Definition 1 *Line integration of a scalar field $f : \mathbb{R}^n \rightarrow \mathbb{R}$ along a smooth curve C*

$$\int_C f(\mathbf{x})\ ds \tag{1}$$

The definition above is derived from the concept of Riemann summation. To illustrate this, we will we take the following toy example.

Example 1 *Given a scalar valued function $f(x,y) = 0.6\cos\left(\sqrt{x^2+y^2}\right)+3$ and a parametric curve C defined by following equations*

$$x = 6t\cos(5t) \qquad\qquad y = 6t^3 - 6$$

Derive the definition of line integration exploiting the concept of Riemann summation. Graph some Riemann approximation of the line integration of $f(x,y)$ along curve C for arbitrary n

Riemann summation performed on $f(x,y)$ along curve C can be achieve by the following steps:

1. Divide the curve C to n small subarcs[1] $\Delta s_1, \ldots \Delta s_n$.
2. For every subarc i, evaluate $f(x,y)$ at some point[2] $(x_i^*, y_i^*) \in \Delta s_i$.
3. Compute $f(x_i^*, y_i^*)\Delta s_i$ for every subarc "i.e the area of a rectangle with length $f(x_i^*, y_i^*)$ and width Δs_i for subarc i".
4. Sum $f(x_i^*, y_i^*)\Delta s_i$ over n "i.e. the rectangles" to approximate the area under $f(x,y)$ along path C.

$$\text{Area under the curve} \approx \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

5. Riemann summation imply that n goes to infinity "i.e. the rectangles become infinitely small". Hence the Riemann summation gives us the line integration[3] of $f(x,y)$ along curve C

$$\int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

For arbitrary chosen n, that is not too large; the Riemann sum would look rectangles that cover the area between curve C and function $f(x,y)$. The plot

[1] small arcs

[2] choice of (x_i^*, y_i^*) will decide which Riemann sum we have. Either Right, Left, Midpoint Riemann sum

[3] recall that $\int \equiv lim_{n\to\infty} \sum_{i=1}^{n}$

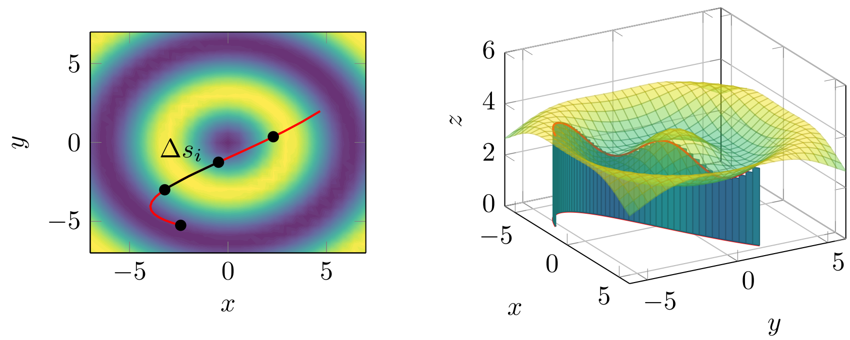

Figure 1: Left: Illustrating Δs_i. Right: Riemann summation for small n

above is the approximated line integration of $f(x, y)$ along curve C using Riemann sum. ■

In context of parametric curves, it is more convenient[4] to reformulate line integration formula "presented in definition (1)" in terms integration operator for the parametric variable "usually referred as t" rather than the arc length "usually referred as s" hence $ds \to dt$.

Definition 2 *Line integration of a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ along a smooth parametric curve curve C that defined by variable t*

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

given that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

This definition can be generalized to $\mathbb{R}^n$

Example 2 *Evaluate $\int_C (2+x^2y)ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.*

The upper half of the unit circle can be expressed using the parametric curve defined by the following equations

$$x = \cos t \qquad\qquad y = \sin t$$

[4]computationally

Hence the line integral reads

$$
\begin{aligned}
\int_C (2 + x^2 y) ds &= \int_0^{\pi} (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^{\pi} (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\
&= \int_0^{\pi} (2 + \cos^2 t \sin t) dt \\
&= \left[2t - \frac{cos^3 t}{3} \right]_0^{\pi} = 2\pi + \frac{2}{3}
\end{aligned}
$$

■