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Introduction

Linear algebra is the branch of mathematics concerning linear equations and functions. Linear algebra is one of the corner stones of computational mathematics. such as Almost all numerical schemes such as the finite element and finite difference methods are in fact techniques that transform, assemble, reduce, rearrange, and/or approximate the differential, integral, or other types of equations to systems of linear algebraic equations.

Solving a system of linear algebraic equations is equivalent to finding the intersection point(s) of all surfaces (lines), represented by these equations, in a space of dimension equals the number of the given equations. If all surfaces happen to pass through a single point, then the solution is unique. If the intersected part is a line or a surface, there are an infinite number of solutions. Otherwise, the solution does not exist.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Systems of Linear Equations

Systems of linear equations lie at the heart of linear algebra. A system of n linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a set of equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where a_{ij}, b_j are real numbers. We may have systems of linear equations where the number of equations is different from the number of unknowns.

This system of equations is called homogeneous if $b_j = 0$ for all $j = 1,2,\cdots,n$ and non-homogeneous otherwise.

Possibility of solving a system of linear equations:

For a system of non-homogeneous linear equations:

(1) the system has a single (unique) solution.

(2) the system has an infinite number of solutions.

(3) the system has no solution at all.

For a system of homogeneous linear equations:

(1) the system has a single solution which is the zero solution.

(2) the system has an infinite number of solutions.

That is, a system of homogeneous linear equations always has at least one solution, the zero-solution. This zero-solution is usually called the trivial solution

A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.

There are two techniques for solving a system of linear algebraic equations: One of them depends on the matrices and determinants algebra, and the other depends on iterations with starting estimates for the solution.

Chapter One

Determinants

A determinant consists of elements listed in rows and columns of the same number between two vertical lines, the number of rows (or columns) is the order of the determinant. The determinant is denoted by $|A|, |B|, |C|, \cdots$. The determinants have many beneficial properties for studying vector spaces, matrices, and systems of equations.

The determinant consists of n rows is of order $n \times n$ or simply we can say of order n.

For examples:

$|A| = |a_{11}|$ of order 1, $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ of order 2, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ of order 3, $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ of order n .

The diagonal of the elements $a_{11}, a_{22}, a_{33}, \cdots, a_{nn}$ is called the main (principal) diagonal.

Determinants have many applications. **In our present study** we are going to use the determinants find the inverse of a square matrix, to solve system of linear algebraic equations (Cramer's Rule), and to evaluate the eigenvalues of square matrix.

Properties of Determinants

The properties of determinants can be very useful, they are complicated to prove.

(1) If we interchange all the rows and the columns, the value of the determinant does not change.

$$\begin{vmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 2 & -5 \\ 7 & -3 & 1 \\ -1 & 4 & -1 \end{vmatrix}$$

(2) If we interchange any two rows (or columns), then sign of the determinant changes.

$$\begin{vmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 4 & 7 & -1 \\ -5 & 1 & -1 \\ 2 & -3 & 4 \end{vmatrix} = -\begin{vmatrix} 7 & 4 & -1 \\ -3 & 2 & 4 \\ 1 & -5 & -1 \end{vmatrix}$$

(3) If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ 2 & -8 & 6 & 8 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ 0 & 4 & 1 & 2 \\ 6 & -24 & 18 & 24 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 4 & 6 & 4 \\ 3 & 6 & 5 & 10 \\ 2 & 4 & 7 & 8 \\ 1 & 2 & 0 & 6 \end{vmatrix} = 0$$

(4) Multiplying a determinant by k means multiplying the elements of only one row (or one column) by k.

$$5 \times \begin{vmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{vmatrix} = \begin{vmatrix} \underline{20} & \underline{35} & \underline{-5} \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 7 & -1 \\ \underline{10} & \underline{-15} & \underline{20} \\ -5 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} \underline{20} & 7 & -1 \\ \underline{10} & -3 & 4 \\ \underline{-25} & 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & \underline{35} & -1 \\ 2 & \underline{-13} & 4 \\ -5 & \underline{5} & -1 \end{vmatrix}$$

Similarly, we can take a common factor from elements of a row (or column):

$$|A| = \begin{vmatrix} 4 & 12 & 20 \\ 2 & -3 & 5 \\ 0 & 1 & 10 \end{vmatrix} = 4 \times \begin{vmatrix} 1 & 3 & 5 \\ 2 & -3 & 5 \\ 0 & 1 & 10 \end{vmatrix} = 2 \times 5 \times \begin{vmatrix} 1 & 3 & 1 \\ 2 & -3 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

(5) If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

$$|A| = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ 1 & -2 & 2 & 0 \\ 1 & -4 & 0 & 6 \end{vmatrix} \xrightarrow[-R_1 + R_4]{-R_1 + R_2} |A| = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 1 & -1 & -1 & 2 \\ 1 & -2 & 2 & 0 \\ -1 & 4 & -6 & -2 \end{vmatrix}$$

$$|B| = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ 1 & -2 & 2 & 0 \\ 1 & -4 & 0 & 6 \end{vmatrix} \xrightarrow[2R_1 + R_3]{R_1 + R_2} |B| = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 5 & -17 & 11 & 18 \\ 5 & -18 & 14 & 16 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

Value of a determinant

- The value of a determinant of order 1 is given as: $|A| = |a_{11}| = a_{11}$

$$\therefore \quad |A| = |5| = 5, \qquad |A| = |-5| = -5$$

- The value of a determinant of order 2 is given as:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

$$\therefore \begin{vmatrix} 5 & 2 \\ 8 & 3 \end{vmatrix} = 5 \times 3 - 2 \times 8 = -1, \qquad \begin{vmatrix} 2 & -4 \\ 6 & 3 \end{vmatrix} = 2 \times 3 - (-4) \times 6 = 30$$

- Evaluation of the value of a determinant of order ≥ 3 needs to know minors and co-factors of the elements of a determinant.

Minor of the element a_{ij} of a determinant $|A|$ is the determinant obtained by deleting i^th row and j^th column, and it is denoted by M_{ij}.

Co-factor of the element a_{ij} of a determinant $|A|$ is given by $A_{ij} = (-1)^{i+j} M_{ij}$, the sign of $(-1)^{i+j}$ is called sign of the element a_{ij} (which is positive if the sum of the numbers of the row and the column of the element is even, and negative if this sum is odd).

For example, the minors of the determinant: $|A| = \begin{vmatrix} 2 & 1 & -2 \\ 3 & 5 & 0 \\ -4 & 8 & 3 \end{vmatrix}$ are:

$M_{11} = \begin{vmatrix} 5 & 0 \\ 8 & 3 \end{vmatrix} = 15 - 0 = 15,$ $\qquad M_{12} = \begin{vmatrix} 3 & 0 \\ -4 & 3 \end{vmatrix} = 9 - 0 = 9,$

$M_{13} = \begin{vmatrix} 3 & 5 \\ -4 & 8 \end{vmatrix} = 24 + 20 = 44,$ $\qquad M_{21} = \begin{vmatrix} 1 & -2 \\ 8 & 3 \end{vmatrix} = 3 + 16 = 19,$

$M_{22} = \begin{vmatrix} 2 & -2 \\ -4 & 3 \end{vmatrix} = 6 - 8 = -2,$ $\qquad M_{23} = \begin{vmatrix} 2 & 1 \\ -4 & 8 \end{vmatrix} = 16 + 4 = 20,$

$M_{31} = \begin{vmatrix} 1 & -2 \\ 5 & 0 \end{vmatrix} = 0 + 10 = 10,$ $\qquad M_{32} = \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 0 + 6 = 6,$

$M_{33} = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 10 - 3 = 7.$

The co-factors of $|A|$ are:

$A_{11} = (-1)^2 M_{11} = 15,$ $\qquad A_{12} = (-1)^3 M_{12} = -9,$ $\qquad A_{13} = (-1)^4 M_{13} = 44,$

$A_{21} = (-1)^3 M_{21} = -19,$ $\qquad A_{22} = (-1)^4 M_{22} = -2,$ $\qquad A_{23} = (-1)^5 M_{23} = -20,$

$A_{31} = (-1)^4 M_{31} = 10, \qquad A_{32} = (-1)^5 M_{32} = -6, \qquad A_{33} = (-1)^6 M_{33} = 7.$

The value of a determinant of order ≥ 3 can be obtained by the sum of products of all elements of a row (or a column) with the corresponding co-factors (we can use any row or any column). That is,

$|A| = \sum_i A_{ij}$ for any row j, or $\quad |A| = \sum_j A_{ij}$ for any column j.

This method is called 'expansion of the determinant'.

Therefore, the value of $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is:

(a) Using the first row, we have

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{31} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{31} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(b) Using the third column, we have

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \overset{+}{a_{13}} \\ a_{21} & a_{22} & \overset{-}{a_{23}} \\ a_{31} & a_{32} & \overset{+}{a_{33}} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Note that the results we obtained by using the first row and the third column are the same.

Remarks:

1- If all the elements of a row (or column) are zeros, then we can use the co-factors of this row (or column) foe which the value of the determinant must be zero.

2- When expanding by co-factors, you do not need to evaluate the co-factors of zero entries, because a zero entry times its co-factor is zero. So that, the row (or column) containing the most zeros is usually the best choice for expansion by co-factors.

Example 1: Find the value of the determinant: $\quad |A| = \begin{vmatrix} 2 & 1 & -2 \\ 3 & 5 & 0 \\ -4 & 8 & 3 \end{vmatrix}$

Solution:

(a) Using the first row, we have:

$$|A| = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 2 & 1 & -2 \\ 3 & 5 & 0 \\ -4 & 8 & 3 \end{vmatrix} \end{array} = 2\begin{vmatrix} 5 & 0 \\ 8 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 0 \\ -4 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 5 \\ -4 & 8 \end{vmatrix}$$

$$= 2(5 \times 3 - 0) - (9 - 0) + (-2)(24 + 20) = -67$$

(b) Using the third column, we have:

$$|A| = \begin{vmatrix} 2 & 1 & -2 \\ 3 & 5 & 0 \\ -4 & 8 & 3 \end{vmatrix} \begin{array}{c} + \\ - \\ + \end{array} = -2\begin{vmatrix} 3 & 5 \\ -4 & 8 \end{vmatrix} - 0 + 3\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}$$

$$= -2(24 + 20) + 3(10 - 3) = -67$$

Note that the results we obtained by using the first row and the third column are the same.

<u>Example 2</u>: Find the value of the determinants:

$$(i)|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 5 \end{vmatrix} \quad (\text{ii})|B| = \begin{vmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{vmatrix} \quad (\text{iii})|C| = \begin{vmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 2 \\ 0 & 3 & -6 & 1 \end{vmatrix}$$

<u>Solution</u>:

(i) Expanding along the first row:

$$\therefore |A| = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 5 \end{vmatrix} \end{array} = \begin{vmatrix} -1 & 2 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$= (-5 - 8) - 2(10 + 2) + 3(8 - 1) = -16$$

(ii) Expanding along the second row:

$$(\text{ii})|B| = \begin{vmatrix} 4 & 7 & -1 \\ \overset{-}{2} & \overset{+}{-3} & \overset{-}{4} \\ -5 & 1 & -1 \end{vmatrix} = -2\begin{vmatrix} 7 & -1 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ -5 & -1 \end{vmatrix} - 4\begin{vmatrix} 4 & 7 \\ -5 & 1 \end{vmatrix}$$

$$= -2(-7 + 1) - 3(-4 - 5) - 4(4 + 35) = -117$$

(iii) Expanding along the first column:

$$|C| = \begin{array}{c} + \\ - \\ + \\ - \end{array}\begin{vmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 2 \\ 0 & 3 & -6 & 1 \end{vmatrix} = 3 \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 8 & 1 & 2 \\ 1 & 8 & 2 \\ 3 & -6 & 1 \end{vmatrix} \end{array} + 6 \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 0 & -4 & 0 \\ 8 & 1 & 2 \\ 3 & -6 & 1 \end{vmatrix} \end{array}$$

$$= 3\left\{8\begin{vmatrix} 8 & 2 \\ -6 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 8 \\ 3 & -6 \end{vmatrix}\right\} + 6 \times 4 \begin{vmatrix} 8 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 3\{8 \times 20 + 5 - 2 \times 30\} + 6 \times 4 \times 2 = 315 + 48$$

<u>The upper triangular form of a determinant</u>

The upper triangular form of a determinant is a determinant in which all the elements below the main diagonal are zeros.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn}$$

We can obtain the triangular form of a determinant by applying the properties of the determinants, as we will see in the following.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$: Interchanging the rows i and j, then sign of the determinant changes.
2. α cf R_i: Taking α as common factor from row i.
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

The above statements are called elementary row operations.

<u>Example 1</u>: Find the upper triangular form of $|A| = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 0 & 2 \\ -1 & 4 & 5 \end{vmatrix}$, and then find its value:

<u>Solution</u>:

$$|A| = \begin{vmatrix} 4 & 2 & 3 \\ 2 & 0 & 2 \\ -1 & 4 & 5 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} - \begin{vmatrix} -1 & 4 & 5 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{vmatrix} \xrightarrow[4R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} - \begin{vmatrix} -1 & 4 & 5 \\ 0 & 8 & 12 \\ 0 & 18 & 23 \end{vmatrix}$$

$$\xrightarrow{-\frac{18}{8}R_2+R_3\to R_3} -\begin{vmatrix} -1 & 4 & 5 \\ 0 & 8 & 12 \\ 0 & 0 & -4 \end{vmatrix} = -(-1)(8)(-4) = -32$$

Example 2: Find the upper triangular form of the determinant, then find its value.

$$(i)|A| = \begin{vmatrix} 2 & 3 & 1 & 0 \\ 1 & 4 & 4 & 6 \\ 3 & 1 & 0 & 3 \\ 5 & 8 & 2 & 6 \end{vmatrix} \qquad (\text{ii})|B| = \begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ -3 & 4 & -5 & 1 \\ 1 & 6 & 0 & 3 \end{vmatrix}$$

Solution: We apply the operations on the determinant as follows:

$$(i)|A| = \begin{vmatrix} 2 & 3 & 1 & 0 \\ 1 & 4 & 4 & 6 \\ 3 & 1 & 0 & 3 \\ 5 & 8 & 2 & 6 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} -\begin{vmatrix} 1 & 4 & 4 & 6 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 3 \\ 5 & 8 & 2 & 6 \end{vmatrix} \xrightarrow[-5R_1+R_4]{\substack{-2R_1+R_2 \\ -3R_1+R_3}}$$

$$-\begin{vmatrix} 1 & 4 & 4 & 6 \\ 0 & -5 & -7 & -12 \\ 0 & -11 & -12 & -15 \\ 0 & -12 & -18 & -24 \end{vmatrix} \xrightarrow[-\frac{12}{5}R_2+R_4\to R_4]{-\frac{11}{5}R_2+R_3\to R_3} -\begin{vmatrix} 1 & 4 & 4 & 6 \\ 0 & -5 & -7 & -12 \\ 0 & 0 & \frac{17}{5} & \frac{57}{5} \\ 0 & 0 & -\frac{6}{5} & \frac{24}{5} \end{vmatrix}$$

$$\xrightarrow{\frac{6}{17}R_3+R_4\to R_4} -\begin{vmatrix} 1 & 4 & 4 & 6 \\ 0 & -5 & -7 & -12 \\ 0 & 0 & \frac{17}{5} & \frac{57}{5} \\ 0 & 0 & 0 & \frac{150}{17} \end{vmatrix} = 150$$

$$(\text{ii})\ |B| = \begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ -3 & 4 & -5 & 1 \\ 1 & 6 & 0 & 3 \end{vmatrix} \xrightarrow[\frac{1}{5}R_1+R_4\to R_4]{-\frac{3}{5}R_1+R_3\to R_3} \begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & \frac{2}{5} & -5 & 1 \\ 0 & \frac{36}{5} & 0 & 3 \end{vmatrix}$$

$$\xrightarrow[\frac{1}{5}cfR_4]{\frac{1}{5}cf\,R_3} \frac{1}{25}\begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -25 & 5 \\ 0 & 36 & 0 & 15 \end{vmatrix} \xrightarrow[-36R_2+R_4\to R_4]{-2R_2+R_3\to R_3}$$

$$\frac{1}{25}\begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -23 & 1 \\ 0 & 0 & 36 & -57 \end{vmatrix} \xrightarrow{\frac{36}{23}R_3+R_4\to R_4} \frac{1}{25}\begin{vmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -23 & 1 \\ 0 & 0 & 0 & \frac{-1275}{23} \end{vmatrix}$$

$$= \frac{1}{25}\left[-5\times 1\times(-23)\times\left(\frac{-1275}{23}\right)\right] = -255$$

Example 3: Find the upper triangular form of the determinant, and then find its value:

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 4 & 2 & 5 & 0 \\ 3 & 0 & 3 & 3 \\ 1 & 4 & 1 & 5 \end{vmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 4 & 2 & 5 & 0 \\ 3 & 0 & 3 & 3 \\ 1 & 4 & 1 & 5 \end{vmatrix} \xrightarrow[-R_1+R_4]{\substack{-4R_1+R_2 \\ -3R_1+R_3}} \begin{vmatrix} 1 & 2 & 1 & 4 \\ 0 & -6 & 1 & -16 \\ 0 & -6 & 0 & -9 \\ 0 & 2 & 0 & 1 \end{vmatrix}$$

$$\xrightarrow[\frac{1}{3}R_2+R_4]{-R_2+R_3} \begin{vmatrix} 1 & 2 & 1 & 4 \\ 0 & -6 & 1 & -16 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & \frac{1}{3} & \frac{-13}{3} \end{vmatrix} \xrightarrow{\frac{1}{3}R_3+R_4} \begin{vmatrix} 1 & 2 & 1 & 4 \\ 0 & -6 & 1 & -16 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -12$$

Cramer’s Rule:

Cramer’s rule, named after Gabriel Cramer (1704–1752), is a formula that uses determinants to solve a system of algebraic linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where the number of equations is equal to the number of unknowns.

According to Cramer's rule, if the system of linear equations has a nonzero coefficient determinant, that is:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

The value of each unknown is the quotient of two determinants. The denominator is the coefficient determinant, and the numerator is the determinant formed by replacing the column corresponding to the unknown being solved for with the column representing the constants. That is:

$$x_k = \frac{\Delta_{x_k}}{\Delta}$$

where: $$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad \Delta_{x_2} = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}$$, and so on.

Possibility of solving a system of linear equations

For a system of homogeneous linear equations ($b_k = 0$ for all $k = 1,2,\cdots,n$):

(1) if $\Delta \neq 0$, the system has only trivial solution (zeros).

(2) if $\Delta = 0$, the system has an infinite number of solutions, including the zero solution. In this case Cramer's rule does not work.

For a system of non-homogeneous linear equations:

(1) If $\Delta \neq 0$, the system has a single solution.

(2) If $\Delta = 0, \Delta_{x_k} = 0$ for all $k = 1,2,\cdots,n$, the system has an infinite number of solutions Cramer's rule does not work.

(3) If $\Delta = 0, \Delta_{x_k} \neq 0$ for some values of k , the system has no solution at all.

Example 1: Using Cramer's rule, discuss the possibility of solving the equations.

$$x + y = 3, \quad 4x - y = 2$$

Solution: We have non-homogeneous system of two equations.

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -5 \neq 0, \qquad \Delta_{\text{x}} = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5, \qquad \Delta_{\text{y}} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10$$

Thus, the given system has a unique solution,

$$x = \frac{\Delta_x}{\Delta} = \frac{-5}{-5} = 1, \quad y = \frac{\Delta_y}{\Delta} = \frac{-10}{-5} = 2.$$

Example 2: Using Cramer's rule, discuss the possibility of solving the equations.

$$x + 2y + 2z = 6, \quad 2x + 4y + z = 7, \quad 3x + 2y + 9z = 14$$

Solution: We have a non-homogeneous system of three equations.

$$\Delta = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = (36 - 2) - 2(18 - 3) + 2(4 - 12) = -12 \neq 0$$

$$\Delta_{\text{x}} = \begin{vmatrix} 6 & 2 & 2 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = 6(36 - 2) - 2(63 - 14) + 2(14 - 56) = 22$$

$$\Delta_{\text{y}} = \begin{vmatrix} 1 & 6 & 2 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = (63 - 14) - 6(18 - 3) + 2(28 - 21) = -27$$

$$\Delta_{\text{z}} = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = (56 - 14) - 2(28 - 21) + 6(4 - 12) = -20$$

Thus, the given system has a unique solution,

$$x = \frac{\Delta_x}{\Delta} = -\frac{22}{12} = -\frac{11}{6}, \quad y = \frac{\Delta_y}{\Delta} = \frac{27}{12}, \quad z = \frac{\Delta_z}{\Delta} = \frac{20}{12} = \frac{5}{3}$$

Example 3: Using Cramer's rule, discuss the possibility of solving the equations.

$$2x + 5y - 3z = 3, \quad x - 2y + z = 2, \quad 7x + 4y - 3z = -4$$

Solution: We have a non-homogeneous system of three equations.

$$\Delta = \begin{vmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 7 & 4 & -3 \end{vmatrix} = 2(6 - 4) - 5(-3 - 7) - 3(4 + 14) = 0$$

$$\Delta_x = \begin{vmatrix} 3 & 5 & -3 \\ 2 & -2 & 1 \\ -4 & 4 & -3 \end{vmatrix} = 3(6-4) - 5(-6+4) - 3(8-8) = 16 \neq 0$$

Thus, the given system has no solution.

Example 4: Using Cramer's rule, discuss the possibility of solving the equations.

$$2x + y + 5z = 0, \quad -2x + y - 3z = 0, \quad 2x - y = 0$$

Solution: We have a homogeneous system of three equations.

$$\Delta = \begin{vmatrix} 2 & 1 & 5 \\ -2 & 1 & -3 \\ 2 & -1 & 0 \end{vmatrix} = 2(0-3) - (0+6) + 5(2-2) = -12 \neq 0$$

Thus, the given system has only trivial solution, $x = 0, \ \ y = 0, \ \ z = 0$

Exercise:

1. Expand each of the following determinants (Find the value of the determinant):

$(i)|A| = \begin{vmatrix} 2 & 3 & -2 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix}$ $\quad (ii)|B| = \begin{vmatrix} -4 & 7 & 1 \\ 5 & 0 & 3 \\ 2 & 0 & -1 \end{vmatrix}$ $\quad (iii)|C| = \begin{vmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

2. Find the upper triangular formula of each of the following determinants, then find its value.

$(i)|A| = \begin{vmatrix} 1 & 0 & 1 & 4 \\ 4 & 2 & 6 & 2 \\ 3 & 0 & 3 & 9 \\ 1 & 4 & 1 & 5 \end{vmatrix}$ $\quad$ (ii)$|B| = \begin{vmatrix} 2 & 2 & 1 & 4 \\ 4 & 2 & 0 & 2 \\ 0 & 0 & 3 & 4 \\ 1 & 4 & 1 & 5 \end{vmatrix}$

3. Using Cramer's rule, discuss the possibility of solving the equations:

i) $-2x + 3y - 5z = -11, \ \ 4x - y + z = -3, \ \ -x - 4y + 6z = 15$

ii) $5x - 2y + z = 15, \ \ 3x - 3y - z = -7, \ \ 2x - y - 7z = 3$

iii) $3x + 2y + 8z = 38, \ \ x + 3y + 9z = 37, \ \ 2x + y + z = 15$

iv) $x + 2z = 1, \ \ 3x - y + z = 2, \ \ 4y + 5z = -1$

v) $4x + 2y - z = 5, \ \ 3x - y + z = -2, \ \ x + y - z = 0$

Chapter Two

Matrices

The importance of matrices has increased because of the human need to display and classify data and store information, as well as to deal with that data in the form of tables, and that organizing the information in a specific way facilitates quick memory and comparison between them. The importance of arrays increased with the use of the computer and its ability to perform long mathematical operations in a short time. The important role of matrices is not limited to mathematics only, but also plays a large role in other sciences such as engineering, physics, chemistry, and others.

A matrix is an array of elements arranged in rows and columns. The numbers that make up a matrix are called the elements of a matrix, and these elements are written in parentheses on the figure $[\quad]$ or on the figure $(\quad)$. A matrix having *m* rows and *n* columns is said to have the order $m \times n$. Usually, we denote the matrix by one of the capital letters, such as A, B, C, $\cdots$ and we denote the elements of a matrix A by the symbol a_{ij} where *i* denotes the row number the element is in and *j* refers to the column number.

Hence a matrix A of order $m \times n$ can be represented in the following forms:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

A matrix having only one column is called **a column vector** or simply a vector; and a matrix with only one row is called **a row vector** and are often written in Boldface lowercase letters.

A (general) row vector is of the form $\boldsymbol{a} = [a_1 \quad a_1 \quad \cdots \quad a_n]$ and a column vector is of the form:

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Equality of two Matrices: Two matrices A and B having the same order $m \times n$ are equal if $a_{ij} = b_{ij}$ for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. In other words, two matrices are said to be equal if they have the same order and their corresponding elements are equal.

Special Matrices:

1. A matrix in which all elements are zeros is called a zero-matrix, denoted by [0]. For example,

$$[0]_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad [0]_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [0]_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. A matrix having the number of rows equal to the number of columns is called **a square matrix**. Thus, its order is $n \times n$ or n only. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where the elements $a_{11}, a_{22}, a_{33}, \cdots, a_{nn}$ are called the diagonal elements and form the principal (main) diagonal of A. The sum of the diagonal elements is called trace of the matrix; $Tr(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

3. A square matrix A is said to be a **diagonal matrix** if $a_{ij} = 0$ for all $i \neq j$ but non-zero elements appear on the principal diagonal. That is,

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

4. A diagonal matrix of order n with $a_{kk} = 1, k = 1,2, \cdots, n$ is called '**Identity matrix**', denoted by I_n (simply we can write I). **For examples**,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \cdots, \qquad I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. A square matrix is said to be **an upper triangular matrix** if $a_{ij} = 0$ for all $i > j$, and it is said to be a lower triangular matrix if $a_{ij} = 0$ for all $i < j$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

Mathematical operations on matrices

1. **Multiplying a scalar to a matrix:** If a scalar (number) is multiplied to a matrix, we multiply it to all the elements of the matrix, we can take a common factor from all the elements of a matrix. For example

$$\text{If} \quad A = \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ 3 & -1 \end{bmatrix} \Rightarrow 4A = \begin{bmatrix} 4 & 0 \\ 16 & 8 \\ 12 & -4 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 4 & 0 \\ 6 & 12 & 20 \\ 8 & 2 & 10 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 10 \\ 4 & 1 & 5 \end{bmatrix}$$

2. **Addition and subtraction of matrices:** To add (or subtract) two matrices A and B, they must be of the same order. If A and B are two matrices of the same order $m \times n$, then $C = A \pm B$ gives a matrix of the same order $m \times n$ such that each element of C is the sum of the corresponding elements of A and B. For examples:

$$\text{i) If} \quad A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \\ 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 5 & 7 \\ -5 & 6 \end{bmatrix}$$

$$\therefore \quad A + B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 5 & 7 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 9 \\ 0 & 7 \end{bmatrix},$$

$$A-B=\begin{bmatrix}1&0\\-3&2\\5&1\end{bmatrix}-\begin{bmatrix}3&0\\5&7\\-5&6\end{bmatrix}=\begin{bmatrix}-2&0\\-8&-5\\10&-5\end{bmatrix}$$

ii) If $A=\begin{bmatrix}2&3&0\\-1&4&0\\3&2&1\end{bmatrix}$, $B=\begin{bmatrix}3&0&0\\2&0&-1\\4&2&1\end{bmatrix}$, then

$$A+B-I=\begin{bmatrix}2&3&0\\-1&4&0\\3&2&1\end{bmatrix}+\begin{bmatrix}3&0&0\\2&0&-1\\4&2&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}4&3&0\\1&3&-1\\7&4&1\end{bmatrix}$$

$$A-2B+3I=\begin{bmatrix}2&3&0\\-1&4&0\\3&2&1\end{bmatrix}-\begin{bmatrix}6&0&0\\4&0&-2\\8&4&2\end{bmatrix}+\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}-1&3&0\\-5&7&2\\-5&-2&2\end{bmatrix}$$

3. **Multiplication of matrices**: The product $A\times B$ of two matrices A and B is defined if and only if the number of columns of A = the number of rows of B. Consider a matrix A of order $m\times r$ and a matrix B of order $r\times n$, then If $C=AB$

$$\therefore \qquad c_{ij}=\sum_{k=1}^{r}a_{ik}b_{kj}$$

When a matrix A multiplies a vector $\boldsymbol{x}$, it transforms $\boldsymbol{x}$ into the vector $A\boldsymbol{x}$.

Remarks:

(a) The matrix product AB may be defined (available), but matrix product BA is not.

(b) Suppose that the matrix products AB and BA are defined, in general $AB\neq BA$.

(c) If A is a square matrix, then $A^2=AA$

(d) Let A be an $m\times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

1. $ABC=A(BC)=(AB)C$ (associative law of multiplication)

2. $A(B+C)=AB+AC$ (left distributive law)

3. $(B+C)A=BA+CA$ (right distributive law)

4 d. $IA=AI=A$ (identity for matrix multiplication)

In (2) and (3) we can find the matrix $D=B+C$ and multiply it by the matrix A.

For examples:

1). If $\boldsymbol{a} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\boldsymbol{b} = [2 \quad 3 \quad 4]$

$$\therefore \quad \boldsymbol{ab} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} [2 \quad 3 \quad 4] = \begin{bmatrix} 1 \times 2 & 1 \times 3 & 1 \times 4 \\ 3 \times 2 & 3 \times 3 & 3 \times 4 \\ 2 \times 2 & 2 \times 3 & 2 \times 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 9 & 12 \\ 4 & 6 & 8 \end{bmatrix},$$

$$\boldsymbol{ba} = [2 \quad 3 \quad 4] \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = [\, 2 \times 1 + 3 \times 3 + 4 \times 2] = [19]$$

Note that: Even the products $\boldsymbol{ab}$ and $\boldsymbol{ba}$ are available, the results of the products are different.

2). If $A = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 2 & 8 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$\therefore \quad A\boldsymbol{b} = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 + 9 + 0 \\ 1 + 6 + 16 \end{bmatrix} = \begin{bmatrix} -8 \\ 23 \end{bmatrix}$$

That is, the matrix $\begin{bmatrix} -1 & 3 & 0 \\ 1 & 2 & 8 \end{bmatrix}$ transforms the vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ into the vector $\begin{bmatrix} -8 \\ 23 \end{bmatrix}$.

3). If $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 5 \\ 2 & 1 & 1 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 3 \times 0 + 4 \times 2 & 1 \times 6 + 3 \times 3 + 4 \times 1 & 1 \times 3 + 3 \times 5 + 4 \times 1 \\ 2 \times 1 + 0 \times 0 + 8 \times 2 & 2 \times 6 + 0 \times 3 + 8 \times 1 & 2 \times 3 + 0 \times 5 + 8 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 19 & 22 \\ 18 & 20 & 14 \end{bmatrix}$$

Note that: The product $B \times A$ is not available (defined).

4) If $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times (-1) - 1 \times 2 & -1 \times 0 - 1 \times 1 \\ 0 \times (-1) + 3 \times 2 & 0 \times 0 + 3 \times 1 \\ 2 \times (-1) + 4 \times 2 & 2 \times 0 + 4 \times 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 6 & 3 \\ 6 & 4 \end{bmatrix}$$

Note that: The product BA is not available (not defined).

5) If $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 1 & 2 \\ 8 & 0 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & 1 & 2 \\ 8 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 9-8 & 1+0 & 2-4 \\ 27+32 & 3+0 & 6+16 \\ 18-8 & 2+0 & 4-4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 59 & 3 & 2 \\ 10 & 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 9 & 1 & 2 \\ 8 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 9+3+4 & -9+4-2 \\ 8+0+8 & -8+0-4 \end{bmatrix} = \begin{bmatrix} 16 & -7 \\ 16 & -12 \end{bmatrix}$$

Note that: Even the products AB and BA are defined but $AB \neq BA$.

6) If $A = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & 3 \\ 2 & 1 & 6 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & 3 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & 3 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1+0-2 & 0+0-1 & -1+0-6 \\ 4+12+6 & 0+9+3 & -4+9+18 \\ 2+4+12 & 0+3+6 & -2+3+36 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -7 \\ 22 & 12 & 23 \\ 18 & 9 & 37 \end{bmatrix}$$

6) If $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 12 & -21 \\ 2 & -5 \end{bmatrix}$$

$$\therefore \quad ABC = (AB)C = \begin{bmatrix} 3 & -3 \\ 12 & -21 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 3 & 57 \\ -1 & 11 \end{bmatrix}$$

$$\text{OR} \quad BC = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 1 & -1 \\ 3 & 7 \end{bmatrix}$$

$$\therefore \quad ABC = A(BC) = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ 1 & -1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 3 & 57 \\ -1 & 11 \end{bmatrix}$$

7) Consider the diagonal matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 5^2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 3^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 5^2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 5^3 \end{bmatrix} =$$

Generally, it is easy to show that:

$$\text{If } A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \text{ then } A^k = \begin{bmatrix} {a_{11}}^k & 0 & 0 & 0 \\ 0 & {a_{22}}^k & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & {a_{nn}}^k \end{bmatrix}$$

4. **Transpose of a matrix**: The transpose of an $m \times n$ matrix A is denoted by A^T and is obtained by letting the rows to be the columns or vice-versa. Thus, the transpose of a row vector is a column vector and vice-versa. **For examples:**

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{a}^T = [1 \quad 0 \quad 5], \quad A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \\ 5 & 1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \quad \Rightarrow \quad B^T = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Let A and B are matrices whose sizes are appropriate for sums and products, then

$a. (A^T)^T = A$

$b. (A + B)^T = A^T + B^T$

$c. (AB)^T = B^T A^T$

5. Determinant of a square matrix: The determinant is a function that takes a square matrix as an input and produces a scalar as an output. The determinant of a square matrix A is denoted by det(A) or $|A|$ (note that $|A|$ is read as determinant of A and not, as modulus of A). The elements of $|A|$ are the same elements of the matrix A with the same order, that is:

If $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$

Invertible matrices

A square matrix A of order n is called invertible (or nonsingular) if there exists a square matrix B of the same order such that $AB = BA = I$. The matrix B is called a multiplicative inverse of A and is denoted by A^{-1}. A matrix that does not have an inverse is called noninvertible (or singular). If A is invertible its inverse is also invertible and gives the matrix A itself. Moreover, if A and B are two invertible matrices, their product AB is also invertible, with $(AB)^{-1} = B^{-1}A^{-1}$. It is worth to mention that not all matrices have inverses. Non-square matrices, for example, do not have inverses by definition. The square matrix A has inverse if $|A| \neq 0$.

For examples:

1. The matrices $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ are invertible, because,

$$AB = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad BA = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. The matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} \frac{12}{11} & -\frac{6}{11} & -\frac{1}{11} \\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22} \\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11} \end{bmatrix}$ are invertible, because:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{12}{11} & -\frac{6}{11} & -\frac{1}{11} \\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22} \\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$BA = \begin{bmatrix} \frac{12}{11} & -\frac{6}{11} & -\frac{1}{11} \\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22} \\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the following we will show how to find the inverse of a square matrix.

Inverse of Matrices

First, it is important to note that:

1. There is no inverse for a non-square matrix.

2. Only non-singular matrices have inverses, a matrix A is non-singular matrix if $|A| \neq 0$.

3. The inverse of a matrix (if exists) is unique. That's why we say "the" inverse matrix of A and denote it by A^{-1}, where both of them of the same order.

There are two ways to obtain the inverse of a square matrix. One of them depends on using the elementary row operations to determine the inverse of a matrix, and the other depends on using the determinants and the adjoint of the matrix.

The inverse of a matrix and determinants

If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be singular. Only non-singular matrices have inverses. Given any non-singular matrix A, its inverse can be found from the formula:

$$A^{-1} = \frac{adjA}{|A|}, \quad |A| \neq 0$$

where $adjA$ is the adjoint of the matrix A and $|A|$ is the determinant of the matrix A.

The procedure for finding the adjoint matrix is given below.

Finding the adjoint of a matrix:

The adjoint of a matrix A is the transpose of the co-factor matrix of A. The co-factor matrix $[A_{ij}]$ of A is the matrix formed by replacing each element of A by its cofactor. $AdjA = [A_{ij}]^T$.

Co-factor of the element a_{ij} is given by $A_{ij} = (-1)^{i+j}M_{ij}$, the sign of $(-1)^{i+j}$ is called sign of the element a_{ij} . Where **Minor** of the element a_{ij} is the determinant obtained by deleting ith row and jth column, and it is denoted by M_{ij}.

For example, the co-factors of the matrix: $A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 5 & 0 \\ -4 & 8 & 3 \end{bmatrix}$ are:

$$A_{11} = \begin{vmatrix} 5 & 0 \\ 8 & 3 \end{vmatrix} = 15, \quad A_{12} = -\begin{vmatrix} 3 & 0 \\ -4 & 3 \end{vmatrix} = -9, \quad A_{13} = \begin{vmatrix} 3 & 5 \\ -4 & 8 \end{vmatrix} = 44,$$

$$A_{21} = -\begin{vmatrix} 1 & -2 \\ 8 & 3 \end{vmatrix} = -19, \quad A_{22} = \begin{vmatrix} 2 & -2 \\ -4 & 3 \end{vmatrix} = -2, \quad A_{23} = -\begin{vmatrix} 2 & 1 \\ -4 & 8 \end{vmatrix} = -20,$$

$$A_{31} = \begin{vmatrix} 1 & -2 \\ 5 & 0 \end{vmatrix} = 10, \quad A_{32} = -\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = -6, \quad A_{33} = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7$$

Example 4: Using the determinants, find the inverse of the matrices:

$$(i)\ A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \quad (ii)\ B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 2 \\ 2 & 0 & 1 \end{bmatrix} \quad (iii)\ C = \begin{bmatrix} 4 & 2 & 3 \\ 7 & 5 & 4 \\ 9 & 2 & 6 \end{bmatrix}$$

Solution:

$$(i)\ |A| = \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8 \neq 0, \quad [A_{ij}] = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow AdjA = [A_{ij}]^T = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{AdjA}{|A|} = \frac{1}{8}\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$(ii)|B| = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 6 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 24 \neq 0$$

$$[B_{ij}] = \begin{bmatrix} +\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 6 \\ 2 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & 0 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 6 & 4 & -12 \\ 0 & 4 & 0 \\ 0 & -8 & 24 \end{bmatrix}$$

$$\Rightarrow \quad AdjB = \begin{bmatrix} 6 & 0 & 0 \\ 4 & 4 & -8 \\ -12 & 0 & 24 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{AdjB}{|B|} = \frac{1}{24}\begin{bmatrix} 6 & 0 & 0 \\ 4 & 4 & -8 \\ -12 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ 1/6 & 1/6 & -1/3 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$(iii)\ |C| = \begin{vmatrix} 4 & 2 & 3 \\ 7 & 5 & 4 \\ 9 & 2 & 6 \end{vmatrix} = 4\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} 7 & 4 \\ 9 & 6 \end{vmatrix} + 3\begin{vmatrix} 7 & 5 \\ 9 & 2 \end{vmatrix} = -17 \neq 0$$

$$[C_{ij}] = \begin{bmatrix} +\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 7 & 4 \\ 9 & 6 \end{vmatrix} & +\begin{vmatrix} 7 & 5 \\ 9 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} & +\begin{vmatrix} 4 & 3 \\ 9 & 6 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 9 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 7 & 4 \end{vmatrix} & +\begin{vmatrix} 4 & 2 \\ 7 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 22 & -6 & -31 \\ -6 & -3 & 10 \\ -7 & 5 & 6 \end{bmatrix}$$

$$\Rightarrow \quad AdjC = \begin{bmatrix} 22 & -6 & -7 \\ -6 & -3 & 5 \\ -31 & 10 & 6 \end{bmatrix}$$

$$\therefore \quad C^{-1} = \frac{AdjC}{|C|} = \frac{-1}{17}\begin{bmatrix} 22 & -6 & -7 \\ -6 & -3 & 5 \\ -31 & 10 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{22}{17} & \frac{6}{17} & \frac{7}{17} \\ \frac{6}{17} & \frac{3}{17} & -\frac{5}{17} \\ \frac{31}{17} & -\frac{10}{17} & -\frac{6}{17} \end{bmatrix}$$

Exercises

1. Find $2A+3B,\ A-2B$ for the matrices: $A=\begin{bmatrix}2 & -13\\ 0 & -12\end{bmatrix},\quad B=\begin{bmatrix}-3 & 0\\ 8 & 5\end{bmatrix}$

2. Find $2A+3B-5I,\ -B+I$ for the matrices:

$$A=\begin{bmatrix}2 & 5 & 0\\ 1 & 3 & 1\\ 7 & -2 & 4\end{bmatrix},\quad B=\begin{bmatrix}0 & 1 & 0\\ 1 & 2 & 4\\ 5 & 4 & 2\end{bmatrix}$$

3. If $A=\begin{bmatrix}1 & 3 & 0\\ 0 & 2 & -1\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 3 & 4\\ 2 & 4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2 & 3\\ 0 & 3 & 4\\ -1 & 5 & 2\end{bmatrix}$. Find, if it is available:

$A+B,\ A-B^T,\ A+C,\ AB,\ BA,\ AB^T,\ AC,\ B^2,\ C^2,\ |A|,\ |B|,\ |C|.$

4. Find the product of the matrices:

$(i)\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix},\quad (ii)\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}\begin{bmatrix}2 & 1 & 3\end{bmatrix},\quad (iii)\begin{bmatrix}2 & 1 & 3\\ -1 & 3 & 0\\ 4 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ -3 & 0 & 1\\ 1 & 3 & 3\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0 & 3\\ 0 & 3 & 2 & -1\end{bmatrix}$

5. Show that matrices A and B are invertible:

$i.\ A=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$

$ii.\ A=\begin{bmatrix}2 & -1 & 3\\ -5 & 3 & 1\\ -3 & 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}-7 & -9 & 10\\ -12 & -15 & 17\\ 1 & 1 & -1\end{bmatrix}$

6. Using determinants, find the inverse of the matrices (if possible):

(i) $A=\begin{bmatrix}3 & 2\\ -1 & 4\end{bmatrix}$ (ii) $B=\begin{bmatrix}1 & -2 & 0\\ 3 & 1 & 5\\ -1 & 2 & 3\end{bmatrix}$ (iii) $C=\begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ -1 & 3 & 0\end{bmatrix}$

Chapter Three

Matrix solutions of linear system of equations

As we know, system of n linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a set of equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where a_{ij}, b_j are real numbers.

The above system of equations can be written in a matrix equation as: $A\boldsymbol{x} = \mathbf{b}$, **w**here:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The matrix A is the coefficient matrix, $\boldsymbol{b}$ is the column representing the constants (constants matrix), and $\boldsymbol{x}$ is the column representing the unknowns.

Also, it can be written as the vector equation:

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 + \cdots + x_n\boldsymbol{a}_n = \mathbf{b}$$

wbhere: $\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{n3} \end{bmatrix}, \quad \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

The augmented matrix for a system of linear equations

The above system of equations has the same solution set as the linear system whose **augmented matrix** is:

$$[A \backslash \boldsymbol{b}] = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3 \quad \cdots \quad \boldsymbol{a}_n \quad \boldsymbol{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Each row of the augmented matrix represents one equation of the system. In each row, the coefficient of x_1 is in the first column, the coefficient of x_2 is in the second column, the coefficient of x_k is in the k^{th} column, and the constants are in the last column.

Important remarks:

I. If the number of equations (n) is greater than the number of unknowns (m), we can introduce $(n-m)$ nonzero artificial unknowns with zero coefficients to all the equations.
II. If the number of equations (n) is smaller than the number of unknowns (m), we can introduce $(m-n)$ zero rows at bottom of the augmented matrix $[A \backslash \boldsymbol{b}]$.

Elementary matrix row operations:

For an augmented matrix, there are three kinds of elementary row operations on matrices:

a) Interchanging two rows i and j (described by the symbolic shorthand $R_i \leftrightarrow R_j$).

b) Multiplying all (elements) of a row i by a nonzero constant (described by the a symbolic shorthand αR_i).

c) Multiply a row i by the scalar α and add to another row j (described by the a symbolic shorthand $\alpha R_i + R_j$).

An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent to the original linear system) system of linear equations. Two equivalent systems have the same set of solutions. Whenever a linear system has an infinite number of solutions, it is possible to obtain a parametric description of the solution set by row reducing the associated augmented matrix.

Remember that: Possibility of solving a system of linear equations can be recognized as:

For a homogeneous system of linear equations either

(1) the system has only the trivial solution (zeros);

(2) the system has an infinite number of solutions.

For a non-homogeneous system either

(1) the system has a single (unique) solution.

(2) the system has an infinite number of solutions.

(3) the system has no solution at all.

We now come to one of the most important use of matrices, that is, using matrices to solve systems of linear equations. If the coefficient matrix A is invertible, we can use its inverse as we will see after below. Many of the matrix solution methods depend on what we call **augmented matrices** and the elementary matrix row operations.

Gauss-Elimination method

This method is based on transforming the augmented matrix $[A\backslash \boldsymbol{b}]$ into a triangular one; write down the corresponding linear system of equations, which can be easily solved by back substitution. In the present approach, we use the upper triangular matrix. The element we use to make the elements below, in the same column, zeros is called pivot element (or leading for that row) and its row is called pivot row. If the pivot element is zero, we must interchange that pivot row with one below to obtain non-zero pivot. So that all rows consisting entirely of zeros occur at

the bottom of the matrix.

Suppose that, applying the elementary row operation on the augmented matrix $[A\backslash \boldsymbol{b}]$ gives the matrix $[C\backslash \boldsymbol{d}]$ such that:

$$[C\backslash \boldsymbol{d}] = \left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{nn} & d_n \end{array}\right] \sim [\boldsymbol{c}_1 \quad \boldsymbol{c}_2 \quad \boldsymbol{c}_3 \quad \cdots \quad \boldsymbol{c}_n \quad \boldsymbol{d}]$$

which equivalent to the system of vector equation:

$$x_1 \begin{bmatrix} c_{12} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} c_{12} \\ c_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} c_{1n} \\ c_{2n} \\ c_{3n} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Which is equivalent to the system of linear equations:

$$\begin{aligned} c_{11}x_1 + c_{12}\,x_2 + c_{13}\,x_3 + \cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + c_{23}\,x_3 + \cdots + c_{2n}\,x_n &= d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ c_{nn}x_n &= d_n \end{aligned}$$

which can be easily solved by back substitution in which we evaluate x_n from the last equation, and then evaluate x_{n-1} using x_n and so on. Practically, it is better to begin with the equation corresponding to the last row in the row reduced form.

Important remarks:

1. If no one of the pivots $c_{11}, c_{22}, c_{33}, \cdots, c_{nn}$ equals zero, we say that: rank of the coefficient matrix A = rank of augmented matrix $[A\backslash \boldsymbol{b}]$ = number of unknowns.

2. If the values of all elements (entries) of the last row(s) of the equivalent augmented matrix $[C\backslash \boldsymbol{d}]$ are zeros, we say that: rank of the coefficient matrix A = rank of augmented matrix $[A\backslash \boldsymbol{b}]$ less than the number of unknowns, and the system has an infinite number of solutions and the variable(s) associated to the zero pivot is (are) a free variable(s), so it (they) can be set arbitrarily, and the others are dependent variables. In this case the given system of equations is reduced to an equivalent system of equations in which the number of equations is less than the number of variables.

3. If the values of all elements (entries) of the last row(s) of the reduced matrix C are zeros but the associated element in $\boldsymbol{d}$ is nonzero (is a false statement), we say that: rank of the coefficient matrix A < rank of augmented matrix $[A\backslash \boldsymbol{b}]$, and the system has no solution at all.

4. If the number of equations (n) is greater than the number of unknowns (m), we can introduce $(n-m)$ nonzero artificial variables (unknowns), whose values are not needed, with zero coefficients to all the equations.

5. If the number of equations (n) is smaller than the number of unknowns (m), the given system of equations has an infinite number of solutions. In this case we can introduce $(m-n)$ zero rows at bottom of the augmented matrix $[A\backslash \boldsymbol{b}]$ which guarantees at least $(m-n)$ free variables.

Note that: Any rows of all zeros in matrix C must be appeared at the bottom of last augmented matrix $[C\backslash \boldsymbol{d}]$.

Example 1: Using Gauss-Elimination method, discuss the solution of the system of equations: $x+y-z=-2, \quad 2x-y+z=5, \quad -x+2y+2z=1$

Solution: The augmented matrix corresponding to the given system is:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right]$$

Now we want zeros for the elements in the first column below the pivot 1. The first zero can be obtained by multiplying the first row by (-2) and adding the results to the second row. The second zero can be obtained by adding the first row to the third row, note that the first row is unchanged. We use elementary row operations to obtain the reduced form as follows:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right] \xrightarrow[R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Thus, the system has a unique solution:

$$4z = 8 \;\Rightarrow\; z = 2, \quad -3y + 3z = 9 \;\Rightarrow\; y = -1, \quad x + y - z = -2 \;\Rightarrow\; x = 1$$

Example 2: Using Gauss-Elimination method, discuss the solution of the system of equations: $3x + 4y + 4z = 7, \quad x - y - 2z = 2, \quad 2x - 3y + 6z = 5$

Solution: It is better to rearrange the given equations to be:

$$x - y - 2z = 2, \qquad 3x + 4y + 4z = 7, \qquad 2x - 3y + 6z = 5$$

The augmented matrix corresponding to them is:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & 4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right] \xrightarrow[-2R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 7 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 7 & 10 & 1 \\ 0 & 0 & 80/7 & 8/7 \end{array}\right]$$

Thus, the system has a unique solution:

$$\frac{80}{7}z = \frac{8}{7} \;\Rightarrow\; z = \frac{1}{10}, \quad 7y + 10z = 1 \;\Rightarrow\; y = 0, \quad x - y - 2z = 2 \;\Rightarrow\; x = 2.2$$

Example 3: Using Gauss-Elimination method, discuss the solution of the system of equations: $x - 3y + z = 1, \quad 2x - y - 2z = 2, \quad x + 2y - 3z = -1$

Solution: The augmented matrix corresponding to the given system is:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 2 & -1 & -2 & 2 \\ 1 & 2 & -3 & -1 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 5 & -4 & 0 \\ 0 & 5 & -4 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 5 & -4 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

Because the third equation is a false statement, the system has no solution.

Example 4: Using Gauss-Elimination method, discuss the solution of the system of equations: $y - z = 0, \ x - 3z = -1, \ -x + 3y = 1$

Solution: The augmented matrix corresponding to the given system is:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -3 & -1 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the elements of the third row are zeros, then the system has an infinite number of solutions. Since the pivot of the third column is zero, then we choose z to be the free variable (arbitrary) and represent it by the parameter $z = c$. The first and second rows give:

$$z = c, \quad y - z = 0 \quad \Rightarrow \quad y = c, \quad x - 3z = -1, \quad x = 3c - 1,$$

where c is any real number.

Example 5: Using Gauss-Elimination method, discuss the solution of the system of equations: $4x + 2y - z = 5\ , \quad 3x - y + z = -2, \quad x + y - z = 0$

Solution: It is better to rearrange the given equations and the augmented matrix becomes:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 4 & 2 & -1 & 5 \\ 3 & -1 & 1 & -2 \end{array}\right] \xrightarrow[-3R_1+R_3]{-4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 5 \\ 0 & -4 & 4 & -2 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 5 \\ 0 & 0 & -2 & -12 \end{array}\right]$$

Then we have:

$$-2z = -12 \;\Rightarrow\; z = 6, \; -2y + 3z = 5 \;\Rightarrow\; y = \frac{13}{2}, \quad x + y - z = 0 \;\Rightarrow\; x = -\frac{1}{2}$$

<u>Example 6</u>: Using Gauss-Elimination method, discuss the solution of the system of equations: $-x + y + z = 0, \quad 3x - y = 0, \; 2x - 4y - 5z = 0$

<u>Solution</u>: The augmented matrix corresponding to the homogenous system is:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \xrightarrow[2R_1 + R_3]{3R_1 + R_2} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the elements of the third row are zeros, then the system has an infinite number of solutions. Since the pivot of the third column is zero and there is no rows below to interchange, then we choose z to be the free variable (arbitrary) and represent it by the parameter $z = c$ (real number). The first and second rows give:

$$z = c, \quad 2y + 3z = 0 \;\Rightarrow\; y = -3c/2, \quad -x + y + z = 0, \quad x = -c/2$$

<u>Example 7</u>: Using Gauss-Elimination method, discuss the solution of the system of equations: $x + y + z = 0, \quad x - y + 2z = 0, \quad 2x - y - z = 0$

<u>Solution</u>: The augmented matrix corresponding to the homogenous system is:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow[-2R_1 + R_3]{-R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{3}{2}R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -9/2 & 0 \end{array}\right]$$

Then, the system has the zero solution as a unique solution: $x = y = z = 0$

<u>Example 8</u>: Using Gauss-Elimination method, discuss the solution of the system of equations: $x_1 + 4x_2 - 5x_3 = 0, \; 2x_1 - x_2 + 8x_3 = 9$

<u>Solution</u>: It is seen that the number of equations is less than the number of variables by 1, so that we construct the augmented matrix as:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then the system has an infinite number of solutions. Since the pivot of the third column is zero, then we choose x_3 to be the free variable (arbitrary) and represent it by the parameter $x_3 = c$. The second and first rows give:

$$x_3 = c, \quad x_2 - x_3 = 0 \quad \Rightarrow \quad x_2 = c, \quad x_1 - 3x_3 = -1, \quad x_1 = 3c - 1,$$

where c is any real number.

Example 9: Using Gauss-Elimination method, discuss the solution of the system of equations:

$$x_1 - 2x_2 + x_3 = 5, \quad -x_1 + 2x_2 + x_3 - 2x_4 = 0, \quad 2x_1 + 3x_2 + x_3 + x_4 = 1$$

Solution: It is seen that the number of equations is less than the number of variables by 1, so that we construct the augmented matrix as:

$$[A|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 5 \\ -1 & 2 & 1 & -2 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-2R_1 + R_3]{R_1 + R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 5 \\ 0 & 0 & 2 & -2 & 5 \\ 0 & 7 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 5 \\ 0 & 7 & -1 & 1 & -9 \\ 0 & 0 & 2 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then the system has an infinite number of solutions. Since the pivot of the forth column is zero, then we choose x_4 to be the free variable (arbitrary) and represent it by the parameter $x_4 = c$ (real number). The third, second, and first rows give:

$$x_4 = c, \quad 2x_3 - 2x_4 = 5 \quad \Rightarrow \quad x_3 = \frac{5}{2} + c,$$

$$7x_2 - x_3 + x_4 = -9 \quad \Rightarrow \quad x_2 = -\frac{13}{17}, \quad x_1 - 2x_2 + x_3 = 5 \quad \Rightarrow \quad x_1 = \frac{3}{34} - c$$

Example 10: Using Gauss-Elimination method, discuss the solution of the system of equations: $x_1 + 2x_2 = 3$, $2x_1 + 3x_2 = 1$, $3x_1 + 2x_2 = 1$

Solution: It is seen that the number of equations is greater than the number of variables by 1, so that we introduce a nonzero artificial variable x_3 with zero coefficients to all the equations and the augmented matrix becomes:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -1 & 0 & -5 \\ 0 & -4 & 0 & -8 \end{array}\right]$$

$$\xrightarrow{-4R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Then the system has no solution at all.

Example 11: Using Gauss-Elimination method, discuss the solution of the system of equations:

$x_1 + x_2 = 3$, $2x_1 + x_2 + 2x_3 = 1$, $x_1 + 2x_2 + 3x_3 = -1$, $3x_1 + 3x_2 + 4x_3 = 1$

Solution: It is seen that the number of equations is greater than the number of variables by 1, so that we introduce a nonzero artificial variable x_4 with zero coefficients to all the equations and the augmented matrix becomes:

$$[A|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 2 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 0 & -1 \\ 3 & 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_4]{\substack{-2R_1+R_2 \\ -R_1+R_3}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & -1 & 2 & 0 & -5 \\ 0 & 2 & 3 & 0 & -4 \\ 0 & 0 & 4 & 0 & -8 \end{array}\right]$$

From the last row we obtain $4x_3 = -8 \Rightarrow x_3 = -2$, from the second or third row we have $2x_2 + 3x_3 = -4 \Rightarrow x_2 = 1$, and from the first row we have $x_1 + x_2 = 3 \Rightarrow x_1 = 2$.

Exercises

Using Gauss-Elimination, discuss the solution of the system of equations:

1) $x+y+z=1, \quad x-2y+2z=4, \quad x+2y-z=2$

2) $2x+y+5z=10, \quad x-3y-z=-2, \quad 2x-y+3z=6$

3) $12x+7y+3z=2, \quad x+5y+2z=-5, \quad 2x+7y-11z=6$

4) $x+y-z=-1, \quad x+z=3, \quad 3x+2y-z=1$

5) $3x_1+5x_2-7x_3=0\,, \quad -6x_1+7x_2+8x_3=0$

6) $x_1-x_2+x_3+x_4=1\,, \quad 2x_1+x_2+x_3-2x_4=2, \quad 3x_1+x_2+2x_3+x_4=0$

7) $x_1-3x_2-x_3+2x_4=2\,, \quad x_1+x_2+x_3-x_4=0, \quad x_2-2x_3+x_4=3$

8) $x+y-z=-1, \quad x+z=3, \quad 3x+2y-z=1$

9) $x_1+4x_2+x_3+2x_4=0\,, \quad x_1+3x_2+x_4=0, \quad 2x_1+x_2+x_3+x_4=0\,,$

$4x_1+9x_2+3x_3+5x_4=0, \quad 5x_1+5x_2+2x_3+3x_1=0$

Gauss–Jordan Elimination method

In this method, we use the matrix row operations to convert the augmented matrix $[A\backslash \boldsymbol{b}]$ into the form $[C\backslash \boldsymbol{d}]$ where C is a diagonal matrix (may be the identity matrix I), and $\boldsymbol{d} = [d_1, d_2, \cdots, d_n]^T$. That is;

$$[C\backslash \boldsymbol{d}] = \left[\begin{array}{cccc|c} c_{11} & 0 & \cdots & 0 & d_1 \\ 0 & c_{22} & \cdots & 0 & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{nn} & d_n \end{array}\right]$$

Then we obtain the solution directly as : $\quad x_k = \dfrac{d_k}{c_{kk}}, \ \ k = 1, 2, \cdots, n.$

Remark:

If $c_{kk} \neq 0, \ k = 1,2,\cdots,n$ the system of equations has unique solution. Otherwise, the system may have an infinite number of solutions or has no solution according to the zero rows in the matrices C and $[C\backslash \boldsymbol{d}]$, by the same manner in Gauss-Elimination method.

Example 1: Using Gauss–Jordan Elimination method, discuss the solution of the system of equations: $\quad x + y + z = 0, \ \ x - 2y + 2z = 4, \ \ x + 2y - z = 2$

Solution: We begin by writing the system as an augmented matrix:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow[-R_1 + R_3]{-R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 1 & 4 \end{array}\right] \xrightarrow[3R_2 + R_3]{-R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & 10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[2R_3 + R_2]{-3R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus, the solution is: $\quad x = 4, \ y = -2, \ z = -2$

Example 2: Using Gauss–Jordan Elimination method, discuss the solution of the system of equations: $\quad 4x + 3y + 5z = 3, \quad 2x + y + 3z = 1, \quad x + 5y + 6z = 4$

Solution: It is better to rearrange the given equations and the augmented matrix becomes:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 5 & 6 & 4 \\ 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 3 \end{array}\right] \xrightarrow[-4R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 5 & 6 & 4 \\ 0 & -9 & -9 & -7 \\ 0 & -17 & -19 & -13 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{9}R_2} \left[\begin{array}{ccc|c} 1 & 5 & 6 & 4 \\ 0 & 1 & 1 & 7/9 \\ 0 & -17 & -19 & -13 \end{array}\right] \xrightarrow[17R_2+R_3]{-5R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/9 \\ 0 & 1 & 1 & 7/9 \\ 0 & 0 & -2 & 2/9 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_3+R_2]{\frac{1}{2}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/9 \\ 0 & 1 & 0 & 8/9 \\ 0 & 0 & -2 & 2/9 \end{array}\right]$$

Thus, the solution is: $x = \frac{2}{9},\ y = \frac{8}{9},\ \text{-2z} = \frac{2}{9} \Rightarrow z = -\frac{1}{9}$

<u>Example 3</u>: Using Gauss–Jordan Elimination method, discuss the solution of the system of equations: $x - 11y + 4z = 3, \quad 2x + 4y - z = 7, \quad 5x - 3y + 2z = 3$

<u>Solution</u>: We begin by writing the system as an augmented matrix:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -11 & 4 & 3 \\ 2 & 4 & -1 & 7 \\ 5 & -3 & 2 & 3 \end{array}\right] \xrightarrow[-5R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -11 & 4 & 3 \\ 0 & 26 & -9 & 1 \\ 0 & 52 & -18 & -12 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_3]{\frac{11}{26}R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{26} & \frac{89}{26} \\ 0 & 26 & -9 & 1 \\ 0 & 0 & 0 & -14 \end{array}\right]$$

Because the third “equation” is a false statement, the system has no solution.

<u>Example 4</u>: Using Gauss–Jordan Elimination method, discuss the solution of the system of equations: $2x + 3z = 0, \quad 4x - 3y + 7z = 0, \quad 8x - 9y + 15z = 0$

<u>Solution</u>: We begin by writing the system as an augmented matrix:

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 4 & -3 & 7 & 0 \\ 8 & -9 & 15 & 0 \end{array}\right] \xrightarrow[-4R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -9 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the elements of the third row are zeros, then the system has an infinite number of solutions. Since the pivot of the third column is zero and there is no rows below to interchange, then we choose z to be the free variable (arbitrary) and represent it by the parameter $z = c$ (real number). The first and second rows give:

$$z = c, \qquad -3y + z = 0 \quad \Rightarrow \quad y = \frac{c}{3}, \qquad 2x + 3z = 0, \quad x = -\frac{3c}{2}$$

Invertible matrices method

If we write the system of equations (1) in a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ then we multiply both sides by A^{-1}, where $A^{-1}A = I,\ I\boldsymbol{x} = \boldsymbol{x}$, we obtain:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

The system has a single solution (which is the trivial solution, zeros, for a homogeneous system of linear equations) if A^{-1} exists. Otherwise, the system may have an infinite number of solutions or has no solution at all, which may be verified by using Gauss-Elimination method.

The inverse of a matrix using the elementary row operations

Starting with the augmented matrix $[A|I]$ and using the elementary row operations to obtain the equivalent matrix $[I|A^{-1}]$, where I is the identity matrix of the same order of A.

If we failed to obtain the identity matrix, then the matrix dos not have inverse.

Example 1: Find the inverse of the matrix (if it exists): $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solution: $|A| = \begin{vmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{vmatrix} = 7 - 3 - 3 = 1 \neq \mathbf{0}$

The augmented matrix is

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1 + R_3]{-R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-3R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

<u>Example</u> <u>2</u>: Find the inverse of the matrix (if it exists): $A = \begin{bmatrix} 2 & 0 & 3 \\ 4 & -3 & 7 \\ 8 & -9 & 15 \end{bmatrix}$

<u>Solution</u>: $|\boldsymbol{A}| = \begin{vmatrix} \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{4} & \mathbf{-3} & \mathbf{7} \\ \mathbf{8} & \mathbf{-9} & \mathbf{15} \end{vmatrix} = 36 - \mathbf{0} - 36 = \mathbf{0}$

Then the given matrix has no inverse.

<u>Example</u> <u>3</u>: Find the inverse of the matrix (if it exists): $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

<u>Solution</u>: $|\boldsymbol{A}| = \begin{vmatrix} \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{6} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} \end{vmatrix} = \mathbf{24} \neq \mathbf{0}$

The augmented matrix is

$$[A|I] = \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1 + R_3} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_3 + R_2} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_1/4,\ \ R_2/6} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & 1/6 & 1/6 & -1/3 \\ 0 & 0 & 1 & -1/2 & 0 & 1 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 1/6 & 1/6 & -1/3 \\ -1/2 & 0 & 1 \end{bmatrix}$$

<u>Example</u> <u>4</u>: Use inverse of matrices method to solve the following equations (if it is possible): $2x + y + 5z = 4, \quad -2x + y - 3z = -2, \quad 2x - y = -1$

Solution: First, we have to obtain the inverse of the matrix of the coefficients:

$$A=\begin{bmatrix}2 & 1 & 5\\ -2 & 1 & -3\\ 2 & -1 & 0\end{bmatrix}$$

$$|A|=\begin{vmatrix}2 & 1 & 5\\ -2 & 1 & -3\\ 2 & -1 & 0\end{vmatrix}=2(0-3)-(0+6)+5(2-2)=-12\neq 0$$

The inverse of the matrix A can be given using the augmented matrix, as following

$$[A|I]=\left[\begin{array}{ccc|ccc}2 & 1 & 5 & 1 & 0 & 0\\ -2 & 1 & -3 & 0 & 1 & 0\\ 2 & -1 & 0 & 0 & 0 & 1\end{array}\right]\xrightarrow[-R_1+R_3]{R_1+R_2}\left[\begin{array}{ccc|ccc}2 & 1 & 5 & 1 & 0 & 0\\ 0 & 2 & 2 & 1 & 1 & 0\\ 0 & -2 & -5 & -1 & 0 & 1\end{array}\right]$$

$$\xrightarrow[R_2+R_3]{-\frac{1}{2}R_2+R_1}\left[\begin{array}{ccc|ccc}2 & 0 & 4 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 2 & 2 & 1 & 1 & 0\\ 0 & 0 & -3 & 0 & 1 & 1\end{array}\right]\xrightarrow[\frac{2}{3}R_3+R_2]{\frac{4}{3}R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc}2 & 0 & 0 & \frac{1}{2} & \frac{5}{6} & \frac{4}{3}\\ 0 & 2 & 0 & 1 & \frac{5}{3} & \frac{2}{3}\\ 0 & 0 & -3 & 0 & 1 & 1\end{array}\right]\xrightarrow[-\frac{1}{3}R_3]{\frac{1}{2}R_1,\ \frac{1}{2}R_2}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{4} & \frac{5}{12} & \frac{2}{3}\\ 0 & 1 & 0 & \frac{1}{2} & \frac{5}{6} & \frac{1}{3}\\ 0 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3}\end{array}\right]$$

$$\therefore A^{-1}=\begin{bmatrix}\frac{1}{4} & \frac{5}{12} & \frac{2}{3}\\ \frac{1}{2} & \frac{5}{6} & \frac{1}{3}\\ 0 & -\frac{1}{3} & -\frac{1}{3}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & \frac{5}{12} & \frac{2}{3}\\ \frac{1}{2} & \frac{5}{6} & \frac{1}{3}\\ 0 & -\frac{1}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}4\\ -2\\ -1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ 0\\ 1\end{bmatrix}$$

$$\therefore\ x=\frac{-1}{2},\quad y=0,\quad z=1$$

Example 5: Use inverse of matrices method to solve the following equations (if it is possible): $x + y + z = 1,\ \ x - 2y + 2z = 4,\ \ x + 2y - z = 2$

Solution: First, we have to obtain the inverse of the matrix of the coefficients:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = (2-4) - (-1-2) + (2+2) = 5 \neq 0$$

The inverse of the matrix A can be given using the augmented matrix, as following

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_2+R_3]{\frac{1}{3}R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & -\frac{5}{3} & -\frac{4}{3} & \frac{1}{3} & 1 \end{array}\right] \xrightarrow[\frac{3}{5}R_3+R_2]{\frac{4}{5}R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 0 & -3 & 0 & -\frac{9}{5} & \frac{6}{5} & \frac{3}{5} \\ 0 & 0 & -\frac{5}{3} & -\frac{4}{3} & \frac{1}{3} & 1 \end{array}\right] \xrightarrow[-\frac{3}{5}R_3]{-\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{4}{5} & -\frac{1}{5} & -\frac{3}{5} \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \\ \frac{4}{5} & -\frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ \frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} \\ \frac{4}{5} & -\frac{1}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{18}{5} \\ -\frac{7}{5} \\ -\frac{6}{5} \end{bmatrix}$$

$$\therefore \quad x = \frac{18}{5}, \quad y = -\frac{7}{5}, \quad z = -\frac{6}{5}$$

Example 6: Use inverse of matrices method to solve the following equations (if it is possible): $4x + 2y + 3z = 2$, $7x + 5y + 4z = 1$, $9x + 2y + 6z = 0$

Solution: First, we have to obtain the inverse of the matrix of the coefficients:

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 7 & 5 & 4 \\ 9 & 2 & 6 \end{bmatrix}$$

Using the determinants technique, we have:

$$|A| = \begin{vmatrix} 4 & 2 & 3 \\ 7 & 5 & 4 \\ 9 & 2 & 6 \end{vmatrix} = 4\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} 7 & 4 \\ 9 & 6 \end{vmatrix} + 3\begin{vmatrix} 7 & 5 \\ 9 & 2 \end{vmatrix} = -17 \neq 0$$

$$[A_{ij}] = \begin{bmatrix} +\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 7 & 4 \\ 9 & 6 \end{vmatrix} & +\begin{vmatrix} 7 & 5 \\ 9 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} & +\begin{vmatrix} 4 & 3 \\ 9 & 6 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 9 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 7 & 4 \end{vmatrix} & +\begin{vmatrix} 4 & 2 \\ 7 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 22 & -6 & -31 \\ -6 & -3 & 10 \\ -7 & 5 & 6 \end{bmatrix}$$

$$\Rightarrow \quad AdjA = \begin{bmatrix} 22 & -6 & -7 \\ -6 & -3 & 5 \\ -31 & 10 & 6 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{AdjA}{|A|} = \frac{-1}{17}\begin{bmatrix} 22 & -6 & -7 \\ -6 & -3 & 5 \\ -31 & 10 & 6 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{22}{17} & \frac{6}{17} & \frac{7}{17} \\ \frac{6}{17} & \frac{3}{17} & -\frac{5}{17} \\ \frac{31}{17} & -\frac{10}{17} & -\frac{6}{17} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{38}{17} \\ \frac{15}{17} \\ \frac{52}{17} \end{bmatrix}$$

$$\therefore \; x = -\frac{38}{17}, \quad y = \frac{15}{17}, \quad z = \frac{52}{17}$$

Exercises

Using Gauss - Jordan elimination methods, and inverse of matrices discuss the solution of each of the following systems.

1) $x + y + z = 1, \quad x - 2y + 2z = 4, \quad x + 2y - z = 2$

2) $2x + y + 5z = 10, \quad x - 3y - z = -2, \quad 2x - y + 3z = 6$

3) $12x + 7y + 3z = 2, \quad x + 5y + 2z = -5, \quad 2x + 7y - 11z = 6$

4) $x + y - z = -1, \quad x + z = 3, \quad 3x + 2y - z = 1$

5) $3x - y + 2z = 1, \quad 2x - 2y + 3z = 2, \quad x + 2y - z = 3$

6) $2x - y + 5z = 4, \quad -6x + 3y - 9z = -6, \quad 4x - 2y = -2$

7) $2x - y + 5z = 4, \quad -2x + y - 3z = -2, \quad 2x - y = -1$

8) $4x - y + 2z = 0, \quad 2x + 3y - z = 0, \quad 3x + y + z = 0$

9) $2x - y + 5z = 0, \quad -2x + y - z = 0, \quad x + y + 3z = 0$

LU - decomposition method

As we know, a system of linear equations can be written in the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In the LU decomposition method the coefficient matrix A (a non-singular matrix) is decomposed into a product of a lower triangular matrix L and an upper triangular matrix U, i.e., $A = LU$.

A version of expressing the LU matrices explicitly, they look like:

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

The LU decomposition is not always guaranteed. It is guaranteed only when the leading principal minors of the coefficient matrix are not zeros.

Leading Principal Minors of A square Matrix

Each square $n \times n$ matrix has n leading principal minors. A leading principal minor is the determinant of the leading principal submatrix obtained by deleting the last $n - k$ rows and columns of the matrix, $k = 1, 2, 3, \cdots, n$. The leading principal minors of a matrix A starts by $|a_{11}|$ and ends by $|A|$.

For example: the matrix $A = \begin{bmatrix} 7 & 6 & 3 \\ -5 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ has three leading principal minors:

1. $|7|$ formed by deleting the last two rows and columns of A.
2. $\begin{vmatrix} 7 & 6 \\ -5 & 4 \end{vmatrix}$ formed by deleting the last row and column of A.
3. $\begin{vmatrix} 7 & 6 & 3 \\ -5 & 4 & 5 \\ 1 & 0 & 6 \end{vmatrix}$ formed by the determinant of the matrix, which is $|A|$.

Hint: Sometimes it is needed to rearrange the given equations to guarantee the nonzero leading principal minors of the coefficient matrix, which makes the LU decomposition method applicable.

Decomposing the coefficient matrix into LU matrices gives:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

However, to perform the calculations of the L and U matrices. First, we obtain the product matrix LU and equating the elements of its first row by those of the first row in the U matrix to get the first row in the upper matrix U. Equating the second rows in both sides yields the second row in each of the matrices L and U. By the same way we can find all the elements of the matrices L and U.

Thus, the system of linear algebraic equations can be decomposed to two systems of linear algebraic equations which can be solved directly.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad LU\boldsymbol{x} = \boldsymbol{b}$$

Consider $\boldsymbol{y} = U\boldsymbol{x}$ then $L\boldsymbol{y} = \boldsymbol{b}$.

Therefore, we must obtain $\mathbf{y}$ by solving the equation $L\boldsymbol{y} = \boldsymbol{b}$, that is:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then we can obtain the solution $\boldsymbol{x}$ by solving the equation $U\boldsymbol{x} = \boldsymbol{y}$, that is:

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Example 1: Using the LU decomposition method solve the equations (if it is possible):

$$x + 2y + 4z = 2, \quad 2x + 3y - 5z = 1, \quad 4x - 5y + 8z = 5$$

Solution: We have

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 3 & -5\\ 4 & -5 & 8\end{bmatrix},\quad \boldsymbol{x}=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad \boldsymbol{b}=\begin{bmatrix}2\\ 1\\ 5\end{bmatrix}$$

The leading principal minors of the matrix A are:

1. $|1|=1\neq 0,$ 2. $\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}=-15\neq 0,$ 3. $|A|=\begin{vmatrix}1 & 2 & 4\\ 2 & 3 & -5\\ 4 & -5 & 8\end{vmatrix}=-161\neq 0$

Decomposing the coefficient matrix as:

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 3 & -5\\ 4 & -5 & 8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$$=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

Then we have

$u_{11}=1,\ \ u_{12}=2,\ \ u_{13}=4$ $\qquad l_{21}u_{11}=2 \ \rightarrow\ l_{21}=2,$

$l_{21}u_{12}+u_{22}=3 \ \rightarrow\ u_{22}=-1,$

$l_{21}u_{13}+u_{23}=-5 \ \rightarrow\ u_{23}=-13,$

$l_{31}u_{11}=4 \ \rightarrow\ l_{31}=4,$

$l_{31}u_{12}+l_{32}u_{22}=-5 \ \rightarrow\ l_{32}=13,$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=8 \ \rightarrow\ u_{33}=161$

Solving the equation $L\boldsymbol{y}=\boldsymbol{b}$, i.e. $\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & 13 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}2\\ 1\\ 5\end{bmatrix}$ gives (forward substitution)

$$y_1=2,\ \ 2y_1+y_2=1 \ \Rightarrow\ y_2=-3,\ \ 4y_1+13y_2+y_3=5 \ \Rightarrow\ y_3=36$$

Applying back substitution to the equation $\boldsymbol{y}=U\boldsymbol{x}$ gives

$$\begin{bmatrix}1 & 2 & 4\\ 0 & -1 & -13\\ 0 & 0 & 161\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ -3\\ 36\end{bmatrix},$$

from which we obtain the equations:

$$161z=36,\quad -y-13z=-3,\quad x+2y+4z=2$$

Therefore, the solution is:

$$z = \frac{36}{161}, y = \frac{15}{161}, x = \frac{148}{161}$$

<u>Example</u> <u>2:</u> Using the LU decomposition method solve the equations (if it is possible):

$$x + y + z = 1, \quad 2x - 3y - z = 1, \quad x - y + z = 0$$

<u>Solution</u>: We have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The leading principal minors of the matrix A are:

1. $|1| = 1 \neq 0,$ 2. $\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5 \neq 0,$ 3. $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -6 \neq 0$

Decomposing the coefficient matrix as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Then we have

$u_{11} = 1, u_{12} = 1, u_{13} = 1$

$l_{21}u_{11} = 2 \rightarrow l_{21} = 2,$

$l_{21}u_{12} + u_{22} = -3 \rightarrow u_{22} = -5,$

$l_{21}u_{13} + u_{23} = -1 \rightarrow u_{23} = -3,$

$l_{31}u_{11} = 1 \rightarrow l_{31} = 1,$

$l_{31}u_{12} + l_{32}u_{22} = -1 \rightarrow l_{32} = 2/5,$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \rightarrow u_{33} = 6/5$

Solving the equation $L\boldsymbol{y} = \boldsymbol{b}$, i.e. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{2}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

gives (forward substitution)

$$y_1 = 1, \quad 2y_1 + y_2 = 1 \quad \Rightarrow \quad y_2 = -1, \quad y_1 + \frac{2}{5}y_2 + y_3 = 0 \;\Rightarrow y_3 = -\frac{3}{5}$$

Applying back substitution to the equation $\boldsymbol{y} = U\boldsymbol{x}$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & \frac{6}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -\frac{3}{5} \end{bmatrix},$$

from which we obtain the equations:

$$\frac{6}{5}z = -\frac{3}{5}, \qquad -5y - 3z = -1, \qquad x + y + z = 1$$

Therefore, the solution is:

$$z = -\frac{1}{2}, \;\; y = \frac{1}{2}, \;\; x = 1$$

Example 3: Using the LU decomposition method solve the equations (if it is possible):

$$y + z = 3, \qquad 2x - 3y - z = 1, \qquad x - y + 2z = 2$$

Solution: We rearrange these equations as:

$$2x - 3y - z = 1, \qquad y + z = 3, \qquad x - y + 2z = 2$$

$$\therefore \quad A = \begin{bmatrix} 2 & -3 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

The leading principal minors of the matrix A are:

1. $|2| = 2 \neq 0,$ 2. $\begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = 2 \neq 0,$ 3. $\begin{vmatrix} 2 & -3 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 4 \neq 0$

Decomposing the coefficient matrix as:

$$\begin{bmatrix} 2 & -3 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Then we have:

$u_{11} = 2, \quad u_{12} = -3, \quad u_{13} = -1$

$l_{21}u_{11} = 0 \quad \to \quad l_{21} = 0,$

$l_{21}u_{12} + u_{22} = 1 \quad \to \quad u_{22} = 1,$

$l_{21}u_{13} + u_{23} = 1 \quad \to \quad u_{23} = 1,$

$l_{31}u_{11} = 1 \quad \to \quad l_{31} = \frac{1}{2},$

$l_{31}u_{12} + l_{32}u_{22} = -1 \quad \to \quad l_{32} = \frac{1}{2},$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \quad \to \quad u_{33} = 3$

The equation $L\boldsymbol{y} = \boldsymbol{b}$ gives: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

Applying forward substitutions gives:

$$y_1 = 1, \quad y_2 = 3 \quad \frac{1}{2}y_1 + \frac{1}{2}y_2 + y_3 = 2 \;\Rightarrow\; y_3 = 0$$

The equation $\boldsymbol{y} = U\boldsymbol{x}$ gives: $\begin{bmatrix} 2 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix},$

from which we obtain the equations:

$$3z = 0, \qquad y + z = 3, \qquad 2x - 3y - z = 1$$

Therefore, the solution is: $x = 5, \quad y = 3, \; z = 0$

Example 4: Using the LU decomposition method, solve the equations (if it is possible):

$$2x + 5y - 3z = 3, \qquad x - 2y + z = 2, \qquad 2x + 4y - 3z = -4$$

Solution: We have

$$A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 2 & 4 & -3 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

The leading principal minors of the matrix A are:

1. $|2| = 2 \neq 0,$ 2. $\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9 \neq 0,$ 3. $|A| = \begin{vmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 2 & 4 & -3 \end{vmatrix} = 5 \neq 0$

Decomposing the coefficient matrix as:

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 2 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 2 & 4 & -3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Then we have:

$u_{11} = 2, \quad u_{12} = 5, \quad u_{13} = -3 \qquad\qquad l_{21}u_{11} = 1 \quad \rightarrow \; l_{21} = 1/2,$

$l_{21}u_{12} + u_{22} = -2 \quad \rightarrow \quad u_{22} = -9/2,$

$l_{21}u_{13} + u_{23} = 1 \quad \rightarrow \quad u_{23} = 5/2,$

$l_{31}u_{11} = 2 \quad \rightarrow \quad l_{31} = 1,$

$l_{31}u_{12} + l_{32}u_{22} = 4 \quad \rightarrow \quad l_{32} = 2/9,$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -3 \quad \rightarrow \quad u_{33} = -5/9$

The equation $L\boldsymbol{y} = \boldsymbol{b}$ gives: $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{2}{9} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$

Applying forward substitutions gives:

$$y_1 = 3, \quad \frac{1}{2}y_1 + y_2 = 2 \;\Rightarrow\; y_2 = \frac{1}{2}, \quad y_1 + \frac{2}{9}y_2 + y_3 = -4 \;\Rightarrow\; y_3 = -64/9$$

The equation $\boldsymbol{y} = U\boldsymbol{x}$ gives: $\begin{bmatrix} 2 & 5 & -3 \\ 0 & -\frac{9}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \\ -\frac{64}{9} \end{bmatrix},$

from which we obtain the equations:

$$-\frac{5}{9}z = -\frac{64}{9}, \quad -\frac{9}{2}y + \frac{5}{2}z = \frac{1}{2} \quad ,2x + 5y - 3z = 3$$

$$\therefore \quad z = \frac{64}{5} = 12.8, \quad y = 7, \quad x = \frac{16}{5} = 3.2$$

Example 5: Using the LU decomposition method, solve the equations (if it is possible):

$$2x + 5y - 3z = 3, \quad x - 2y + z = 2, \quad 7x + 4y - 3z = -4$$

Solution: We have

$$A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 7 & 4 & -3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

The leading principal minors of the matrix A are:

1. $|2| = 2 \neq 0,$ 2. $\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9 \neq 0,$ 3. $|A| = \begin{vmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 7 & 4 & -3 \end{vmatrix} = 0$

Therefore, the coefficient matrix cannot be decomposed into LU form, and the LU decomposition method is not applicable and the given system of equation does not have unique solution.

Exercises

Using the LU decomposition method solve the equations:

1)$x + 2z = 1, \quad 3x - y + z = 2, \quad 4y + 5z = -1$

2)$3x + y - 2z = 4, \quad 2x + 3y + z = 2, \quad x - 3y + 2z = 8$

3)$4x + 2y - z = 0, \quad 6x + 3y + 2z = -16, \quad x + 2y + 9z = -4$

4)$4x + 10y + 8z = 44, \quad 10x + 26y + 26z = 128, \quad 8x + 26y + 61z = 214$

Chapter Four

Iterative methods

If we have a system of large number of linear equations, the above direct methods need many operations which may affect the result due to the round off error. To avoid this problem, we use the iterative methods, in which we start from an approximation to the solution. We apply the iterative methods if the convergence is rapid or there are many zero-coefficients in the system.

Consider the system of equations (2.1) and rewrite them is the fixed-point form as

$$x_k = \frac{1}{a_{kk}}\left(b_k - \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj}\, x_j \right), \qquad k = 1,2,3,\cdots,n$$

Explicitly, we obtain:

$$x_1 = \frac{1}{a_{11}}\{b_1 - (a_{12}\, x_2 \;+\; a_{13}\, x_2 + \; a_{14}\, x_4 \;+\cdots+\; a_{1n}\, x_n)\}$$
$$x_2 = \frac{1}{a_{22}}\{b_2 - (a_{21}\, x_1 \;+\; a_{23}\, x_3 \;+\; a_{24}\, x_4 + \cdots + \; a_{2n}\, x_n)\}$$
$$x_3 = \frac{1}{a_{33}}\{b_2 - (a_{31}\, x_1 \;+\; a_{32}\, x_2 \;+\; a_{24}\, x_4 + \cdots + \; a_{2n}\, x_n)\}$$
$$\vdots$$
$$x_n = \frac{1}{a_{nn}}\{b_n - (a_{n1}\, x_1 \;+\; a_{n2}\, x_2 + \; a_{n3}\, x_3 \;+\cdots+\; a_{n,n-1}\, x_{n-1})\}$$

The numbers a_{kk} are the leading coefficients, where $|a_{kk}|$ must be the largest relative to the $|a_{ki}|$, $k = 1,2,3,\cdots,n$. The sufficient conditions for which it is guaranteed that the iterative methods will converge are:

$$|a_{kk}| \geq \sum_{i\neq k}^{n} |a_{ki}|, \qquad k = 1,2,3,\cdots,n \qquad (*)$$

If (*) is satisfied, then the system of linear equations has a unique solution to which the iterative methods will converge for any initial approximation.

Remark:

Sometimes, it is needed to interchange equations (rearrange the equations) of the given system to obtain convergence of the solution.

Below are two of the most important iterative methods.

Jaccobi method

This method uses the starting approximate solution $x_k^0,\ k = 1,2,\cdots,n$ to obtain x_k^1 , then uses x_k^1 to find x_k^2 and so on until the solution converges, i.e., $\left|x_k^{m+1} - x_k^m\right| \le \varepsilon$.

The iterations of Jaccobi method are written as

$$x_k^{(m+1)} = \frac{1}{a_{kk}}\left(b_k - \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj}\, x_j^{(m)}\right), k = 1,2,3,\cdots,n, \qquad m = 0,1,2,\cdots$$

Explicitly, we obtain:

$$x_1^{(m+1)} = \frac{1}{a_{11}}\left\{b_1 - \left(a_{12}\, x_2^{(m)} + a_{13}\, x_3^{(m)} + a_{14}\, x_4^{(m)} + \cdots + a_{1n}\, x_n^{(m)}\right)\right\}$$
$$x_2^{(m+1)} = \frac{1}{a_{22}}\left\{b_2 - \left(a_{21}\, x_1^{(m)} + a_{23}\, x_3^{(m)} + a_{24}\, x_4^{(m)} + \cdots + a_{2n}\, x_n^{(m)}\right)\right\}$$
$$x_3^{(m+1)} = \frac{1}{a_{22}}\left\{b_2 - \left(a_{31}\, x_1^{(m)} + a_{32}\, x_2^{(m)} + a_{34}\, x_4^{(m)} + \cdots + a_{3n}\, x_n^{(m)}\right)\right\}$$
$$\vdots$$
$$x_n^{(m+1)} = \frac{1}{a_{nn}}\left\{b_n - \left(a_{n1}\, x_1^{(m)} + a_{n2}\, x_2^{(m)} + a_{n3}\, x_3^{(m)} + \cdots + a_{n,n-1}\, x_{n-1}^{(m)}\right)\right\}$$

<u>Example 1</u>: Using Jaccobi method, solve the equations:

$4x + .24y - 0.08z = 8,\quad 0.04x - 0.08y + 4z = 20,\quad 0.09x + 3y - 0.15z = 9$

using $x_0 = 0,\ y_0 = 0,\ z_0 = 0$

<u>Solution</u>: We should rearrange the equations to be:

$4x + 0.24y - 0.08z = 8,\quad 0.09x + 3y - 0.15z = 9,\quad 0.04x - 0.08y + 4z = 20$

Then we have the iterative formula:

$$x_{i+1} = \frac{1}{4}\{8 - 0.24y_i + 0.08z_i\},$$

$$y_{i+1} = \frac{1}{3}\{9 - 0.09x_i + 0.15z_i\},$$

$$z_{i+1} = \frac{1}{4}\{20 - 0.04x_i + 0.08y_i\}$$

The iterations give

$i = 0$:

$$x_1 = \frac{1}{4}\{8 - 0.24y_0 + 0.08z_0\} = 2,$$

$$y_1 = \frac{1}{3}\{9 - 0.09x_0 + 0.15z_0\} = 3.0,$$

$$z_1 = \frac{1}{4}\{20 - 0.04x_0 + 0.08y_0\} = 5.0$$

$i = 1$:

$$x_2 = \frac{1}{4}\{8 - 0.24y_1 + \ 0.08z_1\} = \frac{1}{4}\{8 - 0.24 \times 3 \ - \ 0.08 \times 5\} = 1.92000000178,$$

$$y_2 = \frac{1}{3}\{9 - 0.09x_1 + 0.15z_1\} = \frac{1}{3}\{9 - 0.09 \times 2 + 0.15 \times 5\ \} = 3.19000000754,$$

$$z_2 = \frac{1}{4}\{20 - 0.04x_1 + 0.08y_1\} = \frac{1}{4}\{20 - 0.04 \times 2 + 0.08 \times 3\} = 5.03999999910$$

$i = 2$:

$$x_3 = \frac{1}{4}\{8 - 0.24y_2 + \ 0.08z_2\} = \frac{1}{4}\{8 - 0.24 \times 3.19 + \ 0.08 \times 5.04)\}$$

$$= 1.90940000155$$

$$y_3 = \frac{1}{3}\{9 - 0.09x_2 + 0.15z_2\} = \frac{1}{3}\{9 - 0.09 \times 1.92 + 0.15 \times 5.04\ )\} = 3.19440000762$$

$$z_3 = \frac{1}{4}\{20 - 0.04x_2 + 0.08y_2\} = \frac{1}{4}\{20 - 0.04 \times 1.92 + 0.08 \times 3.19)\}$$

$$= 5.04459999913$$

The following table contains more results:

i	x_{i+1}	y_{i+1}	z_{i+1}
0	2.000000000000000	3.000000000000000	5.000000000000000
1	1.920000001788139	3.190000007549922	5.039999999105930
2	1.909400001554191	3.194400007626415	5.044599999136230
3	1.909228001554052	3.194948007656723	5.044793999135763
4	1.909199001552872	3.194962867657294	5.044806679136087
5	1.909198363552859	3.194964371657406	5.044807266336097
6	1.909198285056854	3.194964420157409	5.044807302796098
7	1.9091982826876054	3.194964424335289	5.044807304551058
8	1.909198282660480	3.194964424488461	5.044807304656424
9	1.909198282653398	3.194964424500197	5.044807304661643
10	1.909198282652798	3.194964424500670	5.044807304661949

Example 2: Using six iterations of Jaccobi method, solve the equations:

$$12x + 7y + 3z = 1, \quad x + 8y + 2z = 2, \quad -2x - 7y + 25z = 1$$

with $x_0 = 0,\ y_0 = 0,\ z_0 = 0$:

Solution: The Jaccobi iterative formula is:

$x_{i+1} = (1 - 7y_i - 3z_i)/12,$

$y_{i+1} = (2 - x_i - 2z_i)/8,$

$z_{i+1} = (1 + 2x_i + 7y_i)/25$

The iterations give:

$i = 0$:

$x_1 = (1 - 7y_0 - 3z_0)/12 = 0.08333334,$

$y_1 = (2 - x_0 - 2z_0)/8 = 0.25,$

$z_1 = (1 + 2x_0 + 7y_0)/25 = 0.04$

$i = 1$:

$x_2 = (1 - 7y_1 - 3z_1)/12 = -0.0725,$

$y_2 = (2 - x_1 - 2z_1)/8 = 0.2295833,$

$z_2 = (1 + 2x_1 + 7y_1)/25 = 0.1166667$

$i = 2$:

$x_3 = (1 - 7y_2 - 3z_2)/12 = -0.07975695,$

$y_3 = (2 - x_2 - 2z_2)/8 = 0.2298958,$

$z_3 = (1 + 2x_2 + 7y_2)/25 = 0.09848333$

The following table contains more results:

i	x_{i+1}	y_{i+1}	z_{i+1}
3	-0.07539340	0.2353488	0.09799027
4	-0.07845103	0.2349266	0.09986619
5	-0.07867374	0.2348398	0.09950337

Example 3: Using five iterations of Jaccobi method, solve the system equations:

$$x + 2y - z = 5, \quad 3x - y + z = -2, \quad x + y + 2z = 1$$

with $x_0 = 1,\ y_0 = 0,\ z_0 = 1.$

Solution: We must rearrange the equations as:

$$3x - y + z = -2, \quad x + 2y - z = 5, \quad x + y + 2z = 1$$

and the iterative formula becomes:

$x_{i+1} = (-2 + y_i - z_i)/3,$

$y_{i+1} = (5 - x_i + z_i)/2,$

$z_{i+1} = (1 - x_i - y_i)/2$

The iterations give:

$i = 0$:

$x_1 = (-2 + y_0 - z_0)/3 = -1$

$y_1 = (5 - x_0 + z_0)/2 = 2.5$

$z_1 = (1 - x_0 - y_0)/2 = 0.0$

$i = 1$:

$x_2 = (-2 + y_1 - z_1)/3 = 1.66667$

$y_2 = (5 - x_1 + z_1)/2 = 3$

$z_2 = (1 - x_1 - y_1)/2 =$ -0.25

$i = 2$:

$x_3 = (-2 + y_2 - z_2)/3 = 0.4166667$

$y_3 = (5 - x_2 + z_2)/2 = 2.291667$

$z_3 = -(1 - x_2 - y_2)/2 = -1.08333$

The following table contains more results:

i	x_{i+1}	y_{i+1}	z_{i+1}
3	$4.583334E - 01$	1.750000	$-8.541667E - 01$
4	$2.013889E - 01$	1.843750	$-6.041667E - 01$
5	$1.493056E - 01$	2.097222	$-5.225694E - 01$

Gauss - Seidel method

This method is a modification of Jaccobi method as it uses the result reached in calculation. It uses $x_j^{m+1},\ j<k$ and $x_j^m,\ j>k$ to calculate x_k^{m+1} until the solution converges, i.e., $\left|x_k^{m+1}-x_k^m\right|\le\varepsilon$. The iterations of Gauss - Seidel method is written as

$$x_k^{m+1}=\frac{1}{a_{kk}}\left(b_k-\sum_{j=1}^{k-1}a_{kj}\,x_j^{m+1}-\sum_{j=k+1}^{n}a_{kj}\,x_j^m\right),m=0,1,2,\cdots$$

Explicitly, we obtain:

$$x_1^{(m+1)}=\frac{1}{a_{11}}\left\{b_1-\left(a_{12}\,x_2^{(m)}+a_{13}\,x_3^{(m)}+a_{14}\,x_4^{(m)}+\cdots+a_{1n}\,x_n^{(m)}\right)\right\}$$
$$x_2^{(m+1)}=\frac{1}{a_{22}}\left\{b_2-\left(a_{21}\,x_1^{(m+1)}+a_{23}\,x_3^{(m)}+a_{24}\,x_4^{(m)}+\cdots+a_{2n}\,x_n^{(m)}\right)\right\}$$
$$x_3^{(m+1)}=\frac{1}{a_{22}}\left\{b_2-\left(a_{31}\,x_1^{(m+1)}+a_{32}\,x_2^{(m+1)}+a_{34}\,x_4^{(m)}+\cdots+a_{3n}\,x_n^{(m)}\right)\right\}$$
$$\vdots$$
$$x_n^{(m+1)}=\frac{1}{a_{nn}}\left\{b_n-\left(a_{n1}\,x_1^{(m+1)}+a_{n2}\,x_2^{(m+1)}+a_{n3}\,x_3^{(m+1)}+\cdots+a_{n,n-1}\,x_{n-1}^{(m+1)}\right)\right\}$$

Example 1: Using nine iterations of Gauss - Seidel method, solve the system equations:

$$x+2y+z=1,\qquad 3x-2y+2z=4,\qquad x+y-5z=2$$

with $x_0=1,y_0=1,z_0=1$.

Solution: We must rearrange the equations and the iterative formula becomes:

$x_{i+1}=(4+2y_i-2z_i)/3$

$y_{i+1}=(1-x_{i+1}-z_i)/2$

$z_{i+1}=-(2-x_{i+1}-y_{i+1})/5$

The iterations give:

$i=0$:

$x_1=(4+2y_0-2z_0)/3=1.33333$

$y_1=(1-x_1-z_0)/2=-0.666666$

$z_1=-(2-x_1-y_1)/5=-0.266666$

$i=1$:

$x_2=(4+2y_1-2z_1)/3=1.066667$

$y_2=(1-x_2-z_1)/2=0.1$

$z_2=-(2-x_2-y_2)/5=-0.1666667$

$i = 2$:

$x_3 = (4 + 2y_2 - 2z_2)/3 = 1.51111$

$y_3 = (1 - x_3 - z_2)/2 = -0.172222$

$z_3 = -(2 - x_3 - y_3)/5 = -0.132222$

The following table contains more results:

i	x_{i+1}	y_{i+1}	z_{i+1}
3	1.306667	$-$ 0.8722220	$-$ 0.1561111
4	1.379259	$-$ 0.1115741	$-$ 0.1464630
5	1.356593	$-$ 0.1050648	$-$ 0.1496944
6	1.363086	$-$ 0.1066960	$-$ 0.1487219
7	1.361351	$-$ 0.1063144	$-$ 0.1489927
8	1.361786	$-$ 0.1063964	$-$ 0.1489222

Example 2: Using six iterations of Gauss - Seidel method, solve the system equations:

$$x + y + 5z = 4, \quad 2x - 3y + z = 2, \quad 5x - 4y + z = 0$$

with $x_0 = 1$, $y_0 = 0$, $z_0 = 0$.

Solution: We must rearrange the equations and the iterative formula becomes:

$x_{i+1} = (4y_i - z_i)/5,$

$y_{i+1} = -(2 - 2x_{i+1} - z_i)/3,$

$z_{i+1} = (4 - x_{i+1} - y_{i+1})/5$

The iterations give:

$i = 0$:

$x_1 = (4y_0 - z_0)/5 = 0$

$y_1 = -(2 - 2x_1 - z_0)/3 = -0.666666$

$z_1 = (4 - x_1 - y_1)/5 = 0.9333333$

$i = 1$:

$x_2 = (4y_1 - z_1)/5 = -0.72$

$y_2 = -(2 - 2x_2 - z_1)/3 = -0.8355556$

$z_2 = (4 - x_2 - y_2)/5 = 1.11111$

$i = 2$:

$x_3 = (4y_2 - z_2)/5 = -0.890666$

$y_3 = -(2 - 2x_3 - z_2)/3 = -0.890074$

$z_3 = (4 - x_3 - y_3)/5 = 1.156148$

The following table contains more results:

i	x_{i+1}	y_{i+1}	z_{i+1}
3	$-9.432889E-01$	$-9.101432E-01$	1.170686
4	$-9.622518E-01$	$-9.179391E-01$	1.176038
5	$-9.695589E-01$	$-9.210265E-01$	1.178117

Example 3: Using five iterations of Gauss - Seidel method, solve the system equations:

$$4x + 0.24y - 0.08z = 8, \quad 0.09x + 3y - 0.15z = 9, \quad 0.04x - 0.08y + 4z = 20$$

with $x_0 = 0,\ y_0 = 0,\ z_0 = 0.$

Solution: The iterative formula is:

$x_{i+1} = \{8 - (0.24y_i - 0.08z_i)\}/4,$

$y_{i+1} = \{9 - (0.09\, x_{i+1} - 0.15\, z_i)\}/3,$

$z_{i+1} = \{20 - (0.04\, x_{i+1} - 0.08\, y_{i+1})\}/4.$

Then we have

$i = 0$:

$x_1 = \{8 - 0.24\, y_0 + 0.08\, z_0\}/4 = \{8 - 0.24 \times 0 + 0.08 \times 0\}/4 = 2.0,$

$y_1 = \{9 - 0.09x_1 + 0.15z_0\}/3 = \{9 - 0.09 \times 2 + 0.15 \times 0)\}/3 = 2.940,$

$z_1 = \{20 - 0.04x_1 + 0.08y_1\}/4 = \{20 - 0.04 \times 2 + 0.08 \times 2.94\}/4 = 5.0388$

$i = 1$:

$x_2 = \{8 - 0.24\, y_1 + 0.08\, z_1\}/4 = \{8 - 0.24 \times 2.94 + 0.08 \times 5.0388\}/4 = 1.924276,$

$y_2 = \{9 - 0.09x_2 + 0.15z_1\}/3 = \{9 - 0.09 \times 1.924276 + 0.15 \times 5.0388\}/3$

$= 3.194209,$

$z_2 = \{20 - 0.04x_2 + 0.08y_2\}/4 = \{20 - 0.04 \times 1.924276 + 0.08 \times 3.194209\}/4$

$= 5.044641$

$i = 2$:

$x_3 = \{8 - 0.24\, y_2 + 0.08\, z_2\}/4 = \{8 - 0.24 \times 3.194209 + 0.08 \times 5.044641\}/4$

$= 1.909240,$

$y_3 = \{9 - 0.09x_3 + 0.15z_2\}/3 = \{9 - 0.09 \times 1.909240 + 0.15 \times 5.044641\}/3$

$= 3.194955,$

$z_3 = \{20 - 0.04x_3 + 0.08y_3\}/4 = \{20 - 0.04 \times 1.909240 + 0.08 \times 3.194955\}/4$

$= 5.044806$

Repeating the iteration yields

i	x_{i+1}	y_{i+1}	z_{i+1}
3	1.909199	3.194964	5.044807
4	1.909198	3.194964	5.044807

Example 4: Using six iterations of Gauss - Seidel method, solve the system equations:

$$12x + 7y + 3z = 2, \quad x + 5y + 2z = -5, \quad 2x + 7y - 11z = 6$$

with $x_0 = 0,\ y_0 = 0,\ z_0 = 0.$

Solution: The iterative formula is:

$x_{i+1} = (2 - 7y_i - 3z_i)/12, \qquad y_{i+1} = (-5 - x_{i+1} - 2z_i)/5,$

$z_{i+1} = -(6 - 2x_{i+1} - 7y_{i+1})/11$

The iterations yield:

i	x_{i+1}	y_{i+1}	z_{i+1}
0	0.1666667	− 1.033333	− 1.172727
1	1.062626	− 1.681616	− 1.422369
2	1.503202	− 1.869588	− 1.461883
3	1.622730	− 1.909299	− 1.465421
4	1.646780	− 1.915524	− 1.465010
5	1.650308	− 1.916066	− 1.464713

Exercises

Using Jaccobi and Gauss - Seidel methods to solve following the equations:

1) $9x - 5z = 10, \quad 20y - 12z = -2, \quad -5x - 12y + 20z = 0$

where $x_0 = 0,\ y_0 = 0,\ z_0 = 0$

2) $2x + 10y - z = -32, \quad -x + 2y + 15z = 17, \quad 10x - y + 2z = 58$

where $x_0 = 1,\ y_0 = 1,\ z_0 = 1$

3) $4x + 2y + z = 14, \quad x + 5y - z = 10, \quad x + y + 8z = 20$

where $x_0 = 1,\ y_0 = 1,\ z_0 = 1$

4) $6x + y + z = 107, \quad x + 9y - 2z = 36, \quad 2x - y + 8z = 121$

where $x_0 = 1,\ y_0 = 1,\ z_0 = 1$

Chapter Five

Eigenvalues and eigenvectors of a square matrix

If the product $A\boldsymbol{x}$ of a square matrix A of order n transforms a nonzero vector $\boldsymbol{x}$ into itself multiplied by a scalar λ, that is $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then λ is called an eigenvalue of the matrix A and the vector $\boldsymbol{x}$ is called an eigenvector of A corresponding to λ. Generally, a square matrix of order n has n eigenvalues and each eigenvalue has a corresponding nonzero vector of order n.

To find the eigenvalues and eigenvectors of a square matrix A of order n, let I be the identity matrix of order n. Writing the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ in the form $A\boldsymbol{x} = \lambda I\boldsymbol{x}$ that produces the system of equations:

$$(A - \lambda I)\boldsymbol{x} = 0$$

$$i.e, \qquad \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (1)$$

This homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(A - \lambda I)$ is not invertible, that is; $det(A - \lambda I) = 0$

$$i.e, \qquad \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad (2)$$

Equation (2) is called the characteristic equation of A and it gives a polynomial of degree n in λ, its n roots are the eigenvalues of the matrix A. For each eigenvalue there exist a corresponding eigenvector (nonzero vector) which can be obtained by solving the homogeneous system (1).

Since the eigenvectors are nonzero vectors, then the homogeneous system (1) has an infinite number of solutions for each value of λ. So that we consider the value 1 for the free variable. This requires row reducing of a matrix where the resulting row reduced form must have at least one row of zeros.

Example 1: Verify that which of the vectors $\boldsymbol{x}_1^T = [4 \quad 1]$, $\boldsymbol{x}_2^T = [3 \quad 1]$ and $\boldsymbol{x}_3^T = [1 \quad 1]$ is an eigenvector of the matrix $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ and find the corresponding eigenvalue.

Solution:

$$A\boldsymbol{x}_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix} = -\begin{bmatrix} 4 \\ 1 \end{bmatrix} = -\boldsymbol{x}_1 \,,$$

then $\boldsymbol{x}_1$ is an eigenvector of the matrix A and the corresponding eigenvalue is $\lambda = -1$.

$$A\boldsymbol{x}_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2\boldsymbol{x}_2,$$

then $\boldsymbol{x}_2$ is an eigenvector of the matrix A , and the corresponding eigenvalue is $\lambda = -2$.

$$A\boldsymbol{x}_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} \neq \lambda\boldsymbol{x}_3 \quad ,$$

then $\boldsymbol{x}_3$ is not an eigenvector of the matrix A.

Example 2: Verify that which of the vectors $\boldsymbol{x}_1^T = [-2 \quad 1 \quad 0]$, $\boldsymbol{x}_2^T = [-10 \quad 3 \quad 1]$, and $\boldsymbol{x}_3^T = [0 \quad 2 \quad 3]$ is an eigenvector of the matrix $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and find the corresponding eigenvalue.

Solution:

$$A\boldsymbol{x}_1 = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \lambda\boldsymbol{x}_1 \,,$$

then $\boldsymbol{x}_1$ is not an eigenvector of the matrix A.

$$A\boldsymbol{x}_2 = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 6 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} = 2\boldsymbol{x}_2 \,,$$

then $\boldsymbol{x}_2$ is an eigenvector of the matrix A and the corresponding eigenvalue is $\lambda = 2$.

$$A\boldsymbol{x}_3 = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ 18 \\ 6 \end{bmatrix} \neq \lambda\boldsymbol{x}_3 \quad ,$$

then $\boldsymbol{x}_3$ is not an eigenvector of the matrix A.

Example 3:

Verify that which of the vectors $\boldsymbol{x}_1^T = [1 \quad 0 \quad 1]$, $\mathbf{x}_2^T = [-1 \quad 1 \quad 0]$, and $\mathbf{x}_3^T = [1 \quad 1 \quad 2]$ is an eigenvector of the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ and find the corresponding eigenvalue.

Solution:

$$A\boldsymbol{x}_1 = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \neq \lambda \boldsymbol{x}_1$$

Then $\boldsymbol{x}_1$ is not an eigenvector of the matrix A.

$$A\boldsymbol{x}_2 = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \neq \lambda \boldsymbol{x}_2$$

Then $\boldsymbol{x}_2$ is not an eigenvector of the matrix A.

$$A\boldsymbol{x}_3 = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 4\boldsymbol{x}_3$$

Then $\boldsymbol{x}_3$ is an eigenvector of the matrix A and the corresponding eigenvalue is $\lambda = 4$.

Example 4:

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

Solution:

The eigenvalues λ of the matrix A are given by solving the characteristic equation:

$$det(A - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = 0$$

$$\therefore \quad \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda+1)(\lambda+2) = 0 \quad \Rightarrow \quad \lambda = -1, -2$$

Consider $\boldsymbol{x}_1^T = [x_1 \quad x_2]$ to be the eigenvector corresponding to the eigenvalue $\lambda_1 = -1$

$$\therefore \quad (A - \lambda_1 I)\boldsymbol{x}_1 = 0 \quad \Rightarrow \quad \begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{cc|c} 3 & -12 & 0 \\ 1 & -4 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1 + R_2} \left[\begin{array}{cc|c} 3 & -12 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad 3x_1 - 12x_2 = 0.$$

Letting the free variable $x_2 = k = 1$ gives $x_1 = 4 \quad \Rightarrow \quad \boldsymbol{x}_1^T = [4 \quad 1]$

Consider $\boldsymbol{x}_2^T = [x_1 \quad x_2]$ to be the eigenvector corresponding to the eigenvalue $\lambda_2 = -2$

$$\therefore \ (A - \lambda_2 I)\boldsymbol{x}_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{cc|c} 4 & -12 & 0 \\ 1 & -3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{4}R_1 + R_2} \left[\begin{array}{cc|c} 4 & -12 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \ 4x_1 - 12x_2 = 0.$$

Letting the free variable $x_2 = k = 1$ gives $x_1 = 3 \quad \Rightarrow \quad \boldsymbol{x}_2^T = [3 \quad 1]$

<u>Example 5</u>: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

<u>Solution</u>: The eigenvalues λ of the matrix A are given by solving the equation:

$$det(A - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \ (1-\lambda)[(2-\lambda)(1-\lambda) - 1] + (\lambda - 1) = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2 - 3\lambda) = 0 \quad \Rightarrow \quad \lambda = 0, 1, 3$$

The eigenvector $\boldsymbol{x}_1^T = [x_1 \quad x_2 \quad x_3]$ corresponding to the eigenvalue $\lambda_1 = 0$ satisfies:

$$(A - \lambda_1 I)\boldsymbol{x}_1 = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \ x_1 = x_2, \qquad x_2 = x_3.$$

Letting the free variable $x_3 = k = 1$ gives $x_1 = 1,\ \ x_2 = 1 \ \Rightarrow \ \ \boldsymbol{x}_1^T = [1 \quad 1 \quad 1]$.

Consider $\boldsymbol{x}_2^T = [x_1 \quad x_2 \quad x_3]$ to be the eigenvector corresponding to the eigenvalue $\lambda_2 = 1$

$$\therefore (A - \lambda_2 I)\boldsymbol{x}_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination s gives the augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad x_2 = 0, \quad x_3 = k, \quad x_1 = -x_3.$$

Letting the free variable $x_3 = k = 1$ gives $x_1 = -1,\ \ x_2 = 0 \ \Rightarrow \ \ \boldsymbol{x}_2^T = [-1 \quad 0 \quad 1]$.

Consider $\boldsymbol{x}_3^T = [x_1 \quad x_2 \quad x_3]$ to be the eigenvector corresponding to the eigenvalue $\lambda_3 = 3$

$$\therefore \ (A - \lambda_3 I)\boldsymbol{x}_3 = 0 \quad \Rightarrow \quad \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1 + R_2} \left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ 0 & -1/2 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ 0 & -1/2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \ 2x_1 = -x_2, \quad x_2 = -2x_3.$$

Letting the free variable $x_3 = k = 1$ gives $x_1 = 1,\ \ x_2 = -2 \ \Rightarrow \ \ \boldsymbol{x}_3^T = [1 \quad -2 \quad 1]$.

<u>**Example 6**</u>: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

<u>**Solution:**</u> The eigenvalues λ of the matrix A are given by solving the equation:

$$det(A - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore \ (3-\lambda)(2-\lambda)(4-\lambda) = 0 \quad \Rightarrow \quad \lambda = 3, 2, 4$$

If the eigenvector corresponding to the eigenvalue $\lambda_1 = 3$ is $\boldsymbol{x}_1^T = [x_1 \quad x_2 \quad x_3]$, then,

$$(A - \lambda_1 I)\boldsymbol{x}_1 = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to see that $x_2 = 0, x_3 = 0$, consider the free variable $x_1 = k = 1$

$$\Rightarrow \boldsymbol{x}_1^T = [1 \quad 0 \quad 0]$$

Consider $\boldsymbol{x}_2^T = [x_1 \quad x_2 \quad x_3]$ to be the eigenvector corresponding to the eigenvalue $\lambda_2 = 2$

$$\therefore (A - \lambda_2 I)\boldsymbol{x}_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to see that $x_1 = 0,\ x_3 = 0$, consider the free variable $x_2 = k = 1$

$$\Rightarrow \boldsymbol{x}_2^T = [0 \quad 1 \quad 0]$$

Consider $\boldsymbol{x}_3^T = [x_1 \quad x_2 \quad x_3]$ to be the eigenvector corresponding to the eigenvalue $\lambda_3 = 4$

$$\therefore (A - \lambda_3 I)\boldsymbol{x}_3 = 0 \quad \Rightarrow \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to see that $x_1 = 0,\ x_2 = 0$, consider the free variable $x_3 = k = 1$

$$\Rightarrow \boldsymbol{x}_3^T = [0 \quad 0 \quad 1]$$

Example 7: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

Solution: The eigenvalues λ of the matrix A are given by solving the equation:

$$det(A - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 5 & -10 \\ 1 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda) \begin{vmatrix} 1-\lambda & 5 & -10 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)^2(2-\lambda)(3-\lambda) = 0 \quad \Rightarrow \quad \lambda = 1, 2, 3$$

Note that $\lambda = 1$ is repeated root.

The eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ is $\boldsymbol{x}_1^T = [x_1 \quad x_2 \quad x_3 \quad x_4]$ such that:

$$(A - \lambda_1 I)\boldsymbol{x}_1 = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & -10 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & -10 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \quad x_1 = -2x_4, \qquad x_3 = 2x_4$$

Since we have two rows of zeros, then we have two eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$. There are two free variables, x_2 and x_4.

Letting $x_2 = k = 1, x_4 = 0$ gives $x_1 = 0, \; x_2 = 1, \; x_3 = 0, \; x_4 = 0$

$$\Rightarrow \; \boldsymbol{x}_1^T = [0 \quad 1 \quad 0 \quad 0]$$

Letting $x_2 = 0, \; x_4 = 1$ gives $x_1 = -2, \; x_2 = 0, \; x_3 = 2, \; x_4 = 1$

$$\Rightarrow \; \boldsymbol{x}_2^T = [-2 \quad 0 \quad 2 \quad 1]$$

The eigenvector corresponding to the eigenvalue $\lambda_2 = 2$ is $\boldsymbol{x}_3^T = [x_1 \quad x_2 \quad x_3 \quad x_4]$ such that:

$$(A - \lambda_2 I)\boldsymbol{x}_3 = 0 \quad \Rightarrow \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & -10 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 5 & -10 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1+R_4]{R_1+R_3} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore x_1 = 0, \quad x_4 = 0, \quad x_2 = 5x_3.$$

Letting the free variable $x_3 = k = 1$ gives $x_2 = 5 \quad \Rightarrow \quad \boldsymbol{x}_3^T = [0 \quad 5 \quad 1 \quad 0]$.

The eigenvector corresponding to the eigenvalue $\lambda_3 = 4$ is $\boldsymbol{x}_4^T = [x_1 \quad x_2 \quad x_3 \quad x_4]$ such that:

$$(A - \lambda_3 I)\boldsymbol{x}_4 = 0 \quad \Rightarrow \quad \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -10 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method gives the augmented matrix:

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 5 & -10 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_1+R_4]{\frac{1}{2}R_1+R_3} \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 5 & -10 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore x_1 = 0, x_3 = 0, x_2 = -5x_4.$$

Letting the free variable $x_4 = k = 1$ gives $x_2 = -5 \Rightarrow \boldsymbol{x}_4^T = [0 \quad -5 \quad 0 \quad 1]$

Exercise

1. Verify that which of the vectors $\boldsymbol{x}_1^T = [1 \quad 0 \quad 0]$, $\boldsymbol{x}_2^T = [0 \quad -1 \quad 0]$, and $\boldsymbol{x}_3^T = [5 \quad 1 \quad 2]$ is an eigenvector of the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ and find the corresponding eigenvalue.
2. Verify that which of the vectors $\boldsymbol{x}_1 = [1 \quad 2 \quad -1]$, $\boldsymbol{x}_2^T = [-2 \quad 1 \quad 0]$, and $\boldsymbol{x}_3^T = [3 \quad 0 \quad 0]$ is an eigenvector of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ and find the corresponding eigenvalue.
3. Verify that which of the vectors $\boldsymbol{x}_1^T = [1 \quad 0 \quad -1]$, $\boldsymbol{x}_2^T = [1 \quad 2 \quad 0]$, and $\boldsymbol{x}_3^T = [-2 \quad 1 \quad 1]$ is an eigenvector of the matrix $A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and find the corresponding eigenvalue.
4. Find the eigenvalues and corresponding eigenvectors of the matrices:

(i) $A = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (v) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Chapter Six

Vectors and Vector Space

Vectors in $\mathbf{R}^n$

As we know, a matrix having only one column is called a vector and is written in the form:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$$

The vectors of n entries are called vectors in $\mathbf{R}^n$, where $u_1, u_2, u_3, \cdots, u_n$ are any real numbers. The following $\boldsymbol{a}, \boldsymbol{b}$ vectors are vectors in $\mathbf{R}^2$ while $\boldsymbol{c}, \boldsymbol{d}$ are vectors in $\mathbf{R}^3$.

$$\boldsymbol{a} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} 0 \\ -1 \\ 8 \end{bmatrix}$$

The vector whose all entries are zeros is called the zero vector and is denoted by $\mathbf{0}$.

Linear Combinations

Given vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \cdots, \boldsymbol{u}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, c_3, \cdots, c_p$, the vector $\boldsymbol{y}$ defined by

$$\boldsymbol{y} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\,\boldsymbol{u}_3 + \cdots + c_p\,\boldsymbol{u}_p$$

is called a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \cdots, \boldsymbol{u}_p$ with weights $c_1, c_2, c_3, \cdots, c_p$. The weights in a linear combination can be any real numbers, including zero. In this case we say that the vector $\boldsymbol{y}$ is generated by a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \cdots, \boldsymbol{u}_p$.

For examples:

(i) Show that the vector $\boldsymbol{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ can be written as a linear combination of the vectors

$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ and $\boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$.

<u>Solution</u>:

It is required to determine whether weights c_1 and c_2 exist such that: $c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 = \boldsymbol{b}$.

Then we need to solve the vector equation:

$$c_1\begin{bmatrix}1\\-2\\-5\end{bmatrix}+c_2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}$$

Introducing an artificial variable $c_3 \neq 0$ with zero coefficients and employing Gauss-Elimination method gives:

$$[A|\boldsymbol{b}]=\left[\begin{array}{ccc|c}1&2&0&7\\-2&5&0&4\\-5&6&0&-3\end{array}\right]\xrightarrow[5R_1+R_3]{2R_1+R_2}\left[\begin{array}{ccc|c}1&2&0&7\\0&9&0&18\\0&16&0&32\end{array}\right]$$

$$\xrightarrow{R_2/9}\left[\begin{array}{ccc|c}1&2&0&7\\0&1&0&2\\0&16&0&32\end{array}\right]\xrightarrow{-16R_2+R_3}\left[\begin{array}{ccc|c}1&2&0&7\\0&1&0&2\\0&0&0&0\end{array}\right]$$

Hence, we can obtain $c_2 = 2$ and $c_1 + 2c_2 = 7 \quad \Longrightarrow \quad c_1 = 3$. Therefore, the vector $\boldsymbol{b}$ is a linear combination of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

(ii) Given the vectors: $\boldsymbol{a}_1=\begin{bmatrix}1\\-2\\0\end{bmatrix}$, $\boldsymbol{a}_2=\begin{bmatrix}0\\1\\2\end{bmatrix}$, $\boldsymbol{a}_3=\begin{bmatrix}5\\-6\\8\end{bmatrix}$, and $\boldsymbol{b}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$

Determine if the vector $\boldsymbol{b}$ is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ or not.

Solution:

It is required to determine whether weights c_1, c_2 and c_3 exist such that: $c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + c_3\boldsymbol{a}_3 = \boldsymbol{b}$ or not. Then we need to solve the vector equation:

$$c_1\begin{bmatrix}1\\-2\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\2\end{bmatrix}+c_3\begin{bmatrix}5\\-6\\8\end{bmatrix}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$$

Employing Gauss-Elimination method gives:

$$[A|\boldsymbol{b}]=\left[\begin{array}{ccc|c}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{array}\right]\xrightarrow{2R_1+R_2}\left[\begin{array}{ccc|c}1&0&5&2\\0&1&4&3\\0&2&8&6\end{array}\right]$$

$$\xrightarrow{-2R_2+R_3}\left[\begin{array}{ccc|c}1&0&5&2\\0&1&4&3\\0&0&0&0\end{array}\right]$$

Then the system of equations has an infinite number of solutions:

$c_3 = c$ (real number) , $c_2 + 4c_3 = 3 \quad \Longrightarrow \quad c_2 = 3 - 4c,\ c_1 + 5c_3 = 2 \ \Longrightarrow \quad c_1 = 2 - 5c$

Therefore, the vector $\boldsymbol{b}$ is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \cdots, \boldsymbol{u}_p\}$ of vectors.

Definition:

If the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_p$ are in $\mathbf{R}^n$, then the set of all linear combinations of them is subset of $\mathbf{R}^n$ and is denoted by:

$$\text{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_p\} = \{\boldsymbol{b} : \ \boldsymbol{b} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_p\,\boldsymbol{u}_p\}$$

with $c_1, c_2, c_3, \cdots, c_p$ scalars. That is, set 'Span $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_p\}$' is generated by the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_p$. Asking if a vector $\boldsymbol{b}$ is in Span $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \cdots, \boldsymbol{u}_p\}$ amounts to asking whether the vector equation:

$$x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + \cdots + x_p\,\boldsymbol{u}_p = \boldsymbol{b}$$

has a solution. In particular, the zero vector must be in Span $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_p\}$.

For examples:

(i) Given the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$

Determine if the vector $\boldsymbol{b}$ is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ or not.

<u>**Solution**</u>: It is required to solve the vector equation: $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\,\boldsymbol{a}_3 = \boldsymbol{b}$

$$\Longrightarrow \quad x_1\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

Employing Gauss-Elimination method gives:

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right] \xrightarrow{2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

Then, we have: $x_3 = -2$, $3c_2 - 2c_3 = 1 \implies c_2 = -1$, and $c_1 - 4c_3 = 4 \implies c_1 = -4$

Therefore, the vector $\boldsymbol{b}$ is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$.

(ii) Given the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$

Determine if the vector $\boldsymbol{b}$ is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ or not.

<u>Solution</u>:

It is required to solve the vector equation: $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\,\boldsymbol{a}_3 = \boldsymbol{b}$

$$\Longrightarrow \quad x_1\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$$

Employing Gauss-Elimination method gives:

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right] \xrightarrow[-R_1/2 + R_3]{R_1/2 + R_2} \left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_2/4 + R_3} \left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then the system of equations has an infinite number of solutions: $c_3 = c$ (real number), $c_2 + c_3 = 1 \Longrightarrow c_2 = 1 - c,\; 2c_1 + 6c_3 = 10 \Longrightarrow c_1 = 5 - 3c$ and the vector $\boldsymbol{b}$ is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$.

Linear independence

A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \cdots, \boldsymbol{u}_p\}$ in $\mathbf{R}^n$ is said to be linearly independent if the vector equation:

$$x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + x_3\boldsymbol{u}_3 + \cdots + x_p\,\boldsymbol{u}_p = \boldsymbol{0}$$

has only the single solution, $x_1 = x_2 = x_3 = \cdots + x_p = 0$. Otherwise,it is said to be linearly dependent. **For examples**:

(1) Given the vectors: $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\; \boldsymbol{u}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},\; \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(i) Determine if they are linearly independent.

(ii) If possible, find a linear dependence relation among them.

<u>Solution</u>:

(i) It is required to solve the vector equation: $x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + x_3\,\boldsymbol{u}_3 = \boldsymbol{0}$

$$\Longrightarrow \quad x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Employing Gauss-Elimination method gives:

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, we have an infinite number of solutions and the given vectors are linearly dependent.

(ii) To find a linear dependence relation among the vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ Choose any nonzero value for the free variable $c_3 = 1$ (say) yields $c_2 + c_3 = 0 \implies c_2 = -1$, and $c_1 + 4c_2 + 2c_3 = 0 \implies c_1 = -6$. Thus, we obtain the linear dependence relation:

$$6\boldsymbol{u}_1 + \boldsymbol{u}_2 - \boldsymbol{u}_3 = \boldsymbol{0}$$

(2) Given the vectors: $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$

(i) Determine if they are linearly independent.

(ii) If possible, find a linear dependence relation among them.

Solution:

(i) It is required to solve the vector equation: $x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + x_3\,\boldsymbol{u}_3 = \boldsymbol{0}$

$$\implies x_1 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Employing Gauss-Elimination method gives:

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & 6 & 0 \end{array}\right] \xrightarrow[-4R_1+R_3]{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & 10 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{5}{2}R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, we have an infinite number of solutions and the given vectors are linearly dependent.

(ii) To find a linear dependence relation among the vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ Choose any nonzero value for the free variable c_3 (say 1) yields $-2c_2 + 4c_3 = 0 \implies c_2 = 2$, and $c_1 + 3c_2 - c_3 = 0 \implies c_1 = -5$. Thus, we obtain the relation: $-5\boldsymbol{u}_1 + 2\boldsymbol{u}_2 + \boldsymbol{u}_3 = \boldsymbol{0}$

Exercise

(1) Determine if the vector $\boldsymbol{b}$ is a linear combination of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ or not for the following vectors:

(i) $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

(ii) $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}$

(iii) $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

(2) Determine if the vector $\boldsymbol{b}$ is in Span $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ or not for the following vectors:

(i) $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(ii) $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

(iii) $\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

(3) Determine if the given vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ are linearly independent. If possible, find a linear dependence relation among them.

(i) $\boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ (ii) $\boldsymbol{u}_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$

(iii) $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 0 \\ -9 \end{bmatrix}$

Vector Space

A vector space over the field K is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\boldsymbol{u}, \boldsymbol{\upsilon}$, and $\mathbf{w}$ in V and for all elements c and d of K..

1. The sum $\boldsymbol{u} + \boldsymbol{\upsilon} \in V$. (i.e., vector addition is Closure)
2. $\boldsymbol{u} + \boldsymbol{\upsilon} = \boldsymbol{\upsilon} + \boldsymbol{u}$. (i.e., vector addition is commutative)
3. $(\boldsymbol{u} + \boldsymbol{\upsilon}) + \mathbf{w} = \boldsymbol{u} + (\boldsymbol{\upsilon} + \mathbf{w})$. (i.e., vector addition is associative)
4. There is a zero vector $\mathbf{0}$ in V such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$.
5. For each $\boldsymbol{u}$ in V, there is a vector $-\boldsymbol{u}$ in V such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$.
6. The scalar multiple of $\boldsymbol{u}$ by c, denoted by $c\boldsymbol{u}$, is in V .
7. $c(\boldsymbol{u} + \boldsymbol{\upsilon}) = c\boldsymbol{u} + c\boldsymbol{\upsilon}$.
8. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$.
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$.
10. $1\boldsymbol{u} = \boldsymbol{u}$, where 1 is the multiplicative identity element of the field K.

A vector space is said to be trivial if it consists of a single element (which must then be the zero vector).

a. The set V of all vectors in $\mathbf{R}^n$ is a vector space over the field $\mathbf{R}$ (the set of real numbers). For any vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$, we have:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}, \text{and} \quad \boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix}$$

Where $\{x_i, y_i, z_i\}$ are real numbers.

Using the properties of addition and scalar multiplication of vectors as matrices, it is easy to identify that (where $c, d \in \mathbf{R}$):

1. $\boldsymbol{x} + \boldsymbol{y}$ is a vector in $\mathbf{R}^n$ $\Longrightarrow$ $\boldsymbol{x} + \boldsymbol{y} \in V$
2. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
3. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
4. There is a zero vector $\mathbf{0}$ in V such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$.
5. For each $\boldsymbol{x}$ in V, there is a vector $-\boldsymbol{x}$ in V such that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.
6. $c\boldsymbol{x}$ gives a vector in $\mathbf{R}^n$, $\quad$ i.e., $c\boldsymbol{x} \in V$.
7. $c\,\boldsymbol{x} + \mathrm{c}\boldsymbol{y} = c(\boldsymbol{x} + \mathbf{y})$
8. $c\boldsymbol{x} + d\boldsymbol{x} = (c + d)\,\boldsymbol{x}$

9. $c(d\boldsymbol{x}) = (cd)\boldsymbol{x}$

10. Since the multiplicative identity element of **R** is 1 (the number one) then $1\boldsymbol{x} = \boldsymbol{x}$.

Simply, we can say that $\mathbf{R}^n$ is a vector space.

b. The set V of all real 2×2 matrices is a vector space over the field **R** (in this case vectors are matrices). All real 2×2 matrices have the form: $\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbf{R}$.

Using the properties of addition and scalar multiplication of matrices, it is easy to show that the axioms of vector spaces are satisfied. Remember that:

$$-\boldsymbol{M} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

c. The set of all $m \times n$ matrices is a vector space over the field **R**.

d. The solution set of linear non-homogeneous equations of n unknowns is not a vector space in $\mathbf{R}^n$ over the field **R**, because it does not contain the zero vector (zero solution). Note that each solution represents a vector in $\mathbf{R}^n$.

In applications of linear algebra, subspaces of $\mathbf{R}^n$ usually arise in one of two ways:

(1) as the set of all solutions to a system of homogeneous linear equations, or

(2) as the set of all linear combinations of certain specified vectors, for example.

Null space of a matrix:

The solution set of linear homogeneous equations of n unknowns, $A\boldsymbol{x} = \mathbf{0}$**,** is called 'Null space of the matrix A', written as '$Null\ A$'.

The set $Null\ A$ is a vector space in $\mathbf{R}^n$ over the field **R**, the zero vector corresponds to the zero solution.

For examples:

1. The null space of the matrix $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ is the set solution of $A\boldsymbol{x} = \mathbf{0}$.

Let $\boldsymbol{x}^T = [x_1, x_2, x_3]$ and using Gauss-Elimination method gives,

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -5 & 9 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{5R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -6 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have a free variable $x_3 = c$ (any real number). The second and first rows give:

$$-6x_2 - 9x_3 = 0 \quad \Rightarrow \quad x_2 = -1.5\,c, \quad x_1 - 3x_2 - 2x_3 = 0, \quad x_1 = -2.5\,c$$

Then null space of the matrix A is $\begin{bmatrix} -2.5\,c \\ -1.5\,c \\ c \end{bmatrix}$ which is a vector space in $\mathbf{R}^3$ over the field $\mathbf{R}$, where the zero vector $\mathbf{0}$ is obtained for $c = 0$.

2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 1 \end{bmatrix}$.

The vector $\boldsymbol{u} = \begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix} \in Null\ A$ because: $\begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

while the vector $\boldsymbol{\upsilon} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \notin Null\ A$ because $\begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Column space of a matrix:

The set of all linear combinations of vectors representing columns of an $m \times n$ matrix A is called '**C**olumn space of the matrix A', written as Col A.

That is, if $A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n]$, That is;

$$Col\ A = \{\boldsymbol{b} \ : \ \boldsymbol{b} = c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_n\boldsymbol{a}_n \ \ (\text{for real numbers } c_1, c_2, \cdots, c_n).\}$$

It is easy to see that Col A is a vector space in $\mathbf{R}^n$ over the field $\mathbf{R}$.

For examples:

1. Consider the matrix $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$, to show that the vector $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is in $Col\ A$ we have to solve the equation:

$$c_1\begin{bmatrix} -8 \\ 6 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -9 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Using Gauss-Elimination method, we obtain the augmented matrix:

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & 0 \end{array}\right] \xrightarrow[2R_1 + R_3]{3R_1/4 + R_2} \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 5/2 & 5/4 & 5/2 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

Then the system of linear equations c_1, c_2, c_3 has single solution, then the vector $\boldsymbol{u}$ is in $Col\ A$.

Subspaces

In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

Definition

A subspace of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, the sum $\boldsymbol{u}+\boldsymbol{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\boldsymbol{u}$ in H and each scalar c, the vector $c\boldsymbol{u}$ is in H.

These properties guarantee that a subspace of vector space is itself a vector space.

Examples:

1. The set of all real $n \times n$ diagonal matrices (including the **zero** matrix) is subspace of the vector space of all real $n \times n$ matrices over the field $\mathbf{R}$. Because the sum of two diagonal matrices gives diagonal matrix of the same order, the scalar product of diagonal matrix gives diagonal matrix of the same order.

2. The solution set of linear homogeneous equations of n unknowns, $A\boldsymbol{x} = \mathbf{0}$ $(Nul\ A)$ is a vector space over the field $\mathbf{R}$, and is a subspace of the vector space $\mathbf{R}^n$ (where n is the number of columns of A).

For example: The solution set H of the linear homogeneous equations:

$$-x_1 + x_2 + x_3 = 0, \quad 3x_1 - x_2 = 0, \quad 2x - 4x_2 - 5x_3 = 0$$

is $H = \left\{ \boldsymbol{x} = \begin{bmatrix} -\frac{1}{2}c \\ -\frac{3}{2}c \\ c \end{bmatrix}, c \in \mathbf{R} \right\}$.

It is easy to show that H is a vector space over the field $\mathbf{R}$, where the zero vector $\mathbf{0}$ is given for $c = 0$. The set H is also subspace of the vector space $\mathbf{R}^3$.

Note that H is a subset of $\mathbf{R}^3$, because many vectors in $\mathbf{R}^3$ do not exist in H, for example $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin H$.

3. The set of vectors defined as: $H = \left\{ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 - 4x_2 + 5x_3 = 3 \right\}$ is a subset of $\mathbf{R}^3$, but it is not subspace of the vector space of all vectors in $\mathbf{R}^3$, because it does not

include the zero vector $\mathbf{0}$. Note that the zero solution is out of the solutions of the equation satisfied by the variables $x_1, x_2,$ and x_3.

4. The set of all linear combinations of certain specified k vectors is a subspace of $\mathbf{R}^k$.

Basis of a space

A basis of a vector space is one of the most useful concepts in linear algebra. It often provides a concise, finite description of an infinite vector space.

Definition: Suppose V is a vector space. Then a subset $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p\} \subseteq V$ is a basis of V if it is linearly independent and spans V.

. According to definition of linear independence of vectors, the vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n$ are linearly independent of the $n \times n$ matrix whose columns are those vectors is invertible.

Then if the matrix $A = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_n]$ is invertible, the set of the vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2, \cdots, \boldsymbol{b}_n$ form a basis of the space $\mathbf{R}^n$ because they are linearly independent and they span $\mathbf{R}^n$ (any vector in $\mathbf{R}^n$ can be written as a linear combination of the given vectors).

Examples:

1. Determine if the set of the vectors $\boldsymbol{b}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $\boldsymbol{b}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ is a basis of the space $\mathbf{R}^3$.

Solution: Since there are exactly three vectors here in $\mathbf{R}^3$, we should determine if the matrix $A = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3]$ is invertible or not.

$$|A| = \begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{vmatrix} = -6 + 24 - 12 = 6 \neq 0$$

Then the given vectors are linearly independent and the given set of vectors is a basis of the space $\mathbf{R}^3$.

2. Determine if the following set of vectors form a bases for $\mathbf{R}^3$.

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \ \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \text{ and } \boldsymbol{b}_3 = \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$$

Solution: Since there are exactly three vectors here in $\mathbf{R}^3$, we should determine if the matrix $A = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3]$ is invertible or not.

$$|A| = \begin{vmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{vmatrix} = -44 + 13 + 7 = -24 \neq 0$$

Then the given vectors are linearly independent and the given set of vectors is a basis of the space $\mathbf{R}^3$.

The standard basis of $\mathbf{R}^n$:

The columns of the identity matrix of order n for a set of unit vectors:

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and } \boldsymbol{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are called standard unit vectors is a basis of the space $\mathbf{R}^n$, called the standard basis for $\mathbf{R}^n$. This is obvious because they are linearly independent and any vector $\boldsymbol{\upsilon}$ in $\mathbf{R}^n$ can be represented as a linear combination of these standard unit vectors:

$$\boldsymbol{\upsilon} = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \vdots \\ \upsilon_n \end{bmatrix} = \upsilon_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \upsilon_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \upsilon_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \upsilon_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise

1. Determine if the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in $Null\ A$, where $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$.

2. Determine if the vector $\mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in $Null\ A$, where $A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}$.

3. Consider $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if $\boldsymbol{u}$ is in each of $Col\ A$ and $Null\ A$.

4. Determine if $\boldsymbol{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ is in the null space of the matrix $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

5. Find the null space of the matrix $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$.

6. Determine if $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in the null space of the matrix $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$.

7. Determine if $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is in the column space of the matrix $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$.

8. Determine which sets in the following are bases for $\mathbf{R}^3$.

i. $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\boldsymbol{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

ii. $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\boldsymbol{b}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

iii. $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, and $\boldsymbol{b}_3 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$