EECI Course on

Nonlinear and Data-based
Model Predictive Control

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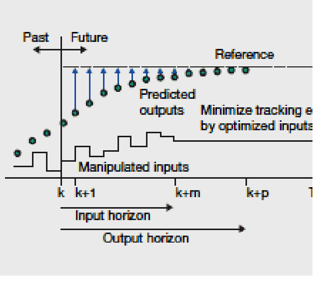


Structure of Course

MONDAY:
introduction, background, nominal stability (Part 1)

TUESDAY:
exercise, nominally stable MPC (Part 2), exercise,
nominally stable MPC (Part 3)

WEDNESDAY:
robust MPC, exercise, economic MPC

THURSDAY:
tracking MPC, data-based MPC; selected advanced topics

Nominal Stability of the Closed Loop

INFINITE HORIZON NMPC

Infinite Horizon NMPC

$$\min_u J(x(t), u)$$

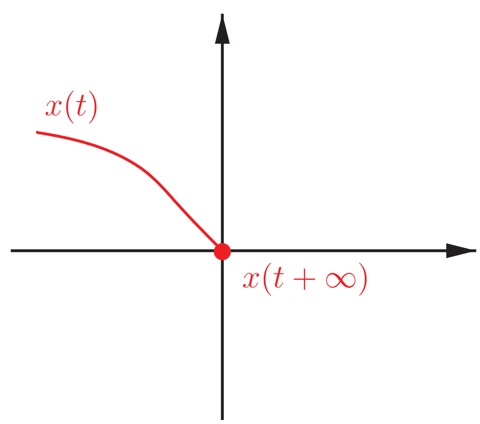
$$J(\cdot) = \int_t^{t+\infty} L\left(x(\tau), u(\tau)\right) d\tau$$

subject to: $\dot{x} = f(x,u)$ system dynamics

$x(t)$ given "state feedback"

$u(\tau) \in \mathcal{U}$ input constraints

$x(\tau) \in \mathcal{X}$ state constraints



+ stability/convergence to origin
+ feasibility guaranteed if initially feasible
+ feasibility $\Leftrightarrow$ constrained controllability

\- **implementation often not possible**

Infinite Horizon NMPC – Sketch of Stability Proof

- **Feasibility** (+optimality) at t_i implies **feasibility** (+optimality) at t_{i+1}:

$$u^\star(\tau; t_{i+1}) = u^\star(\tau; t_i),\ \tau \in [t_{i+1}, \infty)$$

  (Follows from Bellmans optimality principle)

  => **recursive feasibility**

- Use **value function** $V(x) = J^\star(x(\mathrm{t}))$ as Lyapunov function candidate:

$$V(x(t_i+\tau)) - V(x(t_i)) \leq -\int_{t_i}^{t_i+\tau} L(x(s), u(s))ds$$

  Value function is strictly decreasing $\Rightarrow$ **stability**

- **attractivity** follows from Barbalat's Lemma

Infinite Horizon NMPC

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subject to:	$\dot{x} = f(x, u)$	system dynamics
	$x(t)$ given	"state feedback"
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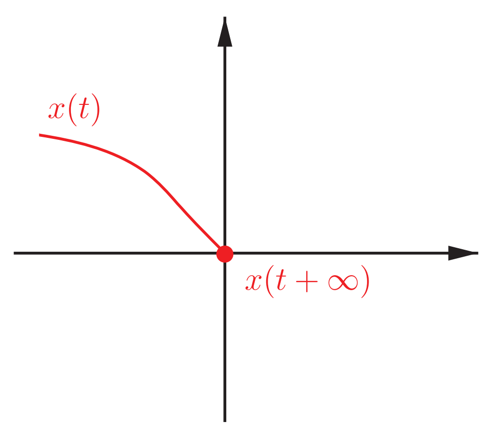


+ stability/convergence to origin
+ feasibility guaranteed if initially feasible
+ feasibility $\Leftrightarrow$ constrained controllability
+ **implementation usually not possible**

Dilemma of NMPC

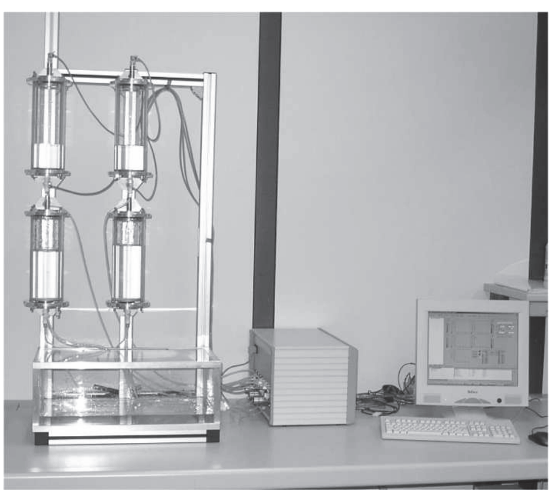
Large horizon $\Rightarrow$ good performance **closed loop stability** $\Rightarrow$ computationally **not feasible**

Need for short horizon formulations that achieves good performance and stability

2 possible approaches:

- choose T, F, ... s.t. closed loop is stable

Experimental stability study of NMPC



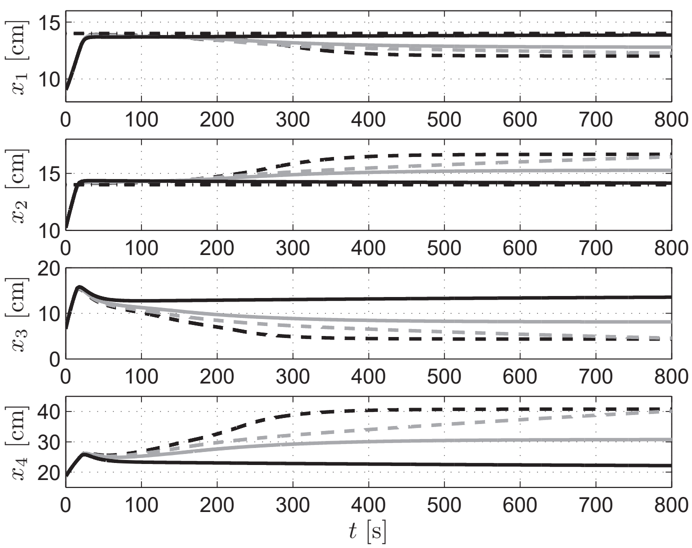
Four tank system at the IST laboratory

Four tank system:

- Nonlinear system
- Input constraints
- State constraints
- “Slow” dynamics

Four tank system is suited for an experimental stability study of NMPC

Four Tank: Simulation and Experiment



- Only simulations
- Setpoint:
  - $x_{1,2s} = 14cm$
  - $\gamma_{1,2} = 0.4$
- Prediction horizon:
  - - -$T = 60s$: unstab.
  - - -$T = 75s$: unstab.
  - —$T = 90s$: unstab.
  - —$T = 135s$: stable
- Stability constraints:
  - **None**
- Input constraints

Closed-loop stability can be achieved with an **long enough prediction horizon**

Problem:
How long is long enough (for an answer see unconstrained NMPC later)

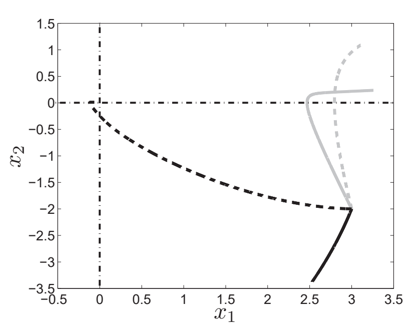
Unstable closed loop for academic example

Example *[Primbs '00]*: Stabilize the nonlinear system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1\left(\frac{\pi}{2} + \arctan(5x_1)\right) - \frac{5x_1^2}{2(1+25x_1^2)} + 4x_2 + 3u$$

via NMPC **without stability constraints** and the cost function $\int_{t_i}^{t_i+T} x_2^2 + u^2 d\tau$.



- — $T = 0.2$: unstable trajectory
- - - $T = 0.3$: stable trajectory
- — $T = 0.4$: unstable trajectory
- - - $T = 1.0$: unstable trajectory

Closed-loop stability can be achieved with an **long enough prediction horizon**
How long is long enough?
This academic example shows that this is not so easy to answer

Dilemma of NMPC
Guaranteed Stability versus Performance

Large horizon	$\Rightarrow$	good performance **closed loop stability**	$\Rightarrow$	computationally **not feasible**

Need for short horizon formulations that achieves good performance and stability

2 possible approaches:

- choose $T\,(, L, \ldots)$ s.t. closed loop is stable
- modify NMPC setup, s.t. closed loop is stable independent of choice of $T, \ldots$
  $\Rightarrow$ **NMPC scheme with "guaranteed stability"**

Nominal Stability of the Closed Loop

ZERO-TERMINAL-CONSTRAINT NMPC

Zero Terminal Constraint NMPC

[Kwon '77], [Keerthi& Gilbert '88], [Mayne& Michalska '90]

Simplest finite horizon NMPC scheme with guaranteed stability

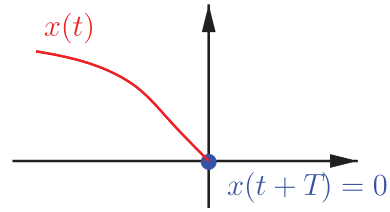
Idea: enforce that predicted state reaches origin in finite time

$$\min_u J(x(t), u)$$

$$J(\cdot) = \int_t^{t+T} L(x(\tau), u(\tau))d\tau$$

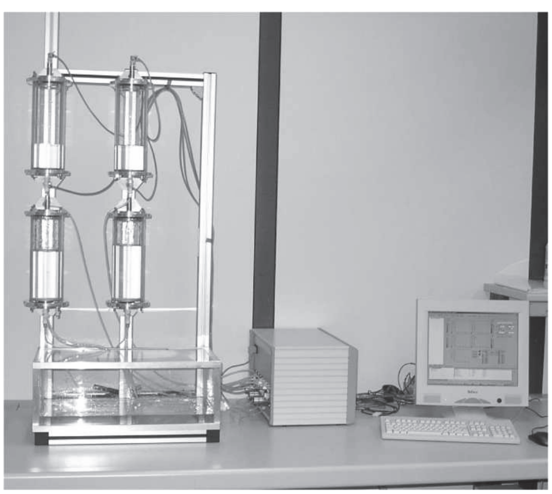
subject to: $\dot{x} = f(x, u)$, system dynamics

$x(t+T) = 0$ zero terminal const.



+ similar to infinite horizon: feasibility once guarantees consecutive decrease of value function and feasibility (supplementing old input with zero) $\Rightarrow$ **stability**
- initial feasibility, performance for short horizons?
- **computationally expensive (boundary value problem)**

Application Example: Four Tank System



Four tank system of the IST laboratory

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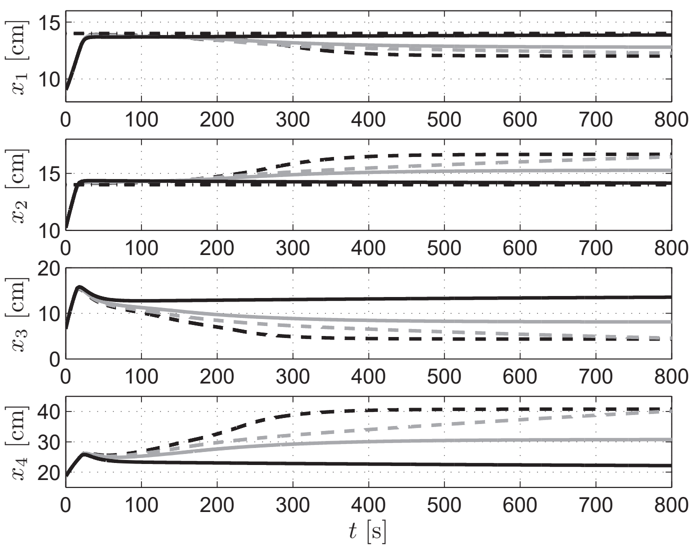
Four tank system is suited for an experimental stability study of NMPC

Recall: Four tank with NMPC with T=60s is unstable



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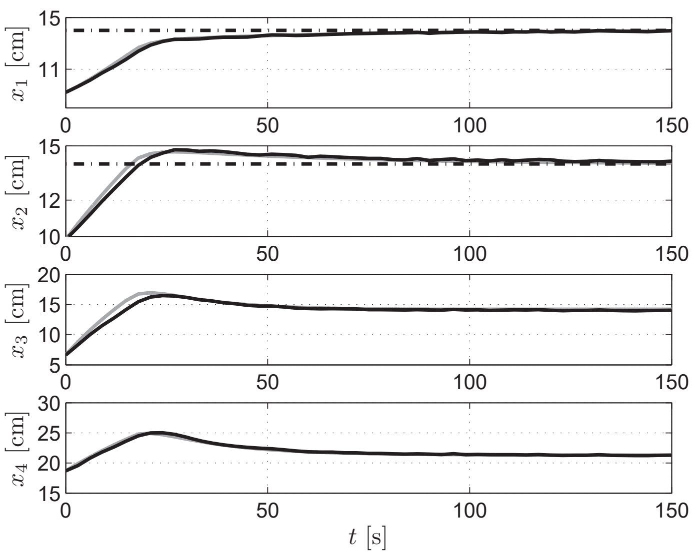
Four Tank: Zero Terminal Constraint MPC



- Sim. —, Exp. —
- Setpoint:
  - $x_{1,2s} = 14cm$
  - $\gamma_{1,2} = 0.4$
- Prediction horizon:
  - $T = 60s$
- Stability constraints:
  Zero Terminal Constrained MPC
-
- Input constraints

  State constraints

Closed-loop is asymptotically stable with **zero terminal state constraint** NMPC

(for prediction horizon T=60s that led to instability without zero terminal state constraint)

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