Vector Calculus

February 3, 2024

1 Motivation

Placeholder

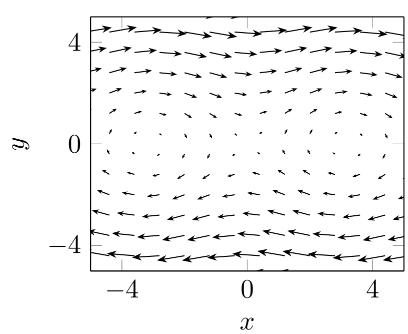
2 Introduction

Placeholder

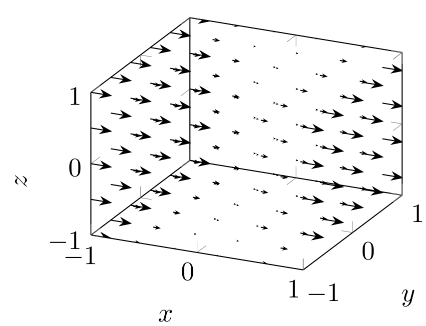
3 Vector Field

Definition 1 *A vector field on* $\mathbb{R}^n$ *is a function* $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *that assign to each point* $\mathbf{x}$ *in domain*[1] D *a vector* $\mathbf{F}(\mathbf{x}) \equiv \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$

For visualisation of vector fields defined in $\mathbb{R}^2$ or $\mathbb{R}^3$ one would plot some vectors samples across the domain to avoid clutter. See plots of vector fields in $\mathbb{R}^2$ or $\mathbb{R}^3$ receptively.



(a) $\mathbf{F} : (x, y) \mapsto \langle y, \sin x \rangle$



(b) $\mathbf{F} : (x, y, z) \mapsto \langle x^2, 0, 0 \rangle$

A beautiful observation from these plots, that is most vector fields can be seen as flowing fluid, which provide a beautiful and helpful interpretation about their behaviour.

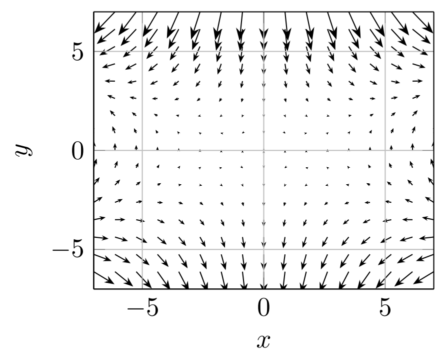
[1] domain could be a subset of $\mathbb{R}^n$ or all of it

A place where vector field might appear is in the representation of gradient of multivariable scalar function. Lets consider the following example.

Example 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function act on $\mathbb{R}^2$, with $f(x, y) = x^2y - y^3$. Write out $\nabla f(x, y)$ and plot it.*

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy \\ x^2 - 3y^2 \end{bmatrix} \equiv 2xy\ \mathbf{i} + (x^2 - 3y^2)\ \mathbf{j}$$

The gradient $\nabla f(x, y)$ is a vector field act on $\mathbb{R}^2$, and its visualization



This vector field hold information of the magnitude and direction of the gradient of $f(x, y)$ at any point $(x, y) \in \mathbb{R}^2$. ■

4 Line Integrals

Anther extension to the concept of integration is computing of the integration of a mathematical form[2] along a curve. This operation is called line integration.

Definition 2 *Line integration of a mathematical form* $\mathbf{F}$ *is a the cumulative change of* $\mathbf{F}$ *along a path defined by some curve* C

As for its computation; inherited from general idea of integration, line integral exploit the concept of Riemann summation in its evaluation of the integral. This is done with the following steps:

1. Divid the curve C to n small subarcs[3] $\Delta s_1, \ldots \Delta s_n$, see figure 3.
2. For every subarc i, evaluate the mathematical form $\mathbf{F}$ at some point[4] $P_i^* \in \Delta s_i$.

[2]function

[3]small arcs

[4]choice of P_i^* will decide which Riemann sum we have. Either Right, Left, Midpoint Riemann sum

3. Compute $\mathbf{F}(P_i^*)\Delta s_i$ for every subarc "i.e the area of a rectangle with length $\mathbf{F}(P_i^*)$ and width Δs_i for subarc i".

4. Sum the rectangles to approximate the area under $\mathbf{F}$ along path C.

Hence you have the general form of line integration of $\mathbf{F}$ along a path defined by some curve C

$$\int_C \mathbf{F} ds = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*)\Delta s_i \tag{1}$$

The following figure illustrate step 1 of computing the line integration of a mathematical form $\mathbf{F}$, which is dividing the curve C into subarcs Δs.

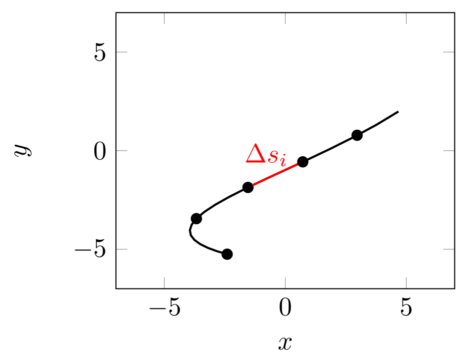


Figure 2: Dividing curve C in to subarcs Δs

As for the mathematical forms that we will address in this script, are scalar valued functions in $\mathbb{R}^2$, $\mathbb{R}^3$ and vector fields in $\mathbb{R}^2$, $\mathbb{R}^3$

4.1 Scalar Function

Let $f : \mathbb{R}^k \to \mathbb{R}$ a scalar valued function that maps set

Definition 3 *Line integration of a scalar valued function* $f(\mathbf{x})$ *along a smooth curve* C

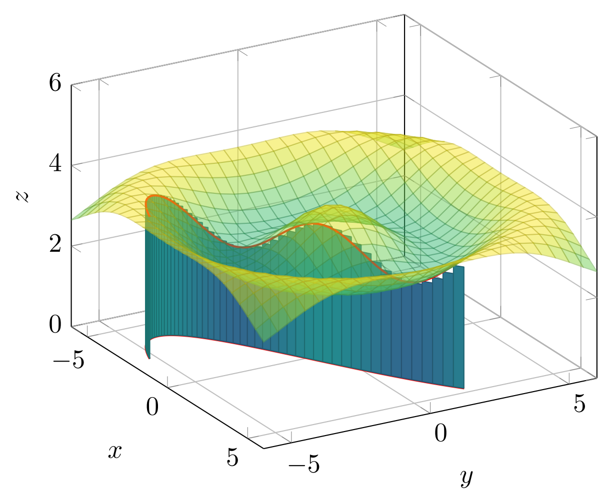
$$\int_C f(\mathbf{x})ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(\mathbf{x}_i^*)\Delta s_i \tag{2}$$

Example 2 *Given a scalar valued function* $f(x,y) = 0.6\cos\left(\sqrt{x^2+y^2}\right) + 3$ *and a parametric curve* C *defined by following equations*

$$x = 6t\cos(5t) \qquad y = 6t^3 - 6$$

,or equivalently by vector equation $r(t) = 6t\cos(5t)\ \mathbf{i} + (6t^3 - 6)\ \mathbf{j}$*; graph the Riemann sum approximation of the line integration of* $f(x,y)$ *along curve* C *for arbitrary* n

For arbitrary chosen n, that is not too large; the Riemann sum would look rectangles that cover the area between curve C and function $f(x,y)$.



The plot above is the approximated line integration of $f(x,y)$ along curve C using Riemann sum. ■

4.2 Vector Function

References

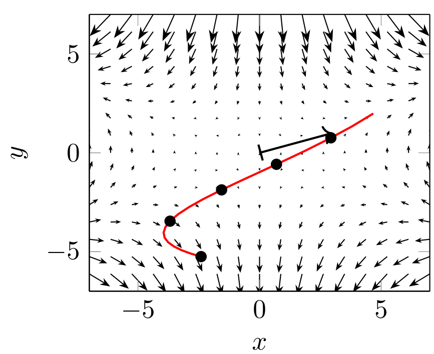


Figure 3: Dividing curve C in to subarcs Δs