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MPC – Exercise 3

The main goals of this exercise are:

- Get used to work with polytopes and the Multi-Parametric Toolbox (MPT).
- Use the minimal Robust Positively Invariant (mRPI) set within the framework of Robust MPC.
- Implement a tube-based robust MPC algorithm.

Consider a DC motor which drives a single axis robot. We assume that the axis is directly coupled to the motor (without a gearbox). The following figures show the equivalent circuit of the motor and the axis.

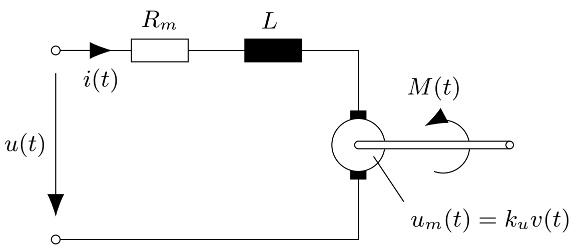

Figure 1: Equivalent circuit of the motor

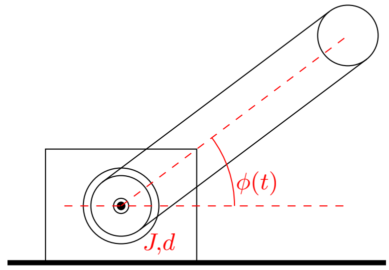

Figure 2: Single axis robot

The voltage applied to the motor is denoted by $u(t)$ and the electrical current is $i(t)$. The resistance and inductance of the circuit are R_m and L, respectively. The motor speed constant is denoted by k_u and the torque constant is given by k_m, i.e., $M(t) = k_m i(t)$, where $M(t)$ is the torque applied to the axis. The inertia of the motor and axis is J. Friction is modeled as viscous friction with coefficient d. The values of the physical constants are given in the MATLAB script `MPC_Exercise_3.m`. Using Kirchhoff's and Newton's laws, the system can be modeled by the differential equations

$$\begin{aligned} u(t) &= R_m i(t) + L\dot{i}(t) + k_u\dot{\phi}(t) \\ J\ddot{\phi}(t) &= k_m i(t) - d\dot{\phi}(t). \end{aligned}$$

We want to control the angular velocity $\dot{\phi}(t)$ and electrical current of the motor. Therefore, we choose the states $x(t) = \begin{bmatrix} i(t) & \dot{\phi}(t) \end{bmatrix}^\top$, which yields the state-space representation

$$\dot{x}(t) = \begin{bmatrix} -\frac{R_m}{L} & -\frac{k_u}{L} \\ \frac{k_m}{J} & -\frac{d}{J} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t) + w(t),$$

where we introduced an additive disturbance $w(t) \in \mathcal{W} = \{w \in \mathbb{R}^2 | \|w\|_\infty \leq 0.05\}$. Moreover, the system is subject to constraints

$$x(t) \in \mathcal{X} = \{x \in \mathbb{R}^2 \mid |i| \leq 3, |\dot{\phi}| \leq 10\} \text{ and } u(t) \in \mathcal{U} = \{u \in \mathbb{R} \mid |u| \leq 10\}.$$

For this system, a tube-based MPC scheme as proposed in [2] shall be implemented with a prediction horizon $T = 0.05\,\text{s}$. The weighting matrices of the cost function are

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1000 \end{bmatrix} \quad \text{and} \quad R = 1.$$

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As a local feedback law, the optimal unconstrained LQR controller $u = Kx$ is used. The terminal cost is chosen to be $x^T Px$, where P is the solution of the corresponding discrete-time algebraic Riccati equation.

Problem 1 In order to reformulate the above control problem, a suitable discretization is needed. To this end, a sampling time of $\delta = 0.01$ is chosen. During each sampling interval, the open-loop input $u(t)$ is chosen to be piecewise constant (zero-order hold), that is,

$$u(k \cdot \delta + \tau) = u_k \ , \quad \forall \tau \in [0, \delta) \ ,$$

where $k = 0, \ldots, N-1$. Discretize the above system using exact discretization with zero-order hold.

Hint: Use the MATLAB command `c2d`.

Problem 2 Determine an invariant approximation of the minimal robust positively invariant (mRPI) set for the system $x^+ = (A + BK)x + w$ as discussed in the lecture. In order to do so, compute the local feedback K and implement the constraint sets.

a) Determine the LQR controller K and the solution P to the corresponding discrete-time algebraic Riccati equation.
Hint: Use the MATLAB command `dlqr(A,B,Q,R)` to compute the discrete-time LQR controller. We define the local feedback as $u = Kx$, whereas the MATLAB function `dlqr` assumes $u = -Kx$. You can use the MATLAB command `dare` to solve the algebraic Riccati equation.

b) Implement the constraint sets $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{W}$. Plot the sets to verify your implementation.
Hint: For a polyhedron `X`, you can type `X.plot` to plot the set.

Next, we determine an outer approximation $\mathcal{S}$ of the mRPI set. Implement the algorithm from the lecture in the file `InvariantApprox_mRPIset_lec.m`.

c) Fix $\alpha \in (0,1)$ and $\kappa \in \mathbb{I} \geq 0$.

d) Check whether $A^\kappa \mathcal{W} \subseteq \alpha \mathcal{W}$ (equation (8) in the robust MPC slides) holds:

1. if yes, return $S(\alpha,\kappa) := (1-\alpha)^{-1} S_\kappa$ where $S_\kappa := \sum_{i=0}^{\kappa-1} A^i \mathcal{W}$
2. if not, set $\kappa = \kappa + 1$ and go try again.

Hint: Set inclusion '$\subseteq$' can be implemented in Matlab using `<=`.

e) Vary the values of α and κ What do you observe?

Problem 3 In order to implement the tube-based MPC algorithm from the lecture, we need to compute a terminal set. We use the function `maxInvSet.m`, an implementation of the algorithm provided in [1]. The function `maxInvSet.m` computes the maximal positively invariant polytope for a system $x^+ = (A + BK)x$, subject to $Hx \leq h$ such that all constraints are satisfied.

a) Determine the tightened state and input constraint sets

$$\mathbb{Z} = \mathcal{X} \ominus \mathcal{S} = \{x \in \mathbb{R}^2 | H_x x \leq h_x\}$$
$$\mathbb{V} = \mathcal{U} \ominus K\mathcal{S} = \{u \in \mathbb{R} | H_u u \leq h_u\}.$$

b) Extract the matrices H_x, h_x, H_u, and h_u.
Hint: Type `S.A` and `S.b` to get the matrices A and b that define the set $S = \{Ax \leq b\}$.

c) Compute the terminal set $\mathbb{Z}^f$ using the function `maxInvSet.m`.
Hint: `[Zf,Hf,Kf] = maxInvSet(A,H,h)` computes the required terminal set for a

system $x^+ = Ax$ subject to $Hx \leq h$. Note that, since $u = Kx$, the matrices H and h have to capture both the state and input constraints. The function returns the terminal set and corresponding matrices $Z_f = \{x \mid H_f x \leq K_f\}$.

Problem 4 Implement a tube-based MPC algorithm for the problem above with a prediction horizon of $N = 5$. Solve the optimal control problem with the Casadi Toolbox. Take the following considerations into account:

1. What are the optimization variables?
2. How can the initial condition for the nominal predictions $x(t) \in z(t|t) \oplus \mathcal{S}$ be implemented?
3. Which input $u(t)$ is applied to the system at each time step, after the optimal control problem is solved?

Implement the MPC scheme by following the steps below.

a) Setup the optimization problem by defining the optimization variables.
b) Implement the dynamic constraints.
c) Implement the given input and state constraints.
d) Implement the initial constraint.
e) Implement the terminal constraint.
f) Implement the cost function, the terminal cost, solve the optimization problem and extract the optimal open-loop trajectories.
g) Implement the input $u(t)$.
h) Provide a warm-start solution for the next time step.
i) Simulate the closed loop.

Problem 5 Replace your approximation of the mRPI set from `InvariantApprox_mRPIset_lec.m` with the MATLAB function `InvariantApprox_mRPIset_lit.m`, which also determines an invariant outer approximation of the mRPI set following the algorithm presented in [3]. You can call the function by setting `epsilon = 0.1` and typing

`S = InvariantApprox_mRPIset_lit(A_K,W,epsilon)`.

Hint: The computation with `epsilon = 0.1` may take some time. You can decrease the computation time by increasing `epsilon`, at the cost of increasing the size of the approximation of the mRPI set.

a) Compare your approximation of the mRPI set from Problem 2 with the approximation obtained from `InvariantApprox_mRPIset_lit.m`. What do you observe?
b) Run the simulation of the tube-based MPC controller with the newly obtained approximation of the mRPI set and compare it to your results from Problem 4. What do you observe?

References

[1] Elmer G Gilbert and K Tin Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic control*, 36(9):1008–1020, 1991.

[2] David Q Mayne, María M Seron, and SV Raković. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2):219–224, 2005.

[3] Sasa V Rakovic, Eric C Kerrigan, Konstantinos I Kouramas, and David Q Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on automatic control*, 50(3):406–410, 2005.