MULTIPLE INTEGRALS

Multiple Integrals over multidimensional Intervals

Theoretical notes

We remind that the motivation for introducing the Riemann integral was the problem of determining the area under the graph of a function. This was based on the idea of approximating this area by summing up areas of rectangles. From this approach, we arrived at the concept of *Riemann integrability.*

The introduction of multiple integrals is motivated by similar geometric. As an example, consider a two-variable, real-valued, positive function defined, for simplicity, on a rectangle with sides parallel to the coordinate axes. The *volume* of the space beneath the graph of the function can be approximated by summing the volumes of rectangular parallelepipeds.

We will see that the approach used to introduce the one-variable Riemann integral can be literally extended to functions of type $\mathbb{R}^n \to \mathbb{R}$, and thus the interpretation of multiple integrals is a *direct* generalization of the definition of the one-variable Riemann integral.

Let us start with the simplest sets in $\mathbb{R}^n$, the so-called n-dimensional intervals. An ***n*-dimensional interval** is defined as Cartesian product

$$(1) \qquad I := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n \qquad (n \in \mathbb{N}^+),$$

where $a_k, b_k \in \mathbb{R}$, $a_k < b_k$ $(k = 1, 2, \ldots, n)$. The number

$$|I| := \mu(I) := \prod_{k=1}^{n} (b_k - a_k)$$

is called the *measure* of the interval I.

- If $n = 1$, we get the *usual* bounded and closed interval $[a, b] \subset \mathbb{R}$, whose measure is the *length* of the interval, so $|I| = b_1 - a_1$.
- If $n = 2$, we get a rectangle on the coordinate plane with sides parallel to the coordinate axes, whose measure is the *area* of the rectangle, so $|I| = (b_1 - a_1) \cdot (b_2 - a_2)$.
- If $n = 3$, we get a rectangular parallelepiped in the spatial right-angle coordinate system with faces parallel to the coordinate planes, whose measure is the *volume* of the parallelepiped, in other words, $|I| = (b_1 - a_1) \cdot (b_2 - a_2) \cdot (b_3 - a_3)$.

A partition of a bounded and closed interval $[a, b] \subset \mathbb{R}$ is a *finite* set $\tau \subset [a, b]$ for which $a, b \in \tau$, that is:

$$\tau := \{a = x_0 < x_1 < x_2 < \cdots < x_m = b\},$$

where m is a given natural number. The set of all partitions of the interval $[a, b]$ is denoted by the symbol $\mathcal{F}[a, b]$. Notice that the partition given by the above division points can also be understood as the set of subintervals $I_j := [x_j, x_{j+1}]$ $(j = 0, 1, \ldots, m - 1)$, that is

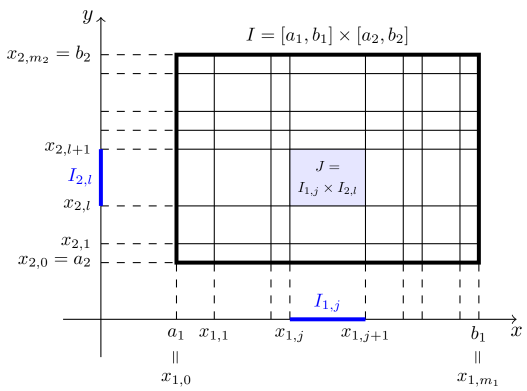
$$\tau = \big\{ I_j = [x_j, x_{j+1}] \;\big|\; j = 0, 1, 2, \ldots, m - 1 \big\}.$$

To define the partition of an n-dimensional interval, we take a partition for each interval $[a_k, b_k]$ in the representation (1):

$$\begin{aligned} \tau_k &= \{a_k = x_{k,0} < x_{k,1} < x_{k,2} < \cdots < x_{k,m_k} = b_k\} \\ &= \big\{ I_{k,j} = [x_{k,j}, x_{k,j+1}] \;\big|\; j = 0, 1, \ldots, m_k - 1 \big\}. \end{aligned}$$

The above partition contains $m_k + 1$ division points and m_k intervals. In this case, a ***partition*** of the n-dimensional interval I in (1) is defined as the set

$$\tau := \tau_1 \times \tau_2 \times \cdots \times \tau_n \subset I,$$



and the set of all such partitions is denoted by the symbol $\mathcal{F}(I)$. The elements of the set τ are the n-dimensional intervals

$$I_{1,j_1} \times I_{2,j_2} \times \cdots \times I_{n,j_n},$$

where $0 \le j_i \le m_i - 1$ $(i = 1, 2, \ldots, n)$. This is illustrated in the case of $n = 2$ in the above figure.

It is straightforward to show that

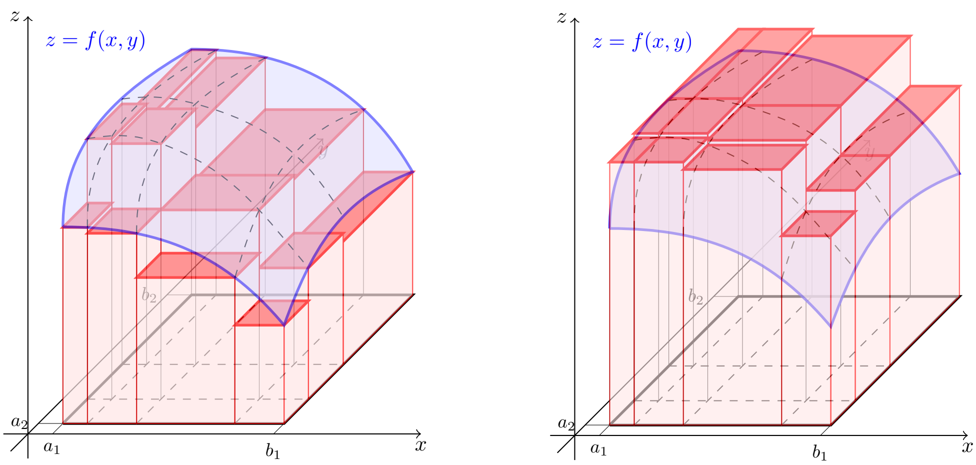
$$I = \bigcup_{J\in\tau} J, \qquad \mu(I) = \sum_{J\in\tau} \mu(J).$$

Similar to the one-variable case, we define the notions of lower and upper approximating sums. Let τ be a partition of the n-dimensional interval I, and let $f : I \to \mathbb{R}$ be a bounded function. Then, the ***lower approximating sum*** of f with respect to the partition τ is given by

$$s(f, \tau) := \sum_{J\in\tau} \inf_{x\in J} f(x) \cdot \mu(J),$$

and the ***upper approximating sum*** of f with respect to the partition τ is given by

$$S(f, \tau) := \sum_{J\in\tau} \sup_{x\in J} f(x) \cdot \mu(J).$$



lower approximating sum — upper approximating sum

Since for any partition $\tau \in \mathcal{F}(I)$, we have

$$\inf_{x\in I} f(x) \cdot \mu(I) \le s(f, \tau) \le S(f, \tau) \le \sup_{x\in I} f(x) \cdot \mu(I),$$

it follows that for any bounded function f, the sets

$$\big\{ s(f, \tau) \;\big|\; \tau \in \mathcal{F}(I) \big\} \qquad \text{and} \qquad \big\{ S(f, \tau) \;\big|\; \tau \in \mathcal{F}(I) \big\}$$

are bounded. The real number

$$I_*(f) := \sup\big\{ s(f, \tau) \;\big|\; \tau \in \mathcal{F}(I) \big\}$$

is called the ***Darboux lower integral*** of the function f, and the real number

$$I^*(f) := \inf\big\{ S(f, \tau) \;\big|\; \tau \in \mathcal{F}(I) \big\}$$

is called the ***Darboux upper integral*** of the function f.

A bounded function $f : I \to \mathbb{R}$ is said to be ***Riemann-integrable*** (or simply ***integrable***) on the interval I (denoted $\boldsymbol{f \in R(I)}$) if $I_*(f) = I^*(f)$. The common value $I_*(f) = I^*(f)$ is called the ***Riemann integral*** of the function f on the interval I (or simply the ***integral***), and it is denoted by one of the following symbols:

$$\int_I f, \quad \int_I f(x)\, dx, \quad \int \cdots \int_I f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Riemann integrability and the Riemann integral itself, in the case of multiple variables, possess the same properties as those observed in the single-variable case.

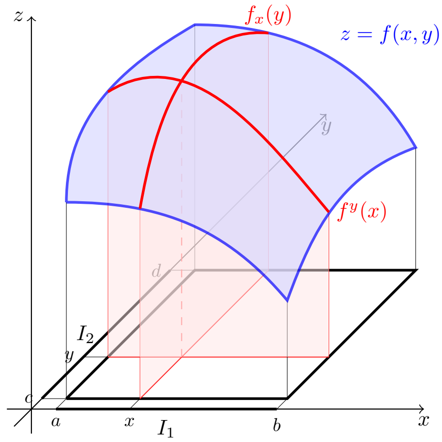
The computation of the integral of a function defined on an n-dimensional interval can be reduced to the successive calculation of integrals of real-valued functions. In the following, we present the theoretical background for this in the case of $n = 2$, known as ***double integrals***, but it can similarly be generalized for $n > 2$.

In the following, we assume that a

$$I := I_1 \times I_2 := [a, b] \times [c, d] \subset \mathbb{R}^2$$

two-dimensional interval and a *bounded* function $f : I \to \mathbb{R}$ are given.

To better understand the behavior of a function of two variables, it can be helpful to fix one of its variables and regard the function as depending on the other variable. The resulting functions are called the ***section functions*** of the original function.



If $f : I_1 \times I_2 \to \mathbb{R}$ is a given function of two variables, then for any fixed $x \in I_1$, the function

$$f_x : I_2 \to \mathbb{R}, \quad f_x(y) := f(x, y) \qquad (y \in I_2),$$

and for any fixed $y \in I_2$, the function

$$f^y : I_1 \to \mathbb{R}, \quad f^y(x) := f(x, y) \qquad (x \in I_1),$$

are called the section functions of f.

Iterated Integrals. Let $I = [a, b] \times [c, d]$ and $f : I \to \mathbb{R}$. Assume that

a) $f \in R(I)$,

b) $\forall x \in [a, b]\colon f_x \in R[c, d]$,

c) $\forall y \in [c, d]\colon f^y \in R[a, b]$.

Then

$$\iint\limits_I f(x, y)\, dx\, dy = \int\limits_a^b \left(\int\limits_c^d f(x, y)\, dy \right) dx = \int\limits_c^d \left(\int\limits_a^b f(x, y)\, dx \right) dy. \tag{2}$$

If the function f is *continuous* on the rectangle I, then $f \in R(I)$, and the section functions f_x $(x \in [a, b])$ and f^y $(y \in [c, d])$ are also continuous and, consequently, Riemann-integrable. Thus, the conditions for using iterated integrals are satisfied. Then, the integral of a two-variable function can be computed by first (arbitrarily) fixing one of the variables and integrating with respect to the other variable, and then integrating the resulting integral (which depends on the fixed variable). This is where the term *iterated* originates. The equality in (2) also asserts that the order of integration can be interchanged.

Remarks

1. If the variables x and y in the integrand are separable in the form

$$f(x, y) := g(x)h(y) \qquad (a \le x \le b,\ c \le y \le d),$$

where $g : [a, b] \to \mathbb{R}$ and $h : [c, d] \to \mathbb{R}$ are continuous functions, then the conditions of the Fubini theorem are clearly satisfied, and the following holds:

$$\iint\limits_I f(x, y)\, dx\, dy = \int\limits_a^b \left(\int\limits_c^d g(x)h(y)\, dy \right) dx = \int\limits_a^b g(x) \left(\int\limits_c^d h(y)\, dy \right) dx = \left(\int\limits_a^b g(x)\, dx \right) \cdot \left(\int\limits_c^d h(y)\, dy \right),$$

which means that the double integral can be decomposed into the product of two real integrals.

2. Iterated integration states that it does not matter in which order the integration is performed, the result will be the same. However, this does *not* mean that the technical difficulties encountered during the calculation in the two different orders will be the same.

Exercise 1. *Integrate the function*

$$f(x,y) := xy^2 + 3x^2y$$

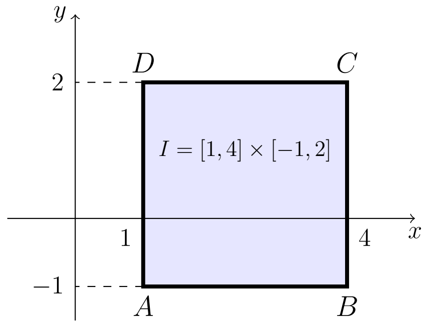
over the rectangle whose vertices are

$$A(1,-1), \quad B(4,-1), \quad C(4,2), \quad D(1,2).$$

Solution

Based on the figure, we see that the rectangle can be written as

$$I = [1,4] \times [-1,2].$$



The integrand is continuous on the rectangle I, so $f \in R(I)$. According iterated integration, the order of integration does not matter, and the result will be the same. This allows us to compute the integral in two ways. If we first fix the variable $x \in [1,4]$ and integrate with respect to y, then we obtain

$$\iint_I \left(xy^2 + 3x^2y\right) dx\,dy = \int_1^4 \left(\int_{-1}^2 (xy^2 + 3x^2y)\,dy \right) dx = \int_1^4 \left[\frac{xy^3}{3} + \frac{3x^2y^2}{2} \right]_{y=-1}^{y=2} dx$$

$$= \int_1^4 \left(\left(\frac{x \cdot 2^3}{3} + \frac{3x^2 \cdot 2^2}{2} \right) - \left(\frac{x(-1)^3}{3} + \frac{3x^2(-1)^2}{2} \right) \right) dx = \int_1^4 \left(3x + \frac{9x^2}{2} \right) dx$$

$$= \left[\frac{3x^2}{2} + \frac{9x^3}{2 \cdot 3} \right]_{x=1}^{x=4} = \frac{3}{2} \cdot \left[x^2 + x^3 \right]_{x=1}^{x=4} = \frac{3}{2} \cdot \left(\left(4^2 + 4^3\right) - \left(1^2 + 1^3\right) \right) = \underline{\underline{117}}\,.$$

On the other hand, if we first fix the variable $y \in [-1,2]$ and integrate with respect to x, we obtain:

$$\iint_I \left(xy^2 + 3x^2y\right) dx\,dy = \int_{-1}^2 \left(\int_1^4 (xy^2 + 3x^2y)\,dx \right) dy = \int_{-1}^2 \left[\frac{x^2y^2}{2} + x^3y \right]_{x=1}^{x=4} dy$$

$$= \int_{-1}^2 \left(\left(\frac{4^2y^2}{2} + 4^3y \right) - \left(\frac{1^2y^2}{2} + 1^3y \right) \right) dy = \int_{-1}^2 \left(\frac{15y^2}{2} + 63y \right) dy$$

$$= \left[\frac{15y^3}{2 \cdot 3} + \frac{63y^2}{2} \right]_{y=-1}^{y=2} = \frac{1}{2} \cdot \left[5y^3 + 63y^2 \right]_{y=-1}^{y=2}$$

$$= \frac{1}{2} \cdot \left(\left(5 \cdot 2^3 + 63 \cdot 2^2\right) - \left(5(-1)^3 + 63(-1)^2\right) \right) = \underline{\underline{117}}\,.$$

Exercise 2. *Evaluate the double integral*

$$\iint\limits_I x^3\sqrt{y}\,dx\,dy$$

where $I := [0,1] \times [0,2]$.

Solution The integrand is continuous on the rectangle I, so $f \in R(I)$. Note that the the variables x and y in the integrand are separated by multiplication. Therefore,

$$\iint\limits_I x^3\sqrt{y}\,dx\,dy = \left(\int_0^1 x^3\,dx\right)\cdot\left(\int_0^2 \sqrt{y}\,dy\right) = \left[\frac{x^4}{4}\right]_0^1 \cdot \left[\frac{y^{3/2}}{3/2}\right]_0^2 = \frac{1}{4}\cdot\frac{2^{3/2}}{3/2} = \underline{\underline{\frac{\sqrt{2}}{3}}}\,.$$

Exercise 3. *Evaluate the double integral*

$$\iint\limits_I x\cdot\sin(xy)\,dx\,dy$$

where $I := \left[1,3\right] \times \left[0,\frac{\pi}{2}\right]$.

Solution The integrand is continuous on the rectangle I, so $f \in R(I)$. If we first fix the variable $x \in [1,3]$ and integrate with respect to y, we need to successively compute the single-variable integrals in

$$(*) \qquad \int_1^3 \left(\int_0^{\pi/2} x\cdot\sin(xy)\,dy\right) dx.$$

If we instead fix the variable $y \in \left[0,\frac{\pi}{2}\right]$ and integrate with respect to x, we obtain:

$$(**) \qquad \int_0^{\pi/2} \left(\int_1^3 x\cdot\sin(xy)\,dx\right) dy.$$

Notice, however, that in case $(**)$, partial integration must be applied first, while in case $(*)$, the inner integral can be computed directly. Therefore, we proceed by integrating in the order shown under $(*)$:

$$\begin{aligned}\iint\limits_I x\cdot\sin(xy)\,dx\,dy &= \int_1^3 \left(\int_0^{\pi/2} x\cdot\sin(xy)\,dy\right) dx = \int_1^3 \Big[-\cos(xy)\Big]_{y=0}^{y=\pi/2}\,dx \\ &= \int_1^3 \Big(-\cos\frac{\pi x}{2} + \cos 0\Big)\,dx = \int_1^3 \Big(-\cos\frac{\pi x}{2} + 1\Big)\,dx = \left[-\frac{\sin\frac{\pi x}{2}}{\frac{\pi}{2}} + x\right]_1^3 \\ &= \left(-\frac{2}{\pi}\sin\frac{3\pi}{2} + 3\right) - \left(-\frac{2}{\pi}\sin\frac{\pi}{2} + 1\right) = \underline{\underline{2 + \frac{4}{\pi}}}\,.\end{aligned}$$

Exercise 4. *Integrate the function*

$$f(x,y) := xy + xz$$

over the cuboid $[0,2]\times[1,2]\times[1,3]$.

Solution We evaluate the triple integral

$$\iint\limits_I xy + xz \, dx\, dy\, dz$$

where $I := [0,2]\times[1,2]\times[1,3]$. The integrand is continuous on the cuboid I, so $f \in R(I)$. We can integrate it in any order, having $3! = 6$ possibilities to do so. One of these is

$$\int_1^3 \left(\int_1^2 \left(\int_0^2 (xy+xz)\, dx\right) dy\right) dz = \int_1^3 \left(\int_1^2 \left(\int_0^2 x(y+z)\, dx\right) dy\right) dz$$

$$= \int_1^3 \left(\int_1^2 (y+z) \left(\int_0^2 x\, dx\right) dy\right) dz = \int_1^3 \left(\int_1^2 (y+z)\left[\frac{x^2}{2}\right]_{x=0}^{x=2} dy\right) dz$$

$$= \int_1^3 \left(\int_1^2 (y+z)(2-0)dy\right) dz = 2\int_1^3 \left(\int_1^2 (y+z)dy\right) dz$$

$$= 2\int_1^3 \left[\frac{y^2}{2} + yz\right]_{y=1}^{y=2} dz = 2\int_1^3 \left(2 + 2z - \left(\frac{1}{2}+z\right)\right) dz = 2\int_1^3 \left(\frac{3}{2}+z\right) dz$$

$$= 2\left[\frac{3z}{2} + \frac{z^2}{2}\right]_{z=1}^{z=3} = \left[3z+z^2\right]_{z=1}^{z=3} = (9+9)-(3+1) = \underline{\underline{14}}\,.$$

Multiple Integrals over bounded sets

Theoretical notes

The concept of integrability can be straightforwardly extended to *arbitrary* bounded functions defined on bounded sets $H \subset \mathbb{R}^n$. Let H be such a set, and let $f : H \to \mathbb{R}$ be a bounded function. Then there exists an n-dimensional interval I such that $H \subset I$. Extend the definition of the function f to the interval I as follows:

$$\tilde{f}(x) := \begin{cases} f(x), & \text{if } x \in H,\\ 0, & \text{if } x \in I \setminus H. \end{cases}$$

We say that the function $f : H \to \mathbb{R}$ is ***(Riemann)-integrable on the set*** H (denoted by $\boldsymbol{f \in R(H)}$) if the extended function $\tilde{f} : I \to \mathbb{R}$ is integrable on the interval I. In this case, let

$$\int\limits_H f := \int\limits_I \tilde{f}.$$

It can be proven that this definition is *independent* of the choice of the interval I containing H.

Geometric Meanings of Double Integrals:

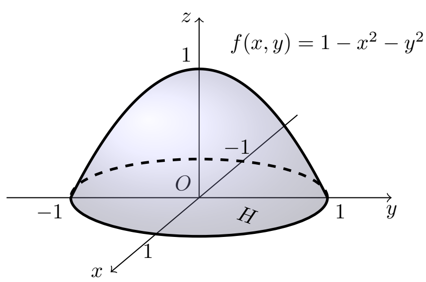
- Let $H \subset \mathbb{R}^2$ be a bounded set and $f : H \to \mathbb{R}$ a non-negative bounded function. We say that the

  $$S := \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; (x,y) \in H,\ 0 \le z \le f(x,y)\right\}$$

  solid region (cylindrical body) ***has a volume*** if $f \in R(H)$. In this case, the double integral

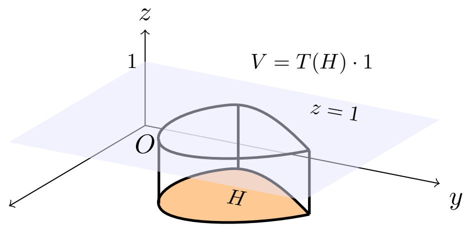
  $$V(S) := \iint\limits_H f = \iint\limits_H f(x,y)\, dx\, dy$$

  is called the ***volume*** of the solid S.



- Let $H \subset \mathbb{R}^2$ be a bounded set, and let $f(x,y) := 1$ $\big((x,y) \in H\big)$. We say that the region H **has an area** if $f \in R(H)$, and in this case, the **area of** H is defined by the double integral

$$T(H) := \iint\limits_H f = \iint\limits_H 1\,dx\,dy.$$



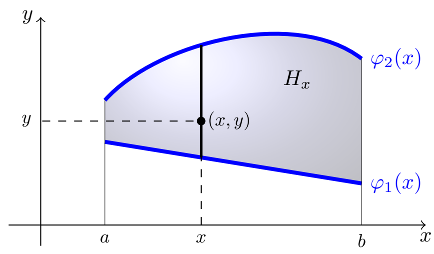
Double Integral over Normal Regions: It often happens that we need to calculate an integral over a region bounded by two functions.

- Let $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ be continuous functions, and assume that $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in [a,b]$. The set

$$H_x := \big\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \ \varphi_1(x) \leq y \leq \varphi_2(x)\big\}$$

is called a ***normal region with respect to the x-axis***. Assume that the function $f : H_x \to \mathbb{R}$ is continuous. Then, $f \in R(H_x)$, and

$$\iint\limits_{H_x} f(x,y)\,dx\,dy = \int\limits_a^b \left(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx.$$

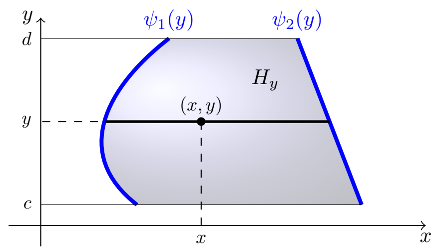


- Let $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ be continuous functions, and suppose that $\psi_1(y) \leq \psi_2(y)$ for all $y \in [c,d]$. The set

$$H_y := \big\{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\big\}$$

is called a ***normal region with respect to the y-axis***. Assume that the function $f : H_y \to \mathbb{R}$ is continuous. Then $f \in R(H_y)$, and

$$\iint\limits_{H_y} f(x,y)\,dx\,dy = \int\limits_c^d \left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy.$$



It may happen that the same region can be considered a normal region with respect to both axes.

Triple Integral over Normal Regions: Similarly to $\mathbb{R}^2$, in $\mathbb{R}^3$ we can also define normal regions in various ways. A "typical" example is the following.

Consider the following normal region in the xy-plane with respect to the x-axis:

$$D := \big\{(x,y) \in \mathbb{R}^2 \;\big|\; a \leq x \leq b,\ \ \varphi_1(x) \leq y \leq \varphi_2(x)\big\},$$

where $[a,b] \subset \mathbb{R}$ is a compact interval, and $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ are continuous functions satisfying the inequality $\varphi_1(x) \leq \varphi_2(x)$ $(x \in [a,b])$. Assume furthermore that $g_1, g_2 : D \to \mathbb{R}$ are continuous functions satisfying the inequality

$$g_1(x,y) \leq g_2(x,y) \quad \big((x,y) \in D\big).$$

With these notations, the set

$$H := \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; a \leq x \leq b,\ \ \varphi_1(x) \leq y \leq \varphi_2(x),\ \ g_1(x,y) \leq z \leq g_2(x,y)\big\}$$

is a **normal region in** $\mathbb{R}^3$. Thus, H is a "cylinder-like" region in space bounded below by the graph of g_1 and above by the graph of g_2.

Assume that the function $f : H \to \mathbb{R}$ is continuous, and thus $f \in R(H)$. Then

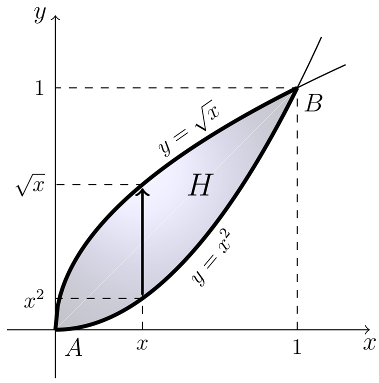
$$\iiint\limits_H f(x,y,z)\,dx\,dy\,dz = \int\limits_a^b \left(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} \left(\int\limits_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) dy \right) dx\,.$$

Exercise 5. *Evaluate the following double integral:*

$$\iint_H xy^2\,dx\,dy,$$

where H is the bounded plane region enclosed by the curves $y = x^2$ and $y = \sqrt{x}$.

Solution The figure represents the set H.



It is clear that H is a normal region with respect to the x-axis. The integrand is continuous over the entire $\mathbb{R}^2$, and therefore also over the normal region H. Consequently, $f \in R(H)$.

To compute the integral, we first determine the coordinates of the intersection points of the curves:

$$\left.\begin{aligned} y &= x^2 \\ y &= \sqrt{x} \end{aligned}\right\} \iff \sqrt{x} = x^2 \iff x = 0 \text{ or } x = 1.$$

Thus, the points of intersection are $A(0,0)$ and $B(1,1)$.

The set H is a normal region with respect to both the x-axis and the y-axis. Therefore, we can use either of the formulas we have learned. Let us consider H as a normal region with respect to the x-axis:

$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}.$$

In this case, we first integrate with respect to y. (The arrow indicates the direction of the "inner" integral.) Thus,

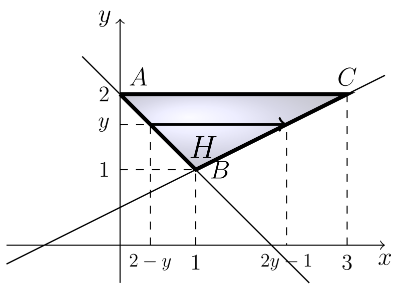
$$\iint_H xy^2\,dx\,dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} xy^2\,dy \right) dx = \int_0^1 \left[x \cdot \frac{y^3}{3} \right]_{y=x^2}^{y=\sqrt{x}} dx = \frac{1}{3} \int_0^1 x \cdot \left(x^{3/2} - x^6 \right) dx =$$

$$= \frac{1}{3} \int_0^1 \left(x^{5/2} - x^7 \right) dx = \frac{1}{3} \cdot \left[\frac{x^{7/2}}{7/2} - \frac{x^8}{8} \right]_0^1 = \frac{1}{3} \cdot \left(\frac{1}{7/2} - \frac{1}{8} \right) = \underline{\underline{\frac{3}{56}}}.$$

Exercise 6. *Let H denote the triangular region with vertices $(0,2)$, $(1,1)$, and $(3,2)$. Evaluate the integral*

$$\iint_H y\,e^x\,dx\,dy.$$

Solution The figure represents the set H.



If we consider H as a normal region with respect to the x-axis, the calculation of the integral must be divided into two parts over the intervals $[0,1]$ and $[1,3]$.

However, if we consider the H as a normal region with respect to the y-axis, it is not necessary to divide H into two parts. To determine the H domain, we need to find the equations of the lines AB and BC.

- For the line AB:
$$\frac{y-2}{x-0} = \frac{1-2}{1-0} = -1 \quad \Longrightarrow \quad y = -x+2 \quad \Longleftrightarrow \quad x = 2-y.$$

- For the line BC:
$$\frac{y-2}{x-3} = \frac{1-2}{1-3} = \frac{1}{2} \quad \Longrightarrow \quad y = \frac{1}{2}x + \frac{1}{2} \quad \Longleftrightarrow \quad x = 2y-1.$$

Thus,
$$H := \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; 1 \le y \le 2,\; 2-y \le x \le 2y-1\Big\}.$$
The integrand
$$f(x,y) := y\,e^x$$
is continuous over the entire $\mathbb{R}^2$, and therefore also over the bounded domain H. Consequently, $f \in R(H)$. We first integrate with respect to x. (The arrow indicates the "inner" direction of the integral.) Thus,

$$\iint\limits_H ye^x\,dx\,dy = \int\limits_1^2 \left(\int\limits_{2-y}^{2y-1} ye^x\,dx \right) dy = \int\limits_1^2 y \cdot \Big[e^x\Big]_{x=2-y}^{x=2y-1} dy = \int\limits_1^2 y \cdot \Big(e^{2y-1} - e^{2-y}\Big) dy$$
$$= e^{-1} \int\limits_1^2 y \cdot e^{2y}\,dy - e^2 \int\limits_1^2 y \cdot e^{-y}\,dy.$$

We will calculate the two resulting integrals using the rule of integration by parts:

$$\int\limits_1^2 y \cdot e^{2y}\,dy = \left[y \cdot \frac{e^{2y}}{2}\right]_1^2 - \int\limits_1^2 1 \cdot \frac{e^{2y}}{2}\,dy = \frac{1}{2}\Big(2e^4 - e^2\Big) - \frac{1}{2}\left[\frac{e^{2y}}{2}\right]_1^2$$
$$= \left(e^4 - \frac{e^2}{2}\right) - \frac{1}{4}\Big(e^4 - e^2\Big) = \frac{e^2}{4}\Big(3e^2 - 1\Big)$$

and

$$\int\limits_1^2 y \cdot e^{-y}\,dy = \left[y \cdot \frac{e^{-y}}{-1}\right]_1^2 - \int\limits_1^2 1 \cdot \frac{e^{-y}}{-1}\,dy = (-1) \cdot \Big(2e^{-2} - e^{-1}\Big) + \left[\frac{e^{-y}}{-1}\right]_1^2$$
$$= \Big(-2e^{-2} + e^{-1}\Big) - \Big(e^{-2} - e^{-1}\Big) = \frac{2e-3}{e^2}.$$

Therefore,
$$\iint\limits_H ye^x\,dx\,dy = \underline{\underline{\frac{3}{4}e^3 - \frac{9}{4}e + 3}}\,.$$

Exercise 7. *Evaluate the integral*

$$\iint\limits_H y \sin x^2 \, dx \, dy \qquad H := \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; 0 \le y \le 1,\; y^2 \le x \le 1\Big\}$$

Solution H is a y-axis normal region (see Figure (a)) defined by the inequalities

$$0 \le y \le 1, \quad y^2 \le x \le 1.$$

Thus,

$$\iint\limits_H y \sin x^2 \, dx \, dy = \int\limits_0^1 \left(\int\limits_{y^2}^1 y \sin x^2 \, dx \right) dy.$$

In the formula above, we need to integrate with respect to x first. However, we encounter the following problem: the function $\sin x^2$ $(x \in \mathbb{R})$ *does* have a primitive function (since it is continuous), but this primitive function is *not an elementary function*, and thus the Newton–Leibniz theorem *cannot be applied* to compute the inner (single-variable) integral.

Let us try swapping the order of integration, that is, integrating with respect to y first. This is possible because H is also an x-axis normal region, as defined by the inequalities below Figure (b).

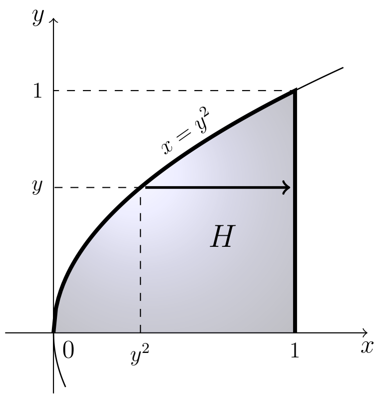


Figure (a)
H as an y normal region
$0 \le y \le 1, \quad y^2 \le x \le 1$

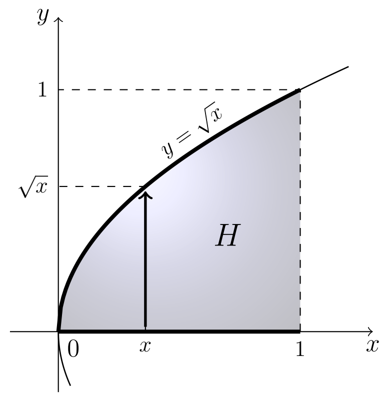


Figure (b)
H as an x normal region
$0 \le x \le 1, \quad 0 \le y \le \sqrt{x}$

Hence

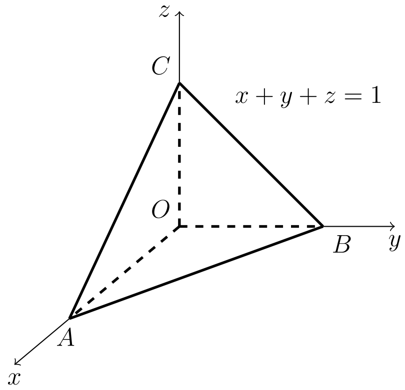
$$\iint\limits_H y \sin x^2 \, dx \, dy = \int\limits_0^1 \left(\int\limits_0^{\sqrt{x}} y \sin x^2 \, dy \right) dx = \int\limits_0^1 (\sin x^2) \cdot \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \frac{1}{2} \int\limits_0^1 x \cdot \sin x^2 \, dx = \frac{1}{2} \cdot \frac{1}{2} \int\limits_0^1 2x \cdot \sin x^2 \, dx = \frac{1}{4} \Big[-\cos x^2 \Big]_0^1 = \underline{\underline{\frac{1}{4} \Big(1 - \cos 1 \Big)}}\,.$$

Exercise 8. *Integrate the function $f(x,y,z) := 2xyz$ over the region*

$$S := \Big\{(x,y,z) \in \mathbb{R}^3 \;\Big|\; x \geq 0, y \geq 0, x+y+z \leq 1\Big\}.$$

Solution Let's first examine the set S.



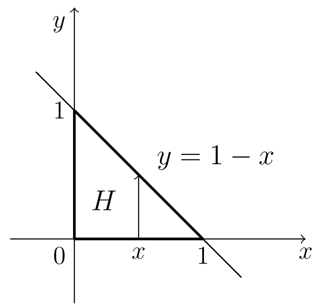
It is easy to notice that the plane defined by the equation $x+y+z=1$ has a single point of intersection with the x-, y-, and z-axes, which are the points $A(1,0,0)$, $B(0,1,0)$, and $C(0,0,1)$, respectively. Therefore, S is the tetrahedron $OABC$.

Let

$$f(x,y,z) := 2xyz \qquad \big((x,y,z) \in H\big).$$

The plane defined by the equation $x+y+z=1$ can be viewed as the graph of the function

$$g_2(x,y) := 1-x-y \quad \text{for} \quad (x,y) \in \mathbb{R}^2.$$



The set S is then a normal region in $\mathbb{R}^3$. Specifically, let H be the closed triangular region OAB in the xy-plane. We can consider H as a normal domain with respect to the x-axis:

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1-x.$$

Based on the previous analysis, the set $S \subset \mathbb{R}^3$ is a cylindrical region above the domain H, which is bounded below by the function $g_1(x,y) = 0$ for $(x,y) \in \mathbb{R}^2$ and above by the function $g_2(x,y) = 1-x-y$ for $(x,y) \in \mathbb{R}^2$. Therefore,

$$S = \Big\{(x,y,z) \in \mathbb{R}^3 \;\Big|\; 0 \leq x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\Big\}.$$

Since $f \in C(H)$, we have $f \in R(H)$, and

$$\iiint\limits_H f(x,y,z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 2xyz\,dz \right) dy \right) dx.$$

First, we deal with the inner integral

$$\begin{aligned}\int_0^{1-x-y} 2xyz\,dz &= \Big[xyz^2\Big]_{z=0}^{z=1-x-y} = xy(1-x-y)^2 = xy(1+x^2+y^2-2x-2y+2xy)\\ &= xy + x^3y + xy^3 - 2x^2y - 2xy^2 + 2x^2y^2\\ &= xy^3 + (2x^2-2x)y^2 + (x^3-2x^2+x)y.\end{aligned}$$

Now we integrate this result with respect to y

$$\int_0^{1-x} \Big(xy^3 + (2x^2-2x)y^2 + (x^3-2x^2+x)y\Big)dy$$

$$= \left[x\,\frac{y^4}{4} + (2x^2-2x)\,\frac{y^3}{3} + (x^3-2x^2+x)\,\frac{y^2}{2}\right]_{y=0}^{y=1-x}$$

$$= x\,\frac{(1-x)^4}{4} + (2x^2-2x)\,\frac{(1-x)^3}{3} + (x^3-2x^2+x)\,\frac{(1-x)^2}{2}$$

$$= \frac{1}{12}\,x^5 - \frac{1}{3}\,x^4 + \frac{1}{2}\,x^3 - \frac{1}{3}\,x^2 + \frac{1}{12}\,x$$

Finally, we integrate this last result with respect to x

$$\int_0^1 \left(\frac{1}{12}\,x^5 - \frac{1}{3}\,x^4 + \frac{1}{2}\,x^3 - \frac{1}{3}\,x^2 + \frac{1}{12}\,x\right)dx$$

$$= \frac{1}{72}\,x^6 - \frac{1}{15}\,x^5 + \frac{1}{8}\,x^4 - \frac{1}{9}\,x^3 + \frac{1}{24}\,x^2 = \frac{1}{72} - \frac{1}{15} + \frac{1}{8} - \frac{1}{9} + \frac{1}{24} = \underline{\underline{\frac{1}{360}}}\,.$$