Definition 1 *Line integration of a vector field* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *along a smooth curve* C

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds \tag{1}$$

where $\mathbf{T}$ *is the tangent vector to curve* C

As the definition of line integration in scalar field, the definition above is derived from the concept of Riemann summation. Let's take the following toy example to illustrate this.

Example 1 *Given a vector field in* $\mathbb{R}^2$ *defined by* $\mathbf{F}(x, y) = (x^2 - y^2 - 12)\,\mathbf{i} + 2xy\,\mathbf{j}$ *and a parametric curve* C *defined by following equations*

$$x = 6t\cos(5t) \qquad\qquad y = 6t^3 - 6$$

Derive the definition of line integration exploiting the concept of Riemann summation. Graph some Riemann approximation of the line integration of $\mathbf{F}(x, y)$ *along curve* C *for arbitrary* n

Riemann summation performed on $\mathbf{F}(x, y)$ along curve C can be achieve by the following steps:

1. Divide the curve C to n small subarcs[1] $\Delta s_1, \dots \Delta s_n$.

2. For every subarc i, evaluate $\mathbf{F}(x, y)$ and the unit tangent vector to the curve direction $\mathbf{T}(t)$ at some point[2] $t_i^* \equiv (x_i^*, y_i^*) \in \Delta s_i$.

3. Compute $[\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(t_i^*)]\ \Delta s_i$ for every subarc " i.e. the alignment between the vector field vector $\mathbf{F}(x_i^*, y_i^*)$ with the unit tangent vector $\mathbf{T}(t_i^*)$ of subarc i".

4. Sum $[\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)]\ \Delta s_i$ over n to approximate how much the vectors of $\mathbf{F}(x_i^*, y_i^*)$; along the path defined by C, align with C.

$$\text{Work} \approx \sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)] \Delta s_i$$

5. Riemann summation imply that n goes to infinity. Hence the Riemann summation gives us the line integration[3] of $\mathbf{F}(x, y)$ along curve C

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \lim_{n\to\infty} \sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)] \Delta s_i$$

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[1] small arcs

[2] choice of (x_i^*, y_i^*) will decide which Riemann sum we have. Either Right, Left, Midpoint Riemann sum

[3] recall that $\int \equiv lim_{n\to\infty} \sum_{i=1}^{n}$

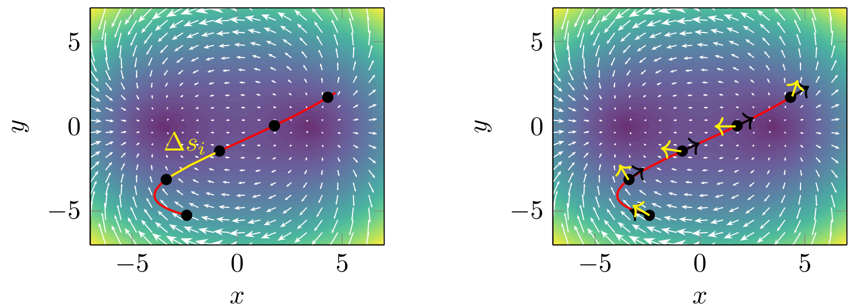

Figure 1: Left: Illustrating Δs_i. Right: Riemann summation for small n where vectors in yellow are $\mathbf{F}(x_i^*, y_i^*)$ and in black are $\mathbf{T}(x_i^*, y_i^*)$

The plot above is the approximated line integration of $\mathbf{F}(x, y)$ along curve C using Riemann sum. ■

In context of parametric curves, it is more convenient[4] to reformulate line integration formula "presented in definition (1)" in terms integration operator for the parametric variable "usually referred as t" rather than the arc length "usually referred as s" hence $ds \to dt$.

Definition 2 *Line integration of a vector field* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *along a smooth parametric curve* C *defined by vector function* $\mathbf{r}(t)$, $a \leq t \leq b$

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Example 2 *Evaluate* $\int_C \mathbf{F} \cdot\, d\mathbf{r}$, *where* C *is a quarter-circle defined through* $\mathbf{r}(t) = \cos t\; \mathbf{i} + \sin t\; \mathbf{j}$, $0 \leq t \leq \pi/2$ *and* $\mathbf{F}$ *is a vector field* $\mathbf{F}(x, y) = x^2\; \mathbf{i} - xy\; \mathbf{j}$.

From the parametric equation $x = \cos t$ and $y = \sin t$, hence we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\; \mathbf{i} - \cos t\; \sin t\; \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\; \mathbf{i} + \cos t\; \mathbf{j}$$

Therefore the line integration reads

$$\begin{aligned}
\int_C \mathbf{F} \cdot\, d\mathbf{r} &= \int_0^{\pi/2} \left(-\cos^2 t \sin t - \cos^2 t \sin t\right) dt \\
&= \int_0^{\pi/2} (-2\cos^2 t \sin t) dt \\
&= \left[2\frac{\cos^3 t}{3}\right]_0^{\pi/2} = -\frac{2}{3}
\end{aligned}$$

■

[4]computationally