Tensor Calculus

January 4, 2024

1 Motivation

2 Introduction

3 Notable Tensors

3.1 Kronecker Delta (δ)

A special type of tensor (can take any order of $2p$ where $p = \{1, \ldots, \infty\}$) that follow the following rule

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

We note that ($\delta_{ij} \equiv \delta^{ij} \equiv \delta^i_j$)

If we assume that δ^i_j is a matrix then it would be equivalent to the identity matrix $\delta^i_j \equiv I$ where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A nice property of δ^i_j is that it acts like index changer (inherited from identity matrix)

$$\delta^i_j A^j = A^i$$
$$\delta^i_j A_i = A_j$$

Example 1 *Let $n = 2$ (i.e cardinality). Evaluate $\delta^i_j x_i$*

We have a dummy index i hence this imply a summation[1]

$$\delta^i_j x_i = \sum_{i=1}^{2} \delta^i_j x_i = \delta^1_j x_1 + \delta^2_j x_2$$

[1] according to Einstein summation convention

We have one live index j, which means that the evaluation of $\delta^i_j x_i$ is a first order tensor x_j. To prove this lets compute the expression for $j \in \{1, 2\}$

$$\begin{aligned}
j = 1 \quad &\rightarrow \quad \cancelto{1}{\delta^1_1} x_1 + \cancelto{0}{\delta^2_1} x_2 = x_1 \\
j = 2 \quad &\rightarrow \quad \cancelto{0}{\delta^1_2} x_1 + \cancelto{1}{\delta^2_2} x_2 = x_2
\end{aligned}$$

As mentioned above δ^i_j acts like index changer ■

Example 2 *Let $n = 2$ (i.e cardinality). Evaluate $\delta^i_j x_i x^j$*

We have a two dummy indices $i,\ j$ hence this imply two summation

$$\begin{aligned}
\delta^i_j x_i x^j = \sum_{j=1}^{2} \sum_{i=1}^{2} \delta^i_j x_i x^j &= \sum_{j=1}^{2} \delta^1_j x_1 x^j + \delta^2_j x_2 x^j \\
&= \cancelto{1}{\delta^1_1} x_1 x^1 + \cancelto{0}{\delta^2_1} x_2 x^1 + \cancelto{0}{\delta^1_2} x_1 x^2 + \cancelto{1}{\delta^2_2} x_2 x^2 \\
&= x_1 x^1 + x_2 x^2 \equiv x_i x^i
\end{aligned}$$

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Example 3 *Suppose that $A,\ B$ are 3×3 inverse matrices (i.e. $A^{-1} = B$) and that $T^i = g^i_r a_{rs} y_s$ with $y_s = b_{sr} x_r$. Express T^i in terms of x_r*

Since $A,\ B$ are inverse matrices then the following is true

$$A_{ij} B_{jk} = \delta_{ik}$$

To express T^i in terms of x_r we will substitute y_s into it, but first will rename the dummy index r in y_s to avoid overlap. Hence $(r \rightarrow k)$ $y_s = b_{sk} x_k$

$$\begin{aligned}
T^i = g^i_r a_{rs} (b_{sk} x_k) &= g^i_r \overbrace{a_{rs} b_{sk}}^{\delta_{rk}} x_k \\
&= g^i_r \overbrace{\delta_{rk} x_k}^{x_r} = g^i_r x_r
\end{aligned}$$

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Example 4 *Calculate $\frac{\partial}{\partial x^k}(a_{ij} x^i x^j)$. Given that a_{ij} is a matrix of constants.*

$$
\begin{aligned}
\frac{\partial}{\partial x^k}(a_{ij}x^i x^j) &= a_{ij} \overbrace{\frac{\partial}{\partial x^k}(x^i x^j)}^{\text{product rule}} \\
&= a_{ij}\left(\frac{\partial x^i}{\partial x^k} x^j + x^i \frac{\partial x^j}{\partial x^k}\right)
\end{aligned}
$$

Remember

From partial differentiation rules

$$
\frac{\partial x}{\partial y} = 0 \qquad\qquad \frac{\partial x}{\partial x} = 1
$$

Which is equivalent to

$$
\frac{\partial x^i}{\partial x^j} = 0 \qquad\qquad \frac{\partial x^i}{\partial x^i} = 1
$$

One can see that the behaviour of partial derivative is equivalent to Kronecker Delta where

$$
\frac{\partial x^i}{\partial x^j} \equiv \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
$$

Hence we replace partial derivative by δ

$$
\begin{aligned}
\frac{\partial}{\partial x^k}(a_{ij}x^i x^j) &= a_{ij}(\delta^i_k x^j + x^i \delta^j_k) \\
&= \delta^i_k a_{ij} x^j + \delta^j_k a_{ij} x^i = a_{kj} x^j + a_{ik} x^i
\end{aligned}
$$

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References