Computer Science BSc Analysis-1 Practices

Semester 2 (Spring) in the academic year 2023/24. Edited by Csörgő István.

The abbreviation „AN-1" denotes the following reference:

Analysis-1 Lecture Schemes (with Homeworks), written by István Csörgő

It can be found in the Digital Library of the Faculty of Informatics or in the Canvas at the subject „Analysis-1"

Remember that $\mathbb{N} = \{1,\, 2,\, 3,\, \ldots\}$.

week 1 (12-16 of February, 2024):

1. *(to Lesson 2 in „AN-1")*

   Determine whether the following $\mathbb{R} \to \mathbb{R}$ type functions are invertible or not. If a function is invertible, determine its inverse (domain and formula).

   a) $f(x) = \dfrac{1}{1 + |\,x - 1|}$ *b)* $f(x) = \dfrac{2x+1}{3x-2}$

   c) $f(x) = \dfrac{x+1}{3x-2} \quad D_f = [2,\, +\infty)$ *d)* $f(x) = x^2 + 4x \quad D_f = [-1,\, +\infty]$

   Homework to this topic: „AN-1" p. 27, ex. 2

2. (to Lesson 2 in „AN-1")

   Determine the compositions $f \circ g$ and $g \circ f$ of the following $\mathbb{R} \to \mathbb{R}$ type functions if they exist (domain and formula):

   a) $f(x) = \sqrt{x+1}$, $g(x) = x^2 - 3x + 1$

   b) $f(x) = \dfrac{1}{x^2 - 4}$, $g(x) = \sqrt{x-1}$

   Homework to this topic: „AN-1" p. 27, ex. 3

week 2 (19-23 of February, 2024):

3. *(to Lesson 1 in „AN-1")*

   Determine (without using the concept of the limit) $\sup H$, $\inf H$, $\max H$, $\min H$ if

   a) $H = \left\{ \dfrac{n+1}{2n+3} \mid n \in \mathbb{N} \right\}$ *b)* $H = \left\{ \sqrt{x+1} - \sqrt{x} \mid x \geq 0 \right\}$

   c) $H = \left\{ \dfrac{1}{x} \mid 0 < x < 1 \right\}$ *d)* $H = \left\{ \dfrac{2x^2+1}{5x^2+2} \mid x \in \mathbb{R} \right\}$

Homework to this topic:

i) „AN-1" p. 16, ex. 3

ii) Determine (without using the concept of the limit) $\sup H$, $\inf H$, $\max H$, $\min H$ if

$$H = \left\{ \sqrt{x+1} - \sqrt{x} \mid x > 0 \right\}$$

4. *(to Lesson 3 in „AN-1")*

Prove by definition of the limit that

$$a) \quad \lim_{n\to\infty} \frac{n}{2n-3} = \frac{1}{2} \qquad\qquad b) \quad \lim_{n\to\infty} \frac{2n^3+10}{n^3+n^2+n+1} = 2$$

$$c) \quad \lim_{n\to\infty} \frac{1+n^2}{2+n-2n^2} = -\frac{1}{2} \qquad\qquad d) \quad \lim_{n\to\infty} (\sqrt{n^2+1} - n) = 0$$

In each question determine a threshold index to $\varepsilon = 0,01$.

Homework to this topic: „AN-1" p. 36, ex. 1

week 3 (26 of February – 1 of March, 2024) and week 4 (4-8 of March, 2024):

5. *(to Lesson 4 in „AN-1")*

Using the results of the sections 4.1 and 4.2 in „AN-1" determine the following limits if they exist:

$$a) \quad \lim_{n\to\infty} \frac{n^3-3n^2+n-1}{1-2n^3+n} \qquad\qquad b) \quad \lim_{n\to\infty} \frac{n^4+n^2+n+1}{2n^5+n-4}$$

$$c) \quad \lim_{n\to\infty} \frac{(2-n)^7+(2+n)^7}{(n^2+n+1)\cdot(2n+1)^5} \qquad\qquad d) \quad \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$$

$$e) \quad \lim_{n\to\infty} (\sqrt{n^2+n} - n) \qquad\qquad f) \quad \lim_{n\to\infty} (\sqrt{n^2+2n+3} - \sqrt{n^2-n+1})$$

$$g) \quad \lim_{n\to\infty} \frac{\sqrt{3n^2+n} - \sqrt{2n^2+3}}{\sqrt{n^2+9}-2} \qquad\qquad gg) \quad \lim_{n\to\infty} \frac{\sqrt{3n^2+5} - \sqrt{3n^2+3}}{\sqrt{n^2+9}-n}$$

$$h) \quad \lim_{n\to\infty} \frac{5^{n+1}+2^n}{3\cdot 5^n - 5^{-n}} \qquad\qquad i) \quad \lim_{n\to\infty} \sqrt[n]{3n^5+2n+1}$$

$$j) \quad \lim_{n\to\infty} \sqrt[n]{\frac{n+1}{2n+3}} \qquad\qquad k) \quad \lim_{n\to\infty} \sqrt[n]{2\cdot 5^n + 7^n}$$

$$l) \quad \lim_{n\to\infty} \sqrt[n]{\frac{6^n + 2^n\cdot n!}{n!}} \qquad\qquad m) \quad \lim_{n\to\infty} \frac{n^2\cdot 3^n + 2^{2n}}{4^{n+1}+2^n}$$

$$n) \quad \lim_{n\to\infty} \sqrt{\frac{(-5)^n+7^n}{7^{n+1}+n^7}} \qquad\qquad o) \quad \lim_{n\to\infty} \frac{(-2)^n+n}{n!+3^n}$$

Homework to this topic: „AN-1” p. 47, ex. 1

week 5 (11-15 of March, 2024):

6. *(to Lesson 5 in „AN-1”)*

   Determine the limits of the following recursive sequences if they exist:

$$a)\quad a_1 = 0, \quad a_{n+1} = \sqrt{2 + a_n} \qquad b)\quad a_1 = \frac{1}{4}, \quad a_{n+1} = 1 - \sqrt{1 - a_n}$$

Homework to this topic: „AN-1” p. 59, ex. 1

7. *(to Lesson 5 in „AN-1”)*

   Determine the following limits if they exist.

$$a)\quad \lim_{n\to\infty} \left(\frac{4n+3}{4n}\right)^{5n} \qquad b)\quad \lim_{n\to\infty} \left(\frac{3n+1}{3n+2}\right)^{2n+3} \qquad c)\quad \lim_{n\to\infty} \left(\frac{6n-7}{6n+4}\right)^{3n+2}$$

Homework to this topic: „AN-1” p. 59, ex. 2

week 6 (18-22 of March, 2024):

8. *(to Lesson 5 in „AN-1”)*

   Prove by definition of the limit that

$$a)\quad \lim_{n\to\infty} \frac{n^4 + 5n^3 + 11n^2 - 3n - 7}{15n^3 + 7n^2 - 21n + 5} = +\infty \qquad b)\quad \lim_{n\to\infty} \frac{n^4 - 5n^3 + 8n + 3}{7 - 3n^2 - 5n^3} = -\infty$$

   In question a) determine a threshold index to $P = 100$. In question b) determine a threshold index to $P = -200$.

Homework to this topic: „AN-1” p. 59, ex. 3

9. *(to Lesson 5 in „AN-1”)*

   Determine the following limits if they exist

$$a)\quad \lim_{n\to\infty} \frac{n^3 - n^2 - 12n - 7}{13n^2 + 20n + 6} \qquad b)\quad \lim_{n\to\infty} \left(2n - \sqrt{5n^2 + 1}\right)$$

$$c)\quad \lim_{n\to\infty} \frac{4n^3 + 6n^2 - 11n - 8}{4 - 15n - 14n^2}$$

Homework to this topic:

i) $\lim\limits_{n\to\infty}\left(3n-\sqrt{7n^2+2}\right)=?$

ii) $\lim\limits_{n\to\infty}\dfrac{\sqrt{3n^2+n}-\sqrt{3n^2-2n+5}}{\sqrt{n^2+9}-\sqrt{n^2-2}}=?$

week 7 (25, 26 of March and 3, 4, 5 of April, 2024) and week 8 (8-12 of April, 2024):

10. (to Lesson 6 in „AN-1")

Determine the sums of the following series.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}$ *b)* $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n+2^{n+2}}{5\cdot 5^{n+1}}$ *c)* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^{2n}}\right)$

d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)}$ *e)* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cdot(2n+1)}$

Homework to this topic: „AN-1" p. 69, ex. 1

11. *(to Lesson 6 and 7 in „AN-1")*

Determine whether the following series are convergent or not. If a series is convergent, then determine its absolute or conditional convergence.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2n^2+n+5}$ *b)* $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2n^3+n+5}$ *c)* $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2n+1}$

d) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ *e)* $\displaystyle\sum_{n=1}^{\infty}\sqrt[n]{0,1}$ *f)* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\cdot(1+n^2)}}$

g) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\cdot(n+3)}}$ *h)* $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ *i)* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2+n+1}$

j) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ *k)* $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{2^{n^2}}$

l) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cdot\left(\sqrt{n+1}-\sqrt{n}\right)$ *m)* $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(n+1)}{2n^2+5n+2}$

n) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n\cdot n!}{n^n}$ *o)* $\displaystyle\sum_{n=1}^{\infty}\frac{4^n\cdot n!}{n^n}$

p) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+3^n}{2n+1}$ *q)* $\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{(-3)^n}$

Homework to this topic: „AN-1” p. 70, ex. 2 and p. 84, ex 1

week 9 (15-19 of April, 2024):

12. *(to Lesson 8 in „AN-1”)*

Determine the radii of convergence and the convergence sets of the following power series. If the radius of convergence is positive, then give the domain of the corresponding analytical function.

a) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n\cdot x^n$ *b)* $\displaystyle\sum_{n=1}^{\infty}\frac{(x-1)^n}{n^p}$ ($p\in\mathbb{R}$ is a parameter)

c) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\cdot(x+2)^n$ *d)* $\displaystyle\sum_{n=0}^{\infty}\frac{4^n+(-3)^n}{n+1}\cdot x^n$

Homework to this topic: „AN-1” p. 93, ex. 1

13. *(to Lesson 8 in „AN-1”)*

Expand the following functions into power series around the center $x_0=0$. Give the radii of convergence.

a) $f(x)=\dfrac{1}{x-5}$ *b)* $f(x)=\dfrac{1}{3x-1}$ *c)* $f(x)=\dfrac{1}{2x+3}$

d) $f(x)=\dfrac{x}{x^2-5x+6}$ *e)* $f(x)=\dfrac{x+3}{5x^2+9x-2}$

Homework to this topic:

i) „AN-1” p. 93, ex. 2

ii) Expand the following function into power series around the center $x_0=0$. Give the radius of convergence.

$$f(x)=\frac{x-20}{12x^2+13x-14}$$

week 10 (22-26 of April, 2024) and week 11 (29 of April – 3 of May, 2024):

14. *(to Lesson 10 in „AN-1”)*

Prove by the definition of the limit that

a) $\displaystyle\lim_{x\to 0}\frac{1}{1+x}=1$ *b)* $\displaystyle\lim_{x\to 1}\frac{x^4+2x^2-3}{x^2-3x+2}=-8$

Homework to this topic: „AN-1” p. 111, ex. 1

15. *This exercise is optional, it can be omitted (one-sided limit by definition).*

(to Lesson 11 in „AN-1")

Prove by the definitions of the one-sided limits that

$$\lim_{x\to -3+} \frac{x^2-9}{|\,x+3|} = -6$$

Homework to this topic: „AN-1" p. 119, ex. 1

16. *(to Lessons 10-12 in „AN-1")*

Determine the following limits:

$a)\quad \displaystyle\lim_{x\to 2} \frac{x^3-8}{x^2+x-6}$ $\qquad b)\quad \displaystyle\lim_{x\to -1} \frac{x^2+x-3}{x^2+2x+1}$

$c)\quad \displaystyle\lim_{x\to +\infty} \frac{x^3+x-3}{-x^2+2x+1}$ $\qquad d)\quad \displaystyle\lim_{x\to -\infty} \frac{x^2+x-3}{x^3+2x+1}$

$e)\quad \displaystyle\lim_{x\to 1} \frac{x^3-1}{x^2-1}$ $\qquad f)\quad \displaystyle\lim_{x\to 2} \frac{x^2+2x-7}{x^2-5x+6}$

$g)\quad \displaystyle\lim_{x\to 2} \frac{x^2+2x-8}{x^2-5x+6}$ $\qquad h)\quad \displaystyle\lim_{x\to +\infty} \frac{2x^3+3x^2+23}{-3x^3-5x^2+31x+1}$

Homework to this topic: „AN-1" p. 122, ex. 1

17. *(to Lessons 10-12 in „AN-1")*

Determine the following limits:

$a)\quad \displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x^2}$ $\qquad b)\quad \displaystyle\lim_{x\to 0} \frac{x^2}{\sqrt[3]{1+5x}-x-1}$

$c)\quad \displaystyle\lim_{x\to +\infty} (\sqrt{x^2-x+1}-\sqrt{x^2-1})$

Homework to this topic: „AN-1" p. 122, ex. 2

week 12 (6-10 of May, 2024):

18. *(to Lesson 13 in „AN-1")*

Using the results in the examples 13.4 in „AN-1" or using power series expansion determine the following limits:

a) $\lim\limits_{x\to 0} \dfrac{e^{7x} - e^{5x}}{x}$ *b)* $\lim\limits_{x\to 0} \dfrac{\sin(8x)}{\sin(3x)}$

c) $\lim\limits_{x\to 0} \dfrac{\sin(5x) - \sin(3x)}{\sin x}$ *d)* $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{x}$

e) $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{1 + x \cdot \sin x} - \sqrt{\cos x}}$ *f)* $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

g) $\lim\limits_{x\to \pi/3} \dfrac{\sin\left(x - \dfrac{\pi}{3}\right)}{1 - 2\cos x}$ *h)* $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

i) $\lim\limits_{x\to 0} \dfrac{1 - \cos^3 x + \sin^2(2x)}{2x^2 - \sin^2 x}$ *j)* $\lim\limits_{x\to 0} \dfrac{\cos(\sqrt{x}) - \dfrac{1}{1-x}}{x + \sin(2x)}$

Homework to this topic: „AN-1" p. 126, ex. 1

week 13 (13-17 of May, 2024):

Consultations.