The manufacturing problem : graphical resolution

Problem formulation

$$(P)\begin{cases} \max\limits_{x,y} & 4x+5y \\ \text{s.t.} & x+y \le 4 \\ & x+2y \le 7 \\ & x \ge 0,\, y \ge 0 \end{cases}$$

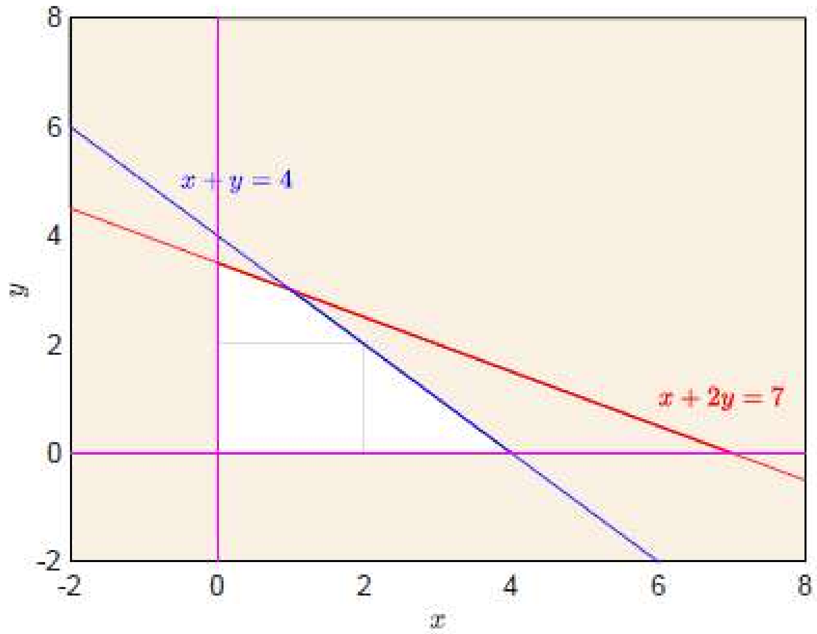
The feasible set

The manufacturing problem : graphical resolution

Problem formulation

$$(P)\begin{cases} \max_{x,y} & 4x+5y \\ \text{s.t.} & x+y \leq 4 \\ & x+2y \leq 7 \\ & x \geq 0, y \geq 0 \end{cases}$$

The feasible set



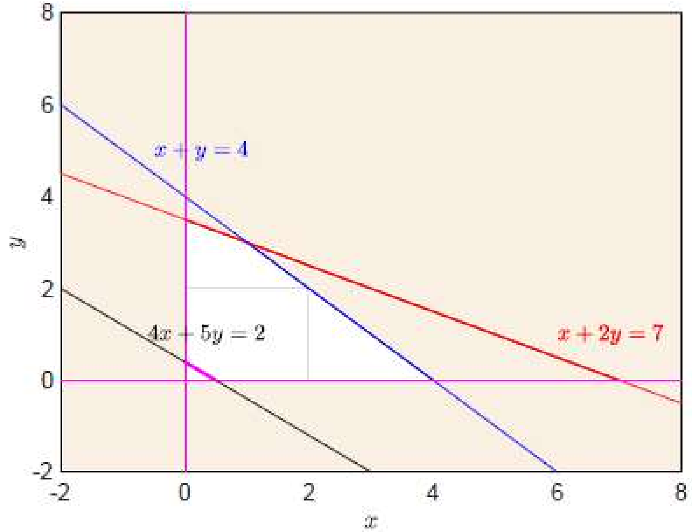
The manufacturing problem : graphical resolution (2)

Objective $= 2$

Maximize the objective

The manufacturing problem : graphical resolution (2)

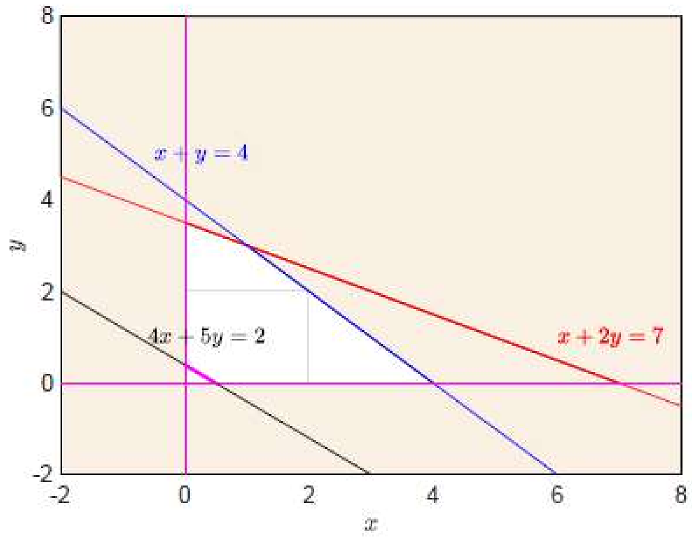
Objective
$= 2$



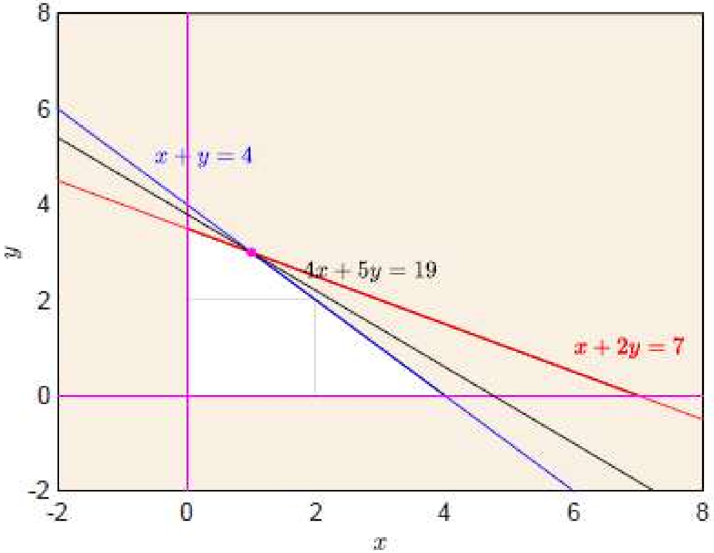
Maximize the objective

The manufacturing problem : graphical resolution (2)

Objective $= 2$

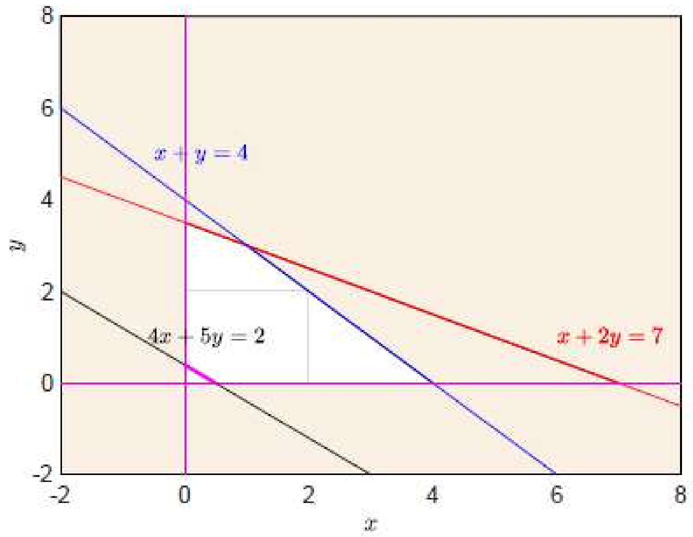


Maximize the objective

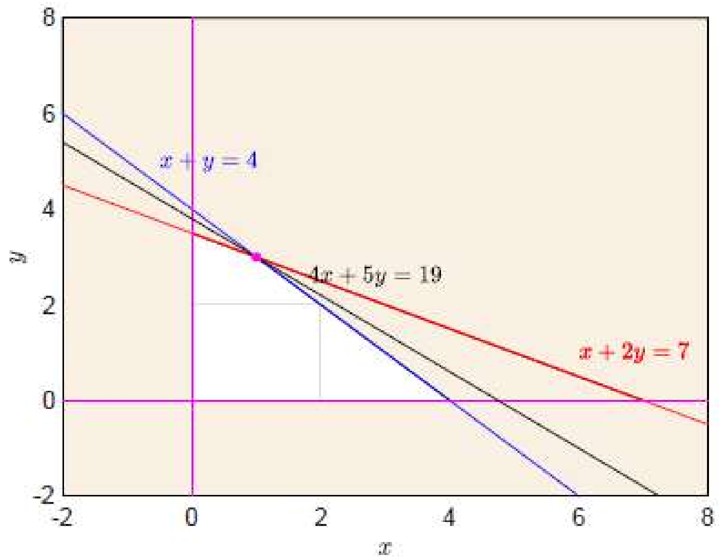


The manufacturing problem : graphical resolution (2)

Objective = 2



Maximize the objective



At the optimal, the two constraints are equalities : they are said to be *binding* or *saturated*.
From a business perspective, they are *bottlenecks*.

Table of Contents

The Simplex algorithm : introduction

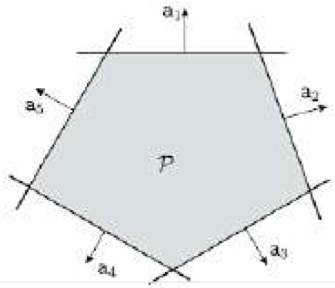
A little history	The simplex algorithm is the most widely used method of operational research. It was G.B. Dantzig who, in an article appeared in 1949, described this algorithm, which forms the backbone operational research. Since then, this algorithm has been the subject of hundreds of scientific articles and has served to solve numerous linear models relating to problems of management, dietary, transportation, assignment...
Complexity	The complexity of the simplex algorithm is an exponential-time algorithm, as proved by Keely and Minty in 1972, by finding one example. However, the simplex algorithm is behaving in the polynomial-time algorithm for solving real-life problems.

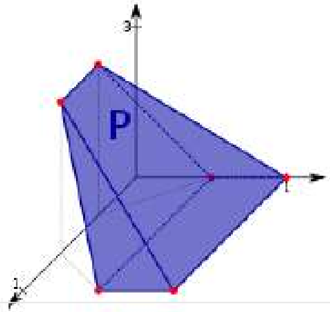
Half-space

A set defined by a set of inequality constraints is a half-space.
Ex : $\{x \in \mathbb{R}^n : a_1^T x \leq b_1, a_2^T x \geq b_2\}$

Polyhedron

A polyhedron is a set defined a the intersection of half-planes.





Linear optimization

Feasible of linear optimization problems are polyhedron.
By varying the level of the objective function, it follows that the **extremal values are attained on the corners of the polyhedron** of the feasible sets.
This simplifies the search of optimizers :

- Start at a corner
- Visit a neighboring corner that improves the objective.

But how to move from one corner to another ?

Linear Program (LP) in standard form

Standard form

$$\begin{cases} \min\limits_x & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0, x \in \mathbb{R}^n \end{cases}$$

with $A \in \mathbb{R}^{m,n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.
We assume that A is full rank m and $m < n$.

Example

$$\begin{cases} \max & x_1 - x_2 \\ \text{s.t.} & x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \geq 3 \\ & x_2 \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -\min\limits_x & -x_1^+ + x_1^- + x_2 \\ \text{s.t.} & x_1^+ - x_1^- + x_2 + s_1 = 1 \\ & x_1^+ - x_1^- + 2x_2 - s_2 = 3 \\ & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0 \end{cases}$$

Linear Program (LP) in standard form - Practice

Initial problem

$$\Leftrightarrow \begin{cases} \min\limits_{x_1, x_2} & \max\{x_1 + x_2 - 1, 2x_1 - x_2\} \\ \text{s.t.} & x_1, x_2 \geq 0 \end{cases}$$

Standard form

Linear Program (LP) in standard form - Practice

Initial problem

$$\Leftrightarrow \begin{cases} \min\limits_{x_1, x_2} & \max\{x_1 + x_2 - 1, 2x_1 - x_2\} \\ \text{s.t.} & x_1, x_2 \geq 0 \end{cases}$$

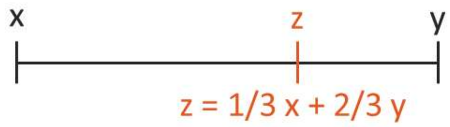
Standard form

$$\Leftrightarrow \begin{cases} \min\limits_{x_1, \cdots, x_5} & x_3 \\ \text{s.t.} & x_1 + x_2 - x_3 + x_4 = 1 \\ & 2x_1 - x_2 - x_3 + x_5 = 0 \\ & x_1, \cdots, x_5 \geq 0 \end{cases}$$

Geometry of Linear Programming (2)

Convex combination

z is a convex combination of x and y if it exists a scalar $\lambda \in [0, 1]$ such that $z = \lambda x + (1 - \lambda) y$.
In other words, z on the segment $[x, y]$



Extreme points

Let $\mathcal{F}$ a polyhedron. x is an **extreme point** if it can not be realized as a convex combination of point of $\mathcal{F}$.
We also call such x a **vertex** of $\mathcal{F}$.

Theorem

Let $\mathcal{F} = \{x \geq 0 : Ax = b\}$ the feasible set of (P).

$x \in \mathcal{F}$ is an extreme point of $\mathcal{F}$ **iff** there exists $J = \{j_1, \cdots, j_m\}$ such that $(A_{\cdot,j})_{j \in J}$ linearly independent and $x_{j'} = 0$ for $j' \notin J$.

x is also said to be **a basic feasible solution (BFS)**.

Construction of a starting corner

Definitions

To each set J corresponds a basic solution that can be or not feasible. Let x be this solution. Denote :

- $B = (A_{.,j})_{j \in J}$
- $N = (A_{.,j})_{j \notin J}$
- $x_B = (x_j)_{j \in J}$
- $x_N = (x_j)_{j \notin J}$

By swapping the rows, it comes : $A = (B, N)$ and $x = (x_B, x_N)$.

Basic principle

By definition of x : $x_N = 0$.

Then $Ax = b$ becomes $Bx_B + Nx_N = Bx_B = b$. It comes that $x_B = B^{-1}b$.

If $x_B \geq 0$ then x is a basic feasible solution (BFS).

Rules of the game

Choose a combination of m indices among n such that the corresponding basic solution is feasible.
Then check if it is optimal.
If not optimal, move on to another BFS that improves the objective.

Reduced costs

Feasible move

Let x a BFS for J and let y be another admissible solution.
Then $d := y - x$ satisfies :

- for $j \notin J$: $d_j = y_j - x_j = y_j$ since $x_j = 0$ for $j \notin J$. So $d_j = y_j \geq 0$ since y is feasible.
- $Ad = Ay - Ax = b - b = 0$ since x and y are both feasible.

$Ad = Bd_B + Nd_N = 0$ so $d_B = -B^{-1}Nd_N$

Cost (Objective) reduction

The objective (or cost) reduction is : $\Delta = c^T y - c^T x = c_B^T d_B + c_N^T d_N$.
As $d_B = -B^{-1}Nd_N$: $\Delta = -c_B^T B^{-1} N d_N + c_N^T d_N = (-N^T B^{-1T} c_B + c_N)^T d_N = \sum_{j \notin J} (c_j - c_B^T B^{-1} A_{.,j}) d_j = \sum_{j \notin J} \bar{c}_j d_j$.

By defining $\bar{c}_j = c_j - c_B^T B^{-1} A_{.,j}$, named **reduced costs**.
Notice that $\bar{c}_j = 0$ for $j \in J$.

Proof

Theorem

x is optimal **iff** reduced costs are non-negative, i.e., $\bar{c}_j \geq 0, \forall j \notin J$.

Feasible move

Let x a BFS for J and let y be another admissible solution.
Then $d := y - x$ satisfies :

- for $j \notin J$: $d_j = y_j - x_j = y_j$ since $x_j = 0$ for $j \notin J$. So $d_j = y_j \geq 0$ since y is feasible.
- $Ad = Ay - Ax = b - b = 0$ since x and y are both feasible.

$Ad = Bd_B + Nd_N = 0$ so $d_B = -B^{-1}Nd_N$

Cost (Objective) reduction

The objective (or cost) reduction is : $\Delta = c^T y - c^T x = c_B^T d_B + c_N^T d_N$.
As $d_B = -B^{-1}Nd_N$: $\Delta = -c_B^T B^{-1}Nd_N + c_N^T d_N =$
$(-N^T B^{-1T} c_B + c_N)^T d_N = \sum_{j \notin J}(c_j - c_B^T B^{-1} A_{.,j}) d_j = \sum_{j \notin J} \bar{c}_j d_j$.

By defining $\bar{c}_j = c_j - c_B^T B^{-1} A_{.,j}$, named **reduced costs**.
Notice that $\bar{c}_j = 0$ for $j \in J$.

Proof

$A_{.,j}$ is a column of B, so $B^{-1}A_{.,j}$ is a row of the identity matrix, where only the j-th component is non-zero and equals 1. So $c_B^T B^{-1} A_{.,j} = c_j$ and $\bar{c}_j = 0$.

Theorem

x is optimal **iff** reduced costs are non-negative, i.e., $\bar{c}_j \geq 0, \forall j \notin J$.

Moving to another corner

Pivoting

Suppose that $j_0 \notin J$ is such that $\bar{c_{j_0}} < 0$. Define d such as :

- $d_{j_0} = 1$
- $d_j = 0$ for $j \neq j_0$, $j \notin J$
- $d_B = -B^{-1}Nd_N = -B^{-1}A_{.,j_0}$

And consider $y = x + \theta d$ with $\theta > 0$. The component y_{j_0} is non-zero so j_0 is said to be the **entering** variable.

Equality constraints

Then $Ad = Bd_B + Nd_N = -BB^{-1}Nd_N + Nd_B = 0$.
As a consequence, for any $y = x + \theta d$ with $\theta > 0$,
$Ay = Ax + \theta Ad = Ax = b$. So y satisfies the equality constraints.
The cost reduction is $c^T y - c^T x = \theta \sum_{j \notin J} \bar{c}_j d_j = \theta \bar{c_{j_0}} < 0$. In order to minimize $c^T x$ we take j_0 with the lowest reduced cost.

Inequality constraints

It remains to verify the inequality constraints. 2 possibilities :

- Either $d \geq 0$: $x + \theta d \geq 0$ for θ arbitrarily large. So $\theta \bar{c_{j_0}}$ can go to infinity, meaning that the problem is unbounded.
- Or $d_j < 0$ for some $j \in J$: so we look for θ such that : $y_j = x_j + \theta d_j \geq 0$.

Moving to another corner (2)

Status

Assume that $d_j < 0$ for some $j \in J$.

We look for θ such that : $y_j = x_j + \theta d_j \geq 0$ for all j, including these j with $d_j < 0$.

Leaving variable

This implies : $\theta \leq -\frac{x_j}{d_j}$, for all $j \in J$ such that $d_j < 0$.
In order to verify this for all $j \in J$ we take :

$$\theta = \min_{j \in J, d_j < 0} -\frac{x_j}{d_j}$$

Let's denote j^* the minimizer.
Then $\theta = -\frac{x_{j^*}}{d_{j^*}}$ so $y_{j^*} = x_{j^*} - \frac{x_{j^*}}{d_{j^*}} d_{j^*} = 0 \rightarrow j^*$ is the **leaving** variable.

Moving to another corner : example

Problem statement

$$
(P)\begin{cases}
\min\limits_{x} & -10x_1 - 12x_2 - 12x_3 \\
\text{s.t.} & x_1 + 2x_2 + 2x_3 \leq 20 \\
 & 2x_1 + x_2 + 2x_3 \leq 20 \\
 & 2x_1 + 2x_2 + x_3 \leq 20 \\
 & x_1, x_2, x_3 \geq 0
\end{cases}
$$

Let's practice

- Write the problem under the standard form
- Find a first BFS by setting $x_1 = x_2 = x_3 = 0$
- Compute the reduced costs
- Select the entering variable
- Compute θ and the leaving variable

Moving to another corner : solution

Problem statement

$$(P)\begin{cases} \min\limits_x & -10x_1 - 12x_2 - 12x_3 \\ \text{s.t.} & x_1 + 2x_2 + 2x_3 + x_4 = 20 \\ & 2x_1 + x_2 + 2x_3 + x_5 = 20 \\ & 2x_1 + 2x_2 + x_3 + x_6 = 20 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

Pivoting

$J = \{4,5,6\}$ so $B = I_3$ and $x_B = B^{-1}b = (20,20,20)^T$. So $x = (0,0,0,20,20,20)^T$

$\bar{c}^T = c^T - c_B^T B^{-1}A = (-10,-12,-12,0,0,0)^T$.

Let's select x_2 to enter in the basis.

Then $y = x + \theta^*(0,1,0,-2,-1,-2)$

$\theta^* = \min_{j\in\{4,5,6\}} -\frac{x_j}{d_j} = \min\{-\frac{20}{-2}, -\frac{20}{-1}, -\frac{20}{-2}\} = 10$

$$\begin{aligned} y &= (0,0,0,20,20,20) + 10 * (0,1,0,-2,-1,-2) \\ &= (0,10,0,0,10,0) \end{aligned}$$

The simplex algorithm

1. Start with
   - basic columns B composed of linearly independent columns $J = (j_1, \ldots, j_m)$ of a,
   - and corresponding Corner x, defined by $x_B = B^{-1}b$ and $x_N = 0$
2. Compute B^{-1}, $p^\intercal := c_B^\intercal B^{-1}$, reduced costs $\bar{c}_j = c_j - p^\intercal a_{.,j}$, $j \notin J$
   - If $\bar{c}_j \geq 0$ for all $j \notin J$, then x is optimal. Stop!
   - Otherwise, select j such that $\bar{c}_j < 0$
3. Compute $d := -B^{-1}a_{.,j}$
   - If $d \geq 0$, then unbounded cost. Stop!
   - Otherwise, continue...
4. $\theta^* := \min_{i \in J : u_i > 0} \frac{-x_i}{d_i} = \frac{-x_{i*}}{d_{i*}}$
5. New basic columns B' with new basic column $a_{.,j}$ replacing $a_{.,i^*}$
6. Move to the next Corner defined by $y_j = \theta^*$, $y_{j_i} = x_{j_i} + \theta^* d_i$

Finding an initial solution

Feasible decision variables $\mathcal{A} = \left\{x \in \mathbb{R}^n : ax = b, x \geq 0\right\} \neq \emptyset$

1. Assume wlog that $b \geq 0$ (otherwise multiply some rows by -1)
2. Consider the auxiliary problem

$$\begin{array}{ll} \min_{x,x'} & x'_1 + \ldots, x'_m \\ \text{s.t.} & (a|I_m)\left(\frac{x}{x'}\right) = ax + x' = b \\ & x, x' \geq 0 \end{array}$$

3. Start from initial Corner $(x, x') = (0, b)$, and apply Simplex Algorithm. Then cost must be zero!
4. If no artificial variable is in the basis, then corner is found!
5. Otherwise, drive artificial variables out of the basis...

Problem

$$(P)\left\{\begin{array}{ll} \min\limits_{x_1,\cdots,x_5} & x_3 \\ \text{s.t.} & x_1 + x_2 - x_3 + x_4 = 1 \\ & 2x_1 - x_2 - x_3 + x_5 = 0 \\ & x_1,\cdots,x_5 \geq 0 \end{array}\right.$$

What you have to do

1. Enumerate all possible independent basic columns $J = \{j_1, j_2\}$
2. Compute the corresponding basic matrices B_J
3. Determine all corners of the Polyhedron of feasible solutions of (P)
4. Deduce the solution of (P)

Hint : recall that $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]$

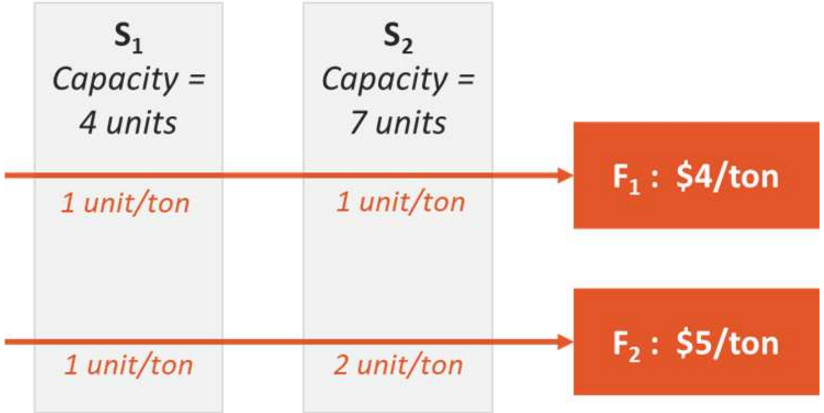
Simplex : let's practice. Solution

2. $(1,2) \longrightarrow \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, $(1,3) \longrightarrow \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$, $(1,4) \longrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, $(1,5) \longrightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$
   $(2,3) \longrightarrow \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, $(2,4) \longrightarrow \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $(2,5) \longrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$,
   $(3,4) \longrightarrow \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $(3,5) \longrightarrow \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$,
   $(4,5) \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
3. $(1,3)$, $(1,5)$, $(2,3)$ and $(3,5)$ do not induce corners, and the other choices of basic columns provide:
$$\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
4. The minimum value of x_3 is 0, and is achieved at five different corners.

Table of Contents

Duality throught the manufacturing example

Problem statement



Problem formulation

$$(P)\begin{cases} \max\limits_{x,y} & 4x + 5y \\ \text{s.t.} & x + y \leq 4 \\ & x + 2y \leq 7 \\ & x \geq 0, y \geq 0 \end{cases}$$

The investor's problem

How much would you be willing to pay to increase the capacity of S_1 or S_2 ?
No more than the resulting increase in profit.

Initial problem

Let's consider the optimal value as a function of the boundaries :

$$\begin{cases} \phi(4,7) = \max_{x,y} & 4x+5y \\ \text{s.t.} & x+y \leq 4 \\ & x+2y \leq 7 \\ & x \geq 0, y \geq 0 \end{cases}$$

Sensitivity of the constraints

$$\begin{cases} \phi(4+1,7) = \max_{x,y} & 4x+5y \\ \text{s.t.} & x+y \leq 4+1 \\ & x+2y \leq 7 \\ & x \geq 0, y \geq 0 \end{cases}$$

Sensitivity

We know that $\phi(4,7) = 19$. $\phi(4+1,7) =$
So the variation of the optimal value is :

$$\frac{\phi(4+1,7) - \phi(4,7)}{1} = 3$$

We don't want to pay more than 3 per unit increase of capacity S_1.
Practice : same exercise for S_2.

Sensitivities of the constraints

Initial problem

Let's consider the optimal value as a function of the boundaries :

$$\begin{cases} \phi(4,7) = \max\limits_{x,y} & 4x+5y \\ \text{s.t.} & x+y \leq 4 \\ & x+2y \leq 7 \\ & x \geq 0, y \geq 0 \end{cases}$$

Sensitivity of the constraints

$$\begin{cases} \phi(4+1,7) = \max\limits_{x,y} & 4x+5y \\ \text{s.t.} & x+y \leq 4+1 \\ & x+2y \leq 7 \\ & x \geq 0, y \geq 0 \end{cases}$$

Sensitivity

We know that $\phi(4,7) = 19$. $\phi(4+1,7) = 22$
So the variation of the optimal value is :

$$\frac{\phi(4+1,7) - \phi(4,7)}{1} = 3$$

We don't want to pay more than 3 per unit increase of capacity S_1.
Practice : same exercise for S_2.

The associated buyer problem

- How much would you be willing to pay to buy back your own industrial process? (the phosphat and the acid of the first firm)
- Offer prices p_1 for phosphat and p_2 for acid, so your objective is:

$$\min 4p_1 + 7p_2$$

- The firm would not accept the offer unless it is more favorable than persuing its industrial process:
  - For the production of F_1, this restricts offered prices by $p_1 + p_2 \geq 4$ (because 1 ton of phosphate and 1 ton of acid allows the firm to produce 1 ton of F_1 which is worth 4)
  - Similarly, for F_2 one should have $p_1 + 2p_2 \geq 5$
- Consequently, you choose your offer so as

$$\begin{aligned} \min \quad & 4p_1 + 7p_2 \\ & p_1 + p_2 \geq 4 \\ & p_1 + 2p_2 \geq 5 \end{aligned}$$

The associated buyer problem

Manufacturing	Offer
$\max_{x,y} 4x + 5y,$	$\min_{p_1,p_2} 4p_1 + 7p_2,$
$x + y \leq 4,$	$p_1 + p_2 \geq 4,$
$x + 2y \leq 7,$	$p_1 + 2p_2 \geq 5,$
$0 \leq x, \quad 0 \leq y.$	$0 \leq p_1, \quad 0 \leq p_2.$
Primal	**Dual**

Practice : solve this problem graphically.

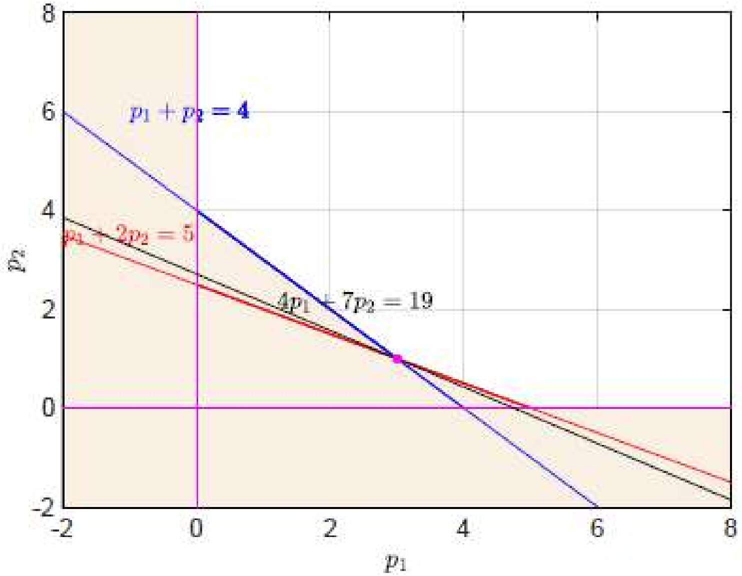
The associated buyer problem : solution

$$\min_{p_1, p_2} 4p_1 + 7p_2,$$

$$p_1 + p_2 \geq 4,$$

$$p_1 + 2p_2 \geq 5,$$

$$0 \leq p_1, \quad 0 \leq p_2.$$



$\Longrightarrow$ Best offer is $p_1 = 3$, $p_2 = 1$

$\Longrightarrow$ Compare with sensitivity analysis...

Linear duality

Primal problem

$$(P)\begin{cases} p^* = \min\limits_{x} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

Dual problem

$$(D)\begin{cases} d^* = \max\limits_{p} & b^T p \\ \text{s.t.} & A^T p \leq c \end{cases}$$

Theorem

Assume p^* or d^* is finite. Then $p^* = d^*$ (**strong duality**) and (x, p) are optimizers of (P) and (D) **iff** :

$$\begin{cases} Ax = b \\ x \geq 0 \\ A^T p \leq c \\ c^T x = b^T p \end{cases}$$

Linear duality : let's play (2)

Exercise

Prove that if (x, p) are feasible, then a necessary and sufficient optimality condition is that :

- Either $x_j = 0$
- Or $A_{.,j} p = c_j$

This condition is named **complementary slackness**.

Solution

Linear duality : let's play (2)

Exercise

Prove that if (x, p) are feasible, then a necessary and sufficient optimality condition is that :

- Either $x_j = 0$
- Or $A_{.,j}p = c_j$

This condition is named **complementary slackness**.

Solution

$c^T x = b^T p$ or equivalently : $c^T x - b^T p = x^T c - b^T p = 0$.

As x is feasible : $b = Ax$ so
$x^T c - x^T A^T p = x^T(c - A^T p) = \sum_{i=1}^n x_j(c_j - A^T_{.,j}p) = 0$.

As both $x_j \geq 0$ and $A^T_{.,j}p \geq 0$ for all j, the only possibility is that at least one of them vanishes for all j.

Exercise

Let's prove that for any feasible (x, p) then $c^T x \geq b^T p$.

Solution

Weak duality

It comes that :

Exercise

Let's prove that for any feasible (x, p) then $c^T x \geq b^T p$.

Solution

$$c^T x - b^T p = c^T x - (Ax)^T p = x^T (c - A^T p) \geq 0$$

Weak duality

It comes that :

Exercise

Let's prove that for any feasible (x, p) then $c^T x \geq b^T p$.

Solution

$$c^T x - b^T p = c^T x - (Ax)^T p = x^T (c - A^T p) \geq 0$$

Weak duality

It comes that : $p^* \geq d^*$

Strong duality

Refresher from simplex

x is optimal **iff** :

- x is a BFS : $x = (x_B, 0)^T$ with $x_B = B^{-1}b$
- its reduced costs $c^T - c_B^T B{-}1A^T \geq 0$

Prove that $d^* \geq p^*$

Strong duality

Let x^* be the optimal solution of the primal problem and B the corresponding basic matrix.

Set $p^T = c_B^T B^{-1}$. Then :

- p is feasible for the dual problem :
- $b^T p = c^T x^*$:

Consequently : $d^* \geq b^T p = c^T x \geq p^*$ so strong duality holds.

Strong duality

Refresher from simplex

x is optimal **iff** :

- x is a BFS : $x = (x_B, 0)^T$ with $x_B = B^{-1}b$
- its reduced costs $c^T - c_B^T B{-}1A^T \geq 0$

Prove that $d^* \geq p^*$

Strong duality

Let x^* be the optimal solution of the primal problem and B the corresponding basic matrix.

Set $p^T = c_B^T B^{-1}$. Then :

- p is feasible for the dual problem :
  $A^T p - c = (p^T A - c^T)^T = -(c^T - c_B^T B{-}1A^T)^T \leq 0$
- $b^T p = c^T x^*$:

Consequently : $d^* \geq b^T p = c^T x \geq p^*$ so strong duality holds.

Refresher from simplex

x is optimal **iff** :

- x is a BFS : $x = (x_B, 0)^T$ with $x_B = B^{-1}b$
- its reduced costs $c^T - c_B^T B{-}1A^T \geq 0$

Prove that $d^* \geq p^*$

Strong duality

Let x^* be the optimal solution of the primal problem and B the corresponding basic matrix.

Set $p^T = c_B^T B^{-1}$. Then :

- p is feasible for the dual problem :
  $A^T p - c = (p^T A - c^T)^T = -(c^T - c_B^T B{-}1A^T)^T \leq 0$
- $b^T p = c^T x^*$: $b^T p = p^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x^*$

Consequently : $d^* \geq b^T p = c^T x \geq p^*$ so strong duality holds.

Primal			**Dual**		
min	$c \cdot x$		max	$b \cdot p$	
s.t.	$a_{i_1,.} \cdot x \geq b_{i_1}$	$i_1 \in M_1$	s.t.	$p_{i_1} \geq 0$	$i_1 \in M_1$
	$a_{i_2,.} \cdot x \leq b_{i_2}$	$i_2 \in M_2$		$p_{i_2} \leq 0$	$i_2 \in M_2$
	$a_{i_3,.} \cdot x = b_{i_3}$	$i \in M_3$		$p_{i_3} \in \mathbb{R}$	$i_3 \in M_3$
	$x_{j_1} \geq 0$	$j_1 \in I_1$		$a_{.,j_1} \cdot p_{j_1} \leq c_{j_1}$	$j_1 \in I_1$
	$x_{j_2} \leq 0$	$j_2 \in I_2$		$a_{.,j_2} \cdot p_{j_2} \geq c_{j_2}$	$j_2 j \in I_2$
	$x_{j_3} \in \mathbb{R}$	$j_3 \in I_3$		$a_{.,j_3} \cdot p_{j_3} = c_{j_3}$	$j_3 \in I_3$

Linear duality : link between primal and dual status

Primal	minimize	maximize	Dual
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

We find the following possibilities for the outcome of a linear program and its dual:

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite Optimum	possible	impossible	impossible
Unbounded	impossible	impossible	possible
Infeasible	impossible	possible	possible