STUDY OF THE ENTROPIC BARRIER (ONGOING)

ABDELLAH AZNAG, YASSINE HAMDI
UNDER THE SUPERVISION OF:
XAVIER ALLAMIGEON, AND STÉPHANE GAUBERT

Abstract. We define the Entropic Barrier as the Cramer Transform of the uniform measure on a convex body in $\mathbb{R}^n$. In this work, we tropicalize the central path of the Interior Point Methods under the Entropic Barrier. We prove that this tropical path satisfies a specific combinatorial property, proving that Entropic Barrier Interior Point Methods are not strongly polynomial. Our proof is based on the approach used by Allamigeon, Benchimol, Gaubert and Joswig for proving a similar result for the Logarithmic Barrier. We also give our result a probabilistic interpretation, where it can be seen as the tropicalization of the average of a certain class of distributions. This Barrier presents a theoretical interest as it was proved by Bubeck and Eldan to be a $(1 + o(1))n$-self-concordant barrier. It is also dual to the universal barrier, which was recently proved to be exactly n-self-concordant (by Yin Tat Lee, Man-Chung Yue).

1. Introduction

1.1. **The context:** One famous open question in Optimization is to determine if solving a linear program can be done by a strongly polynomial algorithm. Interior Point Methods remain widely applied today, as they are considered more efficient for a large panel of situations. In [1], our supervisors (with their co-workers) proved that log-barrier Interior Point Methods are not strongly polynomial, by providing a counter-example with large input entries and slow execution time.

We now provide an intuition of their proof. Consider a family of polytopes containing the *zero* vector included in the positive quadrant of $\mathbb{R}^n$ parametric in a variable t which extreme points behave asymptotically in the form $a_\alpha t^\alpha$, and to each polytope $\mathcal{P}(t)$ of this family, consider $(\mathcal{C}_{\mu,t})_{\mu\geqslant 0}$ the central path generated by solving the linear problem penalized by μlbar the log-barrier term.

We now "Tropicalize" the objects above. By Tropicalizing, we mean studying the limit of $\log_t$ when $t \to +\infty$ of the objects. $\log_t \mathcal{P}(t)$ converges to a "Tropical Polyhedra" $\mathrm{val}(\boldsymbol{\mathcal{P}})$. The most important result is the following: The tropical central path $(\mathcal{C}_{\mu,t})_{\mu\geqslant 0}$ converges to what is defined as the tropical Barycenter path of certain sub-levels. What is important about this limit, is that it lies in the frontier of the tropical polyhedra.

Another result of major importance, bounds by below the number of iterations in the gradient's descent algorithm of the Interior Point Method by the curvature length of the central path, modulo some constants. This bounding is mainly because we restrict the gradient's descent to be confined within a thin pipeline around the central path.

We can combine the two previous arguments (convergence and bounding by below). By taking t large enough, we can make the log_t of the central path as close as possible to the frontier of the tropical frontier, and this would bound by below the number of iterations by the curvature of the tropical frontier, modulo some constants.

This is remarkable, because we can actually control the curvature of the tropical frontier! In particular, we can construct a family of polytopes so that the number of its segment increases exponentially, hence forcing the IPM method to make an exponential number of iterations.

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The counter-example constructed in [1] is a family of polytopes that satisfy the previous property. A simulation of the convergence mentioned above was made by A.Bennouna and Y.El Maazouz in [3].

1.2. **The new problem:** Our supervisors suggested we study another barrier, mainly the Entropic Barrier as introduced by S.Bubeck and R.Eldan in [2]. This barrier is of major theoretical interest since it is log-homogeneous, was proved to be $(1 + o(1))n$-self-corcondant [2], and reveals intriguing duality properties to the universal barrier, which was recently proved [4] to be $o(n)$-self-corcondant.

1.3. **Our Contribution:** Our main result in this work is **Theorem 14**. It proves that the tropical central path of the Entropic Barrier is also in the tropical frontier of the tropical polyhedra. Hence by using the same counter-example in [1], a corollary of this result is that Entropic Barrier Interior Point Methods are not strongly polynomial. We also give a probabilistic interpretation of our result, as we view the tropicalization of the central path as the tropicalization of the average of a random variable following a canonically-exponential distribution. This interpretation could easily be generalized to a larger class of distribution using the core elements of our proof.

1.4. **Our approach:** An intuition of our proof is as follows: We start by viewing our primal penalized problem as the Fenchel transform of a certain function, so we view the central path as the average of $x \in \mathcal{P}$ following the distribution $\mathbf{1}_{\boldsymbol{\mathcal{P}}}\{\exp(\langle x, \theta\rangle - f(\theta))\}$.
Then, we exploit the convergence properties of $log_t\mathcal{P}(t)$ to $\mathbf{val}(\boldsymbol{\mathcal{P}})$ to bound the integral zone (which moves as $t \to +\infty$) by static volumes. When we integrate over these volumes, and after taking the log_t, we obtain a value arbitrarily close to our targeted limit.

2. Theoretical Frame

In this section, we reintroduce the necessary tools from tropical geometry. All references, All definitions below, along with the non-stated proofs of the lemmas/theorems, were already introduced in detail in [1].

2.1. **Fields of real Puiseux series and Puiseux polyhedra.** The field $\mathbb{K}$ of *absolutely convergent generalized real Puiseux series* consists of elements of the form

$$\boldsymbol{f} = \sum_{\alpha\in\mathbb{R}} a_\alpha t^\alpha\,, \tag{1}$$

where $a_\alpha \in \mathbb{R}$ for all α, and such that: (i) the support $\{\alpha \in \mathbb{R} \colon a_\alpha \neq 0\}$ is either finite or has $-\infty$ as the only accumulation point; (ii) there exists $\rho > 0$ such that the series absolutely converges for all $t > \rho$. . The definition of the field $\mathbb{K}$ guarantees that every non-null series has a leading term $a_{\alpha_0}t^{\alpha_0}$, where $\alpha_0 := max(supp(\boldsymbol{f}))$. Moreover, the field is non-archimedian and totally ordered under the definition $\boldsymbol{f} \leqslant \boldsymbol{g}$ if $\boldsymbol{g} - \boldsymbol{f}$ is null or has a positive leading coefficient. Denote $\mathbb{K}_+ := \{f \in \mathbb{K} | f \geqslant 0\}$. Notice that $\boldsymbol{f} \geqslant \mathbf{0}$ if and only if $\boldsymbol{f}(t) \geqslant 0$ for all t sufficiently large.
It was shown that with the previous propreties, $\mathbb{K}$ is a real closed set. This is of great interest, since by *Tarski's Principle*, $\mathbb{K}$ verifies first-order propreties as the reals.
Thus, it is interesting to define linear optimization objects over Puiseux Series. The Puiseux Polyhedra in dimension n is defined as:

$$\boldsymbol{\mathcal{P}} = \{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\}\,, \tag{2}$$

Since *Minkowsi-Weyl* Theorem is a first-order proprety, a Puiseux Polyhedra has an internal representation as a Minkowski sum of finite extreme points and extreme rays in $\mathbb{K}^n$.
A Puiseux Polyhedra can be seen as a family of Real Polyhedras parametrized by t large enough:

$$\boldsymbol{\mathcal{P}}(t) = \{x \in \mathbb{R}^n \colon \boldsymbol{A}(t)x \leqslant \boldsymbol{b}(t)\}\,, \tag{3}$$

One remarkable property is that for t large enough, the internal representation of $\boldsymbol{\mathcal{P}}$ can be seen as the family of internal representations of $\boldsymbol{\mathcal{P}}(\mathrm{t})$:

Proposition 1. *Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_{p+q} \in \mathbb{K}^n$. For all t large enough, the evaluation in t of the Polyhedra generated by the Minkowki sum of the convex hull of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_p$ and the convex cone of $\boldsymbol{u}_{p+1}$, $\boldsymbol{u}_{p+2}$, ..., $\boldsymbol{u}_{p+q}$ is the same as the (Real) Polyhedra generated by the convex hull of $\boldsymbol{u}_1(t)$, $\boldsymbol{u}_2(t)$, ..., $\boldsymbol{u}_p(t)$ and the convex cone of $\boldsymbol{u}_{p+1}(t)$, $\boldsymbol{u}_{p+2}(t)$, ..., $\boldsymbol{u}_{p+q}(t)$.*

2.2. **The tropical space and tropical polyhedra.** The behavior of Puiseux Series motivates us to define the following *valuation* mapping:

$$\mathrm{val} : \boldsymbol{f} \in \mathbb{K} \mapsto \begin{cases} max(sup(\boldsymbol{f})) \text{ if } sup(\boldsymbol{f}) \text{ is nonempty} \\ -\infty \text{ elsewhere} \end{cases} \in \mathbb{R} \cup \{-\infty\}$$

By this definition, we have

$$\mathrm{val}(\boldsymbol{f}) = \lim_{t \to +\infty} \log_t |\boldsymbol{f}(t)| \,,$$

with the convention $\log_t 0 = -\infty$. We immediately obtain the following properties (hence justifying the denomination val). for all $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{K}$:

$$\mathrm{val}(\boldsymbol{f} + \boldsymbol{g}) \leqslant \max(\mathrm{val}(\boldsymbol{f}), \mathrm{val}(\boldsymbol{g})) \quad \text{and} \quad \mathrm{val}(\boldsymbol{f}\boldsymbol{g}) = \mathrm{val}(\boldsymbol{f}) + \mathrm{val}(\boldsymbol{g}) \,. \tag{4}$$

We define $\mathbb{T}$ the tropical space as the semi-field where the elements are $\mathbb{R} \cup \{-\infty\}$, the addition is the maximum, and the multiplication is the addition in the reals:

$$(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, max, +)$$

We can extend easily these definitions and define tropical vectors, tropical half-spaces, then tropical polyhedras. When equipped with the order topology, finite dimensional tropical spaces $\mathbb{T}^n$ have closed half-spaces and closed polyhedras.

DEFINIR LA DISTANCE DE HAUSSDORF

3. The Entropic Barrier and tropicalization of the central path

3.1. **Legendre-Fenchel Duality.**

Definition 2. *for $f : \theta \in D \subset \mathbb{R}^n \mapsto \mathbb{R}$ we define its Legendre-Fenchel Transform as:*

$$f^* : x \in D^* \subset \mathbb{R}^n \mapsto f^*(x) = \sup_{\theta \in D} (\langle x, \theta \rangle - f(\theta))$$

Theorem 3. *If $D \subset \mathbb{R}^n$ is a convex body and $f : x \in D \subset \mathbb{R}^n \mapsto \mathbb{R}$ is convex then $(f^*)^* = f$.*

Finally, we have the following interesting result :

Proposition 4. *If f is differentiable and strictly convex and if $\overline{D^*}$ is compact and f^* is equal to $+\infty$ on its frontier ∂D^* and is continuous on its interior, then for any $\theta \in D$, $\nabla f(\theta)$ is the only value of x maximizing $\langle x, \theta \rangle - f^*(x)$ on D^*.*

Proof. Let us define $F : D^* \times D \to \mathbb{R}$, $(x, \theta) \mapsto \langle x, \theta \rangle - f(\theta)$. Then F is differentiable (on the interior of $D^* \times D$). Since F is continuous and too small outside a certain compact (by strict convexity of f), then for any $x \in D^*$, $f^*(x) = \sup_{\theta \in D}(\langle x, \theta \rangle - f(\theta))$ is reached at a certain point θ, and since $F(x, .)$ is strictly convex (since f is), there is a unique such point. Hence we can denote that point by $\Theta(x)$, the unique θ satisfying $\langle x, \theta \rangle - f(\theta) = f^*(x)$. Since F is differentiable (since f is), we have $\partial_\theta F(x, \Theta(x)) = 0$, that is to say $x = \nabla f(\Theta(x))$.

Let us define $\tilde{F} : D^* \times D \to \mathbb{R}$, $(x, \theta) \mapsto \langle x, \theta \rangle - f^*(x)$. Now, consider any $\theta \in D$. We will show the existence of a (not necessarily unique for now) point $X(\theta)$ such that $\langle X(\theta), \theta \rangle - f^*(X(\theta)) = \sup_{x \in D^*} \tilde{F}(x, \theta)$. Consider a sequence of points $(x_n)_{n \in \mathbb{N}}$ of D^* such that $\tilde{F}(x_n, \theta) \underset{n \to \infty}{\longrightarrow} \sup_{x \in D^*} \tilde{F}(x, \theta)$. Since $\overline{D^*}$ is compact, we can extract a convergent sub-sequence. The limit cannot be on ∂D^* where f^* equals $+\infty$ and $\tilde{F}(., \theta) = -\infty$, so its an interior, which we denote by $X(\theta)$, and by continuity of f^* we have $\tilde{F}(X(\theta), \theta) = \sup_{x \in D^*} \tilde{F}(x, \theta)$. So $\tilde{F}(X(\theta), \theta) = (f^*)^*(\theta) = f(\theta)$

(by theorem 3 since f is convex). By the uniqueness property shown above (applied to the vector $x = X(\theta)$), we get $\theta = \Theta(X(\theta))$, and thus $X(\theta) = \nabla f(\theta)$. In particular this shows the uniqueness of $X(\theta)$.

□

3.2. **The Entropic barrier.** We introduce the barrier which we will be working with by citing the following result from [2] :

Theorem 5. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact convex set with nonempty interior. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined for $\theta \in \mathbb{R}^n$ by*

$$\log\left(\int_{x\in\mathcal{K}} \exp\left(\langle x, \theta\rangle\right) dx\right)$$

Then the Fenchel dual $f^* : int(\mathcal{K}) \to \mathbb{R}$, defined for $x \in int(\mathcal{K})$ is a $(1+o(1))n$-self-concordant barrier on $\mathcal{K}$.

Now, entropic barrier linear programs have the following very interesting property giving us a simple expression of central path points :

Proposition 6. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact convex set with nonempty interior and f^* the entropic barrier on $\mathcal{K}$ defined in theroem 5. Then, for any $c \in \mathbb{R}^n$, for any real $\mu > 0$, the entropic barrier program with parameter μ which states as follows*

$$\textit{maximize} \quad \langle c, x\rangle - \mu f^*(x) \quad \textit{subject to} \quad x \in D^*$$

has a unique solution x equal to $\nabla f(\frac{1}{\mu}c)$.

Proof. This will follow directly proposition 4, with $D = \mathbb{R}^n$ and $D^* = \mathrm{int}(\boldsymbol{\mathcal{K}})$. Since we know that f^* is a barrier on $\mathrm{int}(\boldsymbol{\mathcal{K}})$ and $\boldsymbol{\mathcal{K}}$ is compact, all that remains is to show that f is strictly convex. Since f is defined by an integral of a smooth function over a compact domain, its derivatives can be easily computed and we get that $\forall\theta \in \mathbb{R}^n$, $\forall i, j \in [n]$, $\partial_i\partial_j f(\theta)$ writes as :

$$\frac{\big(\int_{x\in\mathcal{K}} x_i x_j \exp\left(\langle x, \theta\rangle\right) dx\big)\big(\int_{x\in\mathcal{K}} \exp\left(\langle x, \theta\rangle\right) dx\big) - \big(\int_{x\in\mathcal{K}} x_i \exp\left(\langle x, \theta\rangle\right) dx\big)\big(\int_{x\in\mathcal{K}} x_j \exp\left(\langle x, \theta\rangle\right) dx\big)}{\log(t)\big(\int_{x\in\mathcal{K}} \exp\left(\langle x, \theta\rangle\right) dx\big)^2}$$

Hence, if we denote by $H(f)$ the hessian matrix of f, then $\forall\theta \in \mathbb{R}^n$, for all nonzero vectors $y \in \mathbb{R}^n$, we have, up to a strictly positive multiplicative function of θ :

$$\begin{aligned}
y^t H(f)(\theta) y = &\sum_{1\leqslant i,j\leqslant n} y_i y_j \big(\int_{x\in\mathcal{K}} x_i x_j \exp\left(\langle x, \theta\rangle\right) dx\big)\big(\int_{x\in\mathcal{K}} \exp\left(\langle x, \theta\rangle\right) dx\big) \\
&- \sum_{1\leqslant i,j\leqslant n} \big(\int_{x\in\mathcal{K}} x_i \exp\left(\langle x, \theta\rangle\right) dx\big)\big(\int_{x\in\mathcal{K}} x_j \exp\left(\langle x, \theta\rangle\right) dx\big) \\
&= \big(\int_{x\in\mathcal{K}} \big(\sum_{1\leqslant i\leqslant n} x_i y_i\big)^2 \exp\left(\langle x, \theta\rangle\right) dx\big)\big(\int_{x\in\mathcal{K}} \exp\left(\langle x, \theta\rangle\right) dx\big) \\
&- \big(\sum_{1\leqslant i\leqslant n} \int_{x\in\mathcal{K}} x_i y_i \exp\left(\langle x, \theta\rangle\right) dx\big)^2 \\
&> 0
\end{aligned}$$

by the Cauchy Schwarz inequality, and since $\sum_{1\leqslant i\leqslant n} x_i y_i$ is not constant on $\mathcal{K}$ because $y \neq 0$ and $\mathrm{int}(\mathcal{K}) \neq \emptyset$. □

3.3. **Tropicalization of the central path.** From now on we will work with a Puiseux polyhedron $\boldsymbol{\mathcal{K}} \in \mathbb{K}_+^n$ and a Puiseux vector $\boldsymbol{c} \in \mathbb{K}^n$ with negative coordinates and look at the family of linear programs :

$$\mathrm{LP}_t \qquad \text{maximize} \quad \langle \boldsymbol{c}(t), x\rangle \quad \text{subject to} \quad x \in \mathrm{int}(\boldsymbol{\mathcal{K}}(t))$$

We will show that the corresponding family of central paths has a "tropical limit", that is to say a valuation. But first, we need some properties on the polyhedra $\boldsymbol{\mathcal{K}}(t)$.

4. Description of The Tropical Polyhedra

Let us introduce the notation $k(t) := \log_t(\boldsymbol{\mathcal{K}}(t)) = \{(\log_t(x_i))_{1\leqslant i\leqslant n} | x \in \boldsymbol{\mathcal{K}}(t)\}$, the study of which this section is dedicated.

First, we cite the following theorem from [1] :

Theorem 7. *The sequence $(k(t))_t$ of real polyhedra converges to the tropical polyhedron $\mathrm{val}(\boldsymbol{\mathcal{K}})$ with respect to the directed Hausdorff distance d_{H}.*

Then we give a simple extension of this result :

Lemma 8. *There is a finite family $\mathcal{F}$ of hyperplanes of $\mathbb{R}^n$ such that for all $\epsilon > 0$, there is a time T_ϵ such that for any ball B of $\mathbb{R}^n$ such that $d(B, \mathcal{F}) \geqslant \epsilon$ (where d is the canonical euclidean distance), either for all $t \geqslant T_\epsilon$, $B \subset k(t)$ or for all $t \geqslant T_\epsilon$, $B \cap k(t) = \emptyset$.*

Proof. There is a Puiseux matrix $\boldsymbol{A} \in \mathbb{R}^{p\times n}$ and vector $\boldsymbol{b} \in \mathbb{R}^p$ such that

$$\boldsymbol{\mathcal{K}} = \{x \in \mathbb{K}^n \colon \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b}\}$$

Then we have :

$$\boldsymbol{\mathcal{K}}(t) := \{x \in \mathbb{R}^n \colon \boldsymbol{A}(t)x \leqslant \boldsymbol{b}(t)\}$$

and then

$$k(t) := \{y \in \mathbb{R}^n \colon \boldsymbol{A}(t)t^y \leqslant \boldsymbol{b}(t)\}$$

For any $y \in \mathbb{R}^n$ and every $i \in [p]$, there is a finite family of Puiseux series : $\phi_i(y) := (\boldsymbol{b}_i) \bigcup (\boldsymbol{a}_{ij}t^{y_j})_{j\in[n]} \setminus (\boldsymbol{0})$, with the corresponding finite family of valuations, $(\mathrm{val}(\boldsymbol{b}_i)) \bigcup (\mathrm{val}(\boldsymbol{a}_{ij}) + y_j)_{j\in[n]} \setminus (-\infty)$. The elements of the latter are pairwise distinct iff y is outside a certain finite family of hyperplanes : those among $(y_j = \mathrm{val}(\boldsymbol{b}_i) - \mathrm{val}(\boldsymbol{a}_{ij}))_{j\in[n]} \bigcup (y_j - y_k = \mathrm{val}(\boldsymbol{a}_{ik}) - \mathrm{val}(\boldsymbol{a}_{ij}))_{1\leqslant j<k\leqslant n}$ after excluding null series. Let us denote by $\mathcal{F}$ the finite family of those hyperplanes.

Then the hyperplanes in $\mathcal{F}$ delimit a finite family $\mathcal{Z}$ of polyhedral zones, each corresponding to p orders (independant of y) for the p families $(\phi_i(y))_{i\in[p]}$. Thus to each zone $Z \in \mathcal{Z}$ corresponds a function of y in the form of a $\mathbb{K}$-linear combination of the series $(t^{y_j})_{j\in[n]}$:

$$\boldsymbol{g_Z} : \mathbb{R}^n \to \mathbb{K}^p,\ y \mapsto \Big(\text{the element of } \phi_i(y) \text{ having the maximal valuation}\Big)_{i\in[p]}$$

such that for any $y \in Z$, for any $i \in [p]$, $\mathrm{val}\left(\boldsymbol{b}_i - \sum_{j\in[n]} \boldsymbol{a}_{ij}t^{y_j}\right) = \mathrm{val}(\boldsymbol{g_{Zi}(y)})$, with $\boldsymbol{g}_{Zi} \equiv \boldsymbol{0}$ if $\phi_i = \emptyset$, that is to say if $\boldsymbol{b}_i = \boldsymbol{0}$ and $\forall j \in [n]\ \boldsymbol{a}_{ij} = \boldsymbol{0}$. This function comes with a linear function $v_Z : \mathbb{R}^n \to \mathbb{R}^p$, $y \mapsto \mathrm{val}(\boldsymbol{g_Z(y)})$ and a vector (independent from y) $c_Z \in \mathbb{R}^p :=$ (the leading coefficient of $\boldsymbol{g_{Zi}(y)})_{i\in[p]}$.

Now everything is in place to prove the lemma. Note that d_{H} and the canonical euclidean distance d are equivalent because $\forall x, y \in \mathbb{R}^n$, $d_{\mathrm{H}}(x, y) = d_{\mathrm{H}}(x - y, 0)$ and $y \mapsto d_{\mathrm{H}}(y, 0)$ is a norm, and norms in a finite dimension R-vector space are equivalent. So we shall prove the lemma for the canonical euclidean distance d. Let us denote by T_0 a time from which all

Puiseux series in the matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ can be evaluated. Consider any real $\epsilon > 0$. We will show the existence of a time T_ϵ such that for any $y \in \mathbb{R}^n$ such that $d(y, \mathcal{F}) \geqslant \epsilon$ (with d the standard euclidean distance), if Z is the zone in which y lies, then $\forall i \in [p]$, $\forall t \geqslant T_\epsilon$, $\left|\boldsymbol{b}_i(t) - \sum_{j\in[n]} \boldsymbol{a}_{ij}(t)t^{y_j}\right| \in \left[(1 - t^{-\frac{\epsilon}{2}})|\boldsymbol{g_{Zi}(y)}(t)|, (1 + \frac{\epsilon}{2})|\boldsymbol{g_{Zi}(y)}(t)|\right]$. This is true if $\phi_i = \emptyset$, so let us assume it is not the case. Since $d(y, \mathcal{F}) \geqslant \epsilon$, then for any series $\boldsymbol{u(y)} \in \phi_i(y) \setminus (\boldsymbol{g_Z(y)})$, with valuation $v(y)$, we have $v_Z(y) - v(y) \geqslant \epsilon$, so $\forall t \geqslant T_0$:

$$\left|\frac{\boldsymbol{u(y)}(t)}{\boldsymbol{g_{Zi}(y)(t)}}\right| = \left|\frac{c_Z \cdot t^{v_Z(y)}}{\boldsymbol{g_{Zi}(y)(t)}} \cdot \frac{\boldsymbol{u(y)}(t)}{c_Z \cdot t^v}\right| t^{v - v_Z(y)} \leqslant \left|\frac{c_Z \cdot t^{v_Z(y)}}{\boldsymbol{g_{Zi}(y)(t)}} \cdot \frac{\boldsymbol{u(y)}(t)}{c_Z \cdot t^v}\right| t^{-\epsilon}$$

The key property is that both fractions are independent from y, by the form of the series in $\phi_i(y)$. Moreover those fractions are Puiseux series of null valuation. So there is a time $T_{u,Z,\epsilon}$ and a constant $C_{u,Z,\epsilon} > 0$ such that $\forall t \geqslant T_{u,Z,\epsilon}$:

$$\left|\frac{\boldsymbol{u(y)}(t)}{\boldsymbol{g_{Zi}(y)(t)}}\right| \leqslant C_{u,Z,\epsilon} t^{-\epsilon}$$

Since there is a finite number of zones Z and series u, there is a time T_ϵ such that for any $y \in \mathbb{R}^n$ such that $d(y, \mathcal{F}) \geqslant \epsilon$ (with d the standard euclidean distance), if Z is the zone in which y lies, then $\forall i \in [p]$, $\forall t \geqslant T_\epsilon$, $\left|\boldsymbol{b}_i(t) - \sum_{j\in[n]} \boldsymbol{a}_{ij}(t)t^{y_j}\right| \in \left[(1 - t^{-\frac{\epsilon}{2}})|\boldsymbol{g_{Zi}(y)}(t)|, (1 + \frac{\epsilon}{2})|\boldsymbol{g_{Zi}(y)}(t)|\right]$.

Finally, since the sign of $\boldsymbol{g_{Zi}(y)}(t)$ is independent from y (by the shape of the series in $\phi_i(y)$) and is constant after a certain time (independent from y), and there are finitely many zones Z, there is a time T'_ϵ and signs $(e_Z)_{Z\in\mathcal{Z}} \in (\{-,+\}^p)^{\mathcal{Z}}$ such that for any $y \in \mathbb{R}^n$ such that $d(y, \mathcal{F}) \geqslant \epsilon$, if Z is the zone in which y lies, then $\forall i \in [p]$, $\forall t \geqslant T'_\epsilon$, $\boldsymbol{b}_i(t) - \sum_{j\in[n]} \boldsymbol{a}_{ij}(t)t^{y_j}$ is of sign e_{Zi} (the sign $+$ being attributed if null). Hence, for any zone $Z \in \mathcal{Z}$, if $\forall i \in [p]$, $e_{Zi} = +$, then $\forall y \in Z$ such that $d(y, \mathcal{F}) \geqslant \epsilon$, $\forall t \geqslant T'_\epsilon$, $y \in k(t)$(by definition of $k(t)$). And for any zone Z for which $\exists i \in [p]$ such that $e_{Zi} = -$, then $\forall y \in Z$ such that $d(y, \mathcal{F}) \geqslant \epsilon$, $\forall t \geqslant T'_\epsilon$, $y \notin k(t)$. Since a ball B such that $d(B, \mathcal{F}) \geqslant \epsilon$ would have to be included in a unique zone, this proves the lemma. □

5. Tropicalization of the central path

Assumption 9. $\mathrm{val}(\boldsymbol{\mathcal{K}})$ *is a pure and full dimensional polyhedral complex.*

In this section we will work with a real $\mu > 0$, and we will denote by $\boldsymbol{\theta}$ the Puiseux series $\frac{\boldsymbol{c}(t)}{\mu}$, which has negative coordinates (since $\boldsymbol{c}$ does).

First, we define

$$y^* := \text{barycenter of } \{y \in \mathrm{val}(\boldsymbol{\mathcal{K}}) | y \leqslant -val(\boldsymbol{\theta})\}$$

The central path point of parameter μ is $\nabla f(\boldsymbol{\theta}(t))$ by proposition 6, which can be written as :

$$\left(\frac{\int_{x\in\boldsymbol{\mathcal{K}}(t)} x_i \exp\left(\langle x, \boldsymbol{\theta}(t)\rangle\right)dx}{\int_{x\in\boldsymbol{\mathcal{K}}(t)} \exp\left(\langle x, \boldsymbol{\theta}(t)\rangle\right)dx}\right)_{i\in[n]}$$

By the change of variable $x = t^y = (t^{y_i})_{i\in[n]}$, we get :

$$\left(\frac{\int_{y\in k(t)} t^{y_i + \sum_{j\in[n]} y_j} \exp\left(\langle t^y, \boldsymbol{\theta}(t)\rangle\right)dy}{\int_{y\in k(t)} t^{\sum_{j\in[n]} y_j} \exp\left(\langle t^y, \boldsymbol{\theta}(t)\rangle\right)dy}\right)_{i\in[n]}$$

We will bound the numerator and denominator by vectors involving y^*.

5.1. **Upper bound.** Consider T_0^θ such that $\forall i \in [n]$, the Puiseux series $\boldsymbol{\theta}_i(t)$ converges and takes negative values for $t \geqslant T_0^\theta$. Let us define, $\forall t \geqslant T_0^\theta$, $\forall \epsilon > 0$, the sub-level :

$$k_\epsilon(t) := \{y \in k(t) | y \leqslant \epsilon - \log_t(-\boldsymbol{\theta}(t))\}$$

and its complementary $\overline{k_\epsilon(t)} = k(t) \setminus k_\epsilon(t) = \{y \in k(t) | \exists i \in [n], y_i > \epsilon - \log_t(-\boldsymbol{\theta}_i(t))\}$.

We start by proving two lemmas.

Lemma 10. *There exists a time T_1 and a constant C such that the coordinates of any point in $\cup_{t \geqslant T_1} k(t)$ are all less or equal to C.*

Proof. Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., $\boldsymbol{u}_p \in \mathbb{K}^n$ such that $\boldsymbol{\mathcal{K}}$ is the convex hull of $\boldsymbol{u}_1, ..., \boldsymbol{u}_n$. Since $\forall i \in [p], \forall j \in [n], t \mapsto \log_t(\boldsymbol{u}_{ij}(t))$ is either constant equal to $-\infty$ (if $\boldsymbol{u}_{ij} = 0$) or with real values and converging towards $\mathrm{val}(\boldsymbol{u}_{ij})$ when $t \to +\infty$, and thus - in each case - has a finite upper bound. Let us denote by $C \in \mathbb{R}$ a common upper bound to those (finitely many) functions. We know that there exists a time T_1 such that $\forall t \geqslant T$, $\boldsymbol{\mathcal{K}}(t)$ is the convex hull of $\boldsymbol{u}_1(t), ..., \boldsymbol{u}_n(t)$. Consider a time $t \geqslant T_1$ and a vector $y \in k(t)$, which writes as $y = (\log_t(x_j))_{1 \leqslant j \leqslant n}$ for a certain $x \in \boldsymbol{\mathcal{K}}(t)$ Then x can be written as a convex combination of $\boldsymbol{u}_1(t), ..., \boldsymbol{u}_n(t)$. Hence for any $j \in [n]$, $x_j) \leqslant \max_{1 \leqslant i \leqslant p} u_{ij}(t)$, so by monotony of $\log_t$, $y_j = \log_t(x_j) \leqslant \max_{1 \leqslant i \leqslant p} \log_t(\boldsymbol{u}_{ij}(t)) \leqslant C$. ☐

Lemma 11. *For any $\epsilon > 0$, there exists a time $T_2^{\epsilon,\theta}$ such that $\forall t \geqslant T_2^{\epsilon,\theta}$, $\forall y \in k_\epsilon(t)$:*

$$y \leqslant y^* + 4\epsilon \cdot e$$

where $e = (1, 1, ..., 1)$.

Proof. Consider some $\epsilon > 0$ and a time T^ϵ after which $d_{\mathrm{H}}(k(t), k) \geqslant \epsilon$. Denote by $T_2^{\epsilon,\theta} \geqslant T^\epsilon$ a time from which $\mathrm{val}(\boldsymbol{\theta}) \leqslant \log_t(-\boldsymbol{\theta}(t)) + \epsilon \cdot e$. Consider a time $t \geqslant T_2^{\epsilon,\theta}$ and a vector $y \in k_\epsilon(t)$.

Then there exists a $y' \in k$ such that $d_{\mathrm{H}}(y, y') \leqslant \epsilon$, which implies that $\forall i \in [n]$, $y'_i - y_i \leqslant \epsilon$. So since $y \in k_\epsilon(t)$, $\forall i \in [n]$, $y'_i \leqslant 2\epsilon - \log_t(-\boldsymbol{\theta}) \leqslant 3\epsilon - val(\boldsymbol{\theta}_i)$. Consider $y'' := y' - 3\epsilon \cdot e$. We argue that $y'' \leqslant y^*$.

Indeed since $y' \in k = \mathrm{val}(\boldsymbol{\mathcal{K}})$, there exists $\boldsymbol{Y} \in \boldsymbol{\mathcal{K}}$ such that $y' = \mathrm{val}(\boldsymbol{Y})$. Denote by $\lambda \in \mathbb{K}$ the Puiseux series $t \mapsto t^{-3\epsilon}$. Since $\boldsymbol{0} \in \boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}$ is convex then $(1 - \lambda) \cdot \boldsymbol{0} + \lambda \cdot \boldsymbol{Y} \in \boldsymbol{\mathcal{K}}$. Hence its image y'' under the valuation map is in k.

Finally, since $\forall i \in [n]$, $y''_i = y'_i - 3\epsilon \leqslant -val(\boldsymbol{\theta}_i)$, then y'' is in the sub-level, thus we have $y'' \leqslant y^*$ by definition of the tropical barycenter. So $y' \leqslant y^* + 2\epsilon \cdot e$, hence $\forall i \in [n]$, $y_i - y^*_i \leqslant (y_i - y'_i) + 3\epsilon \cdot e \leqslant d_{\mathrm{H}}(y, y') + 3\epsilon \cdot e \leqslant 4\epsilon \cdot e$. ☐

Now we can give an upper bound to the integrals. We use the notation T_0^θ defined before the two lemmas and denote by C and T_1 the constants given by lemma 10 and for any $\epsilon > 0$, we denote by $T_2^{\epsilon,\theta}$ the time given by lemma 11. Finally we define, for any $\epsilon > 0$, $T^{\epsilon,\theta} := max\{T_0^\theta, T_1, T_2^{\epsilon,\theta}$.

Proposition 12. *For any vector $v \in \mathbb{R}^n$ with positive coordinates, if we denote par g the scalar product by v, then for any $\epsilon > 0$, $\forall t \geqslant T^{\epsilon,\theta}$:*

$$\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy \leqslant \exp\left(-t^\epsilon\right) \frac{t^{g(C \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i} + \frac{t^{g(y^* + 4\epsilon \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}$$

Proof. Consider a positive real ϵ and a time $t \geqslant T^{\epsilon,\theta}$.

First, let us deal with the integral on $\overline{k_\epsilon(t)}$. We claim that for any $y \in \overline{k_\epsilon(t)}$, we have $< t^y, \boldsymbol{\theta}(t) > \leqslant -t^\epsilon$. Indeed, consider such a y. Then there is an $i \in [n]$ such that $y_i > \epsilon - \log_t(-\boldsymbol{\theta}_i(t))$. Thus, by negativity of $\theta(t)$'s coordinates (since $t \geqslant T_0^\theta$), we have $< t^y, \boldsymbol{\theta}(t) > \leqslant t^{y_i} \boldsymbol{\theta}_i(t) = -t^{y_i + \log_t(-\boldsymbol{\theta}_i(t))} \leqslant -t^\epsilon$. Hence we get :

$$
\begin{aligned}
\int_{\overline{k_\epsilon(t)}} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy &\leqslant \exp\left(-t^\epsilon\right) \int_{\overline{k_\epsilon(t)}} t^{g(y)} dy \\
&\leqslant \exp\left(-t^\epsilon\right) \int_{y \leqslant C \cdot e} t^{g(y)} dy \\
&= \exp\left(-t^\epsilon\right) \frac{t^{g(C \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}
\end{aligned}
$$

Now let us tackle the integral on $k_\epsilon(t)$. By negativity of $\theta(t)$'s coordinates (since $t \geqslant T_0^\theta$), we can bound the exponential by 1, and by lemma 11 we get :

$$
\begin{aligned}
\int_{k_\epsilon(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy &\leqslant \int_{k_\epsilon(t)} t^{g(y)} dy \\
&\leqslant \int_{y \leqslant y^* + 4\epsilon \cdot e} t^{g(y)} dy \\
&= \frac{t^{g(y^* + 4\epsilon \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}
\end{aligned}
$$

By summing the inequalities on the two integrals, we finally get :

$$
\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy \leqslant \exp\left(-t^\epsilon\right) \frac{t^{g(C \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i} + \frac{t^{g(y^* + 4\epsilon \cdot e)}}{\log(t)^n \prod_{i=1}^n v_i}
$$

□

5.2. **Lower bound.**

Proposition 13. *For any vector $v \in \mathbb{R}^n$ with positive coordinates, if we denote par g the scalar product by v, then for any $\epsilon > 0$, there is a constant $D^{\epsilon,\theta}$ and a time $T'^{\epsilon,\theta}$ such that $\forall t \geqslant T'^{\epsilon,\theta}$:*

$$
\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy \geqslant D^{\epsilon,\theta} t^{g(y^* - \frac{3\epsilon}{2} \cdot e)}
$$

Proof. Unless stated otherwise, all distances considered hereafter are euclidean distances in $\mathbb{R}^n$.

We know that $\forall y \in k$, $\forall \lambda \in \mathbb{R}_{\geqslant 0}$, $y - \lambda \cdot e \in k$ (because it is the valuation of a convex Puiseux combination of $\mathbf{0}$ and a lift of y). Consider a positive real ϵ. Denote by y_ϵ the point $y^* - \lambda \cdot e \in k$. So, since k is a pure and full dimensional polyhedral complex (by assumption 9), there is a nonempty open ball $B \subset k$ whose elements are at a distance of at most $\frac{\epsilon}{2}$ from y_ϵ. Since $\frac{\epsilon}{2} < \frac{\epsilon}{\sqrt{2}}$, then $\forall y \in B$, $\forall i \in [n]$, $y_i < y_{\epsilon i} + \frac{\epsilon}{\sqrt{2}} = y_i^*$, hence B **is a subset of the sub-level**.

Now, denote by F the family of hyperplanes from lemma 8. Since F is finite, then there is a sub-ball $B' \subset B$ such that B' does not intersect any hyperplane in F. Then, by lemma 8, there is a time T such that either for all $t \geqslant T$, $B' \subset k(t)$ or for all $t \geqslant T$, $B' \cap k(t) = \emptyset$. But the second possibility is impossible since the Hausdorff distance of B''s center -which is in k - to $k(t)$ must converge towards 0 by theorem 7. Then for all $t \geqslant T$, $B' \subset k(t)$. Thus, for any $t \geqslant T$, by positivity :

$$
\int_{k(t)} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy \geqslant \int_{B'} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy
$$

then since $B' \subset B$ and B is $\frac{\epsilon}{2}$-close to y_ϵ, which is at distance ϵ from y^* we get, by transitivity and positivity of v :

$$
\int_{B'} t^{g(y)} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy \geqslant t^{g(y^* - \frac{3\epsilon}{2} \cdot e)} \int_{B'} \exp\left(< t^y, \boldsymbol{\theta}(t) >\right) dy
$$

and since B' is in the sub-level and the function in the above integral is decreasing in each coordinate y_i of y, we finally get :

$$t^{g(y^*-\frac{3\epsilon}{2}\cdot e)}\int_{B'}\exp(<t^y,\boldsymbol{\theta}(t)>)dy \geqslant t^{g(y^*-\frac{3\epsilon}{2}\cdot e)}\exp(<t^{y^*},\boldsymbol{\theta}(t)>)|B'|$$

This inequality is valid for $t\geqslant T$. Moreover, since $y^*\leqslant -\mathrm{val}(\boldsymbol{\theta})$, the exponential converges towards a strictly positive constant l when $t\to+\infty$, so there is a certain time $T'\geqslant T$ such that $\forall t\geqslant T'$:

$$\int_{k(t)} t^{g(y)}\exp(<t^y,\boldsymbol{\theta}(t)>)dy \geqslant t^{g(y^*-\frac{3\epsilon}{2}\cdot e)}\frac{l|B'|}{2}$$

□

5.3. **Conclusion.** From the above we will readily derive our main theorem :

Theorem 14. *Let $n\in\mathbb{N}_{\geqslant 2}$, $\boldsymbol{c}\in\mathbb{K}^n$ a Puiseux vector with negative coordinates and $\boldsymbol{\mathcal{K}}\in\mathbb{K}^n_+$ a Puiseux polyhedron of non-empty interior such that $\mathrm{val}(\boldsymbol{\mathcal{K}})$ is a pure and full dimensional polyhedral complex. Then the central paths $(\mathcal{C}_\mu(t))_t$ of the family of linear programs (LP_t) : $\big(\text{maximize } \langle\boldsymbol{c}(t),x\rangle \text{ subject to } x\in int(\boldsymbol{\mathcal{K}}(t))\big)_t$ with respect to the entropic barrier satisfy that $\forall\mu>0$, $\log_t(\mathcal{C}_\mu(t))$ converges towards the tropical barycenter of the intersection between the tropical image $\mathrm{val}(\boldsymbol{\mathcal{K}})$ of $\boldsymbol{\mathcal{K}}$ and the sublevel $\{y\in\mathbb{T}^n|y+\mathrm{val}(\boldsymbol{\theta})\leqslant 0\}$.*

Proof. Consider a real $\mu>0$. We use the notations of section 5. By what was said at the beginning of that section, we have, for t big enough :

$$\mathcal{C}_\mu(t)=\left(\frac{\int_{y\in k(t)} t^{y_i+\sum_{j\in[n]}y_j}\exp(\langle t^y,\boldsymbol{\theta}(t)\rangle)dy}{\int_{y\in k(t)} t^{\sum_{j\in[n]}y_j}\exp(\langle t^y,\boldsymbol{\theta}(t)\rangle)dy}\right)_{i\in[n]}$$

Consider $\epsilon>0$. By applying propositions 12 and 13 to the numerator and denominator for $g_0=\sum_{j\in[n]}y_j$ and $g_i=y_i+\sum_{j\in[n]}y_j$, we get that $\forall t\geqslant max(T^{\epsilon,\theta},T'^{\epsilon,\theta})$, $\forall i\in[n]$:

$$\frac{D^{\epsilon,\theta}t^{g_i(y^*-\frac{3\epsilon}{2}\cdot e)}}{\exp(-t^\epsilon)\dfrac{t^{g_0(C\cdot e)}}{\log(t)^n}+\dfrac{t^{g_0(y^*+4\epsilon\cdot e)}}{\log(t)^n}}\leqslant\mathcal{C}_\mu(t)_i\leqslant\frac{\exp(-t^\epsilon)\dfrac{t^{g_i(C\cdot e)}}{2\log(t)^n}+\dfrac{t^{g_i(y^*+4\epsilon\cdot e)}}{2\log(t)^n}}{D^{\epsilon,\theta}t^{g(y^*-\frac{3\epsilon}{2}\cdot e)}}$$

which after taking the $\log_t$ gives :

$$g_i(y^*-\tfrac{3\epsilon}{2}\cdot e)-g_0(y^*+4\epsilon\cdot e)+\log_t(D^{\epsilon,\theta}\log(t)^n)+\log_t\big(\frac{1}{1+\exp(-t^\epsilon)t^{g_0(C\cdot e)-g_0(y^*+4\epsilon\cdot e)}}\big)\leqslant\log_t(\mathcal{C}_\mu(t)_i)$$

and

$$\log_t(\mathcal{C}_\mu(t)_i)\leqslant g_i(y^*+4\epsilon\cdot e)-g_0(y^*-\tfrac{3\epsilon}{2}\cdot e)+\log_t(2D^{\epsilon,\theta}\log(t)^n)+\log_t\big(\frac{\exp(-t^\epsilon)t^{g_i(C\cdot e)-g_i(y^*+4\epsilon\cdot e)}+1}{D^{\epsilon,\theta}t}\big)$$

that is to say

$$y^*-g_i(\tfrac{3\epsilon}{2}\cdot e)-g_0(4\epsilon\cdot e)+\log_t(D^{\epsilon,\theta}\log(t)^n)+\log_t\big(\frac{1}{1+\exp(-t^\epsilon)t^{g_0(C\cdot e)-g_0(y^*+4\epsilon\cdot e)}}\big)\leqslant\log_t(\mathcal{C}_\mu(t)_i)$$

and

$$\log_t(\mathcal{C}_\mu(t)_i)\leqslant y^*+g_i(4\epsilon\cdot e)+g_0(\tfrac{3\epsilon}{2}\cdot e)+\log_t(2D^{\epsilon,\theta}\log(t)^n)+\log_t\big(\frac{\exp(-t^\epsilon)t^{g_i(C\cdot e)-g_i(y^*+4\epsilon\cdot e)}+1}{D^{\epsilon,\theta}t}\big)$$

By taking the limit when $t\to+\infty$ we get :

$$\liminf_t \left(\log_t(\mathcal{C}_\mu(t)_i) \right) \geqslant y^* - g_i(\tfrac{3\epsilon}{2} \cdot e) - g_0(4\epsilon \cdot e)$$

and

$$\limsup_t \left(\log_t(\mathcal{C}_\mu(t)_i) \right) \geqslant y^* + g_i(4\epsilon \cdot e) + g_0(\tfrac{3\epsilon}{2} \cdot e)$$

Finally by taking the limit when $\epsilon \to 0$ we get the same value for the $\liminf$ and $\limsup$:

$$\log_t(\mathcal{C}_\mu(t)_i) \underset{t\to+\infty}{\longrightarrow} y^*$$

□

6. Probabilistic Interpretation of the result and a generalization

7. Conclusion

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(Xavier Allamigeon, Stéphane Gaubert) INRIA and CMAP, École Polytechnique, CNRS, Université Paris–Saclay, 91128 Palaiseau Cedex, France
firstname.lastname@inria.fr

(Abdellah Aznag, Yassine Hamdi) École Polytechnique, 91120 Palaiseau, France
firstname.lastname@polytechnique.edu