# Lecture 3. Logic

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# Statements

In English grammar, sentences are divided into categories. In a **declarative sentence**, something is being declared or asserted; in an **interrogative sentence**, a question is being asked; in an **imperative sentence**, a command is given; while in an **exclamatory sentence**, an emotional expression is made.

These four kinds of sentences are illustrated below.

1. Los Angeles is the capital of California. (declarative sentence)

2. Broadway is a street in New York City. (declarative sentence)

3. Which state capital has the largest population? (interrogative sentence)

4. Drive to Main Street and then turn right. (imperative sentence)

5. I can’t believe it! (exclamatory sentence)

6. It was then that he arrived there. (declarative sentence)

# Statements

In Logic, we work with ***statements,*** i.e. ***declarative sentences*** that are either true or false but not both.

**For example:**

$1 + 1 = 2$

The year 1996 was a leap year

Go directly to jail!

Romulus and Remus founded New York City

This sentence is false

$x > 5$

# Statements

In Logic, we work with ***statements,*** i.e. ***declarative sentences*** that are either true or false but not both.

**For example:**

1 + 1 = 2 **(statement)**

The year 1996 was a leap year **(statement)**

Go directly to jail! **(not a statement)**

Romulus and Remus founded New York City **(statement)**

This sentence is false **(not a statement)**

$x > 5$ **(an open statement)**

# Predicates(open sentences)

A **predicate** (an **open sentence)** is a declarative sentence containing one or more variables and whose truth or falseness depends on the values of these variables.

**Example**

Consider the open sentence

$$P(x) : 3x - 9 = 0$$

Is a true statement if $x = 3$ and is a false statement otherwise.

# Some more examples of statements / not statements

Is it true that $\frac{-9}{3}+\frac{4}{2}=\frac{-9+4}{3+2}$?

Multiply the numbers 2/3 and 9/10.

There holds an equality $1.414=\sqrt{2}$

There holds an equality $\sqrt{(-3)^2}=-3$.

What a difficult calculus exam!

3x + 1 = 7.

# Logical operations

One can construct new statements from another, using the following **logical operations**:

NOT (“¬” or “∼”, negation)

AND (“Λ”, conjunction)

OR (“v”, disjunction)

If … then (“→” or “⇒”, implication)

For example, I take two statements: “1 + 1 = 2” and “the diagonals of a rectangle have the same length”.

A new statement: “1 + 1 = 2 and the diagonals of a rectangle have the same length”

# Logical operations

One can construct new statements from another, using the following operations:

NOT (“¬” or “∼”, negation)

AND (“Λ”, conjunction)

OR (“v”, disjunction)

If … then (“→” or “⇒”, implication)

Another example: “My name is Yaraslau” and “My cat can fly”.

A new statement: “If my name is Yaraslau then my cat can fly”.

# How does it work?
# The Truth Table: NOT

| $p$ | $\neg p$ |
|---|---|
| $T$ | $F$ |
| $F$ | $T$ |

- Q1 : Los Angeles is the capital of California.
  Q2 : Broadway is a street in New York City.

  ¬Q1 : Los Angeles is not the capital of California.
  ¬Q2 : Broadway is not a street in New York City.

# How does it work?
# The Truth Table: AND

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

- Q1 : Los Angeles is the capital of California.
  Q2 : Broadway is a street in New York City.

  Q1 ∧ Q2: Los Angeles is the capital of California and Broadway is a street in New York City.

# How does it work?
# The Truth Table: OR

| *p* | *q* | *pvq* |
|---|---|---|
| *T* | *T* | *T* |
| *T* | *F* | *T* |
| *F* | *T* | *T* |
| *F* | *F* | *F* |

- Q1 : Los Angeles is the capital of California.
  Q2 : Broadway is a street in New York City.

  Q1 v Q2: Los Angeles is the capital of California or Broadway is a street in New York City.

# How does it work?
# The Truth Table: IF … THEN

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

- Q1 : Los Angeles is the capital of California.
  Q2 : Broadway is a street in New York City.

  Q1 → Q2: If Los Angeles is the capital of California then Broadway is a street in New York City.

# How does it work?
# The Truth Table: IF … THEN

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

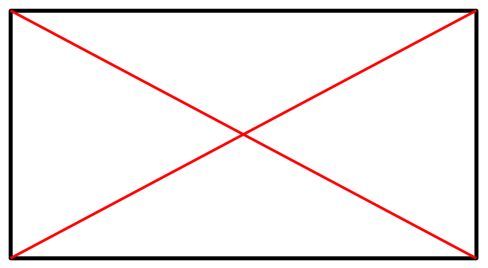
- This may be a bit confusing. For example, the sentence “If a cat can fly then Isaac Newton was using a calculator” turns to be truth!

# A bit more detailed about IF … THEN

Theorem *A*: If *ABCD* is a rectangle then its diagonals are equal.

This quadrilateral is a rectangle, and its diagonals are equal:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

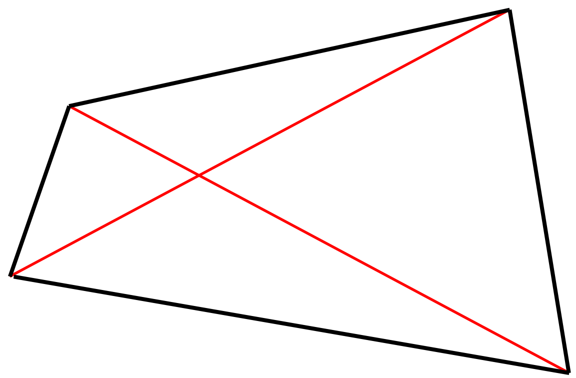


# A bit more detailed about IF … THEN

Theorem *A*: If *ABCD* is a rectangle then its diagonals are equal.

This quadrilateral is not a rectangle, and its diagonals are equal:

| *p* | *q* | *p*→*q* |
|---|---|---|
| *T* | *T* | *T* |
| *T* | *F* | *F* |
| *F* | *T* | *T* |
| *F* | *F* | *T* |

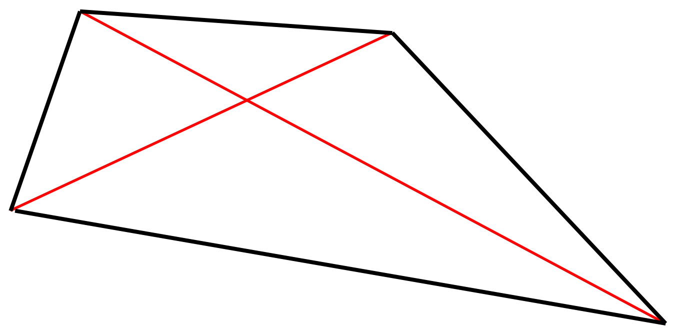


# A bit more detailed about IF … THEN

Theorem *A*: If *ABCD* is a rectangle then its diagonals are equal.

This quadrilateral is not a rectangle, and its diagonals are not equal:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| *T* | *T* | *T* |
| *T* | *F* | *F* |
| *F* | *T* | *T* |
| *F* | *F* | *T* |



# A bit more detailed about IF … THEN

Theorem *A*: If *ABCD* is a rectangle then its diagonals are equal.

If one finds a quadrilateral which is a rectangle, but its diagonals are not equal, then the theorem would turn to be incorrect, but…

| *p* | *q* | *p→q* |
|---|---|---|
| *T* | *T* | *T* |
| *T* | *F* | *F* |
| *F* | *T* | *T* |
| *F* | *F* | *T* |

???

# A Bit more detailed about Implications

For two statements $P$ and $Q$, the implication $P \Rightarrow Q$ is commonly written as “If $P$, then $Q$”.

An implication is also sometimes referred to as a conditional. The statement $P$ in the implication $P \Rightarrow Q$ is the **hypothesis** of $P \Rightarrow Q$, while $Q$ is the **conclusion** of $P \Rightarrow Q$.

# Example

Determine the truth value of each of the following implications.
(a) If $2 + 3 = 5$, then $4 + 6 = 10$.
(b) If $4 + 6 = 10$, then $5 + 7 = 14$.
(c) If $5 + 7 = 14$, then $6 + 9 = 15$.
(d) If $8 + 11 = 21$, then $12 + 14 = 28$.

# Example

Determine the truth value of each of the following implications.
(a) If 2 + 3 = 5, then 4 + 6 = 10.
(b) If 4 + 6 = 10, then 5 + 7 = 14.
(c) If 5 + 7 = 14, then 6 + 9 = 15.
(d) If 8 + 11 = 21, then 12 + 14 = 28.

(a) “If 2 + 3 = 5, then 4 + 6 = 10” reduces to: If T , then T . This is a **true** implication according to the first row of the truth table.
(b) “If 4 + 6 = 10, then 5 + 7 = 14” reduces to: If T , then F . According to the second row of the truth table, this implication is **false**.
(c) “If 5 + 7 = 14, then 6 + 9 = 15” reduces to: If F , then T . By the third row of the truth table, this implication is **true**.
(d) “If 8+11 = 21, then 12+14 = 28” reduces to: If F , then F . This implication is **true**.

# Theorem 1.22 (Commutative Laws)

For every two statements $P$ and $Q$,

$$P \wedge Q \equiv Q \wedge P \text{ and } P \vee Q \equiv Q \vee P.$$

| $p$ | $q$ | $p \wedge q$ | $q \wedge p$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ |

# Theorem (De Morgan’s Laws)

For every two statements $P$ and $Q$,

(a) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$;

(b) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$.

| $p$ | $q$ | $P \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $(\neg p) \vee (\neg q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

# De Morgan’s Laws: Example

For every two statements $P$ and $Q$,

(a) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$;

(b) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$.

**Example**
Use De Morgan’s Laws to express the negation of the following.
(1) Either I get this job or I take another class.
(2) I’m eating dinner out and going to a movie.
**Solution.**
(1) I don’t get this job and I don’t take another class.
(2) I’m not eating dinner out or I’m not going to a movie.

# De Morgan's Laws: Example

**Example**

Use De Morgan's Laws to express the negation of the following.

(1) Either he's not checking his email or he's not answering his email.

(2) This summer, she is neither buying an iPhone nor an iPad.

# De Morgan’s Laws: Example

**Example**

Use De Morgan’s Laws to express the negation of the following.

(1) Either he’s not checking his email or he’s not answering his email.

(2) This summer, she is neither buying an iPhone nor an iPad.

**Solution.**

(1) He checks and answers his email.

(2) The sentence above can also be written as follows: This summer, she is not buying an iPhone and not buying an iPad. Thus its negation is: This summer, she is either buying an iPhone or an iPad.

# De Morgan’s Laws: Example

Let P and Q be two statements. Use De Morgan’s Laws to verify that

$$\neg(P \vee (\neg Q)) \equiv (\neg P) \wedge Q.$$

By De Morgan’s Law, $\neg(P \vee (\neg Q)) \equiv (\neg P) \wedge (\neg(\neg Q))$

Obviously, $\neg(\neg Q) \equiv Q.$

Therefore, $\neg(P \vee (\neg Q)) \equiv (\neg P) \wedge (\neg (\neg Q)) \equiv (\neg P) \wedge Q.$

# Associative and Distributive Laws

Associative Laws:

$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ and $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$.

compare: $x + (y + z) = (x + y) + z$ and $x \times (y \times z) = (x \times y) \times z$

Distributive Laws:

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ and $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.

compare: $x \times (y + z) \equiv (x \times y) + (x \times z)$

# Connection to Arithmetic

Sometimes, if the statement is true we say that its value equals to 1, and if it is false we say that its value equals to 0:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

| $x$ | $y$ | $x \times y$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

# Connection to Arithmetic

Sometimes, if the statement is true we say that its value equals to 1, and if it is false we say that its value equals to 0:

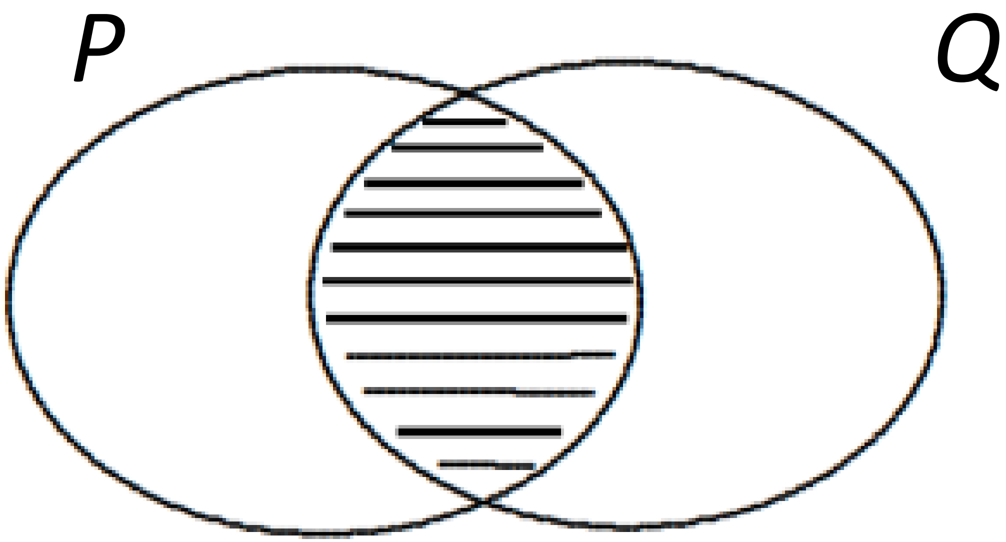
| *p* | *q* | *p*V*q* |
|---|---|---|
| *T* | *T* | *T* |
| *T* | *F* | *T* |
| *F* | *T* | *T* |
| *F* | *F* | *F* |

| *x* | *y* | *x*+*y* |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

# Connection to the Sets

P: There is going to be rain in the evening.

Q: There is going to be hot in the afternoon.



# Connection to the Sets

Thus, $P \wedge Q$ is analogous to $A \cap B$, and $P \vee Q$ is analogous to $A \cup B$.

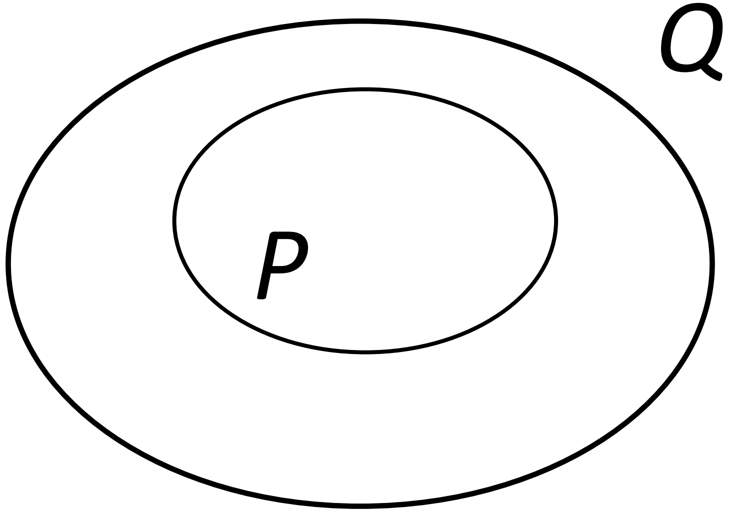
And what is analogous to $\Rightarrow$???

# Connection to the Sets

Thus, P ∧ Q is analogous to A ∩ B, and P ∨ Q is analogous to A ∪ B.

And what is analogous to ⇒???

P ⇒ Q is analogous to $A \subseteq B$!!!




P: There is going to be rain in the evening.

Q: There is going to be hot in the afternoon.

# Predicates

A **predicate** (an **open sentence)** is a declarative sentence containing one or more variables and whose truth or falseness depends on the values of these variables.

# Predicate Logic

The ***existential quantification*** of a predicate $P(x)$ whose variable ranges over a domain set $D$ is the proposition $(\exists x \in D)P(x)$ or $(\exists x)P(x)$ that is true if there is at least one $c$ in $D$ such that $P(c)$ is true. The *existential quantifier symbol*, $\exists$, is read “there exists” or “there is”.

The ***universal quantification*** of a predicate $P(x)$ whose variable ranges over a domain set $D$ is the proposition $(\forall x \in D)P(x)$ or $(\forall x)P(x)$, which is true if $P(c)$ is true for every element $c$ in $D$. The *universal quantifier symbol*, $\forall$, is read “for all”, “for each”, or “for every”.

# Example

The statement “Every day the Sun arises” can be expressed in the following way:

Let $D$ be the set of days, the statement $P$ is “the Sun arises”. Then

$$\forall d \in D = \text{“Every day the Sun arises”.}$$

# Predicate Logic

Examples. Which statements are true?

a) for any natural *n*, there is a natural *m* which is smaller than *n*;

b) for any natural *n*, there is a natural *m* which is not greater than *n*;

c) for any positive rational *p*, there is a positive rational *q* which is smaller than *p.*

Thank you!