STUDY OF THE ENTROPIC BARRIER (ONGOING)

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ABSTRACT. We define the Entropic Barrier as the Cramer Transform of the uniform measure on a convex body in $\mathbb{R}^n$. In this work, we tropicalize the central path of the Interior Point Methods under the Entropic Barrier. We prove that this tropical path satisfies a specific combinatorial property, proving that Entropic Barrier Interior Point Methods are not strongly polynomial. Our proof is based on the approach used by Allamigeon, Benchimol, Gaubert and Joswig for proving a similar result for the Logarithmic Barrier. We also give our result a probabilistic interpretation, where it can be seen as the tropicalization of the average of a certain class of distributions. This Barrier presents a theoretical interest as it was proved by Bubeck and Eldan to be a $(1 + o(1))n$-self-concordant barrier. It is also dual to the universal barrier, which was recently proved to be exactly n-self-concordant (by Yin Tat Lee, Man-Chung Yue).

1. Introduction

1.1. **The problem:** One famous open question in Optimization is to determine if solving a linear program can be done by a strongly polynomial algorithm. Interior Point Methods remain widely applied today, as they are considered more efficient for a large panel of situations. In [1], our supervisors (with their co-workers) proved that log-barrier Interior Point Methods are not strongly polynomial, by providing a counter-example with large input entries and slow execution time. In this paper, we show that their counter-example work all the same to another type of barrier, the Entropic Barrier.

1.2. **Our Contribution:**

1.3. **Our approach:**

2. Theoretical Frame

In this section, we reintroduce the necessary tools from tropical geometry. All references, All definitions below, along with the non-stated proofs of the lemmas/theorems, were already introduced in detail in [1].

2.1. **Fields of real Puiseux series and Puiseux polyhedra.** The field $\mathbb{K}$ of *absolutely convergent generalized real Puiseux series* consists of elements of the form

$$\boldsymbol{f} = \sum_{\alpha \in \mathbb{R}} a_\alpha t^\alpha \,, \tag{1}$$

where $a_\alpha \in \mathbb{R}$ for all α, and such that: (i) the support $\{\alpha \in \mathbb{R} \colon a_\alpha \neq 0\}$ is either finite or has $-\infty$ as the only accumulation point; (ii) there exists $\rho > 0$ such that the series absolutely converges for all $t > \rho$. . The definition of the field $\mathbb{K}$ guarantees that every non-null series has a leading term $a_{\alpha_0} t^{\alpha_0}$, where $\alpha_0 := max(supp(\boldsymbol{f}))$. Moreover, the field is non-archimedian and totally ordered under the definition $\boldsymbol{f} \leqslant \boldsymbol{g}$ if $\boldsymbol{g} - \boldsymbol{f}$ is null or has a positive leading coefficient. Denote $\mathbb{K}_+ := \{f \in \mathbb{K} | f \geqslant 0\}$. Notice that $\boldsymbol{f} \geqslant \boldsymbol{0}$ if and only if $\boldsymbol{f}(t) \geqslant 0$ for all t sufficiently large.
It was shown that with the previous propreties, $\mathbb{K}$ is a real closed set. This is of great interest, since by *Tarski's Principle*, $\mathbb{K}$ verifies first-order propreties as the reals.

Date: December 9, 2019.

Thus, it is interesting to define linear optimization objects over Puiseux Series. The Puiseux Polyhedra in dimension n is defined as:

$$\boldsymbol{\mathcal{P}} = \{\boldsymbol{x} \in \mathbb{K}^n : \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\}\,, \tag{2}$$

Since *Minkowsi-Weyl* Theorem is a first-order proprety, a Puiseux Polyhedra has an internal representation as a Minkowski sum of finite extreme points and extreme rays in $\mathbb{K}^n$.
A Puiseux Polyhedra can be seen as a family of Real Polyhedras parametrized by t large enough:

$$\boldsymbol{\mathcal{P}}(t) = \{x \in \mathbb{R}^n : \boldsymbol{A}(t)x \leqslant \boldsymbol{b}(t)\}\,, \tag{3}$$

One remarkable property is that for t large enough, the internal representation of $\boldsymbol{\mathcal{P}}$ can be seen as the family of internal representations of $\boldsymbol{\mathcal{P}}$(t):

Proposition 1. *Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_{p+q} \in \mathbb{K}^n$. For all t large enough, the evaluation in t of the Polyhedra generated by the Minkowki sum of the convex hull of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_p$ and the convex cone of $\boldsymbol{u}_{p+1}$, $\boldsymbol{u}_{p+2}$, ..., $\boldsymbol{u}_{p+q}$ is the same as the (Real) Polyhedra generated by the convex hull of $\boldsymbol{u}_1(t)$, $\boldsymbol{u}_2(t)$, ..., $\boldsymbol{u}_p(t)$ and the convex cone of $\boldsymbol{u}_{p+1}(t)$, $\boldsymbol{u}_{p+2}(t)$, ..., $\boldsymbol{u}_{p+q}(t)$.*

2.2. **The tropical space and tropical polyhedra.** The behavior of Puiseux Series motivates us to define the following *valuation* mapping:

$$\text{val} : \boldsymbol{f} \in \mathbb{K} \mapsto \begin{cases} max(sup(\boldsymbol{f})) \text{ if } sup(\boldsymbol{f}) \text{ is nonempty} \\ -\infty \text{ elsewhere} \end{cases} \in \mathbb{R} \cup \{-\infty\}$$

By this definition, we have

$$\text{val}(\boldsymbol{f}) = \lim_{t\to+\infty} \log_t |\boldsymbol{f}(t)|\,,$$

with the convention $\log_t 0 = -\infty$. We immediately obtain the following properties (hence justifying the denomination val). for all $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{K}$:

$$\text{val}(\boldsymbol{f} + \boldsymbol{g}) \leqslant \max(\text{val}(\boldsymbol{f}), \text{val}(\boldsymbol{g})) \quad \text{and} \quad \text{val}(\boldsymbol{f}\boldsymbol{g}) = \text{val}(\boldsymbol{f}) + \text{val}(\boldsymbol{g})\,. \tag{4}$$

We define $\mathbb{T}$ the tropical space as the semi-field where the elements are $\mathbb{R}\cup\{-\infty\}$,, the addition is the maximum, and the multiplication is the addition in the reals:

$$(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, max, +)$$

We can extend easily these definitions and define tropical vectors, tropical half-spaces, then tropical polyhedras. When equipped with the order topology, finite dimensional tropical spaces $\mathbb{T}^n$ have closed half-spaces and closed polyhedras.

3. The Entropic Barrier and tropicalization of the central path

3.1. Legendre-Fenchel Duality.

Definition 2. *for $f : \theta \in D \subset \mathbb{R}^n \mapsto \mathbb{R}$ we define its Legendre-Fenchel Transform as:*

$$f^* : x \in D^* \mapsto f^*(x) = \sup_{\theta\in D}(\langle x, \theta\rangle - f(\theta))$$

Theorem 3. *If $D \subset \mathbb{R}^n$ is a convex body and $f : x \in D \subset \mathbb{R}^n \mapsto \mathbb{R}$ is convex then $(f^*)^* = f$.*

Finally, we have the following interesting result :

Proposition 4. *If f is differentiable, strictly convex and big outside of a certain compact, and if $\overline{D^*}$ is compact and f^* is equal to $+\infty$ on ∂D^* and is continuous on $\overline{D^*}$, then for any $\theta \in D$, $\nabla f(\theta)$ is the only value of x maximizing $\langle x, \theta\rangle - f^*(x)$ on D^*.*

Proof. Let us define $F : D^* \times D \to \mathbb{R}$, $(x, \theta) \mapsto \langle x, \theta\rangle - f(\theta)$. Then F is differentiable (on the interior of $D^* \times D$). Since f is continuous and too small outside a certain compact, then for any $x \in D^*$, $f^*(x) = \sup_{\theta\in D}(\langle x, \theta\rangle - f(\theta))$ is reached at a certain point θ, and since $F(x, .)$ is strictly convex (since f is), there is a unique such point. Hence we can denote that point by

$\Theta(x)$, the unique θ satisfying $\langle x,\theta\rangle - f(\theta) = f^*(x)$. Since F is differentiable (since f is), we have $\partial_\theta F(x,\Theta(x)) = 0$, that is to say $x = \nabla f(\Theta(x))$.

Let us define $\tilde{F} : D^* \times D \to \mathbb{R}$, $(x,\theta) \mapsto \langle x,\theta\rangle - f^*(x)$. Now, consider any $\theta \in D$. We will show the existence of a (not necessarily unique for now) point $X(\theta)$ such that $\langle X(\theta),\theta\rangle - f^*(X(\theta)) = \sup_{x\in D^*} \tilde{F}(x,\theta)$. Consider a sequence of points $(x_n)_{n\in\mathbb{N}}$ of D^* such that $\tilde{F}(x_n,\theta) \underset{n\to\infty}{\longrightarrow} \sup_{x\in D^*} \tilde{F}(x,\theta)$. Since $\overline{D^*}$ is compact, we can extract a convergent sub-sequence. The limit cannot be on ∂D^* where f^* equals $+\infty$ and $\tilde{F}(.,\theta) = -\infty$, so its an interior, which we denote by $X(\theta)$, and by continuity of f^* wa have $\tilde{F}(X(\theta),\theta) = \sup_{x\in D^*} \tilde{F}(x,\theta)$. So $\tilde{F}(X(\theta),\theta) = (f^*)^*(\theta) = f(\theta)$ (by theorem 3 since f is convex). By the uniqueness property shown above (applied to the vector $x = X(\theta)$), we get $\theta = \Theta(X(\theta))$, and thus $X(\theta) = \nabla f(\theta)$. In particular this shows the uniqueness of $X(\theta)$.

□

3.2. **The Entropic barrier.** We introduce the barrier which we will be working with by citing the following result from :

Theorem 5. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact convex set with nonempty interior. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined for $\theta \in \mathbb{R}^n$ by*

$$\log\left(\int_{x\in\mathcal{K}} \exp\left(\langle x,\theta\rangle\right) dx\right)$$

Then the Fenchel dual $f^* : int(\mathcal{K}) \to \mathbb{R}$, defined for $x \in int(\mathcal{K})$ is a $(1+o(1))n$-self-concordant barrier on $\mathcal{K}$.

Now, entropic barrier linear programs have the following very interesting property :

Proposition 6. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact convex set with nonempty interior and f^* the entropic barrier on $\mathcal{K}$ defined in theroem 5. Then, for any $c \in \mathbb{R}^n$, for any real $\mu > 0$, the entropic barrier program with parameter μ which states as follows*

$$\textit{maximize} \quad \langle c,x\rangle - \mu f^*(x) \quad \textit{subject to} \quad x \in D^*$$

has a unique solution x equal to $\nabla f(\frac{1}{\mu}c)$.

Proof. By proposition 4... □

3.3. **Tropicalization of the central path.**

4. Description of The Tropical Polyhedra

4.1. **Description of** $\operatorname{val}(\mathcal{K})$**.**

Lemma 7. *There is a finite family $\mathcal{F}$ of hyperplanes of $\mathbb{R}^n$ such that for all $\epsilon > 0$, for any open ball B which does not intersect any of them, there is a time T_ϵ such that either for all $t \geqslant T_\epsilon$, $B \subset k(t)$ or for all $t \geqslant T_\epsilon$, $d(B, k(t)) \geqslant \epsilon$.*

Proof. There is a Puiseux matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ such that

$$\boldsymbol{\mathcal{K}} = \{x \in \mathbb{K}^d \colon \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\}$$

and then

$$\boldsymbol{\mathcal{K}}(t) := \{x \in \mathbb{R}^d \colon \boldsymbol{A}(t)x \leqslant \boldsymbol{b}(t)\}$$

□

Assumption 8. $\operatorname{val}(\boldsymbol{\mathcal{K}})$ *is a polyhedral cell of nonzero volume (a set whose border is composed of polyhedra and which is homeomorphic to a ball of dimension n).*

5. Majoration de l'intégrale

First, we define

$$y^* := \text{barycenter of } \{y \in \text{val}(\boldsymbol{\mathcal{K}}) | y \leqslant -val(\boldsymbol{\theta})\}$$

Secondly, consider T_0^θ such that $\forall i \in [n]$, the Puiseux series $\boldsymbol{\theta}_i(t)$ converges and takes negative values for $t \geqslant T_0^\theta$. Let us define, $\forall t \geqslant T_0^\theta$, $\forall \epsilon > 0$, the sub-level :

$$k_\epsilon(t) := \{y \in k(t) | y \leqslant \epsilon - \log_t(-\boldsymbol{\theta}(t))\}$$

and its complementary $\overline{k_\epsilon(t)} = k(t) \setminus k_\epsilon(t) = \{y \in k(t) | \exists i \in [n], y_i > \epsilon - \log_t(-\boldsymbol{\theta}_i(t))\}$.

We start by proving two lemmas.

Lemma 9. *There exists a time T_1 and a constant C such that the coordinates of any point in $\cup_{t \geqslant T_1} k(t)$ are all less or equal to C.*

Proof. Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_p \in \mathbb{K}^n$ such that $\boldsymbol{\mathcal{K}}$ is the convex hull of $\boldsymbol{u}_1, ..., \boldsymbol{u}_n$. Since $\forall i \in [p], \forall j \in [n], t \mapsto \log_t(\boldsymbol{u}_{ij}(t))$ is either constant equal to $-\infty$ (if $\boldsymbol{u}_{ij} = 0$) or with real values and converging towards $\text{val}(\boldsymbol{u}_{ij})$ when $t \to +\infty$, and thus - in each case - has a finite upper bound. Let us denote by $C \in \mathbb{R}$ a common upper bound to those (finitely many) functions. We know that there exists a time T_1 such that $\forall t \geqslant T$, $\boldsymbol{\mathcal{K}}(t)$ is the convex hull of $\boldsymbol{u}_1(t), ..., \boldsymbol{u}_n(t)$. Consider a time $t \geqslant T_1$ and a vector $y \in k(t)$, which writes as $y = (\log_t(x_j))_{1 \leqslant j \leqslant n}$ for a certain $x \in \boldsymbol{\mathcal{K}}(t)$ Then x can be written as a convex combination of $\boldsymbol{u}_1(t), ..., \boldsymbol{u}_n(t)$. Hence for any $j \in [n]$, $x_j) \leqslant \max_{1 \leqslant i \leqslant p} u_{ij}(t)$, so by monotony of $\log_t$, $y_j = \log_t(x_j) \leqslant \max_{1 \leqslant i \leqslant p} \log_t(\boldsymbol{u}_{ij}(t)) \leqslant C$. □

Lemma 10. *For any $\epsilon > 0$, there exists a time $T_2^{\epsilon,\theta}$ such that $\forall t \geqslant T_2^{\epsilon,\theta}$, $\forall y \in k_\epsilon(t)$:*

$$y \leqslant y^* + 4\epsilon \cdot e$$

where $e = (1, 1, ..., 1)$.

Proof. Consider some $\epsilon > 0$ and a time T^ϵ after which $d_{\text{H}}(k(t), k) \geqslant \epsilon$. Denote by $T_2^{\epsilon,\theta} \geqslant T^\epsilon$ a time from which $\text{val}(\boldsymbol{\theta}) \leqslant \log_t(-\boldsymbol{\theta}(t)) + \epsilon \cdot e$. Consider a time $t \geqslant T_2^{\epsilon,\theta}$ and a vector $y \in k_\epsilon(t)$.

Then there exists a $y' \in k$ such that $d_{\text{H}}(y, y') \leqslant \epsilon$, which implies that $\forall i \in [n]$, $y'_i - y_i \leqslant \epsilon$. So since $y \in k_\epsilon(t)$, $\forall i \in [n]$, $y'_i \leqslant 2\epsilon - \log_t(-\boldsymbol{\theta}) \leqslant 3\epsilon - val(\boldsymbol{\theta}_i)$. Consider $y'' := y' - 3\epsilon \cdot e$. We argue that $y'' \leqslant y^*$.

Indeed since $y' \in k = \text{val}(\boldsymbol{\mathcal{K}})$, there exists $\boldsymbol{Y} \in \boldsymbol{\mathcal{K}}$ such that $y' = \text{val}(\boldsymbol{Y})$. Denote by $\lambda \in \mathbb{K}$ the Puiseux series $t \mapsto t^{-3\epsilon}$. Since $\mathbf{0} \in \boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}$ is convex then $(1 - \lambda) \cdot \mathbf{0} + \lambda \cdot \boldsymbol{Y} \in \boldsymbol{\mathcal{K}}$. Hence its image y'' under the valuation map is in k.

Finally, since $\forall i \in [n]$, $y''_i = y'_i - 3\epsilon \leqslant -val(\boldsymbol{\theta}_i)$, then y'' is in the sub-level, thus we have $y'' \leqslant y^*$ by definition of the tropical barycenter. So $y' \leqslant y^* + 2\epsilon \cdot e$, hence $\forall i \in [n]$, $y_i - y_i^* \leqslant (y_i - y'_i) + 3\epsilon \cdot e \leqslant d_{\text{H}}(y, y') + 3\epsilon \cdot e \leqslant 4\epsilon \cdot e$. □

Now we can give an upper bound to the integrals. We use the notation T_0^θ defined before the two lemmas and denote by C and T_1 the constants given by lemma 9 and for any $\epsilon > 0$, we denote by $T_2^{\epsilon,\theta}$ the time given by lemma 10. Finally we define, for any $\epsilon > 0$, $T^{\epsilon,\theta} := max\{T_0^\theta, T_1, T_2^{\epsilon,\theta}$.

Proposition 11. *For any vector $v \in \mathbb{R}^n$ with positive coordinates, if we denote par g the scalar product by v, then for any $\epsilon > 0$, $\forall t \geqslant T^{\epsilon,\theta}$:*

$$\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy \leqslant \exp\left(-t^\epsilon\right) \frac{t^{g(C \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i} + \frac{t^{g(y^* + 4\epsilon \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}$$

Proof. Consider a positive real ϵ and a time $t \geqslant T^{\epsilon,\theta}$.

First, let us deal with the integral on $\overline{k_\epsilon(t)}$. We claim that for any $y \in \overline{k_\epsilon(t)}$, we have $< t^y, \boldsymbol{\theta}(t) > \leqslant -t^\epsilon$. Indeed, consider such a y. Then there is an $i \in [n]$ such that $y_i > \epsilon - \log_t(-\boldsymbol{\theta}_i(t))$. Thus, by negativity of $\theta(t)$'s coordinates (since $t \geqslant T_0^\theta$), we have $< t^y, \boldsymbol{\theta}(t) > \leqslant t^{y_i} \boldsymbol{\theta}_i(t) = -t^{y_i + \log_t(-\boldsymbol{\theta}_i(t))} \leqslant -t^\epsilon$. Hence we get :

$$\begin{aligned}\int_{\overline{k_\epsilon(t)}} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy &\leqslant \exp\left(-t^\epsilon\right) \int_{\overline{k_\epsilon(t)}} t^{g(y)}dy \\ &\leqslant \exp\left(-t^\epsilon\right) \int_{y\leqslant C\cdot e} t^{g(y)}dy \\ &= \exp\left(-t^\epsilon\right) \frac{t^{g(C\cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}\end{aligned}$$

Now let us tackle the integral on $k_\epsilon(t)$. By negativity of $\theta(t)$'s coordinates (since $t \geqslant T_0^\theta$), we can bound the exponential by 1, and by lemma 10 we get :

$$\begin{aligned}\int_{k_\epsilon(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy &\leqslant \int_{k_\epsilon(t)} t^{g(y)}dy \\ &\leqslant \int_{y\leqslant y^*+4\epsilon\cdot e} t^{g(y)}dy \\ &= \frac{t^{g(y^*+4\epsilon\cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}\end{aligned}$$

By summing the inequalities on the two integrals, we finally get :

$$\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy \leqslant \exp\left(-t^\epsilon\right) \frac{t^{g(C\cdot e)}}{\log(t)^n \prod_{i=1}^n v_i} + \frac{t^{g(y^*+4\epsilon\cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}$$

□

Corollary 12. *For any vector $v \in \mathbb{R}^n$ with positive coordinates, if we denote par g the scalar product by v, then :*

$$\limsup_t \left\{ \log_t \left(\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy \right) \right\} \leqslant g(y^*)$$

6. Minoration de l'intégrale

Proposition 13. *For any vector $v \in \mathbb{R}^n$ with positive coordinates, if we denote par g the scalar product by v, then for any $\epsilon > 0$, there is a constant $D^{\epsilon,\theta}$ and a time $T^{\epsilon,\theta}$ such that $\forall t \geqslant T^{\epsilon,\theta}$:*

$$\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy \geqslant D^{\epsilon,\theta} t^{g(y^*-\frac{3\epsilon}{2}\cdot e)}$$

Proof. Unless stated otherwise, all distances considered hereafter are euclidean distances in $\mathbb{R}^n$.

We know that $\forall y \in k$, $\forall \lambda \in \mathbb{R}_{\geqslant 0}$, $y - \lambda \cdot e \in k$. Consider a positive real ϵ. Denote by y_ϵ the point $y^* - \lambda \cdot e \in k$. So, since k is a polyhedral cell of nonzero volume (by assumption 8), there is a nonempty open ball $B \subset k$ whose elements are at a distance of at most $\frac{\epsilon}{2}$ from y_ϵ. Since $\frac{\epsilon}{2} < \frac{\epsilon}{\sqrt{2}}$, then $\forall y \in B$, $\forall i \in [n]$, $y_i < y_{\epsilon i} + \frac{\epsilon}{\sqrt{2}} = y_i^*$, hence B **is a subset of the sub-level**.

Now, denote by F the family of hyperplanes from lemma 7. Since F is finite, then there is a sub-ball $B' \subset B$ such that B' does not intersect any hyperplane in F. Then, by lemma 7, there is a time T such that either for all $t \geqslant T$, $B' \subset k(t)$ or for all $t \geqslant T$, $B' \cap k(t) = \emptyset$. But the second possibility is impossible since the Hausdorff distance of B''s center to $k(t)$ must converge towards 0. Then for all $t \geqslant T$, $B' \subset k(t)$. Thus, for any $t \geqslant T$, by positivity :

$$\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy \geqslant \int_{B'} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right)dy$$

then since $B' \subset B$ and B is $\frac{\epsilon}{2}$-close to y_ϵ, which is at distance ϵ from y^* we get, by transitivity and positivity of v :

$$\int_{B'} t^{g(y)} \exp(< t^y, \boldsymbol{\theta}(t) >)dy \geqslant t^{g(y^*-\frac{3\epsilon}{2}\cdot e)} \int_{B'} \exp(< t^y, \boldsymbol{\theta}(t) >)dy$$

and since B' is in the sub-level and the function in the above integral is decreasing in each coordinate y_i of y, we finally get :

$$t^{g(y^*-\frac{3\epsilon}{2}\cdot e)} \int_{B'} \exp(< t^y, \boldsymbol{\theta}(t) >)dy \geqslant t^{g(y^*-\frac{3\epsilon}{2}\cdot e)} \exp(< t^{y^*}, \boldsymbol{\theta}(t) >)|B'|$$

This inequality is valid for $t \geqslant T$. Moreover, since $y^* \leqslant -\operatorname{val}(\boldsymbol{\theta})$, the exponential converges towards a strictly positive constant l when $t \to +\infty$, so there is a certain time $T' \geqslant T$ such that $\forall t \geqslant T'$:

$$\int_{k(t)} t^{g(y)} \exp(< t^y, \boldsymbol{\theta}(t) >)dy \geqslant t^{g(y^*-\frac{3\epsilon}{2}\cdot e)} \frac{l|B'|}{2}$$

$\square$

References

[1] X.Allamigeon, P.Benchimol, S.Gaubert, M.Joswig log-barrier interior point methods are not strongly polynomial. siam journal on applied algebra and geometry, 2(1), 140–178, 2018.

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