Here, the three reducts are first defined (where only Form II is concerned) and will be subsequently compared with our approach by using reduction targets and degeneration.

Definition 2.6 [61]. B is a positive-based reduct of C, if (1) $\mathrm{POS}_B^{\alpha,\beta}(D) = \mathrm{POS}_C^{\alpha,\beta}(D)$; (2) $\forall b \in B, \mathrm{POS}_{B-\{b\}}^{\alpha,\beta}(D) \neq \mathrm{POS}_B^{\alpha,\beta}(D)$.

Definition 2.7 [8]. B is a nonnegative-based reduct of C, if (1) $\neg\mathrm{NEG}_B^{\alpha,\beta}(D) = \neg\mathrm{NEG}_C^{\alpha,\beta}(D)$; (2) $\forall b \in B, \neg\mathrm{NEG}_{B-\{b\}}^{\alpha,\beta}(D) \neq \neg\mathrm{NEG}_B^{\alpha,\beta}(D)$. Here, $\neg\mathrm{NEG}^{\alpha,\beta}$ indicates the union of C-POS and C-BND.

Proposition 2.8 [8]. *Let* $\lambda_{PP} = 0 = \lambda_{NN}, p = P(D_{max}([x]_A)|[x]_A)$; *then, the cost to the knowledge A is*

$$COST(A) = \sum_{x \in \mathrm{POS}_A^{\alpha,\beta}(D)} (1-p)\lambda_{PN} + \sum_{x \in \mathrm{BND}_A^{\alpha,\beta}(D)} (p\lambda_{BP} + (1-p)\lambda_{BN}) + \sum_{x \in \mathrm{NEG}_A^{\alpha,\beta}(D)} p\lambda_{NP}.$$

Definition 2.9 [12]. B is a cost-based reduct of C, if (1) $B = \arg\min_{A \subseteq C} COST(A)$; (2) $\forall B' \subset B, COST(B') > COST(B)$.

3. Knowledge preservation (K-Preservation) and KP-Reduct

Attribute reduction is related to a change in knowledge structure. Based on the novel idea that decision attributes provide only structure criteria, the independent study of the knowledge structures of conduction attributes provides a fundamental approach to D-Table reduction. Thus, we study both the K-Preservation target and the reduct in this section.

Suppose that D-Table $S = (U, C \cup D), C' \subseteq C, a \in C$. When deleting attribute set C', C and $C - C'$ correspond to old and new knowledge, respectively. This process is specifically referred to as knowledge coarsening (K-Coarsening) in this paper and is noted as $C \stackrel{-C'}{\to} C - C'$ or $C \stackrel{-}{\to} C - C'$; if $C' = \{a\}$, then only $C \stackrel{-a}{\to} C - \{a\}$ is used. In fact, attribute reduction concerns mainly K-Coarsening, and the latter is directly related to granular computing. Next, Theorem 3.1 and Proposition 3.2 present common properties of K-Coarsening.

Theorem 3.1 (*Knowledge monotonicity*).
(1) $IND(C - C') \supseteq IND(C)$. *i.e.,* $C \Rightarrow C - C'$; *(2)* $|U/IND(C - C')| \leqslant |U/IND(C)|$.

Proposition 3.2.

(1) $IND(C - C') = IND(C) \iff |U/IND(C - C')| = |U/IND(C)|$;
(2) $IND(C - C') \supset IND(C) \iff |U/IND(C - C')| < |U/IND(C)|$.

Theorem 3.1 reflects the most basic feature of K-Coarsening – knowledge monotonicity, i.e., knowledge does not become finer and the knowledge-granular number does not increase. Proposition 3.2 further shows that the K-Coarsening feature can be represented only by the knowledge-granular number.

K-Coarsening involves two cases. First, the knowledge structure may not change when reducing the attributes, where the deleted attributes might be redundant. Second, attribute deletion leads to some changes in the knowledge structure, i.e., old but finer knowledge becomes new but rougher knowledge. Proposition 3.2 actually reflects both cases and provides an identification feature.

For both cases, knowledge preservation and knowledge non-preservation are important properties. For simplicity, these concepts are denoted as KP and KNP, respectively. Suppose that K-Coarsening $\varphi : . \stackrel{-}{\to} .$; then, "$. \stackrel{-}{\to} . \models KP$" indicates that φ preserves knowledge, whereas "$. \stackrel{-}{\to} . \models KNP$" shows the opposite. Next, KP and KNP stability will be discussed. For this purpose, the K-Coarsening sequence is first studied. Here, suppose that $C = B \cup C', C' = \{a_1, a_2, \ldots, a_t\}, t \geqslant 2$.

Lemma 3.3.

$$C \stackrel{-C'}{\to} B \iff C \stackrel{-a_1}{\to} B - \{a_1\} \stackrel{-a_2}{\to} \ldots \stackrel{-a_t}{\to} B - \{a_1, a_2, \ldots, a_t\}, \forall a_1, a_2, \ldots, a_t. \tag{8}$$

Lemma 3.3 reflects the decomposition/construction property and interchangeability for a K-Coarsening sequence, which are clear when considering the attribute deletion action. Moreover, they also hold for a sequence in which the deleted part is not a single attribute but instead is an attribute subset. Next, suppose that $C_0 \subseteq C'$, and $. \stackrel{-C_0}{\to} .$ is K-Coarsening in a sequence T for $C \stackrel{-C'}{\to} B$.

Theorem 3.4 (*Infection Principle*).

$$\begin{aligned} &\exists T, \exists . \stackrel{-C_0}{\to} . \models KNP \iff C \stackrel{-C'}{\to} B \models KNP, \\ &\exists C_0^* \subseteq C', C \stackrel{-C_0^*}{\to} C - C_0^* \models KNP \iff C \stackrel{-C'}{\to} C - C' \models KNP. \end{aligned} \tag{9}$$

Proof. (1) Based on Lemma 3.3, $. \stackrel{-C_0}{\to} .$ can be assumed to be the first K-Coarsening with KNP in the sequence, i.e., $IND(C - C_0) \supset IND(C)$. Based on knowledge monotonicity (Theorem 3.1), the same knowledge on $U/IND(C)$ cannot be obtained in the later K-Coarsening. Hence, $C \stackrel{-C'}{\to} B \models KNP$. The opposite is made clear by $C_0 = C'$ and $T = \{C \stackrel{-C'}{\to} B\}$. (2) $C \stackrel{-C'}{\to} B \iff C \stackrel{-C_0^*}{\to} C - C_0^* \stackrel{-}{\to} B$. As $C \stackrel{-C_0^*}{\to} C - C_0^* \models KNP, C \stackrel{-C'}{\to} B \models KNP$ according to (1). The opposite is made clear by $C_0^* = C'$. □

Theorem 3.5 (*Purity Principle*).

$$\begin{aligned} &C \stackrel{-C'}{\to} B \models KP \iff \forall T, \forall . \stackrel{-C_0}{\to} . \models KP, \\ &C \stackrel{-C'}{\to} C - C' \models KP \iff \forall C_0^* \subseteq C', C \stackrel{-C_0^*}{\to} C - C_0^* \models KP. \end{aligned} \tag{10}$$

The Infection Principle shows that any sub-deletion (in any stages) with the KNP feature will lead to the KNP result for the whole deletion. In other words, KNP exhibits the infection feature from the internal part of the K-Coarsening sequence. In contrast, the Purity Principle demonstrates the KP harmony between the global and local K-Coarsening. Moreover, the Infection and Purity Principles exhibit duality.

Definition 3.6.

$$KP(C) = \{a \in C | IND(C - \{a\}) = IND(C)\},$$
$$KNP(C) = \{a \in C | IND(C - \{a\}) \supset IND(C)\} \tag{11}$$

are called the K-Preservation and KN-Preservation sets of C, respectively.

According to the single attribute deletion, Definition 3.6 describes the two cases with respect to C in K-Coarsening and proposes two classified sets for attributes, i.e., $KP(C)$ and $KNP(C)$.

Corollary 3.7.

$$C \stackrel{-a}{\rightarrow} C - \{a\} \models KP \iff a \in KP(C);$$
$$C \stackrel{-a}{\rightarrow} C - \{a\} \models KNP \iff a \in KNP(C). \tag{12}$$

Corollary 3.7 exhibits the fundamental feature of $KP(C)$, that knowledge C does not change when deleting each attribute, whereas $KNP(C)$ reflects the opposite. Furthermore, the Core Principle will illustrate the deletion property on $KP(C)$ and $KNP(C)$.

Theorem 3.8 (*Core Principle*).

$$(1)\ KNP(C) \cap C' \neq \phi \Rightarrow C \stackrel{-C'}{\rightarrow} B \models KNP;$$
$$(2)\ C \stackrel{-C'}{\rightarrow} B \models KP \Rightarrow C' \subseteq KP(C). \tag{13}$$

Proof. (1) $KNP(C) \cap C' \neq \phi$ indicates that $\exists a_i \in C', C \stackrel{-a_i}{\rightarrow} C - \{a_i\} \models KNP$; based on the Infection Principle, $C \stackrel{-C'}{\rightarrow} B \models KNP$. Moreover, (1) and (2) are dual, and thus, the proof is completed. □

Definition 3.9. $Core(C) = KNP(C)$ is the core on K-Preservation.

The attribute set C has been divided into two classes (i.e., $KP(C)$ and $KNP(C)$) according to the KP feature of the single attribute deletion. The Core Principle (Theorem 3.8) further shows that KP cannot delete attributes in $KNP(C)$ but might delete some attributes in $KP(C)$. Therefore, $KNP(C)$ becomes the K-Preservation core. The opposite of the Core Principle does not hold because both $KP(C)$ and $KNP(C)$ require the specific condition with respect to only one attribute. Example 1 provides a counterexample. Moreover, the Core Principle and its opposite become clear according to the KRS core and reduct, and Theorem 3.12 will establish the basis. For K-Preservation simplification, attribute deletion in $KP(C)$ therefore exhibits both necessity and non-sufficiency.

Example 1. In $S = (U, C \cup D), U = \{x_1, \ldots, x_8\}, C = \{a, b, c\}$.

$$U/IND(C) = \{\{x_1, x_5\}, \{x_2, x_8\}, \{x_3\}, \{x_4\}, \{x_6\}, \{x_7\}\},$$
$$U/IND(\{a\}) = \{\{x_1, x_4, x_5\}, \{x_2, x_8\}, \{x_3\}, \{x_6, x_7\}\};$$
$$U/IND(\{b\}) = \{\{x_1, x_3, x_5\}, \{x_6\}, \{x_2, x_4, x_7, x_8\}\};$$
$$U/IND(\{c\}) = \{\{x_1, x_5\}, \{x_6\}, \{x_2, x_7, x_8\}, \{x_3, x_4\}\}. \tag{14}$$

Thus, $KNP(C) = \{a\}, KP(C) = \{b, c\}$; however, $C \stackrel{-KP(C)}{\rightarrow} \{a\} \models KNP$.

For D-Table, the Infection and Purity Principles play a fundamental role in the independent environment with respect to knowledge C. The principles ensure the K-Preservation's stability and provide the deletion core. The Infection and Purity Principles and the Core Principle harmoniously underlie K-Preservation reduction. Moreover, they all depend on only the D-Table's formal structure (mainly on C) rather than on concrete models. Thus, a type of reduct on K-Preservation can be defined for all models, which is also related only to the formal structure.

Definition 3.10 (*KP-Reduct*). B is a KP-Reduct of C, if (1) $IND(B) = IND(C)$; (2) $\forall B' \subset B, IND(B') \supset IND(B)$.

Corollary 3.11. *If B is a KP-Reduct of C, then $C \stackrel{-(C-B)}{\rightarrow} B \models KP$.*

Note that a KP-Reduct has a unified definition relative to the KRS Pawlak reduct [29], but its descriptive framework is D-Table $S = (U, C \cup D)$. The reduction target of KP-Reduct, i.e., K-Preservation, is the highest requirement for the knowledge structure; thus, it should be naturally suitable for all model reduction. Corollary 3.11 indicates that deleting all of the attributes beyond KP-Reduct do not cause any change in knowledge. Thus, some redundant attributes can be first deleted by KP-Reduct if our purpose is to seek a weaker reduct. Based on the core $KNP(C)$, KP-Reduct provides an approach to searching for concrete deletion-attribute sets in $KP(C)$. To a certain extent, therefore, K-Preservation reduction can be viewed as a pretreatment for D-Table reduction and all model reduction. Based on KP-Reduct, Section 7 will provide valuable information for other reducts.

Theorem 3.12 (*KP-Reduct Promotion*). *Based on the preservation of the knowledge of C, KP-Reduct on D-Table $S = (U, C \cup D)$ promotes the Pawlak reduct on KRS (U, C) from an applicability perspective.*

Theorem 3.12 reflects the applicability promotion of KP-Reduct based on the KRS Pawlak reduct. (1) The latter applies only to the Pawlak model, whereas the former may apply to arbitrary models, although both models are equivalent at the Pawlak model level. (2) The latter applies only to KRS (U, C), whereas the former could apply to D-Table $S = (U, C \cup D)$, with the arbitrary decision set D; as a result, we establish the reduction connection between D-Table and its sub-KRS. From another perspective, the KRS Pawlak reduct has been promoted to describe D-Table and becomes a type of general D-Table reduct, i.e., the proposed KP-Reduct. Therefore, K-Preservation and KP-Reduct exhibit a generalization feature and have applicable significance for all model reduction; of course, they should utilize the results of the KRS Pawlak reduct. Moreover, they correspond to the dependency supremum and thus provide the most natural but the strongest structure. Against this background, the structure targets and general reducts in Section 6 will provide a perfect structural explanation.

4. Consistency preservation (C-Preservation) and CP-Reduct

Consistency/inconsistency is a fundamental feature of decision rules. What the Pawlak reduct aims to preserve in decision logic is rule consistency in both consistent and inconsistent D-Tables. In this section, we explore rule consistency/inconsistency within the two-category DTRS framework; furthermore, we construct the corresponding reduction principle and concept. Thus, a fundamental principle (C-Preservation) will be rationally established and extensively generalized. Next, consistency/inconsistency will be directly described while the rules are omitted.

Definition 4.1. Given $S = (U, C \cup D), U/IND(D) = \{X, \neg X\}, \forall A \subseteq C.\ cst_A, icst_A : 2^U \to 2^U, \forall E \subseteq U,$

$$\begin{aligned} &cst_A(E) = \{x \in E | \forall [x]_A, ([x]_A \subseteq X) \vee ([x]_A \subseteq \neg X)\}, \\ &icst_A(E) = \{x \in E | \forall [x]_A, ([x]_A \cap X \neq \phi) \wedge ([x]_A \cap \neg X \neq \phi)\}. \end{aligned} \tag{15}$$

$cst_A(E)$ and $icst_A(E)$ are called consistent and inconsistent regions of E on A, respectively.

The consistent and inconsistent regions aim to extract objects that are connected with consistent and inconsistent rules, respectively.

Theorem 4.2 (*Consistency Stability*).

$S = (U, C \cup D), cst_C(U) = U - BND_C(X)$, and $icst_C(U) = BND_C(X)$.

The consistency stability demonstrates that both the consistent and inconsistent features of D-Tables are related to only Pawlak-BND. Thus, consistency/inconsistency is completely determined by D-Table (mainly by the formal structure on both C and D) and has no relationship with concrete models or set regions. In other words, the Pawlak-Region structure has been promoted and is related not only to the Pawlak-Region but also to consistency/inconsistency and thus underlies consistency/inconsistency for all of the models. Put simply, Pawlak-BND has been promoted to describe consistency/inconsistency.

Theorem 4.3. *D-Table S is consistent if and only if $BND_C(X) = \phi$; S is inconsistent if and only if $BND_C(X) \neq \phi$.*

Based on the consistency stability, Theorem 4.3 extracts the features of consistent and inconsistent D-Tables by Pawlak-BND. Thus, the consistency/inconsistency of D-Table is determined only by Pawlak-BND.

Next, we explore the applicability of DTRS to the consistent/inconsistent D-Table.

Theorem 4.4. *If three-way decisions exist and do not degenerate, then S is inconsistent. If S is consistent, then the deferred decision does not exist, and thus, only the two-way decisions or the single-way decision exists.*

Proof. $\mathrm{BND}_C^{\alpha,\beta}(X) \subseteq \mathrm{BND}_C(X)$. Based on Theorem 4.3, if three-way decisions exist, then $\mathrm{BND}_C^{\alpha,\beta}(X) \neq \phi$; thus, $\mathrm{BND}_C(X) \neq \phi$ and S is inconsistent; if S is consistent, then $\mathrm{BND}_C^{\alpha,\beta}(X) = \phi$, and thus, the deferred decision does not exist. □

Theorem 4.4 reflects the relationships between the decision number and the consistent/inconsistent D-Table. Thus, the three-way decisions are mainly related to the inconsistent D-Table, i.e., DTRS is more suitable for the inconsistent D-Table. In fact, the inconsistent D-Table can better reflect the DTRS thresholds and extension.

Proposition 4.5. *$cst_C(E) = E - \mathrm{BND}_C(X), icst_C(E) = E \cap \mathrm{BND}_C(X)$.*

Corollary 4.6.

$$\begin{aligned}&cst_C(\mathrm{POS}_C^{\alpha,\beta}(X)) = \mathrm{POS}_C(X), icst_C(\mathrm{POS}_C^{\alpha,\beta}(X)) = \mathrm{POS}_C^{\alpha,\beta}(X) \cap \mathrm{BND}_C(X);\\ &cst_C(\mathrm{BND}_C^{\alpha,\beta}(X)) = \phi, icst_C(\mathrm{BND}_C^{\alpha,\beta}(X)) = \mathrm{BND}_C^{\alpha,\beta}(X);\\ &cst_C(\mathrm{NEG}_C^{\alpha,\beta}(X)) = \mathrm{NEG}_C(X), icst_C(\mathrm{NEG}_C^{\alpha,\beta}(X)) = \mathrm{NEG}_C^{\alpha,\beta}(X) \cap \mathrm{BND}_C(X).\end{aligned} \tag{16}$$

Proposition 4.5 provides the computational formulas for consistent and inconsistent regions. Corollary 4.6 further describes the consistency and inconsistency regions with respect to POS, BND and NEG, where the following formulas are specially utilized: $\mathrm{POS}_C^{\alpha,\beta}(X) \supseteq \mathrm{POS}_C(X), \mathrm{BND}_C^{\alpha,\beta}(X) \subseteq \mathrm{BND}_C(X), \mathrm{NEG}_C^{\alpha,\beta}(X) \supseteq \mathrm{NEG}_C(X)$.

Theorem 4.7 (*Consistency Monotonicity*). *In K-Coarsening, the change in consistency is monotonic, i.e., the consistency region is not enlarged, whereas the inconsistent region is not lessened.*

Pawlak reduction aims to preserve the consistency in decision logic. For extended and quantitative models (such as PRS models), the consistency – an essential factor for attribute reduction – has usually been forgotten and neglected. We start by using this key notion in this paper. Consistency monotonicity (Theorem 4.7) is clear because of Pawlak-BND, and more importantly, it reveals the fact that consistency inherits the monotonicity exhibited by K-Coarsening. Therefore, the C-Preservation Principle is a natural but important requirement. Moreover, consistency monotonicity always exists regardless of the model considered; thus, C-Preservation is a general principle for D-Table reduction.

Principle 4.8 (*C-Preservation Principle*). Attribute reduction should consider C-Preservation; the consistent and inconsistent regions do not change, i.e., the consistency and inconsistency of relevant rules do not change.

The C-Preservation Principle is clearly based on the consistency mechanism and the monotonicity. Preserving consistency is equivalent to preserving inconsistency; thus, C-Preservation indicates dual preservation. In contrast, if consistency is not preserved, then rule consistency or inconsistency necessarily changes. Herein, an example is first presented to provide detailed explanations.

Example 2. Table 1 provides a D-Table with a statistical form, where $S = (U, C \cup D), U = \{x_1, x_2, \ldots, x_{29}\}, C = \{c_1, c_2, c_3\}$ and $U/IND(D) = \{X, \neg X\}$. The original Pawlak-BND, Pawlak-NEG and Pawlak-POS are $[x]_3 \cup [x]_4 \cup [x]_6 \cup [x]_7, [x]_1 \cup [x]_2, \phi$, respectively. First, we can easily verify that the inconsistent rules are truly related to only Pawlak-BND. Next, (1) if c_2 is deleted, then Pawlak-BND is U, and thus, only the changed $[x]_1 \cup [x]_2$ requires analysis. In K-Coarsening $\{c_1, c_2, c_3\} \xrightarrow{-c_2} \{c_1, c_3\}$, $[x]_1 \cup [x]_2$ is transferred from the old Pawlak-NEG to the new Pawlak-BND; the old rule $(c_1 = 1) \wedge (c_2 = 1) \wedge (c_3 = 1) \Rightarrow \neg X$ is consistent, but the new rule $(c_1 = 1) \wedge (c_3 = 1) \Rightarrow \neg X$ becomes inconsistent due to the existing rule $(c_1 = 1) \wedge (c_3 = 1) \Rightarrow X$ that is obtained by $[x]_3$ or $[x]_4$. (2) If c_3 is deleted, i.e., for $\{c_1, c_2, c_3\} \xrightarrow{-c_3} \{c_1, c_2\}$, Pawlak-BND does not change while $[x]_1$ and $[x]_2$ are still in Pawlak-NEG; the sole consistent rule $(c_1 = 1) \wedge (c_2 = 1) \wedge (c_3 = 1) \Rightarrow \neg X$ changes into $(c_1 = 1) \wedge (c_2 = 1) \Rightarrow \neg X$, which is still consistent. Therefore, deleting c_2 is prohibited, whereas deleting c_3 is allowed for the C-Preservation Principle; moreover, the consistency monotonicity is also verified.

Next, we further analyze the scientific nature of the C-Preservation Principle. C-Preservation maintains consistency and, in fact, maintains absolute and complete certainty. In other words, the position of the real certainty cannot change even to a slight degree, i.e., the absolute certainty cannot be fused with approximate certainty because both are completely different notions from a qualitative standpoint. On the other hand, inconsistency also requires a qualitative preservation. In Example 2, $[x]_1$ and $[x]_2$ have already been determined to completely belong to $\neg X$, which exhibits real certainty, or so C-Preservation suggests: they should stay in their group with absolute certainty, whereas they cannot be combined with other knowledge granules (such as $[x]_3$ and $[x]_4$) from the other group with uncertainty. This principle is extremely similar to that of social classes and activities. C-Preservation ensures certainty purity and provides the qualitative requirement; other impure certainties will make the hierarchical division and mutual combination with the quantitative criterion, which is partially addressed in the next section's discussion on quantitative regions. Therefore, the C-Preservation Principle is related to the certainty/uncertainty preservation and becomes rational and effective; furthermore, it should be and can become a generalized criterion, whereas the Pawlak-Regions or Pawlak-BND plays the core role.

Table 1
A statistical decision table.

$[x]_i$	$\|[x]_i\|$	c_1	c_2	c_3	$\|[x]_i \cap X\|$
$[x]_1$	5	1	1	1	0
$[x]_2$	3	1	1	1	0
$[x]_3$	6	1	2	1	1
$[x]_4$	5	1	2	1	1
$[x]_6$	4	2	3	1	3
$[x]_7$	6	2	3	3	5

Theorem 4.9 (*Equivalence Principle*). *In attribute reduction, C-Preservation is equivalent to Pawlak-BND preservation (or Pawlak-Region preservation).*

Theorem 4.9 indicates that C-Preservation means Pawlak-Region preservation. For the Pawlak reduct, this result is clear according to the two approaches on regions and consistency. For quantitative attribute reduction, the Equivalence Principle becomes a novel content, which means that qualitative Pawlak-Region preservation is also necessary according to the definition of C-Preservation. Thus, Pawlak-Region preservation has already been extensively promoted and generalized due to the universality of C-Preservation. In other words, Pawlak-Region preservation becomes a comprehensive requirement due to the certainty/uncertainty qualitative classification.

Thus, the Pawlak reduct has been naturally promoted for other model reductions. In another view, C-Preservation constructs another criterion, and a type of new reduct (which corresponds to the Pawlak reduct) can be established for D-Table or other models. This concept is similar to that of K-Preservation or KP-Reduct.

Definition 4.10 (*CP-Reduct*). B is the CP-Reduct of C if

(1) $cst_B(U) = cst_C(U)$, i.e., $\mathrm{BND}_B(X) = \mathrm{BND}_C(X)$;
(2) $\forall B' \subset B, cst_{B'}(U) \neq cst_B(U)$, i.e., $\mathrm{BND}_{B'}(X) \supset \mathrm{BND}_B(X)$.

Theorem 4.11 (*CP-Reduct Promotion*). *Under the significance of consistency/inconsistency preservation, CP-Reduct promotes the D-Table Pawlak reduct in terms of applicability.*

To preserve consistency, Definition 4.10 proposes CP-Reduct, and its applicable range is associated with D-Table and arbitrary models. Moreover, the core can be normally defined. Theorem 4.11 is clear and demonstrates that CP-Reduct promotes the D-Table Pawlak reduct for all model reduction, whereas C-Preservation becomes a fundamental and necessary requirement for D-Table reduction. From another perspective, C-Preservation and CP-Reduct are related to the D-Table Pawlak reduct; thus, the relevant Pawlak reduct theory underlies and promotes the CP-Reduct system.

Based on the above-discussed two-category results, both the C-Preservation Principle and CP-Reduct can be generalized to attribute reduction theory. In particular, C-Preservation indicates preserving the Pawlak model's C-BND for multiple categories, which corresponds to classical C-POS preservation in the D-Table Pawlak reduct. Thus far, we have discussed two types of reduction commonality, and they have fully utilized the classical Pawlak reduction theory [29]. Therefore, this development approach is scientific, effective and creative because of the promotion study. Next, these results will be gradually introduced into our concrete topic, which is two-category DTRS reduction, and interesting new content will emerge.

5. Double preservation (CR-Preservation) and CRP-Reduct

The above-mentioned studies are based on only the D-Table or the formal structure thereof and can be generally applied to D-Table and other models. To a certain extent, these results provide sufficient or necessary conditions for D-Table regardless of the model technology employed. Next, we investigate two-category DTRS reduction, and the basic reduct concept will be established and compared in this section. Here, two subsections are provided to discuss the reduction criterion and concept.

5.1. Double preservation (CR-Preservation)

Proposition 5.1. *In* $C \xrightarrow{-} B, IND(B) = IND(C) \Rightarrow cst_B(U) = cst_C(U)$.

K-Preservation on C and C-Preservation on C, D have already provided a basic framework for reduction targets. Proposition 5.1 presents the two-target relationship, i.e., K-Preservation deduces C-Preservation, but the opposite does not hold. Thus, K-Preservation is a high and natural reduction target, whereas C-Preservation is a low and basic reduction target. For the reduction targets, the two concepts define the supremum and infimum and provide the largest necessary range. Furthermore, this basic range should be improved according to another basic factor – region preservation (R-Preservation).

Proposition 5.2. *C-Preservation and R-Preservation are equivalent in the Pawlak model.*

Except for C-Preservation, the Pawlak reduct usually uses a region criterion, namely C-POS preservation because of the monotonicity of C-POS; moreover, C-POS preservation is usually related to R-Preservation, especially in the two-category case. Thus, this equivalence conclusion becomes natural in Proposition 5.2. Here, the region target is first analyzed. As is well known, C-POS has non-monotonicity in PRS models, which actually exhibits new uncertainty, and thus, the region criterion becomes a focus and a source of difficulty for quantitative attribute reduction. Here, the R-Preservation criterion will be established according to two-category DTRS, which can also be generalized.

According to the closed word assumption (CWA) [29], the initial attributes are complete in D-Table. Therefore, the initial D-Table is usually perfect and fundamental, whereas attribute reduction can remove redundant information according to reduction targets. Thus, scientific reduct definitions are especially important because unreasonable reduct concepts can wear away and even distort the initial information. Under this idea, K-Preservation is clearly the safest approach that offers high accuracy, but reduction targets must be considered in depth for generalization to other applications. As a result, a reasonable criterion in fact exists in the balance between accuracy and generalization. Regions mainly extract condensed structures for semantics and other applications, thus becoming the basic application units. Hence, R-Preservation is a reasonable strategy because it provides the same region structure with respect to the initial D-Table and thus can act as a fundamental principle and a common requirement.

Principle 5.3 (*R-Preservation Principle*). Attribute reduction should consider R-Preservation, i.e., the three regions POS, BND and NEG do not change.

Regions can link microscopic and macroscopic information; thus, they correspond to application structures and underlie model applications. For attribute reduction, R-Preservation is a natural and normal criterion in the region monotonicity environment, and furthermore, it should be a more necessary requirement in the region non-monotonicity case. In other words, R-Preservation is a natural and certain strategy in a complex quantitative environment. Thus, the R-Preservation Principle is rational and effective. Here, we want to further add powerful evidence for the concrete two-category DTRS. Based on the modeling process related to the Bayesian risk-decision, three-way decisions are already the optimal decisions; thus, we can conclude that the three regions inherit optimal structures and also exhibit equality. Therefore, the R-Preservation Principle becomes a necessary standard for two-category DTRS reduction. In particular, the R-Preservation Principle can be generalized to attribute reduction theory, especially for multi-category cases. Based on the reasonable classification of regions, R-Preservation connotes the preservation of not only C-POS and C-BND but also the preservation of basic structural units that consist of C-POS or C-BND.

The R-Preservation reduct and core can be normally defined, which yields results similar to those for the KP-Reduct and CP-Reduct. Next, an example is provided to illustrate the concept of R-Preservation.

Example 3. For D-Table in Example 2 (i.e., Table 1), let $\alpha = 0.8, \beta = 0.2$. $\mathrm{POS}_C^{\alpha,\beta}(X) = [x]_7, \mathrm{BND}_C^{\alpha,\beta}(X) = [x]_6, \mathrm{NEG}_C^{\alpha,\beta}(X) = [x]_1 \cup [x]_2 \cup [x]_3 \cup [x]_4$. (1) In $\{c_1, c_2, c_3\} \overset{-c_3}{\rightarrow} \{c_1, c_2\}$, only $[x]_6$ and $[x]_7$ are merged; suppose that $[x]_6 \cup [x]_7 = [x]_*$; then, $P(X|[x]_*) = 0.8$, which indicates that $[x]_* \subseteq \mathrm{POS}_C^{\alpha,\beta}(X)$. Then, $\mathrm{POS}_C^{\alpha,\beta}(X) = [x]_6 \cup [x]_7, \mathrm{BND}_C^{\alpha,\beta}(X) = \phi, \mathrm{NEG}_C^{\alpha,\beta}(X) = [x]_1 \cup [x]_2 \cup [x]_3 \cup [x]_4$. Thus, deleting c_3 leads to a region change; concretely, BND is reduced while POS is enlarged, which also suggests that the region has non-monotonicity. Hence, this reduction action is not allowed under the R-Preservation Principle. (2) In $\{c_1, c_2, c_3\} \overset{-c_2}{\rightarrow} \{c_1, c_3\}$, the three regions do not change (although there is a combination with respect to $[x]_1, [x]_2, [x]_3, [x]_4$); thus, it is allowed for the R-Preservation Principle. (3) For R-Preservation, there is only one reduct $\{c_1, c_3\}$, and the core is $\{c_3\}$. (4) According to (1), only $[x]_6$ exhibits a region change when deleting c_3, and it is transferred from the old BND to new POS. This action appears to yield a better result because the deferment decision becomes the acceptance decision for the solely changed $[x]_6$. Thus, can $\{c_1, c_2\}$ be preferentially chosen as a reduct (in spite of the R-Preservation Principle)? If $\{c_1, c_2\}$ becomes a reduct, then a better certainty result appears to be obtained; however, the approach is followed only by deleting an attribute c_3; thus, this hypothesis exhibits a contradiction. According to the optimal DTRS mechanism, the deferment decision is initially the best for $[x]_6$; thus, $[x]_6$ should always use the deferment decision, even in attribute reduction. This analysis also illustrates the rationality of R-Preservation.

Based on the above-discussed mechanism and example analyses, it is observed that regions exhibit the fundamental function and requirement in qualitative classification, which is similar to certainty/uncertainty classification. However, the new qualitative classification requires quantitative thresholds. Thus far, we have established two fundamental principles. C-Preservation and R-Preservation are actually equivalent in Pawlak reduction. However, they are segregated in quantitative models, a fundamental and favorable fact that will yield new situations and novel studies.

Theorem 5.4 (*Preservation Separability*). *C-Preservation and R-Preservation are separate in the two-category DTRS model.*

$\mathrm{BND}_C^{\alpha,\beta}(X) \subseteq \mathrm{BND}_C(X)$; thus, C-Preservation and R-Preservation have been split into two different parts. Therefore, preservation separability is verified. For generalization, C-Preservation and R-Preservation are associated with qualitative and quantitative content, respectively, and thus, the preservation separability still holds for the quantitative PRS models.

Two fundamental requirements are now established, i.e., C-Preservation and R-Preservation. Using only one may not be sufficient; Examples 2 and 3 specifically provide the appropriate direct evidence. In view of both the necessity and separability of the two criteria, the integration approach naturally becomes a scientific and perfect strategy. CR-Preservation (i.e., double preservation), a novel concept, is defined to describe both C-Preservation and R-Preservation and provides the required harmony between these two ideas. Next, the CR-Preservation principle and reduct will be proposed, which underlie the next in-depth investigation.

Principle 5.5 (*CR-Preservation Principle*). Attribute reduction should consider both C-Preservation and R-Preservation, i.e., CR-Preservation.

Theorem 5.6 (*Five-Region Preservation*). *In two-category DTRS reduction, CR-Preservation actually preserves the (complete) five regions:* $\mathrm{POS}_C(X), \mathrm{NEG}_C(X), \mathrm{BND}_C(X) \cap \mathrm{POS}_C^{\alpha,\beta}(X), \mathrm{BND}_C^{\alpha,\beta}(X), \mathrm{BND}_C(X) \cap \mathrm{NEG}_C^{\alpha,\beta}(X)$, *or only the (dependent) three regions:* $\mathrm{BND}_C(X) \cap \mathrm{POS}_C^{\alpha,\beta}(X), \mathrm{BND}_C^{\alpha,\beta}(X), \mathrm{BND}_C(X) \cap \mathrm{NEG}_C^{\alpha,\beta}(X)$, *or only the (dependent) three regions:* $\mathrm{BND}_C(X), \mathrm{POS}_C^{\alpha,\beta}(X), \mathrm{NEG}_C^{\alpha,\beta}(X)$.

The CR-Preservation Principle (Principle 5.5) is natural and scientific and defines a fundamental framework and underlies the in-depth exploration of attribute reduction. According to the two-category DTRS reduction, Theorem 5.6 provides concrete results for CR-Preservation. In the regional view, CR-Preservation is actually the (complete) five-region or the (dependent) three-region preservation. In other words, only the five or three regions can fully determine the CR-Preservation structure. Thus, Theorem 5.6 is specifically called *Five-Region Preservation*. First, Five-Region Preservation can be generalized to the usual quantitative and extended models. The classical Pawlak reduction preserves C-POS or consistency and, in fact, preserves the (complete) three regions or the (dependent) single region: C-BND, where the criterion corresponds to *Three-Region Preservation*. For the qualitative Pawlak model, Five-Region Preservation will naturally degenerate into Three-Region Preservation, whereas for the quantitative models, Three-Region Preservation is no longer applicable. Therefore, Five-Region Preservation is an extension of the classical approach and is a specific feature of the extended and quantitative models. Moreover, based on the calculation mechanism of rough set regions, Five-Region Preservation usually exhibits only a linear difference, and thus, it has the same computational feasibility level as Three-Region Preservation.

Some basic properties are provided for the next reduct construction, and certain symbols are defined. Similarly to KP/KNP, CRP is used to denote CR-Preservation, whereas CRNP denotes the opposite. Note that $CRP(C) = \{a \in C | C \overset{-a}{\to} C - \{a\} \models CRP\}$ and $CRNP(C) = \{a \in C | C \overset{-a}{\to} C - \{a\} \models CRNP\}$. $\forall A \subseteq C, CRP(A)$ and $CRNP(A)$ have a similar expression. Suppose that $C_0 \subseteq C'$ and that $. \overset{-C_0}{\to} .$ is K-Coarsening in a sequence T for $C \overset{-C'}{\to} B$.

Proposition 5.7 (*Infection Principle*).

$$\begin{aligned}
&\exists T, \exists . \overset{-C_0}{\to} . \models CRNP \iff C \overset{-C'}{\to} B \models CRNP,\\
&\exists C_0^* \subseteq C', C \overset{-C_0^*}{\to} C - C_0^* \models CRNP \iff C \overset{-C'}{\to} C - C' \models CRNP.
\end{aligned} \tag{17}$$

Proposition 5.8 (*Purity Principle*).

$$\begin{aligned}
&C \overset{-C'}{\to} B \models CRP \iff \forall T, \forall . \overset{-C_0}{\to} . \models CRP,\\
&C \overset{-C'}{\to} C - C' \models CRP \iff \forall C_0^* \subseteq C', C \overset{-C_0^*}{\to} C - C_0^* \models CRP.
\end{aligned} \tag{18}$$

Proposition 5.9 (*Core Principle*).

$$\begin{aligned}
&CRNP(C) \cap C' \neq \phi \Rightarrow C \overset{-C'}{\to} B \models CRNP,\\
&C \overset{-C'}{\to} B \models CRP \Rightarrow C' \subseteq CRP(C).
\end{aligned} \tag{19}$$

In the DTRS model, a change in region exhibits complexity, such as non-monotonicity. The Infection and Purity Principles ensure the stability of CR-Preservation in the complex environment and address our concern about whether the global CRP can be implemented by the local CRNP in K-Coarsening. More importantly, the Core Principle further reflects the core existence. Thus, the Infection and Purity Principles and the Core Principle harmoniously underlie CR-Preservation attribute reduction, and in turn, the reduction construction can be naturally carried out according to the Pawlak reduction approach.

5.2. CRP-Reduct

Next, we study the CR-Preservation reduct based on CR-Preservation and the relevant properties.

Definition 5.10. If $a \in CRNP(C)$, then a is independent. $\forall b \in B$, if $b \in CRNP(B)$, then B is independent. B is a CRP-Reduct of C, if $C \overset{-}{\to} B \models CRP$ and B is independent. $CRNP(C)$ is the core of C, which is denoted as $Core^{\alpha,\beta}(C)$.

Definition 5.11 (*CRP-Reduct*). B is CRP-Reduct of C, denoted as $B \in Red^{\alpha,\beta}(C)$, if (1) $C \overset{-}{\to} B \models CRP$; (2) $\forall B' \subset B, B \overset{-}{\to} B' \models CRNP$.

In the CR-Preservation Principle, CRP-Reduct is equivalently defined by Definitions 5.10 and 5.11. With respect to joint sufficiency, CRP-Reduct connotes the CR-Preservation target, which is located between K-Preservation and C-Preservation; with respect to individual necessity, set maximality and set independency become equivalent for the reduction target.

Theorem 5.12 (*CRP-Reduct Extension*). *Based on CR-Preservation, CRP-Reduct extends the qualitative Pawlak reduct to the quantitative environment.*

Theorem 5.12 reflects the CRP-Reduct extension of the Pawlak reduct. Thus, the Pawlak reduct-based reduction theory has been expanded to the quantitative models by CRP-Reduct, and the Pawlak reduct acts as only a special case of CRP-Reduct. Therefore, the novel CRP-Reduct becomes a natural and rational expansion of the Pawlak reduct and holds great significance for quantitative attribute reduction.

We finally establish the two-category DTRS-Reduction concept: CRP-Reduct. Next, we summarize some conclusions regarding generalization with respect to the category case. For simplicity, the two-category and multi-category cases are denoted as Cases I and II, respectively. Our approach concerning the preservation and reduct not only directly applies to Case I but is also suitable for Case II. C-Preservation is clear in both Cases I and II. For R-Preservation, the generalization can be implemented by preserving both the classification regions and some of their internal structures in Case II. In fact, this paper's descriptive term (in Case I) concerns only set regions rather than classification regions due to the former's sufficiency. Furthermore, CR-Preservation and CRP-Reduct can be naturally generalized by the integration of C-Preservation and R-Preservation.

Next, our approach will be compared with other existing methods. For DTRS-Reduction, there are three main reducts [8]: a positive-based reduct [61], a nonnegative-based Reduct [8] and a cost-based reduct [8], which are all based on Case II and are reviewed in Section 2. The reduction targets naturally become the subject of comparison; thus, they can be compared in Case I by degeneration (in Proposition 2.5) and in Case II by generalization (from the above analysis). (1) The positive-based reduct aims to preserve C-POS in Case II, whereas in Case I, it cannot necessarily preserve POS $\cup$ NEG and POS and NEG. The nonnegative-based reduct aims to preserve C $-$ POS $\cup$ C $-$ BND in Case II, whereas it preserves U in Case I (which is meaningless). Only R-Preservation aims to preserve POS, BND and NEG in Case I, whereas it preserves not only C-POS and C-BND but also some of their internal components in Case II. Thus, R-Preservation is rational and stable in both the degenerate and general cases. Furthermore, CRP-Reduct can also preserve the consistency/inconsistency in Cases I and II. Thus, CRP-Reduct exhibits improvements for both the positive-based and nonnegative-based reducts. (2) CRP-Reduct and the positive-based and nonnegative-based Reducts employ a structural approach because they consider classification regions or set regions (which are related to application structures), whereas the cost-based reduct employs a measurement method in which the reduction target is the minimum cost. The cost-based reduct utilizes basic cost information and obtains the final optimal measure, but it neglects the basic structure and thus lacks the appropriate link to the classical Pawlak reduct. Therefore, based on CR-Preservation, we will make some improvements by considering both structure and measurement, which constitute our work on the optimal reduct in Section 6.3 (where C-Preservation is clearly a good strategy for considering cost).

Compared to the existing methods regarding DTRS-Reduction, our approach exhibits improvements by introducing many innovative elements. In fact, our work outlined above has already fully verified the advancement and superiority of CR-Preservation and CRP-Reduct, such as their extension to the classical targets and reducts. In particular, C-Preservation, R-Preservation, CR-Preservation and CRP-Reduct are all novel concepts and provide valuable content for the exploration of PRS-Reduction. Of course, many further studies must be conducted, such as on the generalization of the implementation. Next, we discuss concrete constructions for CRP-Reduct.

Theorem 5.13 (*Core Property*).
$Core^{\alpha,\beta}(C) = \bigcap Red^{\alpha,\beta}(C)$.

Proof. (1) Suppose that $a \notin \bigcap Red^{\alpha,\beta}(C)$; therefore, $\exists B \in Red^{\alpha,\beta}(C)$ but $a \notin B$. Thus, $C \overset{-(C-B)}{\to} B \models CRP$ and $a \in C - B \subseteq C$; based on the Purity Principle (Proposition 5.8), $C \overset{-a}{\to} C - \{a\} \models CRP$, i.e., $a \in CRP(C)$ and $a \notin CRNP(C)$. Hence, $a \notin Core^{\alpha,\beta}(C)$, and $Core^{\alpha,\beta}(C) \subseteq \bigcap Red^{\alpha,\beta}(C)$. (2) Suppose that $a \notin Core^{\alpha,\beta}(C)$; therefore, $C \overset{-a}{\to} C - \{a\} \models CRP$. $\exists B \in Red^{\alpha,\beta}(C - \{a\})$; hence, $C - \{a\} \to B \models CRP$ and B is independent. Based on the natural composite property, $C \overset{-}{\to} B \models CRP$. Thus, $B \in Red^{\alpha,\beta}(C)$ but $a \notin B$. Hence, $a \notin \bigcap Red^{\alpha,\beta}(C)$, and $\bigcap Red^{\alpha,\beta}(C) \subseteq Core^{\alpha,\beta}(C)$. $\square$

Theorem 5.13 reflects two points. First, the CRP-Reduct Core has extended the Pawlak reduct core and has a similar fundamental position, such as the construction function for reducts. Second, the Purity Principle is utilized to implement the proof, which becomes fundamental in the quantitative environment.

Definition 5.14. $S = (U, C \cup D), A \subseteq C$, and the region dependency-degrees are defined as follows:

$$\gamma_A^P(D) = \frac{|\mathrm{BND}_A(X) \cap \mathrm{POS}_A^{\alpha,\beta}(X)|}{|U|}, \gamma_A^B(D) = \frac{|\mathrm{BND}_A^{\alpha,\beta}(X)|}{|U|}, \gamma_A^N(D) = \frac{|\mathrm{BND}_A(X) \cap \mathrm{NEG}_A^{\alpha,\beta}(X)|}{|U|}.$$

Furthermore, $\gamma_A^{\alpha,\beta}(D) = (\gamma_A(D), \gamma_A^P(D), \gamma_A^N(D))$ is called the dependency-degree array of D on A.

In the Pawlak model, the measure $\gamma_A(D)$ is the addition fusion of the internal region measures and is monotonic such that it completely describes the Pawlak reduct. For the DTRS model, the region change becomes not only non-monotonic but also uncertain; thus, we utilize the classified description strategy rather than the fused approach. $\gamma_A(D)$ describes $\mathrm{POS}_A(X), \mathrm{NEG}_A(X)$, while $\gamma_A^P(D), \gamma_A^N(D)$ control $\mathrm{BND}_A(X) \cap \mathrm{POS}_A^{\alpha,\beta}(X), \mathrm{BND}_A^{\alpha,\beta}(X), \mathrm{BND}_A(X) \cap \mathrm{NEG}_A^{\alpha,\beta}(X)$. According to Five-Region Preservation (Theorem 5.6), the array $\gamma_A^{\alpha,\beta}(D)$ can fully reflect the dependency of knowledge D on knowledge A. In particular, the region dependency degrees $\gamma_A^P(D), \gamma_A^B(D)$ and $\gamma_A^N(D)$ describe the proportions of the objects that are classified into the three specific regions. Moreover, $\gamma_A^P(D), \gamma_A^B(D), \gamma_A^N(D), \gamma_A(D) \in [0, 1]$, and $\gamma_A^P(D) + \gamma_A^B(D) + \gamma_A^N(D) = 1 - \gamma_A(D)$.

Theorem 5.15. $A_1 \subseteq A_2 \subseteq C$.

(1) *In the Pawlak model,* $\gamma_A^{\alpha,\beta}(D) = (\gamma_A(D), 0, 0)$, *and* $A_2 \bar{\rightarrow} A_1 \models CRP \iff \gamma_{A_1}^{\alpha,\beta}(D) = \gamma_{A_2}^{\alpha,\beta}(D) \iff \gamma_{A_1}(D) = \gamma_{A_2}(D)$;

(2) *In the DTRS model,* $A_2 \bar{\rightarrow} A_1 \models CRP \Rightarrow \gamma_{A_1}^{\alpha,\beta}(D) = \gamma_{A_2}^{\alpha,\beta}(D)$.

Theorem 5.15 describes the property of the dependency-degree array. Compared to $\gamma_A(D), \gamma_A^{\alpha,\beta}(D)$ exhibits the extension feature with dimensions and provides an equivalent description for the Pawlak model CR-Preservation. Moreover, the equality of the dependency-degree array acts as the only necessary condition for DTRS CR-Preservation due to the complexity of the region change. Thus, two interesting problems arise. What is suitable heuristic information for the attributes? How can heuristic algorithms for CRP-Reduct be constructed?

CR-Preservation is located between C-Preservation and K-Preservation; as a result, CRP-Reduct adopts an intermediate position between CP-Reduct and KP-Reduct (which are actually Pawlak reducts in D-Table and KRS, respectively). Thus, the CRP-Reduct calculation could adopt two directional strategies that are based on Pawlak reducts, i.e., CRP-Reduct can be searched from Pawlak reducts in D-Table and KRS, respectively. In this context, two algorithms that are connected to the core are provided for CRP-Reduct computation.

Algorithm 1. CP-Reduct-based algorithm for CRP-Reduct

Input:
D-Table $(U, C \cup D)$, thresholds α, β;
Output:
CRP-Reduct subset $Red_{sub}^{\alpha,\beta}(C)$;
1: Obtain CP-Reduct subset $Red_{sub}(C)$;
2: Calculate the CRP-Reduction core $Core^{\alpha,\beta}(C)$. In $Red_{sub}(C)$, obtain the subset $Red_*^{\alpha,\beta}(C) = \{A \in Red_{sub}(C) | A \supseteq Core^{\alpha,\beta}(C)\}$;
3: In $Red_*^{\alpha,\beta}(C)$, obtain subset $Red_{**}^{\alpha,\beta}(C)$, whose elements satisfy the R-Preservation condition;
4: In $Red_{**}^{\alpha,\beta}(C)$, obtain the subset $Red_{sub}^{\alpha,\beta}(C)$ by considering set independency/maximality;
5: **return** $Red_{sub}^{\alpha,\beta}(C)$.

Algorithm 2. KP-Reduct-based algorithm for CRP-Reduct

Input:
D-Table $(U, C \cup D)$, thresholds α, β;
Output:
CRP-Reduct $B \in Red^{\alpha,\beta}(C)$;
1: Obtain KP-Reduct B_K;
2: Obtain the CRP-Reduction core $Core^{\alpha,\beta}(C)$;
3: In the range $Core^{\alpha,\beta}(C) \subseteq B \subseteq B_K$, completely search B to satisfy both the CR-Preservation and independency conditions;
4: **return** B.

In Algorithm 1, Step 1 obtains CP-Reduct subset $Red_{sub}(C)$ using the D-Table Pawlak reduct. Steps 2, 3 and 4 search the hierarchical subsets $Red_*^{\alpha,\beta}(C), Red_{**}^{\alpha,\beta}(C)$ and $Red_{sub}^{\alpha,\beta}(C)$ by the progressive conditions for the core, R-Preservation and maximality, respectively. Step 5 yields the final CRP-Reduct subset $Red_{sub}^{\alpha,\beta}(C)$. This algorithm uses the layer-by-layer condition reinforcement, which is partially due to the stronger development of the reduction target. Thus, if more Pawlak reducts are initially provided, better results can be obtained. In Algorithm 2, Step 1 yields KP-Reduct B_K by KRS Pawlak reduct. Step 2 computes the CRP-Reduction core. Between the core and KP-Reduct, Step 3 searches the independent CR-Preservation set B, which is the final result. Based on Proposition 7.2, this algorithm is convergent and effective because it necessarily obtains all CRP-Reducts included in the given KP-Reduct. In short, both algorithms utilize the Pawlak reduct platform, but they involve the D-Table and KRS, respectively. The CP-Reduct-based algorithm can obtain a certain subset for all of the CRP-Reducts, whereas a KP-Reduct-based algorithm can obtain all of the included CRP-Reducts for the given KP-Reduct. In particular, both algorithms exploit the hierarchical reduct relationship. Section 7 provides relevant discussion in this respect and further analyzes both algorithms using a D-Table example. Moreover, both algorithms adopt a hierarchical strategy. On the other hand, an integrated technology can also be considered for the CRP-Reduct computation, where CR-Preservation is directly required, and this method is similar to the classical approach in RS theory; furthermore, heuristic attribute information is expected.

6. Structure targets and general reducts

CR-Preservation and CRP-Reduct have already provided the basic reduction results for two-category DTRS. Using granular computing, our objective is to conduct an in-depth exploration from a structural point of view and obtain further general benefits. First, attribute reduction is mainly linked to knowledge structures of condition attributes, which is partially related to K-Coarsening (Section 3). Second, concrete reduction targets (such as C-Preservation, R-Preservation and CR-Preservation) are mainly associated with qualitative classifications (such as Three-Region Preservation and Five-Region Preservation). Thus, attribute reduction with reduction targets has the following operational space – interaction within qualitative classifications. All of these concepts motivate the abstract and universal approach to structural reduction described in this section. In particular, our preliminary work is based on two-category DTRS and CR-Preservation, but the concepts and results thereof lie far beyond the concrete framework.

6.1. Structure targets (ST)

This subsection proposes the structure targets for constructing the structural basis.

Proposition 6.1. *K-Preservation deduces CR-Preservation, whereas CR-Preservation deduces C-Preservation and R-Preservation. However, neither of the opposites hold.*

Proposition 6.1 indicates the clear relationships among the relevant reduction targets. The development line, C-Preservation→CR-Preservation→ K-Preservation, mainly seeks both finer structure for condition knowledge and higher requirements for reduction targets. Here, we introduce *structure targets* to link condition knowledge and reduction targets and to further propose *general reducts*. Note that CR-Preservation and K-Preservation have formed a new infimum and supremum for reduction targets; thus, they finally establish the scientific framework with a more reasonable and necessary reduction range. Thus, our concrete discussion occurs mainly within this framework. Here, the knowledge dependency/deducibility relation "$\Leftarrow$"/"$\Rightarrow$" can be easily generalized to describe the arbitrary granular structure.

Example 4. Let $E/IND(A) = \{[x]_A^1, [x]_A^2, [x]_A^3\}$. According to the three knowledge granules, only five types of granular structures exist, i.e., $st_1 = \{[x]_A^1, [x]_A^2, [x]_A^3\}, st_2 = \{[x]_A^1 \cup [x]_A^2, [x]_A^3\}, st_3 = \{[x]_A^1, [x]_A^2 \cup [x]_A^3\}, st_4 = \{[x]_A^1 \cup [x]_A^3, [x]_A^2\}, st_5 = \{[x]_A^1 \cup [x]_A^2 \cup [x]_A^3\}$. $ST_A(E) = \{st_1, \ldots, st_5\}$ is a complete lattice with respect to "$\Leftarrow$", whereas st_1 and st_5 become the greatest and least elements, respectively. Moreover, $st_1 \Rightarrow st_2, st_3, st_4 \Rightarrow st_5$.

Example 4 provides a vivid example illustrating the granular structure. Based on the dependency relation, the roughest st_5 is called the least element, which corresponds to the lowest structural requirement. Moreover, "$\Rightarrow$" is usually used for simplicity. Next, some descriptions are provided for the one-dimensional case in RS theory. $S = (U, C \cup D)$, $A \subseteq C, E \subseteq U, \underline{apr}_A E = \overline{apr}_A E$. Let st denote the arbitrary structure on $E/IND(A)$, and $ST_A(E) = \{st\}$. $\forall st_1, st_2 \in ST_A(E), st_1 \Rightarrow st_2$ is defined as follows: $\forall g_1 \in st_1, \exists g_2 \in st_2, g_1 \subseteq g_2$ holds. Thus, $(ST_A(E), \Leftarrow)$ is a complete lattice.

Definition 6.2 (*Structure Target*). Let $st^{P_0} \in ST_C(\mathrm{POS}_C(X)), st^{N_0} \in ST_C(\mathrm{NEG}_C(X)), st^P \in ST_C(\mathrm{BND}_C(X) \cap \mathrm{POS}_C^{\alpha,\beta}(X))$, $st^B \in ST_C(\mathrm{BND}_C^{\alpha,\beta}(X))$, $st^N \in ST_C(\mathrm{BND}_C(X) \cap \mathrm{NEG}_C^{\alpha,\beta}(X))$. Thus, $st^C = (st^{P_0}, st^{N_0}, st^P, st^B, st^N)$ is a structure target (ST), and $ST^C = \{st^C\}$ is the ST set. Suppose that

$$\begin{aligned} st_1^C &= (st_1^{P_0}, st_1^{N_0}, st_1^P, st_1^B, st_1^N) \in ST^C, \\ st_2^C &= (st_2^{P_0}, st_2^{N_0}, st_2^P, st_2^B, st_2^N) \in ST^C. \end{aligned} \tag{20}$$

If $st_1^{P_0} \Rightarrow st_2^{P_0}, st_1^{N_0} \Rightarrow st_2^{N_0}, st_1^P \Rightarrow st_2^P, st_1^B \Rightarrow st_2^B, st_1^N \Rightarrow st_2^N$, then we define $st_1^C \Rightarrow st_2^C$ and conclude that st_1^C deduces st_2^C while st_2^C depends on st_1^C.

Based on the premise of CR-Preservation, Definition 6.2 formally defines ST according to the knowledge structure with respect to the five fundamental regions. CR-Preservation, i.e., Five-Region Preservation, determines the entire complete classification with respect to the five regions. Thus, ST aims to describe, in depth, the five internal structures. From another perspective, st^C is a type of specific structure that has the range ST^C. In particular, a partial order $\Rightarrow$ on ST^C is naturally defined by the Cartesian combination of the internal partial orders and reflects the five-structure dependency/deducibility. Moreover, ST – a quintuple form – corresponds to five dimensions, but only one component can be used to conduct a representative and effective analysis. Thus, $(ST_A(E), \Leftarrow)$ and Example 4 provide a simple but powerful illustration.

Theorem 6.3. *$(ST^C, \Leftarrow)$ is a complete lattice, and the greatest and least elements are $st_{KP}^C = (U/C|\mathrm{POS}_C(X),\ U/C|\mathrm{NEG}_C(X)$, $U/C|(\mathrm{BND}_C(X) \cap \mathrm{POS}_C^{\alpha,\beta}(X)), U/C|\mathrm{BND}_C^{\alpha,\beta}(X), U/C|(\mathrm{BND}_C(X) \cap \mathrm{NEG}_C^{\alpha,\beta}(X))), st_{CRP}^C = (\mathrm{POS}_C(X), \mathrm{NEG}_C(X), \mathrm{BND}_C(X) \cap \mathrm{POS}_C^{\alpha,\beta}(X)$, $\mathrm{BND}_C^{\alpha,\beta}(X), \mathrm{BND}_C(X) \cap \mathrm{NEG}_C^{\alpha,\beta}(X))$, respectively, where $U/C|E$ indicates the restriction structure of knowledge $U/IND(C)$ with respect to set E. Thus, $\forall st^C \in ST^C, st_{KP}^C \Rightarrow st^C \Rightarrow st_{CRP}^C$.*

Theorem 6.3 describes the mathematical structural feature of $(ST^C, \Leftarrow)$. The greatest element st^C_{KP} is actually $U/IND(C)$, and it corresponds to the finest structure and the highest target. The least element st^C_{CRP} actually represents the five whole blocks, and it corresponds to the roughest structure and the lowest requirement. st^C_{KP} and st^C_{CRP} provide the ST supremum and infimum; therefore, a general ST is usually located between them and can be obtained by roughening st^C_{KP} or refining st^C_{CRP}. Next, let us analyze the $(ST^C, \Leftarrow)$ significance for attribute reduction. C is the initial complete condition attribute, and thus, arbitrary feasible knowledge (including the knowledge defined by a reduct) can be represented by $U/IND(C)$. Therefore, ST^C includes all possible knowledge structures based on the premise of CR-Preservation. In other words, $(ST^C, \Leftarrow)$ provides a fundamental platform for the CR-Preservation Principle and a relevant discussion thereof (such as CR-Reduction), where the dependency/deducibility underlies the reduction reasoning; hence, the complete lattice establishes a wide and deep background. Moreover, the knowledge also includes the realizability problem, which relies on C.

Next, we will analyze the structural essence of the reduction target. The reduction target essentially corresponds to certain structures with respect to $U/IND(C)$; thus, it can be described by $(ST^C, \Leftarrow)$. Here, ST and the complete lattice are utilized.

Definition 6.4 (*ST Reduction Target*). $\forall st^C \in ST^C, MT_C(st^C) = \{st^C_* \in ST^C | st^C_* \Rightarrow st^C\}$ is defined as the reduction target on st^C.

Proposition 6.5. *st^C is the infimum of set $MT_C(st^C)$, which is denoted as $inf(MT_C(st^C)) = st^C$. Moreover, $\forall st^C_* \in MT_C(st^C), st^C_* \Rightarrow st^C$, which is denoted as $MT_C(st^C) \Rightarrow st^C$.*

Based on the complete lattice, $MT_C(st^C)$ has a unique existence and depends on the given st^C; moreover, st^C is the infimum of $MT_C(st^C)$. By $(ST^C, \Leftarrow)$, Definition 6.4 provides the formal definition of the ST reduction target, where infimum st^C is the lowest structure requirement. $MT_C(st^C) \Rightarrow st^C$ in Proposition 6.5 shows that the ST Reduction Target is a set of structure targets that can deduce the given structure target. Moreover, attribute reduction aims to obtain structure targets in $MT_C(st^C)$ rather than only st^C. Generally, the typical reduction target can be viewed as the set of certain structures that include the infimum (i.e., the lowest structure requirement). For example, the basic CR-Preservation target aims to seek deducible structures for st^C_{CRP}, which is the lowest structure criterion.

Theorem 6.6. *$MT_C : st^C \to MT_C(st^C)$ is an injective mapping on ST^C. Thus, there is a one-to-one mapping between ST^C and $\{MT_C(st^C) | st^C \in ST^C\}$. Moreover, $inf : MT_C(st^C) \to st^C$ is the inverse mapping of MT_C.*

From a mathematical point of view, Theorem 6.6 is clear. More importantly, it exhibits a one-to-one corresponding relationship between ST and the ST Reduction Target. Thus, only the infimum can fully determine and describe the ST Reduction Target. Hence, the lowest structure requirement is important and can be extensively used to complete the description with respect to the ST Reduction Target. This concept will be extensively utilized to describe the reduction targets of C-Preservation, CR-Preservation, ST-Preservation and K-Preservation in Section 7.

6.2. ST-Preservation and STP-Reduct

Next, K-Coarsening is introduced. As in the case of the classical and above-described reduction targets and reducts, we can require preservation for a specific structure target (i.e., ST-Preservation) and develop the corresponding reduct (i.e., STP-Reduct).

Definition 6.7 (*ST-Preservation*). $\forall st^C \in ST^C$, K-Coarsening $C \bar{\to} B$ satisfies ST-Preservation if (1) $C \bar{\to} B \models CRP$; (2) $U/IND(B) \in MT_C(st^C)$, i.e., $U/IND(B) \Rightarrow st^C$.

Definition 6.7 simulates the K-Coarsening's action for preserving the ST Reduction Target, which is called ST-Preservation. Item (1) concerns the premise of CR-Preservation, whereas item (2) requires the deducible implementation for knowledge B. Note that ST-Preservation does not maintain st^C but deduces st^C, and B corresponds to only one stable structure $U/IND(B)$. Moreover, ST-Preservation can be related to the applicable requirement.

Proposition 6.8. *K-Preservation deduces ST-Preservation, whereas ST-Preservation deduces CR-Preservation. The converse does not hold in either case.*

Proposition 6.8 exhibits the basic position of ST-Preservation, which is located between CR-Preservation and K-Preservation. In other words, ST-Preservation is stronger than CR-Preservation; as a result, stronger or deeper reducts usually exist. Moreover, compared to CR-Preservation, ST-Preservation might be a more reliable criterion because it is more closely related to K-Preservation (which is the surest strategy). Furthermore, K-Preservation and ST-Preservation provide a new reduction range. Based on ST-Preservation, the corresponding reduct can be naturally defined.

Definition 6.9 (*STP-Reduct*). $\forall st^C \in ST^C, B$ is STP-Reduct on st^C, if

(1) $C \bar{\to} B \models CRP$, and $U/IND(B) \Rightarrow st^C$;
(2) $\forall B' \subset B$, if $C \bar{\to} B' \models CRP$, but $U/IND(B') \Rightarrow st^C$ does not hold.

Definition 6.9 defines STP-Reduct for ST-Preservation, where the given structure target is viewed as the lowest target. Item (1) indicates that $C \bar{\rightarrow} B$ has the ST-Preservation feature, whereas item (2) requires the set maximality with respect to ST-Preservation. Moreover, the core can be normally defined.

Theorem 6.10 (*STP-Reduct Inclusiveness*).

Based on the premise of CR-Preservation, STP-Reducts include all attribute reducts. In particular, CRP-Reduct and KP-Reduct constitute two special cases of STP-Reducts. When $st^C = st^C_{CRP}, st^C = st^C_{KP}$, STP-Reduct degenerates into CRP-Reduct and KP-Reduct.

Theorem 6.11 (*STP-Reduct Hierarchy*).

For a given ST, STP-Reduct is a middle-level reduct between CRP-Reduct and KP-Reduct.

As discussed in Section 5, CR-Preservation and CRP-Reduct have already established the initial reduction framework for two-category DTRS. In this section, they are further developed by STP-Reduct. In view of the characteristics of the ST parameter, two points of view are provided by Theorems 6.10 and 6.11. (1) STP-Reduct can provide all reducts with the CR-Preservation property due to the complete change in ST. Thus, STP-Reducts become the general reducts for two-category DTRS and finally establish a complete framework. (2) STP-Reduct also exhibits a hierarchy, but not inclusiveness, and is viewed more as a type of middle level between CRP-Reduct and KP-Reduct. Thus, the suitable choice for ST will become a new problem due to the variability. To address this new issue, we next provide a measurement strategy and further propose the optimal reducts.

Herein, the generalization is explained beyond two-category DTRS-Reduction. Based on the above-described studies, we actually simulate the attribute reduction by ST and its lattice structure. *A reduction target is a set whose elements can deduce the given structure target, whereas the objective of attribute reduction is to seek condition attribute subsets whose knowledge structure can deduce the lowest structure requirement given.* Thus, structural reduction theory, which includes the concepts of ST, ST Reduction Target, ST-Preservation and STP-Reduct, can be extensively utilized. Section 7 will provide a D-table example to illustrate STP-Reduct and the reduction approach based on the aforementioned structures.

6.3. Optimal reducts

In accordance with the conventional dependency/deducibility approach, STP-Reduct is established and provides general reducts. In fact, ST carries specific information, which enables the argument to reflect dependent variables, such as concrete measures. For STP-Reduct, the optimal measures can tackle the ST choice problem. Multiple concrete measures have been developed in DTRS [51], which also provide the STP-Reduction value and space. From another perspective, we can pursue optimal measures in attribute reduction, i.e., the optimal reducts can be established for the given measures. In this context, this subsection will propose several optimal reducts based on the results reported in [8,51]. Note that the optimal reducts lie within the CR-Preservation framework, i.e., the pursuit of the optimal measure has a structural premise; thus, the optimal reducts will consider both the structure and measures.

Definition 6.12 (*Optimal Reduct I*).

Given a measure $(m, \leqslant)$ on 2^C, suppose that the optimal measure value concerns the minimum, $m_{opt} = \min(A)$, where $A \subseteq C$ and $C \bar{\rightarrow} A \models CRP$. B is called an optimal reduct with respect to $(m, \leqslant)$ if

(1) $C \bar{\rightarrow} B \models CRP, B = \arg m_{opt}$;

(2) $\forall B' \subset B$, if $C \bar{\rightarrow} B' \models CRP$, but $m(B') > m(B)$.

According to one measure $(m, \leqslant)$ on knowledge structures, Definition 6.12 proposes a type of optimal reduct. Based on the premise of CR-Preservation, m_{opt} is the optimal value. Thus, the optimal reduct aims to search the independent CR-Preservation set to reach the optimal value. Next, a specific optimal reduct is proposed by a concrete measure, which is the cost, and the cost formula is discussed first.

Proposition 6.13. *Let $\lambda_{PP} = 0 = \lambda_{NN}, p = P(D_{max}([x]_A)|[x]_A)$; thus, the cost on knowledge A is $COST(A) =$*

$$\sum_{x \in \mathrm{POS}_A^{\alpha,\beta}(X) \cap \mathrm{BND}_A(X)} (1-p)\lambda_{PN} + \sum_{x \in \mathrm{BND}_A^{\alpha,\beta}(X)} (p\lambda_{BP} + (1-p)\lambda_{BN}) + \sum_{x \in \mathrm{NEG}_A^{\alpha,\beta}(X) \cap \mathrm{BND}_A(X)} p\lambda_{NP}.$$

$COST(A)$ mainly represents the decision cost on knowledge $A \subseteq C$. For the two-category approach, the original formula [8] in Proposition 2.8 concerns only three set regions. Furthermore, the new formula in Proposition 6.13 involves the five regions of CR-Preservation, and the cost on Pawlak-POS and Pawlak-NEG are equal to 0 because $p = 1$ and $p = 0$, respectively. Concretely, the new cost reflects the following three points. (1) The cost denotes the structural requirement for rational CR-Preservation. (2) The computational regions are scientifically reduced because the two certainty regions do not incur

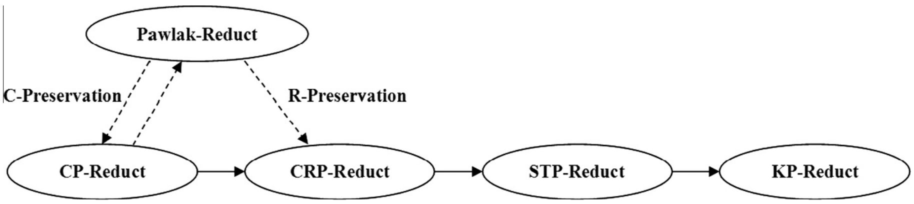

Fig. 1. Reduction hierarchy of four types of reduct.

the cost. (3) The formula originates from the two-category case, which allows it to clearly correspond to the DTRS modeling process. Thus, the new cost formula has improved upon the old one, at least at the two-category level. Moreover, the cost function can well reflect the merits of the qualitative classification with respect to certainty and uncertainty; in other words, the rationality of C-Preservation is also verified by the cost conclusion.

Definition 6.14 (*Optimal Reduct II*).

Let $COST_{opt} = \min_{A \subseteq C, C \overset{-}{\rightarrow} A \models CRP} COST(A)$. B is called an optimal reduct with respect to the cost if

(1) $C \overset{-}{\rightarrow} B \models CRP, B = \arg COST_{opt}$;
(2) $\forall B' \subset B$, if $C \overset{-}{\rightarrow} B' \models CRP$, but $COST(B') > COST(B)$.

Within the framework of Optimal Reduct I, Definition 6.14 further yields a concrete result based on the new cost formula, where $COST_{opt}$ indicates the minimum cost in the CR-Preservation range. Because of the introduction of risk information, this optimal reduct is more closely linked to the Bayesian risk decision of two-category DTRS; thus, it holds important value for semantics applications. In particular, Cost-based Reduct pursues the optimal cost regardless of the structure, whereas Optimal Reduct II considers the minimum cost based on the premise of rational structural; thus, Optimal Reduct II improves upon the cost-based reduct and is more rational, at least at the two-category level.

The above-described optimal reducts involve only a single measure. However, multiple measures can also be considered. Ref. [51] offers many results concerning this topic. Here, a type of optimal reduct is proposed based on the structures.

Definition 6.15 (*Optimal Reduct III*).

Suppose that $M = (m_1, m_2, \ldots)$ is a measure set on 2^C and each measure has the optimal minimum. B is an optimal reduct with respect to M if

(1) $C \overset{-}{\rightarrow} B \models CRP, m(B) \leqslant m(C), \forall m \in M$;
(2) $\forall B' \subset B$, if $C \overset{-}{\rightarrow} B' \models CRP$, then $m(B') \nleqslant m(B), \forall m \in M$.

Based on the initial measure criterion with respect to C, Definition 6.15 searches the optimal reducts by considering the multiple measures given. Of course, CR-Preservation is still the structural premise.

Theorem 6.16 (*Optimal Reduct Merit*).

The optimal reducts can preserve the fundamental structure of CR-Preservation and can exhibit the optimal feature for a measure or multiple measures.

Theorem 6.16 reflects the merit of the three optimal reducts. Based on the premise of CR-Preservation, the reducts can reach the optimal measurement values. Thus, the optimal reducts enhance the basic CRP-Reduct and improve the existing reducts through their structure or measures. Hence, the reducts hold great significance for reduction theory, especially for measurement applications. Furthermore, optimal reducts can be generally explored by adopting the double-requirement concept for the structure and measures. Moreover, the measures can also be considered to be heuristic information for constructing reduction algorithms.

In summary, based on the premise of CR-Preservation, ST concerns the internal structures of the five regions. Within this structural framework, the general reducts are further explored. Based on CR-Preservation, the general reduction makes an adjustment to the feasible five internal structures; thus, it can realize reduction targets with respect to dependency/deducibility or the optimal measure value. In particular, the general reducts are proposed to simulate the classical reducts and improve the applicable reducts. Thus, the general reduction system has also been constructed for two-category DTRS. With respect to generalization, the ST and STP-Reduct effectively reflect the mathematical essence of attribute reduction and thus can be generalized. The objective of attribute reduction is to seek accessible and independent knowledge structures to deduce the lowest structural criterion, and this general approach will be used in the next section. Optimal reducts search the optimal structures for the given measures and thus are more suitable for the DTRS measure environment; this optimal approach based on structure can be generalized as well.

7. Reduct relationships and an example

Up to this point, we have obtained the hierarchical four targets and four reducts, which are C-Preservation, CR-Preservation, ST-Preservation and K-Preservation and CP-Reduct, CRP-Reduct, STP-Reduct and KP-Reduct, respectively. The development line, C-Preservation→CR-Preservation→ST-Preservation→K-Preservation, represents the evolutionary step from a weak reduction target to a strong one. For simplicity, the reduct that is related to a strong reduction target is called *the strong reduct*, and the reduct that is associated with a weak reduction target is called *the weak reduct*. Thus, the corresponding reduct line (from the weak reduct to the strong one) becomes CP-Reduct→CRP-Reduct→ STP-Reduct →KP-Reduct. Moreover, the reduction target relationships have been previously provided. In this section, the four reduct relationships are analyzed, and a D-Table is specifically provided for illustration.

Proposition 7.1. *The core of a strong reduct includes the core of a weak reduct.*

Proposition 7.2. *If B_{strong} is an arbitrary strong reduct, then $\exists B_{weak} \subseteq B_{strong}$, where B_{weak} is a weak reduct. In contrast, if B_{weak} is an arbitrary weak reduct, then there is no necessary conclusion – B_{strong} is a strong reduct and $B_{strong} \supseteq B_{weak}$.*

Propositions 7.1 and 7.2 are easy to prove, and the next example will provide some explanation in this regard. Proposition 7.1 indicates that the core of a weak reduct is necessarily in the core of a strong reduct. Proposition 7.2 indicates that a weak reduct necessarily exists in each strong reduct but could still exist beyond the embedded relationship with respect to all of the strong reducts. Thus, the strong reducts can provide some guidance for the weak reducts. KP-Reduct is the strongest reduct and therefore has some specific functions for all of the reducts, which has been previously noted in Section 3.

The four types of reduction targets have a clear hierarchy. However, the four types of reducts cannot obtain some necessary relationships due to the reduct maximality/independency requirement. Thus, reducts mainly rely on the accessible

Table 2
A statistical decision table that is based on equivalence classes of condition attributes.

$[x]_i$	$\lvert[x]_i\rvert$	c_1	c_2	c_3	c_4	c_5	$\lvert[x]_i \cap X\rvert$
$[x]_1$	5	1	1	1	1	1	0
$[x]_2$	3	1	1	1	2	1	0
$[x]_3$	6	1	2	1	1	3	1
$[x]_4$	5	1	2	1	1	2	1
$[x]_5$	6	2	1	2	2	1	3
$[x]_6$	4	2	3	1	2	2	3
$[x]_7$	6	2	3	3	1	3	5
$[x]_8$	4	3	1	3	2	2	4
$[x]_9$	2	3	2	2	2	1	2

Table 3
Four types of reducts and their cores.

Type	Core	Reduct	Reduct number
CP-Reduct	ϕ	c_1c_2, c_1c_5, c_2c_3	3
CRP-Reduct	ϕ	$c_1c_5, c_2c_3, c_1c_2c_4$	3
STP-Reduct	$\{c_4\}$	$c_1c_2c_4, c_1c_4c_5, c_2c_3c_4$	3
KP-Reduct	$\{c_4, c_5\}$	$c_1c_4c_5, c_2c_3c_4c_5$	2

Table 4
Four types of reducts and their knowledge structures.

Reduct	Knowledge structure	CP-Reduct	CRP-Reduct	STP-Reduct	KP-Reduct
c_1c_2	$\{[x]_1 \cup [x]_2, [x]_3 \cup [x]_4, [x]_5, [x]_6 \cup [x]_7, [x]_8 \cup [x]_9\}$	✓			
c_1c_5	$\{[x]_1 \cup [x]_2, [x]_3, \ldots, [x]_9\}$	✓	✓		
c_2c_3	$\{[x]_1 \cup [x]_2, [x]_3 \cup [x]_4, [x]_5, \ldots, [x]_9\}$	✓	✓		
$c_1c_2c_4$	$\{[x]_1, [x]_2, [x]_3 \cup [x]_4, [x]_5, \ldots, [x]_9\}$		✓	✓	
$c_1c_4c_5$	$\{[x]_1, [x]_2, \ldots, [x]_9\}$			✓	✓
$c_2c_3c_4$	$\{[x]_1, [x]_2, [x]_3 \cup [x]_4, [x]_5, \ldots, [x]_9\}$			✓	
$c_2c_3c_4c_5$	$\{[x]_1, [x]_2, \ldots, [x]_9\}$				✓

substructure of the condition attribute C. Of course, if the maximality/independency condition is weakened, then the hierarchy can provide more reliable conclusions. In particular, Fig. 1 illustrates the relevant reduction hierarchy. The qualitative Pawlak reduct acts as the basis, and its equivalent C-Preservation and R-Preservation are separated into quantitative DRTS-Reduct. Thus, CP-Reduct is based on C-Preservation, whereas CRP-Reduct is based on the dual preservation, and STP-Reduct and KP-Reduct are further related to strong development with high accuracy. Note that DTRS CP-Reduct is also consistent with the Pawlak reduct.

Example 5. In D-Table $S=(U,C\cup D), U=\{x_1,x_2,\ldots,x_{41}\}, C=\{c_1,c_2,c_3,c_4,c_5\}, U/IND(D)=\{X,\neg X\}$. According to the initial D-Table, Table 2 provides statistical information that is based on the equivalence classes of C; moreover, Table 1 is actually a part of Table 2. Here, we utilize the structural approach (including the generalized structure target) for the description and computation.

At first, $U/IND(C)=\{[x]_1,\ldots,[x]_9\}$ and the complete lattice $(ST_C(U),\Leftarrow)$ underlie the entire reduction discussion. For the four types of reduction targets, only one basic computational process is required. (1) C-Preservation concerns only C and D; (2) CR-Preservation also concerns the model, and we assume that $\alpha=0.8,\beta=0.2$; (3) ST-Preservation further involves the given ST, and we assume that $st=\{[x]_1,[x]_2,[x]_3\cup[x]_4,[x]_5\cup[x]_6,[x]_7,[x]_8\cup[x]_9\}$; (4) K-Preservation concerns only C. By calculation, $\mathrm{NEG}_C(X)=[x]_1\cup[x]_2, \mathrm{BND}_C(X)=[x]_3\cup\ldots\cup[x]_7, \mathrm{POS}_C(X)=[x]_8\cup[x]_9; \mathrm{NEG}_C^{\alpha,\beta}(X)=[x]_1\cup\ldots\cup[x]_4, \mathrm{BND}_C^{\alpha,\beta}(X)=[x]_5\cup[x]_6,$ $\mathrm{POS}_C^{\alpha,\beta}(X)=[x]_7\cup[x]_8\cup[x]_9$. Thus, the four reduction targets can be represented by the four target infima (i.e., the lowest structure requirements): (1) $\boldsymbol{inf}(C)=\{[x]_1\cup[x]_2,[x]_3\cup\ldots\cup[x]_7,[x]_8\cup[x]_9\}$, (2) $\boldsymbol{inf}(CR)=\{[x]_1\cup[x]_2,[x]_3\cup[x]_4,[x]_5\cup[x]_6,[x]_7,[x]_8\cup[x]_9\}$, (3) $\boldsymbol{inf}(ST)=\{[x]_1,[x]_2,[x]_3\cup[x]_4,[x]_5\cup[x]_6,[x]_7,[x]_8\cup[x]_9\}$, (4) $\boldsymbol{inf}(K)=U/IND(C)=\{[x]_1,\ldots,[x]_9\}$.

Based on the four target infima, the four types of reducts can be obtained by examining the deducibility and maximality, and this new approach is equivalent to classical processing. Next, the hierarchical results are presented in order from the weakest reduct to the strongest one. (1) For C-Preservation, the core is empty, and there are only three CP-Reducts: c_1c_2, c_1c_5, c_2c_3. (2) For CR-Preservation, the core is also empty, and there are only three CRP-Reducts: $c_1c_5, c_2c_3, c_1c_2c_4$. (3) For ST-Preservation, the core is $\{c_4\}$, and there are only three STP-Reducts: $c_1c_2c_4, c_1c_4c_5, c_2c_3c_4$. (4) For K-Preservation, the core is $\{c_4,c_5\}$, and there are only two KP-Reducts: $c_1c_4c_5, c_2c_3c_4c_5$. Table 3 vividly summarizes the relevant cores and reducts.

There are five condition attributes; therefore, $2^5-1=31$ non-empty subsets exist in theory. The four above-described types of reducts involve eleven reducts in total, but they concern only seven proper subsets. Table 4 provides the seven subsets and their knowledge structures and reduct distributions, where ✔ marks the matched reduct. Thus, the reduct deducibility for the reduction target infima is easily verified.

According to the comprehensive results presented in Table 3, the four cores are $\phi,\phi,\{c_4\}$ and $\{c_4,c_5\}$, which confirms Proposition 7.1. There are only two KP-Reducts. KP-Reduct $c_1c_4c_5$ is also STP-Reduct and further includes CRP-Reduct and CP-Reduct c_1c_5; KP-Reduct $c_2c_3c_4c_5$ includes STP-Reduct $c_2c_3c_4$, and the latter further includes CRP-Reduct and CP-Reduct c_2c_3. For the opposite direction, only one specific case exists, i.e., no KP-Reduct exhibits the inclusiveness for STP-Reduct $c_1c_2c_4$. Thus, Proposition 7.2 is also confirmed. Moreover, KP-Reducts $c_1c_4c_5, c_2c_3c_4c_5$ can provide some guidance in determining other reducts, especially when seeking only one weak reduct.

Here, we choose a structure target: $st=\{[x]_1,[x]_2,[x]_3\cup[x]_4,[x]_5\cup[x]_6,[x]_7,[x]_8\cup[x]_9\}$. In fact, CR-Preservation corresponds to the infimum $\boldsymbol{inf}(CR)=\{[x]_1\cup[x]_2,[x]_3\cup[x]_4,[x]_5\cup[x]_6,[x]_7,[x]_8\cup[x]_9\}$. Thus, $(ST^C,\Leftarrow)$ has only $2^4=16$ structure targets based on the premise of CR-Preservation, including the used $\boldsymbol{inf}(CR), st, \boldsymbol{inf}(K)=U/IND(C)$; moreover, $\boldsymbol{inf}(CR)\Leftarrow st\Leftarrow U/IND(C)$. Of course, not all structure targets can be realized by the knowledge structure of some condition attributes.

Next, this example is utilized to illustrate the two algorithms proposed for CRP-Reduct in Section 5.2. (1) For the CRP-Reduct-based algorithm, suppose that Step 1 yields the usual CP-Reduct result $Red_{sub}(C)=\{c_1c_2,c_1c_5,c_2c_3\}$; Step 2 yields the empty core, which means that $Red_*^{\alpha,\beta}(C)=Red_{sub}(C)$ in the core condition; Step 3 checks the R-Preservation condition, and c_1c_2 is removed, whereas $Red_{**}^{\alpha,\beta}(C)=\{c_1c_5,c_2c_3\}$. Step 4 checks the independency/maximality and obtains the output result $Red_{sub}^{\alpha,\beta}(C)=\{c_1c_5,c_2c_3\}$. Thus, this algorithm provides a CRP-Reduct subset because the complete CRP-Reduct set is $Red^{\alpha,\beta}(C)=\{c_1c_5,c_2c_3,c_1c_2c_4\}$. (2) For the KP-Reduct-based algorithm, Step 2 also yields the empty core. Thus, if Step 1 provides KP-Reduct $B_k=c_1c_4c_5$, then Step 3 will yield CRP-Reduct $B=c_1c_5$; if Step 1 provides KP-Reduct $B_k=c_2c_3c_4c_5$, then Step 3 will yield CRP-Reduct $B=c_2c_3$. Hence, this algorithm provides a concrete CRP-Reduct. This example illustrates the effectiveness of both algorithms. Moreover, note that Proposition 7.2 actually underlies the effectiveness of the KP-Reduct-based algorithm.

In summary, Example 5 clearly shows that CP-Reduct is a D-Table Pawlak reduct, whereas KP-Reduct is a KRS Pawlak reduct. This example and its sub-examples (i.e., Examples 2 and 3) have demonstrated the rationality of both C-Preservation and R-Preservation; thus, CR-Preservation, CP-Reduct and CRP-Reduct are all scientific; moreover, CRP-Reduct is shown to be a novel concept that extends the Pawlak reduct and is mainly applied in quantitative models. Example 5 also demonstrates the generalized structure approach, and STP-Reduct is obtained, which can be viewed as more detailed than CP-Reduct. In particular, Example 5 demonstrates the hierarchy for the four types of targets and reducts, where $\boldsymbol{inf}(K)\Rightarrow\boldsymbol{inf}(ST)\Rightarrow\boldsymbol{inf}(CR)\Rightarrow\boldsymbol{inf}(C)$ actually acts as the hierarchy source. Therefore, Example 5 illustrates well the effectiveness of our whole approach and the relevant results obtained this paper.

Finally, we summarize the structural reduction approach following this vivid example. The reduction target is essentially a family of structure targets that has an infimum, i.e., it has certain structural requirements with the smallest criterion. Thus, attribute reduction considers only the deducibility of the sub-attributes knowledge structures for the reduction target structure. Thus, transforming the reduction target into the structural requirement and focusing on the deducibility of the knowledge structures become a structural reduction characteristic, where the complete lattice acts as the basis and, at the same time, the repeated computation on the decision attributes is avoided. Given a reduction target, attribute reduction mainly removes redundant structures of conditional knowledge and thus realizes optimization and generalization; thus, this structural approach can best reflect the essence of attribute reduction.

8. Conclusions

Attribute reduction plays a central role in RS theory. According to the attribute reduction problem from PRS quantitative extension, this paper concentrates on the hierarchical attribute reduction for two-category DTRS. By promotion and hierarchical analysis, we investigate four targets and reducts. For the reduction targets, K-Preservation implies ST-Preservation, whereas ST-Preservation implies CR-Preservation and CR-Preservation implies C-Preservation. K-Preservation is the highest target, whereas KP-Reduct promotes KRS Pawlak reduct and provides auxiliary information for all of the reducts. Both C-Preservation and R-Preservation are rational and necessary; thus, based on separability, CR-Preservation defines the basic structural premise for two-category DTRS reduction. Moreover, CP-Reduct promotes D-Table Pawlak reduct, whereas CRP-Reduct extends the qualitative Pawlak reduct to the quantitative realm. ST and general reducts (including STP-Reduct and Optimal Reduct) describe the mathematical essence of attribute reduction and provide ample space for application from a structural perspective. In particular, all of the content is effectively explained by the final D-Table example and can be naturally generalized to multiple-category or other quantitative models. Thus, this study demonstrates the promotion, rationality, structure, hierarchy and generalization of attribute reduction and establishes a fundamental reduction framework for two-category DTRS; furthermore, it provides new insights into the problems of Pawlak-Reduction, DTRS-Reduction, and PRS-Reduction. The approaches and results presented are worthy of further verification in practical applications, including experiments on a large database; thus, the structure and hierarchy of attribute reduction must be explored in greater depth based on granular computing, and efforts to generalize the results to multi-category and quantitative models must be undertaken.

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