Le = 21 | 10 | 2016

Série n°(2) :

Exo 1 :

1)
$$\sum_{n=1}^{\infty} \left(\cos\frac{1}{n^2} - \cos\left(\frac{1}{(1+n)^2}\right)\right)$$

$$U_n = \cos\frac{1}{n^2} - \cos\left(\frac{1}{(1+n)^2}\right)$$

$$S_n = U_1 + U_2 + U_3 + \dots + U_n.$$

$$= \left(\cos 1 - \cancel{\cos\frac{1}{4}}\right) + \left(\cancel{\cos\frac{1}{4}} - \cancel{\cos\frac{1}{9}}\right) + \dots +$$

$$\left(\cos\frac{1}{(n-1)^2} - \cancel{\cos\left(\frac{1}{n^2}\right)}\right) + \cos\left(\frac{1}{n^2} - \cos\left(\boxed{\frac{1}{(n+1)^2}}\right)\right)$$

$$S_n = \cos 1 - \cos\left(\frac{1}{(n+1)^2}\right)$$

$$\lim_{n \to +\infty} S_n = \cos 1 - 1 = S$$

donc la série $\sum_{n=1}^{\infty} U_n$ converge. et a pour somme $S = \boxed{\cos 1 - 1.}$

2)
$$\sum_{n=1}^{\infty} \left(\exp\left(\frac{1}{n}\right) - \exp\left(\frac{1}{n+1}\right)\right).$$

$$S_n = U_1 + U_2 + U_3 + \dots + U_n$$

$$S_n = e - e^{1/2} + e^{1/2} - e^{1/3} \; e^{1/4} + \dots , e^{1/n} - e^{\boxed{1/n+1}}$$

$$S_n = \exp - e^{\frac{1}{n+1}}$$

$$\lim_{n \to +\infty} S_n = e - 1$$

donc la série $\sum_{n=1}^{\infty}$ converge et a pour somme

$$S = e - 1.$$

③
$$\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$$

$$U_n = \frac{1}{n^3 - n} = \frac{1}{n(n-1)(n+1)}$$

$$= \frac{a}{n} + \frac{b}{n-1} + \frac{c}{n+1} =$$

$$= \frac{a(n-1)(n+1) + b(n)(n+1) + c(n)(n-1)}{n^3 - n}$$

$$= \frac{a(n^2-1) + b(n^2+n) + c(n^2 - n)}{n^3 - n}$$

$$= \frac{(a+b+c)n^2 + (b-c)n - a.}{n^3 - n}$$

$$\frac{1}{n^3 - m} = \frac{-1}{m} + \frac{1}{2} \times \frac{1}{m-1} + \frac{1}{2} \cdot \frac{1}{m+1}$$

$$= \frac{-1/2 - 1/2}{m} + \frac{1}{2} \times \frac{1}{m-1} + \frac{1}{2}\frac{1}{m+1}$$

$$= 1/2 \cdot \left(\frac{1}{m-1} - \frac{1}{m}\right) + \frac{1}{2}\left(\frac{1}{m+1} - \frac{1}{m}\right)$$

$$\begin{cases} U_n = U_n^1 + U_n^2 \\ U_n^1 = \frac{1}{2}\left(\frac{1}{m-1} - \frac{1}{n}\right) \\ U_n^2 = \frac{1}{2}\left(\frac{1}{m+1} - \frac{1}{m}\right) \end{cases}$$

$$\begin{cases} S_m = U_2 + U_3 + \dots + U_n \\ \quad = S_n^1 + S_n^2 \\ S_n^1 = U_2^1 + U_3^1 + \dots + U_n^1 \\ S_n^2 = U_2^2 + U_3^2 + \dots + U_n^2 \end{cases}$$

$$S_n^2 \cancel{= 1/2 \left(1 - 1/2\right) + 1/2\left(1/2 - 1/3\right) + 1/2}$$

$$\cancel{+ 1/2\left(\frac{1}{3} - 1/4\right) + \dots +}$$

$$\cancel{+ 1/2\left(\frac{1}{m-2} - \frac{1}{m-1}\right) + 1/}$$

$$S_n^1 = \frac{1}{2}\left(1-\cancel{\frac{1}{2}}\right) + 1/2\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \frac{1}{2}\left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n-1}}\right) + 1/2$$

$$+ 1/2\left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$$

$$= 1/2 - \frac{1}{2n}$$

$$S_n^2 = \frac{1}{2}\left(\cancel{\frac{1}{3}} - \frac{1}{2}\right) + 1/2\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{3}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{4}}\right) + \dots + \frac{1}{2}\left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n-1}}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{n+1} - \cancel{\frac{1}{n}}\right)$$

$$= 1/2\left(-\frac{1}{2} - \frac{1}{n+1}\right)$$

$$S_n = \frac{1}{2} - \frac{1}{2n} - \frac{1}{4} - \frac{1}{2} \times \frac{1}{n+1}$$

$$\lim_{n \to +\infty} S_n = \frac{1}{4}$$

donc la serie $\sum V_n$ converge et a pour somme $S = +\frac{1}{4}$

2) $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Exo n°2 :

$$\boxed{\sum_{n=0}^{+\infty} a\, r^n = \frac{a}{1-r} \quad \text{si } |r| < 1.}$$

① $\sum_{n=1}^{+\infty} 6(0{,}9)^{n-1}$ $n = n-1$

$= \sum_{n=0}^{\infty} 6\,(0{,}9)^n$ converge.

$a = 6 \quad r = 0{,}9 \quad |r| < 1$

$= \sum_{n=0}^{+\infty} 6\,(0{,}9)^n = \frac{6}{1-0{,}9} = \frac{6}{0{,}1} = 60$

② $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$

$= \sum_{n=0}^{\infty} \frac{10^{n+1}}{(-9)^n} = \sum_{n=0}^{\infty} 10\left(\frac{-10}{9}\right)$

$|r| \geqslant 1$. diverge.

③ $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n} = \sum_{n=0}^{+\infty} 1 \times \left(\frac{1}{\sqrt{2}}\right)^n$.

$\Rightarrow \begin{cases} a = 1 \\ r = \frac{1}{\sqrt{2}} \rightarrow |r| < 1. \end{cases}$

converge.

$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$

④ $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{+\infty} \frac{\pi^n}{3^n \times 3}$

$= \sum_{n=0}^{\infty} \frac{1}{3} \times \left(\frac{\pi}{3}\right)^n$

$a = \frac{1}{3} \quad , r = \frac{\pi}{3}$

donc elle est divergente $|r| \geqslant 1$

Le 29/10/2016

⑤ $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}}$ $n = n-1$.

$\sum_{n=1}^{\infty} \frac{e^{n+1}}{3^n}$

$\sum_{n=0}^{\infty} \left(\frac{e}{3}\right)^n \cdot e$ $\frac{e}{3} < 1$.

$a = e \quad , r = \frac{e}{3}$

$S = \frac{e}{1-\frac{e}{3}}$ converge.

Exo 3 : $\boxed{\sum \frac{1}{n^p} \text{ converge si } p > 1}$

$\sum \frac{1}{n^2} = \frac{\pi}{6}$

① $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ converge car

$p = \sqrt{2} > 1$.

② $\sum_{n=1}^{\infty} n^{-0{,}9999}$

$\sum_{n=1}^{\infty} \frac{1}{n^{0{,}999}}$

$p = 0{,}999$

diverge car $p \cancel{>} 1$.

③ $\sum_{n=1}^{\infty} \frac{\ln n}{n^3} = \ln n \frac{1}{n^3}$

Test d'intégration

$\sum U_n, \; U_n = f(n)$

f : continue

positive

décroissante

$\sum U_n$ et $\int_1^{\infty} f(x)\,dx$ sont de même nature

③ $\sum_{n=1}^{\infty} \frac{\ln 3}{n^3}$

$f(x) = \frac{\ln x}{x^3}$ continue positive

$f'(x) = \frac{x^2 - 3x^2 \ln x}{x^6} = \frac{1 - 3\ln x}{x^4} < 0$

sur $[e, +\infty[$

$\sum_{n=1}^{+\infty} \frac{\ln n}{n^3}$ et $\int_e^{+\infty} \frac{\ln x}{x^3}\, dx$

sont de même nature

$\int_e^{+\infty} \frac{\ln x}{x^3}\, dx = \lim_{A \to +\infty} \int_e^{A} \frac{\ln x}{x^3}\, dx.$

$u' = \frac{1}{x^3}$ $\quad u = -1/2\, \frac{1}{x^2}$

$v = \ln x.$ $\quad v' = \frac{1}{x}.$

$$\boxed{\int_a^b u'.v = [u\, v]_a^b - \int_a^b u . v'}$$

$= \lim_{A \to +\infty} \left(\left[-\frac{1}{2} \frac{\ln x}{x^2} \right]_e^A - \int_e^A -\frac{1}{2x^3}\, dx. \right.$

$= \lim_{A \to +\infty} \left(-\frac{1}{2} \cdot \frac{\ln A}{A^2} + \frac{1}{2e^2} - \frac{1}{4A^2} + \frac{1}{4e^2} \right)_{n \to +\infty}$ l'intégrale est fini

$= \frac{1}{2e^2} + \frac{1}{4e^2} = \frac{3}{4e^2} < +\infty$

donc $\sum_{n=1}^{+\infty} \frac{\ln n}{n^3}$ converge.

⑥ $\sum_{n=2}^{\infty} n^2 e^{-n^3}$

$f(x) = x^2 e^{-x^3}$ continue positive

$f'(x) = 2x e^{-x^3} - 3x^4 e^{-x^3}$

$= x e^{-x^3} (2 - 3x^3) < 0$

$\int_1^{+\infty} x^2 e^{-x^3}\, dx = \lim_{A \to +\infty} \int_1^A x^2 e^{-x^3}\, dx$

Pour $x \geq 1$.

$= \lim_{A \to +\infty} \left[-1/3\, e^{-x^3} \right]_1^A$

$= \lim_{A \to +\infty} \left(-1/3\, e^{-A^3} + 1/3\, e^{-1} \right) = 1/3\, e^{-1}$

$\boxed{\text{donc } \sum n^2 e^{-n^3}}$ converge.

④ $= \sum_{n=1}^{+\infty} \frac{1}{n^2 + n^3}$

$U_n = \frac{1}{n^2 + n^3}$ $\quad v_n = \frac{1}{n^3}$

$\lim \frac{U_n}{V_n} = \lim_{n \to +\infty} \frac{n^3}{n^3 + n^2} = 1 \neq 0$

donc par le test de comparaison
$\sum U_n$ et $\sum V_n$ sont de même nature
or $\sum \frac{1}{n^3}$ converge donc $\sum \frac{1}{n^2+n^3}$ converge.

$\sum U_n$, $\sum V_n$
positive positive
si $U_n \leq v_n$ et $\sum v_n$ converge
alors $\sum U_n$ converge
si $U_n \leq v_n$ et $\sum U_n$ diverge
alors $\sum v_n$ diverge
* si $\lim \frac{U_n}{V_n} = L \neq 0$ alors
$\sum U_n$ et $\sum V_n$ sont de même nature

Test de comparaison

④ $\sum_{n=1}^{+\infty} \underbrace{\frac{1}{n^2+6n+13}}_{U_n}$

$v_n = \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{U_n}{v_n} = 1 \neq 0$ comme $\sum \frac{1}{n^2}$ converge

$\sum \frac{1}{n^2+6n+13}$ converge.

Exo 6:

① $\sum \frac{n^{1/3}}{\sqrt{n^3+4n+3}}$ U_n

$v_n = \frac{n^{2/3}}{\sqrt{n^3}} = \frac{n^{2/3}}{n^{2/3}}$

$= \frac{1}{n^{3/2 \cdot 1/3}}$

$= \frac{1}{\quad}$

Le . 05/11/2016

① $\sum \frac{n^{1/3}}{\sqrt{n^3+4n+3}}$ U_n

$v_n = \frac{n^{1/3}}{\sqrt{n^3}} = \frac{n^{1/3}}{n^{3/2}} = \frac{1}{n^{3/2-1/3}}$

$\lim_{n\to\infty} \frac{U_n}{v_n} = \lim \frac{n^{1/3} \times n^{7/6}}{\sqrt{n^3+4n+3}} = \frac{1}{n^{7/6}}$

$= \lim_{n\to+\infty} \frac{n^{9/6}}{n^{3/2}} = 1 \neq 0, +\infty$

donc $\sum U_n$ et $\sum v_n$ donc de même manière $\sum \frac{1}{n^{7/6}}$ converge

donc $\sum \frac{n^{1/3}}{\sqrt{n^3+4n+3}}$ converge.

$P = \frac{7}{6} > 1$.

③ $\sum \frac{1}{\sqrt[3]{4n^3+1}}$

$U_n = \frac{1}{\sqrt[3]{4n^3+1}}$

$v_n = \frac{1}{\sqrt[3]{n^3}} = \frac{1}{n}$

$\lim_{n\to+\infty} \frac{U_n}{v_n} = \lim_{n\to\infty} \frac{n}{\sqrt[3]{4n^3+1}}$

$= \lim_{n\to\infty} \frac{\cancel{n}}{4^{1/3}\cancel{n}} = \frac{1}{4^{1/3}} = \frac{1}{2^{2/3}} \neq 0, +\infty$

donc $\sum U_n$ et $\sum v_n$ dont le même nature $\sum v_n = \sum \frac{1}{n}$ diverge donc

$P = 1$

$\sum \frac{1}{\sqrt[3]{4n^3+1}}$ diverge

③ $\sum \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$.

$U_n = \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$

$V_n = \frac{n \times n^2}{n \times n^4} = \frac{1}{n^2}$

$\lim_{n\to+\infty} \frac{2n^5}{n^5} = 2 \neq 0, +\infty$

donc $\sum U_n$ et $\sum V_n$ sont de même ~~ma~~ nature $\sum \frac{1}{n^2}$ converge donc

$\sum \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$ converge.

$P=2$.

③ $\sum \underbrace{\frac{\arctan n}{n^{1,2}}}_{U_n}$

$U_n = \frac{\arctan n}{n^{1,2}} \leq \frac{\frac{\pi}{2}}{n^{1,2}}$

$\sum \frac{\frac{\pi}{2}}{n^{1,2}}$ converge donc $\sum U_n = \sum \frac{\arctan}{n^{1,2}}$ converge

④ $\sum \frac{n \sin^2 n}{1+n^3}$ $\sin^2 n \leq 1$

~~$U_n = \frac{n \sin^2 n}{1+n^3} \leq \frac{n \times 1}{1+n^3} \leq \frac{n}{n^3}$~~

$U_n = \frac{n \sin^2 n}{(1+n^3)} = (n) \times (\sin^2 n) \times \left(\frac{1}{1+n^3}\right)$

$\leq n \times 1 \times \frac{1}{n^3}$

$= 1/n^2$

$\sin^2 n \leq 1$

$1+n^3 \geq n^3$

$\frac{1}{1+n^3} \leq \frac{1}{n^3}$

$\sum \frac{1}{n^2}$ converge donc $\sum \frac{n \sin^2 n}{1+n^3}$ converge.

⑥ $\sum \left(1+\frac{1}{n}\right)^2 e^{-n}$

$U_n = \left(1+\frac{1}{n}\right)^2 e^{-n}$

$U_n = \left(1+\frac{1}{n}\right)^2 e^{-n}$

$\left(1+\left(\frac{1}{n}\right)\right)^2 \leq (1+1)^2 = 2^2 = 4$

$U_n \leq 4e^{-n} \leq \frac{4}{n^2}$

$\sum \frac{4}{n^2}$ converge donc $\sum U_n$ converge.

Exo 8 :

① $\sum_{=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

$U_n = \frac{1}{2n+1}$

$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} \frac{1}{2n} = 0$

$f(x) = \frac{1}{2x+1}$

$f'(x) = \frac{-2}{(2x+1)^2} < 0$

decroissante donc converge.

② $\frac{(-1)^{n-1}}{\ln(n+4)}$

$U_n = \frac{1}{\ln(n+4)}$, $\lim_{n\to+\infty} U_n = 0$

$f(x) = \frac{1}{\ln(x+4)}$, $f'(x) = \frac{-1/x+4}{(\ln(x+4))^2} < 0$

U_n est décroissante donc convergente

$$\sum_{n=1} (-1)^{n+1} n e^{-n}$$

$$U_n = \frac{x}{e^x}$$

$$f(x) = \frac{x}{e^x}$$

$$f'(x) = \frac{(x)'e^x - (e^x)'(x)}{(e^x)^2}$$

$$\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} \frac{n}{e^n} = 0$$

$$\sum (-1)^n \frac{3n-1}{2n+1}$$

$$\lim_{n \to +\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$$

$$\sum (-1)^n \frac{3n-1}{2n+1}$$ diverge

par le test

Série 02 : Les Séries.

Exo 1 :

1) $\sum_{n=1}^{+\infty}\left(\cos(1/n^2) - \cos\left(\frac{1}{(n+1)^2}\right)\right)$

$= \left[\cos(1) - \cancel{\cos\left(\frac{1}{4}\right)}\right] + \left[\cancel{\cos(1/4)} - \cancel{\cos\left(\frac{1}{9}\right)}\right] + \cancel{\dots} + \left[\cancel{\cos(1/n^2)} - \cos\left(\frac{1}{(n+1)^2}\right)\right]$

$\Rightarrow S_n = \cos(1) - \cos\left(\frac{1}{(n+1)^2}\right) \Rightarrow \lim_{n\to+\infty} S_n = \cos(1) - 1$

donc la série converge.

2) $\sum_{n=1}^{+\infty}\left(e^{1/n} - e^{1/(n+1)}\right) = \left(e - \cancel{e^{1/2}}\right) + \left(\cancel{e^{1/2}} - \cancel{e^{1/3}}\right) + \left(\cancel{e^{1/3}} - \cancel{e^{1/4}}\right) + \cancel{\dots} + \left(\cancel{e^{1/n}} - e^{1/(n+1)}\right)$

$\Rightarrow S_n = e - e^{1/(n+1)} \Rightarrow \lim_{n\to+\infty} S_n = e - 1$ donc converge

3) $\sum_{n=2}^{+\infty}\frac{1}{n^3-n}$; $U_n = \frac{1}{n^3-n} = \frac{1}{n(n+1)(n-1)} = \frac{a}{n} + \frac{b}{n-1} + \frac{c}{n+1}$

$= \frac{an^2 + bn^2 + cn^2 - a - cn + bn}{n^3-n} = \frac{(a+b+c)n^2 + (b-c)n - a}{n^3-n}$

$\begin{cases} a+b+c = 0 \\ a = -1 \\ b-c = 0 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = c = 1/2 \end{cases}$

$\frac{1}{n^3-n} = \frac{-1}{n} + \frac{1}{2}\times\frac{1}{n-1} + \frac{1}{2}\times\frac{1}{n+1} = \frac{-1/2 - 1/2}{n} + \frac{1}{2}\times\frac{1}{n-1} + \frac{1}{2}\times\frac{1}{n+1}$

$\Rightarrow \frac{1}{n^3-n} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n}\right)$

$\begin{cases} U_n = U_{n_1} + U_{n_2} \\ U_{n_1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) \\ U_{n_2} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n}\right) \end{cases} \Rightarrow \begin{cases} S_n = U_2 + U_3 + U_4 + \dots + U_n \\ \quad = S_{n_1} + S_{n_2} \\ S_{n_1} = U_{2_1} + U_{3_1} + U_{4_1} + \dots + U_{n_1} \\ S_{n_2} = U_{2_2} + U_{3_2} + U_{4_2} + \dots + U_{n_2} \end{cases}$

$\Rightarrow S_{n_1} = \frac{1}{2}\left(1 - \cancel{1/2}\right) + \frac{1}{2}\left(\cancel{1/2} - \cancel{1/3}\right) + \frac{1}{2}\left(\cancel{1/3} - \cancel{1/4}\right) + \cancel{\dots} + \frac{1}{2}\left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$

$\Rightarrow S_{n_1} = \frac{1}{2}\left(1 - \frac{1}{n}\right) \Rightarrow S_{n_1} = \frac{1}{2} - \frac{1}{2n}$

$\Rightarrow S_{n_2} = \frac{1}{2}\left(\cancel{1/3} - \frac{1}{2}\right) + \frac{1}{2}\left(\cancel{1/4} - \cancel{1/3}\right) + \frac{1}{2}\left(\cancel{1/5} - \cancel{1/4}\right) + \cancel{\dots} + \frac{1}{2}\left(\frac{1}{n+1} - \cancel{\frac{1}{n}}\right)$

$\Rightarrow S_{n_2} = -1/4 - 1/(2n+2)$

$S_n = S_{n_1} + S_{n_2} = \frac{1}{2} - \frac{1}{2n} - \frac{1}{4} \quad \frac{1}{2n+2} \quad \frac{1}{4} - \frac{1}{2}\frac{1}{n} - \frac{1}{2n+2}$

$\Rightarrow \lim_{n \to +\infty} S_n = \frac{1}{4}$ donc la série converge

Exo 2. (série géométrique) $\sum_{n=0}^{+\infty} a r^n$

1) $\sum_{n=1}^{+\infty} 6(0{,}9)^{n-1}$ je fait changement de variable car $n=1$

$= \sum_{m=0}^{+\infty} 6(0{,}9)^m$ $\begin{cases} a = 6 \\ r = 0{,}9 \Rightarrow |r| < 1 \text{ donc converge} \end{cases}$

$\Rightarrow S = \frac{a}{1-r} = \frac{6}{1-0{,}9} \Rightarrow S = 60$

2) $\sum_{n=1}^{+\infty} \frac{(10)^n}{(-9)^{n-1}}$ $m = n-1 \Rightarrow \sum_{m=0}^{+\infty} \frac{(10)^{m+1}}{(-9)^m} = \sum_{m=0}^{+\infty} 10\left(\frac{10}{-9}\right)^m$

$\begin{cases} a = 10 \\ r = \frac{10}{-9} \Rightarrow |r| \geq 1 \end{cases}$

~~diverge~~. converge

3) $\sum_{n=0}^{+\infty} \frac{1}{(\sqrt{2})^n} = \sum_{n=0}^{+\infty} \frac{1}{((2)^{1/2})^n} = 1 \times \left(\frac{1}{\sqrt{2}}\right)^n$ $\begin{cases} a = 1 \\ r = \frac{1}{\sqrt{2}}, |r| < 1 \end{cases}$ converge.

$S = \frac{a}{1-r} = \frac{1}{1-\sqrt{2}} \Rightarrow \boxed{S = -2{,}41}$

4) $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{\pi^n}{3^n \times 3} = \sum_{n=0}^{\infty} \frac{1}{3} \times \left(\frac{\pi}{3}\right)^n$ $\begin{cases} a = 1/3 \\ r = \pi/3, |r| \geq 1 \end{cases}$

donc diverge.

5) $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$, $m = n-1 \Rightarrow \sum_{m=0}^{\infty} \frac{e^{m+1}}{3^m}$

$\Rightarrow \sum_{m=0}^{\infty} \frac{e^m \times e}{3^m} \Rightarrow \sum_{m=0}^{\infty} \left(\frac{e}{3}\right)^m \times e$ $\begin{cases} a = e \\ r = e/3 \to |r| < 1 \end{cases}$

diverge

$S = \frac{e}{1 - e/3} = \frac{e}{3 - e/3} \Rightarrow S = \frac{3e}{3-e}$

Exo 3: 1) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ on $p = \sqrt{2} > 1$ donc converge.

2) $\sum_{n=1}^{\infty} n^{-0{,}999} = \sum_{n=1}^{\infty} \frac{1}{n^{0{,}999}}$ on a: $p = 0{,}999 < 1$ donc diverge.

3) $\sum_{n=1}^{\infty} \cancel{\frac{1}{n^2+n^3}}$ $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$, $f(x) = \frac{\ln x}{x^3} \Rightarrow f'(x) = \frac{x^2 - 3x^2 \ln x}{x^6}$

$\Rightarrow f'(x) = \frac{1 - 3\ln x}{x^4} < 0$

on a f positive, continue sur $[e, +\infty[$ et $f \searrow$ sur $[e, +\infty[$.

$\sum_{n=1}^{+\infty} \frac{\ln n}{n^3}$ et $\int_{e}^{+\infty} \frac{\ln x}{x^3} dx$ sont de même nature.

$$\int_{e}^{+\infty} \frac{\ln x}{x^3} dx = \lim_{A \to +\infty} \int_{e}^{A} \frac{\ln x}{x^3} dx$$

$$U' = \frac{1}{x^3}, \quad U = -\frac{1}{2} \frac{1}{x^2}$$

$$V = \ln x, \quad V' = 1/x$$

$$\int_a^b u'v = [u \times v]_a^b - \int_a^b v'u.$$

$$\Rightarrow \int_{e}^{A} \frac{\ln x}{x^3} dx = \left[\frac{-1}{2x^2} \ln x\right]_e^A - \int_e^A \cancel{\frac{1}{x}} - 1/2 \, 1/x^3 \, dx$$

$$= \lim_{A \to +\infty} \left(-1/2 \underbrace{\frac{\ln A}{A^2}}_{0} - \frac{1}{2e^2} \ln e\right) - \left[\frac{1}{4x^2}\right]_e^A$$

donc $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converge.

3) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$, $f(x) = x^2 e^{-x^3}$, $f'(x) = 2x e^{-x^3} - 3x^4 e^{-x^3}$

$f'(x) = x e^{-x^3} (2 - 3x) < 0$ pour $x \in [1, +\infty[$

donc $\left.\begin{array}{l} f \text{ continue} \\ f \text{ positive} \end{array}\right\}$ sur $[1, +\infty[$ et f décroissante $[1, +\infty[$

$$\int_1^{+\infty} x e^{-x^3} dx = \lim_{A \to +\infty} \int_1^A x^2 e^{-x^3} dx = \lim_{A \to +\infty} \left[-\frac{1}{3} e^{-x^3}\right]_1^A$$

$$= \lim_{A \to +\infty} \left(\underbrace{\frac{-1}{3} e^{-A^3}}_{0} + \frac{1}{3} e^{-1}\right) = 1/3 \, e^{-1}$$

donc $\sum n^2 e^{-n^3}$ converge.

4) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$, $U_n = \frac{1}{n^2 + n^3}$, $V_n = \frac{1}{n^3}$

$\lim_{n \to +\infty} \frac{U_n}{V_n} = \lim_{n \to +\infty} \frac{n^3}{n^2 + n^3} = 1 \neq 0$ donc par le test de comparaison $\sum U_n$, $\sum V_n$ sont les même nature. on $\sum_n \frac{1}{n^3}$ converge car $(P = 3 > 1)$ donc $\sum U_n$ converge.

5) $\sum_{n=1}^{\infty} \frac{1}{n^2 + 6n + 13}$, $U_n = \frac{1}{n^2 + 6n + 13}$, $V_n = \frac{1}{n^2}$

$\Rightarrow \lim_{n \to +\infty} \frac{U_n}{V_n} = \lim_{n \to +\infty} \frac{n^2}{n^2 + 6n + 13} = 1 \neq 0$ sont les même nature on a $\sum V_n = \sum \frac{1}{n^2}$ converge

... ($P = 2 > 1$) donc $\sum U_n$ converge.

Ex 06 :

① $\sum \frac{n^{1/3}}{\sqrt{n^3+4n+3}}$, $U_n = \frac{n^{1/3}}{\sqrt{n^3+4n+3}}$, $\vartheta_n = \frac{n^{1/3}}{\sqrt{n^3}}$.

$\lim\limits_{n\to+\infty} \frac{U_n}{\vartheta_n} = \lim\limits_{n\to+\infty} \frac{n^{1/3} \times n^{3/6}}{\sqrt{n^3+4n+3}} = \lim\limits_{n\to+\infty} \frac{n^{3/6}}{n^{3/2}} = 1 \neq 0$ donc $\sum U_n$, $\sum \vartheta_n$ converge sont même nature.

$\sum \frac{1}{n^{7/6}}$ converge donc $\sum U_n$ converge.

② $\sum \frac{1}{\sqrt[3]{4n^3+1}}$, $U_n = \frac{1}{\sqrt[3]{4n^3+1}}$, $\vartheta_n = \frac{1}{\sqrt[3]{4n^3}} = \frac{1^{1/3}}{4^{1/3}n}$

$\Rightarrow \lim\limits_{n\to+\infty} \frac{U_n}{\vartheta_n} = \lim\limits_{n\to+\infty} \frac{n}{\sqrt[3]{4n^3+1}} = \lim\limits_{n\to+\infty} \frac{\cancel{n}}{4^{1/3}\cancel{n}} = \lim\limits_{n\to+\infty} \frac{1}{4^{1/3}} = \frac{1}{2^{2/3}} \neq 0$

donc : $\sum U_n$ et $\sum \vartheta_n$ sont de même nature. $\sum \vartheta_n = \frac{1}{4^{1/3}}$ diverge donc $\sum \frac{1}{\sqrt[3]{4n^3+1}}$ diverge.

③ $\sum \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$, $U_n = \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$, $\vartheta_n = \frac{n \times n^2}{n \times n^4} = \frac{1}{n^2}$

$\lim\limits_{n\to+\infty} \frac{U_n}{\vartheta_n} = \lim\limits_{n\to+\infty} \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2} \times n^2 = \lim\limits_{n\to+\infty} \frac{2n^5}{n^5} = 2 \neq 0$

donc $\sum U_n$, $\sum \vartheta_n$ sont même nature $\Rightarrow \sum \vartheta_n = \frac{1}{n^2}$, $P = 2 > 0$ converge

donc $\sum U_n$ converge.

④ $\sum\limits_{n=1}^{\infty} \frac{\arctan n}{n^{1,2}}$, $\frac{\operatorname{artg} n}{n^{1,2}} < \frac{\pi/2}{n^{1,2}}$, $\sum \frac{\pi/2}{n^{1,2}}$ converge donc $\sum U_n =$

$\sum \frac{\operatorname{artg} n}{n^{1,2}}$ converge.

④ $\sum \frac{n \sin^2 n}{1+n^3}$, $U_n = \frac{n \sin^2 n}{1+n^3} \leq \frac{n \times 1}{1+n^3} = \frac{1}{n^2}$

$= n \sin^2(n) \times \left(\frac{1}{1+n^3}\right) \leq 1 \times \frac{1}{n^3} \times n$

$\sin^2 n \leq 1$

$n^3 + 1 \geq n^3 \Rightarrow \frac{1}{n^3+1} \leq \frac{1}{n^3}$

$\sum \frac{1}{n^2}$ converge donc $\sum \frac{n \sin^2 n}{1+n^3}$ converge.

⑥ $\sum_{n=1} \left(1+\frac{1}{n^2}\right)^2 e^{-n}$, $U_n = \left(1+\frac{1}{n}\right)^2 e^{-n}$

$$\left(1+\frac{1}{n}\right)^2 \leq (1+1)^2 = 2^2 = 4$$

$$U_n \leq 4e^{-n} \leq \frac{4}{n^2}$$

$\Rightarrow \sum \frac{4}{n^2}$ converge donc $\sum U_n$ converge.

Ex08:

1) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$, $U_n = \frac{1}{2n+1}$

• $\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} \frac{1}{2n+1} = \lim_{n\to+\infty} \frac{1}{2^n} = 0$ -- (1)

• $f(x) = \frac{1}{2x+1}$, $f'(x) = \frac{-2}{2x+1} < 0$ elle decroissante -- (2)

de ① et ② la série est convergente.

2) $\sum \frac{(-1)^{n-1}}{\ln(n+4)}$, $U_n = \frac{1}{\ln(n+4)}$

•) $\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} \frac{1}{\ln(n+4)} = 0$ --- (1)

•) $f(x) = \frac{1}{\ln(x+4)} \Rightarrow^{+\infty}$ $f'(x) = \frac{-1}{x+4(\ln(x+4))^2} < 0$ décroissante -- ②

(1) et ② est convergente.

4) $\sum (-1)^{n+1} n e^{-n}$, $f(x) = xe^{-x} \Rightarrow f'(x) = e^{-x} - xe^{-x} \Rightarrow f'(x) = e^{-x}(1-x) < 0$
décroissante

$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} (n/e^n) = 0$ converge

$\Rightarrow \sum (-1)^n$