EUROPEAN EMBEDDED CONTROL INSTITUTE (EECI) – INTERNATIONAL GRADUATE SCHOOL ON CONTROL

Learning-Based Predictive Control

Chapter 6
Scenario MPC

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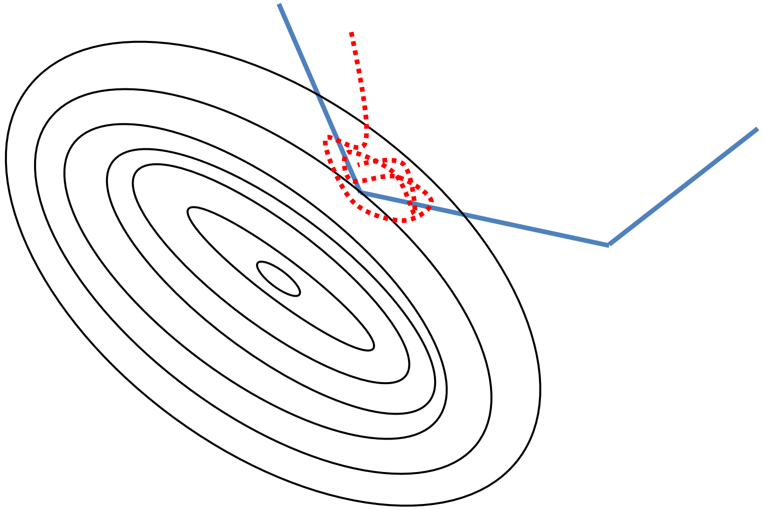
Motivation - type of problems

Scenario MPC is well suited for problems with stochastic uncertainty (variable "$\delta \in \mathbb{R}^{n_\delta}$") where:

- The model features **linear dynamics** and **convex cost and constraint functions** (w.r.t. state and input)
- One wants to control the **actual** (i.e., in close loop) **rate of constraint violations** (average in time n. of violations)
- **Violating constraints is beneficial** for the sake of the cost function
- Out-of-constraint situations can be **always recovered** (no feasibility problems)
- There is a mechanism to obtain i.i.d. uncertainty samples → **data**
- Possible hard (also robust) constraints can still be included

Motivation - example



Convex w.r.t. $x(t), u(t)$

$$\min_{\phi : u = \phi(x, \cdot)} \frac{1}{T} \sum_{t=0}^{T-1} l\big(x(t), u(t)\big)$$

$\delta(0), \delta(1), \ldots$ i.i.d. random variables

subject to

$$x(t+1) = A(\delta(t))x(t) + B(\delta(t))u(t) + w\big(\delta(t)\big)$$

$$\frac{1}{T} \sum_{i=0}^{T-1} I\big(x(t) \notin \mathbb{X}(\delta(t))\big) \leq \varepsilon$$

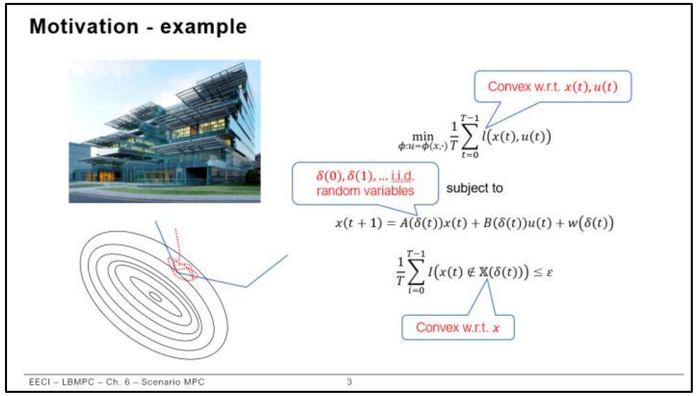
Convex w.r.t. x

Scenario MPC

Good properties (under the working assumptions):

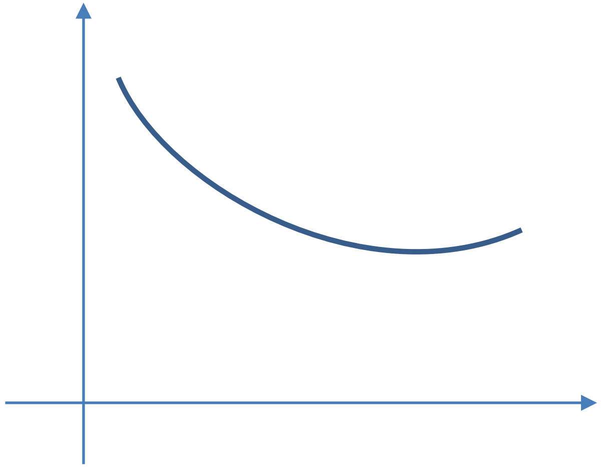
- Tight results (exact control of constraint violations) in many cases
- Uncertainty can enter the problem in any way (nonlinear, discontinuous, etc.) and can have any dimension
- Computational complexity is very low, does not depend on state nor uncertainty dimension

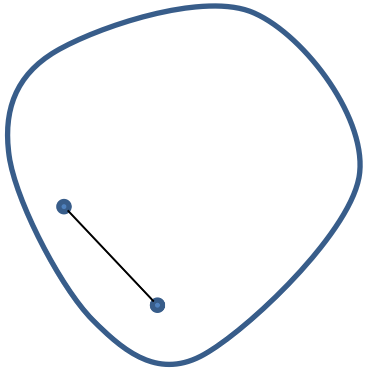
Key point: convexity

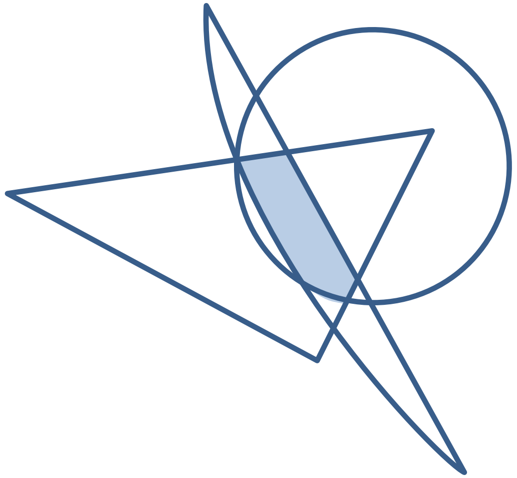


- Convexity of the problem w.r.t. to the **decision variables** is required (**not** w.r.t. uncertainty!!)

- What is a convex optimization program?





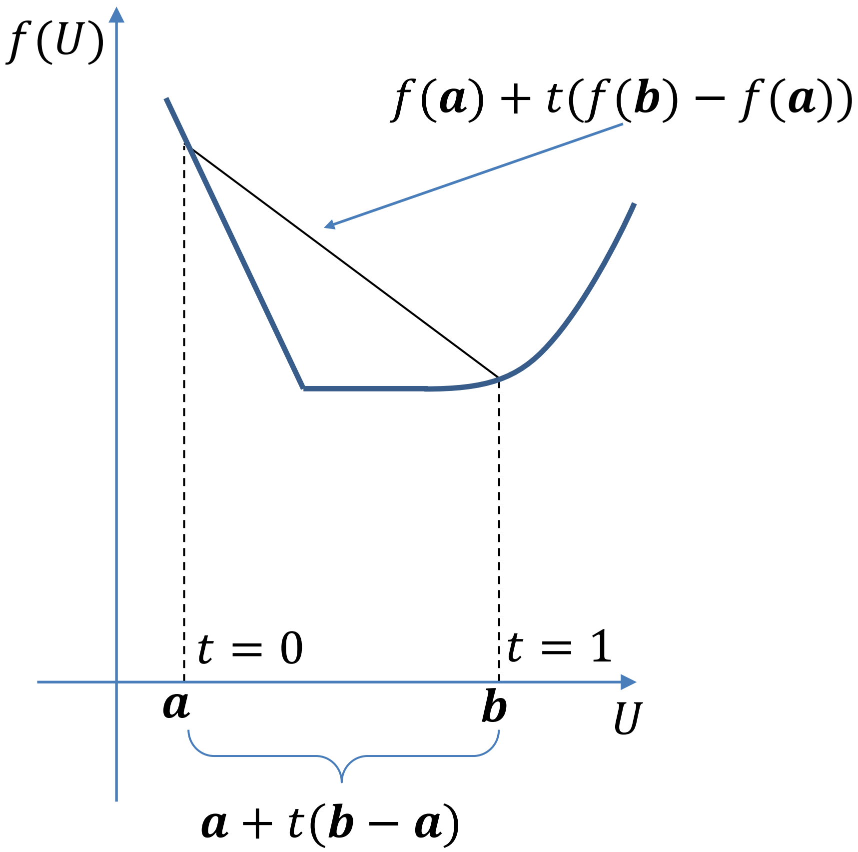


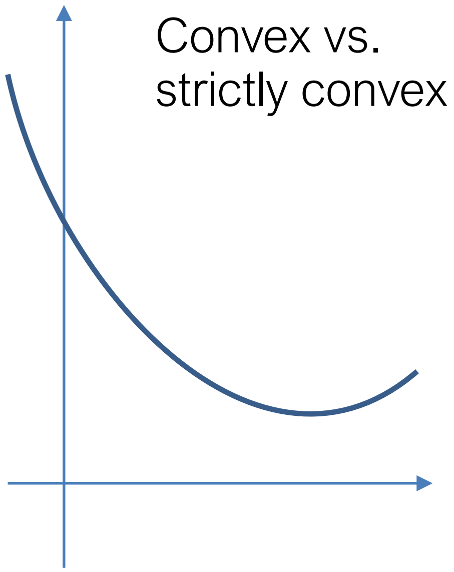
Convex functions

DEF. A function $f: \Omega \to \mathbb{R}$ is convex if and only if it holds:

$$f((1-t)\boldsymbol{a} + t\boldsymbol{b}) \leq (1-t)f(\boldsymbol{a}) + tf(\boldsymbol{b}),$$

$\forall \boldsymbol{a}, \boldsymbol{b} \in \Omega$ and $\forall t \in [0,1]$



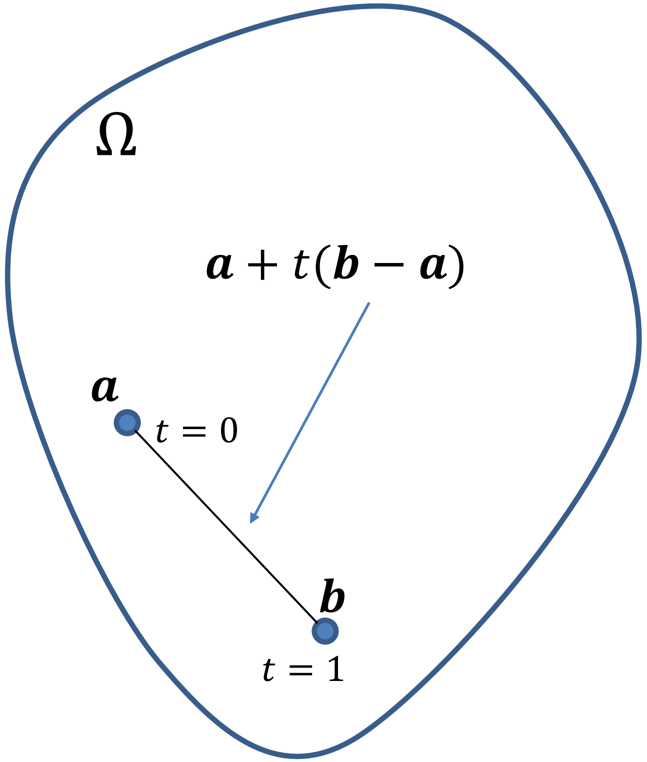


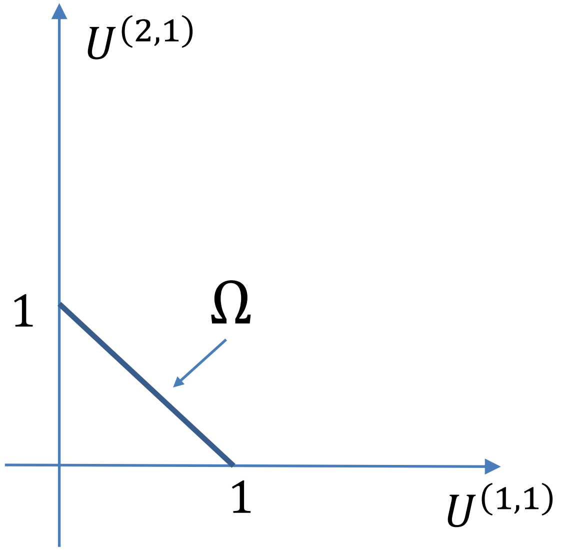
Convex sets

DEF. A set $\Omega \subseteq \mathbb{R}^{n_U}$ is convex if and only if it holds:

$$(1-t)\boldsymbol{a} + t\boldsymbol{b} \in \Omega,$$

$\forall \boldsymbol{a}, \boldsymbol{b} \in \Omega$ and $\forall t \in [0,1]$





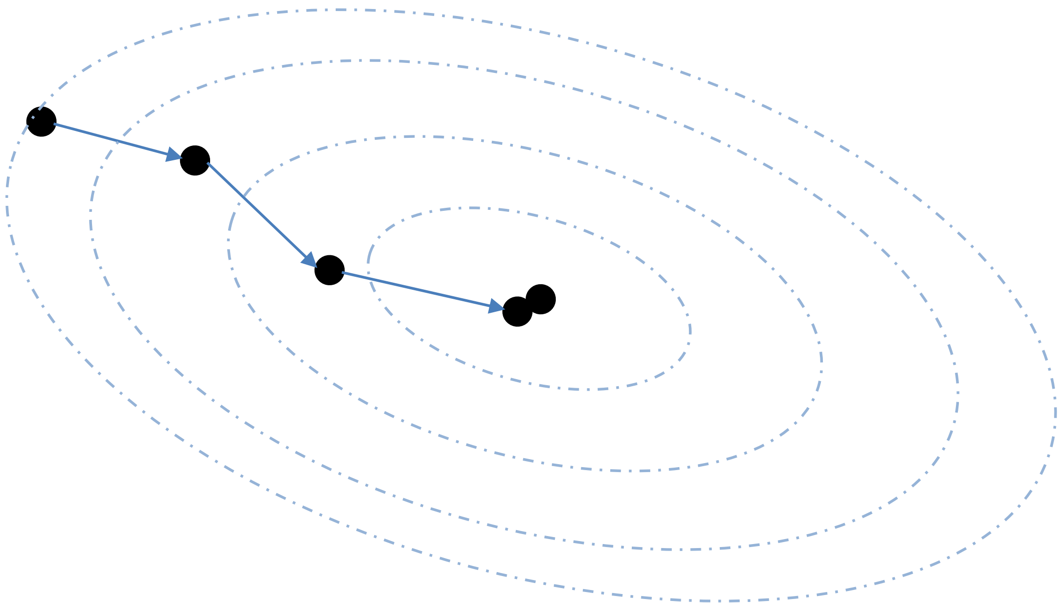
Convex programs

DEF. A convex program is an optimization program where the cost function f is convex, and the constraint set Ω is convex (resp. strictly convex program if f is strictly convex).

$$\min_U f(U)$$
$$\text{subject to } (\text{s.t.})$$
$$U \in \Omega$$

Why is convexity important?

- Widely know reason: in a convex program, every local minimum is a global one (i.e., iterative optimization routines based on local quantities can very efficiently converge to a global solution)



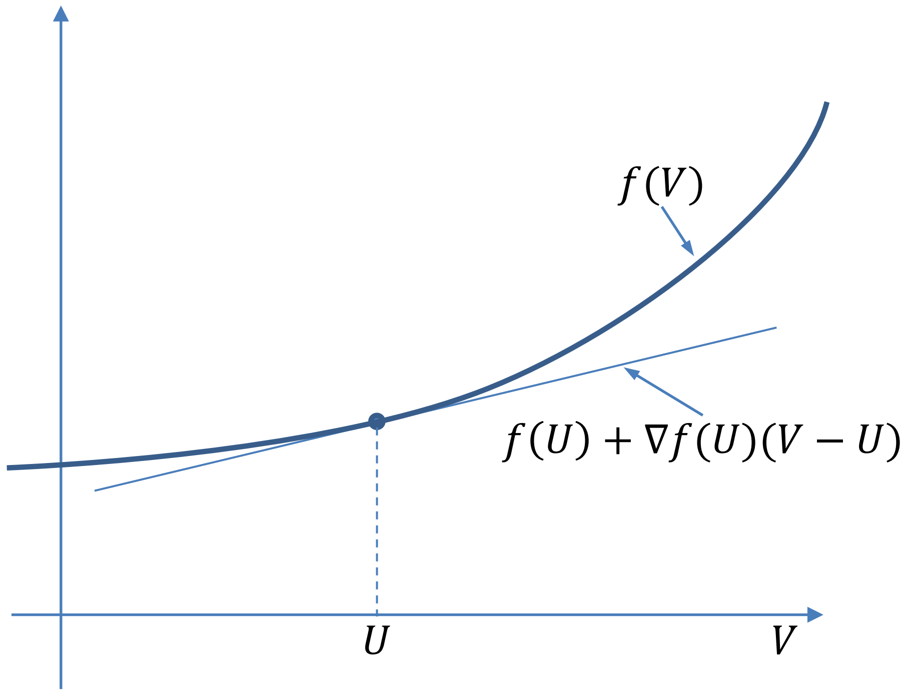
How to check convexity – first order condition

f is convex over Ω

$$\Updownarrow$$

$\forall\, U, V \in \Omega,$

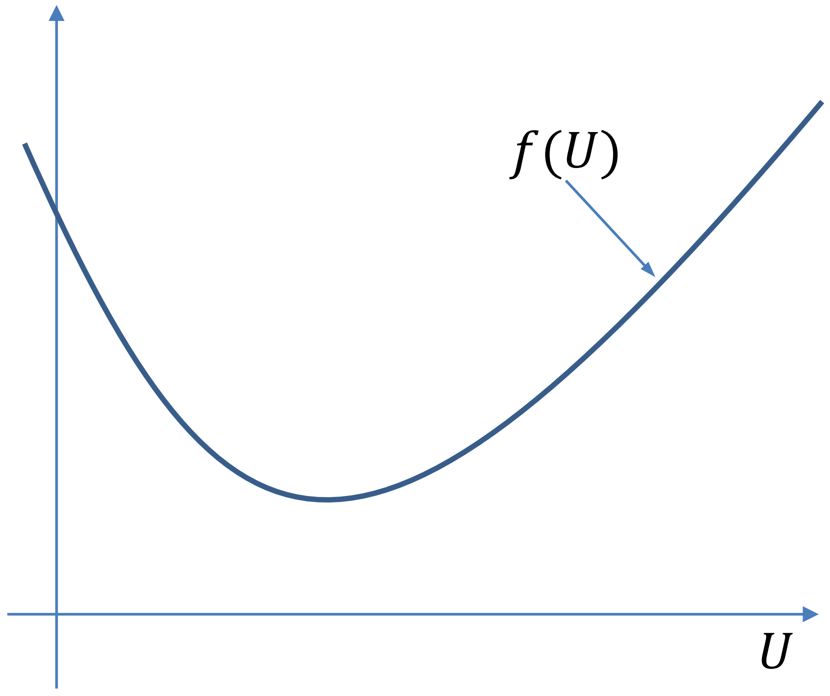
$$f(V) \geq f(U) + \nabla f(U)^T (V - U)$$



e.g., affine functions are convex (actually, both convex and concave)

How to check convexity – second order condition

$$f \text{ is convex over } \Omega$$
$$\Leftrightarrow$$
$$\forall\, U \in \Omega,$$
$$\nabla_U^2 f(U) \geq 0$$



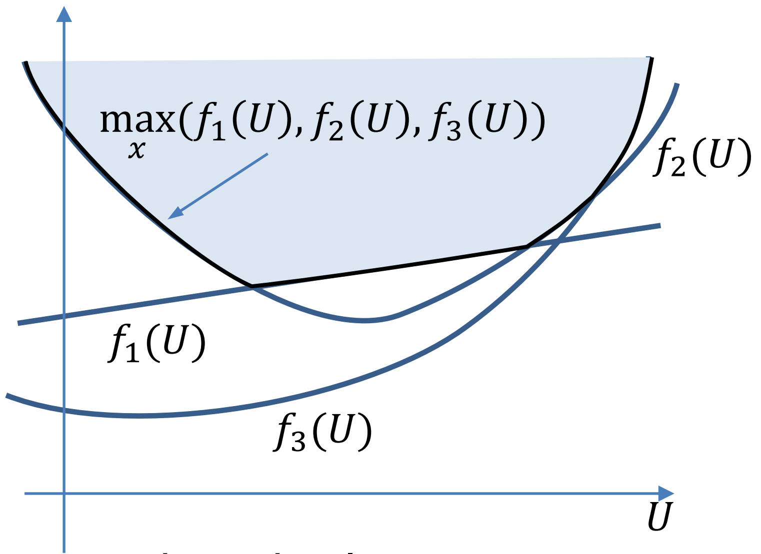
e.g., quadratic functions with positive semidefinite Hessian are convex:

$f(U) = \frac{1}{2} U^T H U + g^T U$ with $H = H^T \geq 0$

Operations that preserve convexity of functions

- Affine input transformation:

  if $f: \Omega \to \mathbb{R}$ convex $\Rightarrow g(V) = f(AV + b)$ is convex over the set $\bar{\Omega} = \{V: AV + b \in \Omega\}$, for given matrix $A \in \mathbb{R}^{n_U \times n_V}$ and vector $b \in \mathbb{R}^{n_U}$.



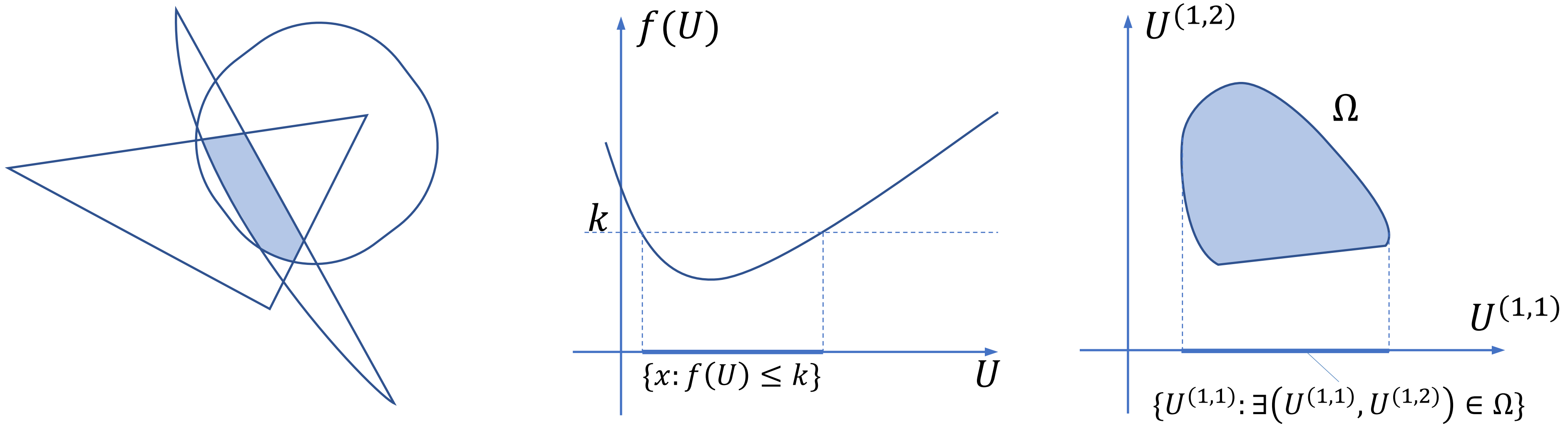
- Non-negative sum (also integrals)

- Maximum of convex functions

- Composition of a convex function with a convex and strictly increasing one, e.g., $g(U) = e^{f(U)}$ with f convex.

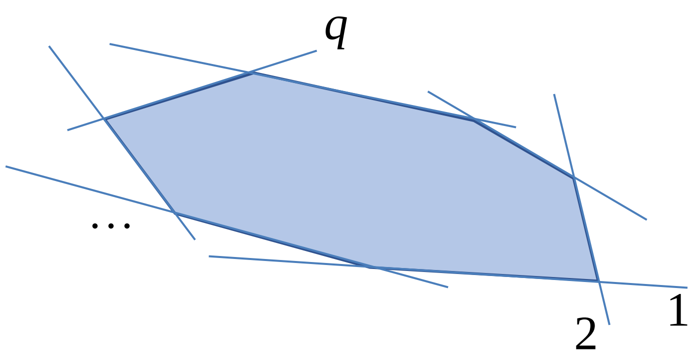
Operations that preserve convexity of sets

- Intersection of convex sets
- Sub-level sets of a convex function (and super-level sets of a concave function)
- Projections of convex sets



Operations that preserve convexity of sets

- Polyhedra



$$\{U: AU + b \geq 0\} = \left\{ \begin{array}{c} U: A^{(1,:)}U + b^{(1,1)} \geq 0 \\ \wedge A^{(2,:)}U + b^{(2,1)} \geq 0 \\ \vdots \\ \wedge A^{(q,:)}U + b^{(q,1)} \geq 0 \end{array} \right\}$$

- Affine image:

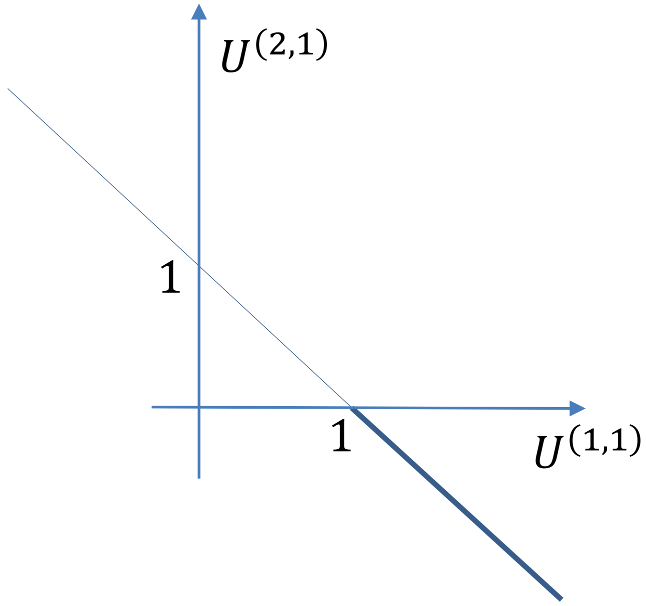
  Ω is convex $\Rightarrow A\Omega + b = \{V = AU + b, U \in \Omega\}$ is convex

- Affine pre-image

  Ω is convex $\Rightarrow \{V: AV + b \in \Omega\}$ is convex for some A, b

Is this a convex set?

$$\Omega = \left\{ \begin{array}{c} U \in \mathbb{R}^2: \\ [1 \quad 1]U - 1 = 0 \\ \wedge\, U^{(1,1)} \geq 0 \\ \wedge\, U^{(2,1)} \leq 0 \end{array} \right\}$$



Important classes of convex programs (for the sake of this course)

- Linear Programs (LP)

$$\begin{gathered}\min_U c^T U \\ s.t. \\ AU = b \\ CU \geq d\end{gathered}$$

Example: Finite Horizon Optimal Control Problems (FHOCP) for LTI systems with polytopic constraints and ℓ_1 or ℓ_∞ stage cost function

Important classes of convex programs (for the sake of this course)

- Quadratic Programs (QP)

$$\min_{U} c^T U + U^T H U$$
$$s.t.$$
$$AU = b$$
$$CU \geq d$$

Example: Finite Horizon Optimal Control Problems (FHOCP) for LTI systems with polytopic constraints and quadratic stage cost function, with stage cost weighting matrices $Q, R, P \geq 0$

Important classes of convex programs (for the sake of this course)

- **LP-type** convex programs

$$\min_{Z} c^T Z$$
$$s.t.$$
$$g(Z) \leq 0$$
$$(g: \mathbb{R}^{n_Z} \to \mathbb{R} \text{ convex})$$

A convex program can be reformulated as LP-type convex program using an **epigraph reformulation**.

Epigraph reformulation

$$\min_{U} c^T U + U^T H U$$
$$s.t.$$
$$CU \geq d$$

$$\min_{\alpha, U} \alpha$$
$$s.t.$$
$$CU \geq d$$
$$c^T U + U^T H U \leq \alpha$$

$$\min_{Z} c^T Z$$
$$s.t.$$
$$g(Z) \leq 0$$
$$(g: \mathbb{R}^{n_Z} \rightarrow \mathbb{R} \text{ convex})$$

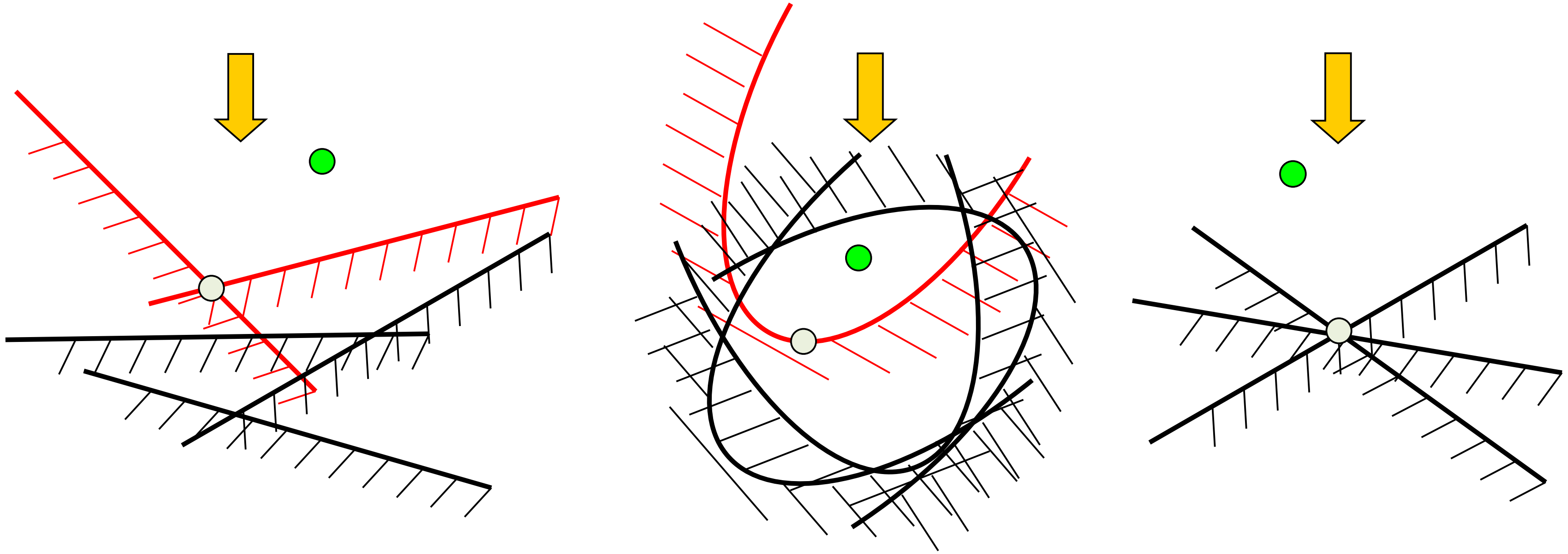
Why is convexity important?

- Widely know reason: in a convex program, **every local minimum is a global one** (i.e., iterative optimization routines based on local quantities can very efficiently converge to a global solution)

- Less widely know reason: **the maximum number of <u>support constraints</u> of a given minimizer is bounded** by the number of optimization variables

Support constraints

- A constraint function g is a support constraint if by removing it from the problem the new minimum is better than the one obtained with the constraint in place.
- **Theorem**: in a LP-type problem with d optimization variables, the maximum number of support constraints is no higher than d.



Now back to scenario MPC

Recall the problem we want to solve:

$$\min_{\phi : u=\phi(x,\cdot)} \frac{1}{T}\sum_{t=0}^{T-1} l\big(x(t),u(t)\big)$$

subject to

$$x(t+1) = A(\delta(t))x(t) + B(\delta(t))u(t) + w\big(\delta(t)\big)$$

$$\frac{1}{T}\sum_{i=0}^{T-1} I\big(x(t) \notin \mathbb{X}(\delta(t))\big) \le \varepsilon$$

Where $l(x,u)$ is convex and δ is a n_δ-dimensional vector of stochastic variables. Remember we only assume to be able to somehow measure (or generate) i.i.d. samples of δ.

Scenario FHOCP

In scenario MPC, the wanted feedback control policy

$$u(t) = \kappa(x(t),\cdot)$$

is given by a receding horizon implementation of the following FHOCP:

$$\min_{u(t|t),\dots,u(t+N-1|t)} \frac{1}{N} \sum_{k=t}^{t+N-1} l\big(x(k|t), u(k|t)\big)$$

Scenario Finite Horizon Optimal Control Problem

subject to

$$x(k+1|t) = A\left(\tilde{\delta}_j(t)\right) x(k|t) + B\left(\tilde{\delta}_j(t)\right) u(k|t) + w\left(\tilde{\delta}_j(t)\right), j = 1, \dots, K$$

$$x(t+k|t) \in \mathbb{X}\left(\tilde{\delta}_j(t)\right), j = 1, \dots, K, k = \mathbf{1}, \dots, N$$

Current step ($k = 0$) not included

Scenario

Sample complexity

Recursive feasibility is assumed

Sample complexity

The K scenarios are i.i.d. samples extracted at each time step.

Key question: **how large shall one pick K in order to meet a given maximum rate ϵ of constraint violations**?

The required value of K is usually referred to as the **sample complexity**.

This problem is studied in the area of “scenario programming” or “random convex programs” (see e.g., papers by Campi and Garatti (2008), Calafiore (2010))

Scenario programming

Denote with $\omega(t) = \left\{\tilde{\delta}_1(t), \dots, \tilde{\delta}_K(t)\right\}$ the "multi-sample" measured at time t.

Consider the LP-type reformulation of the problem, leaving the constraints pertaining to each scenario separate:

$$
\begin{gathered}
\min_{\alpha, U} \alpha \\
s.t. \\
J\big(U, x(t), \omega(t)\big) \leq \alpha \quad \text{(cost function)} \\
g\left(U, \tilde{\delta}_1(t)\right) \leq 0 \\
\vdots \\
g\left(U, \tilde{\delta}_K(t)\right) \leq 0
\end{gathered}
$$

Scenario programming

$$\min_{\alpha,U} \alpha$$
$$s.t.$$
$$J\big(U, x(t), \omega(t)\big) \leq \alpha$$
$$g\left(U, \tilde{\delta}_1(t)\right) \leq 0$$
$$\vdots$$
$$g\left(U, \tilde{\delta}_K(t)\right) \leq 0$$

Key observation: since they are sampled independently, all the scenarios are equally likely to be support constraints, whose number is limited by the n. of optimization variables

→ the probability to be a support constraint does not depend on the underlying distribution, but just on the number of samples

→ the probability that the solution violates a further (unseen) independent constraint equals the probability that the latter is a support constraint of the augmented problem (i.e. with K+1 scenarios)

Scenario FHOCP – main result

Denote with $U^*(t)$ the solution to the scenario FHOCP. This is a function of the multi-sample, thus it is a random variable.

Finally denote the **violation probability** as:

$$V(t) = \mathbb{P}\{\delta : g(U^*(t), \delta) \geq 0\}$$

<u>THM</u> Let β be a reliability level (e.g. $\beta = 10^{-9}$).

If: $$K \geq \frac{2}{\epsilon}(\log(\beta^{-1}) + n_u N + 1)$$

then it holds: $$\mathbb{P}^K\{\omega(t) : V(t) \leq \epsilon\} \geq 1 - \beta$$

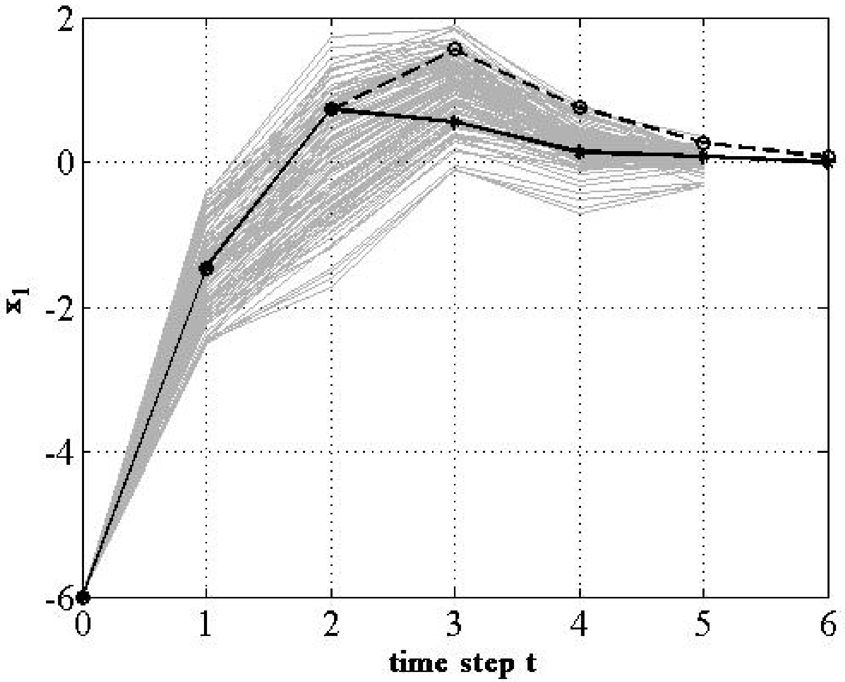
(see, e.g., Campi and Garatti (2008), Calafiore and Fagiano (2013))

A tigher (but implicit) condition exists (and should be used in practice)

Number of optimization variables (n_u inputs)

Scenario FHOCP – some considerations

So far, we are considering the whole predicted trajectory computed by solving the FHOCP at time t. This is useful in open loop.



However, the obtained sample complexity is rather large $\rightarrow$ conservative in close loop. Can we do better?

Scenario MPC - pseudo-algorithm

1. At time t, obtain the state $x(t)$ and generate the multi-sample $\omega(t)$

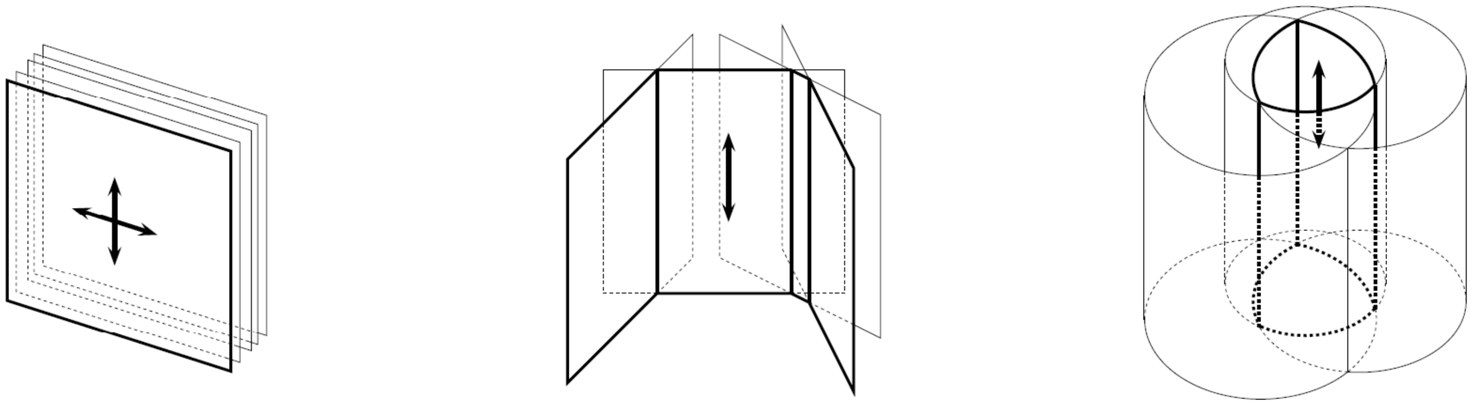
2. Solve the resulting scenario FHOCP to obtain the optimal sequence $U^*(x(t), \omega(t))$

3. Apply the first control input of such a sequence to the plant, go to 1. and repeat by setting $t \leftarrow t + 1$

Bound on the closed loop constraint violation

We can find a much better sample complexity for close loop operation by exploiting two aspects:

- The FHOCP has a **multi-stage structure**: the relevant optimization variables are much less than mN, actually they are only <u>at most</u> n_u



- Close loop operation: the sampling procedure is repeated at each time step → remove the "outer" probability level.

For details: Schildbach, Fagiano, Morari (2013); Schildbach, Fagiano, Frei, Morari (2014)

Scenario MPC – main result

$\rho \leq$ n. of inputs

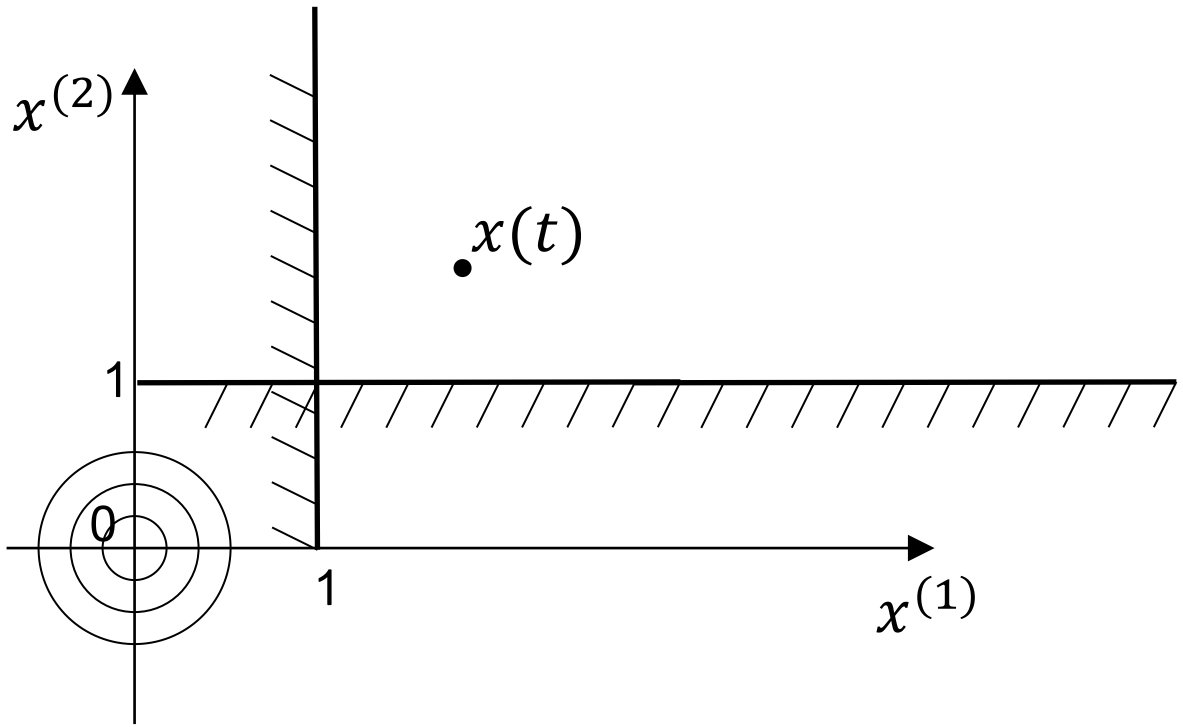
If $K \geq \frac{\rho}{\varepsilon} - 1$ then

$$E\left[\frac{1}{T}\sum_{i=0}^{T-1} I\big(x(t) \notin \mathbb{X}(\delta(t))\big)\right] \leq \varepsilon$$

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{i=0}^{T-1} I\big(x(t) \notin \mathbb{X}(\delta(t))\big) \leq \varepsilon$$

Scenario MPC – example

2 states, 2 inputs, stochastic LPV system + disturbance



Scenario MPC – example

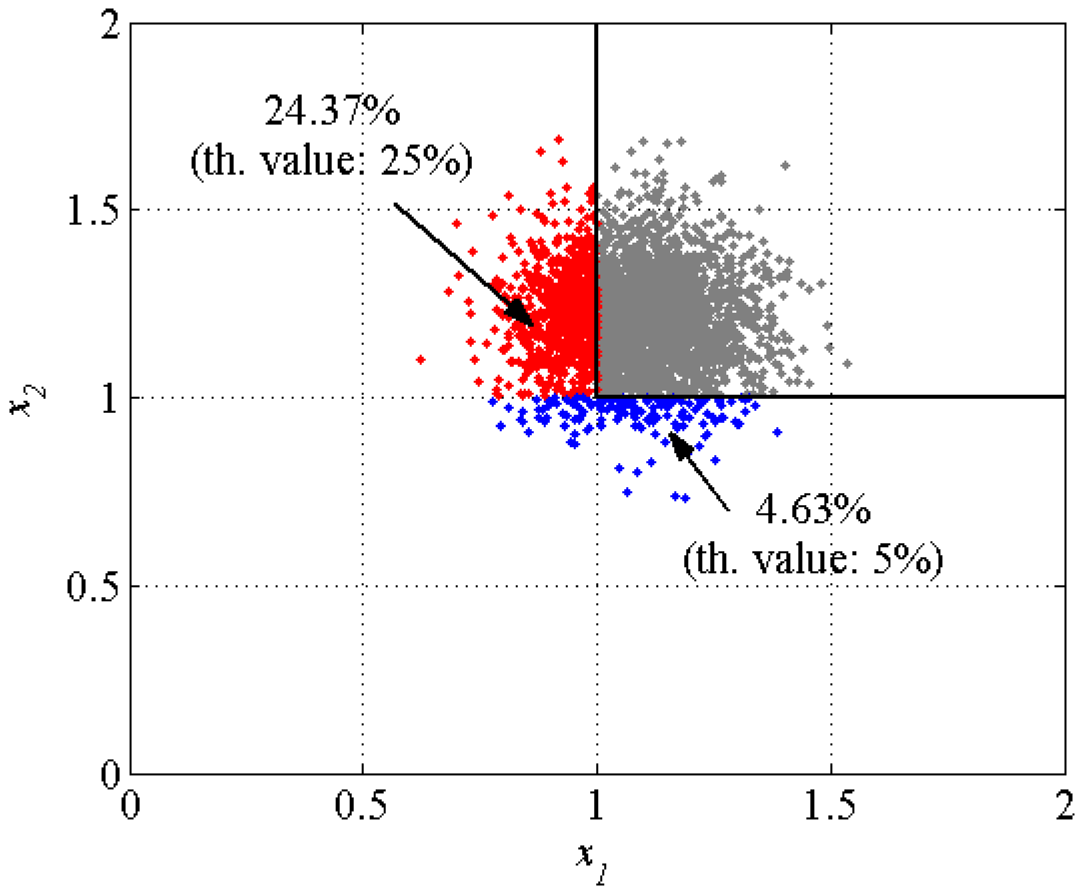
2 states, 2 inputs, stochastic LPV system + disturbance

Very small complexity increase

K	39	19	13	9	7	6
Theoretical bound	0.05	0.1	0.1429	0.2	0.25	0.2857
Ave. constr. viol.	**0.0493**	**0.0927**	**0.126**	**0.1967**	**0.2380**	**0.2793**
Ave. cost	3.995	3.803	3.621	3.561	3.492	3.421

Scenario MPC – example

Can also account for different closed loop violations for the different constraints:



Final considerations

Advantages:

- no need for convexity of the set of uncertain system matrices, no restriction on how uncertainty enters the problem;
- convex problem to be solved at each time step;
- no restrictions on the distribution of uncertainty/disturbance;
- very low sample complexity, depends only on the violation level (does not change with n. of uncertain variables, n. of states, or prediction horizon);
- tight result in most cases (when constraints are non-trivial);
- very intuitive (it's often done in practice)

Limitations:

- i.i.d. stochastic uncertainty assumption
- recursive feasibility assumption

References

[1] M. Campi and S. Garatti, “The Exact Feasibility of Randomized Solutions of Uncertain Convex Programs”, *SIAM J. Optim.*, 19(3), 1211–1230. 2008

[2] G. C. Calafiore, “Random Convex Programs”, *SIAM J. Optim.*, 20(6), 3427–3464. 2010

[3] G.C. Calafiore, L. Fagiano, “Robust Model Predictive Control via Scenario Optimization”, *IEEE Transactions on Automatic Control*, vol. 58, n. 1, pp. 219-224

[4] G. Schildbach, L. Fagiano, M. Morari, “Randomized solutions to convex programs with multiple chance constraints”, *SIAM Journal on Optimization*, vol. 23, n. 4, pp. 2479-2501, 2013

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