Nonlinear and data-driven model predictive control

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Part 1: Stability in Model Predictive Control

Setting

Goal: Guarantee stability and degree of suboptimality without stabilizing terminal constraints and cost

Problem setup

System dynamics: $\dot{x} = f(x,u), \quad x(0) = x_0$

Input constraints: $u \in \mathcal{U} \subseteq \mathbb{R}^m$

For simplicity, no state constraints (but extensions exist)

Recall standing assumptions:

- $f(0,0) = 0 \Rightarrow x_s = 0$ is equilibrium state for $u_s = 0$
- $\mathcal{U}$ is compact
- $0 \in \text{int}(\mathcal{U})$

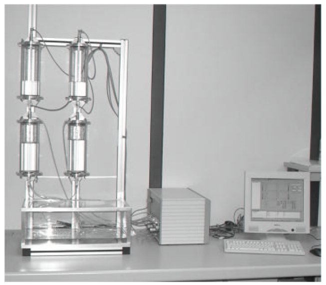
Stability in Model Predictive Control

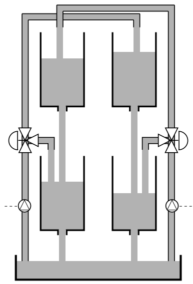
Motivating examples

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1\left(\frac{\pi}{2} + \arctan(5x_1)\right) + 4x_2 - \frac{5x_1^2}{2(1+25x_1^2)} + 3u$$

[Primbs et al. '00]

- unstable for $T = 0.2$
- stable for $T = 0.3$
- unstable for $T = 0.5$





[Raff et al. '06]

- Open-loop stable system
- unstable for $T = 60$ s
- stable for $T = 129$ s

Need to choose prediction horizon suitably in order to guarantee stability without terminal constraints

MPC problem

Finite-horizon cost functional: $J_T(x(t), \bar{u}(\cdot; t)) = \int_t^{t+T} L(\bar{x}(\tau; t), \bar{u}(\tau; t)) d\tau$

MPC problem (MPC without terminal constraints)

$$\begin{aligned} \underset{\bar{u}(\cdot;t)}{\text{minimize}} \quad & J_T(x(t), \bar{u}(\cdot; t)) \\ \text{subject to} \quad & \dot{\bar{x}} = f(\bar{x}, \bar{u}) \\ & \bar{x}(t;t) = x(t) \\ & \bar{u}(\tau; t) \in \mathcal{U} \qquad \forall \tau \in [t, t+T] \\ & \text{no terminal constraint} \end{aligned}$$

MPC input:

$$u(\tau) := \bar{u}^*(\tau; t_i) \qquad \tau \in [t_i, t_i + \delta]$$

with sampling time instants $t_i = i\delta \Rightarrow t_{i+1} = t_i + \delta$

Suboptimality Index

Infinite-horizon cost functional: $J_\infty(x_0, \bar{u}(\cdot; 0)) = \int_0^\infty L(\bar{x}(\tau; 0), \bar{u}(\tau; 0)) d\tau$

$\rightsquigarrow$ optimal cost $J_\infty^*(x_0)$

Assumption: $J_\infty^*(x_0) < \infty$ for all x_0 (system stabilizable)

Infinite-horizon cost resulting from application of MPC controller:

$$J_\infty^{\mathsf{MPC}}(x_0) = \int_0^\infty L(x(\tau), u(\tau)) d\tau \qquad \text{with } x(0) = x_0$$

Definition: Suboptimality Index α

$$\alpha J_\infty^{\mathsf{MPC}}(x_0) \leq J_\infty^*(x_0)$$

$\rightarrow \alpha \leq 1$ by definition

$\rightarrow \alpha > 0$ implies closed-loop stability

Relaxed Dynamic Programming

Proposition 1.1: Relaxed Dynamic Programming

Assume there exists $\alpha \in (0, 1]$ such that

$$J_T^*(x(t+\delta)) \leq J_T^*(x) - \alpha \int_t^{t+\delta} L(x(\tau), u(\tau)) d\tau \tag{1}$$

holds for all $x \in \mathbb{R}^n$ (with $x(t) = x$). Then the estimate

$$\alpha J_\infty^*(x) \leq \alpha J_\infty^{\mathsf{MPC}}(x) \leq J_T^*(x) \leq J_\infty^*(x)$$

holds for all $x \in \mathbb{R}^n$.

Proof of Proposition 1.1

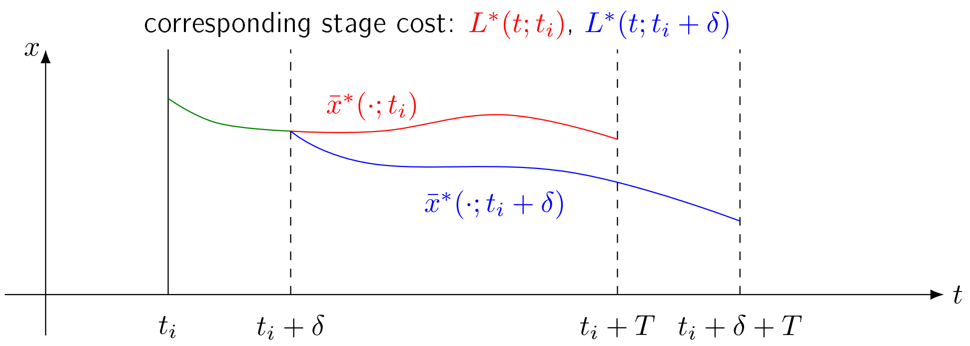
- 1st and 3rd inequality: Due to optimality
- 2nd inequality: Summing up (1) over all sampling instants $t_i = \delta i$, $i = 1, \ldots, N$ gives

$$0 \leq J_T^*(x(N\delta)) \leq J_T^*(x_0) - \alpha \int_0^{N\delta} L(x(\tau), u(\tau)) d\tau$$

$$\rightsquigarrow N \to \infty: \quad J_T^*(x_0) \geq \alpha J_\infty^{\mathsf{MPC}}(x_0)$$

Central idea

Abbreviation: $L^*(t;t_i) := L(\bar{x}^*(t;t_i), \bar{u}^*(t;t_i))$



$$J_T^*(x(t_i+\delta)) \le \frac{1}{\varepsilon}\int_{t_i+\delta}^{t_i+T} L^*(t;t_i)dt \tag{2}$$

$$\int_{t_i+\delta}^{t_i+T} L^*(t;t_i)dt \le \gamma \int_{t_i}^{t_i+\delta} L^*(t;t_i)dt \tag{3}$$

Asymptotic stability and suboptimality

Theorem 1.2: Asymptotic stability and suboptimality

Assume there exists $\varepsilon \in (0,1]$ and $\gamma > 0$ such that (2) and (3) hold.

Then

$$J_T^*(x(t_i+\delta)) - J_T^*(x(t_i)) \le -\alpha \int_{t_i}^{t_i+\delta} L^*(\tau;t_i)d\tau,$$

with $\alpha = 1 - \gamma\frac{1-\varepsilon}{\varepsilon}$.

Furthermore, the estimate $\alpha J_\infty^{\mathrm{MPC}}(x) \le J_\infty^*(x)$ holds and asymptotic stability of the closed loop is guaranteed for $\alpha > 0$.

Proof of Theorem 1.2

$$\begin{aligned}
J_T^*(x(t_i+\delta)) - J_T^*(x(t_i)) &= J_T^*(x(t_i+\delta)) - \int_{t_i}^{t_i+T} L^*(\tau;t_i)d\tau \\
&\overset{(2)}{\le} \frac{1-\varepsilon}{\varepsilon}\int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau - \int_{t_i}^{t_i+\delta} L^*(\tau;t_i)d\tau \\
&\overset{(3)}{\le} \underbrace{\left(\gamma\frac{1-\varepsilon}{\varepsilon} - 1\right)}_{=:-\alpha}\int_{t_i}^{t_i+\delta} L^*(\tau;t_i)d\tau
\end{aligned}$$

→ Suboptimality by Proposition 1.1 (Relaxed Dynamic Programming)

→ Asymptotic stability from Lyapunov arguments and Barbalat's Lemma (for $\alpha > 0$)

Asymptotic controllability

Next: How can ε and γ be computed?

Assumption 1.3: Asymptotic controllability

For all x, there exists a (piece-wise continuous) input trajectory $\hat{u}_x(\cdot)$ with $\hat{u}_x(t) \in \mathcal{U}$ for all $t \ge 0$ such that

$$L(\hat{x}(t), \hat{u}_x(t)) \le \beta(t)\min_{u\in\mathcal{U}} L(x,u) \text{ for all } t \ge 0$$

with $\beta : \mathbb{R} \to \mathbb{R}^+$

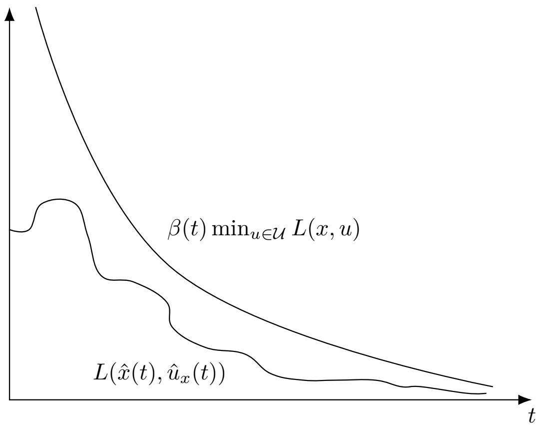
- continuous and positive
- strictly decreasing with $\lim_{t\to\infty}\beta(t) = 0$
- $\int_0^\infty \beta(\tau)d\tau < \infty$

→ denote $B(t) = \int_0^t \beta(\tau)d\tau$

Notation: $\hat{x}(\cdot)$ is the trajectory starting at initial condition x and resulting from application of $\hat{u}_x(\cdot)$

Typical example

$$\beta(t) = Ce^{-\lambda t}, \qquad \begin{array}{rr} C \geq 1: & \text{overshoot constant} \\ \lambda > 0: & \text{decay rate} \end{array}$$



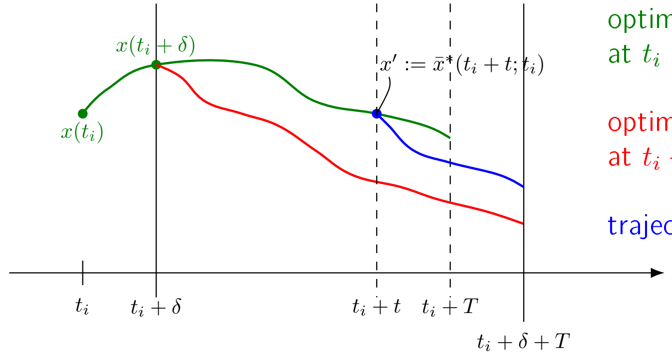
Consequences of Asymptotic Controllability I

Lemma 1.4

Let Assumption 1.3 hold. Then the inequality

$$\underline{J_T^*(x(t_i+\delta))} \leq \underline{\int_{t_i+\delta}^{t_i+t} L^*(\tau;t_i)d\tau} + \underline{B(T+\delta-t)L^*(t_i+t;t_i)} \tag{4}$$

holds for all $t \in [\delta, T]$.



optimal trajectory calculated at t_i

optimal trajectory calculated at $t_i+\delta$

trajectory generated by $\hat{u}_{x'}(\cdot)$

Proof of Lemma 1.4

Consider

$$\bar{u}(\tau;t_i+\delta) = \begin{cases} \bar{u}^*(\tau;t_i) & \tau \in [t_i+\delta, t_i+t] \\ \hat{u}_{x'}(\tau - t_i - t) & \tau \in (t_i+t, t_i+\delta+T] \end{cases}$$

$$\begin{aligned} \Rightarrow \; J_T^*(x(t_i+\delta)) &\leq \bar{J}_T(x(t_i+\delta)) \\ &= \int_{t_i+\delta}^{t_i+t} L^*(\tau;t_i)d\tau + \int_{t_i+t}^{t_i+\delta+T} L(\hat{x}(\tau-t_i-t), \hat{u}_{x'}(\tau-t_i-t))d\tau \\ &\leq \int_{t_i+\delta}^{t_i+t} L^*(\tau;t_i)d\tau + L^*(t_i+t;t_i)\int_0^{T+\delta-t} \beta(\tau)d\tau \\ &= \int_{t_i+\delta}^{t_i+t} L^*(\tau;t_i)d\tau + B(T+\delta-t)L^*(t_i+t;t_i) \end{aligned}$$

Consequences of Asymptotic Controllability II

Lemma 1.5

Let Assumption 1.3 hold. Then the inequality

$$\int_{t_i+t}^{t_i+T} L^*(\tau;t_i)d\tau \leq B(T-t)L^*(t_i+t,t_i) \tag{5}$$

holds for all $t \in [0, T]$.

Proof: Similar to proof of Lemma 1.4.

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)
2. Via an infinite-dimensional linear program

1.1 From (4):

$$
\begin{aligned}
J_T^*(x(t_i+\delta)) &\leq \min_{t\in[\delta,T]} \left\{ \int_{t_i+\delta}^{t_i+t} L^*(\tau;t_i)d\tau + B(T+\delta-t)L^*(t_i+t;t_i) \right\} \\
&\leq \int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau + B(T) \min_{t\in[\delta,T]} L^*(t_i+t;t_i) \\
&\leq \int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau + B(T)\frac{1}{T-\delta}\int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau \\
&= \underbrace{(1+\frac{B(T)}{T-\delta})}_{=:\frac{1}{\varepsilon}} \int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau
\end{aligned}
$$

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)

1.2 From (5):

$$
\begin{aligned}
\int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau &\leq \min_{t\in[0,\delta]} \int_{t_i+t}^{t_i+T} L^*(\tau;t_i)d\tau \\
&\overset{\text{from (5)}}{\leq} \min_{t\in[0,\delta]} \{B(T-t)L^*(t_i+t;t_i)\} \\
&\leq B(T) \min_{t\in[0,\delta]} L^*(t_i+t;t_i) \\
&\leq \underbrace{B(T)\frac{1}{\delta}}_{=:\gamma} \int_{t_i}^{t_i+\delta} L^*(\tau;t_i)d\tau
\end{aligned}
$$

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)
2. Via an infinite-dimensional linear program

Want to compute ε such that (from (2)) $\varepsilon \leq \frac{\int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau}{J_T^*(x(t_i+\delta))}$

Idea: Minimize

$$
\varepsilon = \min_{L_{t_i}(\cdot),\cancelto{1}{J_T^*(x(t_i+\delta))}} \frac{\int_\delta^T L_{t_i}(\tau)d\tau}{\cancelto{1}{J_T^*(x(t_i+\delta))}} \tag{6}
$$

subject to (from (4))

$$
\begin{aligned}
\cancelto{1}{J_T^*(x(t_i+\delta))} &\leq \int_\delta^t L_{t_i}(\tau)d\tau + B(T+\delta-t)L_{t_i}(t) \quad \forall t\in[\delta,T] \\
0 &\leq L_{t_i}(t)
\end{aligned}
$$

Due to linearity in L_{t_i}, set (without loss of generality) $J_T^*(x(t_i+\delta)) = 1$.

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)
2. Via an infinite-dimensional linear program

Want to compute ε such that (from (2)) $\varepsilon \leq \frac{\int_{t_i+\delta}^{t_i+T} L^*(\tau;t_i)d\tau}{J_T^*(x(t_i+\delta))}$

Idea: Minimize

$$
\varepsilon = \min_{L_{t_i}(\cdot)} \int_\delta^T L_{t_i}(\tau)d\tau \tag{6}
$$

subject to (from (4))

$$
\begin{aligned}
1 &\leq \int_\delta^t L_{t_i}(\tau)d\tau + B(T+\delta-t)L_{t_i}(t) \quad \forall t\in[\delta,T] \qquad (7) \\
0 &\leq L_{t_i}(t)
\end{aligned}
$$

$\rightarrow \underbrace{\text{infinite-dimensional}}_{L_{t_i}\text{ is a function}}, \underbrace{\text{linear}}_{\text{in } L_{t_i}}$ problem

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)
2. Via an infinite-dimensional linear program

Idea for Solution: First constraint (i.e., (7)) has to be active all the time!

Differentiate (7) w.r.t. t:

$$0 = L_{t_i}(t) + \frac{dB(T+\delta-t)}{dt} L_{t_i}(t) + B(T+\delta-t)\dot{L}_{t_i}(t)$$

$$\boxed{\dot{L}_{t_i}(t) = \frac{\beta(T+\delta-t)-1}{B(T+\delta-t)} L_{t_i}(t), \quad \text{initial condition: } L_{t_i}(\delta) = \frac{1}{B(T)}}$$

linear, time-varying ODE

Solution: $L^*_{t_i}(t) = \dfrac{1}{B(T+\delta-t)} e^{-\int_\delta^t \frac{1}{B(T+\delta-\tau)} d\tau}$

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)
2. Via an infinite-dimensional linear program

Have to show: $L^*_{t_i}$ is a minimizer of (6), i.e.,

$$\int_\delta^T L_{t_i}(\tau)d\tau \geq \int_\delta^T L^*_{t_i}(\tau)d\tau \qquad \text{for all feasible } L_{t_i}.$$

Proof by contradiction: Assume there exists $\bar{L}_{t_i}$ such that

$$\int_\delta^T L^*_{t_i}(\tau)d\tau > \int_\delta^T \bar{L}_{t_i}(\tau)d\tau.$$

$\rightsquigarrow$ There exists $t \in [\delta, T]$ such that

$$\int_\delta^t L^*_{t_i}(\tau)d\tau \geq \int_\delta^t \bar{L}_{t_i}(\tau)d\tau \quad \text{and} \quad L^*_{t_i}(t) > \bar{L}_{t_i}(t).$$

How to calculate ε and γ

Ways to calculate ε and γ

1. Directly from (4) and (5)
2. Via an infinite-dimensional linear program

$\rightsquigarrow$ Since constraint (7) is active for all $t \in [\delta, T]$:

$$1 = \int_\delta^t L^*_{t_i}(\tau)d\tau + B(T+\delta-t)L^*_{t_i}(t) > \int_\delta^t \bar{L}_{t_i}(\tau)d\tau + B(T+\delta-t)\bar{L}_{t_i}(t) \quad ↯$$

$$\boxed{\Rightarrow \varepsilon = \int_\delta^T L^*_{t_i}(\tau)d\tau = 1 - e^{-\int_0^{T-\delta} \frac{1}{B(T-\tau)} d\tau}}$$

$$= 1 - e^{-\int_\delta^T \frac{1}{B(T+\delta-\tau)} d\tau}$$

Similarly: better estimate for γ can be obtained.

Suboptimality Index

Recall that

$$\alpha J^{\mathrm{MPC}}_\infty(x) \leq J^*_\infty(x)$$

and

$$\alpha = 1 - \gamma^{\frac{1-\varepsilon}{\varepsilon}}.$$

Want: $\alpha \to 1$

For $T \to \infty$: both estimates yield

$$\varepsilon \to 1$$
$$\Rightarrow \alpha \to 1$$

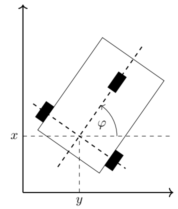
Comparison between MPC algorithms

Quasi-Infinite Horizon MPC	MPC without terminal constraints
with additional terminal constraint and terminal cost	without additional terminal constraints
suboptimality estimate in general not possible	additional suboptimality estimate
stability follows from initial feasibility and terminal region/controllers locally $\rightarrow$ can explicitely be computed	uses controllability assumption to establish stability $\rightarrow$ not (easily) verifiable globally
extensions to more general (e.g., time varying) cost function not always straightforward	extensions that require only local controllability exist
stability proof: feasible solution = shifted previously optimal solution and terminal controller at time T	stability proof: feasible solution = shifted previously optimal solution and controllability input at some time $t \in [\delta, T]$

Example: Brockett-Integrator I

Brockett-Integrator:

$$\dot{x}_1(t) = u_1(t)$$
$$\dot{x}_2(t) = u_2(t)$$
$$\dot{x}_3(t) = x_1(t)\, u_2(t) - x_2(t)\, u_1(t)$$



- Jacobi linearization of the system is not asympt. stabilizable
- Not stabilizable by continuous state feedback
- Design of control Lyapunov function is a difficult task

Open-loop control input $\widehat{u}$ defined as (here for $x_3(0) \geq 0$)

$$\widehat{u}_1(t) = \begin{cases} -x_1(0)/t_1, & 0 \leq t < t_1 \\ \frac{\sqrt{2\pi x_3(0)}}{t_2} \sin(2\pi t/t_2), & t_1 \leq t \leq t_1 + t_2 \end{cases}$$
$$\widehat{u}_2(t) = \begin{cases} -x_2(0)/t_1, & 0 \leq t < t_1 \\ \frac{\sqrt{2\pi x_3(0)}}{t_2} \cos(2\pi t/t_2), & t_1 \leq t \leq t_1 + t_2 \end{cases}$$

steers the system to $x(t_1 + t_2) = 0$.

Example: Brockett-Integrator II

For $\widehat{u}$ and stage cost $L(x,u) = x_1^2 + x_2^2 + \nu_3 \|x_3\| + u_1^2 + u_2^2$

$$J^*_{T'}(x_0) \leq J_{T'}(x_0, \widehat{u}) \leq \underbrace{\left(t_1^* + \frac{3 + 2\pi\nu_3}{6\pi\nu_3} t_2^* + \frac{4\pi + \nu_3}{2\nu_3 t_2^*} + \frac{1}{\pi} \right)}_{=B_{\hat{u}}(T')=\text{const.}} L(x_0, 0)$$

Stability guaranteed for $\nu_3 = 1, T \geq 23.3$ and $\nu_3 = 3, T \geq 15.4$.

Observation 1

Stability can be guaranteed for shorter prediction horizons if a larger ν_3 is chosen.

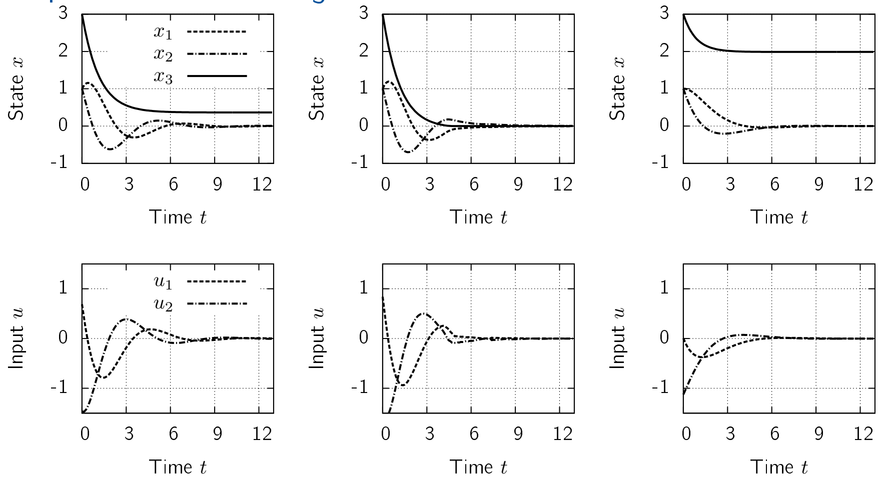
Taking additionally $\tilde{u} \equiv 0$ and $J^*_{T'}(x_0) \leq J_{T'}(x_0, \tilde{u}) = T' L(x_0, 0)$ into account, yields $B(T') = \min\{T', B_{\hat{u}}(T')\}$.

Stability guaranteed for $\nu_3 = 1, T \geq 5.71$ and $\nu_3 = 3, T \geq 4.09$.

Observation 2

Additional information helpful for stability guarantees.

Example: Brockett-Integrator III



$\nu_3 = 3, T = 0.7,$ $\nu_3 = 3, T = 0.8,$ $\nu_3 = 1, T = 0.8$

- Stability conditions can be conservative
- Theoretical analysis gives good guidelines for suitable controller design

MPC without terminal constraints: Summary

- ✓ Numerical efficient formulation
- ✓ Less restrictive than requirement of CLF
- ✓ Performance (*suboptimality*) estimates
- ✗ Controllability in general difficult to verify analytically
- ✗ Possibly conservative estimates on stabilizing prediction horizon

Extension: Additional weighting terms [Reble & Allgöwer '12]

- E.g. terminal cost "similar" to a CLF
- Stability guarantees for **shorter prediction horizons**
- $\Rightarrow$ less computational demanding

Part 2: Robust tube-based MPC

Setting

Uncertain linear discrete-time system

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

or short: $x^+ = Ax + Bu + w$

Constraints:

state constraints:	$x(t) \in \mathcal{X}$	for all $t \in \mathbb{I}_{\geq 0}$
input constraints:	$u(t) \in \mathcal{U}$	for all $t \in \mathbb{I}_{\geq 0}$
bound on w:	$w(t) \in \mathcal{W}$	for all $t \in \mathbb{I}_{\geq 0}$

Assumptions:

1. $\mathcal{W}$ is convex, compact, and contains 0
2. $(0,0) \in \text{int}(\mathcal{X} \times \mathcal{U})$

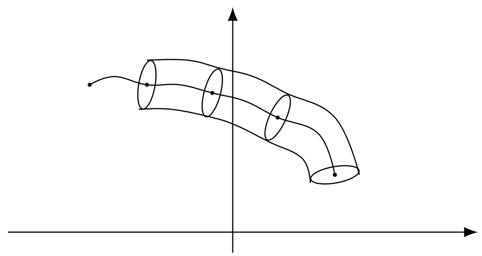
Typical setting in linear MPC: $\mathcal{X}, \mathcal{U}, \mathcal{W}$ polytopic
$\rightsquigarrow$ MPC optimization can be formulated as quadratic program (QP)

Notation: $\mathbb{I}_{\geq 0}$: set of all integers ≥ 0

Main idea of tube-based MPC

Main idea: Use additional local feedback!

"real system state shall be kept in tube around nominal system state"



Nominal MPC optimization problem

Nominal system: $z^+ = Az + Bv$

MPC optimization problem for nominal model

At time t, given $z(t)$, solve

$$\begin{aligned}
&\underset{v(\cdot|t)}{\text{minimize}} && \hat{J}(z(t), v(\cdot|t)) \\
=&\underset{v(\cdot|t)}{\text{minimize}} && \sum_{k=t}^{t+N-1} L(z(k|t), v(k|t)) + F(z(t+N|t)) \\
&\text{such that} && z(k+1|t) = Az(k|t) + Bv(k|t), && t \le k \le t+N-1 \\
& && z(t|t) = z(t) \\
& && z(k|t) \in \mathbb{Z} && t \le k \le t+N \\
& && v(k|t) \in \mathbb{V} && t \le k \le t+N-1 \\
& && z(t+N|t) \in \mathbb{Z}^f \subseteq \mathbb{Z}
\end{aligned}$$

$\rightsquigarrow$ optimizer: $v^*(\cdot|t)$, optimal value function $\hat{J}^*(z(t))$

Notation: $\hat{J}$ (with hat): cost functional for nominal system

Nominal MPC optimization problem

Nominal system: $z^+ = Az + Bv$

MPC optimization problem for nominal model

At time t, given $z(t)$, solve

$$\begin{aligned}
&\underset{v(\cdot|t)}{\text{minimize}} && \hat{J}(z(t), v(\cdot|t)) \\
=&\underset{v(\cdot|t)}{\text{minimize}} && \sum_{k=t}^{t+N-1} L(z(k|t), v(k|t)) + F(z(t+N|t)) \\
&\text{such that} && z(k+1|t) = Az(k|t) + Bv(k|t), && t \le k \le t+N-1 \\
& && z(t|t) = z(t) \\
& && z(k|t) \in \mathbb{Z} && t \le k \le t+N \\
& && v(k|t) \in \mathbb{V} && t \le k \le t+N-1 \\
& && z(t+N|t) \in \mathbb{Z}^f \subseteq \mathbb{Z}
\end{aligned}$$

Definition: Feasible set

$$\mathbb{Z}_N := \{z(t) \in \mathbb{R}^n | \text{a feasible solution to the MPC opt. problem exists}\} \subseteq \mathbb{Z}$$

Quadratic cost and auxiliary controller

Assumption 2.1: Quadratic cost and terminal region/cost

Cost is quadratic: $L(z, v) = \|z\|_Q^2 + \|v\|_R^2, \quad Q, R > 0$

There exists a local auxiliary controller $k^{\text{loc}}(z) = Kz$ such that

(A1) $\mathbb{Z}^f$ is invariant w.r.t. $z^+ = \underbrace{(A + BK)}_{A_K} z$, i.e., $A_K \mathbb{Z}^f \subseteq \mathbb{Z}^f$

(A2) $Kz \in \mathbb{V}$ for all $z \in \mathbb{Z}^f$

(A3) $F(A_K z) - F(z) \le -L(z, Kz)$ for all $z \in \mathbb{Z}^f$

As in the continuous-time case, it follows from Assumption 2.1 that

$$\hat{J}^*(z(t+1)) - \hat{J}^*(z(t)) \le -L(z(t), v(t)).$$

(see Lecture on quasi-infinite-horizon MPC)

Bounded cost

Using the particular (quadratic) cost, there exist constants $c_2 > c_1 > 0$ such that

$$c_1\|z\|^2 \le \hat{J}^*(z) \qquad \forall z \in \mathbb{Z}_N \qquad (1)$$

$$\hat{J}^*(z^+) - \hat{J}^*(z) \le -c_1\|z\|^2 \qquad \forall z \in \mathbb{Z}_N \qquad (2)$$

$$\hat{J}^*(z) \le c_2\|z\|^2 \qquad \forall z \in \mathbb{Z}_N \qquad (3)$$

Why is last inequality true?

Follows from (A3)!

$$\begin{aligned}
\forall z \in \mathbb{Z}^f : \hat{J}^*(z) &\le \hat{J}(z, Kz(\cdot)) \\
&= \sum_{i=0}^{N-1} L(z(i), Kz(i)) + F(z(N)) \\
&\overset{(A3)}{\le} F(z) \overset{\text{quadratic terminal cost}}{=} z^T P z \le \underbrace{\lambda_{\max}(P)}_{=:c_2} \|z\|^2
\end{aligned}$$

One can show that (3) holds for all $z \in \mathbb{Z}_N$ (under some technical conditions)

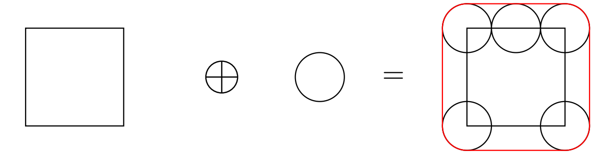
Set addition and set difference

Definition: Set addition and set difference

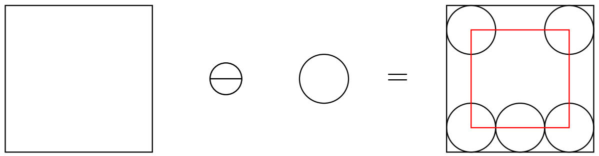
Minkowski set addition $A \oplus B := \{a + b | a \in A, b \in B\}$

(Pontryagin) set difference $A \ominus B := \{a \in \mathbb{R}^n | a + b \in A, \forall b \in B\}$

Example 1:



Example 2:



Robust positively invariant sets I

Definition: Robust positively invariant set (RPI set)

S is called a RPI set for system

$$x^+ = Ax + w$$

iff

$$AS \oplus \mathcal{W} \subseteq S$$

(or, equivalently, $Ax + w \in S$ for all $x \in S$ and $w \in \mathcal{W}$).

Robust positively invariant sets II

Example: $x^+ = \frac{1}{2}x + w, \quad \|w\| \leq 5$

RPI set: $S = [-20, 20]$
minimal RPI set: $S = [-10, 10]$

How to compute the minimal RPI set? Here easy to see:
What is largest x such that $\frac{1}{2}x + 5 = x \rightsquigarrow x = 10$

or: $S_\infty = \mathcal{W} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \mathcal{W} \frac{1}{1-\frac{1}{2}} = 2\mathcal{W} = [-10, 10]$

Minimal RPI set

Minimal RPI set

$$S_\infty := \sum_{i=0}^{\infty} A^i \mathcal{W}$$

S_∞ is called the minimal RPI set for system $x^+ = Ax + w$, $w \in \mathcal{W}$.

S_∞ exists and is bounded if A is Schur stable (all EV in unit disc).

Why? Justification:

current state at time t: x
possible states at time $t+1$: $Ax \oplus \mathcal{W}$
possible states at time $t+2$: $A(Ax \oplus \mathcal{W}) \oplus \mathcal{W} = A^2x \oplus A\mathcal{W} \oplus \mathcal{W}$
in general at time $t+j$: $A^jx \oplus \sum_{k=0}^{j-1} A^k \mathcal{W}$

$\Rightarrow$ „By choosing j large enough": can reach any state in S_∞
$\Rightarrow$ For any RPI set S it holds that $S_\infty \subseteq S$.

S_∞ in general difficult to compute!
$\rightsquigarrow$ can compute invariant outer approximation of S_∞ (see later)

Part 2: Robust tube-based MPC

Central idea for robust tube-based MPC

Use additional local feedback around nominal predicted trajectory:

$$u = v + K(x - z)$$

Proposition 2.2

Let $x^+ = Ax + Bu + w$ with $w \in \mathcal{W}$
and $z^+ = Az + Bv$.
If $x \in z \oplus S$ and $u = v + K(x - z)$, then

$$x^+ \in z^+ \oplus S$$

with S : RPI set for $x^+ = \underbrace{(A + BK)}_{A_K} x + w$ with $w \in \mathcal{W}$.

Proof of Proposition 2.2

Proof of Proposition 2.2: Let $e(t) = x(t) - z(t)$.

$$\begin{aligned}\Rightarrow e^+ = x^+ - z^+ &= Ax + B(v + K(x - z)) + w - Az - Bv \\ &= (A + BK)e + w\end{aligned}$$

As S is RPI for $e^+ = A_K e + w$, it follows that $e \in S \Rightarrow e^+ \in S$.
Hence $x \in z \oplus S \Rightarrow x^+ \in z^+ \oplus S$.

MPC algorithm for robust MPC

MPC optimal control problem

At time t, given $x(t)$, solve

$$\begin{aligned}
&\underset{z(t|t),v(\cdot|t)}{\text{minimize}} && J(x(t), v(\cdot|t)) \\
&= \underset{z(t|t),v(\cdot|t)}{\text{minimize}} && \sum_{k=t}^{t+N-1} L(z(k|t), v(k|t)) + F(z(t+N|t)) \\
&\text{such that} && z(k+1|t) = Az(k|t) + Bv(k|t) && t \le k \le t+N-1 \\
& && x(t) \in z(t|t) \oplus S \\
& && z(k|t) \in \mathbb{Z} := \mathcal{X} \ominus S && t \le k \le t+N \\
& && v(k|t) \in \mathbb{V} := \mathcal{U} \ominus KS && t \le k \le t+N-1 \\
& && z(t+N|t) \in \mathbb{Z}^f \subseteq \mathbb{Z}
\end{aligned}$$

$\rightsquigarrow$ optimizer: $z^*(t|t)$, $v^*(\cdot|t) \rightsquigarrow$ optimal value function $J^*(x(t))$
$\rightarrow$ applied input $u(t) = v^*(t|t) + K(x(t) - z^*(t|t))$
Important: Tightened constraints for nominal predicted system ensure fulfillment of original input/state constraints by the real (disturbed) system!

Properties of robust MPC algorithm

- feasible set $\mathbb{X}_N = \mathbb{Z}_N \oplus S \subseteq \mathcal{X}$
- $J^*(x) = \hat{J}^*(z^*(x))$ by definition of J^* and $\hat{J}^*$
- $J^*(x) = 0 \quad \forall x \in S$ („S serves as origin for the disturbed system and the cost is 0 there")

Why? If $x \in S$, then $z(x) = 0$ and $v(\cdot|t) = 0$ is a feasible solution.
Hence,

$$J^*(x) \leq \hat{J}(0,0) = 0$$

$\Rightarrow J^*(x) = 0$ and $z^*(x) = 0$.

Notation: $z^*(t|t) =: z^*(x(t))$

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Stability and feasibility

Theorem 2.3: Convergence and feasibility

Suppose that Assumption 2.1 holds and the robust MPC problem is feasible (i.e., $x_0 \in \mathbb{X}_N$) at $t = 0$. Then,

(i) the robust MPC problem is recursively feasible and

(ii) the closed-loop system robustly exponentially converges to S.

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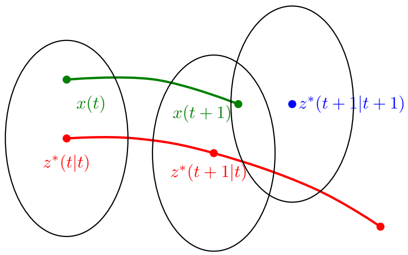
Proof: Feasibility

(i) Consider the feasible candidate solution at time $t+1$

$$\tilde{v}(k|t+1) = \begin{cases} v^*(k|t) & t+1 \leq k \leq t+N-1 \\ Kz^*(t+N|t) & k = t+N \end{cases}$$

$$\tilde{z}(t+1|t+1) = z^*(t+1|t)$$

$\Rightarrow \tilde{z}(t+1|t+1)$ is feasible because $x(t+1) \in z^*(t+1|t) \oplus S$ by Proposition 2.2.



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Proof: Convergence I

(ii) from (1)-(3) we obtain for all $x \in \mathbb{X}_N$

$$J^*(x) = \hat{J}^*(z^*(x)) \overset{(1)}{\geq} c_1 \|z^*(x)\|^2 \tag{4}$$

$$J^*(x) = \hat{J}^*(z^*(x)) \overset{(3)}{\leq} c_2 \|z^*(x)\|^2 \tag{5}$$

$$\begin{aligned} J^*(x(t+1)) - J^*(x(t)) &= \hat{J}^*(z^*(x(t+1))) - \hat{J}^*(z^*(x(t))) \\ &\leq \hat{J}^*(z^*(t+1|t)) - \hat{J}^*(z^*(x(t))) \\ &\overset{(2)}{\leq} -c_1 \|z^*(x(t))\|^2 \overset{(5)}{\leq} -\frac{c_1}{c_2} J^*(x(t)) \end{aligned}$$

$\Rightarrow J^*(x(t+1)) \leq (1 - \frac{c_1}{c_2}) J^*(x(t))$ define $\gamma := 1 - \frac{c_1}{c_2} \in (0,1)$

$$\Rightarrow J^*(x(t)) \leq \gamma^t J^*(x(0)) \overset{(5)}{\leq} c_2 \gamma^t \|z^*(x(0))\|^2$$

$$\overset{(4)}{\Rightarrow} \boxed{\|z^*(x(t))\| \leq \sqrt{\frac{c_2}{c_1}} \sqrt{\gamma}^t \|z^*(x(0))\|}$$

$\Rightarrow z^*(x(\cdot))$ converges exponentially fast to 0

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Proof: Convergence II

Recall:

$$x(t) \in z^*(x(t)) \oplus S$$
$$\Rightarrow \|x(t)\|_S \le \|z^*(x(t))\| \le \sqrt{\frac{c_2}{c_1}} \sqrt{\gamma}^t \|z^*(x(0))\| \quad (6)$$

Notation: $\|x\|_S$ point-to-set distance of point x to set S

Extensions

Linear systems with parametric uncertainties:

⇝ see, e.g., [Kouvaritakis & Cannon, *Model Predictive Control: Classical, Robust and Stochastic*]

Nonlinear Systems: (more) difficult to compute RPI set

⇝ approaches based on ISS/δISS

⇝ approach applying MPC „two times":

[Rawlings et al., *Model Predictive Control: Theory, Computation, and Design*, Chapter 3.6]

- for nominal state sequence
- for local error feedback

⇝ Recent approach with simple tube parameterization:

J. Köhler, R. Soloperto, M. A. Müller, F. Allgöwer, *A computationally efficient robust model predictive control framework for uncertain nonlinear systems*, IEEE Transactions on Automatic Control, vol. 66, no. 2, pp. 794-801, 2021.

Part 2: Robust tube-based MPC

Invariant approximations of minimal RPI set I

Recall: S_∞ in general difficult to compute!

⇝ invariant outer approximations of S_∞ can be computed efficiently

Define $S_k := \sum_{i=0}^{k-1} A^i \mathcal{W}, \quad k \ge 1$

Note that $S_k \to S_\infty$ as $k \to \infty$, but S_k for fixed k is in general not an RPI set!

Invariant approximations of minimal RPI set II

Theorem 2.4: Invariant approximations of minimal RPI set

If $0 \in \text{int}(\mathcal{W})$ and A is Schur stable, then there exists $\kappa \in \mathbb{I}_{\geq 0}$ and $\alpha \in [0,1)$ such that

$$A^\kappa \mathcal{W} \subseteq \alpha \mathcal{W}. \tag{7}$$

If (7) holds, then

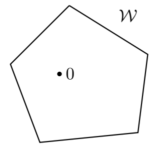
$$S(\alpha,\kappa) := (1-\alpha)^{-1} S_\kappa$$

is an RPI set for the system $x^+ = Ax + w$.

Proof

Proof: (7) follows since A is Schur stable (all EV strictly inside unit disc) and $0 \in \text{int}(\mathcal{W})$.
Next, we want to show that

$$AS(\alpha,\kappa) \oplus \mathcal{W} \subseteq S(\alpha,\kappa)$$



$$\begin{aligned}
\rightsquigarrow AS(\alpha,\kappa) \oplus \mathcal{W} &= (1-\alpha)^{-1} \sum_{i=1}^{\kappa} A^i \mathcal{W} \oplus \mathcal{W} \\
&= (1-\alpha)^{-1} A^\kappa \mathcal{W} \oplus (1-\alpha)^{-1} \sum_{i=1}^{\kappa-1} A^i \mathcal{W} \oplus \mathcal{W} \\
&\overset{(7)}{\subseteq} \underbrace{(1-\alpha)^{-1}\alpha \mathcal{W} \oplus \mathcal{W}}_{\substack{=[(1-\alpha)^{-1}\alpha+1]\mathcal{W} \\ =(1-\alpha)^{-1}\mathcal{W}}} \oplus (1-\alpha)^{-1} \sum_{i=1}^{\kappa-1} A^i \mathcal{W} \\
&= (1-\alpha)^{-1} \sum_{i=0}^{\kappa-1} A^i \mathcal{W} = S(\alpha,\kappa)
\end{aligned}$$

Remarks

- for given κ such that (7) can be satisfied, one wants to find smallest α such that (7) holds
- for given α, one wants to find smallest κ such that (7) holds
- one can determine „how good" $S(\alpha,\kappa)$ is compared to S_∞
  $\rightsquigarrow$ algorithm below can be adapted suitably.

Possible algorithm to determine RPI set:

1. fix $\alpha \in (0,1)$ and $\kappa \in \mathbb{I}_{\geq 0}$
2. check whether (7) holds
   if yes: $S(\alpha,\kappa)$ is RPI set
   if not: set $\kappa := \kappa + 1$ and go to 2.

Part 3: Economic MPC

Motivation

- In standard (stabilizing) MPC, we assume that the stage cost $L(x,u)$ is positive definite with respect to the setpoint to be stabilized.
- However: different control objective is of interest in many applications

- Maximization of product in process industry
- Minimization of energy consumption in building climate control
- Efficient scheduling of production process in manufacturing industry

$\Rightarrow$ Setpoint stabilization is not primary control objective

$\Rightarrow$ more general MPC framework termed economic MPC

Setting

Stage cost L can be general cost function, need not be positive definite

$\Rightarrow$ Closed-loop system does not necessarily converge to a steady state

Setting:

Nonlinear discrete-time systems	$x^+ = f(x,u)$
State and input constraints	$x \in \mathcal{X}$, $u \in \mathcal{U}$
General stage cost function	$L(x,u)$

Assumptions: $\mathcal{X}$ and $\mathcal{U}$ are compact
$L(x,u)$ is continuous

Optimal steady state

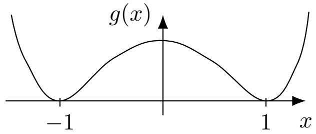
Definition: Optimal steady state

Optimal steady state $(x_s,u_s) := \underset{\substack{x=f(x,u)\\ x\in\mathcal{X},\, u\in\mathcal{U}}}{\operatorname{argmin}} \; L(x,u)$

Example:

System dynamics:	$x^+ = xu$
Constraints:	$\mathcal{X} = \mathcal{U} = [-5,5]$
Cost function:	$L(x,u) = g(x) + (u+1)^2$

$\rightsquigarrow$ Steady state is not optimal operating behavior:



Trajectories

$$x = (1,-1,1,-1,\dots)$$
$$u = (-1,-1,-1,-1,\dots)$$

yield optimal performance!

Economic MPC problem

MPC optimization problem $\mathcal{P}$

At time t, given $x(t)$, solve

$$\begin{aligned}
\underset{u(\cdot|t)}{\text{minimize}} \quad & J(x(t),u(\cdot|t)) = \sum_{k=t}^{t+N-1} L(x(k|t),u(k|t)) \\
\text{subject to} \quad & x(k+1|t) = f(x(k|t),u(k|t)) && t \le k \le t+N-1 \\
& x(t|t) = x(t) \\
& (x(k|t),u(k|t)) \in \mathcal{X}\times\mathcal{U} && t \le k \le t+N-1 \\
& x(t+N|t) = x_s
\end{aligned}$$

Optimizer $u^*(\cdot|t)$, optimal value function $J^*(x(t))$

Remark: Also economic MPC schemes available with terminal cost/region (instead of terminal equality constraint) and without terminal constraints.

Problems in economic MPC

The closed-loop system does not necessarily converge to a steady-state (other types of operation might be better)

- ⇝ Are there guarantees for the closed-loop performance?
- ⇝ Under what conditions is steady-state operation optimal?
- ⇝ If steady-state operation is optimal, does the closed loop converge to the optimal steady state?

Part 3: Economic MPC

Performance guarantees in economic MPC

Theorem 3.1: Closed-loop average performance

The closed-loop asymptotic average performance is at least as good as operation at the optimal steady state, i.e.

$$\limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} \leq L(x_s,u_s)$$

Proof of Theorem 3.1

As in standard MPC, we obtain

$$J^*(x(t+1)) - J^*(x(t)) \leq -L(x(t),u(t)) + L(x_s,u_s)$$

$$\begin{aligned}\Rightarrow \liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} J^*(x(t+1)) - J^*(x(t)) &\leq \liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} L(x_s,u_s) - L(x(t),u(t)) \\ &= L(x_s,u_s) - \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} L(x(t),u(t))\end{aligned}$$

$$\begin{aligned}\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} J^*(x(t+1)) - J^*(x(t)) &= \liminf_{T\to\infty} \frac{1}{T}\left(J^*(x(T)) - J^*(x(0))\right) \\ &\geq \liminf_{T\to\infty} \frac{1}{T}\left(\min_{x\in\mathcal{X},u\in\mathcal{U}} NL(x,u) - J^*(x(0))\right) \\ &= 0\end{aligned}$$

The last equality holds since L is continuous and $\mathcal{X}$, $\mathcal{U}$ are compact.

Part 3: Economic MPC

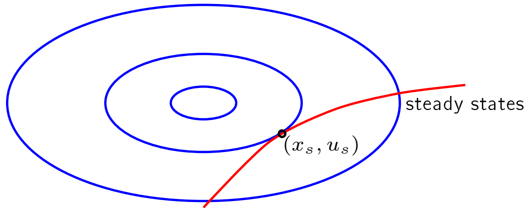
Optimal operation at steady state

Definition: Optimal operation at steady state

A system is optimally operated at steady state if

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} \geq L(x_s,u_s)$$

for all feasible sequences $(x(\cdot),u(\cdot))$.



Optimal operation at steady state

Definition: Optimal operation at steady state

A system is optimally operated at steady state if

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} \geq L(x_s,u_s)$$

for all feasible sequences $(x(\cdot),u(\cdot))$.

Definition: Strict dissipativity

A system is strictly dissipative with respect to the supply rate $s(x,u)$ if there exists a nonnegative storage function $\lambda : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u) - \rho(|x-x_s|), \qquad \rho \in \mathcal{K}_\infty$$

Continuous-time: $\dot{\lambda} \leq s$

Under what conditions is steady-state operation optimal?

Theorem 3.2: Optimal operation at steady state

A system is optimally operated at steady state if it is dissipative with respect to the supply rate $s(x,u) = L(x,u) - L(x_s,u_s)$.

Proof of Theorem 3.2:

$$\begin{aligned} 0 &\leq \liminf_{T\to\infty} \frac{\lambda(x(T)) - \lambda(x(0))}{T} = \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \lambda(x(t+1)) - \lambda(x(t))}{T} \\ &\leq \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} s(x(t),u(t))}{T} = \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} - L(x_s,u_s) \end{aligned}$$

Remark: Under an additional controllability assumption, dissipativity with supply rate $s(x,u) = L(x,u) - L(x_s,u_s)$ is also necessary for optimal steady-state operation.

Example

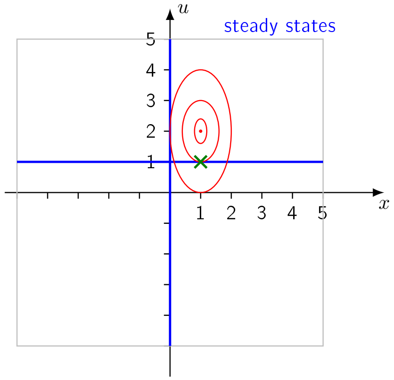
Dynamics $x(t+1) = x(t)u(t)$

Constraints $(x,u) \in \begin{bmatrix}-5 & 5\end{bmatrix} \times \begin{bmatrix}-5 & 5\end{bmatrix}$

Cost function $L(x,u) = (x-1)^2 + \delta(u-2)^2$
$0 < \delta < 1$

Steady states $S = \{(x,u) : x = 0, u \in \mathcal{U}\}$
$\cup\{(x,u) : x \in \mathcal{X}, u = 1\}$

Optimal steady state $(x_s, u_s) = (1,1)$
with $L(x_s, u_s) = \delta$



$\Rightarrow$ system is dissipative with $\lambda(x) = 10\delta - 2\delta x$. Why?

$$\lambda(f(x,u)) - \lambda(x) = -2\delta xu + 2\delta x \overset{!}{\leq} (x-1)^2 + \delta(u-2)^2 - \delta$$

$$0 \leq (x-1)^2 + \delta(u-2)^2 + 2\delta xu - 2\delta x - \delta$$

$$\nabla_{|(x,u)} = 0 \Rightarrow \begin{bmatrix} 2x - 2 - 2\delta + 2\delta u \\ 2\delta u - 4\delta + 2\delta x \end{bmatrix} = 0 \quad \Rightarrow (x,u) = (1,1)$$

Example

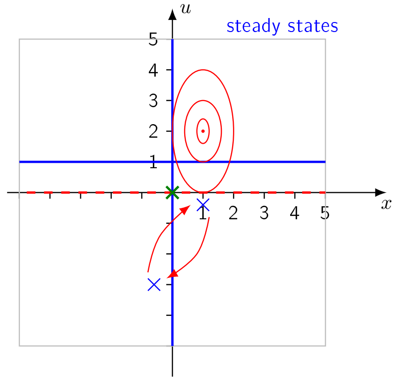
Dynamics $x(t+1) = x(t)u(t)$

Constraints $(x,u) \in \begin{bmatrix}-5 & 5\end{bmatrix} \times \begin{bmatrix}-5 & 0\end{bmatrix}$

Cost function $L(x,u) = (x-1)^2 + \delta(u-2)^2$
$0 < \delta < 1$

Steady states $S = \{(x,u) : x = 0, u \in \mathcal{U}\}$
$\cup\{(x,u) : x \in \mathcal{X}, u = 1\}$

New optimal steady state $(x_s, u_s) = (0,0)$
with $L(0,0) = 1 + 4\delta$



$\to$ system is not optimally operated at steady state (and hence not dissipative) anymore for small δ!

Why? Consider trajectory

$$u = (-\frac{1}{3}, -3, -\frac{1}{3}, -3, \dots)$$

$$x = (1, -\frac{1}{3}, 1, -\frac{1}{3}, \dots)$$

$$\to \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} = \frac{(\frac{4}{3})^2}{2} + \delta \frac{(-\frac{7}{3})^2 + (-5)^2}{2} < L(0,0)$$

Part 3: Economic MPC

Does the closed-loop system converge if steady-state operation is optimal?

Theorem 3.3: Convergence

Suppose the system is strictly dissipative w.r.t. the supply rate $s(x,u) = L(x,u) - L(x_s,u_s)$. Then the closed-loop system asymptotically converges to the optimal steady state x_s.

Remark: x_s is asymptotically stable if aditionally $J^*(x)$ and $\lambda(x)$ are continuous at x_s.

Proof of Theorem 3.3

Define „rotated" cost function

$$\tilde{L}(x,u) = L(x,u) + \lambda(x) - \lambda(f(x,u)) - L(x_s,u_s)$$

Auxiliary optimization problem $\tilde{\mathcal{P}}$

$$\underset{u(\cdot|t)}{\text{minimize}} \quad \tilde{J}(x(t),u(\cdot|t)) = \sum_{k=t}^{t+N-1} \tilde{L}(x(k|t),u(k|t))$$

subject to same constraints as in Problem $\mathcal{P}$

Claim: $\mathcal{P}$ and $\tilde{\mathcal{P}}$ have the same optimizer (proof: see next slide).

$\rightsquigarrow$ Use $\tilde{\mathcal{P}}$ to analyze stability of the closed-loop system

$\tilde{L}(x,u) \geq \rho(\|x-x_s\|)$ by strict dissipativity

$\Rightarrow$ We can apply standard MPC stability theory to conclude that the closed loop converges to x_s

Proof of Claim

Claim: Problems $\mathcal{P}$ and $\tilde{\mathcal{P}}$ share the same optimizer(s)!

$\rightsquigarrow$ Feasible sets coincide

$$\begin{aligned}
&\tilde{J}(x(t),u(\cdot|t)) \\
&= \sum_{k=t}^{t+N-1} \Big[L(x(k|t),u(k|t)) + \lambda(x(k|t)) - \lambda(f(x(k|t),u(k|t))) - L(x_s,u_s) \Big] \\
&= \lambda(x(t|t)) - \lambda(x(t+N|t)) - NL(x_s,u_s) + \sum_{k=t}^{t+N-1} L(x(k|t),u(k|t)) \\
&= \lambda(x(t)) - \lambda(x_s) - NL(x_s,u_s) + J(x(t),u(\cdot|t))
\end{aligned}$$

$\rightsquigarrow$ J and $\tilde{J}$ only differ by constant terms

$\Rightarrow$ Problems $\mathcal{P}$ and $\tilde{\mathcal{P}}$ share the same optimizer(s)

Dissipativity in economic MPC I

Strict dissipativity and optimal steady-state operation

optimal operation at steady state (strict form) $\xRightarrow{\text{local controllability at } x_s \text{ [Müller et al. 15]}}$ Strict dissipativity w.r.t. supply rate $s(x,u) = L(x,u) - L(x_s,u_s)$ $\xRightarrow{\text{[Diehl et al. '11, Angeli et al. '12, Grüne '13, Müller et al. '14]}}$ Closed-loop convergence to x_s

optimal operation at steady state (strict form) $\Longleftarrow$ Strict dissipativity w.r.t. supply rate $s(x,u) = L(x,u) - L(x_s,u_s)$

Discussion

- Closed-loop system "does the right thing", i.e., "finds" optimal operating behavior
- Can be concluded without having to compute storage function λ

Dissipativity in economic MPC II

Results can be extended to optimal periodic behavior:

Dissipativity and optimal periodic operation

Optimal periodic operation $\xRightarrow{\text{local controllability at } \Pi^* \text{ [Müller et al. 15]}}$ Strict dissipativity of P-step system w.r.t. $s(\tilde{x},\tilde{u}) = \tilde{L}(\tilde{x},\tilde{u}) - \tilde{L}(\tilde{x}_s,\tilde{u}_s)$ $\xRightarrow{\text{[Müller\&Grüne'15, Zanon et al. '15]}}$ Closed-loop convergence to Π^*

Optimal periodic operation $\Longleftarrow$ Strict dissipativity of P-step system w.r.t. $s(\tilde{x},\tilde{u}) = \tilde{L}(\tilde{x},\tilde{u}) - \tilde{L}(\tilde{x}_s,\tilde{u}_s)$

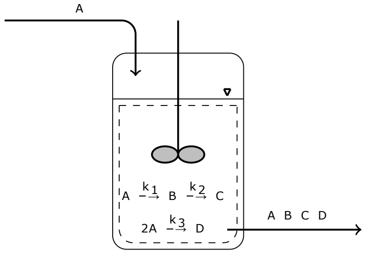
Discussion

- Dissipativity plays central role in economic MPC
- Closed-loop system "does the right thing", i.e., "finds" optimal operating behavior
- Can be concluded without having to compute storage function
- Results hold for both optimal steady-state and periodic behavior

Example - chemical reactor with dissipativity I

Van de Vusse reactor:

- Reactions $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ and $2A \xrightarrow{k_3} D$, with A: reactant, B: desired product, C, D: waste products

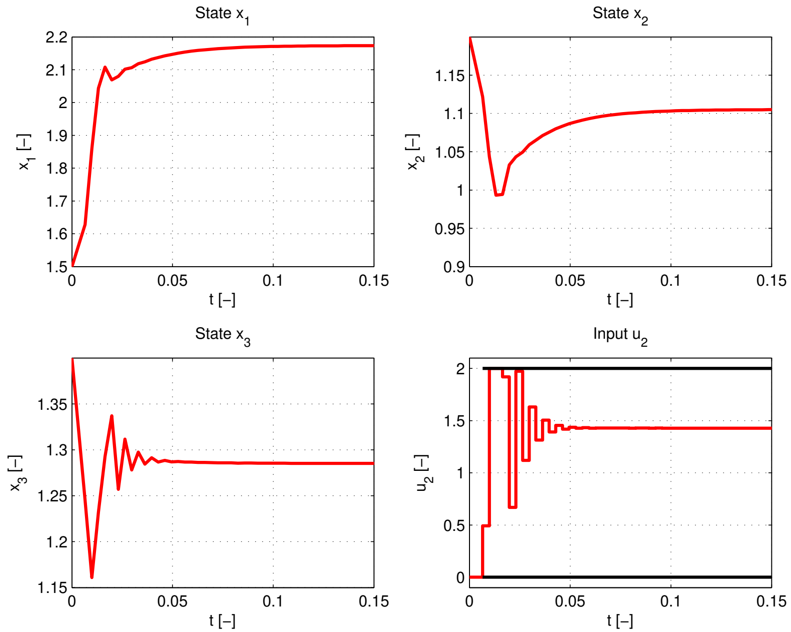


$$\dot{c}_A = r_A(c_A, \vartheta) + (c_{in} - c_A)u_1$$
$$\dot{c}_B = r_B(c_A, c_B, \vartheta) - c_B u_1$$
$$\dot{\vartheta} = h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,$$

ϑ: temperature in the reactor, u_1: normalized flow rate of A, u_2: temperature in cooling jacket

- Control objective: maximize production rate of $B \rightarrow L(x,u) = -c_B u_1$
- System is strictly dissipative w.r.t. supply rate $s(x,u) = L(x,u) - L(x_s, u_s)$

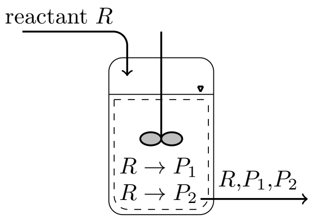
Example - chemical reactor with dissipativity II



Example - chemical reactor without dissipativity III

Continuous flow stirred-tank reactor with parallel reactions

- Reactions $R \rightarrow P_1$ and $R \rightarrow P_2$, with R: reactant, P_1: desired product, P_2: waste product

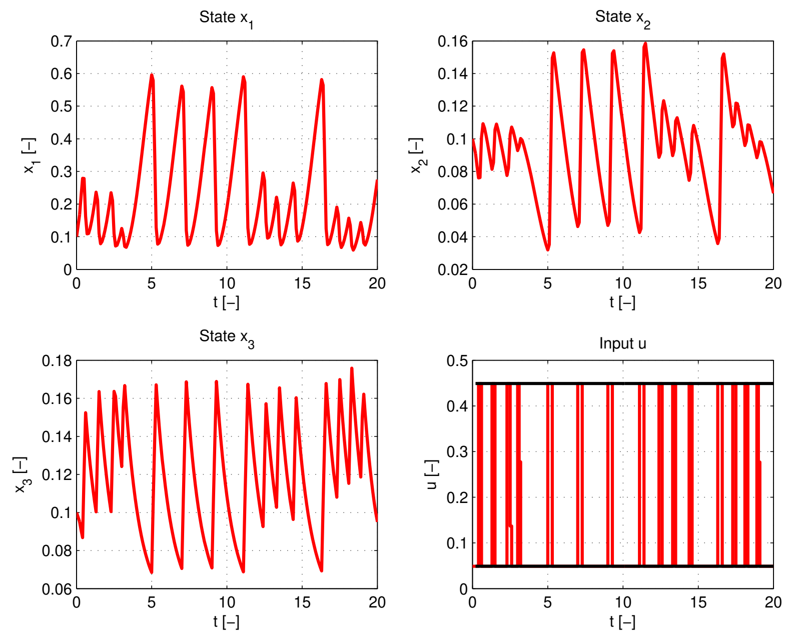


$$\dot{x}_1 = 1 - 10^4 x_1^2 e^{-1/x_3} - 400 x_1 e^{-0.55/x_3} - x_1$$
$$\dot{x}_2 = 10^4 x_1^2 e^{-1/x_3} - x_2$$
$$\dot{x}_3 = u - x_3$$

x_1: concentration of R, x_2: concentration of P_1, x_3: temperature in the reactor, u: proportional to heat flux through cooling jacket

- Control objective: maximize product $P_1 \rightarrow L(x,u) = -x_2$

Example - chemical reactor without dissipativity IV

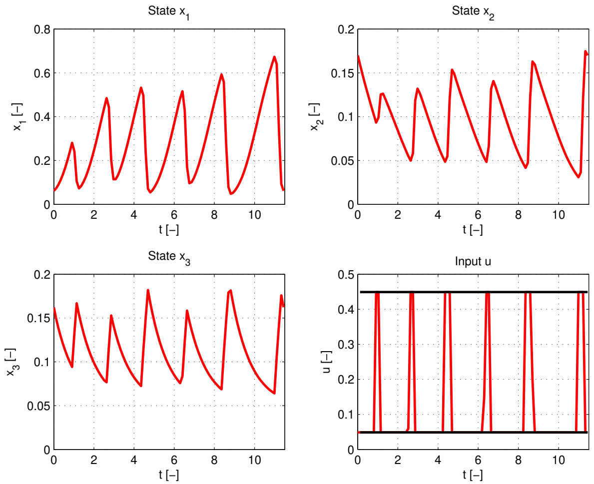


Example - chemical reactor without dissipativity V

Optimal periodic orbit length: $T^\star \approx 11.444$

$$\min_{u(\cdot),T} \frac{1}{T}\int_0^T -x_2(\tau)\mathrm{d}\tau$$

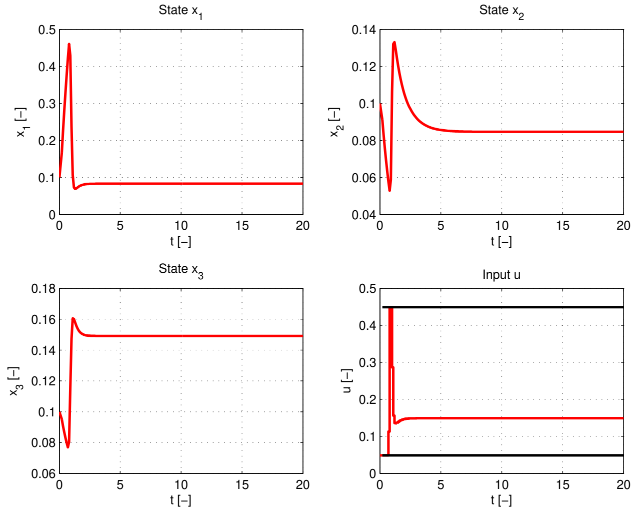
$$\text{subject to} \quad x(0) = x(T), \quad T \in [5, 20].$$



Example - chemical reactor without dissipativity VI

Recovering steady-state optimality through regularization:

$$L(x,u) = -x_2 + \omega(u - u_s)^2, \quad \omega > 0$$



Part 3: Economic MPC

Motivation

So far: Terminal constraint (and knowledge of x_s) necessary to show stability

Now: Economic MPC without terminal constraints

MPC optimization problem

At time t, given $x(t)$, solve

$$\begin{aligned}
\underset{u(\cdot|t)}{\text{minimize}} \quad & J(x(t),u(\cdot|t)) = \sum_{k=t}^{t+N-1} L(x(k|t),u(k|t)) \\
\text{subject to} \quad & x(t|t) = x(t) \\
& x(k+1|t) = f(x(k|t),u(k|t)) \qquad t \le k \le t+N-1 \\
& (x(k|t),u(k|t)) \in \mathcal{X}\times\mathcal{U} \qquad t \le k \le t+N-1 \\
& \cancel{x(t+N|t) = x_s}
\end{aligned}$$

Assumption: $L(x,u)$ is locally Lipschitz continuous, i.e., there exists $L_l > 0$ such that $\|L(x_1,u_1) - L(x_2,u_2)\| \le L_l\|(x_1,u_1)-(x_2,u_2)\|$ for all (x_1,u_1), $(x_2,u_2) \in \mathcal{X}\times\mathcal{U}$.

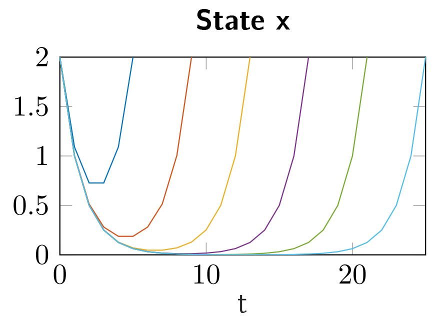
Example

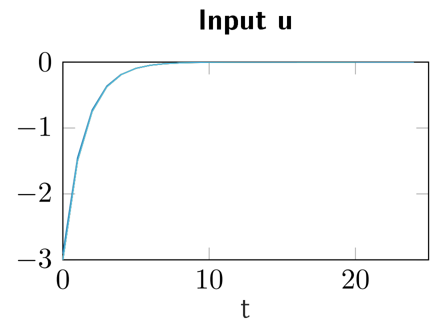
Linear system $x^+ = 2x + u$
Constraints $(x,u) \in \mathcal{X} \times \mathcal{U} = [-2,\ 2] \times [-3,\ 3]$
Stage cost $L(x,u) = u^2$

$\rightsquigarrow$ Optimal steady state $(x_s, u_s) = (0,0)$

$\rightsquigarrow$ Strictly dissipative on $\mathcal{X}$ with storage function $\lambda_c(x) = c - \frac{1}{2}x^2$, $c \geq 2$

Solutions of the MPC problem at time $t = 0$ with $x(0) = 2$ for $N = 5, 9, \dots, 25$:





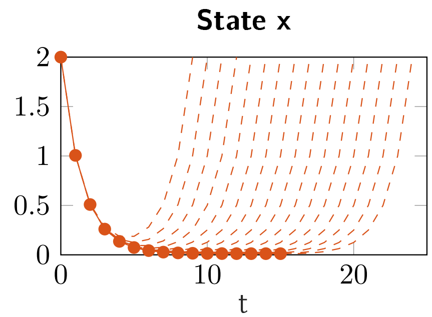
Example

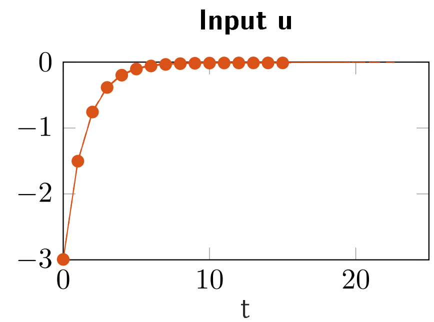
Linear system $x^+ = 2x + u$
Constraints $(x,u) \in \mathcal{X} \times \mathcal{U} = [-2,\ 2] \times [-3,\ 3]$
Stage cost $L(x,u) = u^2$

$\rightsquigarrow$ Optimal steady state $(x_s, u_s) = (0,0)$

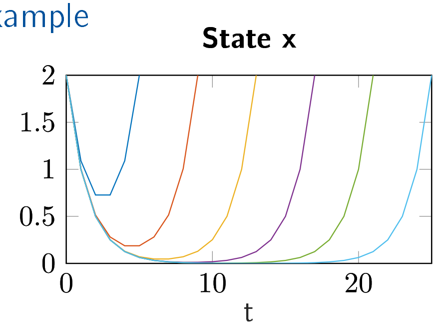
$\rightsquigarrow$ Strictly dissipative on $\mathcal{X}$ with storage function $\lambda_c(x) = c - \frac{1}{2}x^2$, $c \geq 2$

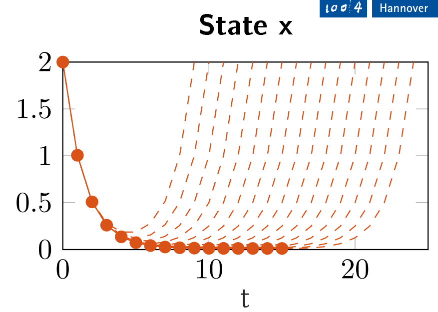
Solutions of the MPC problem at time $t = \{0, ..., 15\}$ with $x(0) = 2$ for $N = 9$:





Example





Observations

- Optimal open-loop state sequences first converge to a neighborhood of the optimal steady state
- Optimal open-loop state sequences leave this neighborhood towards the end of the prediction horizon ("leaving arc")
- Number of time steps in which the optimal open-loop state sequence is close to the optimal steady-state increases with increasing prediction horizon N
- Closed-loop state sequences converge to and stay in a neighborhood of the optimal steady state

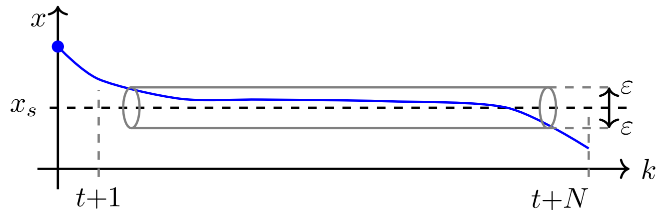
Turnpike property

Definition: Turnpike property

The MPC problem is said to have the turnpike property, if there exists $C > 0$ and $\alpha \in \mathcal{K}_\infty$ such that for all $x(t) \in \mathcal{X}$, we have

$$\#\mathcal{Q}_\varepsilon \geq N - \frac{C}{\alpha(\varepsilon)}, \tag{1}$$

where $\mathcal{Q}_\varepsilon := \left\{k \in \{t, \dots, t+N-1\} \middle| \|(x^*(k|t), u^*(k|t)) - (x_s, u_s)\| \leq \varepsilon\right\}$ and $\#\mathcal{Q}_\varepsilon$ is the cardinality of $\mathcal{Q}_\varepsilon$.



Relation between strict dissipativity and turnpike properties

Definition: Exponential reachability of x_s

The steady state x_s is called exponentially reachable, if there exist $c > 0$ and and $\sigma \in [0,1)$ such that for all $x(0) \in \mathcal{X}$, there exists an infinite-horizon feasible input $\hat{u}(\cdot)$ such that

$$\|(\hat{x}(\tau), \hat{u}(\tau)) - (x_s, u_s)\| \le c\sigma^\tau.$$

Proposition 3.4: Strict dissipativity implies turnpike properties

Let the system be strictly dissipative with respect to the supply rate $L(x,u) - L(x_s,u_s)$ and a storage function bounded on $\mathcal{X}$ and let the steady state x_s be exponentially reachable. Then, the MPC problem has the turnpike property (1).

Remark: With proper controllability and technical assumptions, it is also true that turnpike properties imply strict dissipativity.

Proof of Proposition 3.4

Proof: Let $J_N^*(x(t))$ denote the optimal value function of the MPC problem and assume without loss of generality $L(x_s,u_s) = 0$. Strict dissipativity implies

$$J_N^*(x(t)) \ge \lambda(x^*(N|t)) - \lambda(x(t)) + \sum_{k=t}^{t+N-1} \rho(\|x^*(k|t) - x_s\|).$$

Since $N - \#\mathcal{Q}_\varepsilon$ denotes the number of time instances an optimal pair $(x^*(\cdot|t), u^*(\cdot|t))$ spends outside of an ε-neighborhood of x_s, we have

$$\begin{aligned}
\sum_{k=t}^{t+N-1} \rho(\|x^*(k|t) - x_s\|) &\ge (N - \#\mathcal{Q}_\varepsilon)\rho(\varepsilon) \\
\Rightarrow J_N^*(x(t)) &\ge \lambda(x^*(N|t)) - \lambda(x(t)) + (N - \#\mathcal{Q}_\varepsilon)\rho(\varepsilon) \\
&\ge -\bar{\lambda} + (N - \#\mathcal{Q}_\varepsilon)\rho(\varepsilon), \qquad (2)
\end{aligned}$$

where $\bar{\lambda} := \sup_{x\in\mathcal{X}} |\lambda(x)| < \infty$.

Proof of Proposition 3.4

Using the input sequence $\hat{u}(\cdot)$ from the definition of exponential reachability with $\hat{x}(0) = x(t)$ yields

$$\begin{aligned}
J_N^*(x(t)) &= \sum_{k=t}^{t+N-1} L(x^*(k|t), u^*(k|t)) - L(x_s,u_s) \\
&\le \sum_{\tau=0}^{N-1} L(\hat{x}(\tau), \hat{u}(\tau)) - L(x_s,u_s) \\
&\le L_l \sum_{\tau=0}^{N-1} \|(\hat{x}(\tau), \hat{u}(\tau)) - (x_s,u_s)\| \\
&\le L_l c \sum_{\tau=0}^{N-1} \sigma^\tau \le \frac{L_l c}{1-\sigma}.
\end{aligned}$$

Using (2)

$$\begin{aligned}
-\bar{\lambda} + (N - \#\mathcal{Q}_\varepsilon)\rho(\varepsilon) &\le \frac{L_l c}{1-\sigma} \\
\Rightarrow \#\mathcal{Q}_\varepsilon &\ge N - \frac{L_l c(1-\sigma)^{-1} + \bar{\lambda}}{\rho(\varepsilon)}
\end{aligned}$$

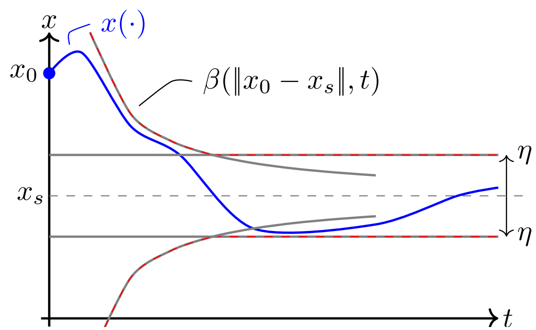
Practical stability

Definition: Practical asymptotic stability

The point x_s is said to be practically asymptotically stable with respect to $\eta > 0$ on a set $\mathcal{X}$, if there exists $\beta \in \mathcal{KL}$ such that

$$\|x(t) - x_s\| \le \max\{\beta(\|x_0 - x_s\|, t), \eta\}$$

holds for all $t \in \mathbb{N}_0$ and $x(0) = x_0 \in \mathcal{X}$.



Recursive feasibility and convergence

Theorem 3.5: Convergence and recursive feasibility

Let the system be strictly dissipative with respect to the supply rate $L(x,u) - L(x_s,u_s)$ and a storage function bounded on $\mathcal{X}$. Let the steady state x_s be exponentially reachable and the linearization of the system at x_s be controllable. Let $\mathcal{X}$ be compact. Then, there exists a sufficiently large prediction horizon $N \in \mathbb{N}$ such that the closed-loop system arising from economic MPC without terminal constraints has the following properties:

(i) The MPC optimization problem is recursively feasible, if it is feasible for $t = 0$ and

(ii) the closed-loop system is practically asymptotically stable for all $x(0) \in \mathcal{X}$. Furthermore, $\eta = \eta(N)$ where $\eta(N) \to 0$ for $N \to \infty$ if some additional technical assumptions hold.

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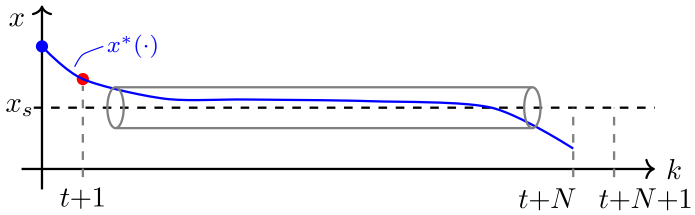
Proof of Theorem 3.5: Recursive Feasibility



Proof: Suppose the MPC optimization problem is feasible at time t and let $x^*(\cdot|t)$ and $u^*(\cdot|t)$ be the optimal state and input sequence at time t.

By Proposition 3.4: For all $x(t) \in \mathcal{X}$ and any $\varepsilon > 0$ we can find $N < \infty$ and k_1, k_2 with $k_1 + 2n \le k_2 \le N$, such that $\|x^*(t+k_1|t) - x_s\| \le \varepsilon$ and $\|x^*(t+k_2|t) - x_s\| \le \varepsilon$.

By controllability close to x_s: For ε sufficiently small, there exist feasible inputs

$u_1^\varepsilon(\cdot)$ of length $n+1$ starting at $x^*(t+k_1|t)$ and ending at x_s,

$u_2^\varepsilon(\cdot)$ of length n starting at x_s and ending at $x^*(t+k_2|t)$.

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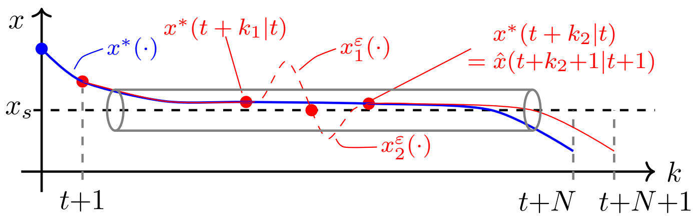
Proof of Theorem 3.5: Recursive Feasibility



Proof: Assume $k_2 = k_1 + 2n$. Then, the candidate input

$$\hat{u}(k|t+1) = \begin{cases} u^*(k|t) & k = t+1, \dots, t+k_1-1 \\ u_1^\varepsilon(k-t-k_1) & k = t+k_1, \dots, t+k_1+n \\ u_2^\varepsilon(k-t-k_1-n-1) & k = t+k_1+n+1, \dots, t+k_2 \\ u^*(k-1|t) & k = t+k_2+1, \dots, N \end{cases}$$

is feasible at time $t+1$!

$\Rightarrow$ If $k_2 > k_1 + 2n$ we can simply add $\hat{u}(k|t+1) = u_s$ between $u_1^\varepsilon(\cdot)$ and $u_2^\varepsilon(\cdot)$

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Proof of Theorem 3.5: Practical stability

Sketch of Proof: The rotated cost function

$$\tilde{J}(x,u) = \lambda(x(t|t)) - \lambda(x(t+N|t)) - NL(x_s,u_s) + J(x(t),u(t))$$

now depends on the chosen input sequence due to $\lambda(x(t+N|t))$
$\Rightarrow$ Rotating the cost function alters optimal solutions

But: Under additional technical assumptions it holds that

$$\tilde{J}_N^*(x) = J_N^*(x) + \lambda(x) - J_N^*(x_s) + R(x,N), \tag{3}$$

with $|R(x,N)| \le \nu(\|x - x_s\|) + \omega(N)$ where $\nu \in \mathcal{K}$, $\omega \in \mathcal{L}$

Consider $\hat{V}_N(x) = \lambda(x) + J_N^*(x) - J_N^*(x_s)$. Then, using (3) and strict dissipativity, one can show that

$$\hat{V}_N(x(t+1)) - \hat{V}_N(x(t)) \le -\rho(\|x(t) - x_s\|) + c(N),$$

where $c(N) > 0$ and $c(N) \to 0$ as $N \to \infty$.

$\Rightarrow$ From here, one can use Lyapunov arguments to show practical asymptotic stability.

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