Learning-Based Predictive Control

Chapter 5
Stochastic MPC

Dr. Lukas Hewing

ETH Zurich

The Exploration Company

2023

Coauthor: Prof. Melanie Zeilinger

Motivation: 'Soft' Constraints

- Allow occasional (but controlled) violations $\rightarrow$ improve performance
- Quantify associated risks $\rightarrow$ allow risk-performance tradeoff
- Explicitly consider consider violation $\rightarrow$ recover gracefully

MPC for bounded uncertainties - Robust setting

Uncertain constrained system

$$x(k+1) = f(x(k), u(k)) + w(k) \qquad\qquad x, u \in \mathcal{X}, \mathcal{U} \qquad w \in \mathcal{W}$$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satisfies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for **all** disturbance realizations
2. Is stable: Converges to a neighborhood of the origin
3. Optimizes (nominal/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

MPC for additive disturbances - Stochastic setting

Uncertain constrained system
$x(k+1) = f(x(k), u(k)) + w(k) \quad \Pr(x(k) \in \mathcal{X}) \geq p, \Pr(u(k) \in \mathcal{U}) \geq p, \quad w(k) \sim \mathcal{Q}^w,$ i.i.d.

Design control law $u(k) = \pi(x(k))$ such that the system:

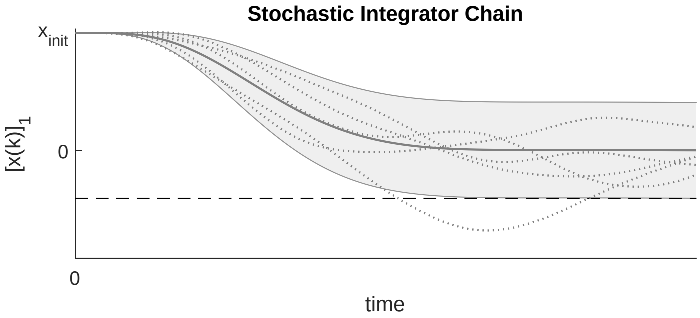
1. Satisfies constraints : $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$ **with given probability p**
2. Is 'stable': Converges to the origin **in a suitable sense**
3. Optimizes (nominal/**expected**) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Outlook

Chance constraint: Constrain probability of constraint satisfaction:

$$\Pr(x(k) \in \mathcal{X}) \geq p, \ \forall k$$

1. 'Open-Loop' chance-constrained optimal control problems
2. Receding horizon control: Feasibility & closed-loop chance constraint satisfaction
3. Next Lecture: Data-driven SMPC using scenario optimization



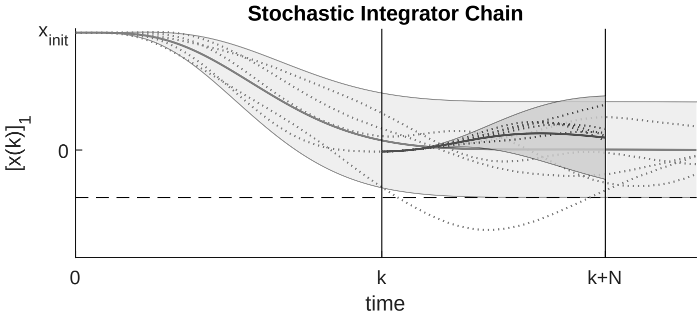
solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Outlook

Chance constraint: Constrain probability of constraint satisfaction:

$$\Pr(x(k) \in \mathcal{X}) \geq p, \ \forall k$$

1. 'Open-Loop' chance-constrained optimal control problems
2. Receding horizon control: Feasibility & closed-loop chance constraint satisfaction
3. Next Lecture: Data-driven SMPC using scenario optimization



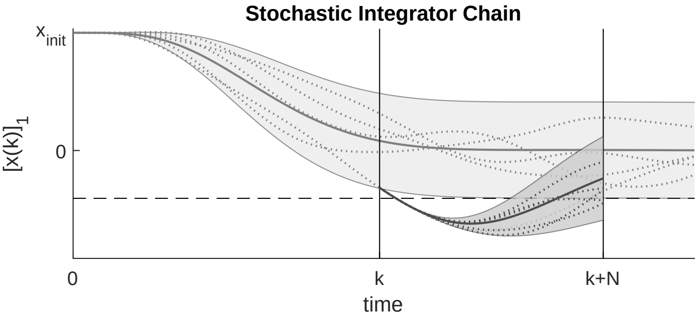
solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Outlook

Chance constraint: Constrain probability of constraint satisfaction:

$$\Pr(x(k) \in \mathcal{X}) \geq p, \ \forall k$$

1. 'Open-Loop' chance-constrained optimal control problems
2. Receding horizon control: Feasibility & closed-loop chance constraint satisfaction
3. Next Lecture: Data-driven SMPC using scenario optimization



solid line: mean trajectory
dashed lines: trajectory samples
shaded area: confidence region

Learning Objectives

- Know how to re-formulate a simple linear chance-constrained stochastic optimal control problem into a deterministic optimization problem
  - Mean variance dynamics
  - Deterministic reformulation of quadratic cost
  - Deterministic reformulation of chance-constraints via constraint backoff

- Understand feasibility issues arising in stochastic receding horizon control
  - Understand recovery mechanism and resulting theoretical issues
  - Derive a recursively feasible indirect feedback formulation
  - Derive a recursively feasible direct feedback formulation for bounded random disturbances

Outline

1. The (Linear) 'Open Loop' Chance Constrained Optimal Control Problem

2. Stochastic Receding Horizon Control: Feasibility & Constraint Satisfaction

3. Discussion

Chance Constrained Linear Quadratic Optimal Control

$$
\begin{aligned}
J^*(x) = \min_{\{\pi_k\}} \quad & \mathbb{E}\left(\|x(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 \right) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k) + w(k), \\
& u(k) = \pi_k(x(k)), \\
& w(k) \sim \mathcal{N}(0, \Sigma_w), \text{ i.i.d.}, \\
& \Pr(h_{x,j}^\mathsf{T} x(k) \leq b_{x,j}) \geq p_{x,j} \ \forall j = 1, \ldots, N_{c,x}, \\
& \Pr(h_{u,j}^\mathsf{T} u(k) \leq b_{u,j}) \geq p_{u,j} \ \forall j = 1, \ldots, N_{c,u}, \\
& x(0) = x
\end{aligned}
$$

Approach

1. Restrict policy class: $\pi_k(x) = K_k x + v_k$
2. Derive mean-variance dynamics
3. Reformulate cost function
4. Reformulate constraints

Mean-Variance Dynamics under Affine Policy

$$x(k+1) = Ax(k) + Bu(k) + w(k), \qquad \mathbb{E}(w(k)) = 0,\ \mathrm{var}(w(k)) = \Sigma_w \text{ i.i.d.}$$

To simplify, we introduce some notation:

$$\begin{aligned} \bar{x}(k) &:= \mathbb{E}(x(k)) & \qquad d(k) &:= x(k) - \bar{x}(k) \\ \bar{u}(k) &:= \mathbb{E}(u(k)) & \bar{d}(k) &:= \mathbb{E}(d(k)) = 0 \\ \Sigma^x(k) &:= \mathrm{var}(x(k)) = \mathrm{var}(d(k)) \end{aligned}$$

where the expectations are understood conditioned on the initial state $x(0)$

Choosing an affine tube policy class $\pi_k(x) = K_k(x - \bar{x}(k)) + v_k$ we have

$$\begin{aligned} u(k) &= K_k d(k) + v_k \\ \bar{u}(k) &= v_k \end{aligned} \qquad \text{resulting in } \rightarrow \qquad \begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) + B\bar{u}(k) \\ \Sigma^x(k+1) &= (A + BK_k)\Sigma^x(k)(A + BK_k)^\mathsf{T} + \Sigma_w \end{aligned}$$

Reformulation of Cost Function

$$\begin{aligned}\bar{x}(k+1) &= A\bar{x}(k) + B\bar{u}(k) \\ \Sigma^x(k+1) &= (A + BK_k)\Sigma^x(k)(A + BK_k)^\mathsf{T} + \Sigma_w\end{aligned}$$

Expected value of quadratic form:

$$\mathbb{E}_z\left(\|z\|_Q^2\right) = \|\mathbb{E}_z(z)\|_Q^2 + \mathrm{tr}(Q\,\mathrm{var}_z(z))$$

Using this allows us to reformulate cost function in term of mean and variance!

$$\mathbb{E}\left(\|x(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2\right)$$

$$= \|\bar{x}(\bar{N})\|_P^2 + \mathrm{tr}(P\Sigma^x(\bar{N})) + \sum_{k=0}^{\bar{N}-1} \|\bar{x}(k)\|_Q^2 + \mathrm{tr}(Q\Sigma^x(k)) + \|\bar{u}(k)\|_R^2 + \mathrm{tr}(RK_k\Sigma^x(k)K_k^\mathsf{T})$$

Gaussian Half-Space Chance Constraint I

Given affine policy, state distributions remain Gaussian. For a half-space chance constraint

$$x(k) \sim \mathcal{N}(\bar{x}(k), \Sigma^x(k))$$
$$\mathcal{X} = \{x \mid h^\top x \leq b\}$$

we can construct the marginal distribution in direction of the constraint

$$h^\top x(k) \sim \mathcal{N}(h^\top \bar{x}(k), h^\top \Sigma^x(k) h) \textbf{ (scalar!)}$$

$$\Pr(x(k) \in \mathcal{X}) = \Pr(h^\top x(k) \leq b) = \phi\left(\frac{b - h^\top \bar{x}(k)}{\sqrt{h^\top \Sigma^x(k) h}}\right)$$

- ϕ is the cumulative distribution function of the standard normal distribution (available)

$$\phi(\tilde{x}) := \Pr(x \leq \tilde{x}) \text{ with } x \sim \mathcal{N}(0, 1)$$

- depends only on $\bar{x}(k)$ and $\Sigma^x(k)$ (available)

Gaussian Half-Space Chance Constraint II

$$
\begin{aligned}
\Pr(x(k) \in \mathcal{X}) \geq p \Leftrightarrow \phi\left(\frac{b - h^\top \bar{x}(k)}{\sqrt{h^\top \Sigma^x(k) h}}\right) &\geq p \\
\Leftrightarrow \frac{b - h^\top \bar{x}(k)}{\sqrt{h^\top \Sigma^x(k) h}} &\geq \phi^{-1}(p) \\
\Leftrightarrow -h^\top \bar{x}(k) &\geq -b + \sqrt{h^\top \Sigma^x(k) h}\, \phi^{-1}(p) \\
\Leftrightarrow h^\top \bar{x}(k) &\leq b \underbrace{-\sqrt{h^\top \Sigma^x(k) h}\, \phi^{-1}(p)}_{\text{tightening/backoff term}}
\end{aligned}
$$

with ϕ^{-1} the inverse cumulative distribution function of the standard normal distribution (available)

$$\boxed{\text{Tightened half-space constraint when optimizing over } \bar{x}(k)}$$

Gaussian Linear Quadratic Control with Half-Space Constraint

$$
\begin{aligned}
J^*(x) = \min_{\{\pi_k\}} \quad & \mathbb{E}\left(\|x(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 \right) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k) + w(k), \\
& u(k) = \pi_k(x(k)), \\
& w(k) \sim \mathcal{N}(0, \Sigma_w), \ \text{i.i.d.}, \\
& \Pr(h_{x,j}^\top x(k) \le b_{x,j}) \ge p_{x,j} \ \forall j = 1, \dots, N_{c,x}, \\
& \Pr(h_{u,j}^\top u(k) \le b_{u,j}) \ge p_{u,j} \ \forall j = 1, \dots, N_{c,u}, \\
& x(0) = x
\end{aligned}
$$

Gaussian Linear Quadratic Control with Half-Space Constraint

$$
\begin{aligned}
J^*_{\text{det}}(x) = \min_{\{v_k, K_k\}} \quad & \|\bar{x}(\bar{N})\|_P^2 + \text{tr}(P\Sigma^x(\bar{N})) + \sum_{k=0}^{\bar{N}-1} \|\bar{x}(k)\|_Q^2 + \|\bar{u}(k)\|_R^2 + \text{tr}((Q + K_k^\top R K_k)\Sigma^x(k)) \\
\text{s.t.} \quad & \bar{x}(k+1) = A\bar{x}(k) + Bv_k, \\
& \Sigma^x(k+1) = (A + BK_k)\Sigma^x(k)(A + BK_k)^\top + \Sigma_w, \\
& h_{x,j}^\top \bar{x}(k) \leq b_{x,j} - \sqrt{h_{x,j}^\top \Sigma^x(k) h_{x,j}}\, \phi^{-1}(p) \ \forall j = 1, \ldots, N_{c,x}, \\
& h_{u,j}^\top v_k \leq b_{u,j} - \sqrt{h_{u,j}^\top K_k \Sigma^x(k) K_k^\top h_{u,j}}\, \phi^{-1}(p) \ \forall j = 1, \ldots, N_{c,u}, \\
& \bar{x}(0)) = x, \ \Sigma^x(0) = 0
\end{aligned}
$$

- With affine policy: deterministic optimization problem over mean and variances (non-convex)

Gaussian Linear Quadratic Control with Half-Space Constraint

$$
\begin{aligned}
\tilde{J}^*_{\text{pre}}(x) = \min_{\{v_k\}} \quad & \|\bar{x}(\bar{N})\|_P^2 + \sum_{k=0}^{\bar{N}-1} \|\bar{x}(k)\|_Q^2 + \|\bar{u}(k)\|_R^2 \\
\text{s.t.} \quad & \bar{x}(k+1) = A\bar{x}(k) + Bv_k, \\
& h_{x,j}^\top \bar{x}(k) \le b_{x,j} - \sqrt{h_{x,j}^\top \Sigma^x(k) h_{x,j}}\, \phi^{-1}(p)\ \forall j = 1, \dots, N_{c,x}, \\
& h_{u,j}^\top v_k \le b_{u,j} - \sqrt{h_{u,j}^\top K_k \Sigma^x(k) K_k^\top h_{u,j}}\, \phi^{-1}(p)\ \forall j = 1, \dots, N_{c,u}, \\
& \bar{x}(0) = x
\end{aligned}
$$

- With affine policy: deterministic optimization problem over mean and variances (non-convex)
- Fixing K_k and only optimizing over v_k reduces problem to 'standard' MPC QP
- Variance dynamics independent of v_k → tightening can be precomputed

Segway: Continuous States & Inputs

States: Position $p \in \mathbb{R}$
Velocity $v \in \mathbb{R}$

Input: Acceleration $a \in \mathbb{R}$

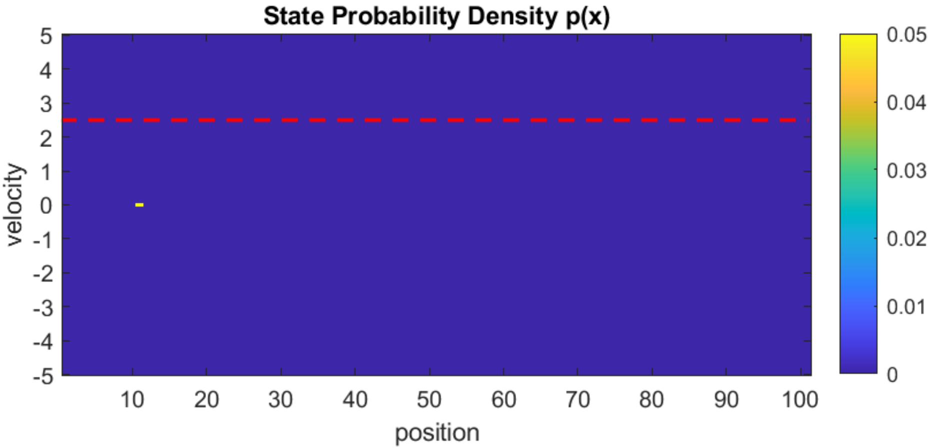
Disturbances $w \sim \mathcal{N}(0, 0.1)$

Dynamics
$$p(k+1) = p(k) + v(k)$$
$$v(k+1) = v(k) + a(k) + w(k)$$

[freepik.com]

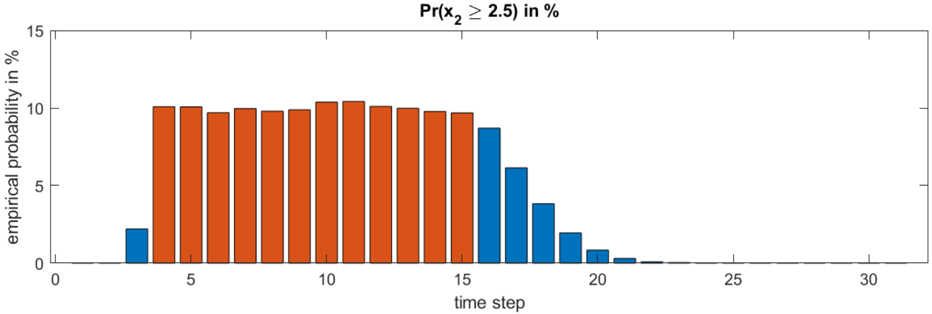
Linear time invariant (LTI) system: $x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k)$

Segway: Open-Loop Sequence with Predefined Tube Controller



Chance Constraint with $p = 0.9$: $\begin{bmatrix} 0 & 1 \end{bmatrix} \mathbb{E}(x(k)) \leq 2.5 - \sqrt{\mathrm{var}(x_2(k))}\,\phi^{-1}(p)$

Segway: Open-Loop Sequence with Predefined Tube Controller



Red bars: Time steps with active constraints $\approx 10\%$

Outline

1. The (Linear) 'Open Loop' Chance Constrained Optimal Control Problem
2. Stochastic Receding Horizon Control: Feasibility & Constraint Satisfaction
3. Discussion

Receding Horizon Control

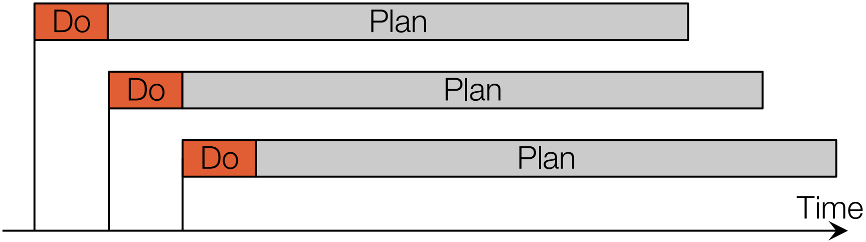
So far: Optimization carried out offline, directly returning feedback control law

$$\{\pi_k(x)\}_0^{\bar{N}-1} = \operatorname{argmin} J(x_{\text{init}})$$

and feedback typically reduced to linear feedback.

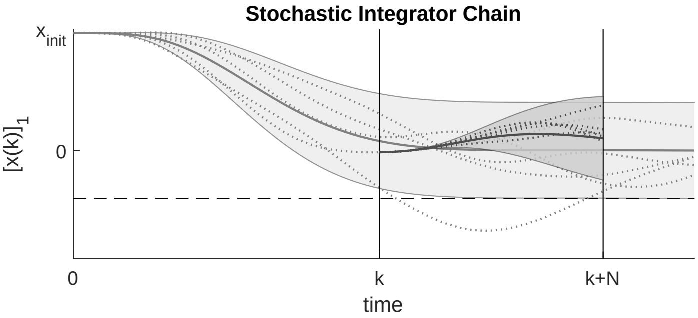
In MPC (receding/shrinking horizon control) the optimization is instead repeated at each time step, with only the first computed control input applied to the system

$$\pi_k(x) = \text{"first element of"}\ \operatorname{argmin} J_k(x)$$



Feasibility Issues in SMPC

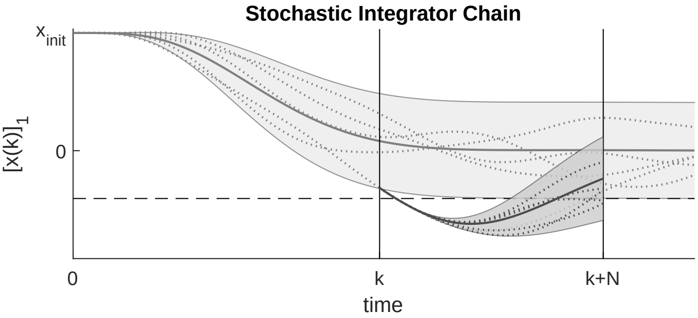
Under general stochastic disturbances, feasibility can usually **not** be guaranteed



solid line: mean trajectory
dashed lines: trajectory samples
shaded area: confidence region

Feasibility Issues in SMPC

Under general stochastic disturbances, feasibility can usually **not** be guaranteed



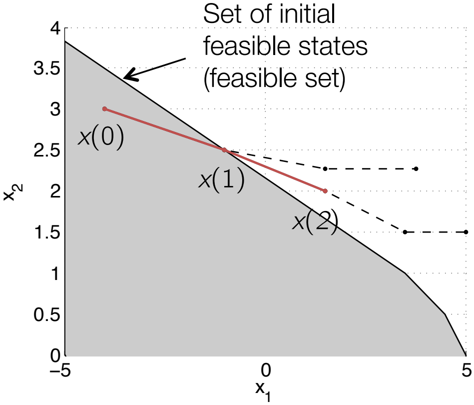
solid line: mean trajectory
dashed lines: trajectory samples
shaded area: confidence region

Feasibility Issues in SMPC

Problem: (Potentially unbounded) stochastic disturbances can drive state initial state $x_0 = x(k)$ into infeasible region.

Several strategies to handle this problem

1. Make use of recovery mechanisms
2. Alternative forms of feedback
3. Assume a maximum size of disturbance $\rightarrow$ use robust techniques



Outline

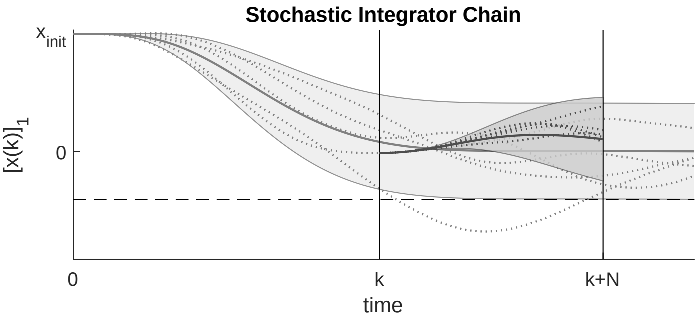
2. Stochastic Receding Horizon Control: Feasibility & Constraint Satisfaction

    Recovery Mechanisms

    Indirect Feedback

    Constraint Tightening SMPC

Idea: Introduce Feedback whenever Feasible



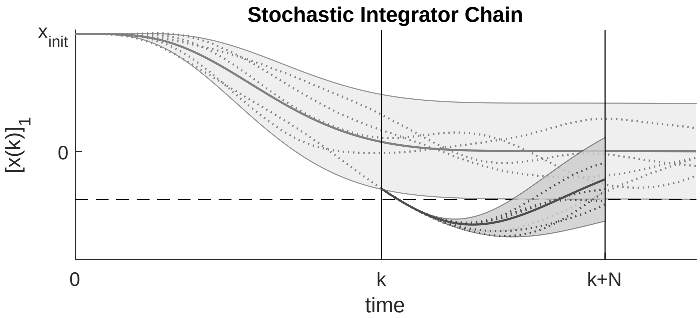
solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Recovery Initialization [1]

Case 1: $\bar{x}_0 = x(k)$, $\Sigma_0^x = 0$ (Whenever feasible)

Case 2: $\bar{x}_0 = \bar{x}_{1|k-1}$, $\Sigma_0^x = \Sigma_{1|k-1}^x$ (Otherwise, guaranteed feasible)

Idea: Introduce Feedback whenever Feasible



solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Recovery Initialization [1]

Case 1: $\bar{x}_0 = x(k), \quad \Sigma_0^x = 0$ (Whenever feasible)

Case 2: $\bar{x}_0 = \bar{x}_{1|k-1}, \quad \Sigma_0^x = \Sigma_{1|k-1}^x$ (Otherwise, guaranteed feasible)

Stochastic MPC with Recovery Initialization

Aims to combine "best of both worlds":

- Feasibility guarantees from 'recovery initialization'
- Use feedback whenever possible

But theoretical analysis proves challenging!

Theoretical Issues

$\rightarrow$ What are the implications for stability/performance?
  - Candidate solution does not remain feasible (from measured state)
  - Cost from measured state may increase (over "shifted" solution) [1]

$\rightarrow$ What are implications for closed-loop constraint satisfaction?
  - No direct guarantees on closed-loop [2]
  - Can also be conservative! [3]

Outline

2. Stochastic Receding Horizon Control: Feasibility & Constraint Satisfaction

   Recovery Mechanisms

   Indirect Feedback

   Constraint Tightening SMPC

Indirect feedback SMPC [4]: Idea

Introduce **nominal state** $z(k)$ and let it evolve according to **nominal input** $v(k) = v_0^*$

$$
\begin{aligned}
z(k+1) &= Az(k) + Bv(k) \\
e(k+1) &= (A + BK)e(k) + w(k)
\end{aligned}
$$

$\rightarrow$ no **direct feedback** from measurement $x(k)$ on $z(k)$
$\rightarrow$ error state $e(k)$ evolves linearly and independent of $v(k)$

Straightforward to formulate constraints on closed-loop (similar to 'open-loop' problem)

$$
h^\mathsf{T} z(k) \leq b - \sqrt{h^\mathsf{T} \Sigma^e(k) h}\, \phi^{-1}(p) \Rightarrow \Pr(h^\mathsf{T} x(k) \leq b) \geq p
$$

As opposed to "open-loop" optimization, we nevertheless introduce feedback by optimizing over cost given measured state $x(k)$, i.e. $x_0 = x(k)$ (**indirect feedback**)

$$
\min_{\{v_i\}} \mathbb{E} \left(\|x_N\|_P^2 + \sum_{i=0}^{N-1} \|x_i\|_Q^2 + \|u_i\|_R^2 \,\middle|\, x_0 = x(k) \right)
$$

Indirect Feedback SMPC [4]: Resulting Formulation

$$
\begin{aligned}
\tilde{J}^*(x(k)) = \min_{\{v_i\}} \quad & \|\bar{x}_N\|_P^2 + \sum_{i=0}^{N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 \\
\text{s.t.} \quad & \bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \\
& \bar{u}_i = K(\bar{x}_i - z_i) + v_i, \\
& z_{i+1} = Az_i + Bv_i, \\
& h_{x,j}^\top z_i \leq b_{x,j} - \sqrt{h_{x,j}^\top \Sigma^e(k+i) h_{x,j}}\, \phi^{-1}(p)\ \forall j = 1, \ldots, N_{c,x}, \\
& h_{u,j}^\top v_i \leq b_{u,j} - \sqrt{h_{u,j}^\top K \Sigma^e(k+i) K^\top h_{u,j}}\, \phi^{-1}(p)\ \forall j = 1, \ldots, N_{c,u}, \\
& z_N \in \mathcal{Z}_f, \\
& \bar{x}_0 = x(k),\ z_0 = z(k) = z_{1|k-1}
\end{aligned}
$$

- Applied input: $u(k) = K(x(k) - z(k)) + v_0^*$
- $\mathcal{Z}_f$ terminal (nominal) control invariant set
- $\Sigma^e(k+i)$ can be precomputed from $e(k+1) = (A + BK)e(k) + w(k)$

Indirect Feedback SMPC [4]: Main Properties

Candidate sequence remains feasible

$$\begin{aligned} \bar{V} &= \{v_1^*, \dots, v_{N-1}^*, \pi_f(z_N^*)\} \\ \rightarrow \bar{Z} &= \{z_1^*, \dots, z_N^*, Az_N^* + B\pi_f(z_N^*)\} \end{aligned}$$

From this follows recursive feasibility and

- Closed-loop chance constraint satisfaction

$$\Pr(x(k) \in \mathcal{X}) \geq p,\ \Pr(u(k) \in \mathcal{U}) \geq p$$

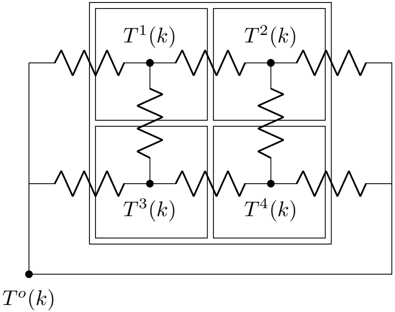
- Asymptotic average performance bound

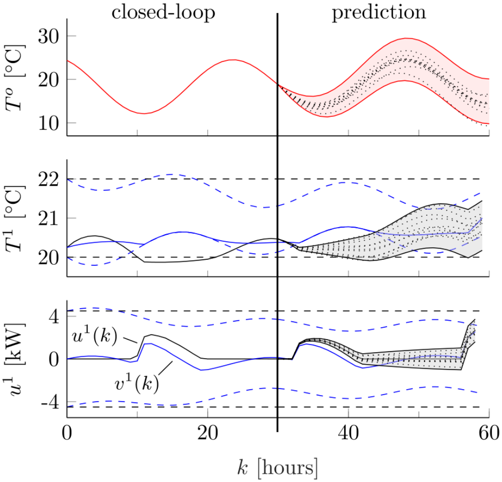
$$\lim_{\bar{N} \to \infty} \frac{1}{\bar{N}} \sum_{k=0}^{\bar{N}-1} \mathbb{E}(l(x(k), u(k)) \leq \mathrm{tr}(P\Sigma_w)$$

  when choosing $\pi_f(x) = Kx$ and $P = (A + BK)^\top P(A + BK) + Q + K^\top RK$

Indirect Feedback SMPC [4]: Building Control

- Modeled as resistance network (linear)
- Heating of 4 different Rooms
- Comfort constraint $\Pr(T_j \in [20, 22]) \geq 0.9$
- Sparsity inducing input cost $\|u\|_1$ (1-norm)





Outline

2. Stochastic Receding Horizon Control: Feasibility & Constraint Satisfaction

    Recovery Mechanisms

    Indirect Feedback

    Constraint Tightening SMPC

Stochastic MPC with Bounded Disturbances

Setup:

$$x(k+1) = Ax(k) + Bu(k) + w(k)$$

where $w(k) \sim \mathcal{Q}^w$ i.i.d. **and** all $w(k) \in \mathcal{W}$ with $\mathcal{W}$ a compact set.

- Stochastic disturbance $w(k)$ has **bounded support** $\mathcal{W}$
- Enables use of robust techniques for recursive feasibility

Outline:

1. Intuition, potential advantages over purely robust formulation
2. Techniques to ensure recursive feasibility
3. Recursive feasibility $\Rightarrow$ closed-loop constraint satisfaction

Recursive feasibility in SMPC for bounded disturbances

Different techniques exist to ensure recursive feasibility based on robust arguments

Essential difference to robust MPC:

$$\underbrace{\text{Robustly ensuring } \textbf{chance} \text{ constraints}}_{\text{Stochastic MPC with bounded disturbances}} \not\Leftrightarrow \underbrace{\text{Robustly ensuring } \textbf{deterministic} \text{ constraints}}_{\text{Robust MPC}}$$

In the following, we discuss one possible approach related to "constraint-tightening" robust MPC

- Enforce **chance** constraints w.r.t. **all** possible previous disturbances
  ($i-1$-steps robust, 1-step stochastic)
- Enforce terminal robust invariant set (within constraints) robustly
- For simplicity, we neglect input constraints for now, extension is straightforward
- References: [4, 5]

Robust Constraint-Tightening MPC

$$
\begin{aligned}
\min_{\{v_i\}} \quad & \|z_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|v_i\|_R^2 \\
\text{s.t.} \quad & z_{i+1} = Az_i + Bv_i, \\
& z_i \in \mathcal{X} \ominus \mathcal{F}_i, \\
& z_N \in \mathcal{Z}_f, \\
& z_0 = x(k)
\end{aligned}
$$

- Reachable set $\mathcal{F}_i = \bigoplus_{j=1}^{i} A_K^{j-1}\mathcal{W}$, with $A_K := A + BK$
- Robustly ensure satisfaction of constraints at each time step
- Terminal (robust) invariant set under tube controller $\mathcal{Z}_f \subseteq \mathcal{X} \ominus \mathcal{F}_N$

Now: Stochastic Constraint-Tightening MPC [5]

$$
\begin{aligned}
\min_{\{v_i\}} \quad & \|z_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|v_i\|_R^2 \\
\text{s.t.} \quad & z_{i+1} = Az_i + Bv_i, \\
& \Pr(z_i + w_{i-1} \in \mathcal{X} \ominus A_K \mathcal{F}_{i-1}) \geq p, \\
& z_N \in \mathcal{Z}_f, \\
& z_0 = x(k)
\end{aligned}
$$

- Reachable set $\mathcal{F}_i = \bigoplus_{j=1}^{i} A_K^{j-1} \mathcal{W}$, with $A_K := A + BK$
- Robustly ensure satisfaction of chance constraints at each time step
- Terminal (robust) invariant set under tube controller $\mathcal{Z}_f \subseteq \mathcal{X} \ominus \mathcal{F}_N$
- Tightening (half-space case): $\mathcal{X} := \{x \,|\, h^\mathsf{T} x \leq b\}$

$$\Pr(z_i + w_{i-1} \in \mathcal{X} \ominus A_K \mathcal{F}_{i-1}) \geq p \Leftrightarrow h^\mathsf{T} z_i \leq \tilde{b}_i - F_w(p) \leftarrow \text{"constraint backoff"}$$

Illustration: SMPC with Bounded Uncertainties [5]

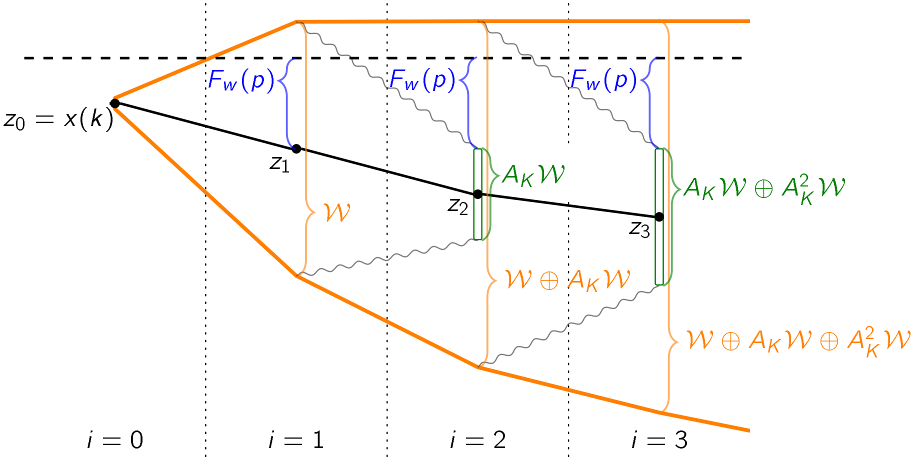


Figure adapted from H. Schlüter, F. Allgöwer, "A Constraint-Tightening Approach to Nonlinear Stochastic Model Predictive Control for Systems under General Disturbances", 2019

Stochastic vs. Robust MPC: Feasible Region

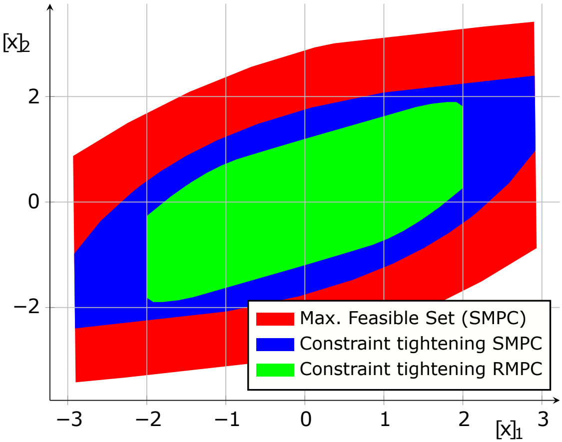


Figure adapted from M. Lorenzen et al, "Constraint-Tightening and Stability in Stochastic Model Predictive Control", Trans. Automatic Control, 2017

Outline

1. The (Linear) 'Open Loop' Chance Constrained Optimal Control Problem

2. Stochastic Receding Horizon Control: Feasibility & Constraint Satisfaction

3. Discussion

Chance Constraints in Closed-Loop vs. in Prediction

We want to show:

$$(*)\ Pr(x(k) \in \mathcal{X} \,|\, x(0)) \quad \text{(conditioned on initial state } x(0))$$

But typically enforce

$$(**)\ Pr(x_i \in \mathcal{X} \,|\, x_0 = x(k)) \quad \text{(conditioned on measured state } x(k))$$

We can then (sometimes) guarantee that $(**)$ always holds, and then $(**) \Rightarrow (*)$

- Constraint tightening SMPC for bounded disturbances

In general, enforcing $(**)$ for all k is much stricter than $(*)$

- Leads to feasibility issues
- Can be conservative, if $(*)$ is the required condition

Chance Constraints in Closed-Loop vs. in Prediction

We want to show:

$$(*)\ Pr(x(k) \in \mathcal{X} \,|\, x(0)) \quad \text{(conditioned on initial state } x(0))$$

But typically enforce

$$(**)\ Pr(x_i \in \mathcal{X} \,|\, x_0 = x(k)) \quad \text{(conditioned on measured state } x(k))$$

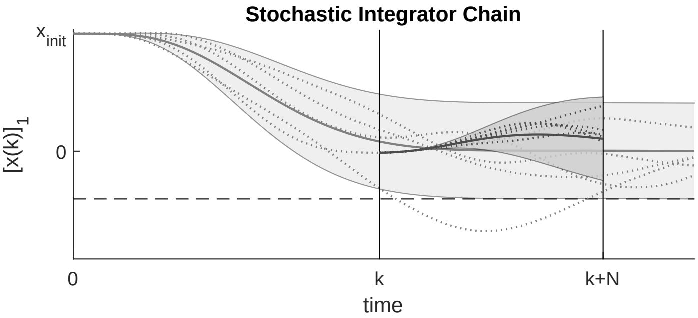
We can then (sometimes) guarantee that $(**)$ always holds, and then $(**) \Rightarrow (*)$

- Constraint tightening SMPC for bounded disturbances

In general, enforcing $(**)$ for all k is much stricter than $(*)$

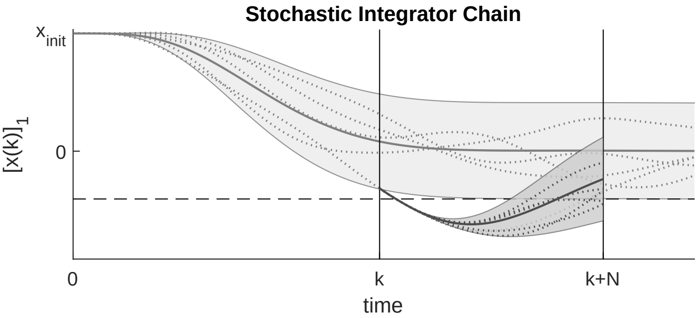
- Leads to feasibility issues
- **Can be conservative, if $(*)$ is the required condition**

Chance Constraints in Closed-Loop vs. in Prediction



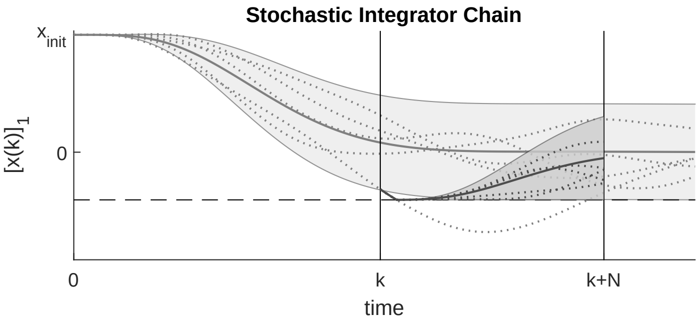
solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Chance Constraints in Closed-Loop vs. in Prediction



solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Chance Constraints in Closed-Loop vs. in Prediction



solid line:	mean trajectory
dashed lines:	trajectory samples
shaded area:	confidence region

Chance Constraints in Closed-Loop vs. in Prediction [3]

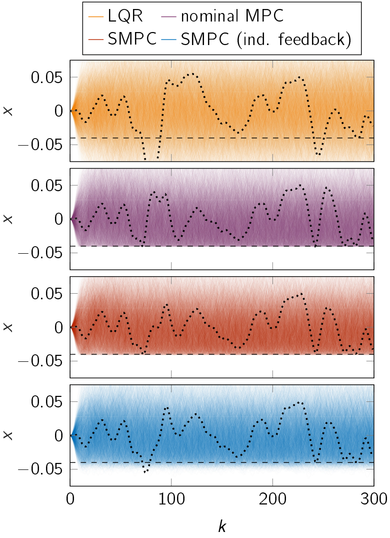
Third order integrator chain:

$$x(k+1) = \begin{bmatrix} 1 & 0.1 & 0.1^2/2 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.1^3/6 \\ 0.1^2/2 \\ 0.1 \end{bmatrix} (u(k) + w(k))$$

with i.i.d. $w(k) \sim \mathcal{N}(0,1)$ and chance constraint

$$\Pr\left([x]_1 \geq -\sqrt{[\Sigma_\infty]_{1,1}}\right) \geq 0.84$$

- Unconstrained (LQR) solution satisfies constraint
- Main effect of disturbance on $[x]_1$ is delayed (needs to propagate through system)
- SMPC results in virtually no "constraint softening"
- Side effect: Aggressive control inputs to ensure feasibility



Summary

- Stochastic MPC remains active research topic, in particular
  - unbounded disturbance distributions
  - nonlinear stochastic MPC (not discussed)
  - data-driven formulations (scenario approach, next lecture)
  - parametric uncertainty & model learning

- Common formulation (satisfying chance constraints in prediction) may need critical re-evaluation
  - indirect feedback as possible alternative
  - connection to stochastic reference governors

References and further reading

[1] M. Farina et al, "A probabilistic approach to Model Predictive Control", Conf. Decision Control, 2013

[2] L. Hewing, M. N. Zeilinger, "Stochastic Model Predictive Control for Linear Systems Using Probabilistic Reachable Sets," Conf. Decision Control, 2018

[3] L. Hewing, M. N. Zeilinger, "Performance Analysis of Stochastic Model Predictive Control with Direct and Indirect Feedback," Conf. Decision Control, 2020

[4] L. Hewing, K. P. Wabersich, M. N. Zeilinger, "Recursively feasible stochastic model predictive control using indirect feedback", Automatica, 2020

[5] M. Lorenzen et al, "Constraint-Tightening and Stability in Stochastic Model Predictive Control", Trans. Automatic Control, 2017

[6] J. Köhler, F. Geuss, M. N. Zeilinger, "On stochastic MPC formulations with closed-loop guarantees: Analysis and a unifying framework", Arxiv preprint, 2023

- B. Kouvaritakis, M. Cannon, "Model Predictive Control: Classical, Robust and Stochastic", 2016
- M. Farina et al., "Stochastic linear Model Predictive Control with chance constraints – A review", J. Process Control, 2016
- A. Mesbah, "Stochastic model predictive control: An overview and perspectives for future research", IEEE Control Systems Magazine, 2016