Appendix A

MATHEMATICAL METHODS

1. Sums of Powers of Integers
2. Logarithms
3. Permutations, Combinations, Factorials
4. Fibonacci Numbers
5. Catalan Numbers

Sums of Powers of Integers

Theorem A.1

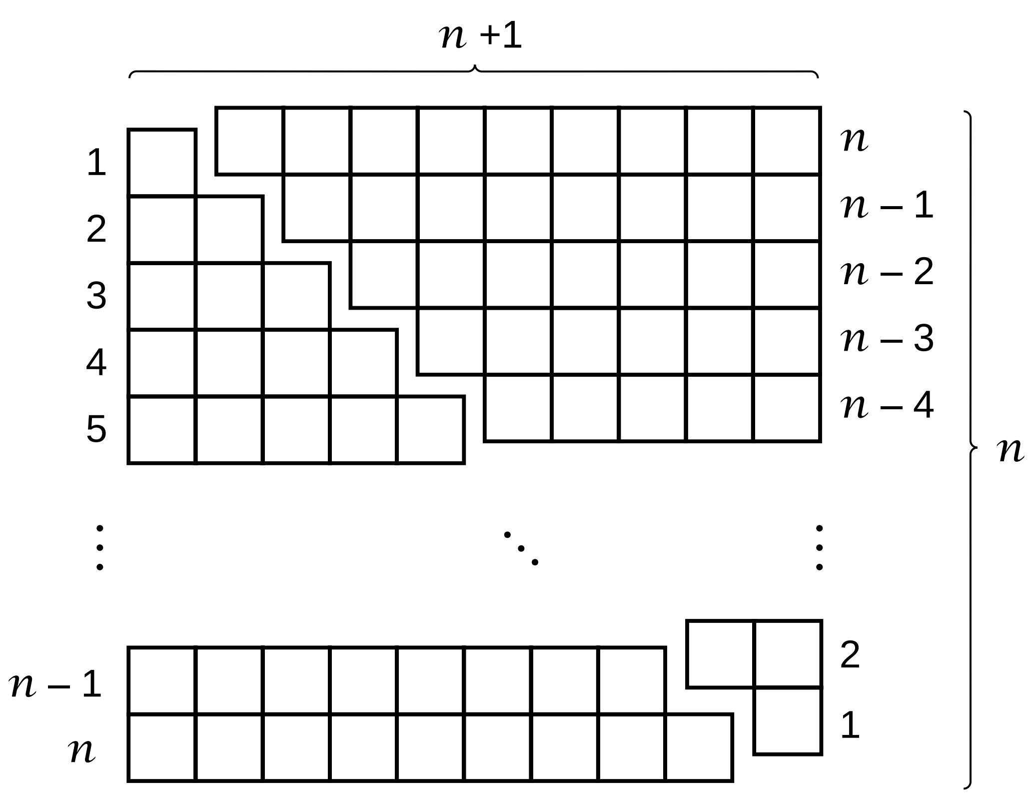
$$1 + 2 + \cdots + n = n(n+1)/2.$$

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6.$$

Proof:

$$\begin{array}{rcrcrcccrcrcr}
1 & + & 2 & + & 3 & + & \cdots & + & n-1 & + & n & = & S \\
n & + & n-1 & + & n-2 & + & \cdots & + & 2 & + & 1 & = & S \\
\hline
n+1 & + & n+1 & + & n+1 & + & \cdots & + & n+1 & + & n+1 & = & 2S
\end{array}$$

There are n columns on the left; hence $n(n+1) = 2S$ and the first formula follows.



Other Sums

THEOREM A.2

$$1 + 2 + 4 + \cdots + 2^{m-1} = 2^m - 1.$$

$$1 \times 1 + 2 \times 2 + 3 \times 4 + \cdots + m \times 2^{m-1} = (m-1) \times 2^m + 1.$$

In summation notation these equations are

$$\sum_{k=0}^{m-1} 2^k = 2^m - 1.$$

$$\sum_{k=1}^{m} k \times 2^{k-1} = (m-1) \times 2^m + 1.$$

THEOREM A.3 If $|x| < 1$ then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{and} \qquad \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.$$

Logarithms

Logarithms are defined in terms of a real number $a > 1$, which is called the ***base*** of the logarithms. For any number $x > 0$, we define $\log_a x = y$, where y is the real number such that $a^y = x$. The logarithm of a negative number, and the logarithm of 0, are not defined.

$$\begin{aligned}
\log_a 1 &= 0, \\
\log_a a &= 1, \\
\log_a x &< 0 \quad \text{for all } x \text{ such that } 0 < x < 1. \\
0 < \log_a x &< 1 \quad \text{for all } x \text{ such that } 1 < x < a. \\
\log_a x &> 1 \quad \text{for all } x \text{ such that } a < x. \\
\log_a(xy) &= (\log_a x) + (\log_a y) \\
\log_a(x/y) &= (\log_a x) - (\log_a y) \\
\log_a x^z &= z \log_a x \\
\log_a a^z &= z \\
a^{\log_a x} &= x,
\end{aligned}$$

where x, y, and z are real numbers, $x > 0$ and $y > 0$.

As x grows large,
$\log x$ grows more slowly than x^c, for any $c > 0$.

The Base for Logarithms

- Base 10 gives ***common logarithms***; often used for hand computation and for expressing very large or small numbers.
- Base $e = 2.7182818284 59\ldots$ gives ***natural logarithms***; it appears often in the mathematical analysis of algorithms; denoted $\ln x$.
- Base 2 is the most common for computer applications; denoted $\lg x$.

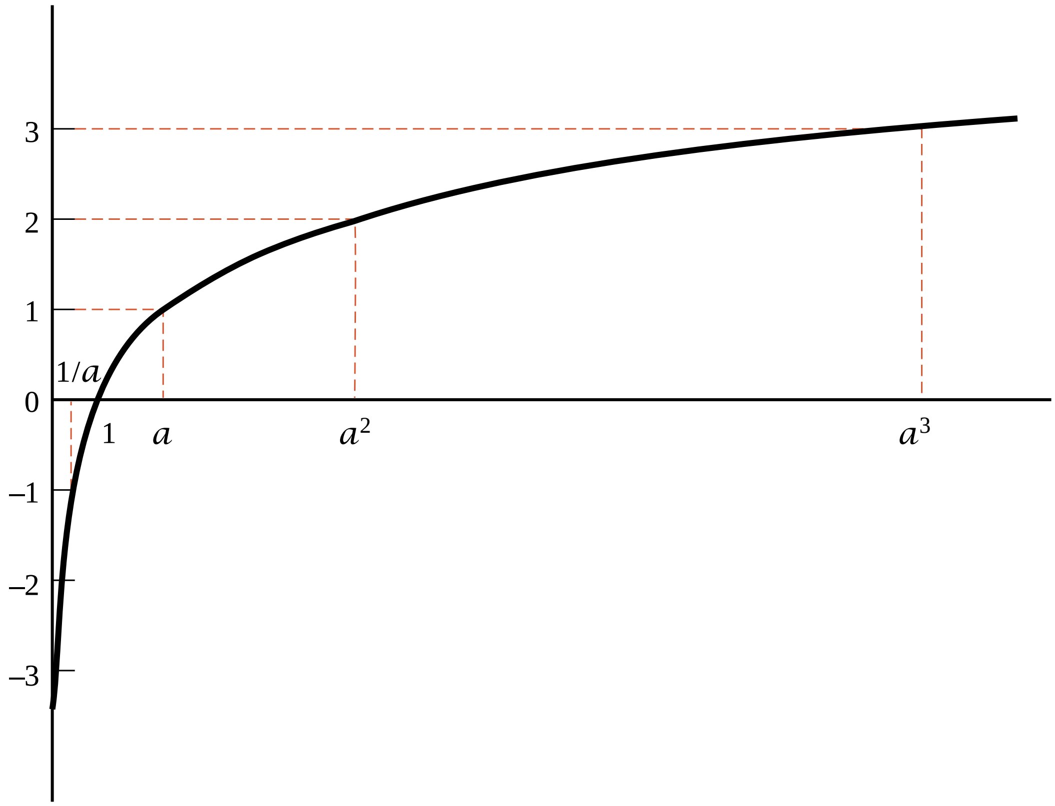
Convention

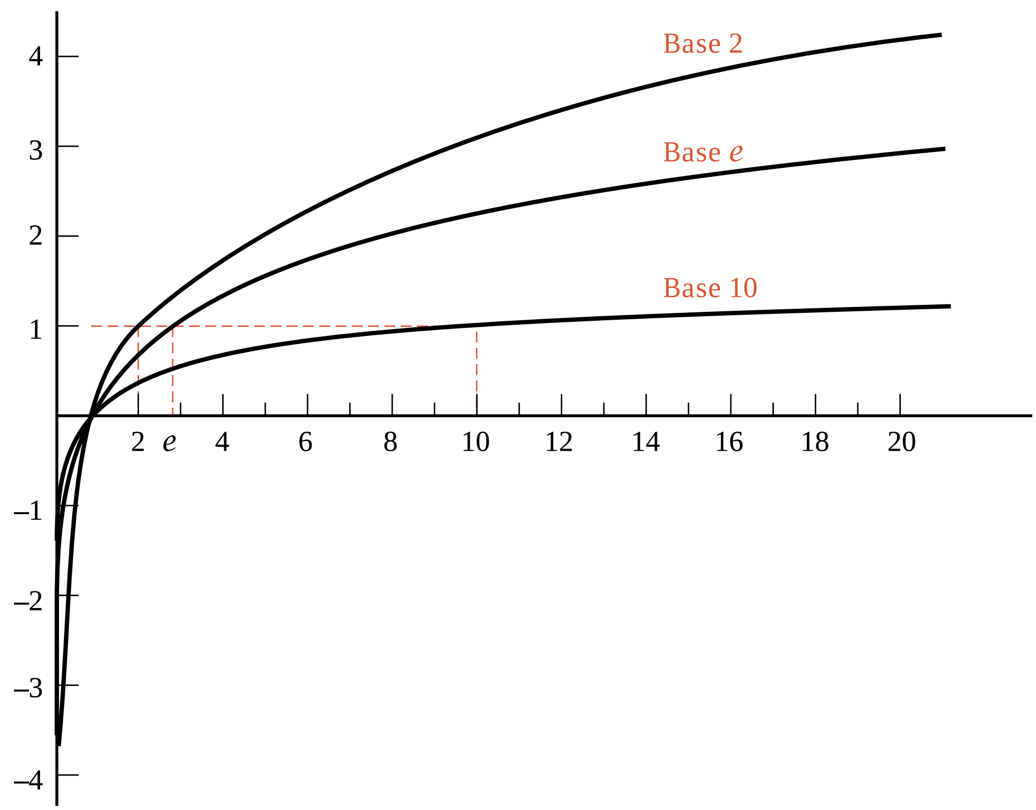
Unless stated otherwise, all logarithms will be taken with base 2.
The symbol $\lg$ denotes a logarithm with base 2,
and the symbol $\ln$ denotes a natural logarithm.
If the base for logarithms is not specified or makes no difference,
then the symbol $\log$ will be used.

To convert logarithms from one base to another, multiply by a constant factor, the logarithm of the first base with respect to the second.

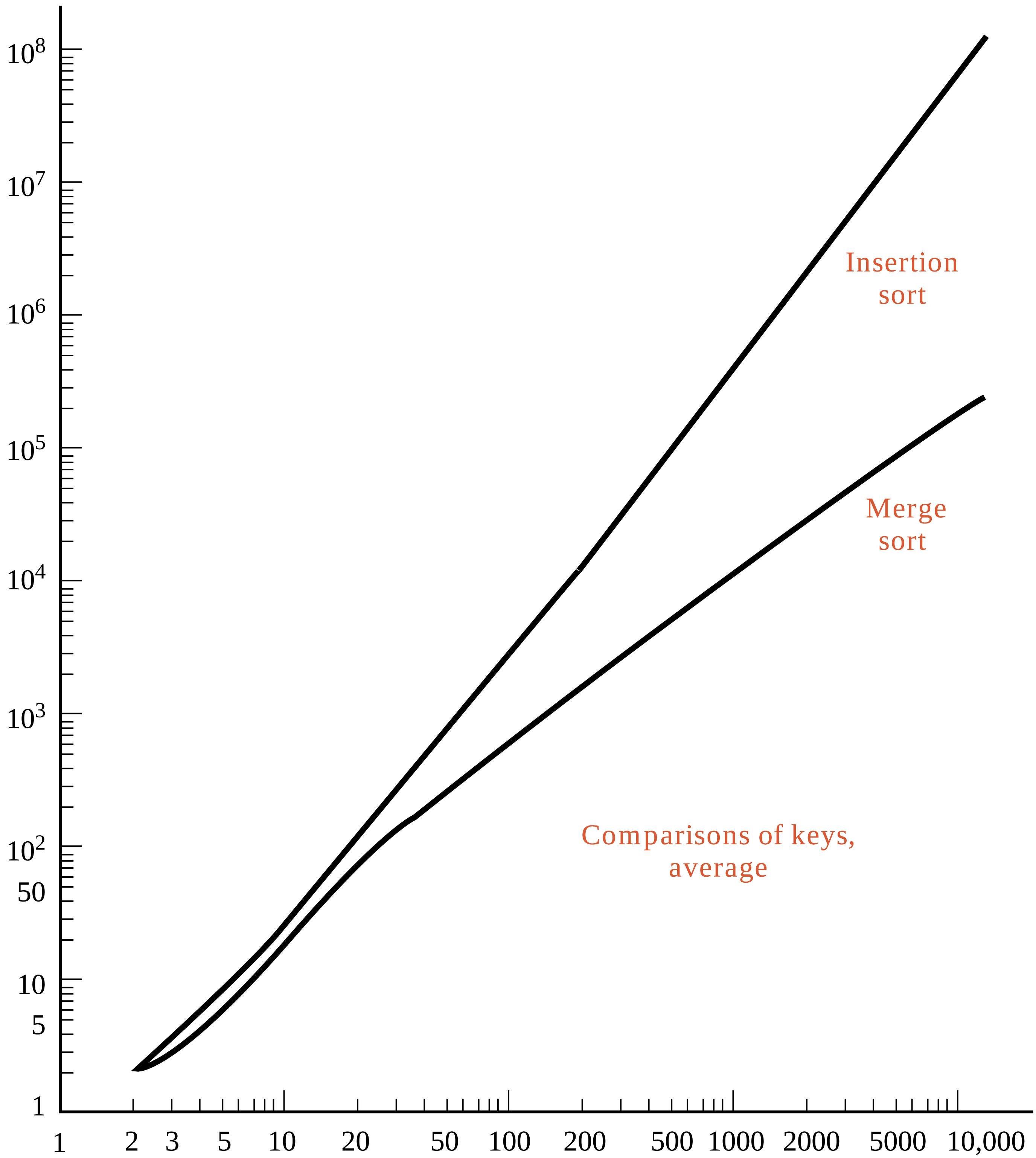
$$\begin{aligned} \lg e &\approx 1.442695041, \\ \ln 2 &\approx 0.693147181, \\ \ln 10 &\approx 2.302585093. \end{aligned}$$

Graphs of Logarithm Functions





Log-Log Graph, Insertion and Merge Sorts



Harmonic Numbers

The n^{th} ***harmonic number*** is defined to be the sum

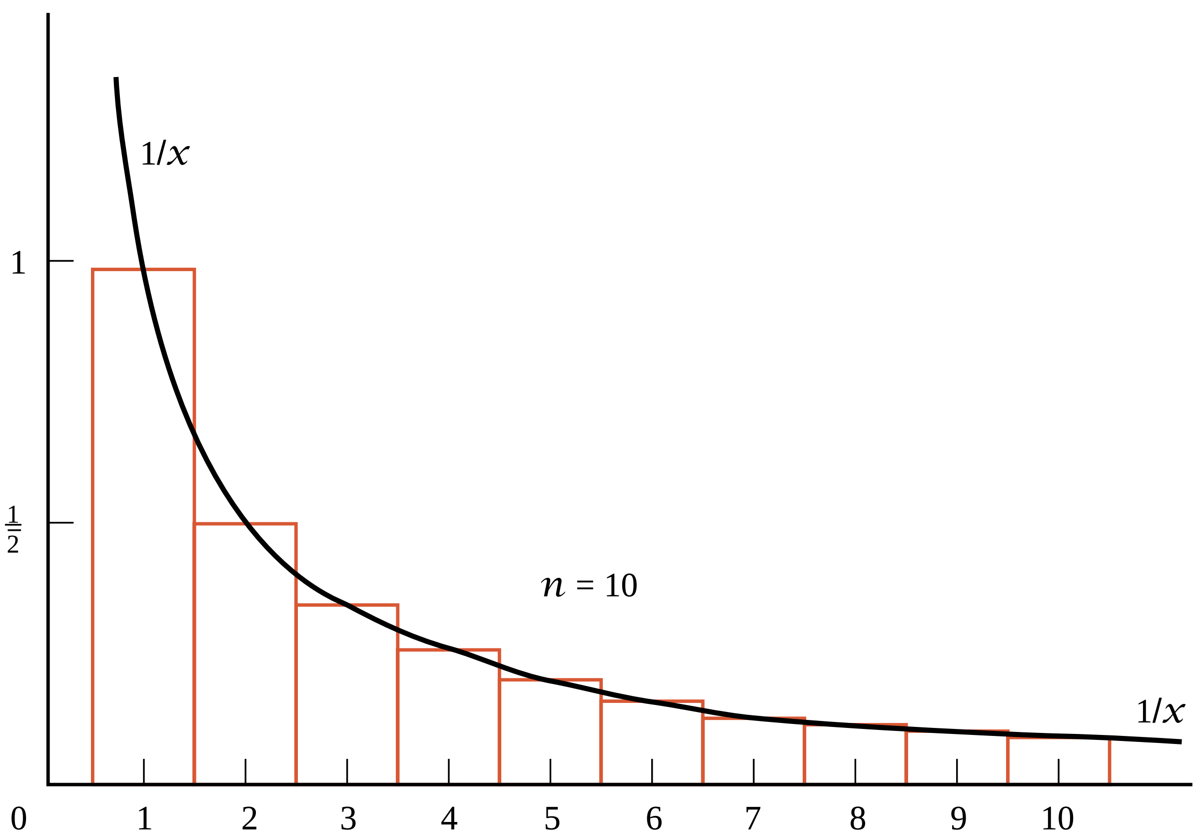
$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

of the reciprocals of the integers from 1 to n.

THEOREM A.4 The harmonic number H_n, $n \geq 1$, satisfies

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \epsilon,$$

where $0 < \epsilon < 1/(252n^6)$, and $\gamma \approx 0.577215665$ is known as ***Euler's constant***.



Permutations and Combinations

A ***permutation*** of objects is an ordering or arrangement of the objects in a row.

Objects to permute: a b c d

Choose a *first*:	a b c d	a b d c	a c b d	a c d b	a d b c	a d c b
Choose b *first*:	b a c d	b a d c	b c a d	b c d a	b d a c	b d c a
Choose c *first*:	c a b d	c a d b	c b a d	c b d a	c d a b	c d b a
Choose d *first*:	d a b c	d a c b	d b a c	d b c a	d c a b	d c b a

A ***combination*** of n objects taken k at a time is a choice of k objects out of the n, without regard for the order of selection. The number of such combinations is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Objects from which to choose: a b c d e f

a b c	a c d	a d f	b c f	c d e
a b d	a c e	a e f	b d e	c d f
a b e	a c f	b c d	b d f	c e f
a b f	a d e	b c e	b e f	d e f

The number of combinations $C(n,k)$ is called a ***binomial coefficient***, since it appears as the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.

Stirling's Approximation to Factorials

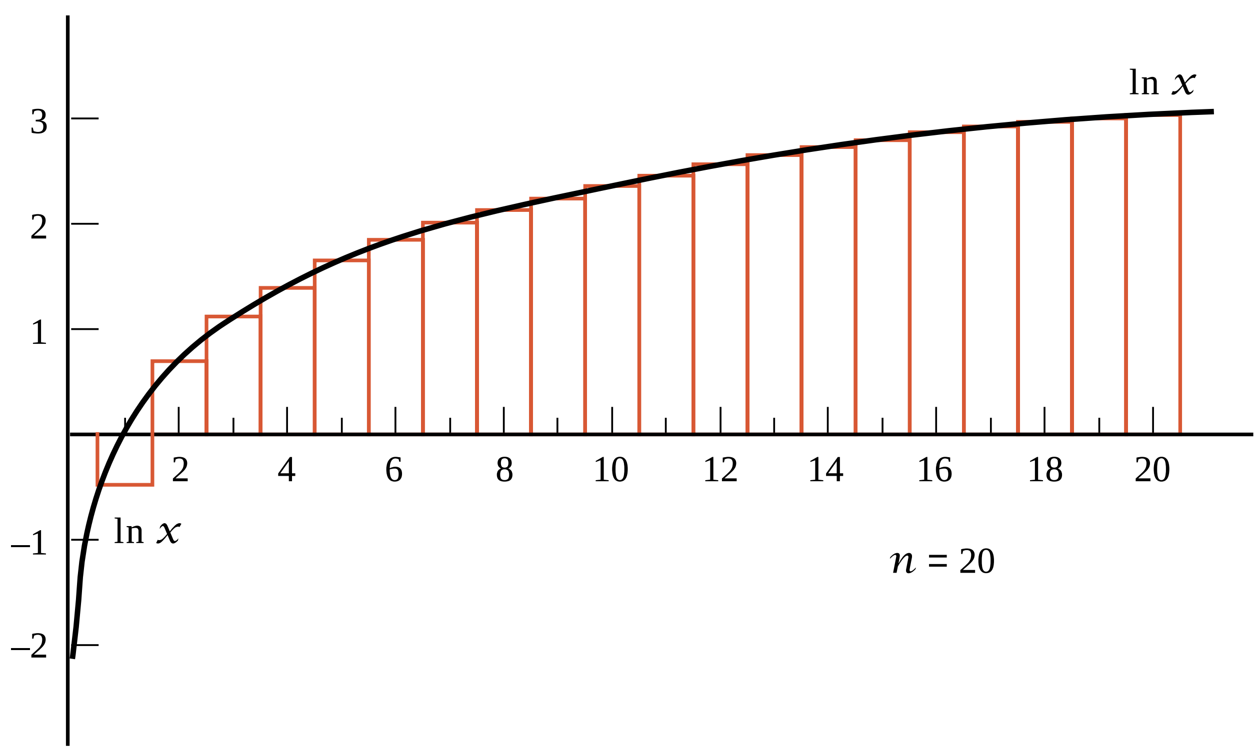
The following results give good approximations to the ***factorial*** of a nonnegative integer, $n! = n \times (n-1) \times \cdots \times 1$.

Theorem A.5

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right].$$

Corollary A.6

$$\ln n! = (n + \tfrac{1}{2})\ln n - n + \tfrac{1}{2}\ln(2\pi) + \frac{1}{12n} + O\left(\frac{1}{n^2}\right).$$



Fibonacci Numbers

The Fibonacci numbers originated as an exercise in arithmetic proposed by LEONARDO FIBONACCI in 1202:

> How many pairs of rabbits can be produced from a single pair in a year? We start with a single newly born pair; it takes one month for a pair to mature, after which they produce a new pair each month, and the rabbits never die.

The ***Fibonacci numbers*** are formally defined by the *recurrence relation*,

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_n - 1 + F_{n-2} \quad \text{for} \quad n \geq 2.$$

By the method of ***generating functions***, the Fibonacci numbers are evaluated as

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n),$$

where

$$\phi = \tfrac{1}{2}(1 + \sqrt{5}) \text{ and } \psi = 1 - \phi = \tfrac{1}{2}(1 - \sqrt{5}).$$

Approximate values for ϕ and ψ are

$$\phi \approx 1.618034 \qquad \text{and} \qquad \psi \approx -0.618034.$$

The absolute value of ψ is sufficiently small that F_n is always $\phi^n/\sqrt{5}$ rounded to the nearest integer.

Catalan Numbers

DEFINITION For $n \geq 0$, the n^{th} ***Catalan number*** is defined to be

$$Cat(n) = \frac{C(2n, n)}{n+1} = \frac{(2n)!}{(n+1)!n!}.$$

THEOREM A.7 The number of distinct binary trees with n vertices, $n \geq 0$, is the n^{th} Catalan number $\mathrm{Cat}(n)$.

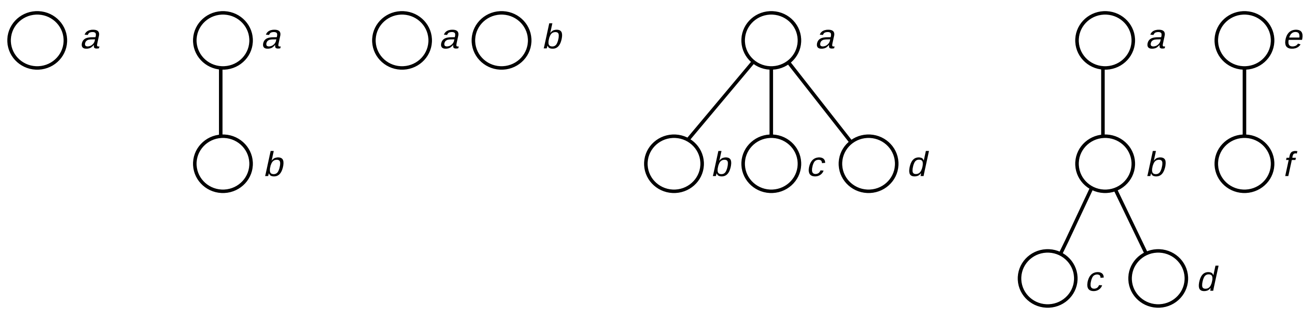
LEMMA A.8 There is a one-to-one correspondence between the orchards with n vertices and the well-formed sequences of n left parentheses and n right parentheses, $n \geq 0$.

LEMMA A.9 The sequences of n left and n right parentheses that are not well formed correspond exactly to all sequences of $n-1$ left parentheses and $n+1$ right parentheses (in all possible orders).

COROLLARY A.10 The number of well-formed sequences of n left and n right parentheses, the number of permutations of n objects obtainable by a stack, the number of orchards with n vertices, and the number of binary trees with n vertices are all equal to the n^{th} Catalan number $Cat(n)$.

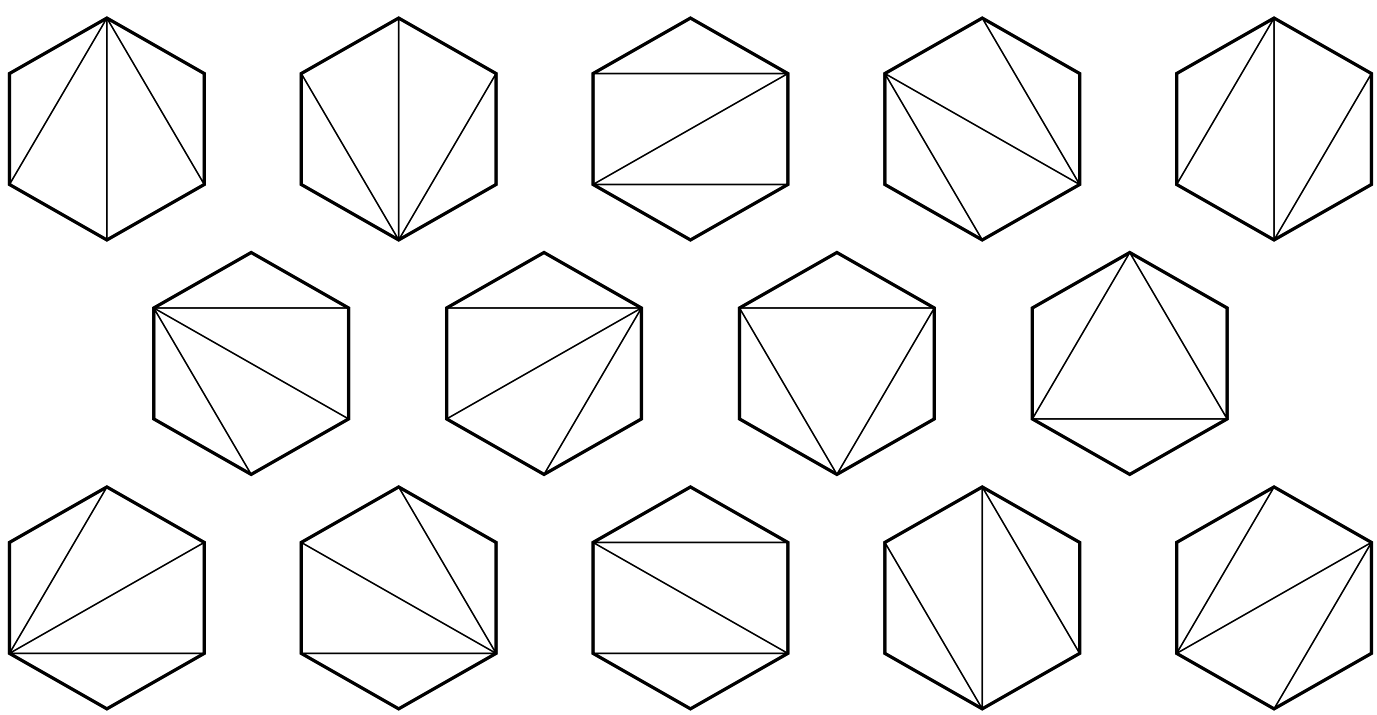
Problems Related to Catalan Numbers

Bracketed form of orchards:



(a) $(a(b))$ $(a)(b)$ $(a(b)(c)(d))$ $(a(b(c)(d)))(e(f))$

Triangulations of a hexagon:



Numerical Results

Approximation to Catalan numbers:

Stirling's approximation gives:

$$\mathrm{Cat}(n) \approx \frac{4^n}{(n+1)\sqrt{\pi n}}$$

The first twenty Catalan numbers:

n	$\mathrm{Cat}(n)$	n	$\mathrm{Cat}(n)$
0	1	10	16,796
1	1	11	58,786
2	2	12	208,012
3	5	13	742,900
4	14	14	2,674,440
5	42	15	9,694,845
6	132	16	35,357,670
7	429	17	129,644,790
8	1,430	18	477,638,700
9	4,862	19	1,767,263,190