Invariant Sets from Lyapunov Functions

Consider the system $x(k+1) = Ax(k)$, and assume $P \succ 0$ satisfies the condition

$$A^T PA - P \prec 0$$

Then the function $V(x(k)) = x(k)^T Px(k)$ is a Lyapunov function.

Our goal is to find the largest α such that the invariant set Y_α is contained in the system constraints $\mathcal{X}$:

$$Y_\alpha := \left\{x \,\middle|\, x^T Px \leq \alpha\right\} \subset \mathcal{X} := \{x \,|\, Fx \leq f\}$$

Equivalently, we want to solve the problem:

$$\begin{aligned} &\max_{\alpha} \alpha \\ \text{subj. to } &h_{Y_\alpha}(F_i) \leq f_i \text{ for all } i \in \{1, \dots, n\} \end{aligned} \tag{1}$$

Maximum Ellipsoidal Invariant Sets

Support of an ellipse:

$$\begin{aligned} h_{Y_\alpha}(\gamma) = \max_x \ & \gamma^T x \\ \text{subj. to } & x^T P x \le \alpha \end{aligned} \tag{2}$$

Change of variables $y := P^{1/2}x$

$$\begin{aligned} h_{Y_\alpha}(\gamma) = \max_y \ & \gamma^T P^{-1/2} y \\ \text{subj. to } & y^T y \le \sqrt{\alpha}^2 \end{aligned} \tag{3}$$

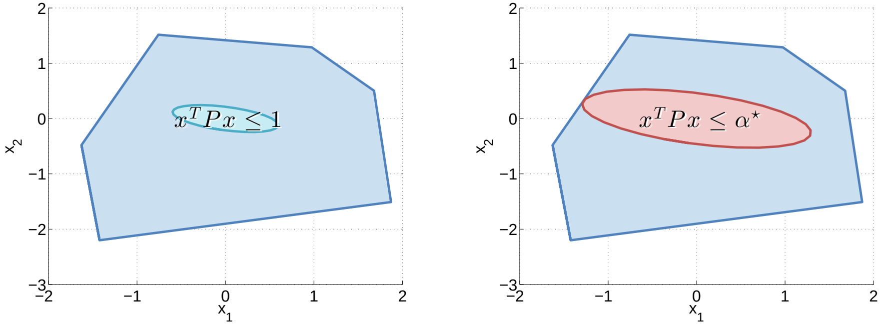
which can be solved by inspection:

$$h_{Y_\alpha}(\gamma) = \gamma^T P^{-1/2} \frac{P^{-1/2}\gamma}{\|P^{-1/2}\gamma\|}\sqrt{\alpha} = \|P^{-1/2}\gamma\|\sqrt{\alpha}$$

Maximum Ellipsoidal Invariant Sets

Largest ellipse in a polytope is now a one-dimensional optimization problem:

$$\begin{aligned}\alpha^\star &= \max_{\alpha}\ \alpha \quad \text{s.t.}\ \ \|P^{-1/2}F_i^T\|^2\alpha \le f_i^2 \text{ for all } i \in \{1,\dots,n\} \\ &= \min_{i\in\{1,\dots,n\}} \frac{f_i^2}{F_i P^{-1} F_i^T}\end{aligned}$$



It is possible to optimize over P, maximizing the volume of the ellipse, subject to stability and containment constraints (convex semi-definite program)