An Introduction to Nonlinear Model Predictive Control

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Abstract

While linear model predictive control is popular since the 70s of the past century, the 90s have witnessed a steadily increasing attention from control theoretists as well as control practitioners in the area of nonlinear model predictive control (NMPC). The practical interest is driven by the fact that today's processes need to be operated under tighter performance specifications. At the same time more and more constraints, stemming for example from environmental and safety considerations, need to be satisfied. Often these demands can only be met when process nonlinearities and constraints are explicitly considered in the controller. Nonlinear predictive control, the extension of well established linear predictive control to the nonlinear world, appears to be a well suited approach for this kind of problems. In this note the basic principle of NMPC is reviewed, the key advantages/disadvantages of NMPC are outlined and some of the theoretical, computational, and implementational aspects of NMPC are discussed. Furthermore, some of the currently open questions in the area of NMPC are outlined.

1 Principles, Mathematical Formulation and Properties of Nonlinear Model Predictive Control

Model predictive control (MPC), also referred to as moving horizon control or receding horizon control, has become an attractive feedback strategy, especially for linear processes. Linear MPC refers to a family of MPC schemes in which linear models are used to predict the system dynamics, even though the dynamics of the closed-loop system is nonlinear due to the presence of constraints. Linear MPC approaches have found successful applications, especially in the process industries. A good overview of industrial linear MPC techniques can be found in [64, 65], where more than 2200 applications in a very wide range from chemicals to aerospace industries are summarized. By now, linear MPC theory is quite mature. Important issues such as online computation, the interplay between modeling/identification and control and system theoretic issues like stability are well addressed [41, 52, 58].

Many systems are, however, in general inherently nonlinear. This, together with higher product quality specifications and increasing productivity demands, tighter environmental regulations and demanding economical considerations in the process industry require to operate systems closer to the boundary of the admissible operating region. In these cases, linear models are often inadequate to describe the process dynamics and nonlinear models have to be used. This motivates the use of nonlinear model predictive control.

This paper focuses on the application of model predictive control techniques to nonlinear systems. It provides a review of the main principles underlying NMPC and outlines the key advantages/disadvantages of NMPC and some of the theoretical, computational, and implementational aspects. Note, however, that it is not intended as a complete review of existing NMPC techniques. Instead we refer to the following list for some excellent reviews [4, 16, 22, 52, 58, 68].

In Section 1.1 and Section 1.2 the basic underlying concept of NMPC is introduced. In Section 2 some of the system theoretical aspects of NMPC are presented. After an outline of NMPC schemes that achieve stability one particular NMPC formulation, namely quasi-infinite horizon NMPC (QIH-NMPC) is outlined to exemplify the basic ideas to achieve stability. This approach allows a (computationally) efficient formulation of NMPC while guaranteeing stability and performance of the closed-loop.

Besides the basic question of the stability of the closed-loop, questions such as robust formulations of NMPC and some remarks on the performance of the closed-loop are given in Section 2.3 and Section 2.2. Section 2.4 gives some remarks on the output-feedback problem in connection with NMPC. After a short review of existing approaches one specific scheme to achieve output-feedback NMPC using high-gain observers for state recovery is outlined. Section 3 contains some remarks and descriptions concerning the numerical solution of the open-loop optimal control problem. The applicability of NMPC to real processes is shown in Section 4 considering the control of a high purity distillation column. This shows, that using well suited optimization strategies together with the QIH-NMPC scheme allow real-time application of NMPC even with todays computing power. Final conclusions and remarks on future research directions are given in Section 5.

In the following, $\|\cdot\|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ (where the dimension n follows from context) or the associated induced matrix norm. Vectors are denoted by boldface symbols. Whenever a semicolon ";" occurs in a function argument, the following symbols should be viewed as additional parameters, i.e. $f(x;\gamma)$ means the value of the function f at x with the parameter γ.

1.1 The Principle of Nonlinear Model Predictive Control

In general, the model predictive control problem is formulated as solving on-line a finite horizon open-loop optimal control problem subject to system dynamics and constraints involving states and controls. Figure 1 shows the basic principle of model predictive control. Based on measurements obtained at time t, the controller predicts the future

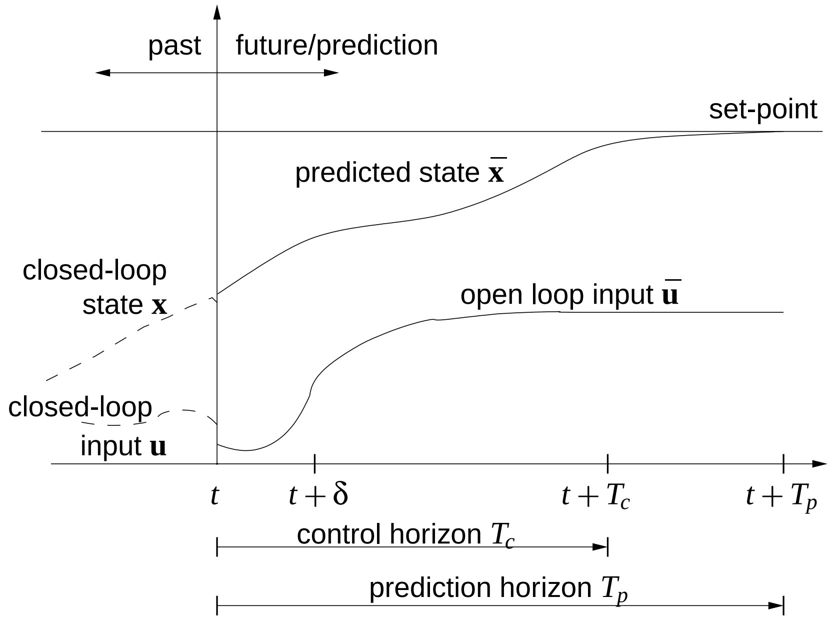

Figure 1: Principle of model predictive control.

dynamic behavior of the system over a prediction horizon T_p and determines (over a control horizon $T_c \leq T_p$) the input such that a predetermined open-loop performance objective functional is optimized. *If* there were no disturbances and no model-plant mismatch, and *if* the optimization problem could be solved for infinite horizons, then one could apply the input function found at time $t = 0$ to the system for all times $t \geq 0$. However, this is not possible in general. Due to disturbances and model-plant mismatch, the true system behavior is different from the predicted behavior. In order to incorporate some feedback mechanism, the open-loop manipulated input function obtained will be implemented only until the next measurement becomes available. The time difference between the recalculation/measurements can vary, however often it is assumed to be fixed, i.e the measurement will take place every δ sampling time-units. Using the new measurement at time $t+\delta$, the whole procedure – prediction and optimization – is repeated to find a new input function with the control and prediction horizons moving forward.

Notice, that in Figure 1 the input is depicted as arbitrary function of time. As shown in Section 3, for numerical solutions of the open-loop optimal control problem it is often necessary to parameterize the input in an appropriate way. This is normally done by using a finite number of basis functions, e.g. the input could be approximated as piecewise constant over the sampling time δ.

As will be shown, the calculation of the applied input based on the predicted system behavior allows the inclusion of constraints on states and inputs as well as the optimization of a given cost function. However, since in general the predicted system behavior will differ from the closed-loop one, precaution must be taken to achieve closed-loop stability.

1.2 Mathematical Formulation of NMPC

We consider the stabilization problem for a class of systems described by the following nonlinear set of differential equations[1]

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

subject to input and state constraints of the form:

$$\mathbf{u}(t) \in \mathcal{U}, \, \forall t \geq 0 \quad \mathbf{x}(t) \in \mathcal{X}, \, \forall t \geq 0, \tag{2}$$

where $\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ denotes the vector of states and inputs, respectively. The set of feasible input values is denoted by $\mathcal{U}$ and the set of feasible states is denoted by $\mathcal{X}$. We assume that $\mathcal{U}$ and $\mathcal{X}$ satisfy the following assumptions:

Assumption 1 *$\mathcal{U} \subset \mathbb{R}^p$ is compact, $\mathcal{X} \subseteq \mathbb{R}^n$ is connected and $(0,0) \in \mathcal{X} \times \mathcal{U}$.*

In its simplest form, $\mathcal{U}$ and $\mathcal{X}$ are given by box constraints of the form:

$$\mathcal{U} := \{\mathbf{u} \in \mathbb{R}^m | \mathbf{u}_{min} \leq \mathbf{u} \leq \mathbf{u}_{max}\}, \tag{3a}$$

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}_{min} \leq \mathbf{x} \leq \mathbf{x}_{max}\}. \tag{3b}$$

Here $\mathbf{u}_{min}$, $\mathbf{u}_{max}$ and $\mathbf{x}_{min}$, $\mathbf{x}_{max}$ are given constant vectors.
With respect to the system we additionally assume, that:

Assumption 2 *The vector field $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfies $\mathbf{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$. In addition, it is locally Lipschitz continuous in $\mathbf{x}$.*

Assumption 3 *The system* (1) *has an unique continuous solution for any initial condition in the region of interest and any piecewise continuous and right continuous input function $\mathbf{u}(\cdot) : [0, T_p] \to \mathcal{U}$.*

In order to distinguish clearly between the real system and the system model used to predict the future "within" the controller, we denote the internal variables in the controller by a bar (for example $\bar{\mathbf{x}}, \bar{\mathbf{u}}$).
Usually, the finite horizon open-loop optimal control problem described above is mathematically formulated as follows:
Problem 1 *Find*

$$\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_c, T_p)$$

with

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p, T_c) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)) d\tau \tag{4}$$

subject to:

$$\dot{\bar{\mathbf{x}}}(\tau) \; = \mathbf{f}(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t) \tag{5a}$$

$$\bar{\mathbf{u}}(\tau) \; \in \mathcal{U}, \quad \forall \tau \in [t, t+T_c] \tag{5b}$$

$$\bar{\mathbf{u}}(\tau) \; = \bar{\mathbf{u}}(\tau + T_c), \quad \forall \tau \in [t+T_c, t+T_p] \tag{5c}$$

$$\bar{\mathbf{x}}(\tau) \; \in \mathcal{X}, \quad \forall \tau \in [t, t+T_p] \tag{5d}$$

where T_p and T_c are the prediction and the control horizon with $T_c \leq T_p$.
The bar denotes internal controller variables and $\bar{\mathbf{x}}(\cdot)$ is the solution of (5a) driven by the input $\bar{\mathbf{u}}(\cdot) : [t, t+T_p] \to \mathcal{U}$ with initial condition $\mathbf{x}(t)$. The distinction between the real system and the variables in the controller is necessary, since the predicted values, even in the nominal undisturbed case, need not, and in generally will not, be the same as the actual closed-loop values, since the optimal input is recalculated (over a moving finite horizon T_c) at every sampling

[1] In this paper only the continuous time formulation of NMPC is considered. However, notice that most of the presented topics have dual counterparts in the discrete time setting.

instance.
The function F, in the following called stage cost, specifies the desired control performance that can arise, for example, from economical and ecological considerations. The standard quadratic form is the simplest and most often used one:

$$F(\mathbf{x},\mathbf{u}) = (\mathbf{x}-\mathbf{x}_s)^T Q(\mathbf{x}-\mathbf{x}_s) + (\mathbf{u}-\mathbf{u}_s)^T R(\mathbf{u}-\mathbf{u}_s), \tag{6}$$

where $\mathbf{x}_s$ and $\mathbf{u}_s$ denote given setpoints; Q and R denote positive definite, symmetric weighting matrices. In order for the desired reference $(\mathbf{x}_s,\mathbf{u}_s)$ to be a feasible solution of Problem 1, $\mathbf{u}_s$ should be contained in the interior of $\mathcal{U}$. As already stated in Assumption 2 we consider, without loss of generality that $(\mathbf{x}_s,\mathbf{u}_s) = (\mathbf{0},\mathbf{0})$ is the steady state that should be stabilized. Note the initial condition in (5a): The system model used to predict the future in the controller is initialized by the actual system state; thus they are assumed to be measured or must be estimated. Equation (5c) is not a constraint but implies that beyond the control horizon the predicted control takes a constant value equal to that at the last step of the control horizon.
In the following an optimal solution to the optimization problem (existence assumed) is denoted by $\bar{\mathbf{u}}^*(\cdot;\mathbf{x}(t), T_p, T_c) : [t, t+T_p] \to \mathcal{U}$. The open-loop optimal control problem will be solved repeatedly at the sampling instances $t = j\delta$, $j = 0,1,\cdots$, once new measurements are available . The *closed-loop* control is defined by the optimal solution of Problem 1 at the sampling instants:

$$\mathbf{u}^*(\tau) := \bar{\mathbf{u}}^*(\tau;\mathbf{x}(t),T_p,T_c)\ ,\ \tau\in[t,\delta]\,. \tag{7}$$

The optimal value of the NMPC open-loop optimal control problem as a function of the state will be denoted in the following as value function:

$$V(\mathbf{x};T_p,T_c) = J(\mathbf{x},\bar{\mathbf{u}}^\star(\cdot;\mathbf{x}(t));T_p,T_c). \tag{8}$$

The value function plays an important role in the proof of the stability of various NMPC schemes, as it serves as a Lyapunov function candidate.

1.3 Properties, Advantages, and Disadvantages of NMPC

In general one would like to use an infinite prediction and control horizon, i.e. T_p and T_c in Problem 1 are set to ∞. 5case) to minimize the performance objective determined by the cost However as mentioned, the open-loop optimal control Problem 1, that must be solved on-line, is often formulated in a finite horizon manner and the input function is parameterized finitely, in order to allow a (real-time) numerical solution of the nonlinear open-loop optimal control problem. It is clear, that the shorter the horizon, the less costly the solution of the on-line optimization problem. Thus it is desirable from a computational point of view to implement MPC schemes using short horizons. However, when a finite prediction horizon is used, the actual closed-loop input and state trajectories will differ from the predicted open-loop trajectories, even if no model plant mismatch and no disturbances are present [4]. This fact is depicted in Figure 2 where the system can only move inside the shaded area as state constraints of the form $\mathbf{x}(\tau)\in\mathcal{X}$ are assumed. This makes the key difference between standard control strategies, where the feedback law is obtained a priori and

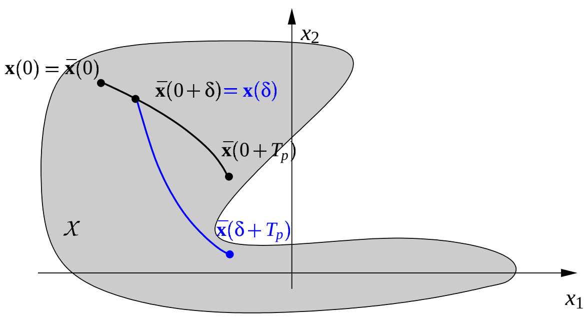

Figure 2: The difference between open-loop prediction and closed-loop behavior.

NMPC where the feedback law is obtained on-line and has two immediate consequences. Firstly, the actual goal to compute a feedback such that the performance objective over the *infinite horizon* of the closed loop is minimized is not achieved. In general it is by no means true that a repeated minimization over a *finite horizon objective* in a receding horizon manner leads to an optimal solution for the infinite horizon problem (with the same stage cost F) [10]. In fact, the two solutions differ significantly if a short horizon is chosen. Secondly, if the predicted and the actual trajectories

differ, there is no guarantee that the closed-loop system will be stable. It is indeed easy to construct examples for which the closed-loop becomes unstable if a (small) finite horizon is chosen. Hence, when using finite horizons in standard NMPC, the stage cost cannot be chosen simply based on the desired physical objectives.
The overall basic structure of a NMPC control loop is depicted in Figure 3. As can be seen, it is necessary to estimate

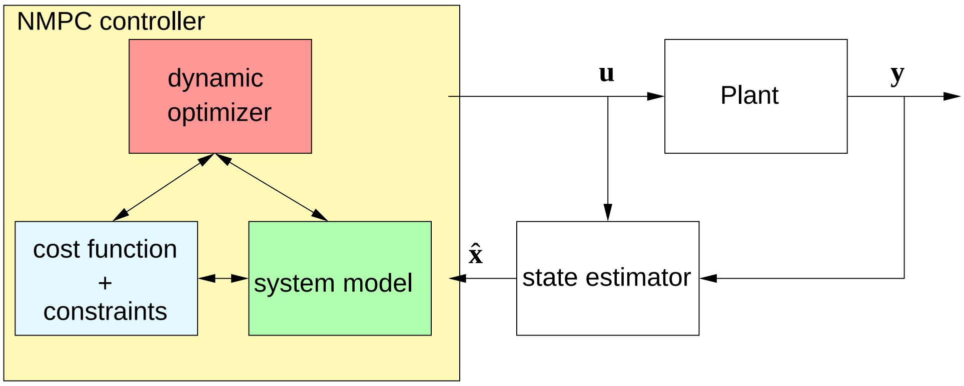

Figure 3: Basic NMPC control loop.

the system states from the output measurements.
Summarizing the basic NMPC scheme works as follows:

1. obtain measurements/estimates of the states of the system
2. compute an optimal input signal by **minimizing** a given **cost function** over a certain **prediction horizon** in the future using a **model of the system**
3. **implement the first part of the optimal input signal** until new measurements/estimates of the state are available
4. continue with 1.

From the remarks given so far and from the basic NMPC setup, one can extract the following key characteristics of NMPC:

- NMPC allows the use of a nonlinear model for prediction.
- NMPC allows the explicit consideration of state and input constraints.
- In NMPC a specified performance criteria is minimized on-line.
- In NMPC the predicted behavior is in general different from the closed loop behavior.
- The on-line solution of an open-loop optimal control problem is necessary for the application of NMPC.
- To perform the prediction the system states must be measured or estimated.

In the remaining sections various aspects of NMPC regarding these properties will be discussed. The next section focuses on system theoretical aspects of NMPC. Especially the questions on closed-loop stability, robustness and the output feedback problem are considered.

2 System Theoretical Aspects of NMPC

In this section different system theoretical aspects of NMPC are considered. Besides the question of nominal stability of the closed-loop, which can be considered as somehow mature today, remarks on robust NMPC strategies as well as the output-feedback problem are given.

2.1 Stability

One of the key questions in NMPC is certainly, whether a finite horizon NMPC strategy does lead to stability of the closed-loop. As pointed out, the key problem with a finite prediction and control horizon stems from the fact that the predicted open and the resulting closed-loop behavior is in general different. Ideally one would seek for a NMPC strategy that achieves closed-loop stability independent of the choice of the performance parameters in the cost functional and, if possible, approximates the infinite horizon NMPC scheme as good as possible. A NMPC strategy that achieves closed-loop stability independent of the choice of the performance parameters is usually referred to a NMPC approach *with guaranteed stability*. Different possibilities to achieve closed-loop stability for NMPC using finite horizon length have been proposed. After giving a short review about these approaches we exemplary present on specific approach that achieves guaranteed stabilit, the so called quasi-infinite horizon approach to NMPC (QIH-NMPC) . This approach achieves guaranteed closed loop stability while being computationally feasible.
Here only the key ideas are reviewed and no detailed proofs are given. Furthermore notice, that we will not cover all existing NMPC approaches, instead we refer the reader to the overview papers [4, 22, 52].
For all the following sections it is assumed that the prediction horizon is set equal to the control horizon, $T_p = T_c$.

2.1.1 Infinite Horizon NMPC

The most intuitive way to achieve stability is the use of an infinite horizon cost [10, 39, 54], i.e. T_p in Problem 1 is set to ∞. In the nominal case feasibility at one sampling instance also implies feasibility and optimality at the next sampling instance. This follows from Bellman‘s Principle of Optimality [7], i.e. the input and state trajectories computed as the solution of the NMPC optimization Problem 1 at a specific instance in time, are in fact equal to the closed-loop trajectories of the nonlinear system, i.e. the remaining parts of the trajectories after one sampling instance are the optimal solution at the next sampling instance. This fact also implies closed-loop stability.
Key ideas of the stability proof: Since nearly all stability proofs for NMPC follow along the same basic steps as for the infinite horizon proof, the key ideas are shortly outlined. In principle the proof is based on the use of the value function as a Lyapunov function. First it is shown, that feasibility at one sampling instance does imply feasibility at the next sampling instance for the nominal case. In a second step it is established that the value function is strictly decreasing and by this the state and input converge to the origin. Utilizing the continuity of the value function at the origin and the monotonicity property, asymptotic stability is established in the third step. As feasibility thus implies asymptotic stability, the set of all states, for which the open-loop optimal control problem has a solution does belong to the region of attraction of the origin.

2.1.2 Finite Horizon NMPC Schemes with Guaranteed Stability

Different possibilities to achieve closed-loop stability for NMPC using a finite horizon length have been proposed, see for example [3, 17, 20, 23, 34, 35, 38, 39, 44, 51, 53, 55, 56, 60, 62, 63, 70]. Most of these approaches modify the NMPC setup such that stability of the closed-loop can be guaranteed independently of the plant and performance specifications. This is usually achieved by adding suitable equality or inequality constraints and suitable additional penalty terms to the cost functional. These additional constraints are usually not motivated by physical restrictions or desired performance requirements but have the sole purpose to enforce stability of the closed-loop. Therefore, they are usually termed *stability constraints* [49, 50, 52].
The simplest possibility to enforce stability with a finite prediction horizon is to add a so called *zero terminal equality constraint* at the end of the prediction horizon [39, 51, 53], i.e. to add the equality constraint

$$\bar{\mathbf{x}}(t+T_p; \mathbf{x}(t), t, \bar{\mathbf{u}}) = \mathbf{0} \tag{9}$$

to Problem 1. This leads to stability of the closed-loop, if the optimal control problem possesses a solution at $t = 0$, since the feasibility at one time instance does also lead to feasibility at the following time instances and a decrease in the value function. One disadvantage of a zero terminal constraint is that the system must be brought to the origin in finite time. This leads in general to feasibility problems for short prediction/control horizon lengths, i.e. a small region of attraction. Additionally, from a computational point of view, an exact satisfaction of a zero terminal equality constraint does require an infinite number of iterations in the nonlinear programming problem [17]. On the other hand, the main advantages are the straightforward application and the conceptual simplicity.

Many schemes have been proposed (i.e. [17, 20, 34, 38, 51, 56, 60, 63]), that try to overcome the use of a zero terminal constraint of the form (9). Most of them either use a so called *terminal region constraint*

$$\bar{\mathbf{x}}(t+T_p) \in \Omega \subseteq \mathcal{X} \tag{10}$$

and/or a *terminal penalty term* $E(\bar{\mathbf{x}}(t+T_p))$ which is added to the cost functional:

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) \quad = \quad \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau))\, d\tau + E(\bar{\mathbf{x}}(t+T_p)). \tag{11}$$

Note that the terminal penalty term is not a performance specification that can be chosen freely. Rather E and the terminal region Ω in (10) are determined off-line such that stability is "enforced". We do not review all these methods here. Instead we exemplify the basic idea considering one specific approach, the so called quasi-infinite horizon NMPC approach [17].

2.1.3 Quasi-Infinite Horizon NMPC

In the quasi-infinite horizon NMPC method [15, 17] a terminal region constraint of the form (10) and a *terminal penalty term* $E(\bar{\mathbf{x}}(t+T_p))$ as in (11) are added to the standard setup. As mentioned the terminal penalty term is not a performance specification that can be chosen freely. Rather E and the terminal region Ω are determined off-line such that the cost functional *with* terminal penalty term (11) gives an upper approximation of the infinite horizon cost functional with stage cost F. Thus closed-loop performance over the infinite horizon is addressed. Furthermore, as is shown later, stability is achieved, while only an optimization problem over a finite horizon must be solved. The resulting open-loop optimization problem is formulated as follows:

Problem 2 [Quasi-infinite Horizon NMPC]:
Find

$$\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) \tag{12}$$

with:

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)) d\tau + E(\bar{\mathbf{x}}(t+T_p)). \tag{13}$$

subject to:

$$\dot{\bar{\mathbf{x}}}(\tau) = \mathbf{f}(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t) \tag{14a}$$

$$\bar{\mathbf{u}}(\tau) \in \mathcal{U}, \quad \forall \tau \in [t, t+T_p] \tag{14b}$$

$$\bar{\mathbf{x}}(\tau) \in \mathcal{X}, \quad \forall \tau \in [t, t+T_p] \tag{14c}$$

$$\bar{\mathbf{x}}(t+T_p) \in \Omega. \tag{14d}$$

If the terminal penalty term E and the terminal region Ω are chosen suitably, stability of the closed-loop can be guaranteed. To present the stability results we need that the following holds for the stage cost-function.

Assumption 4 *The stage cost* $F : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ *is continuous in all arguments with* $F(\mathbf{0}, \mathbf{0}) = 0$ *and* $F(\mathbf{x}, \mathbf{u}) > 0 \ \forall (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^n \times \mathcal{U} \setminus \{\mathbf{0}, \mathbf{0}\}$.

Given this assumption, the following result, which is a slight modification of Theorem 4.1 in [14], can be established:

Theorem 1 *Suppose*

(a) that Assumptions 1-4 are satisfied,

(b) E *is* C^1 *with* $E(\mathbf{0}, \mathbf{0}) = 0$, $\Omega \subseteq \mathcal{X}$ *is closed and connected with the origin contained in* Ω *and there exists a continuous local control law* $\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m$ *with* $\mathbf{k}(\mathbf{0}) = \mathbf{0}$*, such that:*

$$\frac{\partial E}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{k}(\mathbf{x})) + F(\mathbf{x}, \mathbf{k}(\mathbf{x})) \leq 0, \quad \forall \mathbf{x} \in \Omega \tag{15}$$

with $\mathbf{k}(\mathbf{x}) \in \mathcal{U} \ \forall \mathbf{x} \in \Omega$

(c) the NMPC open-loop optimal control problem has a feasible solution for $t = 0$.

Then for any sampling time $0 < \delta < T_p$ the nominal closed-loop system is asymptotically stable with the region of attraction $\mathcal{R}$ being the set of states for which the open-loop optimal control problem has a feasible solution.

A formal proof of Theorem 1 can be found in [14, 16] and for a linear local controller as described below in [17].
Loosely speaking E is a local Lyapunov function of the system under the local control $\mathbf{k}(\mathbf{x})$ in Ω. As will be shown, Equation (15) allows to upper bound the optimal infinite horizon cost inside Ω by the cost resulting from a local feedback $\mathbf{k}(\mathbf{x})$.
Notice, that the result in Theorem 1 is nonlocal in nature, i.e. their exists a region of attraction $\mathcal{R}$ of at least the size of Ω. The region of attraction is given by all states for which the open-loop optimal control problem has a feasible solution.
Obtaining a terminal penalty term E and a terminal region Ω that satisfy the conditions of Theorem 1 is not easy. If the linearized system is stabilizable and the cost function is quadratic with weight matrices Q and R, a locally *linear* feedback law $\mathbf{u} = K\mathbf{x}$ can be used and the terminal penalty term can be approximated as quadratic of the form $E(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$. For this case, a procedure to systematically compute the terminal region and a terminal penalty matrix *off-line* is available [17]. Assuming that the Jacobian linearization (A, B) of (1) is stabilizable, where $A := \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{0},\mathbf{0})$ and $B := \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{0},\mathbf{0})$, this procedure can be summarized as follows:

Step 1 : Solve the linear control problem based on the Jacobian linearization (A, B) of (1) to obtain a locally stabilizing linear state feedback $\mathbf{u} = K\mathbf{x}$.

Step 2 : Choose a constant $\kappa \in [0, \infty)$ satisfying $\kappa < -\lambda_{max}(A_K)$ and solve the Lyapunov equation

$$(A_K + \kappa I)^T P + P(A_K + \kappa I) = -\left(Q + K^T RK\right) \tag{16}$$

to get a positive definite and symmetric P, where $A_K := A + BK$.

Step 3 : Find the largest possible α_1 defining a region

$$\Omega_1 := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x}^T P\mathbf{x} \leq \alpha_1\} \tag{17}$$

such that $K\mathbf{x} \in \mathcal{U}$, for all $\mathbf{x} \in \Omega_1 \subseteq \mathcal{X}$.

Step 4 : Find the largest possible $\alpha \in (0, \alpha_1]$ specifying a terminal region

$$\Omega := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x}^T P\mathbf{x} \leq \alpha\} \tag{18}$$

such that the optimal value of the following optimization problem is non-positive:

$$\max_{\mathbf{x}} \{\mathbf{x}^T P\varphi(\mathbf{x}) - \kappa \cdot \mathbf{x}^T P\mathbf{x} \,|\, \mathbf{x}^T P\mathbf{x} \leq \alpha\} \tag{19}$$

where $\varphi(\mathbf{x}) := \mathbf{f}(\mathbf{x}, K\mathbf{x}) - A_K\mathbf{x}$.

This procedure allows to calculate E and Ω if the linearization of the system at the origin is stabilizable. If the terminal penalty term and the terminal region are determined according to Theorem 1, the open-loop optimal trajectories found at each time instant approximate the optimal solution for the *infinite* horizon problem.
The following reasoning make this plausible: Consider an infinite horizon cost functional defined by

$$J^\infty(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) := \int_t^\infty F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau))\, d\tau \tag{20}$$

with $\bar{\mathbf{u}}(\cdot)$ on $[t, \infty)$. This cost functional can be split up into two parts

$$\min_{\bar{\mathbf{u}}(\cdot)} J^\infty(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) = \min_{\bar{\mathbf{u}}(\cdot)} \left(\int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau))\, d\tau + \int_{t+T_p}^\infty F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau))\, d\tau \right). \tag{21}$$

The goal is to upper approximate the second term by a terminal penalty term $E(\bar{\mathbf{x}}(t + T_p)$. Without further restrictions, this is not possible for general nonlinear systems. However, if we ensure that the trajectories of the closed-loop system remain within some neighborhood of the origin (terminal region) for the time interval $[t + T_p, \infty)$, then an upper bound on the second term can be found. One possibility is to determine the terminal region Ω such that a local state feedback law $\mathbf{u} = \mathbf{k}(\mathbf{x})$ asymptotically stabilizes the nonlinear system and renders Ω positively invariant for the closed-loop. If an additional terminal inequality constraint $\mathbf{x}(t+T_p) \in \Omega$ (see (14d)) is added to Problem 1, then the second term of equation (21) can be upper bounded by the cost resulting from the application of this local controller $\mathbf{u} = \mathbf{k}(\mathbf{x})$. Note that the predicted state will not leave Ω after $t+T_p$ since $\mathbf{u} = \mathbf{k}(\mathbf{x})$ renders Ω positively invariant. Furthermore the feasibility at the next sampling instance is guaranteed dismissing the first part of $\bar{\mathbf{u}}$ and replacing it by the nominal open-loop input resulting from the local controller. Requiring that $\mathbf{x}(t+T_p) \in \Omega$ and using the local controller for $\tau \in [t+T_p, \infty)$ we obtain:

$$\min_{\bar{\mathbf{u}}(\cdot)} J^{\infty}(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) \leq \min_{\bar{\mathbf{u}}(\cdot)} \left(\int_t^{t+T_p} F\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right) d\tau + \int_{t+T_p}^{\infty} F\left(\bar{\mathbf{x}}(\tau), \mathbf{k}\left(\bar{\mathbf{x}}(\tau)\right)\right) d\tau \right). \tag{22}$$

If, furthermore, the terminal region Ω and the terminal penalty term are chosen according to condition b) in Theorem 1 (as for example achieved by the procedure given above), integrating (15) leads to

$$\int_{t+T_p}^{\infty} F\left(\bar{\mathbf{x}}(\tau), \mathbf{k}\left(\bar{\mathbf{x}}(\tau)\right)\right) d\tau \leq E\left(\bar{\mathbf{x}}(t+T_p)\right). \tag{23}$$

Substituting (23) into (22) we obtain

$$\min_{\bar{\mathbf{u}}(\cdot)} J^{\infty}\left(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)\right) \leq \min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); t+T_p). \tag{24}$$

This implies that the optimal value of the finite horizon problem bounds that of the corresponding infinite horizon problem. Thus, the prediction horizon can be thought of as extending *quasi* to infinity which gives this approach its name. Equation (24) can be exploited to prove Theorem 1.

Like in the dual-mode approach [56], the use of the terminal inequality constraint gives the quasi-infinite horizon nonlinear MPC scheme computational advantages. Note also, that as for dual-mode NMPC, it is not necessary to find optimal solutions of Problem 1 in order to guarantee stability. Feasibility also implies stability here [17, 70]. In difference to the dual-mode controller, however, the local control law $\mathbf{u} = \mathbf{k}(\mathbf{x})$ is never applied. It is only used to compute the terminal penalty term E and the terminal region Ω.

Many generalizations and expansions of QIH-NMPC exist. For example discrete time variants can be found in [21, 33]. If the nonlinear system is affine in $\mathbf{u}$ and feedback linearizable, then a terminal penalty term can be determined such that (23) is exactly satisfied with equality [14], i.e. the infinite horizon is recovered exactly. In [18, 19, 44] robust NMPC schemes using a min-max formulation are proposed, while in [27] an extension to index one DAE systems is considered. A variation of QIH-NMPC for the control of varying setpoints is given in [28, 30].

2.2 Performance of Finite Horizon NMPC Formulations

Ideally one would like to use an infinite horizon NMPC formulation, since in the nominal case, the closed-loop trajectories do coincide with the open-loop predicted ones (principle of optimality). The main problem is, that infinite horizon schemes can often not be applied in practice, since the open-loop optimal control problem cannot be solved sufficiently fast. Using finite horizons, however, it is by no means true that a repeated minimization over a *finite horizon objective* in a receding horizon manner leads to an optimal solution for the infinite horizon problem (with the same stage cost F). In fact, the two solutions will in general differ significantly if a short horizon is chosen.

From this discussion it is clear that short horizons are desirable from a computational point of view, but long horizons are required for closed-loop stability and in order to achieve the desired performance.

The QIH-NMPC strategy outlined in the previous section allows in principle to recover the performance of the infinite horizon scheme without jeopardizing the closed-loop stability. The value function resulting from Problem 2 can be seen as an upper bound of the infinite horizon cost. To be more precise, if the terminal penalty function E is chosen such that a corresponding local control law is a good approximation of the control resulting from the infinite horizon control law in a neighborhood of the origin, the performance corresponding to Problem 2 can recover the performance of the infinite horizon cost even for short horizons (assuming the terminal region constraint can be satisfied).

2.3 Robustness

So far only the nominal control problem was considered. The NMPC schemes discussed before do require that the actual system is identical to the model used for prediction, i.e. that no model/plant mismatch or unknown disturbances are present. Clearly this is a very unrealistic assumption for practical applications and the development of a NMPC framework to address robustness issues is of paramount importance. In this note the nonlinear uncertain system is assumed to be given by:

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t)) \tag{25}$$

where the uncertainty $\mathbf{d}(\cdot)$ satisfies $\mathbf{d}(\tau) \in \mathcal{D}(x,u)$ and $\mathcal{D}$ is assumed to be compact. Like in the nominal stability and performance case, the resulting difference between the predicted open-loop and actual closed-loop trajectory is the main obstacle. As additional problem the uncertainty $\mathbf{d}$ hitting the system now leads not only to one single future trajectory in the prediction, instead a whole tree of possible solutions must be analyzed.
Even though the analysis of robustness properties in *nonlinear* NMPC must still be considered as an unsolved problem in general, some preliminary results are available. In principle one must distinguish between two approaches to consider the robustness question. Firstly one can examine the robustness properties of the NMPC schemes designed for nominal stability and by this take the uncertainty/disturbances only indirectly into account [40, 47]. Secondly one can consider to design NMPC schemes that directly take into account the uncertainty/disturbances.

2.3.1 Inherent Robustness of NMPC

As mentioned above, inherent robustness corresponds to the fact, that nominal NMPC can cope with input model uncertainties without taking them directly into account. This fact stems from the close relation of NMPC to optimal control. Assuming that the system under consideration is of the following (input affine) form

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t))\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

and the cost function takes the form:

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \int_t^{t+T_p} \frac{1}{2}||\mathbf{u}||^2 + \mathbf{q}(\mathbf{x})d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{27}$$

where $\mathbf{q}$ is positive definite, that there are no constraints on the state and the input and the resulting control law and the value function satisfies further assumptions ($\mathbf{u}^\star$ being continuously differentiable and the value function being twice continuously differentiable). Then one can show [47] that the NMPC control law is inverse optimal, i.e. it is also optimal for a modified optimal control problem spanning over an infinite horizon. Due to this inverse optimality, the NMPC control law inherits the same robustness properties as infinite horizon optimal control assuming that the sampling time δ goes to zero. In particular, the closed-loop is robust with respect to sector bounded input uncertainties; the nominal NMPC controller also stabilizes systems of the form:

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t))\phi(\mathbf{u}(t)), \tag{28}$$

where $\phi(\cdot)$ is a nonlinearity in the sector $(1/2, \infty)$.

2.3.2 Robust NMPC Schemes

At least three different robust NMPC formulations exist:

- **Robust NMPC solving an open-loop min-max problem [18, 45]**:
  In this formulation the standard NMPC setup is kept, however now the cost function optimized is given by the worst case disturbance "sequence" occurring, i.e.

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \max_{\mathbf{d}(\cdot)} \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau))d\tau + E(\bar{\mathbf{x}}(t+T_p)). \tag{29}$$

  subject to

$$\dot{\bar{\mathbf{x}}}(t) \quad = \quad \mathbf{f}\left(\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t), \bar{\mathbf{d}}(t)\right), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t). \tag{30}$$

The resulting open-loop optimization is a min-max problem. The key problem is, that adding stability constraints like in the nominal case, might lead to the fact that no feasible solution can be found at all. This mainly stems from the fact, that one input signal must "reject" all possible disturbances and guarantee the satisfaction of the stability constraints.

- **H_∞-NMPC [11, 18, 45, 46]:** Another possibility is to consider the standard H_∞ problem in a receding horizon framework. The key obstacle is, that an infinite horizon min-max problem must be solved (solution of the nonlinear Hamilton-Jacobi-Isaacs equation). Modifying the NMPC cost functions similar to the H_∞ problem and optimizing over a sequence of control laws robustly stabilizing finite horizon H_∞–NMPC formulations can be achieved. The main obstacle is the prohibitive computational time necessary. This approach is in close connection to the first approach.
- **Robust NMPC optimizing a feedback controller used during the sampling times [45]:**
The open-loop formulation of the robust stabilization problem can be seen as very conservative, since only open-loop control is used during the sampling times, i.e. the disturbances are not directly rejected in between the sampling instances. To overcome this problem it has been proposed not to optimize over the input signal. Instead of optimizing the open-loop input signal directly, a feedback controller is optimized, i.e. the decision variable $\bar{\mathbf{u}}$ is not considered as optimization variable instead a "sequence" of control laws $\mathbf{u}_i = \mathbf{k}_i(\mathbf{x})$ applied during the sampling times is optimized. Now the optimization problem has as optimization variables the parameterizations of the feedback controllers $\{\mathbf{k}_1, \ldots, \mathbf{k}_N\}$. While this formulation is very attractive since the conservatism is reduced, the solution is often prohibitively complex.

2.4 Output Feedback NMPC

So far it was assumed, that the system state necessary for prediction is (perfectly) accessible through measurements. In general this is not the case and a state observer, as already shown in Figure 3 must be implicitly or explicitly included in the control loop. Two main questions arise from the use of a state observer. Firstly the question occurs, if the closed-loop including the state observer possesses the same stability properties as the state feedback contribution alone. Secondly the question arises, what kind of observer should be used to obtain a good state estimate and good closed loop performance. The second point is not considered in detail here. It is only noted, that a dual of the NMPC approach for control does exist for the state estimation problem. It is formulated as an on-line optimization similar to NMPC and is named moving horizon estimation (MHE). It is dual in the sense, that a moving window of old measurement data is used to obtain an optimization based estimate of the system state, see for example [1, 57, 66, 67, 69, 75].

2.4.1 Possible Solutions to the Output Feedback NMPC Problem

The most often used approach for output-feedback NMPC is based on the "certainty equivalence principle". The estimate state $\hat{\mathbf{x}}$ is measured via a state observer and used in the model predictive controller. Even assuming, that the observer error is exponentially stable, often only local stability of the closed-loop is achieved [42, 43, 71], i.e. the observer error must be small to guarantee stability of the closed-loop and in general nothing can be said about the necessary degree of smallness. This is a consequence of the fact that no general valid separation principle for nonlinear systems exists. Nevertheless this approach is applied successfully in many applications.
To achieve non-local stability results of the observer based output-feedback NMPC controller, different possibilities to attack the problem exist:

- **Direct consideration of the observer error in the NMPC controller**: One could in principle consider the observer error as disturbance in the controller and design a NMPC controller that can reject this disturbance. The hurdle of this approach is the fact, that so far an applicable robust NMPC scheme is not available.
- **Separation of observer error from the controller [31, 37, 57]:** In this approach the observer error is "decoupled"/separated from the controller by either a time scale separation, i.e. making the observer much faster than the other system parts or by projection of the observer error. For example using special separation principles based on high-gain observers, semi-regional stability results for the closed-loop can be established. The key component is, that the speed of the observer can be made as fast as necessary.
- **Usage of I/O models [65]:** One could use suited I/O models that have no internal states for prediction.

In the following we shortly review one possible approach for output-feedback NMPC using a time-scale separation of the observer and controller.

2.4.2 Output Feedback NMPC using High-Gain Observers

We propose to combine high-gain observers with NMPC to achieve semi-regional stability. I.e. if the observer gain is increased sufficiently, the stability region and performance of the state feedback is recovered. The closed loop system is semi-regionally stable in the sense, that for any subset $\mathcal{S}$ of the region of attraction $\mathcal{R}$ of the state-feedback controller (compare Theorem 1, Section 2.1.3) there exists an observer parameter (gain), such that $\mathcal{S}$ is contained in the region of attraction of the output-feedback controller.
The results are based on "nonlinear separation principles [6, 72]" and it is assumed, that the NMPC feedback is instantaneous (see below). We will limit the presentation to a special SISO systems class and only give the main result. The more general MIMO case considering the NMPC inherent open loop control parts (i.e. no instantaneous feedback) can be found [31, 37].
In the following we consider the stabilization of SISO systems of the following form:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}\phi(\mathbf{x}, u) \tag{31a}$$

$$y = x_1. \tag{31b}$$

with $u(t) \in \mathcal{U} \subset \mathbb{R}$ and $y(t) \in \mathbb{R}$. The output y is given by the first state x_1. The $n \times n$ matrix A and the $n \times 1$ vector $\mathbf{b}$ have the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{n\times n}, \qquad \mathbf{b} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T_{n\times 1}, \tag{32a}$$

Additional to the Assumptions 1-4 we assume, that:

Assumption 5 *The function $\phi : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ is locally Lipschitz in its arguments over the domain of interest. Furthermore $\phi(\mathbf{0}, 0) = 0$ and ϕ is bounded in $\mathbf{x}$ everywhere.*

Note that global boundedness can in most cases be achieved by saturating ϕ outside a compact region of $\mathbb{R}^n$ of interest.
The proposed output feedback controller consists of a high-gain observer to estimate the states and an instantaneous variant of the full state feedback QIH-NMPC controller as outlined in Sect. 2.1.3. By instantaneous we mean that the system input at *all* times (i.e. not only at the sampling instances) is given by the instantaneous solution of the open-loop optimal control problem:

$$u(\mathbf{x}(t)) := u^\star(\tau = 0\,; \mathbf{x}(t), T_p). \tag{33}$$

This feedback law differs from the standard NMPC formulation in the sense that no open-loop input is implemented over a sampling time δ. Instead $u(t)$ is considered as a "function" of $\mathbf{x}(t)$.
To allow the subsequent result to hold, we have to require that the feedback resulting from the QIH-NMPC is locally Lipschitz.

Assumption 6 *The instantaneous state feedback* (33) *is locally Lipschitz.*

The observer used for state recovery is a high-gain observer [6, 72, 73] of the following form:

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + \mathbf{b}\phi(\hat{\mathbf{x}}, u) + H(x_1 - \hat{x}_1) \tag{34}$$

where $H^\top = \left[\alpha_1/\varepsilon,\ \alpha_2/\varepsilon^2, \ldots, \alpha_n/\varepsilon^n\right]$. The α_i's are chosen such that the polynomial

$$s^n + \alpha_1 \quad s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0,$$

is Hurwitz. Here $\frac{1}{\varepsilon}$ is the high-gain parameter and can be seen as a time scaling for the observer dynamics (34). A, $\mathbf{b}$ and ϕ are the same as in (31).
Notice that the use of an observer makes it necessary that the input also be defined (and bounded) for (estimated) states that are outside the feasible region of the state feedback controller. We simply define the open-loop input for $\mathbf{x} \notin \mathcal{R}$ as fixed to an arbitrary value $u_f \in \mathcal{U}$:

$$u(\mathbf{x}) = u_f, \qquad \forall \mathbf{x} \notin \mathcal{R}. \tag{35}$$

This together with the assumption that $\mathcal{U}$ is bounded separates the peaking of the observer from the controller/system [26]. Using the high-gain observer for state recovery, the following result, which establishes semi-regional stability of the closed-loop can be obtained [36, 37]:

Theorem 2 *Assume that the conditions a)-c) of Theorem 1 and Assumption 5-6 hold. Let $\mathcal{S}$ be any compact set contained in the interior of $\mathcal{R}$ (region of attraction of the state feedback). Then there exists a (small enough) $\varepsilon^* > 0$ such that for all $0 < \varepsilon \leq \varepsilon^*$, the closed-loop system is asymptotically stable with a region of attraction of at least $\mathcal{S}$. Further, the performance of the state feedback NMPC controller is recovered as ε decreases.*

By performance recovery it is meant that the difference between the trajectories of the state feedback and the output feedback can be made arbitrarily small by decreasing ε. The results show that the performance of the state feedback scheme can be recovered in the output feedback case, if a state observer with a suitable structure and a fast enough dynamics is used.
Figure 5 shows the simulation result for an illustrative application of the proposed output feedback scheme to a two dimensional pendulum car system as depicted in Figure 4 and presented in [36]. The angle between the pendulum

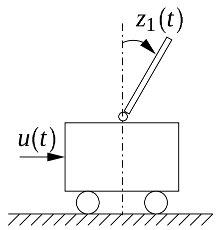

Figure 4: Sketch of the inverted pendulum/car system

and the vertical is denoted by z_1, while the angular velocity of the pendulum is given by z_2. The input u is the force applied to the car. The control objective is to stabilize the upright position of the pendulum. To achieve this objective, a QIH-NMPC scheme with and without (state-feedback case) a high-gain observer is used. For the results shown in Figure 5 the pendulum is initialized with an offset from the upright position, while the high-gain observer is started with zero initial conditions. The figure shows the closed loop trajectories for state feedback QIH-NMPC controller and the output-feedback controller with different observer gains. The gray ellipsoid around the origin is the terminal region of the QIH-NMPC controller. The outer "ellipsoid" is an estimate of the region of attraction of the state-feedback controller. As can be seen for small enough values of the observer parameter ε the closed loop is stable. Furthermore the performance of the state feedback is recovered as ε tends to zero. More details can be found in [36].
Furthermore the recovery of the region of attraction and the performance of the state-feedback is possible up to any degree of exactness. In comparison to other existing output-feedback NMPC schemes [42, 71] the proposed scheme is thus of non-local nature. However, the results are based on the assumption that the NMPC controller is time continuous/instantaneous. In practice, it is of course not possible to solve the nonlinear optimization problem instantaneously. Instead, it will be solved only at some sampling instants. A sampled version of the given result, in agreement with the "usual" sampled NMPC setup can be found in [31]. Notice also, that the use of a high gain observer is critical, if the output measurements are very noise, since the noise will be amplified due to the high gain nature of the observer.

3 Computational Aspects of NMPC

NMPC requires the repeated on-line solution of a *nonlinear* optimal control problem. In the case of linear MPC the solution of the optimal control problem can be cast as the solution of a (convex) quadratic program and can be solved

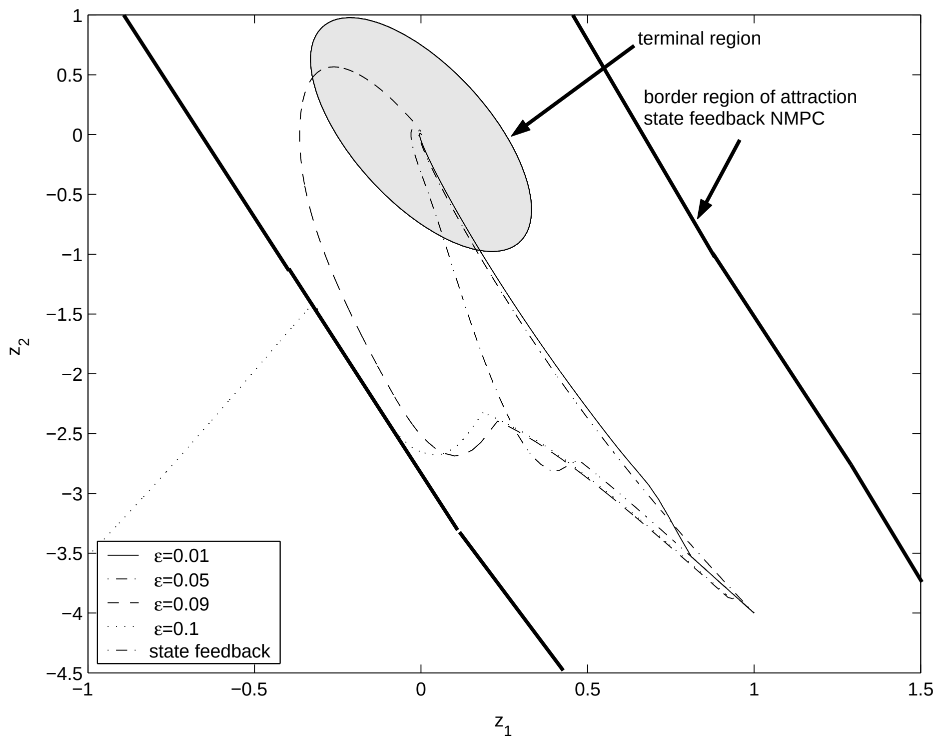

Figure 5: Phase plot of the pendulum angle (z_1) and the angular velocity (z_2)

efficiently even on-line. This can be seen as one of the reasons why linear MPC is widely used in industry. For the NMPC problem the solution involves the solution of a *nonlinear* program, as is shown in the preceding sections. In general the solution of a nonlinear (non-convex) optimization problem can be computational expensive. However in the case of NMPC the nonlinear program shows special structure that can be exploited to still achieve a real-time feasible solution to the NMPC optimization problem.
For the purpose of this Section the open-loop optimal control Problem 2 of Section 2.1.2 will be considered in a more optimization focused setting. Especially it is considered, that the state and input constraints $\mathbf{x} \in \mathcal{X}$, $\mathbf{u} \in \mathcal{U}$ can be recasted as a nonlinear inequality constraint of the form $\mathbf{l}(\mathbf{x},\mathbf{u}) \leq 0$. Furthermore for simplicity of exposition it is assumed that the control and prediction horizon coincide and that no final region constraint is present, i.e. we consider the following deterministic optimal control problem in Bolza form that must be solved at every sampling instance:
Problem 3: *Find*

$$\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t),\bar{\mathbf{u}}(\cdot);T_p) \tag{36}$$

$$\textit{with} \quad J(\mathbf{x}(t),\bar{\mathbf{u}}(\cdot);T_p) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau),\bar{\mathbf{u}}(\tau))d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{37}$$

subject to:

$$\dot{\bar{\mathbf{x}}}(\tau) = \mathbf{f}(\bar{\mathbf{x}}(\tau),\bar{\mathbf{u}}(\tau)), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t) \tag{38a}$$

$$\mathbf{l}(\bar{\mathbf{u}}(\tau),\bar{\mathbf{x}}(\tau)) \leq \mathbf{0}, \ \ \forall \tau \in [t,t+T_p]. \tag{38b}$$

3.1 Solution Methods for the Open-Loop Optimal Control Problem

In principle three different approaches to solve the optimal control Problem 3 exist (see for example [9, 48]):

- **Hamilton-Jacobi-Bellmann partial differential equations/dynamic programming:** This approach is based on the direct solution of the so called Hamilton-Jacobi-Bellmann partial differential equations. Rather than just seeking for the optimal $\mathbf{u}(\tau)$ trajectory the problem is approach as finding a solution for all $\mathbf{x}(t)$. The solution derived is a state feedback law of the form $\mathbf{u}^{\star} = \mathbf{k}(\mathbf{x})$ and is valid for every initial condition. The key obstacle of this approach is, that since the “complete” solution is considered at once, it is in general computationally intractable and suffers from the so called curse of dimensionality, i.e. can be only solved for small systems. Ideally one would like to obtain such a closed loop state feedback law. In principle the intractability of the solution can be seen as the key motivation of receding horizon control.

- **Euler-Lagrange differential equations/calculus of variations/maximum principle:** This method employs classical calculus of variations to obtain an explicit solution of the input as a function of time $\mathbf{u}(\tau)$ and not as feedback law. Thus it is only valid for the specified initial condition $\mathbf{x}(t)$. The approach can be thought of as the application of the necessary conditions for constrained optimization with the twist, that the optimization is infinite dimensional. The solution of the optimal control problem is cast as a boundary value problem. Since an infinite dimensional problem must be solved, this approach can normally not be applied for on-line implementation.

- **Direct solution using a finite parameterization of the controls and/or constraints:** In this approach the input and/or the constraints are parametrized finitely, thus an approximation of the original open-loop optimal control problem is seeked. The resulting finite dimensional dynamic optimization problem is solved with "standard" static optimization techniques.

For an on-line solution of the NMPC problem only the last approach is normally used. Since no feedback is obtained, the optimization problem must be solved at every sampling instance with the new state information. In the following only the last solution method is considered in detail.

3.2 Solution of the NMPC Problem Using a Finite Parameterization of the Controls

As mentioned the basic idea behind the direct solution using a finite parameterization of the controls is to approximate/transcribe the original infinite dimensional problem into a finite dimensional nonlinear programming problem. In this note the presentation is limited to a parameterization of the input signal as piecewise constant over the sampling times. The controls are piecewise constant on each of the $N = \frac{T_p}{\delta}$ predicted sampling intervals: $\bar{\mathbf{u}}(\tau) = \bar{\mathbf{u}}_i$ for $\tau \in [\tau_i, \tau_{i+1})$, $\tau_i = t + i\delta$, compare also Figure 6. Thus in the optimal control problem the "input vector" $\{\bar{\mathbf{u}}_1, \dots \bar{\mathbf{u}}_N\}$ is

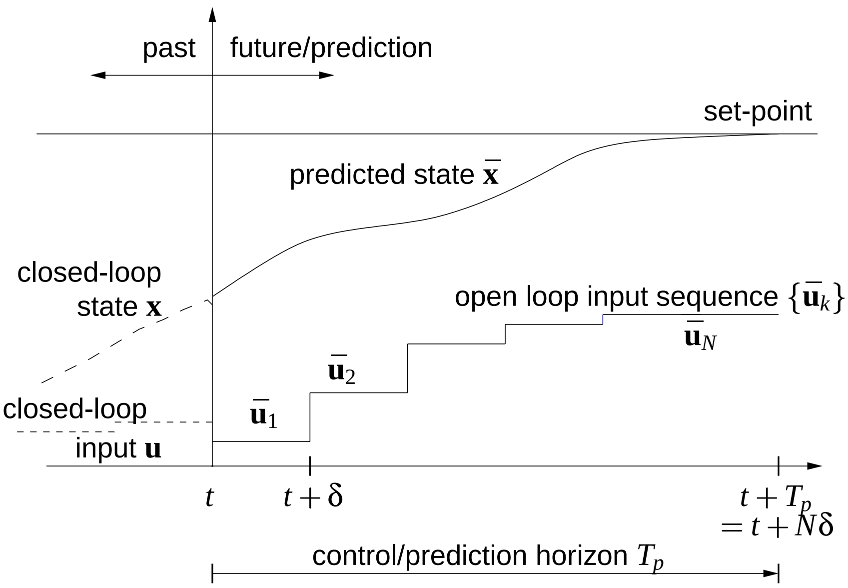

Figure 6: Piecewise constant input signal for the direct solution of the optimal control problem.

optimized, i.e. the optimization problem takes the form

$$\min_{\{\bar{\mathbf{u}}_1,\dots\bar{\mathbf{u}}_N\}} J(\mathbf{x}(t), \{\bar{\mathbf{u}}_1, \dots \bar{\mathbf{u}}_N\}; T_p) \tag{39}$$

subject to the state and input constraints and the system dynamics. Basically two different solution strategies to this optimization problem exist [8, 9, 13, 48, 74]:

- **Sequential approach:** In this method in every iteration step of the optimization strategy the differential equations (or in the discrete time case the difference equation) are solved exactly by a numerical integration, i.e. the solution of the system dynamics is implicitly done during the integration of the cost function and only the input vector $\{\bar{\mathbf{u}}_1, \dots \bar{\mathbf{u}}_N\}$ appears directly in the optimization problem.

- **Simultaneous approach:** In this approach the system dynamics (38a) at the sampling points enter as nonlinear constraints to the optimization problems, i.e. at every sampling point the following equality constraint must be satisfied:

$$\bar{\mathbf{s}}_{i+1} = \bar{\mathbf{x}}(t_{i+1}; \bar{\mathbf{s}}_i, \bar{\mathbf{u}}_i). \tag{40}$$

  Here $\bar{\mathbf{s}}_i$ is introduced as additional degree in the optimization problem and describes the "initial" condition for the sampling interval i, compare also Figure 7. This constraint requires, once the optimization has converged,

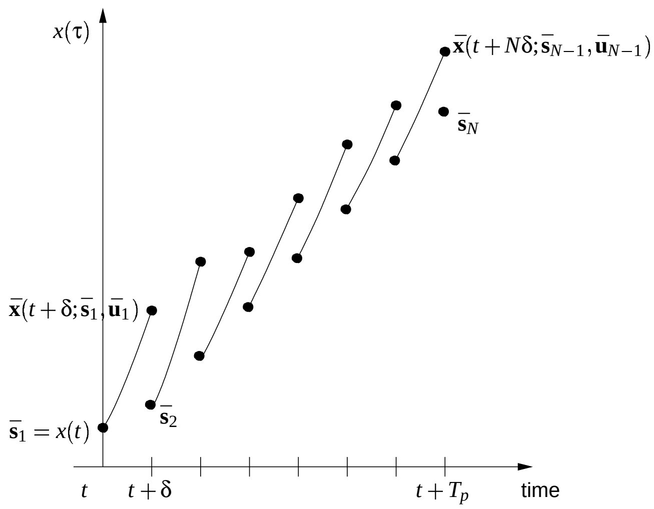

Figure 7: Simultaneous approach.

  that the state trajectory pieces fit together. Thus additionally to the input vector $\{\bar{\mathbf{u}}_1, \dots \bar{\mathbf{u}}_N\}$ also the vector of the $\bar{\mathbf{s}}_i$ appears as optimization variables.

For both approaches the resulting optimization problem is often solved using sequential quadratic programming techniques (SQP). Both approaches have different advantages and disadvantages. For example the introduction of the "initial" states $\bar{\mathbf{s}}_i$ as optimization variables does lead to a special banded-sparse structure of the underlying QP-problem. This structure can be taken into account to lead to a fast solution strategy [8, 24, 74]. In comparison the matrices for the sequential approach are often dense and thus the solution is expensive to obtain. A drawback of the simultaneous approach is, that only at the end of the iteration a valid state trajectory for the system is available. Thus if the optimization cannot be finished in time, nothing can be said about the feasibility of the trajectory at all.

3.2.1 Remarks on State and Input Equality Constraints

In the description given above, the state and input constraints were not taken into account. The key problem is, that they should be satisfied for the whole state and input vector. While for a suitable parametrized input signal (e.g. parametrized as piecewise constant) it is not a problem to satisfy the constraints since only a finite number of points must be checked, the satisfaction of the state constraints must in general be enforced over the whole state trajectory. Different possibilities exist to consider them during the optimization:

- **Satisfaction of the constraints at the sampling instances:** An approximated satisfaction of the constraints can be achieved by requiring, that they are at least satisfied at the sampling instances, i.e. at the sampling times it is required:

$$\mathbf{l}(\bar{\mathbf{u}}(t_i), \bar{\mathbf{x}}(t_i)) \leq \mathbf{0}. \tag{41}$$

Notice, that this does not guarantee that the constraints are satisfied for the predicted trajectories in between the sampling instances. However, since this approach is easy to implement it is often used in practice.

- **Adding a penalty in the cost function:** An approach to enforce the constraint satisfaction exactly for the whole input/state trajectory is to add an additional penalty term to the cost function. This term is zero as long as the constraints are satisfied. Once the constraints are not satisfied the value of this term increases significantly, thus enforcing the satisfaction of the constraints. The resulting cost function may look as following:

$$J(\mathbf{x}(t),\bar{\mathbf{u}}(\cdot);T_p) := \int_t^{t+T_p} \left(F(\bar{\mathbf{x}}(\tau),\bar{\mathbf{u}}(\tau)) + \mathbf{p}(\mathbf{l}(\bar{\mathbf{x}}(\tau)),\bar{\mathbf{u}}(\tau))\right) d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{42}$$

where **p** in the case that only one nonlinear constraint is present might look like shown in Figure 8. A drawback

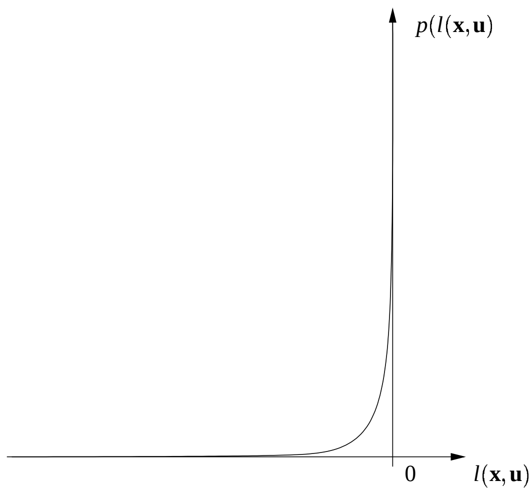

Figure 8: Constraint penalty function for one nonlinear constraint.

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of this formulation is, that the resulting optimization problem is in general difficult to solve for example due to the resulting non-differentiability of the cost function outside the feasible region of attraction.

3.2.2 Efficient NMPC Formulations

One should notice, that besides an efficient solution strategy of the occurring open-loop optimal control problem the NMPC problem should be also formulated efficiently. Different possibilities for an efficient NMPC formulation exist:

- **Use of short horizon length without loss of performance and stability [17, 34, 63]:** As was outlined in Section 2 short horizons are desirable from a computational point of view, but long horizons are required for closed-loop stability and in order to achieve the desired performance in closed-loop. The general NMPC scheme outlined in Section 2.1.2 offers a way out of this dilemma. It uses a terminal region constraint in combination with a terminal penalty term. The terminal penalty term can be used to given a good approximation of the infinite horizon cost utilizing a local control law. Additionally the terminal region constraint is in general not very restrictive, i.e. does not complicate the dynamic optimization problem in an unnecessary manner, as for example in the zero terminal constraint approach. In some cases, e.g. stable systems, feedback linearizable systems or systems for which a globally valid control Lyapunov function is known it can even be removed. Thus such an approach offers the possibility to formulate a computationally efficient NMPC scheme with a short horizon while not sacrificing stability and performance.

- **Use of suboptimal NMPC strategies, feasibility implies stability [17, 34, 56, 70]:** In general no global minima of the open-loop optimization must be found. It is sufficient to achieve a decrease in the value function at every time to guarantee stability. Thus stability can be seen as being implied by feasibility. If one uses an optimization strategy that delivers feasible solutions at every sub-iteration and a decrease in the cost function, the optimization can be stopped if no more time is available and still stability can be guaranteed. The key obstacle is that optimization strategies that guarantee a feasible and decreasing solution at every iteration are normally computationally expensive.

- **Taking the system structure into account [2, 60, 61]:** It is also noticeable, that the system structure should be taken into account. For example for systems for which a flat output is known the dynamic optimization problem can be directly reduced to a static optimization problem. This results from the fact that for flat systems the input and the system state can be given in terms of the output and its derivatives as well as the system initial conditions. The drawback however is, that the algebraic relation between the output and the derivatives to the states and inputs must be known, which is not always possible.

Combining the presented approaches for an efficient formulation of the NMPC problem and the efficient solution strategies of the optimal control problem, the application of NMPC to realistically sized applications is possible even with nowadays computational power. Besides the problem of stability of the closed-loop and the output-feedback problem, the efficient solution of the resulting open-loop optimal control problem is important for any application of NMPC to real processes. Summarizing, a real-time application of NMPC is possible [8, 29, 59] if: a) NMPC schemes that do not require a high computational load and do not sacrifice stability and performance, like QIH-NMPC, are used and b) the resulting structure of the open-loop optimization problem is taken into account during the numerical solution.

4 Application Example–Real-Time Feasibility of NMPC

To show that nonlinear predictive control can be applied to even rather large systems if efficient NMPC schemes and special tailored numerical solution methods are used, we give some results from a real-time feasibility study of NMPC for a high-purity distillation column as presented in [5, 24, 25, 59]. Figure 9 shows the in this study considered 40 tray high-purity distillation column for the separation of Methanol and n-Propanol. The binary mixture is fed in the

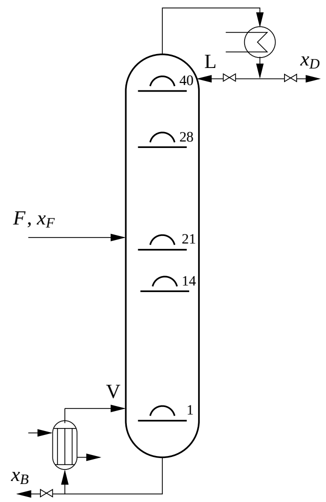

Figure 9: Scheme of the distillation column

column with flow rate F and molar feed composition x_F. Products are removed at the top and bottom of the column with concentrations x_B and x_D respectively. The column is considered in L/V configuration, i.e. the liquid flow rate L and the vapor flow rate V are the control inputs. The control problem is to maintain the specifications on the product concentrations x_B and x_D. For control purposes, models of the system of different complexity are available. As usual in distillation control, x_B and x_D are not controlled directly. Instead an inferential control scheme which controls the deviation of the concentrations on tray 14 and 28 from the setpoints is used, i.e. only the concentration deviations from the setpoint on trays 14 and 28 plus the inputs are penalized in the cost-function. The QIH-NMPC control scheme is used for control. The terminal region and terminal penalty term have been calculated as suggested in Sect. 2.1.3.
In Table 1 the maximum and average CPU times necessary to solve one open-loop optimization problem for the QIH-NMPC scheme in case of a disturbance in x_F with respect to different model sizes are shown. Considering that the

Table 1: Comparison of the average and maximum CPU time in seconds necessary for the solution of one open-loop optimal control problem. The results are obtained using MUSCOD-II [12] and QIH-NMPC for models of different size. The prediction horizon of is 10 minutes and a controller sampling time $\delta = 30$sec is used

model size	max	avrg
42	1.86s	0.89s
164	6.21s	2.48s

sampling time of the process control system connected to the distillation column is 30sec, the QIH-NMPC using the appropriate tool for optimization is even real-time feasible for the 164^{th} order model. Notice, that a straightforward solution of the optimal control problem for the 42^{nd} order model using the optimization-toolbox in Matlab needs in average 620sec to find the solution and is hence not real-time implementable. Also a numerical approximation of the infinite horizon problem by increasing the prediction horizon sufficiently enough is not real-time feasible as shown in [32]. More details and simulation results for the distillation column example can be found in [5, 24, 59]. First experimental results on a pilot scale distillation column are given in [25].

The presented case study underpins, that NMPC can be applied in practice already nowadays, if efficient numerical solution methods and efficient NMPC formulations (like QIH-NMPC) are used.

5 Conclusions

Model predictive control for linear constrained systems has been shown to provide an excellent control solution both theoretically and practically. The incorporation of nonlinear models poses a much more challenging problem mainly because of computational and control theoretical difficulties, but also holds much promise for practical applications. In this note an overview over the theoretical and computational aspects of NMPC is given. As outlined some of the challenges occurring in NMPC are already solvable. Nevertheless many unsolved questions remain. Here only a few are noticed as a guide for future research:

- **Output feedback NMPC:** While some first results in the area of output feedback NMPC exist, none of them seem to be applicable to real processes. Especially the incorporation of suitable state estimation strategies in the NMPC formulation must be further considered.

- **Robust NMPC Formulations:** By now a few robust NMPC formulations exist. While the existing schemes increase the general understanding they are computationally intractable to be applied in practice. Further research is required to develop implementable robust NMPC strategies.

- **Industrial Applications of NMPC:** The state of industrial application of NMPC is growing rapidly and seems to follow academically available results more closely than linear MPC. However, none of the NMPC algorithms provided by vendors include stability constraints as required by control theory for nominal stability; instead they rely implicitly upon setting the prediction horizon long enough to effectively approximate an infinite horizon. Future developments in NMPC control theory will hopefully contribute to making the gap between academic and industrial developments even smaller.

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