Learning-Based Predictive Control

Chapter 1
Introduction & Fundamentals of MPC

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2023

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Outline

1. Motivation
2. Review: Nominal MPC theory
3. Review: Robust MPC

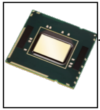
Outline

1. Motivation

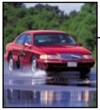
   MPC

   Learning

Successful industrial applications


Computer control — ns

μs — Power systems



Traction control — ms

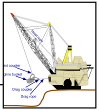
Seconds — Buildings



Refineries — Minutes

Hours — Nurse rostering


Train scheduling — Days

Weeks — Production planning


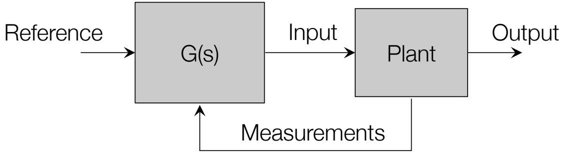
Examples: MPC for Automotive

Autonomous racing and obstacle avoidance
ORCA Racer @ IfA, ETH Zurich
`https://www.youtube.com/watch?v=JoHfJ6LEKVo`

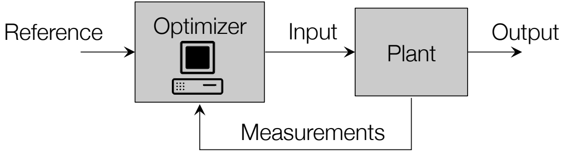
Dynamic Motion Planning
Embotech (w/ thyssenkrupp)
`www.embotech.com/automotive`

Optimization in the loop

Classical control loop:



The classical controller is replaced by an optimization algorithm:



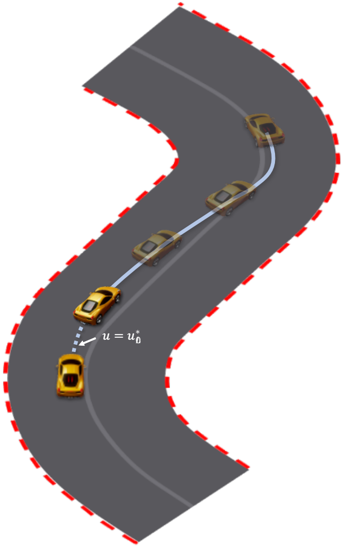
The optimization uses predictions based on a model of the plant.

Basic MPC Problem Formulation

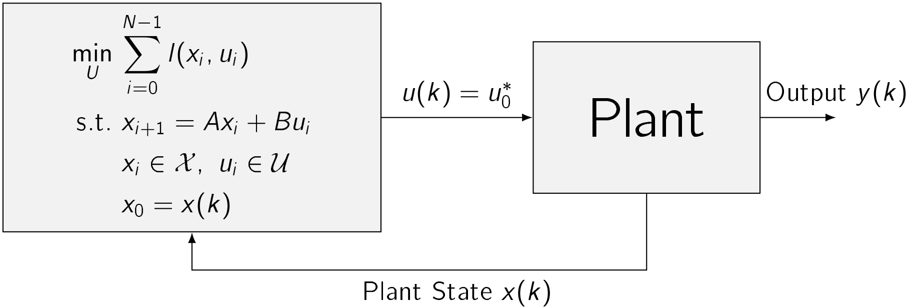
$$
\begin{aligned}
\min_{U} \; & \sum_{i=0}^{N-1} l(x_i, u_i) & \\
\text{s.t.} \; & U = \{u_0, u_1, \ldots, u_{N-1}\} & \text{optimization variables} \\
& x_{i+1} = f(x_i, u_i) & \text{system model} \\
& x_i \in \mathcal{X} & \text{state constraints} \\
& u_i \in \mathcal{U} & \text{input constraints} \\
& x_0 = x(k) & \text{measurement/initialization}
\end{aligned}
$$

Problem is defined by

- **Objective** that is minimized
- Internal **system model** to predict system behavior
- **Constraints** that have to be satisfied



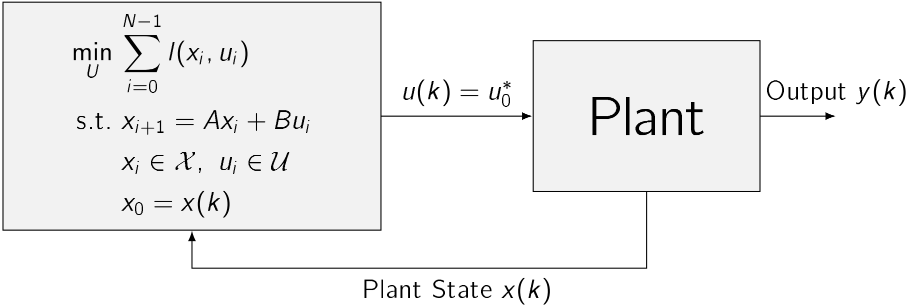
Optimization in the Loop: Receding horizon control

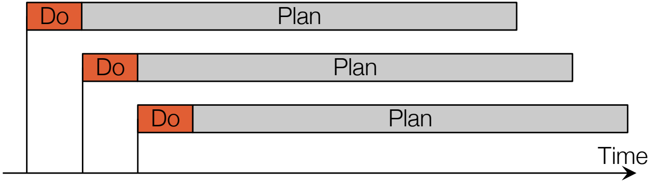


At each sample time:

- Measure / estimate current state $x(k)$
- Find the optimal input sequence for the entire planning window N: $U^* = \{u_0^*, u_1^*, \ldots, u_{N-1}^*\}$
- Implement only the **first** control action $u(k) = u_0^*$

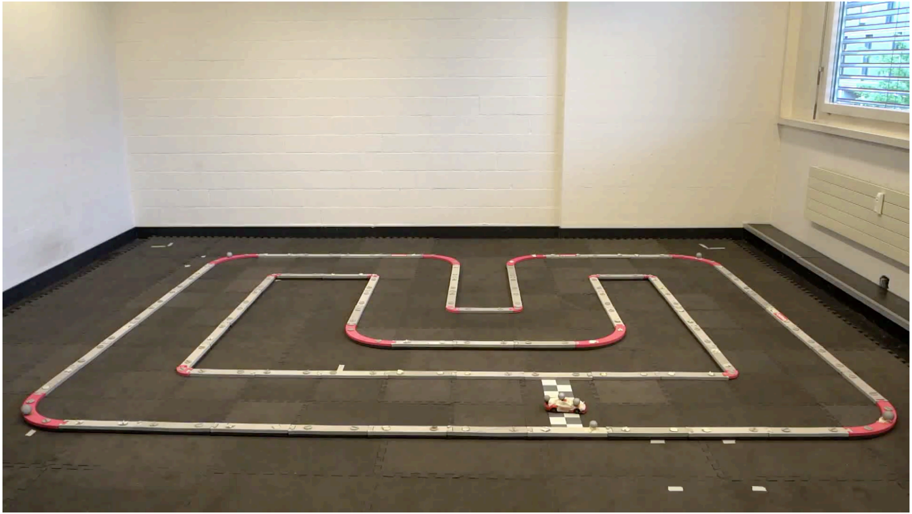
Optimization in the Loop: Receding horizon control





Receding horizon strategy introduces **feedback**.

Examples: MPC for Autonomous Racing



Autonomous racing, CRS @ IDSC, ETH Zurich

Common MPC Assumptions

- Regularization, set point tracking
- Linear systems
- No model uncertainties or bounded additive uncertainties

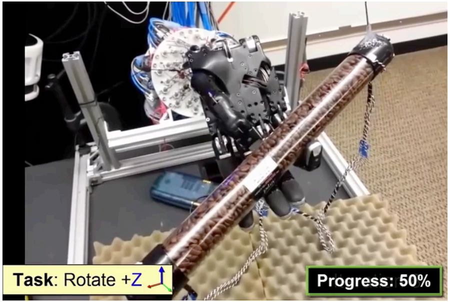
$\rightarrow$ MPC requires good model and suitable cost function.

$$
\begin{aligned}
\min_{U} \; & \sum_{i=0}^{N-1} l(x_i, u_i) & \\
\text{s.t. } & U = \{u_0, u_1, \ldots, u_{N-1}\} & \text{optimization variables} \\
& x_{i+1} = f(x_i, u_i) & \text{system model} \\
& x_i \in \mathcal{X} & \text{state constraints} \\
& u_i \in \mathcal{U} & \text{input constraints} \\
& x_0 = x(k) & \text{measurement/initialization}
\end{aligned}
$$

Challenges

Which problems can we not address with basic MPC yet?

Following complex references / goals



[*ADROIT Manipulation Platform*]
`https://www.youtube.com/watch?v=IN7VFquI7ow`

[*Boston Dynamics*]
`https://www.youtube.com/watch?v=kHBcVlqpvZ8`

Uncertainties and variation

Outline

1. Motivation

    MPC

    Learning

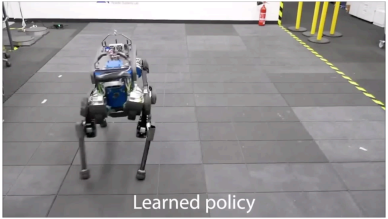
Learning-based Control

Various learning-based concepts have been considered in the literature.

For example: Adaptive Control, Iterative Learning Control, Reinforcement Learning

Example Demonstrations of Learning-based Control

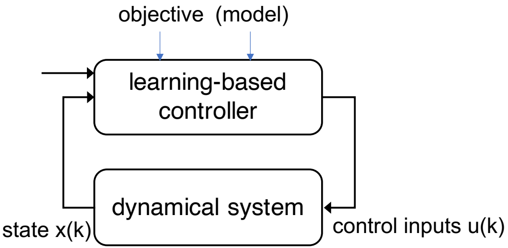
[*Google Deep Mind*]
https://www.youtube.com/watch?v=gn4nRCC9TwQ



[*Robotic Systems Lab, ETH Zurich*]
https://www.youtube.com/watch?v=aTDkYFZFWug&feature=emb_logo

[*UZH Robotics and Perception Group*]
https://www.youtube.com/watch?v=2N_wKXQ6MXA

Challenge of Learning-based Control: Safety



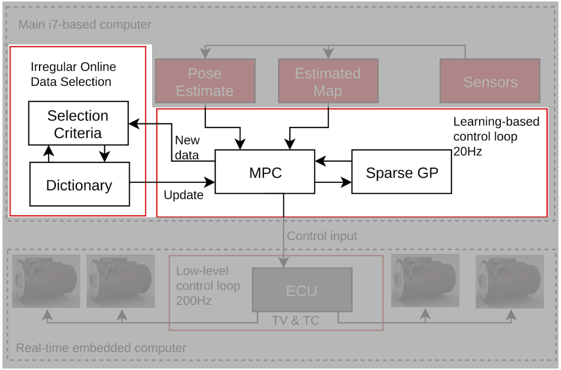
Dilemma:

Need to act to collect data & explore

But: Every action can be safety-critical

> Goal of Learning-based MPC:
>
> Combine constraint satisfaction, i.e. safety, with performance gained through learning

Example: Learning-based MPC for AMZ driverless race car



→ Continuous model improvement while racing

Example: Learning-based MPC for AMZ driverless race car

https://www.youtube.com/watch?v=aCDPwZZm9C4

Learning Objectives Lecture 1: MPC Review

- Understand theoretical challenges of MPC
- Learn how to address these challenges (nominal MPC theory)
  - Prove recursive feasibility
  - Prove asymptotic stability
- Modify MPC formulation to deal with bounded disturbances (robust MPC)

Outline

1. Motivation

2. Review: Nominal MPC theory

3. Review: Robust MPC

Outline

1. Motivation

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3. Review: Robust MPC

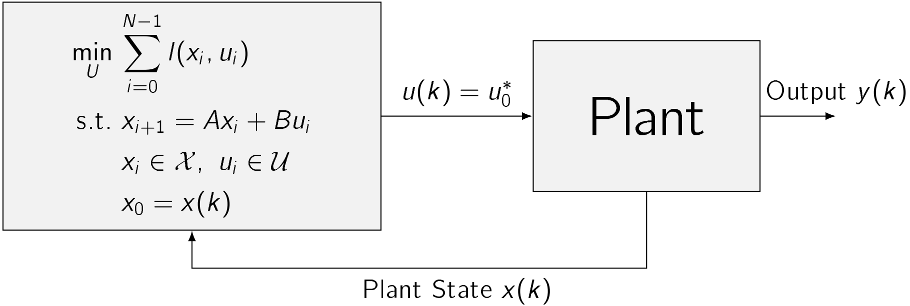
Recall: Nominal MPC - Main goals

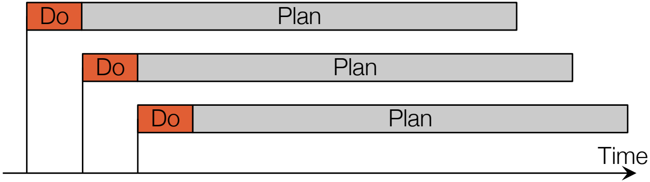
Nominal system
$x(k+1) = f(x(k), u(k))$ $\qquad x \in \mathcal{X}, u \in \mathcal{U}$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satifies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for all k
2. Is stable: $\lim_{k\to\infty} x(k) = 0$
3. Optimizes “performance”
4. Maximizes the set $\{x_0 \mid \text{Conditions 1-3 are met}\}$

Receding Horizon Control

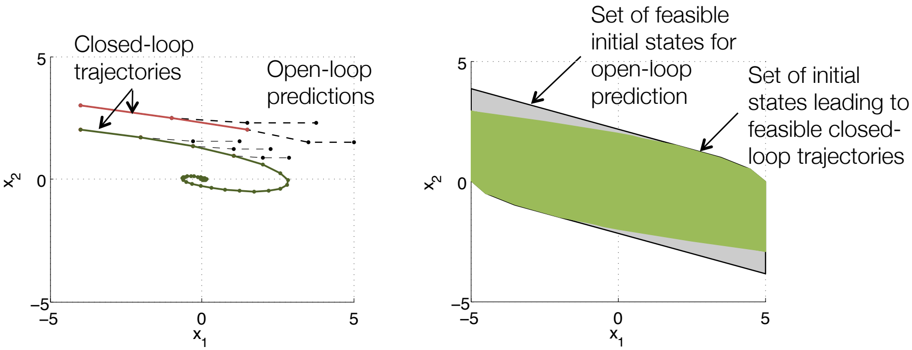




Receding horizon strategy introduces **feedback**.

Challenges in MPC

Theoretical challenges in MPC result from the finite-horizon problem and receding horizon implementation: mismatch between prediction and closed-loop



- Recursive feasibility, i.e. the MPC problem may not have a solution
- Stability / Convergence, i.e. trajectories may grow unbounded or not converge

Feasibility and stability in MPC - Main idea

Main idea: Select terminal cost and constraints to ensure feasibility and stability by design:

$$
\begin{aligned}
V_N^\star(x(k)) = \quad \min_{U} \quad & l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) && \text{Terminal Cost} \\
\text{subj. to} \quad & \\
& x_{i+1} = f(x_i, u_i),\ \ i = 0, \ldots, N-1 \\
& x_i \in \mathcal{X},\ \ u_i \in \mathcal{U},\ \ i = 0, \ldots, N-1 \\
& x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& x_0 = x(k)
\end{aligned}
$$

- $l_f(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon
    - $l_f(\cdot)$ approximates the 'tail' of the cost
    - $\mathcal{X}_f$ approximates the 'tail' of the constraints
- $\mathcal{X}_f$ denotes the feasible set

Feasibility and Stability of MPC: Proof

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Outline

2. Review: Nominal MPC theory

   Invariance and Lyapunov Stability

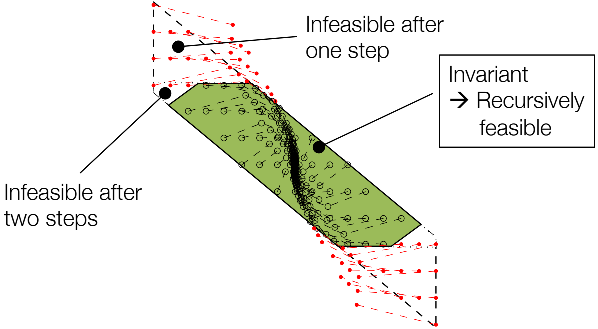
   Main Assumptions and Proof

Property 1: Invariance

Positive Invariant set
A set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system $x(k+1) = f(x(k))$ if $$x(k) \in \mathcal{O} \;\Rightarrow\; x(k+1) \in \mathcal{O} \;,\quad \forall k \in \{0, 1, \dots\}$$

The positively invariant set that contains every closed positively invariant set is called the maximal invariant set $\mathcal{O}_\infty$.

If the invariant set is within the constraints, it provides a set of initial states from which the trajectory will never violate the system constraints.

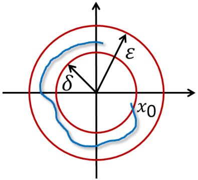


Property 2: Lyapunov and Asymptotic Stability

Lyapunov stability

Formally, the equilibrium point $\bar{x}$ of a system $x(k+1) = f(x(k))$ is **Lyapunov stable** if for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\|x(0) - \bar{x}\| < \delta(\epsilon) \rightarrow \|x(k) - \bar{x}\| < \epsilon, \forall k \geq 0$$



Asymptotic stability

Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded positive invariant set of system $x(k+1) = f(x(k))$ containing the origin. An equilibrium point $\bar{x} \in \Omega$ is

- **asymptotically stable** in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and **attractive**, i.e.

$$\lim_{k \to \infty} \|x(k) - \bar{x}\| = 0, \ \forall x(0) \in \Omega$$

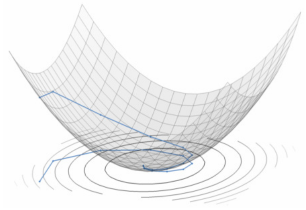
- **globally asymptotically stable** if it is asymptotically stable and $\Omega = \mathbb{R}^n$

Lyapunov Function

Definition: Lyapunov function

Consider the equilibrium point $x = 0$. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded positive invariant set of system $x(k+1) = f(x(k))$ containing the origin. A function $V : \mathbb{R}^n \to \mathbb{R}$, continuous at the origin, finite for every $x \in \Omega$, and such that

$$V(0) = 0 \text{ and } V(x) > 0 \ \forall x \in \Omega \setminus \{0\}$$
$$V(f(x)) - V(x) \leq -\alpha(x) \ \forall x \in \Omega$$



where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite,

is called a **Lyapunov function**.

Theorem: Lyapunov stability (asymptotic stability)

If a system admits a Lyapunov function $V(x)$, then $x = 0$ is **asymptotically stable** in Ω.

Outline

2. Review: Nominal MPC theory

   Invariance and Lyapunov Stability

   Main Assumptions and Proof

Stability of MPC - Main Assumptions

Assumptions

1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $\pi_f(x_i)$:

$$x_{i+1} = Ax_i + B\pi_f(x_i) \in \mathcal{X}_f, \quad \text{for all } x_i \in \mathcal{X}_f$$

   All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ \pi_f(x_i) \in \mathcal{U}, \quad \text{for all } x_i \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$l_f(x_{i+1}) - l_f(x_i) \leq -l(x_i, \pi_f(x_i)), \quad \text{for all } x_i \in \mathcal{X}_f$$

Under those 3 assumptions:

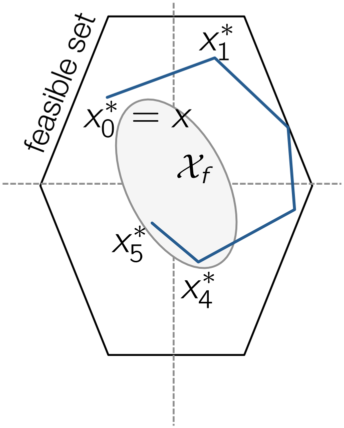
Theorem

The closed-loop system under the MPC control law $u_0^\star(x)$ is asymptotically stable and the feasible set $\mathcal{X}_N$ is positive invariant for the system

$$x(k+1) = Ax(k) + Bu_0^\star(x(k)).$$

Outline of the Proof of Recursive Feasibility

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \dots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and $\{x(k),\ x_1^\star,\ \dots,\ x_N^\star\}$ the corresponding state trajectory



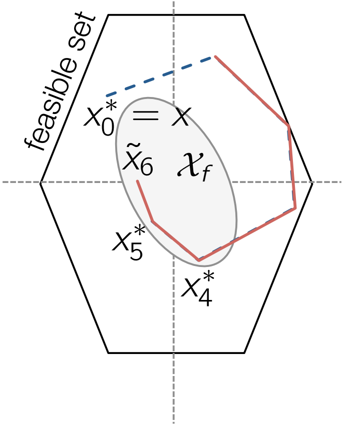
Outline of the Proof of Recursive Feasibility

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \dots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and $\{x(k),\ x_1^\star,\ \dots,\ x_N^\star\}$ the corresponding state trajectory

- At $x(k+1) = x_1^\star$, the control sequence $\tilde{U} = \{u_1^\star,\ u_2^\star,\ \dots,\ \pi_f(x_N^\star)\}$ is feasible:

$$x_N^\star \text{ is in } \mathcal{X}_f \rightarrow \pi_f(x_N^\star) \text{ is feasible}$$
$$\text{and } Ax_N^\star + B\pi_f(x_N^\star) \text{ in } \mathcal{X}_f$$

$\Rightarrow$ **Terminal constraint provides recursive feasibility**



Asymptotic Stability of MPC - Outline of the Proof

$$J^\star(x(k)) = \sum_{i=0}^{N-1} l(x_i^\star, u_i^\star) + l_f(x_N^\star)$$

At $x(k+1) = x_1^\star$, $\tilde{U} = \{u_1^\star,\ u_2^\star,\ \ldots,\ \pi_f(x_N^\star)\}$ is feasible & sub-optimal

$$\begin{aligned}
J^\star(x(k+1)) &\le \sum_{i=1}^{N-1} l(x_i^\star, u_i^\star) + l(x_N^\star, \pi_f(x_N^\star)) + l_f(Ax_N^\star + B\pi_f(x_N^\star)) \\
&= \underbrace{\sum_{i=0}^{N-1} l(x_i^\star, u_i^\star)}_{J^\star(x(k)) - l_f(x_N^\star)} - l(x_0^\star, u_0^\star) + l(x_N^\star, \pi_f(x_N^\star)) + l_f(Ax_N^\star + B\pi_f(x_N^\star)) \\
&= J^\star(x(k)) - l(x(k), u_0^\star) + \underbrace{l_f(Ax_N^\star + B\pi_f(x_N^\star)) - l_f(x_N^\star) + l(x_N^\star, \pi_f(x_N^\star))}_{\le 0 \text{ by Assumption 3}}
\end{aligned}$$

$$\Longrightarrow J^\star(x(k+1)) - J^\star(x(k)) \le -l(x(k), u_0^\star), \quad l(x,u) > 0 \text{ for } x, u \ne 0$$

$J^\star(x)$ is a Lyapunov function

$\Rightarrow$ The closed-loop system under the MPC control law is asymptotically stable

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

$$
\begin{aligned}
J^\star(x(k)) = \quad & \min_{U} \quad x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i && \text{Terminal Cost} \\
& \text{subj. to} \\
& \quad x_{i+1} = A x_i + B u_i, \;\; i = 0, \ldots, N-1 \\
& \quad x_i \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \;\; i = 0, \ldots, N-1 \\
& \quad x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& \quad x_0 = x(k)
\end{aligned}
$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- Design unconstrained LQR control law

$$F_\infty = -(B^\mathsf{T} P_\infty B + R)^{-1} B^\mathsf{T} P_\infty A$$

  where P_∞ is the solution to the discrete-time algebraic Riccati equation:

$$P_\infty = A^\mathsf{T} P_\infty A + Q - A^\mathsf{T} P_\infty B (B^\mathsf{T} P_\infty B + R)^{-1} B^\mathsf{T} P_\infty A$$

- Choose the terminal weight $P = P_\infty$
- Choose the terminal set $\mathcal{X}_f$ to be an invariant set for the closed-loop system $x_{k+1} = (A + BF_\infty)x_k$

$\rightarrow$ All the assumptions for feasibility and stability theorem are satisfied.

Invariant Sets from Lyapunov Functions

Lemma: Invariant set from Lyapunov function

If $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system $x(k+1) = f(x(k))$, then

$$Y := \{x \,|\, V(x) \leq \alpha\}$$

is an invariant set for all $\alpha \geq 0$.

"Proof": We have the basic properties:

- $V(x) \geq 0$ for all x
- $V(f(x)) - V(x) < 0$

The second property implies that once $V(x(k)) \leq \alpha$, $V(x(j))$ will be less than α for all $j \geq k$
$\rightarrow$ Invariance $\square$

We often want the largest invariant set contained in our constraints.

If V is a Lyapunov function for the system $x(k+1) = f(x(k))$, and our constraints are given by the set $\mathcal{X}$, then we maximize α such that

$$Y_\alpha := \{x \,|\, V(x) \leq \alpha\} \subseteq \mathcal{X}$$

Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for feasibility and stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attraction for small N
- Efficient design available for linear systems with quadratic cost
- With larger horizon length N, region of attraction approaches maximum control invariant set

Outline

1. Motivation

2. Review: Nominal MPC theory

3. Review: Robust MPC

Robust MPC: Motivation

1. MPC relies on a model, but models are far from perfect

2. Noise and model inaccuracies can cause:
    - Constraint violation
    - Sub-optimal behaviour

3. Persistent noise prevents the system from converging to a single point

4. Can incorporate some noise models into the MPC formulation

Goals of Robust Constrained Control

Uncertain Constrained System
$x(k+1) = f(x(k), u(k), w(k); \theta) \qquad x, u \in \mathcal{X}, \mathcal{U} \quad w \in \mathcal{W} \quad \theta \in \Theta$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satifies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for all disturbance realizations
2. Is stable: Converges to a neighbourhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Meeting these goals requires some knowledge/assumptions about the uncertainties w and θ.

Goals of Robust Constrained Control

Constrained system with additive uncertainty
$x(k+1) = f(x(k), u(k)) + w(k) \qquad x \in \mathcal{X}, u \in \mathcal{U} \qquad w \in \mathcal{W}$

Design control law $u(k) = \pi(x(k))$ such that the system:

1. Satifies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$ for all disturbance realizations
2. Is stable: Converges to a neighbourhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Challenge: Cannot predict where the state of the system will evolve
We can only compute a set of trajectories that the system *may* follow

Idea: Design a control law that will satisfy constraints and stabilize the system *for all possible disturbances*

Learning Objectives: Robust MPC

- Linear Robust MPC using constraint tightening
  - Learn a simple robust MPC method based on constraint tightening
  - (Proof of recursive feasibility in Bonus exercise)
- Linear Tube-MPC
  - Understand concept of tubes
  - Proof recursive feasibility and robust stability

Outline

3. Review: Robust MPC

    Tools: Robust Invariance, Set Difference & Sum

    Linear Constraint Tightening Robust MPC

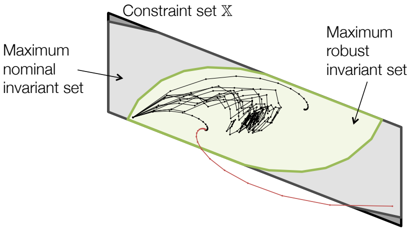
    Linear tube-MPC

    Remarks

Robust Invariance

Robust constraint satisfaction, for **autonomous** system $x(k+1) = f(x(k), w(k))$,
or **closed-loop** system $x(k+1) = f(x(k), \pi(x(k)), w(k))$ for a **given** controller π.

Robust Positive Invariant set
A set $\mathcal{O}^{\mathcal{W}}$ is said to be a robust positive invariant set for the autonomous system $x(k+1) = f(x(k), w(k))$ if $$x \in \mathcal{O}^{\mathcal{W}} \quad \Rightarrow \quad f(x, w) \in \mathcal{O}^{\mathcal{W}} \quad , \quad \text{for all } w \in \mathcal{W}$$



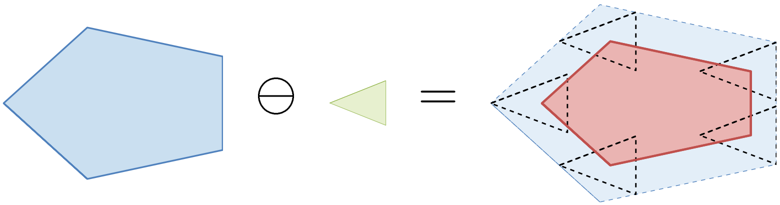
Minkowski Sum and Pontryagin Difference

Minkowski Sum

Let A and B be subsets of $\mathbb{R}^n$. The Minkowski Sum is $A \oplus B := \{x + y \,|\, x \in A, y \in B\}$

Pontryagin Difference

Let A and B be subsets of $\mathbb{R}^n$. The Pontryagin Difference is $A \ominus B := \{x \,|\, x + e \in A \ \forall e \in B\}$



A $\qquad$ B $\qquad$ $A \ominus B$ (Red)

Outline

3. Review: Robust MPC

    Tools: Robust Invariance, Set Difference & Sum

    Linear Constraint Tightening Robust MPC

    Linear tube-MPC

    Remarks

Uncertain State Evolution

Nominal system

$$x(k+1) = Ax(k) + Bu(k)$$

$$
\begin{aligned}
x_1 &= Ax_0 + Bu_0 \\
x_2 &= A^2x_0 + ABu_0 + Bu_1 \\
&\vdots \\
x_i &= A^ix_0 + \sum_{j=0}^{i-1} A^jBu_{i-1-j}
\end{aligned}
$$

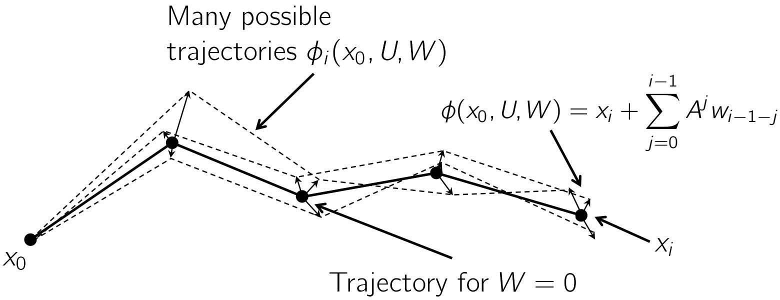
Uncertain system

$$x(k+1) = Ax(k) + Bu(k) + w(k) \ , w \in \mathcal{W}$$

$$
\begin{aligned}
\phi_1 &= Ax_0 + Bu_0 + w_0 \\
\phi_2 &= A^2x_0 + ABu_0 + Bu_1 + Aw_0 + w_1 \\
&\vdots \\
\phi_i &= A^ix_0 + \sum_{j=0}^{i-1} A^jBu_{i-1-j} + \sum_{j=0}^{i-1} A^jw_{i-1-j} \\
\phi_i &= x_i + \sum_{j=0}^{i-1} A^jw_{i-1-j}
\end{aligned}
$$

Uncertain evolution is the nominal system + offset caused by the disturbance (Follows from linearity)

Uncertain State Evolution



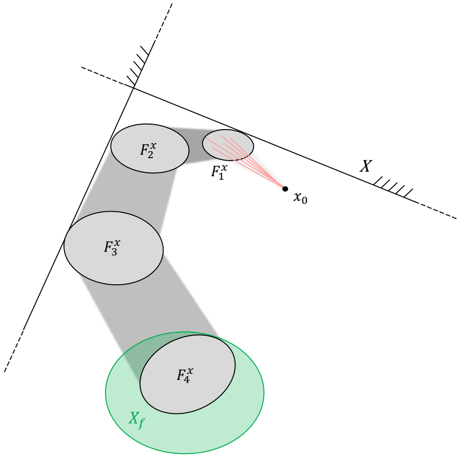
Robust Constraint Satisfaction – Notation

Goal: Ensure that constraints are satisfied for the sequence of sets of possible states

Given input sequence $U = \{u_0, \dots, u_{N-1}\}$ and initial condition x_0 the set of **reachable states** of system $x(k+1) = f(x(k), u(k)) + w(k)$ can be iteratively computed as

$$\mathcal{F}_0^x = \{x_0\},\ \mathcal{F}_{i+1}^x = f(\mathcal{F}_i^x, u_i) \oplus \mathcal{W},$$

where $f(\mathcal{F}_i^x, u_i) = \{x^+ \mid x^+ = f(x,u), u = u_i, x \in \mathcal{F}_i\}$



Ensure robust satisfaction of constraints

$$\mathcal{F}_i^x \subseteq \mathcal{X}$$

Error System Decomposition for Linear Systems

Linear systems $f(x,u) = Ax + Bu$ allow for an error system decomposition

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ z(k+1) &= Az(k) + Bu(k) \\ e(k+1) &= x(k) - u(k) = Ae(k) + w(k) \end{aligned}$$

with $x(0) = z(0)$.

Reachable set computation can be simplified: $\mathcal{F}_k^x = \{z(k)\} \oplus \mathcal{F}_k$

where $\mathcal{F}_k$ is the **disturbance reachable set** for the error system initialized at $e(0) = 0$

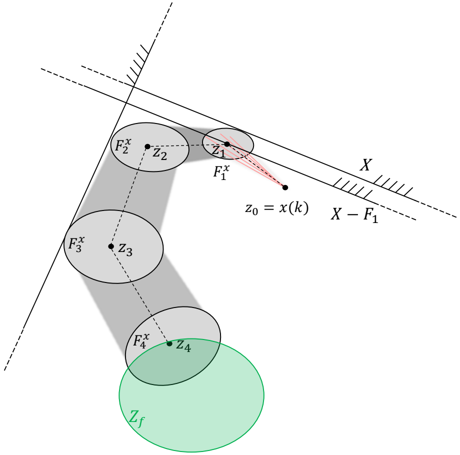
$$\mathcal{F}_{k+1} = A\mathcal{F}_k \oplus \mathcal{W} \text{ or equivalently } \mathcal{F}_k = \bigoplus_{j=1}^{k} A^{j-1}\mathcal{W},\ \mathcal{F}_0 = \{0\}$$

which is the minimum set containing all possible error states: $e(k) \in \mathcal{F}_k$

Linear Constraint Tightening MPC: Main Idea

Optimizing **nominal cost** subject to **tightened constraints** on the nominal system

$$
\begin{aligned}
\min_{\{u_i\}} \quad & \|z_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|u_i\|_R^2 \\
\text{s.t.} \quad & z_{i+1} = Az_i + Bu_i, \\
& z_i \in \mathcal{X} \ominus \mathcal{F}_i, \\
& u_i \in \mathcal{U}, \\
& z_N \in \mathcal{X}_f \ominus \mathcal{F}_N, \\
& z_0 = x(k)
\end{aligned}
$$



Ancillary/Tube Controller

- Disturbance reachable sets $\mathcal{F}_i$ can grow quickly (due to the lack of feedback in the prediction)
- Typical approach: Introduce feedback into the prediction, e.g. $u_i = K(x_i - z_i) + v_i$
  - **Nominal input** v steers the nominal system state z
  - Predefined **tube controller** K ensures that x remains close to z
  - Assume: $A + BK$ is Schur stable, i.e., all eigenvalues inside the unit circle

Error decomposition remains valid

$$\begin{aligned} x_{i+1} &= Ax_i + B(K(x_i - z_i) + v_i) + w_i \\ z_{i+1} &= Az_i + Bv_i \\ e_{i+1} &= (A + BK)e_i + w_i \end{aligned}$$

$\rightarrow$ Resulting disturbance reachable sets: $\mathcal{F}_i = \bigoplus_{j=1}^{i} (A + BK)^{j-1}\mathcal{W}, \ \mathcal{F}_0 = \{0\}$

But we now need to ensure $u_i = Ke_i + v_i \in \mathcal{U} \Leftarrow v_i \in \mathcal{U} \ominus K\mathcal{F}_i$

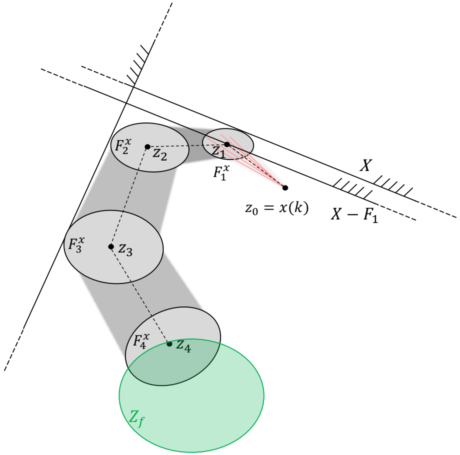
Terminal Set

Ensuring recursive feasibility under disturbances requires stronger conditions on terminal set $\mathcal{X}_f$:

- Robust invariant w.r.t. $f(x, w) = (A + BK)x + w$ and $w \in \mathcal{W}$
- Within state constraints $\mathcal{X}_f \subseteq \mathcal{X}$
- Within input constraints $K\mathcal{X}_f \subseteq \mathcal{U}$

Resulting Formulation: Constraint-Tightening Robust MPC

$$
\begin{aligned}
\min_{\{v_i\}} \quad & \|z_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_Q^2 + \|v_i\|_R^2 \\
\text{s.t.} \quad & z_{i+1} = Az_i + Bv_i, \\
& z_i \in \mathcal{X} \ominus \mathcal{F}_i, \\
& v_i \in \mathcal{U} \ominus K\mathcal{F}_i, \\
& z_N \in \mathcal{X}_f \ominus \mathcal{F}_N, \\
& z_0 = x(k)
\end{aligned}
$$



- Only optimize over a nominal trajectory
- Predefined K: Tradeoff state vs input tightening
- Similar (online) complexity to nominal MPC
- (Proof of recursive feasibility see Bonus exercise)

Outline

3. Review: Robust MPC

   Tools: Robust Invariance, Set Difference & Sum

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   Linear tube-MPC

   Remarks

Linear tube-MPC

Uncertain constrained linear system
$x(k+1) = f(x(k), u(k)) + w(k) \qquad x, u \in \mathcal{X}, \mathcal{U} \qquad w \in \mathcal{W}$

Main idea:

- Seperate the true system (x, u) into a 'nominal' system (z, v) and error $e = x - z$.
- Drive 'nominal' (disturbance free) system to the origin $z(k+1) = Az(k) + Bv(k)$
- Ensure that the real trajectory "tracks" the nominal trajectory by using a linear controller K, which stabilizes the nominal system $u(k) = K(x(k) - z(k)) + v(k)$ (K fixed offline)
- Bound maximum error, or how far the 'real' trajectory can be from the nominal **at any time**
- Tighten constraints by maximum error set

$\rightarrow$ Online complexity similar to nominal MPC

Further reading: D.Q. Mayne, M.M. Seron and S.V. Rakovic, Robust model predictive control of constrained linear systems with bounded disturbances, Automatica, Volume 41, Issue 2, February 2005

Error Set: Minimum Robust Invariant Set

- Error dynamics:

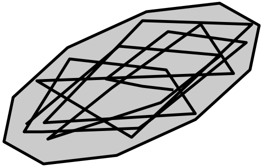
$$\begin{aligned} e_{i+1} &= x_{i+1} - z_{i+1} \\ &= (A + BK)e_i + w_i \end{aligned}$$

- Compute maximum error set for

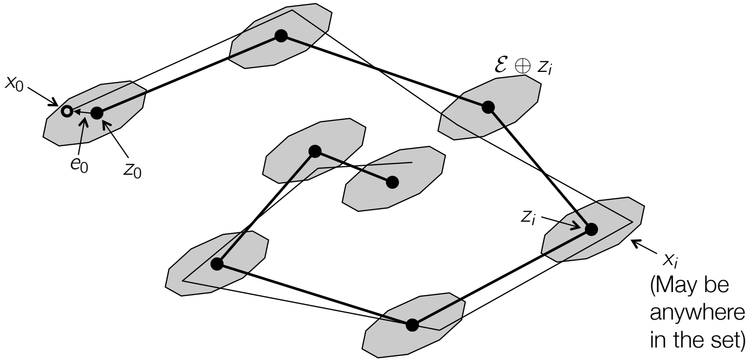
$$e_{i+1} = (A + BK)e_i + w_i, \quad w_i \in \mathcal{W} \quad \forall i$$

  Dynamics $A + BK$ are stable, and the set $\mathcal{W}$ is bounded, so there is some set $\mathcal{E}$ that e will stay inside for all time.
  $\rightarrow$ Ideally we want the smallest such set, i.e. the minimal robust invariant set

$$\mathcal{F}_\infty = \bigoplus_{j=1}^{\infty} A^{j-1}\mathcal{W}, \ \mathcal{F}_0 = \{0\}$$

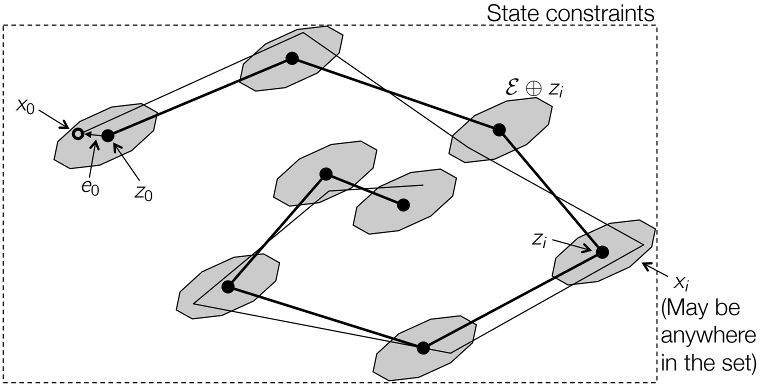


Tube-MPC : The Idea



We know that the real trajectory stays 'nearby' the nominal one: $x_i \in z_i \oplus \mathcal{E}$
because we plan to apply the controller $u_i = K(x_i - z_i) + v_i$ **in the future**
(we won't actually do this, but it's a valid sub-optimal plan)

Tube-MPC : The Idea



We must ensure that all possible state trajectories satisfy the constraints:

$$z_i \oplus \mathcal{E} \subseteq \mathcal{X} \Leftrightarrow z_i \in \mathcal{X} \ominus \mathcal{E}, \quad u_i \in v_i \oplus K\mathcal{E} \subseteq \mathcal{U} \Leftrightarrow v_i \in \mathcal{U} \ominus K\mathcal{E}$$

Tube-MPC

$$
\begin{aligned}
J^\star(x(k)) = \min_{V,Z} \quad & J(Z,V) = \sum_{k=0}^{N-1} l(z_k, v_k) + l_f(z_N) \\
\text{s.t.} \quad & z_{i+1} = Az_i + Bv_i, \quad i \in [0, N-1], \\
& z_i \in \overline{\mathcal{X}} = \mathcal{X} \ominus \mathcal{E}, \quad i \in [0, N-1], \\
& v_i \in \overline{\mathcal{U}} = \mathcal{U} \ominus K\mathcal{E}, \quad i \in [0, N-1], \\
& z_N \in \mathcal{Z}_f, \\
& x(k) \in \{z_0\} \oplus \mathcal{E}
\end{aligned}
$$

- Nominal state and input predictions $Z = (z_0, \dots, z_N)$, $V = (v_0, \dots, v_{N-1})$
- Tightened nominal constraints $\overline{\mathcal{X}}$, $\overline{\mathcal{U}}$
- Terminal set $\mathcal{Z}_f$ invariant set for the nominal system within tightened constraints
- Initial state constraint: $x_0 \in z_0 \oplus \mathcal{E}$
- Control law: $\mu_{\text{tube}}(x) = K(x - z_0^\star(x)) + v_0^\star(x)$

Tube-MPC - Theoretical Properties

Robust Stability of tube-MPC

The tube-MPC is recursively feasible, $(x(k), u(k)) \in \mathcal{X} \times \mathcal{U}$, $\forall k \geq 0$, and the state $x(k)$ converges to the set $\mathcal{E}$ for the closed-loop system $x(k+1) = Ax(k) + B\mu_{\mathrm{tube}}(x(k)) + w(k)$.

- Let $(\{(v_0^\star, \ldots, v_{N-1}^\star\}, \{z_0^\star, \ldots, z_N^\star\})$ be the optimal solution at time k.
- Consider candidate $\{v_1^\star, \ldots, v_{N-1}^\star, \kappa_f(z_N^\star)\}$, $\{z_1^\star, \ldots, z_N^\star, Az_N + B\kappa_f(z_N^\star)\}$
- Feasible solution (nominal proof + $\mathcal{E}$ RPI)
- Constraint satisfaction follows from tightened constraint sets
- Cost $J^\star$ is a Lyapunov function for nominal system $\Rightarrow$ z converges to the origin
- x converges to $\mathcal{E}$

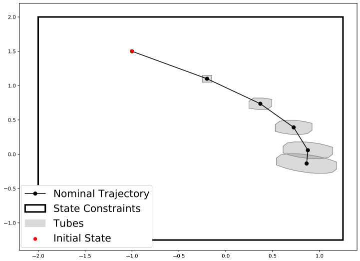
$\Rightarrow$ Strong stability guarantees and clearly defined region of attraction.

Summary: From Error Bounds to Robust MPC - Two Variants

Consider error dynamics $e_{i+1} = (A + BK)e_i + w_i = A_K e_i + w_i \quad w_i \in \mathcal{W}$

Disturbance reachable sets (DRS):

$$e_i \in \mathcal{F}_i = \bigoplus_{j=0}^{i-1} A_K^j \mathcal{W}, \text{ if } e_0 = 0$$

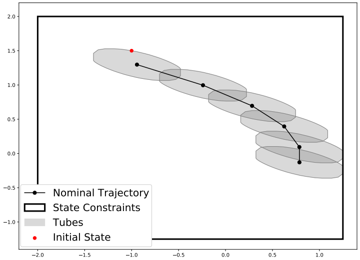


→ Constraint-tightening MPC

Robust positively invariant (RPI) set:

$$e_i \in \mathcal{E} \Rightarrow e_{i+1} \in \mathcal{E}$$

for any $e_0 \in \mathcal{E}$, we have $e_i \in \mathcal{E}$



→ Tube MPC

Outline

3. Review: Robust MPC

   Tools: Robust Invariance, Set Difference & Sum

   Linear Constraint Tightening Robust MPC

   Linear tube-MPC

   Remarks

Remarks

Extension to nonlinear systems:

- Concepts for constrained tightening and tube MPC extend to nonlinear systems
- But: Clear separation into nominal and error system not possible (error evolution depends on x, z)
  $\rightarrow$ requires different definitions and tools
- Computation of reachable sets and tracking controllers more involved

Robust stability of nominal MPC:

- The nominal MPC controller has robust stability properties for linear systems and for nonlinear systems under continuity assumptions
- Benefit: No explicit knowledge of noise set $\mathcal{W}$ required
- But: Robust constraint satisfaction only in robust invariant set, which is difficult to compute
- Hard to tune, no obvious way to trade off robustness against performance