EECI Course on

Nonlinear and Data-based Model Predictive Control

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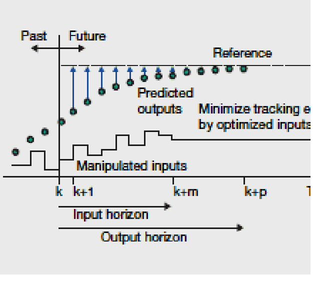


Structure of Course

MONDAY:
introduction, background, nominal stability (Part 1)

TUESDAY:
exercise, nominally stable MPC (Part 2), exercise,
nominally stable MPC (Part 3)

WEDNESDAY:
robust MPC, exercise, economic MPC

THURSDAY:
tracking MPC, data-based MPC; selected advanced topics

Nominal Stability of the Closed Loop

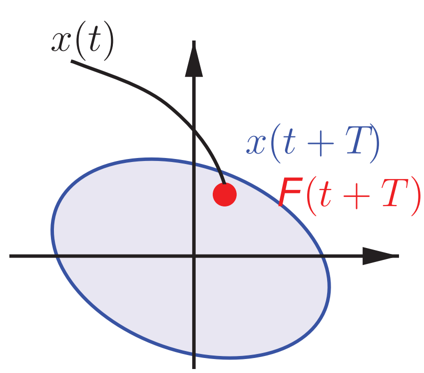
QUASI-INFINITE HORIZON NMPC

Expanding the Horizon Quasi to Infinity

Can the computational demand of ZTC NMPC and infinite horizon NMPC be avoided without jeopardizing stability?

Idea:

- **Approximate infinite horizon cost** inside of terminal region $\mathcal{X}^f$ via **terminal penalty term** $F(x(t+T))$ that correponds to **virtual control law** that
  - stabilizes system in $\mathcal{X}^f$
  - renders $\mathcal{X}^f$ invariant
  - achieves certain decrease
- **Enforce** last predicted state to lie in terminal region



Expanding the Horizon Quasi to Infinity – Setup

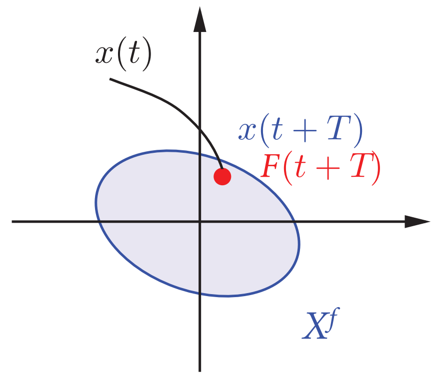
Modify setup via suitably computed

- terminal region constraint $x(t+T) \in X^f$
- terminal penalty term $F(x(t+T))$

$$\min_u J(x(t),u)$$

$$J(\cdot)=\int_t^{t+T} L\,(x(\tau),u(\tau))d\tau+F(x(t+T))$$

subject to:	$\dot{x} = f(x,u),$	system dynamics
	$x(t)$ given	"state feedback"
	$u(\tau) \in \mathcal{U}$	input constraints
	$x(\tau) \in \mathcal{X}$	state constraints
	$x(t+T) \in X^f$	terminal constraint



Additional terms computed such that $F(x(t+T))$ approximates **infinite horizon cost** in terminal region

Generalized Guaranteed Stability Result

[Chen&Allgöwer '96], [Mayne et al. '00], [Fontes '00]

$$\min_u J(x(t),u)$$

with: $J(\cdot)=\int_t^{t+T} L\,(x(\tau),u(\tau))d\tau+F(x(t+T))$

and: $x(t+T) \in X^f$

Theorem (Nominal Stability): If

a) $F(\cdot)$ and X^f are determined s.t.:

$\forall x \in X^f\ \exists u \in \mathcal{U}$ with $\dfrac{\partial F}{\partial x} f(x,u) + L(x,u) < 0$

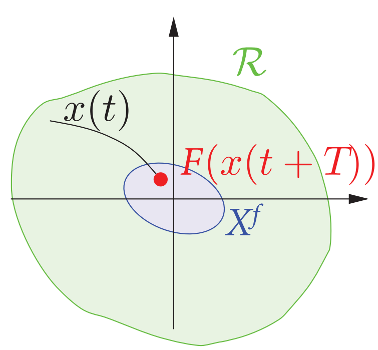
b) optimization feasible for $t=0$

$$\Downarrow$$

Asymptotic Stability

Guaranteed Region of Attraction:

Set $\mathcal{R}$ of states satisfying b)



Comments

- Many schemes fit into this setup:
  - Quasi-infinite horizon NMPC *[Chen&Allgöwer '97]*
  - Simulation-approximated infinite horizon NMPC *[De Nicolao et.al.'97]*
  - CLF approaches *[Jadbabaie et. al. '99, Primbs et.al. '00]*
  - Zero terminal constraint NMPC *[Keerthi&Gilbert'88], [Mayne&Michalska '90]*
  - ...
- Possible to use short horizon length without loss of performance and stability
  - Good performance can be expected if F approximates infinite horizon cost in X^f sufficiently well
  - Size of terminal region and prediction horizon length influence size of region of attraction
- Main differences between schemes:
  - Feasibility
  - Computational burden
  - Performance

Determining X^f, F - Quasi-infinite Horizon Approach

How does one determine X^f, F?

- Based on locally stabilizing controller
- Based on CLF
- Semidefinite programming + PLDI
- ...

Exemplary: Quasi-infinite horizon NMPC

- Jacobian linearization stabilizable
- quadratic cost functional $L(x,u) = x^T Q x + u^T R u$

$$\Rightarrow F(x) = x^T P x$$

- based on local controller $u = Kx$ that renders X^f invariant
- invariance property in $X^f \Rightarrow$ feasibility
- suitable upper bound of the infinite horizon cost
  $\Rightarrow$ decrease of value function

Procedure to Determine X^f, F
Quasi-infinite Horizon Approach

1. Choose Q and R for desired performance
2. Based on Jacobian linearization, obtain a **linear feedback** $u = Kx$ such that $A_K := A + BK$ is **asymptotically stable**.
3. Choose $\kappa < -\lambda_{max}(A_K)$ and solve Lyapunov eq.
   $(A_K + \kappa I)^T P + P(A_K + \kappa I) = -(Q + K^T RK)$
   to get a positive definite, symmetric P
4. Finde the largest possible $\alpha_1 \in (0, \infty)$ such that $u = Kx$ satisfies constraints in $\mathcal{E}_1 := \{x \in \mathbb{R}^n | x^T Px \leq \alpha_1\}$

5: Find the largest possible $\alpha \in (0, \alpha_1]$ such that

$$L_\phi \leq \frac{\kappa \cdot \lambda_{min}(P)}{\|P\|} \quad \text{in } X^f := \{x \in \mathbb{R}^n | x^T Px \leq \text{\textbackslash alpha}\}$$

$$L_\phi := \sup \left\{ \frac{\|f(x, Kx) - A_K x\|}{\|x\|} \middle| \; x \in X^f \; x \neq 0 \right\}$$

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Clarification Upper Bounding of Infinite Cost

see also: *[Chen&Allgöwer '97]*

Goal: minimize
$$J^\infty(x(t), u) = \int_t^{\infty} \left(x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\right) d\tau$$

Idea:

$$\min_u J^\infty(x(t), u) = \min_u \left\{ \int_t^{t+T} (x^T Qx + u^T Ru) \, d\tau + \int_{t+T}^{\infty} (x^T Qx + u^T Ru) \, d\tau \right\}$$

if $x \in X^f \quad \forall \tau \in [t + T, \infty)$, then

$$\min_u J^\infty(x(t), u) \leq \min_u \left\{ \int_t^{t+T} (x^T Qx + u^T Ru) \, d\tau + \int_{t+T}^{\infty} (x^T Qx + x^T K^T RKx) \, d\tau \right\}$$

if F, X^f are chosen according to procedure, then

$$\int_{t+T}^{\infty} (x^T Qx + x^T K^T RKx) \, d\tau \leq F(x(t + T_P)), \forall x \in X^f$$

$$\Rightarrow \quad \min_u J^\infty(x(t), u) \overset{\approx}{\leq} \min_u J(x(t), u; T)$$

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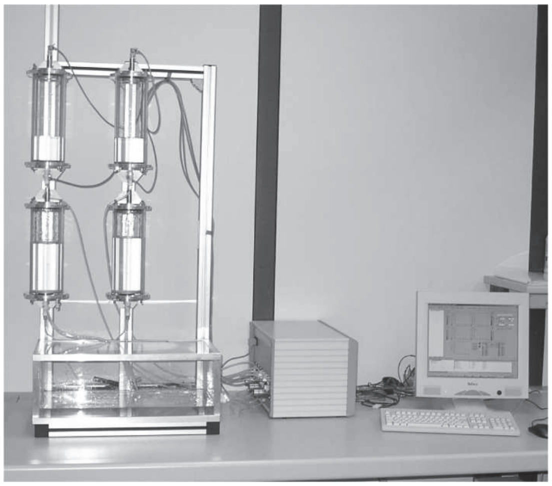
QIH/NPC for Four Tank System



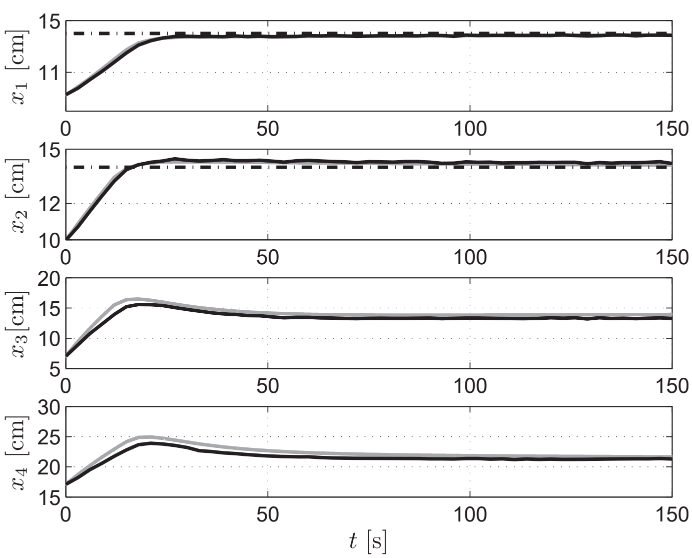
Four tank system of the IST laboratory

Four tank system:

- Nonlinear system
- Input constraints
- State constraints
- "Slow" dynamics

Four tank system is suited for an experimental stability study of NMPC

Simulation and Experiment (4)



- Sim. —, Exp. —
- Setpoint:
  - $x_{1,2s} = 14cm$
  - $\gamma_{1,2} = 0.4$
- Prediction horizon:
  - $T = 60s$
- Stability constraints:
  - **QIH constraints**
- Input constraints
- State constraints

Closed-loop is asymptotically stable with **quasi infinite horizon NMPC**

Performance/Feasibility of QIH NMPC is better than of ZTSC NMPC

Discussion Nominal Stability

- Many NMPC schemes that achieve guaranteed stability exist
- Often employ terminal penalty term + terminal region constraint
- Possible to consider small horizon lengths without loss of stability/performance
- Value function is not necessarily continuous

  $\Rightarrow$ Possibly no inherent robustness to small disturbances

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- Optimality not necessary for stability, sufficient if cost function decreases

  Suboptimal NMPC: feasibility implies stability *[Scokaert et al. '99]*

  - Can break optimization once feasible decreasing solution found
  - For many approaches remaining input guarantees decrease

    $\Rightarrow$ **fallback strategy**

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Discussion Nominal Stability II

- Only minor modifications necessary for stabilization of systems that require discontinuous feedbacks, e.g. nonholonomic systems *[Fontes '03]*
- Possible to consider special input parameterizations/quantizations

  Stability results applicable, however conditions difficult to check
  $\Rightarrow$ Question of feasibility/controllability under quantized control