Artificial Intelligence MSc

– Exam Answers –

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Answers 1: Propositional Logic

(a)

Truth tables have been practised in the tutorial.

(i) Below are the truth tables for the three formulas

1. $p \vee \neg p$

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

(1 Point)

2. $(p \rightarrow \neg p) \leftrightarrow (\neg p \rightarrow p)$

p	$\neg p$	$p \rightarrow \neg p$	$\neg p \rightarrow p$	$(p \rightarrow \neg p) \leftrightarrow (\neg p \rightarrow p)$
F	T	T	F	F
T	F	F	T	F

(1 Point)

3. $(p \rightarrow \neg q) \leftrightarrow (\neg q \rightarrow p)$

p	q	$\neg q$	$p \rightarrow \neg q$	$\neg q \rightarrow p$	$(p \rightarrow \neg q) \leftrightarrow (\neg q \rightarrow p)$
F	F	T	T	F	F
F	T	F	T	T	T
T	F	T	T	T	T
T	T	F	F	T	F

(2 Points)

(ii) Based on the truth tables, we find that the formulas have the following properties

1. $p \vee \neg q$: satisfiable, not falsifiable, not unsatisfiable, valid

(1 Point)

2. $(p \rightarrow \neg p) \leftrightarrow (\neg p \rightarrow p)$: not satisfiable, falsifiable, unsatisfiable, not valid

(1 Point)

3. $(p \rightarrow \neg q) \leftrightarrow (\neg q \rightarrow p)$: satisfiable, falsifiable, not unsatisfiable, not valid

(1 Point)

(b)

This problem and other similar ones have been discussed in the lectures and the tutorial.

(i) If $F \vee G$ is satisfiable, then one of F and G is satisfiable.

(2 Points)

(ii) If $F \vee G$ is unsatisfiable, then both F and G are unsatisfiable.

(2 Points)

(iii) If $F \vee G$ is falsifiable, then both F and G are falsifiable.

(2 Points)

(iv) $F \vee G$ is valid, then either one of F and G is valid, or both are satisfiable.

(2 Points)

(c)

This is a type of problem that has been practised in the tutorial. The problem itself is new to the students.

(i) We let the proposition `d` stand for "david is elected president", `e` for "eric is elected vice-president", `f` for "fred is elected treasurer".

Then we can write the three sentences as follows:

S_1: $\mathtt{d} \rightarrow \neg \mathtt{e} \vee \neg \mathtt{f}$

(1 Point)

S_2: $\mathtt{d} \wedge \mathtt{f}$

(1 Point)

C: $\neg \mathtt{e}$

(1 Point)

(ii) One can check either one of the following

- $(S_1 \wedge S_2) \rightarrow C$ is valid
- $\neg(S_1 \wedge S_2) \wedge \neg C$ is unsatisfiable

(2 Points)

(iii) We check whether $F = (S_1 \wedge S_2) \rightarrow C$ is valid. To this end, we set up a truth table for

$$F = (\mathtt{d} \rightarrow \neg\mathtt{e} \vee \neg\mathtt{f}) \wedge (\mathtt{d} \wedge \mathtt{f}) \rightarrow (\neg\mathtt{e})$$

d	e	f	¬e	¬f	¬e ∨ ¬f	d → ¬e ∨ ¬f	d ∧ f	(d → ¬e ∨ ¬f) ∧ (d ∧ f)	F
F	F	F	T	T	T	T	F	F	T
F	F	T	T	F	T	T	F	F	T
F	T	F	F	T	T	T	F	F	T
F	T	T	F	F	F	T	F	F	T
T	F	F	T	T	T	T	F	F	T
T	F	T	T	F	T	T	T	T	T
T	T	F	F	T	T	T	F	F	T
T	T	T	F	F	F	F	T	F	T

All entries in the column F are T. Therefore, F is valid and C follows from S_1 and S_2.

1 point for the correct approach
1 point for the correct layout of the truth table
3 points for the correct values in the table

(5 Points)

Answers 2: Resolution

(a)

The computation of conjunctive normal forms, including the underlying theory, has been discussed thoroughly in the lectures. We have also briefly discussed in which way disjunctive normal form is analogous to conjunctive normal form.

(i) The method is to check whether the disjunction contains two complementary literals, that is, literals of the form p and $\neg p$.

Clearly, $p \vee \neg p$ is true under every interpretation. If there are no two such literals, then one can find an interpretation that makes every literal false and thus makes the formula false.

1 point for the method
1 point for the explanation

(2 Points)

(ii) The method is to check whether the conjunction contains two complementary literals, that is, literals of the form p and $\neg p$.

Clearly, $p \wedge \neg p$ is false under every interpretation. If there are no two such literals, then one can find an interpretation that makes every literal true and thus makes the formula true.

1 point for the method
1 point for the explanation

(2 Points)

(iii) The method is to check whether each disjunct, which is a conjunction of literals, contains two complementary literals.

As seen in the previous question, a disjunction is unsatisfiable if all of the disjuncts are unsatisfiable.

As seen in part (ii) of this question, a conjunction of literals is unsatisfiable if it contains two complementary literals.

1 point for the method
2 points for the explanation

(3 Points)

(b)

Resolution, including the soundness of the rule, has been discussed in the lectures.

(i) The resolution rule allows one to derive from two given clauses a new clause, which is called the resolvent. It is applicable if the two given clauses contain two complementary literals.

Two literals are complementary if one has the form p and the other one has the form $\neg p$. If the resolution rule is applicable, then one clause has the form $C \vee p$ and the other clause has the form $D \vee \neg p$. The resolvent is $C \vee D$.

1 point for the correct definition of the applicability
2 points for the correct definition of the application

(3 Points)

(ii) The resolution rule is sound. This means that if $\mathcal{I}$ is an interpretation that satisfies both $C \vee p$ and $D \vee \neg p$, then $\mathcal{I}$ satisfies also the resolvent $C \vee \mathcal{D}$.

Suppose that $\mathcal{I}$ satisfies both $C \vee p$ and $D \vee \neg p$. Clearly, $\mathcal{I}$ satisfies either p or $\neg p$. If $\mathcal{I}$ satisfies p, then it cannot satisfy $\neg p$ and therefore must satisfy D. Hence, $\mathcal{I}$ satisfies the resolvent $C \vee D$. If $\mathcal{I}$ satisfies $\neg p$, then it cannot satisfy p and therefore must satisfy C. Hence again, $\mathcal{I}$ satisfies $C \vee D$.

(3 Points)

1 point for the correct answer
2 points for the correct argument

(c)

Background: We have discussed in the lectures how to formalise Knights and Knaves puzzles. As part of their coursework, each student had to formalise three puzzles, generated automatically by a puzzle generator on the Web, and solve it using a propositional logic applet available on the Web.

(i) *The students are familiar with the following approach to formalising this kind of puzzle:*

- *Each character x makes a statement S_x.*
- *Let the atom x stand for "x is a knight". If x is a knight, then the statement is true. If x is not a knight, then the statement is false. This leads to a formula $x \leftrightarrow S_x$ for each character.*

We use two atoms, b and p, standing for "Bob is a knight" and "Peggy is a knight," respectively.

The statement of Bob can be formalised as "$\neg p$". The statement of Peggy can be formalised as "$p \vee b$".

The whole puzzle can be formalised as

$$(b \leftrightarrow \neg p) \wedge (p \leftrightarrow p \vee b)$$

1 point for the correct approach
1 point for each equivalence

(3 Points)

(ii) *In the lecture and in the tutorials, we have discussed and practised how to transform formulas into conjunctive normal form.*

1. First, we replace equivalences by implications. This leads to the formula

$$(b \to \neg p) \land (\neg p \to b) \land (p \to p \lor b) \land (p \lor b \to p)$$

2. Next, we eliminate implications. This leads to the formula

$$(\neg b \lor \neg p) \land (\neg\neg p \lor b) \land (\neg p \lor p \lor b) \land (\neg(p \lor b) \lor p)$$

3. Then we push negations. This leads to the formula

$$(\neg b \lor \neg p) \land (p \lor b) \land (\neg p \lor p \lor b) \land ((\neg p \land \neg b) \lor p)$$

4. Then we use the distributive law to push disjunctions. This leads to the formula

$$(\neg b \lor \neg p) \land (p \lor b) \land (\neg p \lor p \lor b) \land (\neg p \lor p) \land (\neg b \lor p)$$

5. Finally, we replace complementary literals by *true* and simplify. This leads to the formula

$$(\neg b \lor \neg p) \land (p \lor b) \land (\neg b \lor p)$$

1 point for each transformation step

(5 Points)

(iii) We start with the three clauses in the CNF formula:

1. $\neg b \lor \neg p$
2. $p \lor b$
3. $\neg b \lor p$.

From these clauses we derive new clauses:

4. $\neg b$ from 1. and 3.
5. p from 2. and 4.

Since resolution is a sound inference rule, the derived clauses are logical consequences of the puzzle formula.

We have derived the formulas $\neg b$ and p. According to our original encoding, this means that Bob is a knave and Peggy is a knight.

1 point for translating the CNF into three clauses
1 point for each resolution step
1 point fo the correct interpretation of the result

(4 Points)