Analysis I Test 2

① $$\sum_{n=1}^{+\infty} \frac{2}{(2n+1)(2n+3)} = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \frac{2}{(2k+1)(2k+3)}}_{s_n} \Rightarrow$$

$$s_n = \sum_{k=1}^{n} \frac{(2k+3)-(2k+1)}{(2k+1)(2k+3)} =$$

$$= \sum_{k=1}^{n} \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right) = \frac{1}{3} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{9}} + \dots + \cancel{\frac{1}{2n+1}} - \frac{1}{2n+3} =$$

$$= \frac{1}{3} - \frac{1}{2n+3} \to \boxed{\frac{1}{3}} \text{ as } (n\to\infty). \text{ So } \boxed{\sum_{n=1}^{+\infty} \frac{2}{(2n+1)(2n+3)} = \frac{1}{3}}$$

② a) $$\sum_{n=0}^{+\infty} \frac{n+2}{(n^2+1)^2} = \sum_{n=0}^{+\infty} \frac{n+2}{n^4+2n^2+1} \approx \sum_{n=1}^{+\infty} \frac{n}{n^4} = \sum_{n=1}^{+\infty} \frac{1}{n^3}$$

with $\alpha = 3 > 1$ so it is conv.

Proof:

$$0 \le \frac{n+2}{n^4+2n^2+1} \underset{\text{if } n\ge 1}{\le} \frac{n+2n}{n^4} = \frac{3}{n^3}$$ and

$$\sum_{n=1} \frac{3}{n^3} = 3 \cdot \sum_{n=1} \frac{1}{n^3}$$ is convergent

$\Rightarrow$ Comp. Test.

given series is $\boxed{\text{conv}}$.

b) $$\sum_{n=0}^{!} \frac{(3n)!}{8^n (n!)^3} =: a_n$$

Ratio Test:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(3n+3)!}{8^{n+1}((n+1)!)^3}\cdot\frac{8^n(n!)^3}{(3n)!} =$$

$$= \lim_{n\to\infty}\frac{(3n+1)(3n+2)(3n+3)}{8\cdot(n+1)^3} = \frac{27}{8} > 1 \implies$$ Ratio Test.

$\sum_{n=0} a_n$ is divergent.

c) $\sum_{n=1}^{+\infty}\left(\frac{7n-5}{7n+2}\right)^{n^2} =: a_n$ Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{7n-5}{7n+2}\right)^{n^2}\right|} =$$

$\oplus$ if $1 \le n \in \mathbb{N}$

$$= \lim_{n\to+\infty}\left(\frac{7n-5}{7n+2}\right)^n = \lim_{n\to\infty}\left[\frac{\cancel{7n}\left(1-\frac{5}{7n}\right)}{\cancel{7n}\left(1+\frac{2}{7n}\right)}\right]^n =$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{-\frac{5}{7}}{n}\right)^n}{\left(1+\frac{\frac{2}{7}}{n}\right)^n} = \frac{e^{-\frac{5}{7}}}{e^{\frac{2}{7}}} = e^{-1} = \frac{1}{e} < 1 \implies$$

(Root test): $\sum_{n=1} a_n$ is absolutely conv. $\implies$ is conv. too.

(3) $\boxed{\sum_{n=0}^{+\infty} \frac{1}{(3n+2)\cdot 2^n}\cdot(x-3)^n}$ $\boxed{(x\in\mathbb{R})}$

$$= \sum_{n=0}^{+\infty} a_n\cdot(x-a)^n \Rightarrow \boxed{a=3} \text{ Center.} \quad a_n=\frac{1}{(3n+2)\,2^n}$$

- Radius of cnv:

$$\boxed{R}=\frac{1}{\lim_{n\to\infty}\sqrt[n]{\left|\underbrace{\frac{1}{(3n+2)\,2^n}}_{(+)}\right|}} = \lim_{n\to\infty} 2\cdot\sqrt[n]{3n+2} =$$

$$=2\cdot\lim_{n\to\infty}\left(\sqrt[n]{3n+2}\right) \underset{\text{since}}{=} 2\cdot 1 = \boxed{2}.$$

$$\sqrt[n]{2} \underset{\substack{\text{for}\\ n\in\mathbb{N}}}{\le} \sqrt[n]{3n+2} \underset{\text{if } n\ge 1}{\le} \sqrt[n]{5n} = \sqrt[n]{5}\cdot\sqrt[n]{n}$$

(as $n\to\infty$) $\sqrt[n]{2}\to\boxed{1}$; $\sqrt[n]{3n+2}\to 1$ by sandwich theorem; $\sqrt[n]{5}\cdot\sqrt[n]{n}\to 1\cdot 1 = 1$ (as $n\to\infty$)

- By Cauchy-Hadamard theorem:

1) If $|x-3|<2 \iff x\in(1,5) \Rightarrow$ the power series is abs. cnv. $\Downarrow$ is cnv. too.

$$\underset{1}{(}\overset{-2}{\frown}\ \underset{3}{\cdot}\ \overset{+2}{\frown}\underset{5}{)}$$

2) If $|x-3|>2 \iff x\in(-\infty,1)\cup(5;+\infty)$

Then p. series is div.

For

3) $\boxed{x=1} \Rightarrow \sum_{n=0}^{+\infty} \frac{1}{(3n+2)\cdot 2^n}\cdot(1-3)^n = \sum_{n=0}^{\infty} (-1)^n \cdot \underbrace{\frac{1}{3n+2}}_{A_n}$

with $0 \le \frac{1}{3(n+1)+2} \le \frac{1}{3n+2}$ ✓ $(\forall n \in \mathbb{N})$

$0 \le A_{n+1} \le A_n$

and $\lim_{n\to\infty}(A_n) = \lim_{n\to\infty}\left(\frac{1}{3n+2}\right) = 0$ $\Rightarrow \sum_0 \frac{(-1)^n}{3n+2}$ is a conv. Leibniz series.

4) $\boxed{x=5} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{(3n+2)\cdot 2^n}\cdot(5-3)^n = \sum_{n=0}^{\infty} \frac{1}{3n+2}$

which is divergent, since

$$\frac{1}{3n+2} \underset{\text{as } n\ge 1}{\ge} \frac{1}{3n+2n} = \frac{1}{5n} \ge 0$$

and $\sum_{n=1}^{+\infty} \frac{1}{5n} = \frac{1}{5}\cdot \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$.

divergent harmonic series $\underset{\text{comparison test}}{\Longrightarrow}$

$\sum_{n=0}^{\infty} \frac{1}{3n+2}$ is divergent.

So Set of convergence: $\boxed{S=[1,5)}$

0

14) $\boxed{\lim_{x\to 1}\left(\frac{2x^2+1}{3x^2+2}\right)=\frac{3}{5}}$ $\overset{def}{\iff}$ Need to prove:

$$\boxed{\forall \varepsilon>0\ \exists \delta>0\ \forall x\in D_f=\mathbb{R}: 0<|x-1|<\delta \Rightarrow \left|f(x)-\frac{3}{5}\right|<\varepsilon}$$

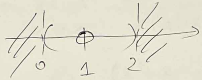
Fix an $\varepsilon>0$; for all $x\in\mathbb{R}$:

$$\left|f(x)-\frac{3}{5}\right| = \left|\frac{2x^2+1}{3x^2+2}-\frac{3}{5}\right| = \frac{|10x^2+5-9x^2-6|}{|\underbrace{15x^2+10}_{+}|} =$$

$$= \frac{|x^2-1|}{15x^2+10} = |x-1|\cdot\frac{|x+1|}{15x^2+10} \le (*) \le$$

Assume, that: $0<|x-1|<1$ so

$x\in(0,2)\setminus\{1\}$.



If $0<x<2 \Rightarrow$

$1<x+1<3 \Rightarrow$

$1<\boxed{|x+1|<3}$

And $0<x<2 \Rightarrow 0<x^2<4 \quad |\cdot 15$

$0<15x^2<60 \quad |+10$

$10<15x^2+10<70$

$\Downarrow$

$$\boxed{\frac{1}{15x^2+10}<\frac{1}{10}}$$

$(*) \le |x-1| \cdot |x+1| \cdot \frac{1}{15x^2+10} \le |x-1| \cdot 3 \cdot \frac{1}{10}$

if: $0 < |x-1| < 1$

$= \frac{3}{10} \cdot |x-1| \overset{?}{<} \varepsilon \iff |x-1| < \frac{10\varepsilon}{3}$

So if we choose $\boxed{\delta := \min\left\{1 \, ; \, \frac{10\varepsilon}{3}\right\} > 0}$
the definition is satisfied.

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5 a) $\lim\limits_{x\to 1} \frac{\sqrt{x^2+x} - \sqrt{2}}{\sin(2x-2)} = \frac{0}{0} =$

$= \lim\limits_{x\to 1} \frac{(\sqrt{x^2+x} - \sqrt{2})(\sqrt{x^2+x} + \sqrt{2})}{\sin(2x-2)\cdot(\sqrt{x^2+x} + \sqrt{2})} =$

$= \lim\limits_{x\to 1} \frac{(x^2+x-2)}{\sin(2x-2)\cdot(\sqrt{x^2+x} + \sqrt{2})} =$

$= \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+2)}{\frac{\sin(2x-2)}{2x-2} \cdot \frac{\cancel{(2x-2)}}{2} \cdot (\sqrt{x^2+x} + \sqrt{2})} =$

with $2x-2 \to 0$

$\frac{\sin(2x-2)}{2x-2} \to 1$

$= \frac{1+2}{1 \cdot 2 \cdot (\sqrt{2}+\sqrt{2})} = \boxed{\frac{3}{4\sqrt{2}}}$

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b) $\lim\limits_{x\to 0} \frac{1-\cos(4x)}{e^{2x} - 2e^x + 1} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{(1-\cos(4x))\cdot(4x)^2}{(4x)^2 \cdot (e^x-1)^2}$

$$= \lim_{\substack{x \to 0 \\ 4x \to 0}} \left(\frac{1-\cos(4x)}{(4x)^2} \right) \cdot \lim_{x \to 0} \frac{1}{\left(\frac{e^x - 1}{x}\right)^2 \cdot \underset{1}{\cancel{x^2}}} \cdot \left(4^2 \cdot \underset{1}{\cancel{x^2}}\right) =$$

$$= \frac{1}{2} \cdot \frac{1}{1^2} \cdot 16 = \frac{16}{2} = \boxed{8}$$