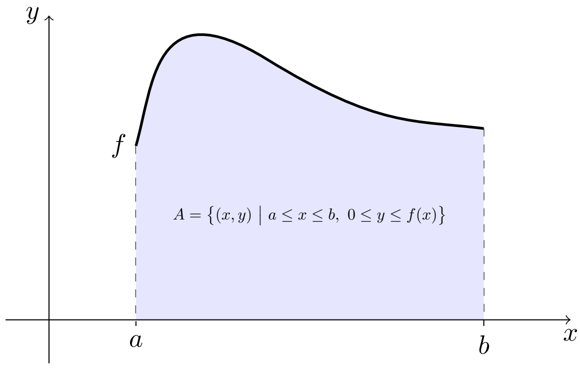
INDEFINITE INTEGRALS

Motivation and basic concepts

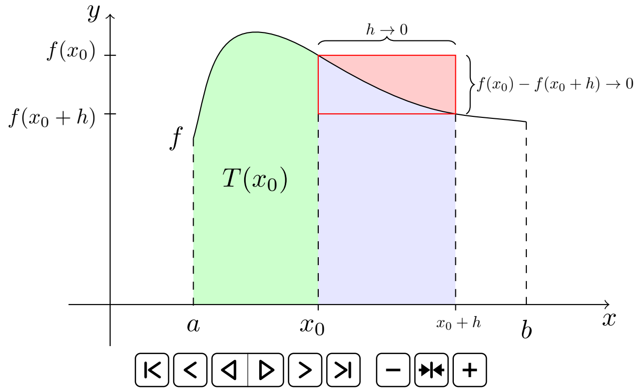
Theoretical notes In the following, we will illustrate the importance of antiderivation to calculate areas. Note that here we use an intuitive approach to area, such as the one commonly used in everyday life, which is sufficient for our illustration.

Let f be a continuous function that takes positive values on a given closed interval $[a, b]$. We aim to determine the area under the curve of the function shown in the figure, that is, the area of the bounded region A enclosed by the graph of the function, the lines $x = a$ and $x = b$, and the x-axis.



To solve this problem, let us introduce another function! For each $x \in [a, b]$, we assign the area under the curve of f from a to x, which we denote by $T(x)$. We call this function T the ***area-measuring function*** of f starting from the point a. By knowing the area-measuring function, we can easily determine the area of the desired region, since it is sufficient to evaluate the area-measuring function at the point $x = b$. Therefore, from now on, our focus will be on determining the area-measuring function.

Let us fix a value $x_0 \in (a, b)$. In the next figure, the area of the region marked in green is precisely the value of the area-measuring function starting from a at the point x_0. Let us take a sufficiently small number $h > 0$ such that $x_0 + h < b$, and consider the region under the curve of f from x_0 to $x_0 + h$. This region is marked in blue in the figure. With knowledge of the area-measuring function T, it is not difficult to calculate the area of the blue region, as it is simply the area under the curve from a to $x_0 + h$ minus the area under the curve from a to x_0, so the measure of the blue region is:



$$T(x_0 + h) - T(x_0).$$

On the other hand, if we combine the blue and red area sections shown in the figure, we obtain a rectangle whose one side has a length of h and the other side has a length of $f(x_0)$, thus its area is $f(x_0)h$. Therefore, if we denote the area of the red shape by $\varepsilon(h)$, the following relation holds:

$$T(x_0 + h) - T(x_0) = f(x_0)h - \varepsilon(h). \tag{1}$$

The area of the red shape is part of a rectangle whose one side has a length of h and the other side has a length of $f(x_0) - f(x_0 + h)$, thus its area is $\big(f(x_0) - f(x_0 + h)\big)h$. Therefore,

$$\varepsilon(h) \le \big(f(x_0) - f(x_0 + h)\big)h.$$

From this, it follows that

$$0 \le \lim_{h\to 0+0} \frac{\varepsilon(h)}{h} \le \lim_{h\to 0+0} \big(f(x_0) - f(x_0 + h)\big) = 0 \qquad \Longrightarrow \qquad \lim_{h\to 0+0} \frac{\varepsilon(h)}{h} = 0,$$

since the continuity of the function f at the point x_0 means that $f(x_0) - f(x_0 + h) \to 0$ as $h \to 0$. Thus, if we divide both sides of the equation (1) by h and take the limit as h approaches zero, we obtain

$$T'_+(x_0) = \lim_{h\to 0+0} \frac{T(x_0 + h) - T(x_0)}{h} = f(x_0).$$

We obtain a similar result if we use a small negative number $h < 0$ and perform similar calculations, namely $T'_{-}(x_0) = f(x_0)$, thus $T'(x_0) = f(x_0)$. This leads us to conclude a very interesting property of the area measuring function T, namely that T is a function whose derivative is the function f at every point in the interval (a, b), provided that f is a continuous function.

It is important to emphasize that the previous reasoning only illustrates but does not rigorously prove the preceding conclusion. Nevertheless, it shows why it is necessary to deal with antiderivatives. We will start the discussion of the topic with the concepts of primitive function and indefinite integral.

Let $f : I \to \mathbb{R}$ be a function defined on an open interval $I \subset \mathbb{R}$. We say that the function $F : I \to \mathbb{R}$ is a ***primitive function*** of f if $F \in D(I)$ and $F'(x) = f(x)$ for all $x \in I$. If the function $f : I \to \mathbb{R}$ defined on the open interval $I \subset \mathbb{R}$ has one primitive function F, then there are infinitely many, but they differ from F by only a constant. This statement is not true if the domain of f is *not an interval*.

The set of all primitive functions of a function f defined on an open interval $I \subset \mathbb{R}$ is called the ***indefinite integral*** of f, and it is denoted as:

$$\int f := \int f(x)\,dx := \big\{F : I \to \mathbb{R} \,\big|\, F \in D \text{ and } F' = f\big\}.$$

In this case, we also use the terms ***integrand*** and ***function to be integrated*** for f. If $F \in \int f$, we will write this in the following form:

$$\int f(x)\,dx = F(x) + c \qquad (x \in I).$$

We always indicate the domain of the given function f—that is, the interval I—the condition $c \in \mathbb{R}$ is *implicitly included* in the formula, but we do not write it out.

Basic Integrals, Linearity of the Indefinite Integral

Theoretical notes The basic integrals are listed in the following table.

$\mathcal{D}_f$	$f(x)$	$\int f(x)\,dx$
$\mathbb{R}$	x^n $(n \in \mathbb{N})$	$\dfrac{x^{n+1}}{n+1} + c$
$(0, +\infty)$	$\dfrac{1}{x}$	$\ln x + c$
$(-\infty, 0)$	$\dfrac{1}{x}$	$\ln(-x) + c$
$(0, +\infty)$	x^α $\big(\alpha \in \mathbb{R} \setminus \{-1\}\big)$	$\dfrac{x^{\alpha+1}}{\alpha+1} + c$
$\mathbb{R}$	e^x	$e^x + c$
$\mathbb{R}$	a^x $\big(a \in (0,1) \cup (1, +\infty)\big)$	$\dfrac{a^x}{\ln a} + c$
$\mathbb{R}$	$\sin x$	$-\cos x + c$
$\mathbb{R}$	$\cos x$	$\sin x + c$
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\dfrac{1}{\cos^2 x}$	$\tan x + c$
$(0, \pi)$	$\dfrac{1}{\sin^2 x}$	$-\cot x + c$
$\mathbb{R}$	$\dfrac{1}{1+x^2}$	$\arctan x + c$
$(-1, 1)$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$

Linearity of the Indefinite Integral. Let $I \subset \mathbb{R}$ be an open interval. If the functions $f, g : I \to \mathbb{R}$ have primitive functions, then for any $\alpha, \beta \in \mathbb{R}$, the function $(\alpha f + \beta g)$ also has a primitive function, and

$$\int \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx \qquad (x \in I).$$

The problems belonging to this category are often not simple because it is necessary to manipulate the integrand so that we can express it as a linear combination of basic integrals.

Exercise 1. *Evaluate the following indefinite integrals.*

a) $\displaystyle\int (6x^2 - 8x + 3)\, dx \quad (x \in \mathbb{R}),$

b) $\displaystyle\int \left(2x + \frac{5}{\sqrt{1-x^2}}\right) dx \quad (x \in (-1, 1)),$

c) $\displaystyle\int \frac{x^2}{x^2+1}\, dx \quad (x \in \mathbb{R}),$

d) $\displaystyle\int \frac{\cos^2 x - 5}{1 + \cos 2x}\, dx \quad \big(x \in (-\pi/2, \pi/2)\big),$

e) $\displaystyle\int \frac{1}{\sqrt{x}}\, dx \quad \big(x \in (0, +\infty)\big),$

f) $\displaystyle\int x \cdot \sqrt{x}\, dx \quad \big(x \in (0, +\infty)\big),$

g) $\displaystyle\int \frac{x^3 + 5x - 3}{x}\, dx \quad \big(x \in (-\infty, 0)\big),$

h) $\displaystyle\int \frac{(x+1)^2}{\sqrt{x}}\, dx \quad \big(x \in (0, +\infty)\big).$

Solution After an "appropriate" transformation of the integrands, we obtain basic integrals by using the theorem on the linearity of the indefinite integral.

a) For $x \in \mathbb{R}$ we have

$$\begin{aligned}\int (6x^2 - 8x + 3)\, dx &= 6 \int x^2\, dx - 8 \int x\, dx + 3 \int 1\, dx = 6\,\frac{x^3}{3} - 8\,\frac{x^2}{2} + 3x + c \\ &= 2x^3 - 4x^2 + 3x + c.\end{aligned}$$

b) For $-1 < x < 1$ we have

$$\begin{aligned}\int \left(2x + \frac{5}{\sqrt{1-x^2}}\right) dx &= 2 \int x\, dx + 5 \int \frac{1}{\sqrt{1-x^2}}\, dx = 2\,\frac{x^2}{2} + 5 \arcsin x + c \\ &= x^2 + 5 \arcsin x + c.\end{aligned}$$

c) For $x \in \mathbb{R}$ we have

$$\int \frac{x^2}{x^2+1}\, dx = \int \frac{(x^2+1) - 1}{x^2+1}\, dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \arctan x + c.$$

d) For $-\pi/2 < x < \pi/2$ we have

$$\begin{aligned}\int \frac{\cos^2 x - 5}{1 + \cos 2x}\, dx &= \int \frac{\cos^2 x - 5}{\cos^2 x + \sin^2 x + \cos^2 x - \sin^2 x}\, dx = \int \frac{\cos^2 x - 5}{2\cos^2 x}\, dx \\ &= \frac{1}{2} \int \left(1 - \frac{5}{\cos^2 x}\right) dx = \frac{1}{2}(x - 5 \tan x) + c = \frac{1}{2}\, x - \frac{5}{2}\, \tan x + c\end{aligned}$$

e) For $x > 0$ we have

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{-1/2+1}}{-1/2+1} + c = \frac{x^{1/2}}{1/2} + c = 2\sqrt{x} + c.$$

f) For $x > 0$ we have

$$\int x \cdot \sqrt{x}\,dx = \int x \cdot x^{1/2}\,dx = \int x^{3/2}\,dx = \frac{x^{3/2+1}}{3/2+1} + c = \frac{x^{5/2}}{5/2} + c = \frac{2}{5}\sqrt{x^5} + c.$$

g) For $x < 0$ we have

$$\int \frac{x^3 + 5x - 3}{x}\,dx = \int \left(x^2 + 5 - \frac{3}{x}\right) dx = \frac{x^3}{3} + 5x - 3\ln(-x) + c.$$

h) For $x > 0$ we have

$$\int \frac{(x+1)^2}{\sqrt{x}}\,dx = \int \frac{x^2 + 2x + 1}{x^{1/2}}\,dx = \int \left(x^{3/2} + 2x^{1/2} + x^{-1/2}\right) dx$$
$$= \frac{x^{5/2}}{5/2} + 2\frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c = \frac{2}{5}\sqrt{x^5} + \frac{4}{5}\sqrt{x^3} + 2\sqrt{x} + c.$$

Reverse of Differentiation of Composite Functions and Special Cases

Theoretical notes The rule of differentiation of composite functions may be reversed as follows.

Reverse of Differentiation of Composite Functions. Let $I, J \subset \mathbb{R}$ be open intervals and $g : I \to \mathbb{R}$, $f : J \to \mathbb{R}$ be functions. Suppose that $g \in D(I)$, $\mathcal{R}_g \subset J$, and that the function f has a primitive function. Then the function $(f \circ g) \cdot g'$ also has a primitive function and

$$\int f\big(g(x)\big) \cdot g'(x)\,dx = F\big(g(x)\big) + c \qquad (x \in I),$$

where F is a primitive function of f.

The first substitution rule can be applied when we need to compute the integral $\int f \circ g \cdot g'$, and we know a primitive function of f. However, we often do not realize that we are dealing with this type of integral, so it is worth noting a few special cases separately.

- $\int \frac{f'}{f}$ ***integrals***: If $f : I \to \mathbb{R}$, $f > 0$, and $f \in D(I)$, then

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c \quad (x \in I)}.$$

- $\int f^\alpha \cdot f'$ ***integrals***: If $f : I \to \mathbb{R}$, $f > 0$, $f \in D(I)$, and $\alpha \in \mathbb{R} \setminus \{-1\}$, then

$$\boxed{\int f^\alpha(x) f'(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \quad (x \in I)}.$$

  If $\alpha \in \mathbb{N}$, then the condition $f > 0$ is not necessary.

- $\int f(ax+b)\,dx$ ***integrals (linear substitution)***: If the function $f : I \to \mathbb{R}$ has a primitive function $F : I \to \mathbb{R}$, with $a, b \in \mathbb{R}$ and $a \neq 0$, then

$$\boxed{\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + c \quad (ax+b \in I)}.$$

Exercise 2. *Evaluate the following indefinite integrals.*

a) $\displaystyle\int (3x+2)^4\,dx \quad (x\in\mathbb{R}),$

b) $\displaystyle\int \frac{1}{1-2x}\,dx \quad \Big(x\in\big(-\infty,\tfrac{1}{2}\big)\Big),$

c) $\displaystyle\int \frac{1}{\sqrt{1-2x}}\,dx \quad \Big(x\in\big(-\infty,\tfrac{1}{2}\big)\Big),$

d) $\displaystyle\int \frac{1}{\sqrt{1-2x^2}}\,dx \quad \Big(x\in\big(-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\big)\Big),$

e) $\displaystyle\int \frac{2}{3+2x^2}\,dx \quad (x\in\mathbb{R}),$

f) $\displaystyle\int \frac{1}{4x^2-4x+2}\,dx \quad (x\in\mathbb{R}),$

g) $\displaystyle\int 5^{2-3x}\,dx \quad (x\in\mathbb{R}),$

h) $\displaystyle\int \sin^2 x\,dx \quad (x\in\mathbb{R}).$

Solution In this exercise, we will apply linear substitution to determine the integrals:

$$\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + c \quad (ax+b\in I)\,.$$

a) We use linear substitution for $f(x) := x^4$ $(x\in\mathbb{R})$, $a=3$, and $b=2$:

$$\int (3x+2)^4\,dx = \frac{(3x+2)^5}{5\cdot 3} + c = \frac{1}{15}\,(3x+2)^5 + c \quad (x\in\mathbb{R}).$$

b) We use linear substitution for $f(x) := 1/x$ $(x>0)$, $a=-2$, and $b=1$:

$$\int \frac{1}{1-2x}\,dx = \frac{\ln(1-2x)}{-2} + c = -\frac{1}{2}\,\ln(1-2x) + c \qquad (x<\tfrac{1}{2}).$$

c) We use linear substitution for $f(x) := x^{-1/2}$ $(x>0)$, $a=-2$, and $b=1$:

$$\int \frac{1}{\sqrt{1-2x}}\,dx = \int (1-2x)^{-1/2}\,dx = \frac{(1-2x)^{1/2}}{1/2\cdot(-2)} + c = -\sqrt{1-2x} + c \qquad (x<\tfrac{1}{2}).$$

d) We use linear substitution for $f(x) := 1/(\sqrt{1-x^2})$ $(-1<x<1)$, $a=\sqrt{2}$, and $b=0$:

$$\int \frac{1}{\sqrt{1-2x^2}}\,dx = \int \frac{1}{\sqrt{1-(\sqrt{2}x)^2}}\,dx = \frac{\arcsin(\sqrt{2}x)}{\sqrt{2}} + c \qquad (-\tfrac{1}{\sqrt{2}}<x<\tfrac{1}{\sqrt{2}}).$$

e) We use linear substitution for $f(x) := 1/(1+x^2)$ $(x\in\mathbb{R})$, $a=\sqrt{2/3}$, and $b=0$:

$$\int \frac{2}{3+2x^2}\,dx = \frac{2}{3}\int \frac{1}{1+\frac{2}{3}x^2}\,dx = \frac{2}{3}\int \frac{1}{1+(\sqrt{\frac{2}{3}}\,x)^2}\,dx = \frac{2}{3}\cdot\frac{\arctan(\sqrt{\frac{2}{3}}\,x)}{\sqrt{\frac{2}{3}}} + c$$

$$= \sqrt{\frac{2}{3}}\,\arctan(\sqrt{\tfrac{2}{3}}\,x) + c \qquad (x\in\mathbb{R}).$$

f) We use linear substitution for $f(x) := 1/(1+x^2)$ $(x\in\mathbb{R})$, $a=2$, and $b=-1$:

$$\int \frac{1}{4x^2-4x+2}\,dx = \int \frac{1}{1+(2x-1)^2}\,dx = \frac{\arctan(2x-1)}{2} + c \qquad (x\in\mathbb{R}).$$

g) We use linear substitution for $f(x) := 5^x$ $(x \in \mathbb{R})$, $a = -3$, and $b = 2$:

$$\int 5^{2-3x}\,dx = \frac{5^{2-3x}}{-3\ln 5} + c \qquad (x \in \mathbb{R}).$$

h) To use linear substitution we write the integrand as follows, called ***linearisation***:

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad (x \in \mathbb{R}).$$

This is easy to prove from the identities

$$1 = \cos^2 x + \sin^2 x \qquad \text{and} \qquad \cos 2x = \cos^2 x - \sin^2 x.$$

Then

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{1}{2}\int \cos 2x\,dx$$

We can use now linear substitution for $f(x) := \cos x$ $(x \in \mathbb{R})$, $a = 2$, and $b = 0$:

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{2}\int \cos 2x\,dx = \frac{x}{2} - \frac{1}{2}\,\frac{\sin 2x}{2} + c = \frac{x}{2} - \frac{\sin 2x}{4} + c \qquad (x \in \mathbb{R}).$$

Exercise 3. *Evaluate the following indefinite integrals.*

a) $\displaystyle\int x^2(2x^3+4)^{10}\,dx \quad (x \in \mathbb{R})$, *b)* $\displaystyle\int \frac{x^2}{\sqrt[3]{6x^3+1}}\,dx \quad (x > 0)$,

c) $\displaystyle\int \frac{x}{x^2+3}\,dx \quad (x \in \mathbb{R})$, *d)* $\displaystyle\int \frac{e^x}{1-2e^x}\,dx \quad (x > 0)$,

e) $\displaystyle\int \frac{1}{x\ln x}\,dx \quad (0 < x < 1)$, *f)* $\displaystyle\int \frac{1}{x\ln^2 x}\,dx \quad (x > 0)$,

g) $\displaystyle\int \tan x\,dx \quad (-\pi/2 < x < \pi/2)$, *h)* $\displaystyle\int \sin^3 x\,dx \quad (x \in \mathbb{R})$,

i) $\displaystyle\int \sin^2 x \cdot \cos^3 x\,dx \quad (x \in \mathbb{R})$, *j)* $\displaystyle\int \sin^2 x \cdot \cos^4 x\,dx \quad (x \in \mathbb{R})$.

Solution In this exercise, we will apply the rules

$$\int f^\alpha(x) f'(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \quad (\alpha \neq -1) \qquad \text{or} \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

to determine the integrals:

a) Note that $(2x^3+4)' = 6x^2$, so the integral is of type $\int f^\alpha f'$, where $\alpha = 10$:

$$\int x^2(2x^3+4)^{10}\,dx = \frac{1}{6}\int 6x^2(2x^3+4)^{10}\,dx = \frac{1}{6}\cdot\frac{(2x^3+4)^{11}}{11} + c = \frac{(2x^3+4)^{11}}{66} + c,$$

where $x \in \mathbb{R}$.

b) Note that $(6x^3+1)' = 18x^2$, so the integral is of type $\int f^\alpha f'$, where $\alpha = -1/3$:

$$\int \frac{x^2}{\sqrt[3]{6x^3+1}} = \frac{1}{18}\int 18x^2(6x^3+1)^{-1/3}\,dx = \frac{1}{18}\cdot\frac{(6x^3+1)^{2/3}}{2/3} + c = \frac{\sqrt[3]{(6x^3+1)^2}}{12} + c,$$

where $x \in \mathbb{R}$.

c) Note that $(x^2+3)' = 2x$, so the integral is of type $\int f'/f$:

$$\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx = \frac{1}{2}\,\ln|x^2+3| + c = \frac{1}{2}\,\ln(x^2+3) + c,$$

where $x \in \mathbb{R}$.

d) Note that $(1-2e^x)' = -2e^x$, so the integral is of type $\int f'/f$:

$$\int \frac{e^x}{1-2e^x}\,dx = -\frac{1}{2}\int \frac{-2e^x}{1-2e^x}\,dx = -\frac{1}{2}\,\ln|1-2e^x| + c = -\frac{1}{2}\,\ln(2e^x-1) + c,$$

where $x > 0$.

e) Note that $(\ln x)' = 1/x$, so the integral is of type $\int f'/f$:

$$\int \frac{1}{x\ln x}\,dx = \int \frac{\frac{1}{x}}{\ln x}\,dx = \ln|\ln x| + c = \ln(-\ln x) + c,$$

where $0 < x < 1$, since in this case $\ln x < 0$.

f) Note that $(\ln x)' = 1/x$, so the integral is of type $\int f^\alpha f'$, where $\alpha = -2$:

$$\int \frac{1}{x\ln^2 x}\,dx = \int \frac{1}{x}(\ln x)^{-2}\,dx = \frac{(\ln x)^{-1}}{-1} + c = -\frac{1}{\ln x} + c,$$

where $x > 0$.

g) Note that $(\cos x)' = -\sin x$, so the integral is of type $\int f'/f$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + c = -\ln(\cos x) + c,$$

where $-\pi/2 < x < \pi/2$, since in this case $\cos x > 0$.

h) First we use the transformation

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin x\cdot\sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx\\ &= \int \sin x\,dx - \int \sin x\cdot\cos^2 x\,dx = -\cos x + \int(-\sin x)\cos^2 x\,dx\end{aligned}$$

Note that $(\cos x)' = -\sin x$, so the integral above is of type $\int f^\alpha f'$, where $\alpha = 2$:

$$\int(-\sin x)\cos^2 x\,dx = \frac{\cos^3 x}{3} + c,$$

hence

$$\int \sin^3 x\,dx = \frac{\cos^3 x}{3} - \cos x + c,$$

where $x \in \mathbb{R}$.

i) First we use the transformation

$$\int \sin^2 x \cdot \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$
$$= \int \cos x \cdot \sin^2 x \, dx - \int \cos x \cdot \sin^4 x \, dx$$

Note that $(\sin x)' = \cos x$, so both integrals above are of type $\int f^\alpha f'$, where $\alpha = 2$ and $\alpha = 4$:

$$\int \cos x \cdot \sin^2 x \, dx - \int \cos x \cdot \sin^4 x \, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c,$$

where $x \in \mathbb{R}$.

j) Using linearisation we obtain

$$\int \sin^2 x \cdot \cos^4 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos^2 x \, dx = \frac{1}{4} \int (2 \sin x \cdot \cos x)^2 \cdot \cos^2 x \, dx$$
$$= \frac{1}{4} \int \sin^2 2x \cdot \frac{1 + \cos 2x}{2} \, dx = \frac{1}{8} \int \sin^2 2x \, dx + \frac{1}{8} \int \sin^2 2x \cdot \cos 2x \, dx$$

Let's solve the integrals separately. To solve the first one, we will use one of the results of the previous exercise:

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c \qquad (x \in \mathbb{R}).$$

Then, by linear substitution we have

$$\int \sin^2 2x \, dx = \frac{\frac{(2x)}{2} - \frac{\sin\big(2(2x)\big)}{4}}{2} + c = \frac{x}{2} - \frac{\sin 4x}{8} + c \qquad (x \in \mathbb{R}).$$

For the second integral note that $(\sin 2x)' = 2 \cos 2x$, so it is of type $\int f^\alpha f'$, where $\alpha = 2$:

$$\int \sin^2 2x \cdot \cos 2x \, dx = \frac{1}{2} \int \sin^2 2x \cdot 2 \cos 2x \, dx = \frac{1}{2} \frac{\sin^3 2x}{3} + c \qquad (x \in \mathbb{R}).$$

Therefore

$$\int \sin^2 x \cdot \cos^4 x \, dx = \frac{1}{8} \left(\frac{x}{2} - \frac{\sin 4x}{8} \right) + \frac{1}{8} \left(\frac{1}{2} \frac{\sin^3 2x}{3} \right) + c = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + c.$$

where $x \in \mathbb{R}$.

Integration by parts

Theoretical notes The following statement expresses the "inverse" of the theorem regarding the differentiation of a product function.

Integration by Parts or Partial Integration. Let $I \subset \mathbb{R}$ be an open interval. Suppose that $f, g \in D(I)$ and the function $f'g$ has a primitive function on I. Then the function fg' also has a primitive function, and

$$\int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx \qquad (x \in I).$$

Partial integration is recommended if the right-hand side $\int f'g$ can be evaluated by some method, even by using partial integration again. There are some basic types of integrals for which it may be worthwhile to integrate by parts.

$\boxed{\textbf{1.}}$ The integrals of the form

$$\int P(x)\cdot T(ax+b)\,dx \qquad \big(x\in\mathbb{R},\ a,b\in\mathbb{R},\ a\neq 0\big)$$

where P is an arbitrary polynomial and $T\in\{\exp,\sin,\cos\}$. In this case, we choose

$$f(x):=P(x) \quad \text{and} \quad g'(x):=T(ax+b) \qquad (x\in\mathbb{R}).$$

$\boxed{\textbf{2.}}$ The integrals of the form

$$\int P(x)\cdot G^n(ax+b)\,dx \qquad \big(ax+b\in\mathcal{D}_G,\ a,b\in\mathbb{R},\ a\neq 0, n\in\mathbb{N}^+\big)$$

where P is an arbitrary polynomial and $G\in\{\ln,\ \arctan,\ \arcsin\}$. In this case, we choose

$$f(x):=G^n(ax+b) \quad \text{and} \quad g'(x):=P(x) \qquad (ax+b\in\mathbb{R}).$$

Exercise 4. *Evaluate the following indefinite integrals.*

a) $\displaystyle\int xe^{2x}\,dx \quad (x\in\mathbb{R}),$ *b)* $\displaystyle\int x^2e^{-x}\,dx \quad (x\in\mathbb{R}),$

c) $\displaystyle\int x^2\sin(5x+1)\,dx \quad (x\in\mathbb{R}),$ *d)* $\displaystyle\int \ln x\,dx \quad (x>0),$

e) $\displaystyle\int (x^3+2x+2)\ln x\,dx \quad (x>0),$ *f)* $\displaystyle\int \arctan(3x)\,dx \quad (x\in\mathbb{R}).$

Solution In this exercise, we will apply integration by parts to determine the integrals.

a) For $x\in\mathbb{R}$ we have

$$\int xe^{2x}\,dx=\int x\left(\frac{e^{2x}}{2}\right)'\,dx=x\cdot\frac{e^{2x}}{2}-\int 1\cdot\frac{e^{2x}}{2}\,dx=\frac{xe^{2x}}{2}-\frac{1}{2}\int e^{2x}\,dx$$
$$=\frac{xe^{2x}}{2}-\frac{1}{2}\cdot\frac{e^{2x}}{2}+c=\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+c=\frac{(2x-1)e^{2x}}{4}+c.$$

b) For $x\in\mathbb{R}$ we have

$$\int x^2e^{-x}\,dx=\int x^2\left(\frac{e^{-x}}{-1}\right)'\,dx=x^2\cdot\frac{e^{-x}}{-1}-\int 2x\cdot\frac{e^{-x}}{-1}\,dx=-x^2e^{-x}+2\left[\int xe^{-x}\,dx\right].$$

Applying integration by parts again:

$$\int xe^{-x}\,dx=\int x\left(\frac{e^{-x}}{-1}\right)'\,dx=x\cdot\frac{e^{-x}}{-1}-\int 1\cdot\frac{e^{-x}}{-1}\,dx=-xe^{-x}+\int e^{-x}\,dx$$
$$=-xe^{-x}-e^{-x}+c.$$

In summary

$$\int x^2e^{-x}\,dx=-x^2e^{-x}+2\left[-xe^{-x}-e^{-x}\right]+c=-(x^2+2x+2)e^{-x}+c.$$

c) For $x \in \mathbb{R}$ we have

$$\begin{aligned}\int x^2 \sin(5x+1)\,dx &= \int x^2 \left(\frac{-\cos(5x+1)}{5}\right)' dx \\ &= x^2 \cdot \frac{-\cos(5x+1)}{5} - \int 2x \cdot \frac{-\cos(5x+1)}{5}\,dx \\ &= -\frac{x^2\cos(5x+1)}{5} + \frac{2}{5}\left[\int x\cos(5x+1)\,dx\right].\end{aligned}$$

Applying integration by parts again:

$$\int x\cos(5x+1)\,dx = \int x\left(\frac{\sin(5x+1)}{5}\right)' dx = x\cdot\frac{\sin(5x+1)}{5} - \int 1\cdot\frac{\sin(5x+1)}{5}\,dx$$
$$= \frac{x\sin(5x+1)}{5} - \frac{1}{5}\int \sin(5x+1)\,dx = \frac{x\sin(5x+1)}{5} + \frac{\cos(5x+1)}{25} + c$$

In summary

$$\begin{aligned}\int x^2\sin(5x+1)\,dx &= -\frac{x^2\cos(5x+1)}{5} + \frac{2}{5}\left[\int x\cos(5x+1)\,dx\right] \\ &= -\frac{x^2\cos(5x+1)}{5} + \frac{2}{5}\left[\frac{x\sin(5x+1)}{5} + \frac{\cos(5x+1)}{25}\right] + c \\ &= \frac{(2-25x^2)\cos(5x+1) + 10x\sin(5x+1)}{125} + c.\end{aligned}$$

d) For $x > 0$ we have

$$\int \ln x\,dx = \int 1\cdot\ln x\,dx = \int (x)'\cdot\ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx$$
$$= x\ln x - \int 1\,dx = x\ln x - x + c.$$

e) For $x > 0$ we have

$$\begin{aligned}\int (x^3+2x+2)\ln x\,dx &= \int\left(\frac{x^4}{4} + x^2 + 2x\right)'\cdot\ln x\,dx \\ &= \left(\frac{x^4}{4}+x^2+2x\right)\cdot\ln x - \int\left(\frac{x^4}{4}+x^2+2x\right)\cdot\frac{1}{x}\,dx \\ &= \frac{(x^4+4x^2+8x)\ln x}{4} - \int\left(\frac{x^3}{4}+x+2\right)dx \\ &= \frac{(x^4+4x^2+8x)\ln x}{4} - \frac{x^4}{16} - \frac{x^2}{2} + 2x + c.\end{aligned}$$

f) For $x \in \mathbb{R}$ we have

$$\int \arctan(3x)\,dx = \int 1 \cdot \arctan(3x)\,dx = \int (x)' \cdot \arctan(3x)\,dx$$
$$= x \cdot \arctan(3x) - \int x \cdot \frac{1}{1+(3x)^2} \cdot 3\,dx = x \cdot \arctan(3x) - \int \frac{3x}{1+9x^2}\,dx.$$

Note that $(1+9x^2)' = 18x$, so the integral is of type $\int f'/f$:

$$\int \frac{3x}{1+9x^2}\,dx = \frac{1}{6}\int \frac{18x}{1+9x^2}\,dx = \frac{1}{6}\ln|1+9x^2| + c = \frac{1}{6}\ln(1+9x^2) + c,$$

In summary

$$\int \arctan(3x)\,dx = x \cdot \arctan(3x) - \frac{1}{6}\ln(1+9x^2) + c.$$

Integration of Rational Functions

Theoretical notes

A rational function is defined as the quotient of two polynomials, i.e., functions of the form $\frac{P}{Q}$, where P and $Q \not\equiv 0$ are algebraic polynomials. We also assume that P and Q have no common roots. In previous exercises, we were already able to integrate some rational functions using elementary techniques. These methods led to quick results, for instance

$$\int \frac{2x^7+x^3}{x^8+x^4+1}\,dx = \frac{1}{4}\int \frac{8x^7+4x^3}{x^8+x^4+1}\,dx = \left(\frac{f'}{f}\text{ type}\right) = \frac{1}{4}\ln(x^8+x^4+1)+c \qquad (x \in \mathbb{R}),$$

but these techniques cannot be used in all cases. Then, we can use a multi-step procedure that, while often lengthy, can lead to the solution without the need for major tricks.

$\boxed{\textit{Step 1}}$ ***Decomposition by division with remainder.*** Before starting the decomposition, check that the degree of the numerator P is less than the degree of the denominator Q. If not, we decompose the rational function as follows

$$\frac{P(x)}{Q(x)} = T(x) + \frac{P^*(x)}{Q(x)} \qquad (x \in \mathcal{D}_Q).$$

where T and P^* are polynomials such that the degree of P^* is now less than the degree of Q. The decomposition can be obtained through long division with polynomials or, in some cases, through simple transformations. For example,

$$\frac{2x^4+x^2+2x+1}{x^3+1} = \frac{2x^4+2x+x^2+1}{x^3+1} = \frac{2x(x^3+1)+x^2+1}{x^3+1} = 2x + \frac{x^2+1}{x^3+1}.$$

Since the integration of the polynomial T is a simple task, our new objective is to find out how we can integrate rational functions with a numerator degree less than that of the denominator.

$\boxed{\textit{Step 2}}$ ***Factor the Denominator.*** The polynomial Q in the denominator is factored (as much as possible) into polynomials with real coefficients. For example:

$$Q(x) = x^2-5x+4 = (x-1)(x-4),$$
$$Q(x) = x^3-1 = (x-1)(x^2+x+1),$$
$$Q(x) = x^3+6x^2+12x+8 = (x+2)^3,$$
$$Q(x) = x^4-1 = (x^2)^2-1^2 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1),$$
$$Q(x) = x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1).$$

Notice that the factors in the decomposition are linear factors and quadratic factors with no real roots.

$\boxed{\textit{Step 3}}$ ***Decomposition into a sum of elementary fractions.*** Here we consider only fractions of the form $\frac{P}{Q}$ where the **degree of the numerator is less than the degree of the denominator** and we have successfully performed the ***Q factorization***; that is, we have completed the first two steps. Such fractions can be decomposed into a sum of elementary fractions. The decomposition is determined using the method of *undetermined coefficients*. For example:

$$\frac{1}{(x-1)(x-4)} = \frac{A_1}{x-1} + \frac{A_2}{x-4}, \qquad \frac{x^2+3}{(x-1)(x^2+x+1)} = \frac{A_1}{x-1} + \frac{B_1x+C_1}{x^2+x+1},$$

$$\frac{x+1}{(x-2)^3} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3}, \qquad \frac{x^3}{(x^2+1)^2} = \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2},$$

$$\frac{4x^2-8x}{(x-1)^2(x^2+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}.$$

Notice that for linear factors, the numerator must be a *constant*, and for quadratic factors, the numerator must be a *first-degree polynomial.* Also, note that if a linear factor in the denominator has an exponent greater than one, then *all* lower power terms must also be "included". The same rule applies to quadratic factors."

To determine the coefficients, bring the expression on the right side to a common denominator, and then the numerator of the resulting fraction will be equal to the numerator of the fraction on the left side. Then, arrange the numerator on the right side according to the powers of x. Two polynomials are equals if and only if their corresponding coefficients are equal. From the equality of the coefficients in the numerators on both sides, we obtain a system of linear equations for the undetermined coefficients. Another possibility is to substitute suitable values of x into the numerators of the two fractions, which we know are equals for every $x \in \mathbb{R}$.

$\boxed{\textit{Step 4}}$ ***Integration of Each Term Separately and Combination of Results.*** Integrate each partial fraction term separately using the elementary techniques showed in previous exercises. Finally, we combine all the obtained results according to the linearity of the integral.

Exercise 5. *Evaluate the following indefinite integrals.*

a) $\displaystyle\int \frac{x^3+x^2-x+3}{x^2-1}\,dx \quad \Big(x \in (-1,1)\Big)$, b) $\displaystyle\int \frac{1}{x^2-6x+8}\,dx \quad \Big(x \in (2,4)\Big)$,

c) $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx \quad \Big(x \in (0,1)\Big)$, d) $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx \quad \Big(x \in (-\infty,-1)\Big)$.

Solution

a) The degree of the numerator is greater than the degree of the denominator, so we first need to perform polynomial long division:

$$(*) \qquad \frac{x^3+x^2-x+3}{x^2-1} = \frac{x\Big(x^2-1\Big) + \Big(x^2-1\Big) + 4}{x^2-1} = x+1+\frac{4}{x^2-1}.$$

We then decompose the remaining fraction into partial fractions:

$$\frac{4}{x^2-1} = \frac{4}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}.$$

The numerators on the left and right sides are equals for every $x \in \mathbb{R}$.
If $x=1$, then $4 = A\cdot 2 + B\cdot 0 \quad \Longrightarrow \quad A=2.$
If $x=-1$, then $4 = A\cdot 0 + B\cdot(-2) \quad \Longrightarrow \quad B=-2.$
Therefore,

$$(**) \qquad \frac{4}{x^2-1} = \frac{2}{x-1} - \frac{2}{x+1}.$$

Based on $(*)$ and $(**)$, we have that if $-1 < x < 1$, then

$$\int \frac{x^3+x^2-x+3}{x^2-1}\,dx = \int \left(x+1+\frac{2}{x-1}-\frac{2}{x+1}\right)\,dx =$$
$$= \frac{x^2}{2}+x+2\ln(1-x)-2\ln(x+1)+c = \frac{x^2}{2}+x+\ln\left(\frac{1-x}{x+1}\right)^2+c.$$

b) The degree of the numerator is smaller than the degree of the denominator, and the denominator is a product of first-degree factors. Based on the partial fraction decomposition, we find a common denominator:

$$\frac{1}{x^2-6x+8} = \frac{1}{(x-2)(x-4)} = \frac{A}{x-2}+\frac{B}{x-4} = \frac{A(x-4)+B(x-2)}{(x-2)(x-4)}.$$

The numerators of the left and right sides are equal. By rearranging the numerator on the right side in powers of x, we obtain the equality of numerators:

$$1 = (A+B)x-4A-2B \qquad (x\in\mathbb{R}).$$

Two polynomials are equals if and only if their corresponding coefficients are equal, so we have:

$$\begin{cases} A+B=0 \\ -4A-2B=1 \end{cases} \qquad \Longrightarrow \qquad -2A=1 \qquad \Longrightarrow \qquad A=-\tfrac{1}{2} \text{ and } B=\tfrac{1}{2}.$$

Consequently,

$$\frac{1}{(x-2)(x-4)} = -\frac{1}{2}\cdot\frac{1}{x-2}+\frac{1}{2}\cdot\frac{1}{x-4}.$$

Thus, if $2 < x < 4$, we have

$$\int \frac{1}{x^2-6x+8}\,dx = \int \frac{1}{(x-2)(x-4)}\,dx = -\frac{1}{2}\int\frac{1}{x-2}\,dx+\frac{1}{2}\int\frac{1}{x-4}\,dx$$
$$= -\frac{1}{2}\ln|x-2|+\frac{1}{2}\ln|x-4|+c = -\frac{1}{2}\ln(x-2)+\frac{1}{2}\ln(4-x)+c$$
$$= \frac{1}{2}\cdot(\ln(4-x)-\ln(x-2)) = \frac{1}{2}\ln\frac{4-x}{x-2}+c = \ln\sqrt{\frac{4-x}{x-2}}+c.$$

c) Based on the partial fraction decomposition

$$\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{4x^2+13x-9}{x(x^2+2x-3)} = \frac{4x^2+13x-9}{x(x+3)(x-1)} = \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$$
$$= \frac{A(x+3)(x-1)+Bx(x-1)+Cx(x+3)}{x(x+3)(x-1)}.$$

The numerators on the left and right sides are equals for every $x\in\mathbb{R}$.
If $x=0$, then $-9 = A\cdot 3\cdot(-1)+B\cdot 0+C\cdot 0 \quad\Longrightarrow\quad A=3$.
If $x=-3$, then $4(-3)^2+13(-3)-9=-12=A\cdot 0+B\cdot(-3)(-4)+C\cdot 0 \Longrightarrow B=-1$.
If $x=1$, then $4(1)^2+13(1)-9=8=A\cdot 0+B\cdot 0+C\cdot 1\cdot 4 \Longrightarrow C=2$.

Therefore,

$$\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{3}{x} - \frac{1}{x+3} + \frac{2}{x-1}.$$

Hence, for $0 < x < 1$ we have

$$\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = 3\int \frac{1}{x}\,dx - \int \frac{1}{x+3}\,dx + 2\int \frac{1}{x-1}\,dx$$
$$= 3\ln x - \ln(x+3) + 2\ln(1-x) + c = \ln x^3 - \ln(x+3) + \ln(1-x)^2 + c$$
$$= \ln \frac{x^3(1-x)^2}{x+3} + c.$$

d) Based on the partial fraction decomposition

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}.$$

The numerators on the left and right sides are equals for every $x \in \mathbb{R}$.
If $x = 0$, then $1 = A \cdot 1 + B \cdot 0 + C \cdot 0 \quad \Longrightarrow \quad A = 1$.
If $x = -1$, then $1 = -12 = A \cdot 0 + B \cdot 0 + C \cdot (-1) \implies C = -1$.
If $x = 1$, then $1 = A \cdot 4 + B \cdot 2 + C \cdot 1 \implies 4A + 2B + C = 1$.
Then $2B = 1 - 4A - C = 1 - 4 \cdot 1 - (-1) = -2 \implies B = -1$. Therefore,

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}.$$

Hence, for $x < -1$ we have

$$\int \frac{1}{x^3+2x^2+x}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$$
$$= \ln(-x) - \ln(-x-1) + \frac{1}{x+1} + c = \ln \frac{-x}{-x-1} + \frac{1}{x+1} + c$$
$$= \ln \frac{x}{x+1} + \frac{1}{x+1} + c.$$

Substitution Rule

Theoretical notes

Substitution Rule. Let $I, J \subset \mathbb{R}$ be open intervals. Assume that $f : I \to \mathbb{R}$, $g : J \to I$, $\mathcal{R}_g = I$, $g \in D(J)$, $g' > 0$ on J (or $g' < 0$ on J), and that the function $(f \circ g) \cdot g' : J \to \mathbb{R}$ has an antiderivative. Then the function f also has an antiderivative, and we have

$$\int f(x)\,dx \underset{x=g(t)}{=} \int f\big(g(t)\big) \cdot g'(t)\,dt\Big|_{t=g^{-1}(x)} \qquad (x \in I).$$

We typically apply substitution rule in the following way. Suppose we want to compute an indefinite integral $\int f(x)\,dx$. Using a suitable function g that satisfies the rule's conditions, we introduce the ***new*** variable $t = g^{-1}(x)$ in place of the ***old*** variable x by setting $x = g(t)$. This replaces x with t in the function f, and we ***multiply*** by $g'(t)\,dt$. We then evaluate the resulting integral (***if possible***), and afterward, we substitute back from t to x.

Exercise 6. *Evaluate the following indefinite integrals.*

a) $\displaystyle\int \frac{e^{3x}}{e^x+2}\,dx \quad (x\in\mathbb{R}),$ *b)* $\displaystyle\int \sqrt{e^x-1}\,dx \quad (x>0),$

c) $\displaystyle\int x\sqrt{3x-1}\,dx \quad (x>1/3),$ *d)* $\displaystyle\int \frac{1}{x^2}\cdot\sqrt[3]{\frac{x+1}{x}}\,dx \quad (x>0).$

Solution

a) We apply the substitution $t=e^x$. Then

$$x=\ln t=:g(t).$$

Since $x\in\mathbb{R}$, we have $\mathcal{R}_g=\mathbb{R}$, and consequently $\mathcal{D}_g=(0,+\infty)$. The function g is differentiable, and

$$g'(t)=\frac{1}{t}>0 \qquad (t>0),$$

which implies that g is strictly increasing, hence invertible, were

$$g^{-1}(x)=e^x=t \qquad (x\in\mathbb{R}).$$

Using substitution rule, we get

$$\begin{aligned}\int \frac{e^{3x}}{e^x+2}\,dx &= \int \frac{t^3}{t+2}\cdot\frac{1}{t}\,dt=\int\frac{t^2}{t+2}\,dt=\int\frac{t^2-4+4}{t+2}\,dt=\\ &=\int\left(\frac{(t+2)(t-2)}{t+2}+\frac{4}{t+2}\right)dt=\int(t-2)\,dt+4\int\frac{1}{t+2}\,dt=\\ &=\frac{t^2}{2}-2t+4\ln(t+2)+c\Big|_{t=e^x}=\frac{e^{2x}}{2}-2e^x+4\ln(e^x+2)+c \quad (x\in\mathbb{R}).\end{aligned}$$

b) Let us define

$$t=\sqrt{e^x-1} \qquad \big(\text{since } x>0,\ \text{we have } t>0\big).$$

To obtain the substitution function g, we first need to express x from this equation:

$$t=\sqrt{e^x-1} \qquad\Longrightarrow\qquad t^2+1=e^x \qquad\Longrightarrow\qquad x=\ln(1+t^2)=:g(t)\quad(t>0).$$

Since $\mathcal{R}_g=(0,+\infty)$, the function g is differentiable, and

$$g'(t)=\frac{2t}{1+t^2}>0 \qquad (t>0),$$

so g is strictly increasing, hence invertible, and

$$g^{-1}(x)=t=\sqrt{e^x-1} \qquad \big(x>0\big).$$

The conditions for the second substitution rule are satisfied. Therefore, if $x>0$, then

$$\begin{aligned}\int\sqrt{e^x-1}\,dx&=\int t\cdot\frac{2t}{1+t^2}\,dt=2\int\frac{t^2+1-1}{1+t^2}\,dt=2\int\left(1-\frac{1}{1+t^2}\right)dt=\\ &=2t-2\arctan t+c\Big|_{t=\sqrt{e^x-1}}=2\sqrt{e^x-1}-2\arctan\sqrt{e^x-1}+c.\end{aligned}$$

c) We apply the substitution $t = \sqrt{3x-1}$. Then

$$t^2 = 3x - 1 \qquad \Longrightarrow \qquad x = \frac{t^2+1}{3} =: g(t).$$

Since $x > 1/3$, we have $\mathcal{R}_g = (0, +\infty)$, and consequently $\mathcal{D}_g = (1/3, +\infty)$. The function g is differentiable, and

$$g'(t) = \frac{2}{3}\,t > 0 \qquad (t > 0),$$

which implies that g is strictly increasing, hence invertible, were

$$g^{-1}(x) = \sqrt{3x-1} = t \qquad (x > 1/3).$$

Using substitution rule, we get

$$\int x\sqrt{3x-1}\,dx = \int \frac{t^2+1}{3} \cdot t \cdot \frac{2}{3}\,t\,dt = \frac{2}{9}\int (t^4+t^2)\,dt = \frac{2}{9}\left(\frac{t^5}{5} + \frac{t^3}{3}\right) + c =$$

$$= \frac{2t^5}{45} + \frac{2t^3}{27} + c\Big|_{t=\sqrt{3x-1}} = \frac{2(\sqrt{3x-1})^5}{45} + \frac{2(\sqrt{3x-1})^3}{27} + c \qquad (x > 1/3).$$

d) We apply the substitution

$$t = \sqrt[3]{\frac{x+1}{x}}.$$

By solving for x in this equation, we get

$$t^3 = \frac{x+1}{x} = 1 + \frac{1}{x} \qquad \Longrightarrow \qquad t^3 - 1 = \frac{1}{x} \qquad \Longrightarrow \qquad x = \frac{1}{t^3-1}.$$

If $x \in (0, +\infty)$, then $t \in (1, +\infty)$. Thus, the substitution function g is given by

$$g(t) = \frac{1}{t^3-1} = x \qquad (t > 1).$$

Since g is differentiable and

$$g'(t) = -\frac{1}{\left(t^3-1\right)^2} \cdot 3t^2 < 0, \quad \text{for } t > 1,$$

g is strictly decreasing, and thus invertible, with

$$g^{-1}(x) = t = \sqrt[3]{\frac{x+1}{x}} \qquad (x > 0).$$

Using the second substitution rule, we obtain

$$\int \frac{1}{x^2} \cdot \sqrt[3]{\frac{x+1}{x}}\,dx = \int \frac{1}{\left(\frac{1}{t^3-1}\right)^2} \cdot t \cdot \left(\frac{-3t^2}{\left(t^3-1\right)^2}\right) dt = -3\int t^3\,dt =$$

$$= -\frac{3}{4}t^4 + c\Big|_{t=\sqrt[3]{\frac{x+1}{x}}} = -\frac{3}{4}\sqrt[3]{\left(\frac{x+1}{x}\right)^4} + c \qquad (x > 0).$$