Vector Calculus

March 17, 2024

1 Motivation

Placeholder

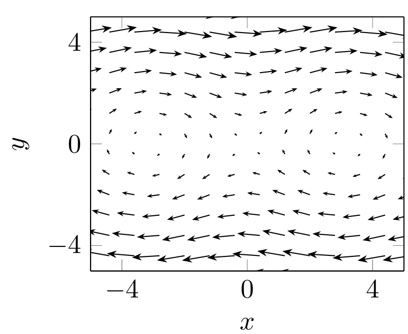
2 Introduction

Placeholder

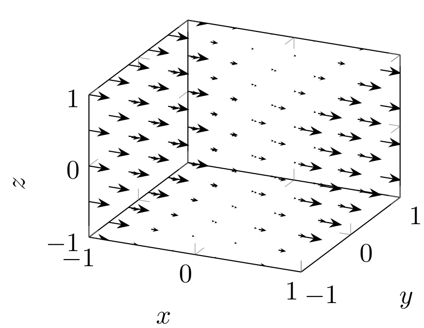
3 Vector Field

Definition 1 *A vector field on* $\mathbb{R}^n$ *is a function* $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *that assign to each point* $\mathbf{x}$ *in domain*[1] D *a vector* $\mathbf{F}(\mathbf{x}) \equiv \langle x_1, \dots, x_n \rangle \in \mathbb{R}^n$

For visualisation of vector fields defined in $\mathbb{R}^2$ or $\mathbb{R}^3$ one would plot some vectors samples across the domain to avoid clutter. See plots of vector fields in $\mathbb{R}^2$ or $\mathbb{R}^3$ receptively.



(a) $\mathbf{F} : (x, y) \mapsto \langle y, \sin x \rangle$



(b) $\mathbf{F} : (x, y, z) \mapsto \langle x^2, 0, 0 \rangle$

A beautiful observation from these plots, that is most vector fields can be seen as flowing fluid, which provide a beautiful and helpful interpretation about their behaviour.

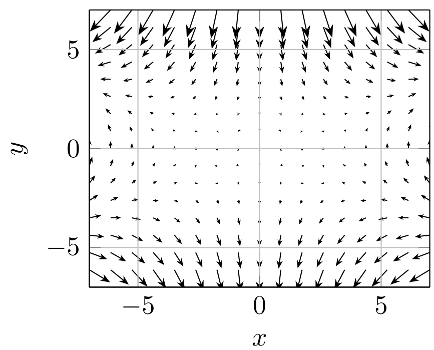
[1] domain could be a subset of $\mathbb{R}^n$ or all of it

A place where vector field might appear is in the representation of gradient of multivariable scalar function. Lets consider the following example.

Example 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function act on $\mathbb{R}^2$, with $f(x, y) = x^2y - y^3$. Write out $\nabla f(x, y)$ and plot it.*

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy \\ x^2 - 3y^2 \end{bmatrix} \equiv 2xy\ \mathbf{i} + (x^2 - 3y^2)\ \mathbf{j}$$

The gradient $\nabla f(x, y)$ is a vector field act on $\mathbb{R}^2$, and its visualization



This vector field hold information of the magnitude and direction of the gradient of $f(x, y)$ at any point $(x, y) \in \mathbb{R}^2$. ∎

4 Line Integrals

Anther extension to the concept of integration is computing of the integration of a mathematical form[2] along a curve. This operation is called line integration.

Definition 2 *Line integration of a mathematical form* $\mathbf{F}$ *is a the cumulative change of* $\mathbf{F}$ *along a path defined by some curve* C

The mathematical forms that we will address in this script, are scalar fields and vector fields

4.1 Scalar Fields

Definition 3 *Line integration of a scalar field* $f : \mathbb{R}^n \to \mathbb{R}$ *along a smooth curve* C

$$\int_C f(\mathbf{x})\, ds \tag{1}$$

The definition above is derived from the concept of Riemann summation. To illustrate this, we will we take the following toy example.

Example 2 *Given a scalar valued function* $f(x, y) = 0.6\cos\left(\sqrt{x^2 + y^2}\right) + 3$ *and a parametric curve* C *defined by following equations*

$$x = 6t\cos(5t) \qquad\qquad y = 6t^3 - 6$$

Derive the definition of line integration exploiting the concept of Riemann summation. Graph some Riemann approximation of the line integration of $f(x, y)$ *along curve* C *for arbitrary* n

Riemann summation performed on $f(x, y)$ along curve C can be achieve by the following steps:

1. Divide the curve C to n small subarcs[3] $\Delta s_1, \ldots \Delta s_n$.
2. For every subarc i, evaluate $f(x, y)$ at some point[4] $(x_i^*, y_i^*) \in \Delta s_i$.
3. Compute $f(x_i^*, y_i^*)\Delta s_i$ for every subarc "i.e the area of a rectangle with length $f(x_i^*, y_i^*)$ and width Δs_i for subarc i".
4. Sum $f(x_i^*, y_i^*)\Delta s_i$ over n "i.e. the rectangles" to approximate the area under $f(x, y)$ along path C.

$$\text{Area under the curve} \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

[2]function

[3]small arcs

[4]choice of (x_i^*, y_i^*) will decide which Riemann sum we have. Either Right, Left, Midpoint Riemann sum

5. Riemann summation imply that n goes to infinity "i.e. the rectangles become infinitely small". Hence the Riemann summation gives us the line integration[5] of $f(x, y)$ along curve C

$$\int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

For arbitrary chosen n, that is not too large; the Riemann sum would look rectangles that cover the area between curve C and function $f(x, y)$. The plot

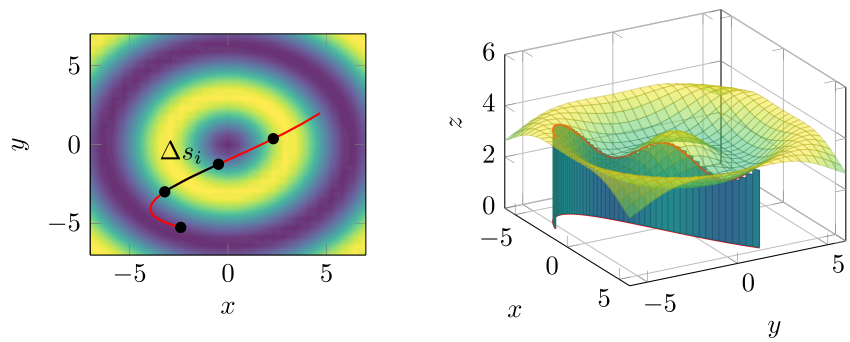


Figure 2: Left: Illustrating Δs_i. Right: Riemann summation for small n

above is the approximated line integration of $f(x, y)$ along curve C using Riemann sum. ■

In context of parametric curves, it is more convenient[6] to reformulate line integration formula "presented in definition (5)" in terms integration operator for the parametric variable "usually referred as t" rather than the arc length "usually referred as s" hence $ds \to dt$.

Definition 4 *Line integration of a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ along a smooth parametric curve curve C that defined by variable t*

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

given that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

This definition can be generalized to $\mathbb{R}^n$

[5] recall that $\int \equiv lim_{n\to\infty} \sum_{i=1}^{n}$
[6] computationally

Example 3 *Evaluate* $\int_C (2+x^2y)ds$*, where* C *is the upper half of the unit circle* $x^2+y^2=1$.

The upper half of the unit circle can be expressed using the parametric curve defined by the following equations

$$x = \cos t \qquad\qquad y = \sin t$$

Hence the line integral reads

$$\begin{aligned}
\int_C (2+x^2y)ds &= \int_0^\pi (2+\cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \\
&= \int_0^\pi (2+\cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}dt \\
&= \int_0^\pi (2+\cos^2 t \sin t)dt \\
&= \left[2t - \frac{cos^3 t}{3}\right]_0^\pi = 2\pi + \frac{2}{3}
\end{aligned}$$

■

4.2 Vector Fields

Definition 5 *Line integration of a vector field* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *along a smooth curve* C

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{2}$$

where $\mathbf{T}$ *is the tangent vector to curve* C

As the definition of line integration in scalar field, the definition above is derived from the concept of Riemann summation. Let's take the following toy example to illustrate this.

Example 4 *Given a vector field in* $\mathbb{R}^2$ *defined by* $\mathbf{F}(x,y) = (x^2-y^2-12)\,\mathbf{i} + 2xy\,\mathbf{j}$ *and a parametric curve* C *defined by following equations*

$$x = 6t\cos(5t) \qquad\qquad y = 6t^3 - 6$$

Derive the definition of line integration exploiting the concept of Riemann summation. Graph some Riemann approximation of the line integration of $\mathbf{F}(x,y)$ *along curve* C *for arbitrary* n

Riemann summation performed on $\mathbf{F}(x,y)$ along curve C can be achieve by the following steps:

1. Divide the curve C to n small subarcs[7] $\Delta s_1, \ldots \Delta s_n$.
2. For every subarc i, evaluate $\mathbf{F}(x, y)$ and the unit tangent vector to the curve direction $\mathbf{T}(t)$ at some point[8] $t_i^* \equiv (x_i^*, y_i^*) \in \Delta s_i$.
3. Compute $[\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(t_i^*)]\ \Delta s_i$ for every subarc " i.e. the alignment between the vector field vector $\mathbf{F}(x_i^*, y_i^*)$ with the unit tangent vector $\mathbf{T}(t_i^*)$ of subarc i".
4. Sum $[\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)]\ \Delta s_i$ over n to approximate how much the vectors of $\mathbf{F}(x_i^*, y_i^*)$; along the path defined by C, align with C.

$$\text{Work} \approx \sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)] \Delta s_i$$

5. Riemann summation imply that n goes to infinity. Hence the Riemann summation gives us the line integration[9] of $\mathbf{F}(x, y)$ along curve C

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*) \cdot \mathbf{T}(x_i^*, y_i^*)] \Delta s_i$$

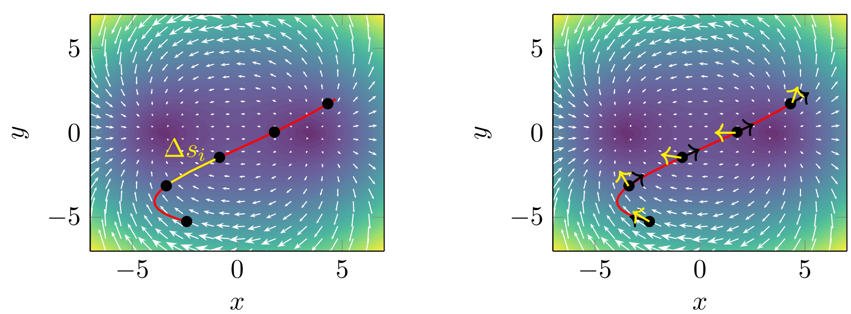


Figure 3: Left: Illustrating Δs_i. Right: Riemann summation for small n where vectors in yellow are $\mathbf{F}(x_i^*, y_i^*)$ and in black are $\mathbf{T}(x_i^*, y_i^*)$

The plot above is the approximated line integration of $\mathbf{F}(x, y)$ along curve C using Riemann sum. ■

In context of parametric curves, it is more convenient[10] to reformulate line integration formula "presented in definition (5)" in terms integration operator for the parametric variable "usually referred as t" rather than the arc length "usually referred as s" hence $ds \to dt$.

[7] small arcs

[8] choice of (x_i^*, y_i^*) will decide which Riemann sum we have. Either Right, Left, Midpoint Riemann sum

[9] recall that $\int \equiv lim_{n\to\infty} \sum_{i=1}^{n}$

[10] computationally

Definition 6 *Line integration of a vector field* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *along a smooth parametric curve* C *defined by vector function* $\mathbf{r}(t)$, $a \le t \le b$

$$\int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

Example 5 *Evaluate* $\int_C \mathbf{F} \cdot \, d\mathbf{r}$, *where* C *is a quarter-circle defined through* $\mathbf{r}(t) = \cos t \; \mathbf{i} + \sin t \; \mathbf{j}$, $0 \le t \le \pi/2$ *and* $\mathbf{F}$ *is a vector field* $\mathbf{F}(x, y) = x^2 \; \mathbf{i} - xy \; \mathbf{j}$.

From the parametric equation $x = \cos t$ and $y = \sin t$, hence we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \; \mathbf{i} - \cos t \; \sin t \; \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t \; \mathbf{i} + \cos t \; \mathbf{j}$$

Therefore the line integration reads

$$\begin{aligned} \int_C \mathbf{F} \cdot \, d\mathbf{r} &= \int_0^{\pi/2} \left(-\cos^2 t \sin t - \cos^2 t \sin t \right) dt \\ &= \int_0^{\pi/2} (-2\cos^2 t \sin t) dt \\ &= \left[2 \frac{\cos^3 t}{3} \right]_0^{\pi/2} = -\frac{2}{3} \end{aligned}$$

■

References