MULTIVARIATE DIFFERENTIATION

Partial derivatives of $\mathbb{R}^2 \to \mathbb{R}$ functions

Theoretical notes

In the linear space $\mathbb{R}^2$, the ***Euclidean norm*** of a vector $(x, y) \in \mathbb{R}^2$ is defined as:

$$\|(x,y)\| := \sqrt{x^2 + y^2}.$$

The ***distance*** between two points $x, y \in \mathbb{R}^2$ is defined using the norm as:

$$d(x,y) := \|x - y\| \qquad (x \in \mathbb{R}^2).$$

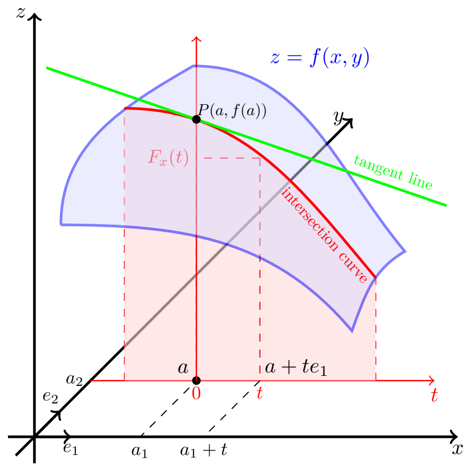
The ***neighborhood*** of a point $a \in \mathbb{R}^2$ with radius $r > 0$ is the set:

$$K_r(a) := \{x \in \mathbb{R}^2 \mid \|x - a\| < r\}.$$

$K_r(a)$ is the open disk of radius r centered at the point a.

Let

$$f \in \mathbb{R}^2 \to \mathbb{R} \quad \text{and} \quad a = (a_1, a_2) \in \operatorname{int} \mathcal{D}_f.$$



The graph of the function in space is the surface given by the equation $z = f(x, y)$. Let us draw a line passing through the point a and parallel to the x-axis (the t-axis). The points of this line in the xy-plane are:

$$(a_1 + t, a_2) \qquad (t \in \mathbb{R}).$$

Let us take the function values at these points and form the real-valued function:

$$F_x(t) := f(a_1 + t, a_2) \qquad \big((a_1 + t, a_2) \in \mathcal{D}_f\big).$$

This function is defined in a neighborhood $K(0)$ of the point $t = 0$ because $a \in \operatorname{int} \mathcal{D}_f$. The image of the function F_x is a curve (intersection curve) on the surface, i.e., the intersection of the surface given by $z = f(x, y)$ and the plane $y = a_2$ (x, z arbitrary). ***The partial derivative of the function f with respect to x at the point a*** (denoted by $\boldsymbol{\partial_x f(a)}$ or $\boldsymbol{\partial_1 f(a)}$) is defined as ***the derivative of the function F_x at the point*** $\mathbf{0}$, provided that the derivative exists, i.e.,

$$\partial_x f(a) := \partial_1 f(a) := F_x'(0) = \lim_{t \to 0} \frac{F_x(t) - F_x(0)}{t} = \lim_{t \to 0} \frac{f(a_1 + t, a_2) - f(a_1, a_2)}{t}.$$

$F_x'(0)$ represents the slope of the tangent line to the intersection curve at the point $P(a, f(a))$.

The partial derivative with respect to y is defined similarly: $F_y(t) := f(a_1, a_2 + t)$ $(t \in K(0))$, and

$$\partial_y f(a) := \partial_2 f(a) := F_y'(0) = \lim_{t \to 0} \frac{F_y(t) - F_y(0)}{t} = \lim_{t \to 0} \frac{f(a_1, a_2 + t) - f(a_1, a_2)}{t}.$$

Exercise 1. *Find the partial derivatives of the following functions.*

$$a) \quad f(x,y) := \ln(xy^2) - x^3y^2\cos(x^2 + y^2) \qquad (x > 0, y \neq 0),$$

$$b) \quad f(x,y) := \arctan\left(\frac{y}{x}\right) \qquad (x \neq 0, y \in \mathbb{R}).$$

Solution To compute the partial derivative of a function $f \in \mathbb{R}^2 \to \mathbb{R}$ with respect to the variable x, we fix the variable y as a constant, and differentiate the expression as a real-valued function which depends only on x (if it is differentiable). We proceed in a similar way for computing the partial derivative with respect to the variable y, but now we consider x a constant and differentiate the expression with respect to y.

a) $f(x, y) := \ln(xy^2) - x^3y^2\cos(x^2 + y^2)$

$$\begin{aligned}
\partial_x(x, y) &= \frac{1}{xy^2}(xy^2)'_x - \Big((x^3y^2)'_x\cos(x^2+y^2) + x^3y^2\Big(\cos(x^2+y^2)\Big)'_x\Big)\\
&= \frac{1}{xy^2}\,y^2 - 3x^2y^2\cos(x^2+y^2) - x^3y^2\Big(-\sin(x^2+y^2)\Big)(x^2+y^2)'_x\\
&= \frac{1}{xy^2}\,y^2 - 3x^2y^2\cos(x^2+y^2) + x^3y^2\sin(x^2+y^2)2x\\
&= \frac{1}{x} - 3x^2y^2\cos(x^2+y^2) + 2x^4y^2\sin(x^2+y^2),
\end{aligned}$$

$$\begin{aligned}
\partial_y(x, y) &= \frac{1}{xy^2}(xy^2)'_y - \Big((x^3y^2)'_y\cos(x^2+y^2) + x^3y^2\Big(\cos(x^2+y^2)\Big)'_y\Big)\\
&= \frac{1}{xy^2}\,2xy - 2x^3y\cos(x^2+y^2) - x^3y^2\Big(-\sin(x^2+y^2)\Big)(x^2+y^2)'_y\\
&= \frac{1}{xy^2}\,2xy - 2x^3y\cos(x^2+y^2) + x^3y^2\sin(x^2+y^2)2y\\
&= \frac{2}{y} - 2x^3y\cos(x^2+y^2) + 2x^3y^3\sin(x^2+y^2).
\end{aligned}$$

b) $f(x, y) := \arctan(\frac{y}{x})$

$$\partial_x(x, y) = \frac{1}{1+(\frac{y}{x})^2}\cdot\left(\frac{y}{x}\right)'_x. = \frac{1}{1+(\frac{y}{x})^2}\cdot y\left(\frac{1}{x}\right)'_x = \frac{1}{1+(\frac{y}{x})^2}\cdot y\left(-\frac{1}{x^2}\right) = -\frac{y}{x^2+y^2},$$

$$\partial_x(x, y) = \frac{1}{1+(\frac{y}{x})^2}\cdot\left(\frac{y}{x}\right)'_y. = \frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}\,(y)'_y = \frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}\cdot 1 = \frac{x}{x^2+y^2},$$

Theoretical notes

Totally differentiable function: A function $f \in \mathbb{R}^2 \to \mathbb{R}$ is ***totally differentiable*** (or differentiable) at a point $a \in \operatorname{int}\mathcal{D}_f$ (denoted as $\boldsymbol{f \in D\{a\}}$) if

$$\exists A \in \mathbb{R}^{1\times 2}\colon\ \lim_{h\to 0}\frac{|f(a+h) - f(a) - A\cdot h|}{\|h\|} = 0.$$

In this case, $f'(a) := A$ is the ***derivative matrix*** of the function f at the point a.

If $f \in D\{a\}$, then the derivative matrix $f'(a)$ is uniquely determined.

Construction of the derivative matrix Let $f \in \mathbb{R}^2 \to \mathbb{R}$ be a function, and let $a \in \operatorname{int}\mathcal{D}_f$. If $f \in D\{a\}$, then $\exists\partial_1 f(a)$, $\exists\partial_2 f(a)$, and

$$f'(a) = \begin{pmatrix}\partial_1 f(a) & \partial_2 f(a)\end{pmatrix}$$

is the so-called ***Jacobian matrix***.

The existence of partial derivatives *does not imply* total differentiability. However, if we assume slightly more than the existence of partial derivatives, we can guarantee total differentiability.

Sufficient condition for total differentiability: Let $f \in \mathbb{R}^2 \to \mathbb{R}$ and $a \in \operatorname{int} \mathcal{D}_f$. Assume that there exists a neighborhood $K(a) \subset \mathcal{D}_f$ of the point a such that the following conditions hold:

a) $\exists\, \partial_1 f(x)$ and $\exists\, \partial_2 f(x)$ at every point $x \in K(a)$,

b) the partial derivative functions $\partial_1 f, \partial_2 f \colon K(a) \to \mathbb{R}$ are continuous at the point a.

Then the function f is totally differentiable at the point a.

Continuously differentiable function: A function $f \in \mathbb{R}^2 \to \mathbb{R}$ is ***continuously differentiable*** at the point $a \in \operatorname{int} \mathcal{D}_f$ (denoted as $\boldsymbol{f \in C^1\{a\}}$) if:

a) There exists a neighborhood $K(a) \subset \mathcal{D}_f$ such that $f \in D\big(K(a)\big)$, and

b) every partial derivative is continuous at the point a.

Partial derivatives of second order: Let $f \in \mathbb{R}^2 \to \mathbb{R}$ and $a \in \operatorname{int} \mathcal{D}_f$. If for a fixed $i = 1, 2$, the partial derivative $\partial_i f$ exists in a neighborhood of the point a, and the partial derivative of the function $\partial_i f$ with respect to the j-th variable ($j = 1, 2$) exists at a, then the number $\partial_{ij} f(a) := \partial_i \partial_j f(a) := \partial_j\big(\partial_i f\big)(a)$ (interpreted as the j-th partial derivative of the function $\partial_i f \in \mathbb{R}^n \to \mathbb{R}$ at a) is called the ***second-order partial derivative*** of the function at a with respect to ij.

Twice differentiable function: The function $f \in \mathbb{R}^2 \to \mathbb{R}$ is ***twice differentiable*** at the point $a \in \operatorname{int} \mathcal{D}_f$ (denoted as $f \in D^2\{a\}$) if:

a) There exists a neighborhood $K(a) \subset \mathcal{D}_f$ such that $f \in D\{x\}$ at every point $x \in K(a)$, and

b) $\forall i = 1, 2, \ldots, n$ index, $\partial_i f \in D\{a\}$.

Then, the matrix

$$f''(a) := \begin{pmatrix} \partial_{11} f(a) & \partial_{12} f(a) \\ \partial_{21} f(a) & \partial_{22} f(a) \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

is called ***the Hessian matrix of the function f at the point a***.

Young's theorem: If $f \in \mathbb{R}^2 \to \mathbb{R}$ and $f \in D^2\{a\}$, then

$$\partial_{12} f(a) = \partial_{21} f(a).$$

Therefore, the Hessian matrix is symmetric.

Twice continuously differentiable function: A function $f \in \mathbb{R}^2 \to \mathbb{R}$ is ***twice continuously differentiable*** at the point $a \in \operatorname{int} \mathcal{D}_f$ (denoted as $\boldsymbol{f \in C^2\{a\}}$) if:

a) There exists a neighborhood $K(a) \subset \mathcal{D}_f$ such that $f \in D^2\big(K(a)\big)$, and

b) every partial derivative of second order is continuous at the point a.

Local extrema of $\mathbb{R}^2 \to \mathbb{R}$ functions

Theoretical notes We say that the function $f \in \mathbb{R}^2 \to \mathbb{R}$ has a ***local maximum*** at the point $a \in \operatorname{int} \mathcal{D}_f$ (or equivalently, that a is a ***local maximum point*** of f) if there exists a neighborhood $K(a) \subset \mathcal{D}_f$ such that

$$\forall x \in K(a) \colon f(x) \le f(a).$$

In this case, the value $f(a)$ is called the ***local maximum*** of f.

Similarly, the notions of ***local minimum point*** and ***local minimum*** are defined analogously.

Necessary condition for local extrema. Assume that $f \in \mathbb{R}^2 \to \mathbb{R}$ and $a \in \operatorname{int} \mathcal{D}_f$. Additionally,

- $f \in D\{a\}$, and
- the function f has a local extremum at the point a.

Then $f'(a) = 0$, i.e., $f'(a) = \begin{pmatrix} \partial_1 f(a) & \partial_2 f(a) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$.

Sufficient condition for local extrema. Let $f \in \mathbb{R}^2 \to \mathbb{R}$, $a \in \operatorname{int} \mathcal{D}_f$, and $f \in C^2\{a\}$. Assume that

$$\partial_1 f(a) = 0 \quad \text{and} \quad \partial_2 f(a) = 0.$$

Define

$$D(a) := \det \begin{pmatrix} \partial_{11} f(a) & \partial_{12} f(a) \\ \partial_{21} f(a) & \partial_{22} f(a) \end{pmatrix}.$$

Then:

1. If $D(a) > 0$ and $\partial_{11}f(a) > 0$ [or $\partial_{11}f(a) < 0$], then f has a local minimum [or maximum] at a.
2. If $D(a) < 0$, then f has no local extremum at a (this is called a saddle point).
3. If $D(a) = 0$, this test does not determine whether a is a local extremum or not.

Exercise 2. *Find the local extreme values and their places of the following functions.*

$$\text{a)}\quad f(x,y) := x^2 - 4xy + y^3 + 4y \qquad \big((x,y) \in \mathbb{R}^2\big),$$

$$\text{b)}\quad f(x,y) := x^4 - 4xy + y^4 \qquad \big((x,y) \in \mathbb{R}^2\big).$$

Solution The functions in the exercise are twice continuously differentiable on $\mathbb{R}^2$ because they are bivariate polynomials.

a) $f(x,y) := x^2 - 4xy + y^3 + 4y \qquad \big((x,y) \in \mathbb{R}^2\big)$

Necessary condition:

$$\left.\begin{array}{rcccl} \partial_1 f(x,y) & = & 2x - 4y & = & 0 \\ \partial_2 f(x,y) & = & -4x + 3y^2 + 4 & = & 0 \end{array}\right\} \quad \Longrightarrow \quad x = 2y \quad \Longrightarrow \quad -4(2y)+3y^2+4 = 0.$$

Hence

$$3y^2 - 8y + 4 = (y-2)(3y-2) = 0 \quad \Longrightarrow \quad y = 2 \ \text{ or } \ y = \frac{2}{3},$$

Therefore, the stationary points of the f function, i.e., the possible locations of local extrema, are:

$$P_1(4,2), \quad P_2\Big(\tfrac{4}{3}, \tfrac{2}{3}\Big).$$

Sufficient condition: The second-order partial derivatives at a point $(x,y) \in \mathbb{R}^2$ are

$$\partial_{11}f(x,y) = 2, \quad \partial_{12}f(x,y) = -4 = \partial_{21}f(x,y), \quad \partial_{22}f(x,y) = 6y.$$

Hence

$$f''(x,y) = \begin{pmatrix} \partial_{11}f(x,y) & \partial_{12}f(x,y) \\ \partial_{21}f(x,y) & \partial_{22}f(x,y) \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & 6y \end{pmatrix},$$

$$D(x,y) = \det f''(x,y) = 12y - 16, \quad \partial_{11}f(x,y) = 2.$$

At the point $P_1(4,2)$, $D(4,2) = 12 \cdot 2 - 16 = 8 > 0$ and $\partial_{11}f(4,2) = 2 > 0$. Therefore P_1 is a *local minimum point*, and the minimum value is $f(4,2) = 0$.

At the point $P_2\big(\frac{4}{3}, \frac{2}{3}\big)$, $D\big(\frac{4}{3}, \frac{2}{3}\big) = 12 \cdot 2/3 - 16 = -8 < 0$. Therefore the function f has *no local extremum* at P_2.

b) $f(x,y) := x^4 - 4xy + y^4 \qquad \big((x,y) \in \mathbb{R}^2\big)$

Necessary condition:

$$\left.\begin{array}{rcccl} \partial_1 f(x,y) & = & 4x^3 - 4y & = & 0 \\ \partial_2 f(x,y) & = & -4x + 4y^3 & = & 0 \end{array}\right\} \quad \Longrightarrow \quad y = x^3 \quad \Longrightarrow \quad -x + (x^3)^3 = 0.$$

Hence
$$-x + x^9 = x(1 - x^8) = 0 \qquad \Longrightarrow \qquad x = 0, \quad x = 1 \text{ or } x = -1,$$
Therefore, the stationary points of the f function, i.e., the possible locations of local extrema, are:
$$P_1(0,0), \quad P_2(1,1), \quad P_3(-1,-1).$$

Sufficient condition: The second-order partial derivatives at a point $(x, y) \in \mathbb{R}^2$ are
$$\partial_{11} f(x,y) = 12x^2, \quad \partial_{12} f(x,y) = -4 = \partial_{yx} f(x,y), \quad \partial_{22} f(x,y) = 12y^2.$$

Hence
$$f''(x,y) = \begin{pmatrix} \partial_{11} f(x,y) & \partial_{12} f(x,y) \\ \partial_{21} f(x,y) & \partial_{22} f(x,y) \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix},$$
$$D(x,y) = \det f''(x,y) = 144x^2y^2 - 16, \quad \partial_{11} f(x,y) = 12x^2.$$

At the point $P_1(0,0)$, $D(0,0) = -16 < 0$. Therefore the function f has *no local extremum* at P_0.

At the point $P_2(1,1)$, $D(1,1) = 144 - 16 < 0$ and $\partial_{11} f(1,1) = 12 > 0$. Therefore P_1 is a *local minimum point*, and the minimum value is $f(1,1) = -2$.

At the point $P_3(-1,-1)$, $D(-1,-1) = 144 - 16 < 0$ and $\partial_{11} f(-1,-1) = 12 > 0$. Therefore P_3 is a *local minimum point*, and the minimum value is $f(1,1) = -2$.

Global extrema of $\mathbb{R}^2 \to \mathbb{R}$ functions on compact sets

Theoretical notes

The statements about finding global extrema for real-valued functions can be generalized to the case of multi-variable functions.

Weierstrass theorem: Assume that H is a bounded and closed set in the Euclidean space $\mathbb{R}^2$, and the function $f \colon H \to \mathbb{R}$ is continuous. Then the function f has global extremum points, i.e.,

- $\exists x_1 \in H,\ \forall x \in H \colon f(x) \le f(x_1)$ $\quad$ (x_1 is an global maximum point),
- $\exists x_2 \in H,\ \forall x \in H \colon f(x) \ge f(x_2)$ $\quad$ (x_2 is an global minimum point).

Let $H \subset \mathbb{R}^2$ be a compact, i.e. a bounded and closed set. Assume that the function $f \colon H \to \mathbb{R}$ is continuous, and also differentiable at every interior point of H. Then f attains its largest (smallest) value either on the boundary of the set H, or at an stationary point a ($\partial_1 f(a) = 0$ and $\partial_2 f(a) = 0$) in the interior of H.

Exercise 3. *Find the global extreme values and their places of the following function on the set H.*
$$f(x,y) := 2xy - 3y, \qquad H := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 0 \le x \le 2,\ 0 \le y \le x^2 \right\}$$

Solution The set H is the plane region enclosed by the parabola $y = x^2$ and the lines $y = 0$ and $x = 0$, as shown in the figure. This set is bounded and closed. The function f is continuous on H, so by the Weierstrass theorem, the values of the function on H include both a maximum and a minimum.

The function f is differentiable in the interior of the set H. The global extrema of f may occur either at interior points of H (which must then also be stationary points) or on the boundary of H. First, we determine the stationary points of the function within the interior of H. Since

$$\left.\begin{array}{rclcl} \partial_x f(x,y) & = & 2y & = & 0 \\ \partial_y f(x,y) & = & 2x-3 & = & 0 \end{array}\right\} \quad \Longrightarrow \quad x = 3/2 \;\text{ and }\; y = 0,$$

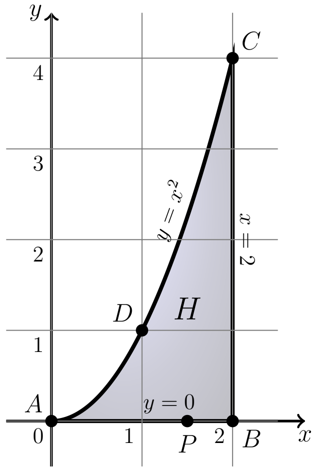
the stationary point of f is

$$P(\tfrac{3}{2}, 0).$$

However, P is not in the interior of H. Therefore, the global extrema of f on H are on the boundary of H.

Next, we examine the values of f along the boundary of H. The boundary of the set H can be divided into three parts. The task is to determine the global extrema of f on each part. To do this, we express the values of f along each part of the boundary using a real-valued function g defined on a closed interval. The possible global extrema of g are either at stationary points within the interval (where $g' = 0$) or at the endpoints of the interval. Denote

$$A(0,0), \qquad B(2,0), \qquad C(4,2).$$



Segment AB: $y = 0$, where $0 \le x \le 2$. Then, the function

$$g_1(x) := f(x,0) = 2x \cdot 0 - 3 \cdot 0 = 0 \qquad \big(x \in [0,2]\big)$$

is constant. Therefore, a global extremum for f may occur at any point along the segment AB.

Segment BC: $x = 2$, where $0 \le y \le 4$. Then, the function

$$g_2(y) := f(2,y) = 2 \cdot 2 \cdot y - 3 \cdot y = y \qquad \big(y \in [0,4]\big)$$

is increasing, so it has global extrema at $y = 0$ and $z = 4$. Therefore, a global extremum for f may occur at the points B and C.

Piece of a parabola AB: $y = x^2$, where $0 \le x \le 2$. Then, the function

$$g_3(x) := f(x,x^2) = 2x \cdot x^2 y - 3x^2 = 2x^3 - 3x^2 \qquad \big(x \in [0,2]\big)$$

has stationary point where the condition $g_3'(x) = 0$ holds:

$$g_3'(x) = 6x^2 - 6x = 6x(x-1) = 0 \qquad \Longrightarrow \qquad x = 0 \quad \text{or} \quad x = 1,$$

but $x = 0$ is not in the interior of the interval $[0,2]$. The value $x = 1$ leads us to point $D(1,1)$, where a global extremum for f may occur. The values $x = 0$ and $x = 2$ may also be considered because they are the boundaries of the interval $[0,2]$, but these lead us to points A and C, which have already been considered.

Finally, we compare all the values of f at the previously mentioned points where global extrema might occur.

- $f(x,0) = 0 \qquad \big(x \in [0,2]\big),$
- $f(4,2) = 4 \quad \Longrightarrow \quad f$ has a global maximum at the point C.
- $f(1,1) = -1 \quad \Longrightarrow \quad f$ has a global minimum at the point D.

Exercise 4. *Determine the absolute extremum points and the absolute extrema of the function*

$$f(x,y) := x^3 - 12x + y^3 - 3y \qquad \big((x,y) \in \mathbb{R}^2\big)$$

on the following set:

$$H := \Big\{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 3, \quad -x \le y \le 2\Big\}.$$

Solution The set H is a bounded and closed triangle with vertices $A(-2,2)$, $B(3,-3)$, and $C(3,2)$. The function f is continuous on H, so by the Weierstrass theorem, the function attains both a maximum and a minimum value on H.

The function f is differentiable in the interior of H because it is differentiable at every point in $\mathbb{R}^2$. The absolute extremum points can either be in the interior of H (in which case they are also stationary points) or on the boundary of H.

First, we determine the stationary points of the function in the interior of H. Since

$$\left.\begin{array}{rcccl} \partial_x f(x,y) & = & 3x^2 - 12 & = & 0 \\ \partial_y f(x,y) & = & 3y^2 - 3 & = & 0 \end{array}\right\} \quad \Longrightarrow \quad x = \pm 2 \ \text{ and } \ y = \pm 1,$$

the stationary points of f are:

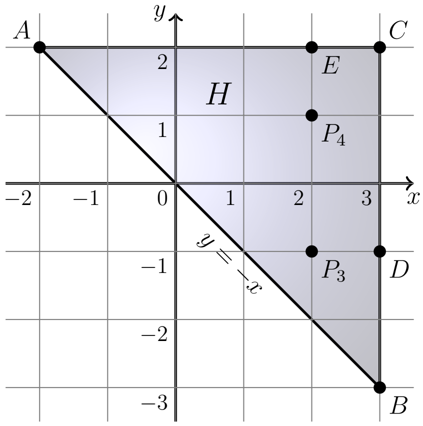
$$P_1(-2,-1), \quad P_2(-2,1), \quad P_3(2,-1), \quad P_4(2,1),$$

but only P_3 and P_4 are in the interior of H. It is not necessary to determine which of these is a local extremum point since there are only two points. It is sufficient to calculate

$$f(P_3) = f(2,-1) = -14 \quad \text{and} \quad f(P_4) = f(2,1) = -18$$

and compare these values with those at other possible absolute extremum points.

Next, we examine the values of f on the boundary of H. The boundary of H can be divided into three segments. The task is to determine the absolute extremum points of f on each segment. For this, we represent the values of f on a given segment using a real-valued function g defined on a closed interval. The possible absolute extremum points of g are either the stationary points within the interval (where $g' = 0$) or the endpoints of the interval.



$\underline{\text{Segment } AB}$: $y=-x$, where $-2 \le x \le 3$. In this case,

$$g_1(x) := f(x,-x) = -9x \qquad \big(x \in [-2,3]\big)$$

is a strictly decreasing function. Therefore, $x=-2$ and $x=3$ are the absolute extremum points of g_1, corresponding to points A and B:

$$f(A) = f(-2,2) = 18 \quad \text{and} \quad f(B) = f(3,-3) = -27.$$

$\underline{\text{Segment } BC}$: $x=3$, where $-3 \le y \le 2$. In this case,

$$g_2(y) := f(3,y) = y^3 - 3y - 9 \qquad \big(y \in [-3,2]\big).$$

Since

$$g_2'(y) = 3y^2 - 3 = 0 \qquad \Longrightarrow \qquad y = \pm 1,$$

the possible absolute extremum points of g_2 are $y=-3$, $y=-1$, $y=1$, and $y=2$. Their values are:

$$g_2(-3) = -27, \quad g_2(-1) = -7, \quad g_2(1) = -11, \quad g_2(2) = -7.$$

Therefore, $y=-3$, $y=-1$, and $y=2$ are the absolute extremum points of g_2, corresponding to points B, $D(3,-1)$, and C, respectively:

$$f(D) = f(3,-1) = -7 \quad \text{and} \quad f(C) = f(3,2) = -7.$$

$\underline{\text{Segment } AC}$: $y=2$, where $-2 \le x \le 3$. In this case,

$$g_3(x) := f(x,2) = x^3 - 12x + 2 \qquad \big(x \in [-2,3]\big).$$

Since

$$g_3'(x) = 3x^2 - 12 = 0 \qquad \Longrightarrow \qquad x = \pm 2,$$

the possible absolute extremum points of g_3 are $x=-2$, $x=2$, and $x=3$. Their values are:

$$g_3(-2) = 18, \quad g_3(2) = -14, \quad g_3(3) = -7.$$

Therefore, $x=-2$ and $x=2$ are the absolute extremum points of g_3, corresponding to points A and $E(2,2)$, respectively:

$$f(E) = f(2,2) = -14.$$

Comparing the obtained values

$$f(P_3) = -14,\; f(P_4) = -18,\; \underline{\underline{f(A) = 18}},\; \underline{\underline{f(B) = -27}},\; f(C) = -7,\; f(D) = -7,\; f(E) = -14$$

we see that the absolute maximum of f on H occurs at $A(-2,2)$ with $f(-2,2) = 18$, and the absolute minimum occurs at $B(3,-3)$ with $f(3,-3) = -27$. Thus,

$$\boxed{\min_{(x,y)\in H} f(x,y) = f(3,-3) = -27}, \qquad \boxed{\max_{(x,y)\in H} f(x,y) = f(-2,2) = 18}.$$