Recurrences

Recurrence Equation

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le c \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

- Technicalities
  1. Ignoring the assumption of integer arguments to function
  2. Ignoring boundary conditions
- Merge-Sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n > 1 \end{cases}$$

➢ After Ignoring 1& 2

$$T(n) = 2T(n/2) + \Theta(n)$$

Solving Recurrences

1. Substitution Method
2. Recursion-tree Method
3. Master Method

1. Substitution Method

- Guess the solution
- ✓ Use Mathematical induction to show that the solution works.

Examples:

1. $T(n) = 2T(\lfloor n/2 \rfloor) + n$
2. $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$

2. The Recursion-Tree Method

- Each node represents the cost of a single sub-problem
- ✓ Summing the cost within each level of the tree give a per-level costs
- ✓ Summing the pre-level cost gives the total costs of the recurrence.

Examples:

1. $T(n) = 3T(n/4)+cn^2$
2. $T(n) = T(n/3)+T(2n/3)+O(n)$

3. Master Method

Solve the recurrence of the form:

$T(n) = a\,T(n/b) + f(n)$,

Where: $a \geq 1$, $b > 1$, $f(n)$ asymptotically $+ve$ function.

1. If $f(n) = O(n^{\log_b a - \varepsilon})$, ε > 0, then: $T(n) = \Theta(n^{\log_b a})$

2. If $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$, ε > 0, and if

   $af(n/b) \leq cf(n)$ for $c < 1$, then: $T(n) = \Theta(f(n))$

Case 1:

If $f(n) = O(n^{\log_b a - \varepsilon})$, ε > 0, then: $T(n) = \Theta(n^{\log_b a})$

If $n^{\log_b a} / f(n) = n^{\varepsilon}$ then: $T(n) = \Theta(n^{\log_b a})$

If $n^{\log_b a} > f(n)$ and $n^{\log_b a} / f(n) \neq n^{\varepsilon}$

then: $T(n) = !!!$

Case 2:

If $f(n) = \Theta(n^{\log_b a})$,then: $T(n) = \Theta(n^{\log_b a} \lg n)$

If $n^{\log_b a} = f(n)$ then: $T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3:

If $f(n) = \Omega(n^{\log_b a+\varepsilon})$, $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for $c < 1$, then: $T(n) = \Theta(f(n))$

If $f(n) / n^{\log_b a} = n^{\varepsilon}$ and $af(n/b) \leq cf(n)$

then: $T(n) = \Theta(f(n))$

If $f(n) > n^{\log_b a}$ and $f(n) / n^{\log_b a} \neq n^{\varepsilon}$

then: $T(n) =!!!$

Examples:

1. $T(n) = 9T(n/3)+n$
2. $T(n) = T(2n/3)+1$
3. $T(n) = 4T(n/2)+n^3$
4. $T(n) = 2T(n/2)+n \lg n$