APPLICATIONS OF INTEGRATION

The definite integral

Theoretical notes In the introductory part of indefinite integrals we referred to the area under the graph of functions, but we only used an intuitive approach to the concept of area. With differential calculus, we gave an analytical interpretation to several geometric concepts (e.g., the tangent line). It is a valid question whether we could proceed similarly with the area under the graph of functions.

The precise task is as follows: let f be a non-negative, bounded (***not necessarily continuous***) function on a closed interval $[a, b]$, where $a, b \in \mathbb{R}$, and consider the region under the graph of f

$$A := \big\{(x, y) \;\big|\; x \in [a, b],\; 0 \le y \le f(x)\big\}$$

How should we interpret the area of A, and how could we calculate it?

We start from the rather ”natural” idea already used by the Greek mathematicians: approximating the area of the region in question by the sum of the areas of rectangles. Let us see how Archimedes determined the area under a parabola. We will use modern notation and concepts we have already learned.

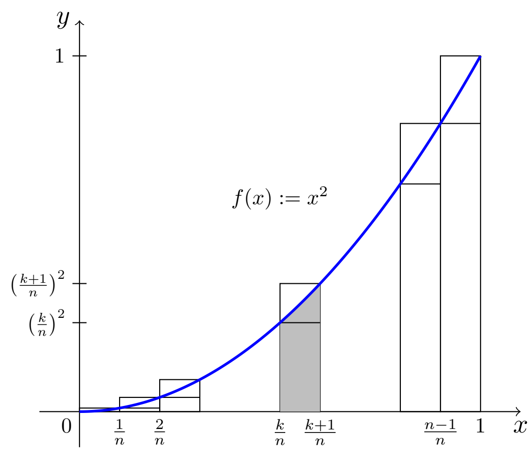
Consider the function

$$f(x) := x^2 \qquad \big(x \in [0, 1]\big)$$

Fix a number $1 \le n \in \mathbb{N}$, and divide the interval $[0, 1]$ using the points

$$x_k := \frac{k}{n} \qquad (k = 0, 1, 2, \dots, n)$$

We will approximate the area under the graph of f with the ”inscribed” and ”circumscribed” rectangles illustrated in the figure.



For a fixed index $0 \le k < n$, let us now focus on the rectangles related to the interval

$$I_k := [x_k, x_{k+1}] = \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Due to the fact that f is strictly increasing, we notice that

$$m_k := \inf\big\{f(x) \;\big|\; x \in I_k\big\} = x_k^2 = \left(\frac{k}{n}\right)^2 \qquad \text{and} \qquad M_k := \sup\big\{f(x) \;\big|\; x \in I_k\big\} = x_{k+1}^2 = \left(\frac{k+1}{n}\right)^2.$$

Therefore, the areas of the inscribed and circumscribed rectangle related to I_k are respectively

$$m_k(x_{k+1} - x_k) = \frac{k^2}{n^2} \cdot \frac{1}{n} = \frac{k^2}{n^3} \qquad \text{and} \qquad M_k(x_{k+1} - x_k) = \frac{(k+1)^2}{n^2} \cdot \frac{1}{n} = \frac{(k+1)^2}{n^3}.$$

We sum separately the area of the inscribed and circumscribed rectangles related to all the intervals I_k.

$$s_n := \sum_{k=0}^{n-1} \frac{k^2}{n^3} = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2 = \frac{1}{n^3}\sum_{k=1}^{n-1} k^2 \qquad \text{and} \qquad S_n := \sum_{k=0}^{n-1} \frac{(k+1)^2}{n^3} = \frac{1}{n^3}\sum_{k=0}^{n-1} (k+1)^2 = \frac{1}{n^3}\sum_{k=1}^{n} k^2.$$

Then, from the identity

$$\sum_{k=1}^{m} k^2 = 1^2 + 2^2 + 3^2 + \dots + m^2 = \frac{m(m+1)(2m+1)}{6} \qquad (1 \le m \in \mathbb{N})$$

we obtain

$$s_n := \frac{1}{n^3}\sum_{k=1}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6n^3} \xrightarrow[n\to+\infty]{} \frac{1}{3} \qquad \text{and} \qquad S_n := \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \xrightarrow[n\to+\infty]{} \frac{1}{3}.$$

Denote by $T(A)$ the area of the set A, that is the area under the graph of f between 0 and 1. The inscribed rectangles only ”touch” each other, and they are all contained in A. On the other hand, A is contained in the union of the circumscribed rectangles. This implies that

$$s_n \le T(A) \le S_n,$$

hence by the squeeze theorem for sequences we deduce that $T(A) = 1/3$. It is therefore natural to say that the set A has an area, and it is $1/3$.

The above construction inspires us to do the same with any bounded function that is defined on a closed interval of finite length. Let $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a bounded function which is not necessarily to be continuous or positive. By a ***partition*** of the interval $[a, b]$, we mean the set

$$\tau := \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

The elements of a partition do not have to be equidistant. Denote $I_k := [x_k, x_{k+1}]$, and

$$m_k := \inf\{f(x) \mid x \in I_k\}, \qquad \text{and} \qquad M_k := \sup\{f(x) \mid x \in I_k\}$$

for all index $k = 0, 1, \ldots, n-1$. The sums

$$s(f, \tau) := \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k), \qquad \text{and} \qquad S(f, \tau) := \sum_{k=0}^{n-1} M_k(x_{k+1} - x_k)$$

are called the ***lower*** and ***upper Riemann sums*** of the function f with respect to the partition τ. Since $m_k \le M_k$ for all $k = 0, 1, \ldots, n-1$, we have $s(f, \tau) \le S(f, \tau)$ for any partition τ.

Denoted the set of all partitions of the interval $[a, b]$ by $\boldsymbol{\mathcal{F}[a, b]}$. From the boundedness of the function f follows that

$$A := \inf\{f(x) \mid x \in [a, b]\} > -\infty \qquad \text{and} \qquad B := \sup\{f(x) \mid x \in [a, b]\} < +\infty.$$

It is not hard to see, that $m_k \ge A$ and $M_k \le B$ for all $k = 0, 1, \ldots, n-1$. Thus

$$-\infty < A(b-a) = A\sum_{k=0}^{n-1}(x_{k+1} - x_k) = \sum_{k=0}^{n-1} A(x_{k+1} - x_k) \le \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k) = s(f, \tau)$$

$$\le S(f, \tau) = \sum_{k=0}^{n-1} M_k(x_{k+1} - x_k) \le \sum_{k=0}^{n-1} B(x_{k+1} - x_k) = B\sum_{k=0}^{n-1}(x_{k+1} - x_k) = B(b-a) < +\infty,$$

since the sum of the length of all intervals $I_k = (x_k, x_{k+1})$ is $(b - a)$. Then, the sets

$$\{s(f, \tau) \mid \tau \in \mathcal{F}[a, b]\} \qquad \text{and} \qquad \{S(f, \tau) \mid \tau \in \mathcal{F}[a, b]\}$$

are bounded, consequently, they have a finite infimum and supremum. The real numbers

$$I_*(f) := \sup\{s(f, \tau) \mid \tau \in \mathcal{F}[a, b]\}, \quad \text{and} \quad I^*(f) := \inf\{S(f, \tau) \mid \tau \in \mathcal{F}[a, b]\}$$

are called the ***Darboux lower integral*** and ***Darboux upper integral*** of the function f, respectively. It can be shown that

$$I_*(f) \le I^*(f).$$

We say that f is **Riemann-integrable** on the interval $[a, b]$ (in short, ***integrable*** on $[a, b]$) if $I_*(f) = I^*(f)$. In this case, the number

$$\int_a^b f := \int_a^b f(x)\, dx := I_*(f) = I^*(f)$$

is called the ***Riemann integral*** (or ***definite integral***) of the function f over the interval $[a, b]$. The set of all Riemann-integrable functions on the interval $[a, b]$ will be denoted by the symbol $\boldsymbol{R[a, b]}$.

If the bounded function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable on the interval $[a, b]$ and $f(x) \ge 0$ ($x \in [a, b]$), then we define ***the area under the graph of f***, as follows:

$$T(A) := \int_a^b f(x)\, dx.$$

It is easy to give an example of a bounded function f for which $I_*(f) < I^*(f)$, meaning that the function is ***not integrable***. For example, let

$$f(x) := \begin{cases} 0 & \big(x \in [0, 1] \cap \mathbb{Q}\big) \\ 1 & \big(x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})\big). \end{cases}$$

In this case, $I_*(f) = 0$ and $I^*(f) = 1$, so $f \notin R[0, 1]$. However, if a function f is continuous on an interval $[a, b]$ of finite length, then it is integrable on this interval.

Newton–Leibniz Formula: Let $a, b \in \mathbb{R}$ and suppose that

- f is integrable on $[a, b]$ and
- the function f has a primitive function on the interval $[a, b]$.

Then

$$\int_a^b f(x)\,dx = F(b) - F(a) =: \big[F(x)\big]_a^b,$$

where F is a primitive function of f.

Exercise 1. *Find the following definite integrals.*

$$a)\ \int_0^{\pi/3} \sin 8x\,dx, \qquad b)\ \int_0^1 \frac{1}{3x-5}\,dx, \qquad c)\ \int_1^e \frac{\ln^2 x}{x}\,dx, \qquad d)\ \int_0^1 xe^{-x^2}\,dx.$$

Solution The usual way for computing a definite integral is to find first a primitive function of the integrand, that is, to evaluate the indefinite integral, and then we use Newton–Leibniz formula to obtain the result. In simple cases the primitive function is not difficult to find, so it is not necessary to evaluate the indefinite integral separately.

a) Using linear transformation

$$\int_0^{\pi/3} \sin 8x\,dx = \left[\frac{-\cos 8x}{8}\right]_0^{\pi/3} = -\frac{1}{8}\Big(\cos(\tfrac{8\pi}{3}) - \cos 0\Big) = \frac{1}{8}\Big(\cos 0 - \cos(\tfrac{2\pi}{3})\Big)$$

$$= \frac{1}{8}\Big(1 - (-\tfrac{1}{2})\Big) = \frac{1}{8}\cdot\frac{3}{2} = \underline{\underline{\frac{3}{16}}}\ .$$

b) Using linear transformation

$$\int_0^1 \frac{1}{3x-5}\,dx = \left[\frac{\ln|3x-5|}{3}\right]_0^1 = \frac{1}{3}\Big(\ln|-2| - \ln|-5|\Big) = \frac{1}{3}\Big(\ln 2 - \ln 5\Big) = \underline{\underline{\frac{1}{3}\,\ln\tfrac{2}{5}}}\ .$$

c) Note that the integrand is of type $f^\alpha f'$ where $\alpha = 2$. Thus

$$\int_1^e \frac{\ln^2 x}{x}\,dx = \int_1^e (\ln x)^2\,\frac{1}{x}\,dx = \left[\frac{\ln^3 x}{3}\right]_1^e = \frac{1}{3}\Big(\ln^3 e - \ln 1\Big) = \frac{1}{3}(1^3 - 0) = \underline{\underline{\frac{1}{3}}}\ .$$

d) We use the general formula for reversing differentiation of composite Functions:

$$\int f\big(g(x)\big)\cdot g'(x)\,dx = F\big(g(x)\big) + c$$

where F is a primitive function of f. In our case, we take $f(x) = F(x) = e^x$ and $g(x) = -x^2$, $g'(x) = -2x$. Then

$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 e^{-x^2}\cdot(-2x)\,dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1 = -\frac{1}{2}\Big(e^{-1} - e^0\Big) = \underline{\underline{\frac{1}{2}\Big(1 - \frac{1}{e}\Big)}}\ .$$

Exercise 2. *Find the following definite integrals.*

$$a)\ \int_0^{\pi} x\cos x\,dx, \qquad b)\ \int_e^{e^2} x\ln x\,dx, \qquad c)\ \int_{\ln 4}^{\ln 8} \frac{e^x}{e^{2x}-4}\,dx, \qquad d)\ \int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx.$$

Solution Integration by parts and the use of substitution rule transform one integral into another, so it may be convenient to evaluate the indefinite integral first. However, it is also possible to use the definite integral in these transformations, as we can see in the solution of this exercise.

a) Using integration by parts we have

$$\begin{aligned}\int_0^{\pi} x\cos x\,dx &= \int_0^{\pi} x(\sin x)'\,dx = \Big[x\sin x\Big]_0^{\pi} - \int_0^{\pi} 1\cdot\sin x\,dx \\ &= (\pi\sin\pi - 0\cdot\sin 0) + \int_0^{\pi}(-\sin x)\,dx = 0 + \Big[\cos x\Big]_0^{\pi} \\ &= \cos\pi - \cos 0 = -1-1 = \underline{\underline{-2}}\ .\end{aligned}$$

b) Using integration by parts we have

$$\begin{aligned}\int_e^{e^2} x\ln x\,dx &= \int_e^{e^2}\left(\frac{x^2}{2}\right)'\ln x\,dx = \left[\frac{x^2}{2}\,\ln x\right]_e^{e^2} - \int_e^{e^2}\frac{x^2}{2}\cdot\frac{1}{x}\,dx \\ &= \frac{1}{2}\left(e^4\ln e^2 - e^2\ln e\right) - \frac{1}{2}\int_e^{e^2} x\,dx = \frac{1}{2}\left(2e^4-e^2\right) - \frac{1}{2}\left[\frac{x^2}{2}\right]_e^{e^2} \\ &= \frac{1}{2}\left(2e^4-e^2\right) - \frac{1}{4}\left(e^4-e^2\right) = \underline{\underline{\frac{3e^4}{4} - \frac{e^2}{4}}}\ .\end{aligned}$$

c) We apply the substitution $t = e^x$. Then

$$x = \ln t =: g(t).$$

and

$$\ln 4 \le x \le \ln 8 \qquad \Longrightarrow \qquad 4 = \ln e^{\ln 4} \le t \le e^{\ln 8} = 8.$$

The function g is differentiable, and

$$g'(t) = \frac{1}{t} > 0 \qquad (4 < t < 8),$$

which implies that g is strictly increasing, hence invertible, were

$$g^{-1}(x) = e^x = t \qquad (\ln 4 \le x \le \ln 8).$$

Using substitution rule, we get

$$\int_{\ln 4}^{\ln 8}\frac{e^x}{e^{2x}-4}\,dx = \int_4^8\frac{t}{t^2-4}\cdot\frac{1}{t}\,dt = \int_4^8\frac{1}{t^2-4}\,dt = \int_4^8\frac{1}{(t-2)(t+2)}\,dt.$$

By partial fraction decomposition

$$\frac{1}{(t-2)(t+2)} = \frac{A}{t-2} + \frac{B}{t+2} = \frac{A(t+2)+B(t-2)}{(t-2)(t-4)} = \frac{(A+B)t+2A-2B}{(t-2)(t-4)}.$$

Hence

$$\begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \qquad \Longrightarrow \qquad A=\frac{1}{4} \text{ and } B=-\frac{1}{4}.$$

Therefore

$$\int_4^8 \frac{1}{(t-2)(t+2)}\,dt = \int_4^8 \left(\frac{1}{4}\cdot\frac{1}{t-2} - \frac{1}{4}\cdot\frac{1}{t+2}\right)dt = \left[\frac{1}{4}\ln|t-2| - \frac{1}{4}\ln|t+2|\right]_4^8$$

$$= \frac{1}{4}\Big((\ln 6 - \ln 10) - (\ln 2 - \ln 6)\Big) = \frac{1}{4}\Big(2\ln 6 - \ln 10 - \ln 2\Big)$$

$$= \frac{1}{4}\ln\tfrac{36}{20} = \underline{\underline{\frac{1}{4}\ln\tfrac{9}{5}}}\ .$$

d) We apply the substitution $t = \arcsin x$. Then

$$x = \sin t =: g(t).$$

and

$$\frac{1}{2} \le x \le 1 \qquad \Longrightarrow \qquad \frac{\pi}{6} = \arcsin\tfrac{1}{2} \le t \le \arcsin 1 = \frac{\pi}{2}.$$

The function g is differentiable, and

$$g'(t) = \cos t > 0 \qquad \left(\tfrac{\pi}{6} < t < \tfrac{\pi}{2}\right),$$

which implies that g is strictly increasing, hence invertible, were

$$g^{-1}(x) = \arcsin x = t \qquad \left(\tfrac{1}{2} \le x \le 1\right).$$

Using substitution rule, we get

$$\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx = \int_{\pi/6}^{\pi/2} \frac{\sqrt{1-\sin^2 t}}{\sin^2 t}\cdot\cos t\,dt = \int_{\pi/6}^{\pi/2} \frac{\sqrt{\cos^2 t}}{\sin^2 t}\cdot\cos t\,dt = (\ \cos t > 0\ ) =$$

$$= \int_{\pi/6}^{\pi/2} \frac{\cos^2 t}{\sin^2 t}\,dt = \int_{\pi/6}^{\pi/2} \frac{1-\sin^2 t}{\sin^2 t}\,dt = \int_{\pi/6}^{\pi/2} \left(\frac{1}{\sin^2 t} - 1\right)dt$$

$$= \Big[-\cot t - t\Big]_{\pi/6}^{\pi/2} = \left(-\cot\tfrac{\pi}{2} - \frac{\pi}{2}\right) - \left(-\cot\tfrac{\pi}{6} - \frac{\pi}{6}\right)$$

$$= \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \underline{\underline{\sqrt{3} - \frac{\pi}{3}}}\ .$$

Improper Integrals

When defining the Riemann integral, our starting point was to consider only those functions f for which the following two conditions hold:

a) the domain of f is a ***bounded and closed*** $[\boldsymbol{a},\boldsymbol{b}]$ ***interval***,

b) the function $\boldsymbol{f}$ ***is bounded on*** $[\boldsymbol{a},\boldsymbol{b}]$.

The question arises whether the concept of the integral can be defined for functions that do not satisfy these conditions. A certain extension is provided by so-called improper integrals.

Definition *Let* $-\infty \le a < b < +\infty$ *and* $f:(a,b] \to \mathbb{R}$. *Suppose that* $f \in R[t,b]$ *for every* $t \in (a,b)$. *Consider the following limit:*

$$\lim_{t\to a+0} \int_t^b f(x)\,dx.$$

- *If the above limit exists and is finite, then we say that* **the function $\boldsymbol{f}$ is improperly integrable on $[\boldsymbol{a},\boldsymbol{b}]$** (*or* **the improper integral of $\boldsymbol{f}$ on $[\boldsymbol{a},\boldsymbol{b}]$ exists and converges**). *In this case, the number*

$$\int_a^b f := \int_a^b f(x)\,dx := \lim_{t\to a+0} \int_t^b f(x)\,dx$$

  is called **the improper integral of $\boldsymbol{f}$ on $[\boldsymbol{a},\boldsymbol{b}]$**.

- *If the above limit equals* $+\infty$ (*or* $-\infty$), *then we say that* **the improper integral of $\boldsymbol{f}$ on $[\boldsymbol{a},\boldsymbol{b}]$ exists but diverges**, *and in this case,* **the improper integral of $\boldsymbol{f}$ on $[\boldsymbol{a},\boldsymbol{b}]$** *is*

$$\int_a^b f := \int_a^b f(x)\,dx := \lim_{t\to a+0} \int_t^b f(x)\,dx = +\infty \qquad (or\ -\infty).$$

- *If the above limit does not exist, then we say that* **the improper integral of $\boldsymbol{f}$ on $[\boldsymbol{a},\boldsymbol{b}]$ diverges**.

Remarks

1. If $a = -\infty$, then the right-hand limit of a refers to the limit taken at $(-\infty)$.

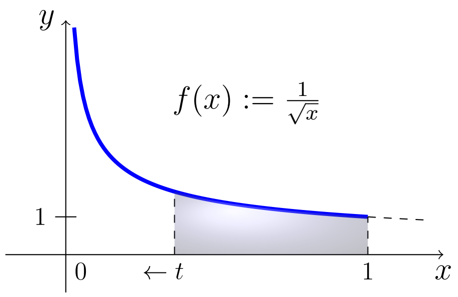
2. ***Caution!*** From the notation of the improper integral, it is not apparent that it is not the usual definite integral if $a \in \mathbb{R}$. However, it is not difficult to see that if $a \in \mathbb{R}$ and $f \in R[a,b]$, then the improper integral coincides with the usual definite integral.

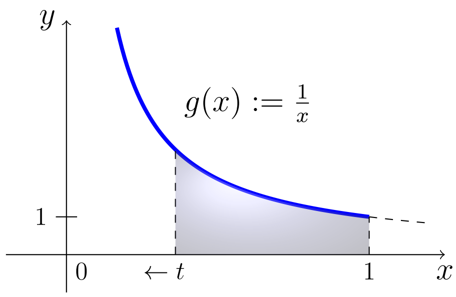
3. The improper integral can also be used to define the area of certain unbounded plane regions.

Examples. Let

$$f(x) := \frac{1}{\sqrt{x}} \quad \big(x \in (0,1]\big) \qquad \text{and} \qquad g(x) := \frac{1}{x} \quad \big(x \in (0,1]\big).$$

Consider the following diagrams:





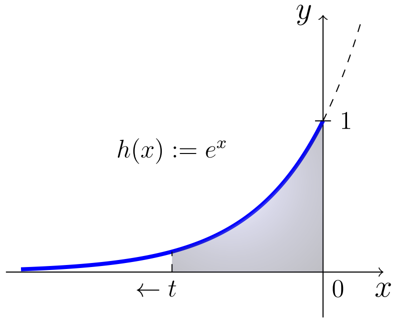
Then

$$\int\limits_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0+0} \int\limits_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0+0} \Big[2\sqrt{x}\Big]_t^1 = \lim_{t\to 0+0} \Big(2 - 2\sqrt{t}\Big) = 2 - 0 = 2,$$

$$\int\limits_0^1 \frac{1}{x}\,dx = \lim_{t\to 0+0} \int\limits_t^1 \frac{1}{x}\,dx = \lim_{t\to 0+0} \Big[\ln x\Big]_t^1 = \lim_{t\to 0+0} \Big(\ln 1 - \ln t\Big) = 0 - (-\infty) = +\infty.$$

On the other hand, let

$$h(x) := e^x \quad \Big(x \in (-\infty, 0]\Big).$$



Then

$$\int\limits_{-\infty}^0 e^x\,dx = \lim_{t\to -\infty} \int\limits_t^0 e^x\,dx = \lim_{t\to -\infty} \Big[e^x\Big]_t^0$$

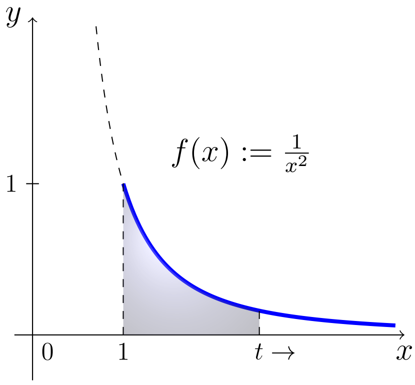
$$= \lim_{t\to -\infty} \Big(e^0 - e^t\Big) = 1 - 0 = 1.$$

The improper integral can similarly be defined for functions of the type $f : [a,b) \to \mathbb{R}$, where $-\infty < a < b \leq +\infty$, and $f \in R[a,t]$ for every $t \in (a,b)$. In this case,

$$\int_a^b f := \int_a^b f(x)\,dx := \lim_{t\to b-0} \int_a^t f(x)\,dx.$$

Example. Let

$$f(x) := \frac{1}{x^2} \qquad \Big(x \in [1, +\infty)\Big).$$



Then

$$\int\limits_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to +\infty} \int\limits_1^t \frac{1}{x^2}\,dx = \lim_{t\to +\infty} \left[-\frac{1}{x}\right]_1^t =$$

$$= \lim_{t\to +\infty} \left(-\frac{1}{t} - (-1)\right) = 0 + 1 = 1.$$

There is yet a third case where it is necessary to define the improper integral.

Definition *Let $-\infty \leq a < b \leq +\infty$ and $f : (a,b) \to \mathbb{R}$. Suppose that $f \in R[x,y]$ for every $a < x < y < b$. We say that* **the function $\boldsymbol{f}$ is improperly integrable on $[\boldsymbol{a}, \boldsymbol{b}]$** *if for every $c \in (a,b)$, the function f is simultaneously improperly integrable on $[a,c]$ and $[c,b]$. In this case,*

$$\int\limits_a^b f := \int\limits_a^c f + \int\limits_c^b f.$$

It is not difficult to reason that the value of c does not affect the result of $\int_a^b f$.

Example. Let

$$f(x) := \frac{1}{1+x^2} \qquad \Big(x \in (-\infty, +\infty)\Big).$$

It holds that

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to+\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to+\infty} \Big[\arctan x\Big]_0^t = \lim_{t\to+\infty} (\arctan t - 0) = \frac{\pi}{2}.$$

Similarly,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \Big[\arctan x\Big]_t^0 = \lim_{t\to-\infty} (0 - \arctan t) = \frac{\pi}{2}.$$

Therefore,

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{+\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Exercise 3. *Find the improper integrals*

$$a)\quad \int_1^2 \frac{1}{\sqrt{x-1}}\,dx, \qquad b)\quad \int_0^{+\infty} xe^{-2x}\,dx, \qquad c)\quad \int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx, \qquad d)\quad \int_0^1 \ln x\,dx.$$

Solution

a) The integrand is not bounded on a right neighborhood of 1, but it is continuous on $(1,2]$, so it is integrable on $[t,2]$ for all $1<t<2$. Then

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to1+0} \int_t^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{t\to1+0} \int_t^2 (x-1)^{-1/2}\,dx = \lim_{t\to1+0} \left[\frac{(x-1)^{1/2}}{1/2}\right]_t^2$$

$$= 2\lim_{t\to1+0} \Big[\sqrt{x-1}\Big]_t^2 = 2\lim_{t\to1+0} \left(\sqrt{1}-\sqrt{t-1}\right) = 2(1-0) = \underline{\underline{2}}\,.$$

b) The integrand is continuous on $[0,+\infty)$, so it is integrable on $[0,t]$ for all $0<t<+\infty$. On the other hand, using integration by parts

$$\int xe^{-2x}\,dx = \int x\cdot\left(\frac{e^{-2x}}{-2}\right)'\,dx = x\cdot\left(\frac{e^{-2x}}{-2}\right) - \int (x)'\cdot\left(\frac{e^{-2x}}{-2}\right)\,dx$$

$$= -\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + c = -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}} + c \qquad (x\in\mathbb{R}).$$

Then

$$\int_0^{+\infty} xe^{-2x}\,dx = \lim_{t\to+\infty} \int_0^t xe^{-2x}\,dx = \lim_{t\to+\infty} \left[-\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}}\right]_0^t = -\lim_{t\to+\infty} \left[\frac{x}{2e^{2x}} + \frac{1}{4e^{2x}}\right]_0^t$$

$$= -\lim_{t\to+\infty} \left(\frac{t}{2e^{2t}} + \frac{1}{4e^{2t}} - \left(\frac{1}{4}\right)\right) = -\left(0+0-\frac{1}{4}\right) = \underline{\underline{\frac{1}{4}}},$$

since

$$\lim_{t\to+\infty} \frac{t}{2e^{2t}} = \left(\frac{+\infty}{+\infty}\right) \overset{\text{L'Hospital}}{=} \lim_{t\to+\infty} \frac{1}{4e^{2t}} = \frac{1}{+\infty} = 0.$$

c) The integrand is not bounded on a right neighborhood of 0 and on a left neighborhood of 2, but it is continuous on $(0,2)$, so it is integrable on $[t,1]$ for all $0<t<1$ and it is also integrable on $[1,t]$ for all $1<t<2$. On the other hand,

$$\int \frac{1}{\sqrt{x(2-x)}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \arcsin(x-1)+c \qquad (0<x<2).$$

Then

$$\int_0^1 \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{t\to 0+0}\int_t^1 \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{t\to 0+0}\Big[\arcsin(x-1)\Big]_t^1$$
$$= \lim_{t\to 0+0}\Big(0-\arcsin(t-1)\Big) = -\arcsin(-1) = \frac{\pi}{2}$$

and

$$\int_1^2 \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{t\to 2-0}\int_1^t \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{t\to 2-0}\Big[\arcsin(x-1)\Big]_1^t$$
$$= \lim_{t\to 2-0}\Big(\arcsin(t-1)-0\Big) = \arcsin(1) = \frac{\pi}{2}.$$

Hence

$$\int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx = \int_0^1 \frac{1}{\sqrt{x(2-x)}}\,dx + \int_1^2 \frac{1}{\sqrt{x(2-x)}}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

d) The integrand is not bounded on a right neighborhood of 0, but it is continuous on $(0,1]$, so it is integrable on $[t,1]$ for all $0<t<1$. On the other hand, we are already proved that (see integration by parts)

$$\int \ln x\,dx = x\ln x - x + c \qquad (x>0).$$

Then

$$\int_0^1 \ln x\,dx = \lim_{t\to 0+0}\int_t^1 \ln x\,dx = \lim_{t\to 0+0}\Big[x\ln x - x\Big]_t^1 = \lim_{t\to 0+0}\Big((\ln 1-1)-(t\ln t - t)\Big)$$
$$= (0-1)-(0-0) = \underline{\underline{-1}},$$

since

$$\lim_{t\to 0+0} t\ln t = \lim_{t\to 0+0}\frac{\ln t}{\frac{1}{t}} = \left(\frac{-\infty}{+\infty}\right) \overset{\text{L'Hospital}}{=} \lim_{t\to 0+0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0+0}(-t) = 0.$$

Applications

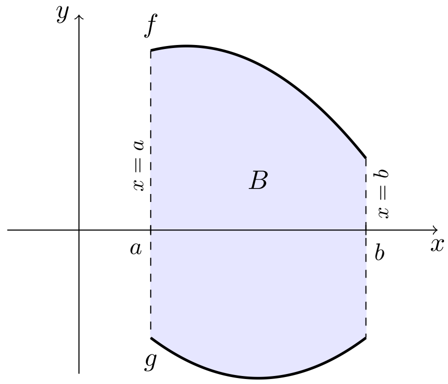
Theoretical notes

Area of a Plane Figure: For two bounded and Riemann-integrable functions $f, g : [a, b] \to \mathbb{R}$, such that $g(x) \le f(x)$ for all $x \in [a, b]$, the area of the plane figure enclosed by the functions and the lines $x = a$ and $x = b$,

$$B = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ g(x) \le y \le f(x)\},$$

is defined by the following definite integral

$$T(B) = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

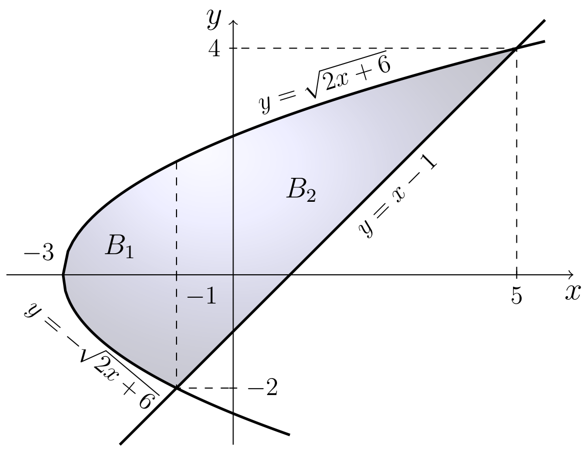


Exercise 4. *Calculate the area of the bounded plane figure enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.*

Solution Draw a diagram.

The area enclosed by the two curves is determined by finding first their points of intersection. For this, we need to solve the system of equations

$$y^2 = 2x + 6$$
$$y = x - 1$$



This is satisfied for those x values where

$$(x - 1)^2 = 2x + 6,$$

that is, $x^2 - 4x - 5 = 0$. This equation has two solutions: $x = -1$ and $x = 5$.

The equation $y^2 = 2x + 6$ represents a parabola whose axis of symmetry coincides with the x-axis, and its vertex is at the point $(-3, 0)$. The equation of the upper branch of the parabola is $y = \sqrt{2x + 6}$, while that of the lower branch is $y = -\sqrt{2x + 6}$. Thus, the region of interest is not defined by just two functions, so we will divide it as follows:

$$B_1 = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; -3 \le x \le -1,\ -\sqrt{2x + 6} \le y \le \sqrt{2x + 6}\right\},$$
$$B_2 = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; -1 \le x \le 5,\ x - 1 \le y \le \sqrt{2x + 6}\right\}.$$

The areas of the above regions can now be calculated using integral calculus:

$$T(B_1) = \int_{-3}^{-1} \left(\sqrt{2x + 6} - (-\sqrt{2x + 6})\right) dx = 2\int_{-3}^{-1} \sqrt{2x + 6}\, dx = 2\int_{-3}^{-1} (2x + 6)^{1/2}\, dx =$$

$$= 2\left[\frac{(2x + 6)^{3/2}}{3/2 \cdot 2}\right]_{-3}^{-1} = \frac{2}{3}\left[\sqrt{(2x + 6)^3}\right]_{-3}^{-1} = \frac{2}{3}\left(\sqrt{4^3} - \sqrt{0^3}\right) = \frac{16}{3},$$

$$T(B_2) = \int_{-1}^{5} \left(\sqrt{2x+6} - (x-1)\right) dx = \int_{-1}^{5} \left((2x+6)^{1/2} - x + 1\right) dx =$$
$$= \left[\frac{(2x+6)^{3/2}}{3/2 \cdot 2} - \frac{x^2}{2} + x\right]_{-1}^{5} = \left[\frac{1}{3}\sqrt{(2x+6)^3} - \frac{x^2}{2} + x\right]_{-1}^{5} =$$
$$= \left(\frac{1}{3}\sqrt{16^3} - \frac{25}{2} + 5\right) - \left(\frac{1}{3}\sqrt{4^3} - \frac{1}{2} - 1\right) = \frac{38}{3}.$$

The sum of these two areas gives the area of the region enclosed by the line and the parabola:
$$T(B_1) + T(B_2) = 18.$$

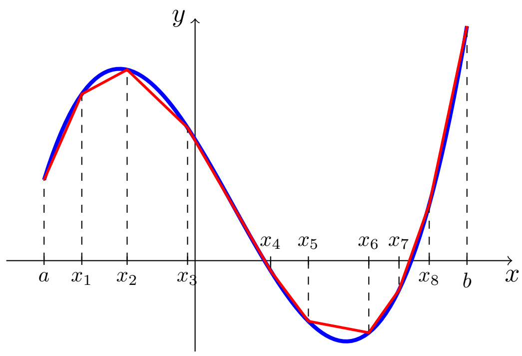
Remark The previous calculations are significantly simplified if we swap the roles of the variables x and y.

Express the variable x in terms of y for both equations. The resulting system
$$x = y + 1, \qquad x = \frac{y^2}{2} - 3$$
has solutions only for $y = -2$ and $y = 4$. With this, the region in question can be determined by two functions as follows:
$$B = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; -2 \le y \le 4,\ \frac{y^2}{2} - 3 \le x \le y + 1\right\}.$$

Thus, the area of this region is:
$$T(B) = \int_{-2}^{4} \left((y+1) - \left(\frac{y^2}{2} - 3\right)\right) dy = \int_{-2}^{4} \left(-\frac{y^2}{2} + y + 4\right) dy = \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right]_{-2}^{4} =$$
$$= \left(-\frac{4^3}{6} + \frac{4^2}{2} + 4 \cdot 4\right) - \left(-\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4 \cdot (-2)\right) = \frac{40}{3} - \left(-\frac{14}{3}\right) = 18.$$

Theoretical notes

Length of a Plane Curve: Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$. The graph of the function f is defined as
$$\Gamma_f := \left\{(x, f(x)) \;\middle|\; x \in [a,b]\right\}.$$



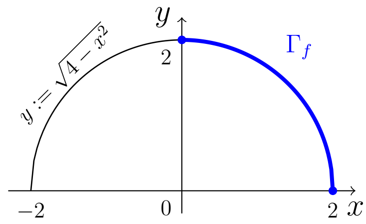
To define the length of the curve formed by the graph, we follow the reasoning introduced during the definition of area. The curve is approximated by a polygonal line corresponding to a partition of the interval $[a,b]$. Based on intuition, we expect that a "sufficiently fine" inscribed polygonal line will approximate the curve so closely that its length will also be close to the arc length of the curve. From this, we can conclude that the arc length of the curve equals the supremum of the lengths of all inscribed polygonal lines. We will adopt this statement as the definition.

It can be proven that if f is continuously differentiable on $[a,b]$, then the arc length of its graph is finite, and it can be computed by the following formula
$$\ell(\Gamma_f) = \int_{a}^{b} \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

Exercise 5. *Compute the arc length of the graph of the function* $f(x) := \sqrt{4-x^2}$ *bounded by the points* $(0,2)$ *and* $(2,0)$.

Solution

The equation of the curve corresponding to the graph of the function is $y = \sqrt{4-x^2}$, which describes a semicircle. The part between the points $(0,2)$ and $(2,0)$ is determined by the values of x in the interval $[0,2]$. For this reason



$$\ell\big(\Gamma_f\big) = \int_0^2 \sqrt{1+\big[f'(x)\big]^2}\,dx.$$

Moreover, for every $0 < x < 2$ we have

$$f'(x) = \frac{1}{2\sqrt{4-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{4-x^2}},$$

and

$$\sqrt{1+\big[f'(x)\big]^2} = \sqrt{1+\frac{x^2}{4-x^2}} = \frac{2}{\sqrt{4-x^2}} = \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,.$$

Note that this function is not bounded on a left neighborhood of 2, in other words, we are dealing with an improper integral.

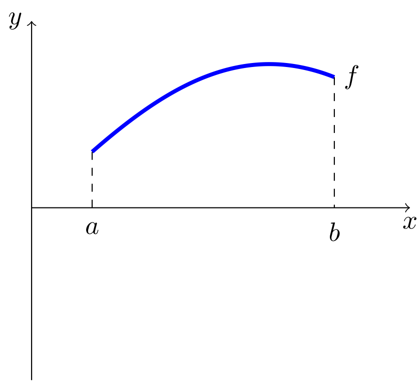
$$\begin{aligned}\ell\big(\Gamma_f\big) &= \int_0^2 \sqrt{1+\big[f'(x)\big]^2}\,dx = \int_0^2 \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx = \lim_{t\to 2-0}\int_0^t \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,dx \\ &= \lim_{t\to 2-0}\left[\frac{\arcsin\frac{x}{2}}{1/2}\right]_0^t = 2\lim_{t\to 2-0}\Big(\arcsin(\tfrac{t}{2}) - \arcsin(0)\Big) \\ &= 2\Big(\arcsin(1) - 0\Big) = 2\cdot\frac{\pi}{2} = \underline{\underline{\pi}}\,.\end{aligned}$$

Theoretical notes

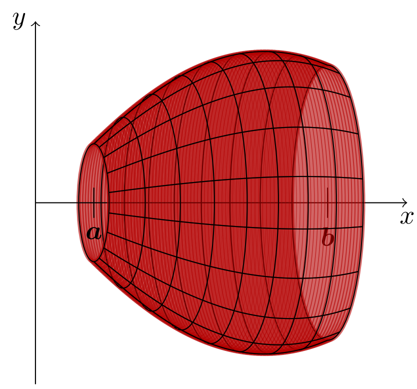
Consider a non-negative function f on the bounded and closed interval $[a,b]$. Then the set

$$A_f := \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; a \le x \le b,\; y^2+z^2 \le f^2(x)\big\}$$

is called the ***solid of revolution*** determined by the function f.



$\longrightarrow$



If f is integrable on $[a,b]$, then the solid of revolution obtained by rotating the graph of f around the x-axis has a volume equal to the integral

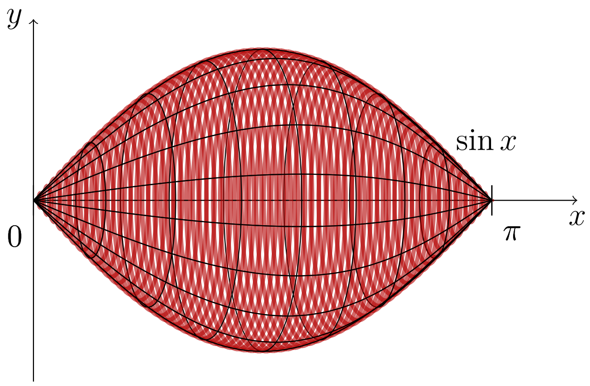
$$\pi \int_a^b f^2(x)\,dx.$$

Exercise 6. *Revolve the curve*

$$y = \sin x \qquad (0 \le x \le \pi)$$

around the x-axis. Determine the volume of this solid of revolution.

Solution We are looking for the volume of the solid of revolution shown in the figure.



The function $f(x) = \sin x$ is continuous, and therefore it is integrable on the interval $[0,\pi]$. Hence, the solid of revolution has a volume given by

$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_0^\pi \sin^2 x\,dx = \pi \int_0^\pi \frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi =$$

$$= \frac{\pi}{2}\left(\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin(2\cdot 0)}{2}\right)\right) = \frac{\pi^2}{2}.$$