Learning-Based Predictive Control | EECI 2023
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# 1 Pen & Paper

## 1.1 Invariant Terminal Set

Consider the linear time invariant system $x(k+1) = Ax(k) + Bu(k)$, which is to be controlled using the standard nominal MPC:

$$
\begin{aligned}
J^\star(x(k)) = \min_{u_i} \; & x_4^T P x_4 + \sum_{i=0}^{3} x_i^T x_i + u_i^T u_i \\
\text{s.t. } & x_{i+1} = Ax_i + Bu_i, \; i = 0, \dots, 3 \\
& x_i \in \mathcal{X}, \; i = 0, \dots, 3 \\
& u_i \in \mathcal{U}, \; i = 0, \dots, 3 \\
& x_4 \in \mathcal{X}_f \\
& x_0 = x(k).
\end{aligned}
$$

Let $K$ be the infinite horizon LQR controller, $P$ the solution of the associated algebraic Riccati equation, and $\mathcal{X}_f$ an invariant set for the closed-loop system $x(k+1) = (A + BK)x(k)$ such that $\mathcal{X}_f \subseteq \mathcal{X}$ and $K\mathcal{X}_f := \{Kx \,|\, x \in \mathcal{X}_f\} \subseteq \mathcal{U}$.

At time $k = 0$, the system is in the state $x(0)$ and the optimal solution to this problem is given by the state and input sequences $\{x_0^\star, x_1^\star, \dots, x_4^\star\}$ and $\{u_0^\star, \dots, u_3^\star\}$. We apply the optimal control input $u(0) = u_0^\star$ and the system moves to state $x(1) = Ax(0) + Bu(0)$.

1. Given the information above, design an *approximate MPC* scheme that does not require *solving any further optimization problems*.

   *Hint: Use the techniques from the stability proof of nominal MPC.*

   a) State a control sequence of horizon $N = 4$, i.e. $\{u_0, u_1, \dots, u_3\}$, that satisfies all constraints starting from the current state at time $k = 1$.

   b) Following this scheme, give an infinite sequence of control inputs $\{u(0), u(1), \dots\}$ that makes the state converge to the origin while satisfying all constraints.

2. If you can write down such a scheme given only the solution at time $k = 0$, why do we bother re-optimizing every time-step in MPC? Is the resulting closed-loop system under this scheme guaranteed to be stable? What about under a control law defined by the nominal MPC scheme above?

## 1.2 (Bonus) Recursive Feasibility of Linear Constraint Tightening MPC

Consider the linear time invariant system $x(k+1) = Ax(k) + Bu(k) + w(k)$ subject to bounded disturbances $w(k) \in \mathcal{W}$. The system is to be controlled using a robust constraint-tightening MPC:

$$\begin{aligned}
\underset{v_i}{\text{minimize}} \quad & \|z_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|v_i\|_R^2 \\
\text{subject to} \quad & z_{i+1} = Az_i + Bv_i, \\
& z_i \in \mathcal{X} \ominus \mathcal{F}_i, \\
& z_N \in \mathcal{X}_f \ominus \mathcal{F}_N, \\
& z_0 = x(k)
\end{aligned}$$

where the applied control input is $u(k) = v_0^*$.

Constraints are tightened using $i$-step disturbances reachable sets: $\mathcal{F}_{i+1} = A_K \mathcal{F}_i \oplus \mathcal{W} = \mathcal{F}_i \oplus A_K^i \mathcal{W}$ with $\mathcal{F}_0 = \{0\}$ and $A_K := A + BK$, where $K$ is a predefined auxiliary control gain. The terminal set $\mathcal{X}_f \subseteq \mathcal{X}$ is chosen as a robust invariant set under the controller $K$ w.r.t. $w \in \mathcal{W}$, i.e., it holds that $A_K \mathcal{X}_f \oplus \mathcal{W} \subseteq \mathcal{X}_f$.

Given an optimal solution at time step $k$

$$\begin{aligned}
V &= [v_0^* = u(k), v_1^*, \ldots, v_{N-1}^*] \\
Z &= [z_0^* = x(k), z_1^*, \ldots, z_N^*]
\end{aligned}$$

show recursive feasibility of the MPC scheme following these main steps:

1. Construct the candidate state sequence $\bar{Z}$ corresponding to the candidate control sequence

$$\bar{V} = [v_1^* + Kw(k), v_2^* + KA_K w(k), \ldots, v_N^* + KA_K^{N-1} w(k), K(z_N^* + A_K^{N-1} w(k))]$$

2. Show that this sequence satisfies the state constraints.

3. Show that this sequence satisfies the terminal constraint.
   *Hint:* You can assume that the set $\mathcal{X}_f \ominus \mathcal{F}_N$ is robust invariant itself, i.e., it holds that $A_K(\mathcal{X}_f \ominus \mathcal{F}_N) \oplus A_K^N \mathcal{W} \subseteq \mathcal{X}_f \ominus \mathcal{F}_N$[1].

*Note: For simplicity, we consider the problem without input constraints, but the same arguments can be applied to that case.*

[1] This property is shown in Chisci et al.

# 2 Programming Exercise

## Nominal & Robust MPC of Mass-Spring-Damper

### Preliminaries

You are provided with the following coding template files:

| | | |
|---|---|---|
| `E01_Fundamentals.mlx` | Main (live) script | (to be edited throughout) |
| `NominalMPC.m` | Nominal MPC Class | (to be edited in Section 3) |
| `RobustMPC.m` | Robust MPC Class | (to be edited in Section 4) |
| `LinearSystem.m` | Helper Class | |
| `SimulateMSD.m` | Helper Function | |

Areas in the code to be modified are indicated by

```
% ----- Start Modifying Code Here ------
%
% The existing (incomplete) code in the modifiable areas serves as a hint
%
% ----- End Modifying Code Here ------
```

### Problem Description

In this exercise we investigate nominal and (constraint-tightening) robust linear MPC for a simple mass-spring damper system with spring constant $k = 0.3$ and damping coefficient $d = 0.2$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

which is brought into our standard form $x(k+1) = Ax(k) + Bu(k)$ by Euler discretization with sampling time $T_s = 0.25$.

The system is subject to polytopic state and input constraints

$$\mathcal{X} = \left\{ x \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{U} = \left\{ u \;\middle|\; \begin{bmatrix} 1 \\ -1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

1. Familiarize yourself with the `LinearSystem` class and the `SimulateMSD` function. Plot the systems' state constraints using MPT, design your own controller and simulate the system.

   *Hint: Use `plot(P)` to plot a (1D or 2D) MPT polyhedron object `P`.*
   *For more information see: `https://www.mpt3.org/`*

2. Familiarize yourself with the basics of Yalmip. Solve the quadratic program

   $$\min_x x^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} x \text{ subject to } [1\ 0]x \geq 3,\ [0\ 1]x \geq 1$$

   *For more information see: `https://yalmip.github.io/`*

3. Implement a nominal model predictive controller for the mass-spring-damper system

   a) Open the nominal MPC class `NominalMPC.m` and complete the implementation of the class constructor setting up the MPC optimization problem.

      *Hints: Use P.A and P.b to access the defining matrix/vector of an MPT Polyhedron.*
      *Try successively adding features (e.g. state constraints) while testing your controller*

b) Design the nominal MPC controller with the following terminal components: Using terminal cost $l_f(x) = x^T P x$ as the infinite horizon unconstrained optimal cost and terminal set $\mathcal{X}_f = \{x \,|\, x^T P x \leq \alpha\}$. Compute $\alpha$ such that $\mathcal{X}_f \subseteq \mathcal{X}$ and $K\mathcal{X}_f \subseteq \mathcal{U}$, where $K$ is the infinite horizon unconstrained LQR controller (corresponding to cost matrix $P$).

*Hint: Use the additional slides* `EllipseXf.pdf`.

c) Simulate the System with the nominal MPC controller

4. (Bonus) Consider now a bounded disturbance $w(k) \in \mathcal{W} = \{Bw \,|\, |w| \leq 0.2\}$ acting on the system as

$$x(k+1) = Ax(k) + Bu(k) + w(k).$$

Implement a robust (constraint-tightening) model predictive controller for the system.

a) Using MPT, compute the $i$-step disturbance reachable sets $\mathcal{F}_i$ for $i = 1, \ldots, N$ as

$$\mathcal{F}_{i+1} = (A + BK)\mathcal{F}_i \oplus \mathcal{W}$$

with the LQR tube controller $K$ and $\mathcal{F}_0 = \{0\}$.

b) Recompute the terminal set[2] scaling $\bar{\alpha}$, such that $\mathcal{X}_f \ominus \mathcal{F}_N = \{z \,|\, z^T P z \leq \bar{\alpha}\} \subseteq \mathcal{X} \ominus \mathcal{F}_N$ and $K(\mathcal{X}_f \ominus \mathcal{F}_N) \subseteq \mathcal{U} \ominus K\mathcal{F}_N$.

c) Open the robust MPC class `robustMPC.m` and complete the implementation of the skeleton class. Compute the resulting tightened constraint sets $\bar{\mathcal{X}}_i = \mathcal{X} \ominus \mathcal{F}_i$ and $\bar{\mathcal{U}}_i = \mathcal{U} \ominus K\mathcal{F}_i$ and adjust the constraint tightening.

d) Simulate your system for $w(k)$ drawn i.i.d. from a uniform random distribution. Compare to the nominal MPC for different noise realization.

[2] *Note: To guarantee recursive feasibility, $\mathcal{X}_f \ominus \mathcal{F}_N$ needs to be robust invariant w.r.t. $w \in (A + BK)^N \mathcal{W}$. Here you can assume this is the case.*