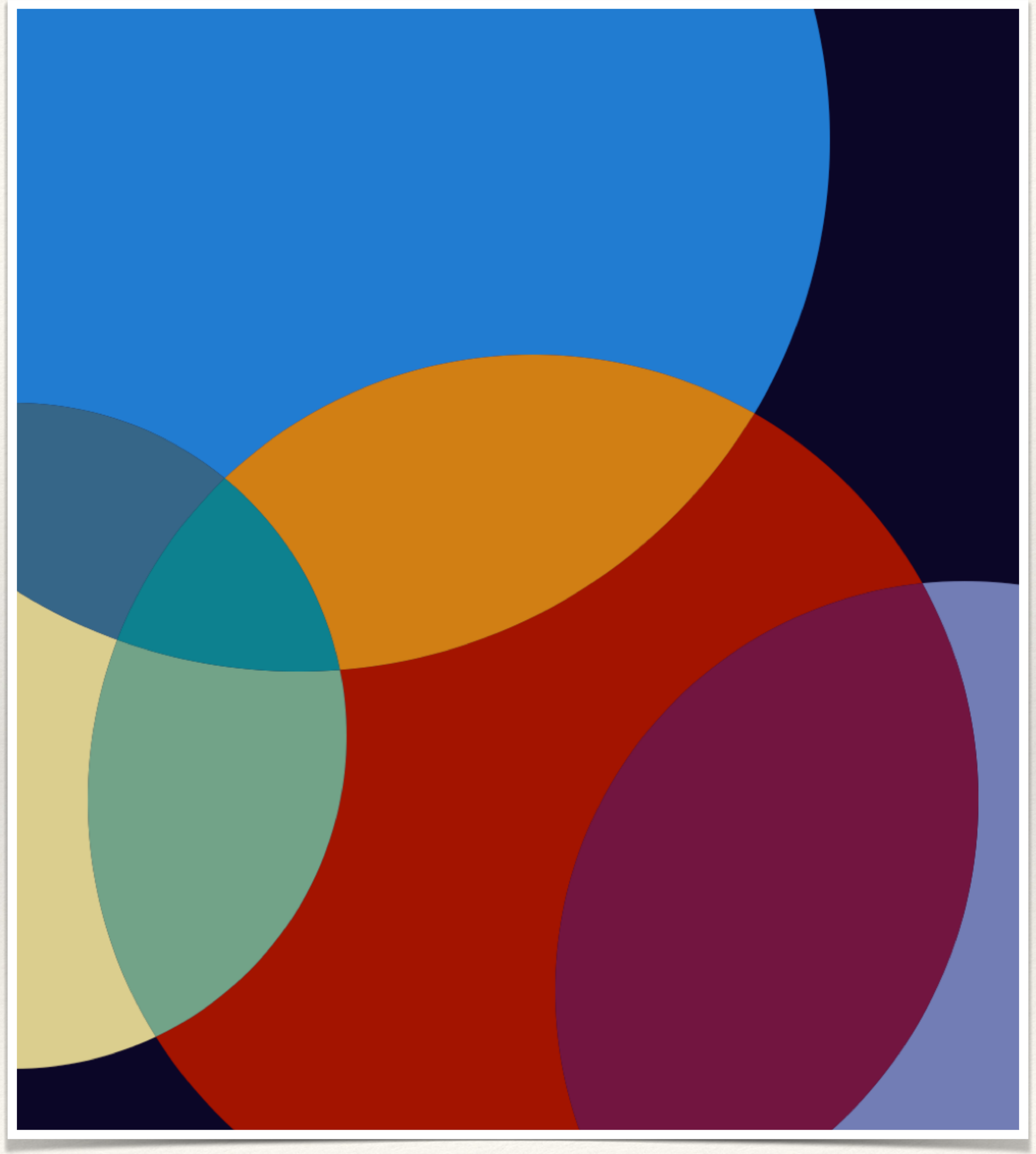
Lecture 15

Conditional Probability

Dr. David Zmiaikou



Example 1

We have seen that the probability of rolling a 7 with a pair of dice is $1/6 \approx 0.167$.

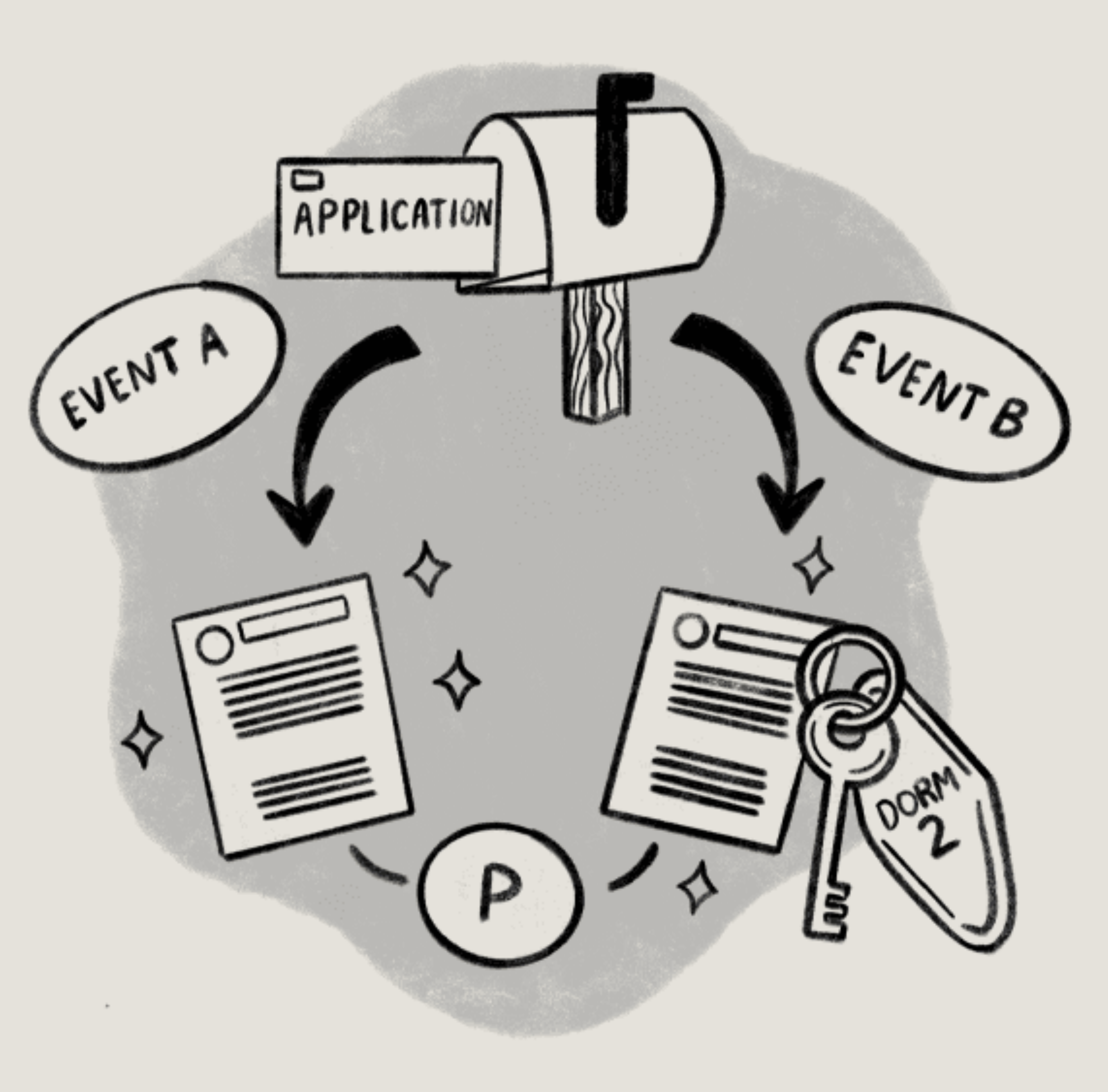
Suppose that a woman is watching a man roll a pair of dice. It is important that the man rolls a 7 in the game he is playing. Because the woman is not wearing her glasses, she is unable to see the outcome of the throw except that she can tell there is more than one spot on each die. With this information, is it more or less likely that the man rolled a 7?

Example 1

Suppose that a woman is watching a man roll a pair of dice. It is important that the man rolls a 7 in the game he is playing. Because the woman is not wearing her glasses, she is unable to see the outcome of the throw except that she can tell there is more than one spot on each die. With this information, is it more or less likely that the man rolled a 7?

Solution. Since 1 does not occur on either die, there are now only $5 \cdot 5 = 25$ possibilities for the pair of dice. That is, the **sample space** in this case now consists of 25 ordered pairs (instead of 36), namely all those ordered pairs (a, b) for which $a, b \neq 1$. The only ways to roll a 7 without a 1 on either die: (2, 5), (3, 4), (4, 3), (5, 2). Hence, the probability that the man has rolled a 7 is now **4/25 = 0.16** and it is less likely that he rolled a 7.

$\underline{\underline{\text{def}}}$ Conditional Probability of an Event?



- Let E and F be events in a sample space with $p(F) > 0$. Then the **conditional probability** of the event E given that event F has occurred is

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

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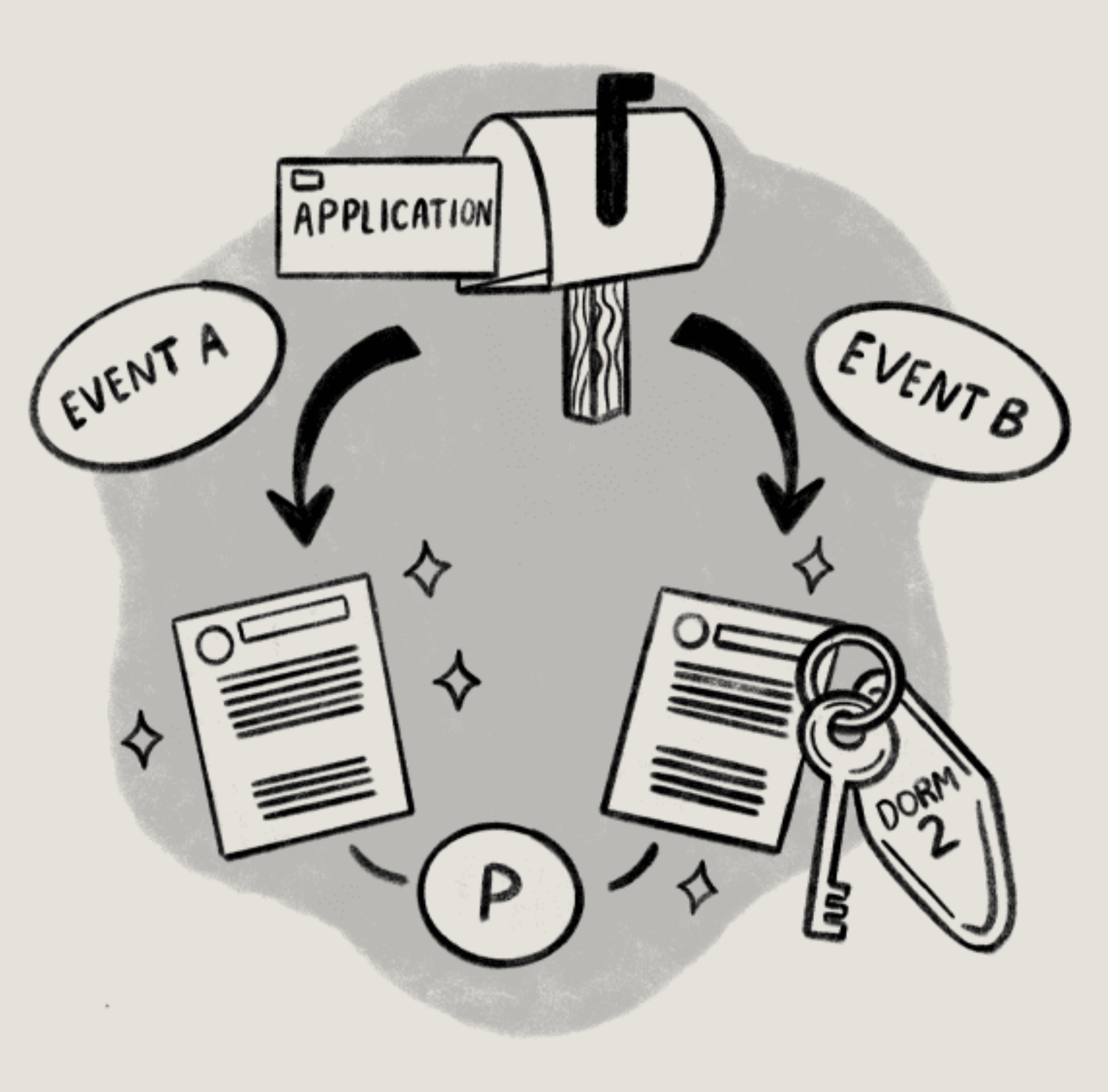

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- If p is a *uniform probability function*, then we also have

$$p(E \mid F) = \frac{|E \cap F|}{|F|}.$$

Example 2

When a coin is flipped 4 times, the **sample space** S consists of 16 **outcomes**, namely,

$$S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}.$$

The probability of getting an equal number of heads and tails when a coin is flipped 4 times is $6/16 = 3/8$. Suppose that a coin is flipped 4 times but we don't get the same result on the first and last flips. Does this information affect the probability of getting an equal number of heads and tails?

Example 2

Solution. If F is the event that a different result is obtained on the first and last flips of a coin, then

$$F = \{\text{HHHT, HHTT, HTHT, HTTT, THHH, THTH, TTHH, TTTH}\}.$$

If we let E be the event that there is an equal number of heads and tails, then

$$E = \{\text{HHTT, HTHT, HTTH, THHT, THTH, TTHH}\}.$$

Thus $E \cap F = \{\text{HHTT, HTHT, THTH, TTHH}\}$ and so

$$p(E \mid F) = \frac{|E \cap F|}{|F|} = \frac{4}{8} = \frac{1}{2}.$$

Thus under the condition that the results of the first and last flips are different on four flips of a coin, we now see that it is more likely that there will be an equal number of heads and tails.

Example 3

A box contains 2 red balls and 3 blue balls. Four balls are selected from the box, one at a time and when a ball is selected, it is returned to the box before the next ball is selected.

(a) What is the probability that two of the four balls selected are red and the other two are blue?

(b) What is the probability that two of the four balls selected are red and the other two are blue given that the first two balls selected are of different colors?

Example 3

Solution. Let E be the event that two of the four balls selected are red and the remaining two balls are blue.
(a) Since there are 5 choices for each ball selected, the number of possible outcomes is $5^4 = 625$ and the **sample space** S has 625 elements. One way of selecting 2 red balls and 2 blue balls is to select them in the order RRBB. There are two ways to select each red ball and three ways to select each blue ball, so the number of ways to select two red balls and two blue balls in this order is $2 \cdot 2 \cdot 3 \cdot 3 = 36$. However, the number of orders in which two red balls can be selected is $\binom{4}{2} = 6$. By the Multiplication Principle, the number of ways to select two red balls and two blue balls is $6 \cdot 36 = 216$ and so $|E| = 216$. Thus,

$$p(E) = |E|/|S| = 216/625 \approx 0.346.$$

Example 3

Solution. (b) Let F be the set of outcomes in which the first two balls selected have different colors. The number of ways of selecting 4 balls where the first ball is red and the second ball is blue is $2 \cdot 3 \cdot 5 \cdot 5 = 150$. There are also 150 ways of selecting 4 balls where the first ball is blue and the second ball is red. Therefore, there are 300 ways of selecting 4 balls where the first two balls selected are of different colors and so $|F| = 300$. Since there are 12 ways of selecting the first two balls if they are of different colors and 12 ways of selecting the last two balls if they are of different colors, there are 144 ways to select two red balls and two blue balls where the first two balls selected are of different colors. Hence $|E \cap F| = 144$ and so

$$p(E|F) = |E \cap F|/|F| = 144/300 \approx 0.48.$$

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Solution. Let E be the event that heads comes up exactly twice when a coin is flipped four times. Therefore,

$$E = \{\text{HHTT, HTHT, HTTH, THHT, THTH, TTHH}\}.$$

Let F be the event that heads comes up on the first flip. Then

$$F = \{\text{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT}\}.$$

Thus, $E \cap F = \{\text{HHTT, HTHT, HTTH}\}$.

Hence the probability of E given F is $p(E \mid F) = \dfrac{|E \cap F|}{|F|} = \dfrac{3}{8}$.

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$$F = \{\text{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT}\}.$$

Thus, $E \cap F = \{\text{HHTT, HTHT, HTTH}\}$.

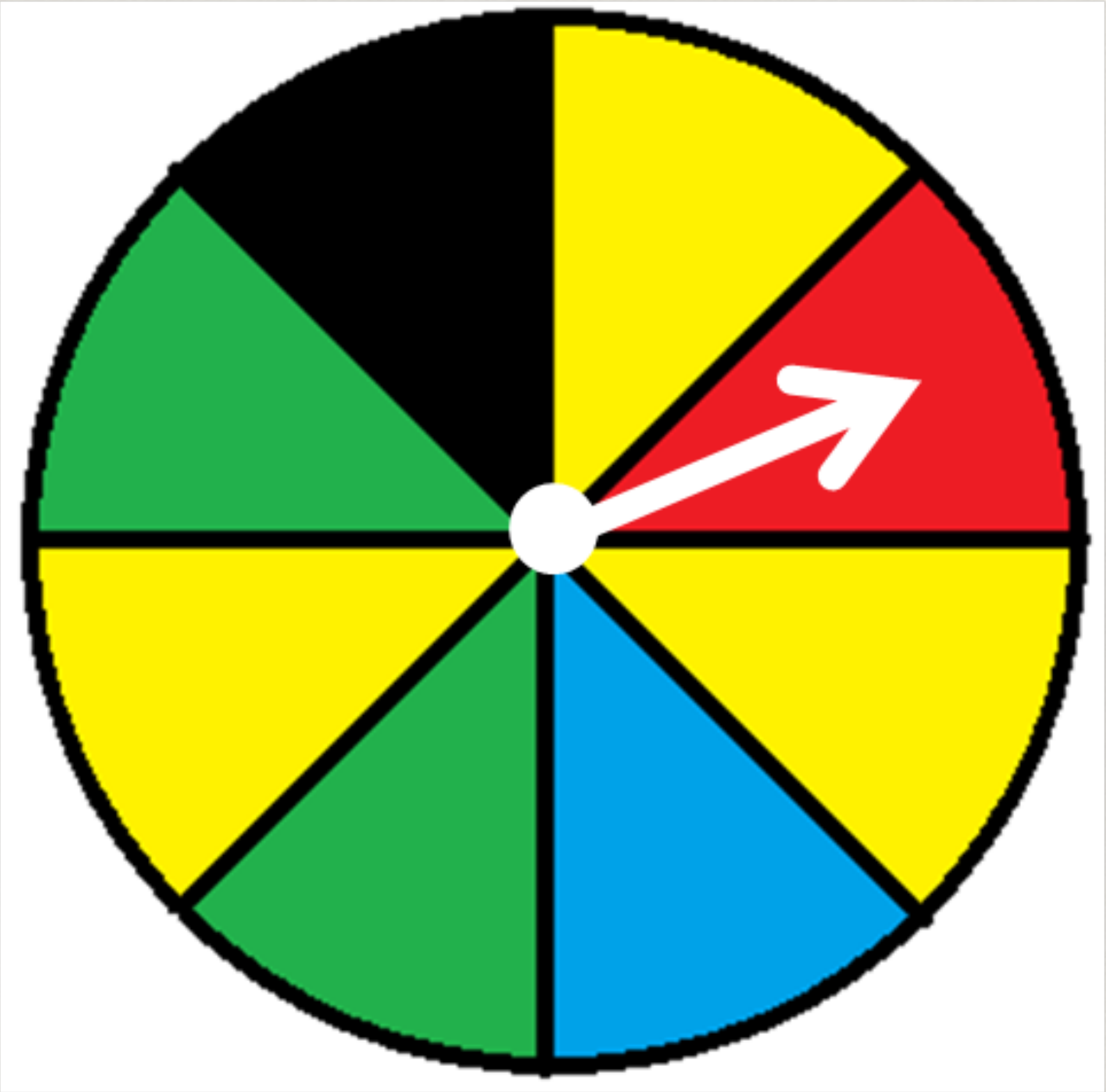
Hence the probability of E given F is $p(E \mid F) = |E \cap F| / |F| = 3/8$.

Note that in this case $p(E \mid F) = p(E)$**, that is** $\dfrac{p(E \cap F)}{p(F)} = p(E)$**,**

which implies that $p(E \cap F) = p(E) \cdot p(F)$**.**

Independent Events

$\stackrel{\text{def}}{=}$ Independent Events?



- Two events E and F in the same sample space are called **independent** if

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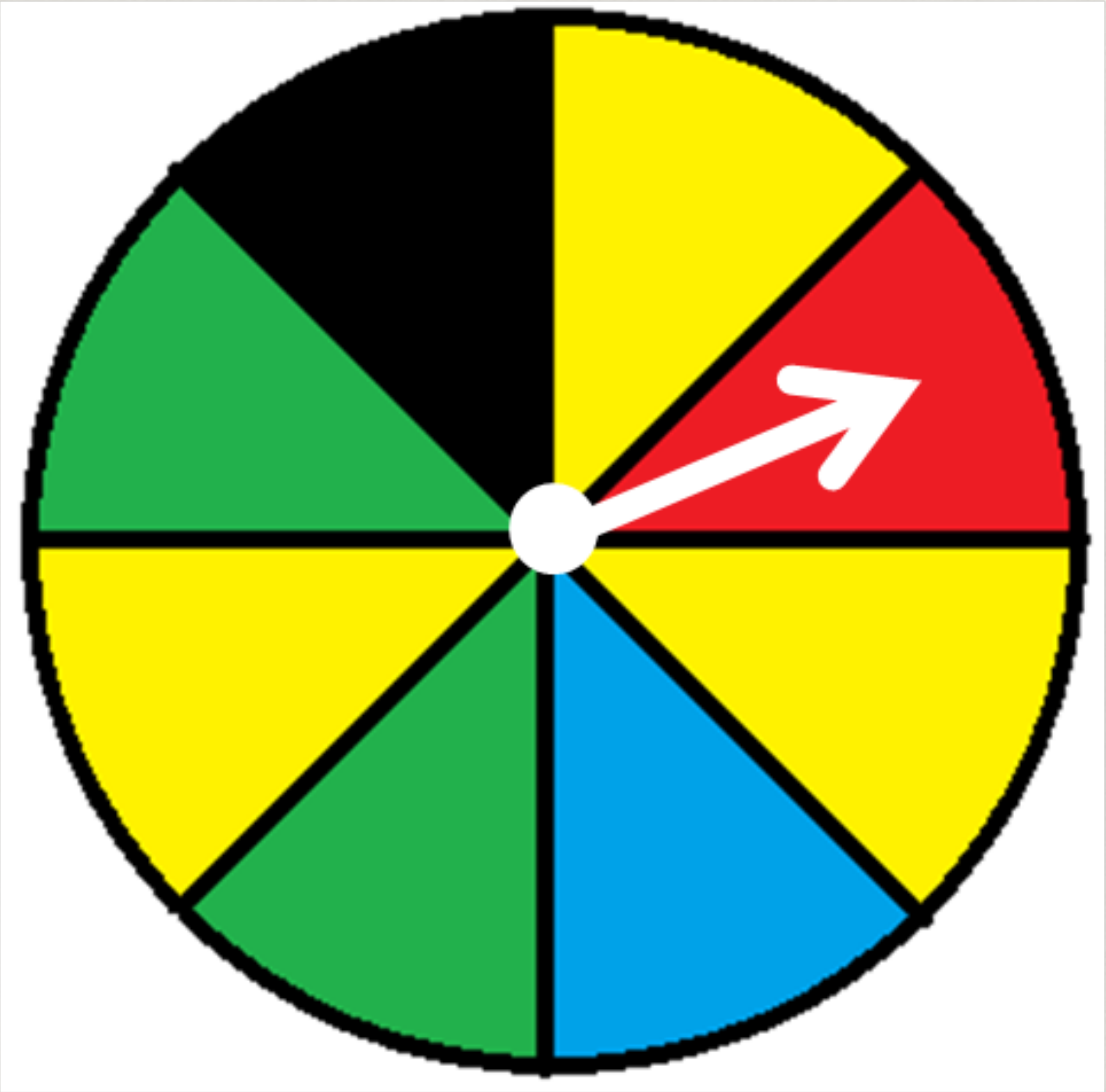

- Two events E and F in the same sample space are called **independent** if

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- It follows that if E and F are *independent*, then

$$p(E \mid F) = p(E) \quad \text{and} \quad p(F \mid E) = p(F).$$

Example 5

Two dice are rolled. Let E be the event that (the sum) 7 is rolled on the two dice and let F be the event that the first die is 1. Are E and F independent events?

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Solution. Here $E \cap F$ is the event that 1 is rolled on the first die and that 7 is the outcome of the two dice. Thus $E \cap F$ consists of the single outcome in which 1 occurs on the first die and 6 occurs on the second die. So $p(E \cap F) = 1/36$. Since $p(E) = p(F) = 1/6$, it follows that

$$p(E \cap F) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = p(E) \cdot p(F)$$

and so E and F are *independent*.

Example 6

A coin is flipped 6 times. Let E be the event that heads and tails come up an equal number of times. Let F be the event that heads and tails come up once each during the first two flips of the coin. Are E and F independent events?

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Solution. The sample space S in this case has $2^6 = 64$ elements. One of the outcomes in E is HHHTTT. Since there are 20 possible locations for the three heads, $|E| = 20$. Since there are 2 possibilities for the first two flips for each outcome in F, we have $|F| = 2 \cdot 2^4 = 32$. Since there are two possibilities for the first 2 flips in $E \cap F$ and 6 possible outcomes for the two heads in the last four flips, $|E \cap F| = 12$. So,

$$p(E \cap F) = |E \cap F|/|S| = 12/64 = 3/16 \quad \text{and}$$

$$p(E)p(F) = |E|/|S| \cdot |F|/|S| = 20/64 \cdot 32/64 = 5/32.$$

Since $p(E \cap F) \neq p(E)p(F)$, the events E and F are *not independent*.

Example 7

If a coin is flipped three times, then it is unlikely that heads comes up all three times or that tails comes up all three times. Let E be the event that when a coin is flipped three times, we don't get all heads or all tails. Let F be the event that when a coin is flipped three times, heads comes up at most once. Are E and F independent events?

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Solution. Observe that
$E = \{HHT, HTH, HTT, THH, THT, TTH\}$ and $F = \{HTT, THT, TTH, TTT\}$.
We have $|S| = 8$ and $p(E) = 3/4$ and $p(F) = 1/2$.
Now $E \cap F = \{HTT, THT, TTH\}$ and so $p(E \cap F) = 3/8$.
Since $p(E \cap F) = p(E)p(F)$, the events E and F are *independent*.

Example 8

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Solution. Since the outcome on the second flip of a coin is not influenced by the first flip of the coin, it follows that the probability of getting heads both times is the product of probabilities of getting heads on the first flip and getting heads on the second flip, which is

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Example 9

It's Friday and the TV weather forecaster states that there is a 50% chance of rain on Saturday and a 50% chance of rain on Sunday. If we were to assume that having rain on Saturday and having rain on Sunday are independent events, then what is the probability of having no rain over the weekend?

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Solution. Observing that the probability of having no rain on Saturday is $1/2$ and no rain on Sunday is $1/2$, we see that the probability of having no rain over the weekend is

$$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}.$$

Example 10

Three jars, denoted by J_1, J_2 and J_3, contain three balls each, which are identical except for their colors. The jar J_1 contains a red ball (R), a blue ball (B) and a green ball (G); the jar J_2 contains two red balls and a blue ball; and the jar J_3 contains three red balls. The jars are not labeled and the balls cannot be seen from the outside. A jar is selected at random and a ball is selected at random from that jar. If the jar J_2 was selected and the blue ball, say, was selected from J_2, then this outcome is denoted by (J_2, B). What is the probability that J_1 was the jar selected given that a red ball was selected from a jar?

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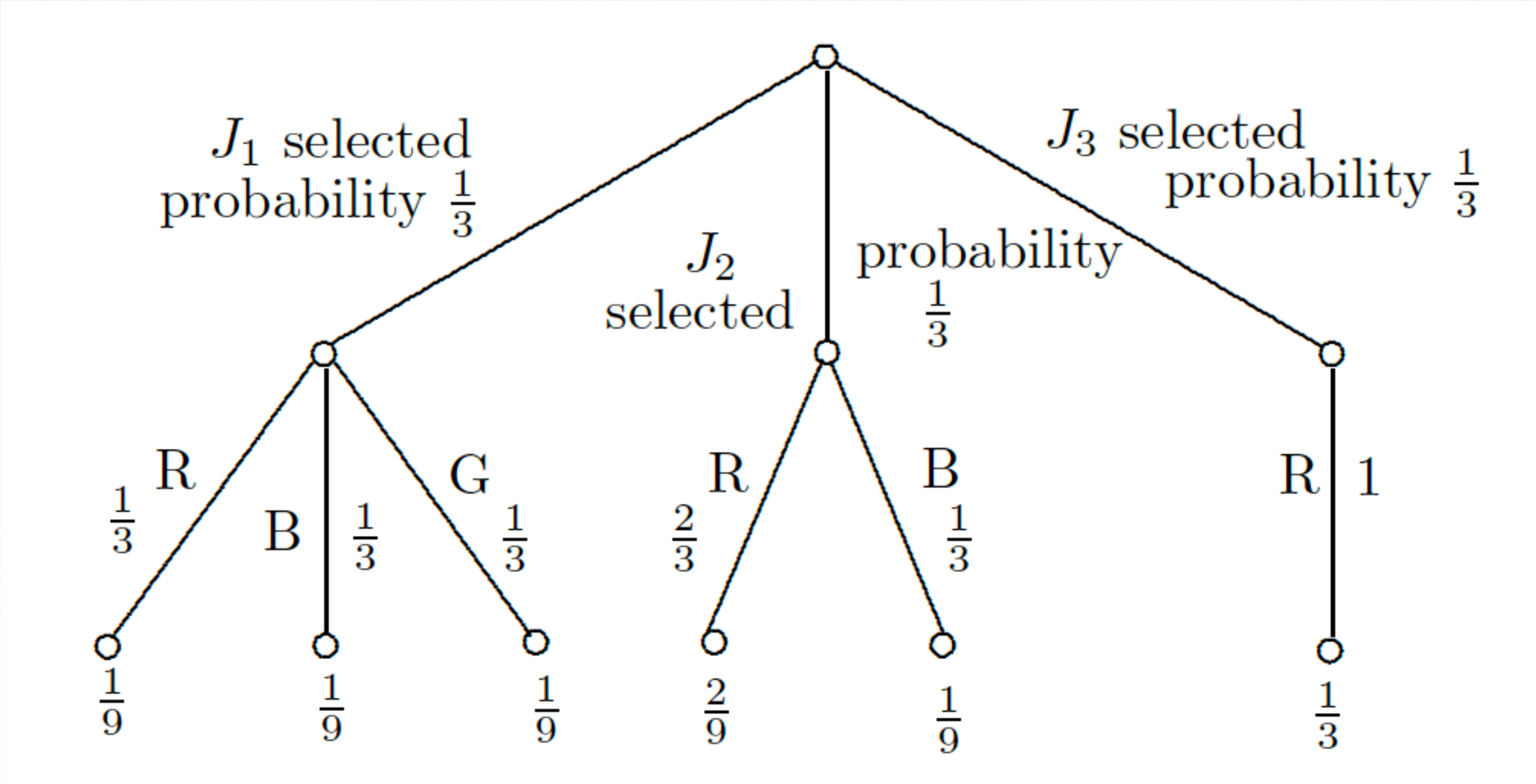
Solution. Let E be the event that jar J_1 was selected and let F be the event that a red ball was selected from a jar. Hence we seek the conditional probability $p(E \mid F)$, which equals $p(E \cap F)/p(F)$. Thus $E \cap F = \{(J_1, R)\}$ and $F = \{(J_1, R), (J_2, R), (J_3, R)\}$. To determine $p(E \cap F)$ and $p(F)$, see the tree diagram in Figure. Therefore,

$$p(E \cap F) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \text{ and } p(F) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot 1 = \frac{2}{3}.$$

Thus $p(E \mid F) = \dfrac{1/9}{2/3} = \dfrac{1}{6}$.



Example 11

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Answer. 5

Thank you!