Analysis I, 1st midterm test, Nov 4, 2022

Problem 1. [20%] Let us define $H := \left\{ \sqrt{n^2+2} - \sqrt{n^2+1} : n \in \mathbb{N}^+ \right\}$. Find max H, min H, sup H and inf H, and prove your results. (The concept of the limit in this exercise *cannot* be used.)

Problem 2. [20%] Let us define a function $f(x) := x^2 + 1$ for $x < 0$. Determine its inverse together with its domain of definition.

Problem 3. [20%] Let us define $a_n := \frac{n^2+n+1}{n^3+n+1}$. First conjecture $\lim_{n\to\infty} a_n$. Then prove this by using only the *definition* of the limit ($\forall\, \epsilon > 0\ \exists \ldots$).

Problem 4. [10+10%] Evaluate the following limits:

a) $\lim_{n\to\infty} \left(\frac{2n+3}{2n+4}\right)^{4n+5}$

a) $\lim_{n\to\infty} \frac{\sqrt{2n^3+2} - \sqrt{2n^2+n+1}}{n^{3/2}}$

Problem 5. [20%] Let us define a sequence a_n (for $n = 0, 1, 2, \ldots$) as follows: $a_{n+1} := {a_n}^2 + \frac{1}{4}$, $a_0 = 0$. Prove that a_n is convergent and find its limit. (Hint: first conjecture what the limit is, then apply a monotonicity and boundedness argument.)