Lecture 14

The Probability of an Event

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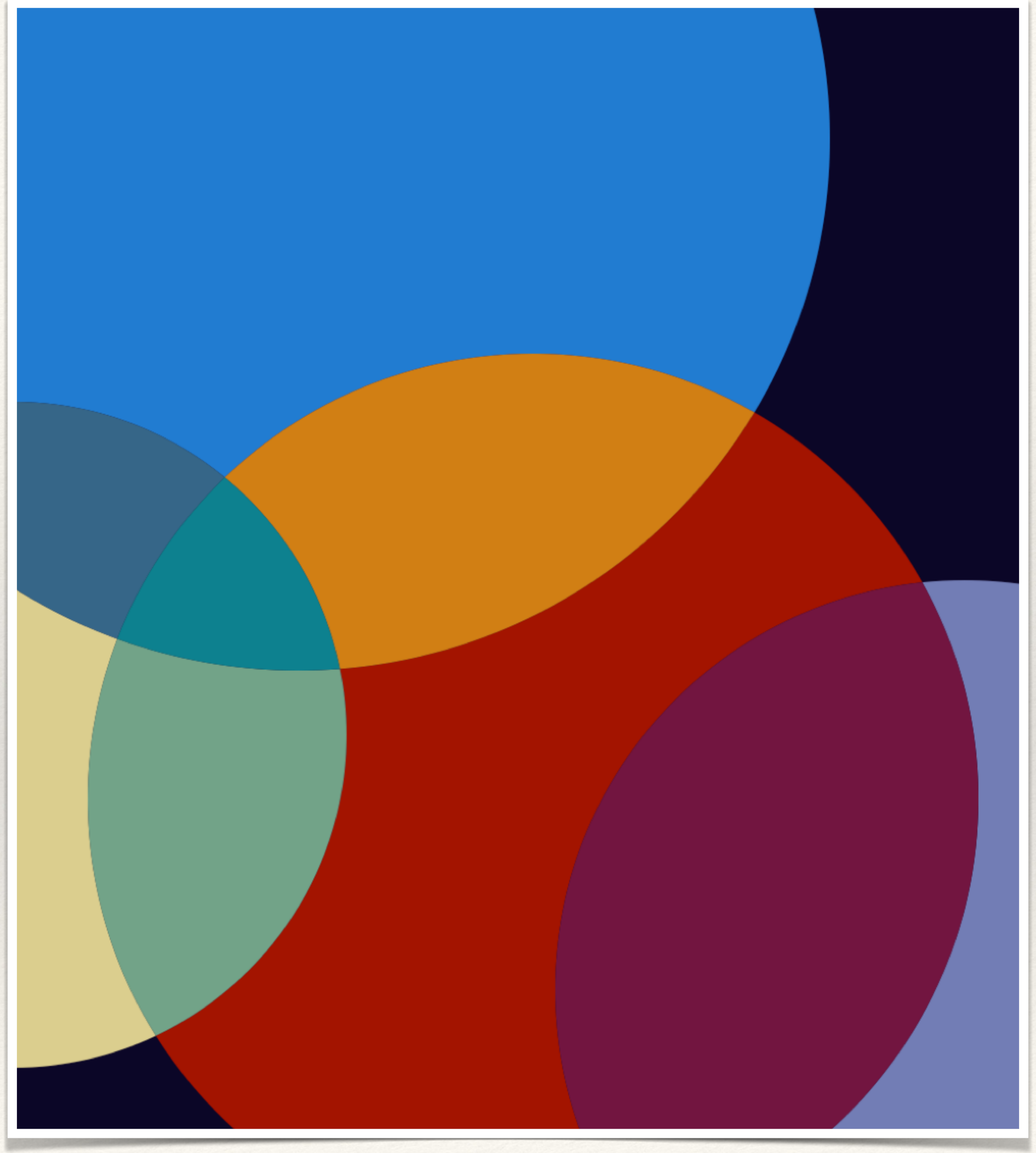

⌛ A bit of history…

- The Frenchman **Chevalier de Méré** (17th century) was known to gamble often.
- From his own experiences, he had observed that if a single die was rolled four times, then it was more likely than not that 6 would occur at least once. Consequently, he would bet with others that this would happen. Having grown tired of this game, de Méré decided to try a new game.
- He bet with others that if a pair of dice was rolled 24 times, a total of 12 (6 on each die) would occur at least once.

⌛ A bit of history…

- He soon learned, however, that he was not winning as often with the second game as he did with the first game, so de Méré then asked his friend **Blaise Pascal** (after whom the Pascal triangle is named) why this was happening.
- Pascal determined that de Méré should expect to win the first game 51.8% of the time and the second game only 49.1% of the time.

⌛ A bit of history…

- The story goes that de Méré's problem initiated correspondence between Pascal and another famous French mathematician, **Pierre de Fermat** (between 31 October and 6 December 1607 – 12 January 1665).
- The communication between Pascal and Fermat led them to define the probability of winning certain kinds of games. Specifically, if a game has n equally likely outcomes, m of which are winning outcomes, then the probability of winning the game was defined to be m/n.
- While this definition requires the outcomes of a game to be equally likely, this is not always the case. Indeed, it's not always clear when the outcomes of a game are equally likely.

⌛ A bit of history…

- Pascal and Fermat also defined probability when a game is repeated a large number of times under the same conditions.
- In 1933 the Russian mathematician **Andrey Kolmogorov** (25 April 1903 – 20 October 1987) developed the first rigorous approach to probability.

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- The set of all possible outcomes of an experiment is called the **sample space** for the experiment.
- Each subset of a sample space is called an **event**.

Example 1

Flipping a coin once is an **experiment**. If we're interested in whether heads or tails occurs, then the two **outcomes** of this experiment are heads (H) and tails (T).

The **sample space** here is therefore $S = \{\mathrm{H}, \mathrm{T}\}$ so that the possible **events** in this case are $\varnothing$, $\{\mathrm{H}\}$, $\{\mathrm{T}\}$ and $\{\mathrm{H}, \mathrm{T}\}$.

Example 2

A coin is flipped n times and we are interested in the possible sequences of heads and tails that can occur. What is the number of outcomes in the sample space S of this experiment? For $n = 3$, what is S?

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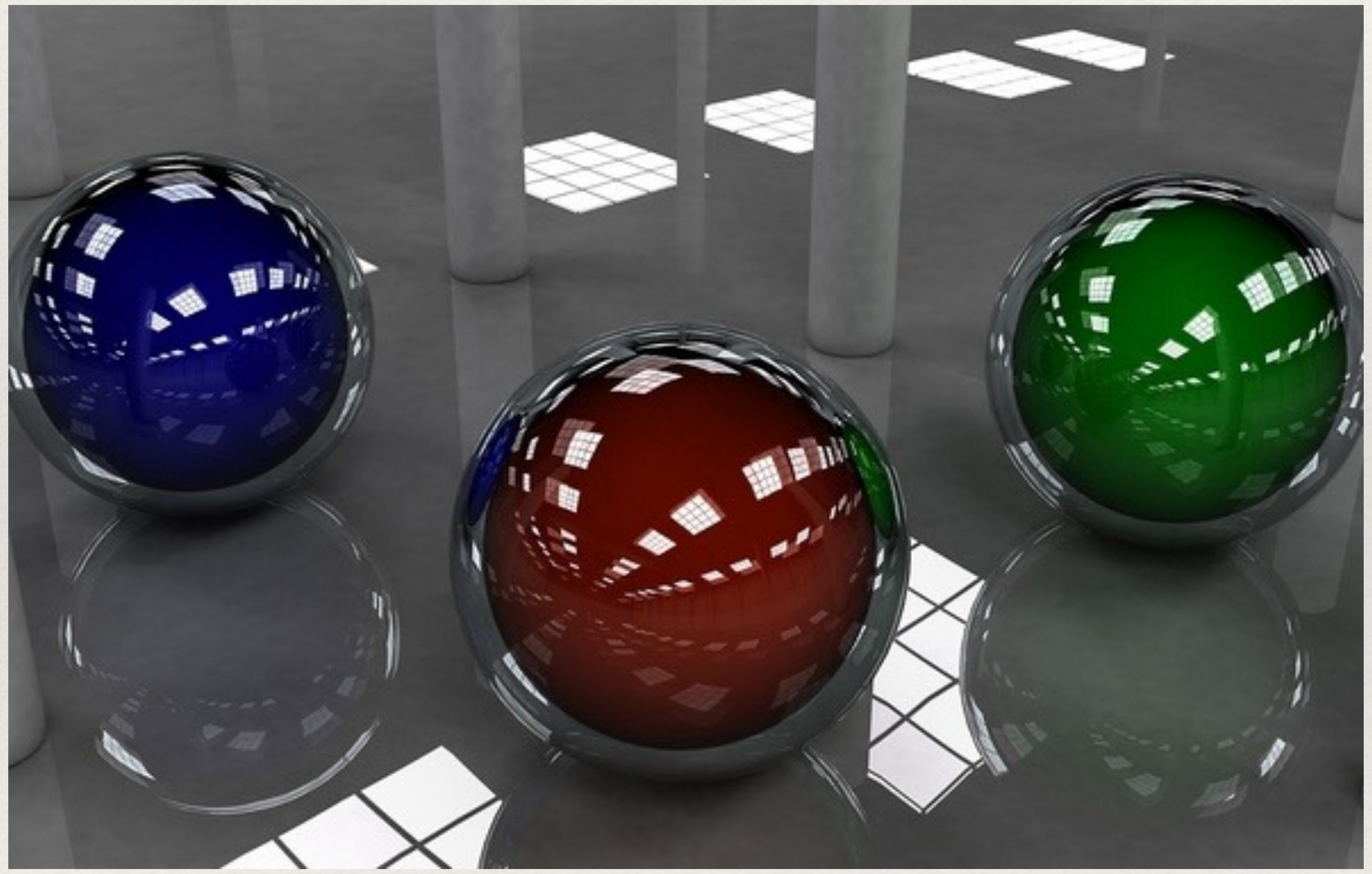
A coin is flipped n times and we are interested in the possible sequences of heads and tails that can occur. What is the number of outcomes in the sample space S of this experiment? For $n = 3$, what is S?

Solution. Since there are two possible results with each flip of a coin, it follows by the Multiplication Principle that $|S| = 2^n$.
If $n = 3$, then the **sample space** is

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

so $|S| = 2^3 = 8$. There are $2^8 = 256$ possible events in this case, including $\varnothing$, {HHH}, {HTH, TTH} and S.

Example 3



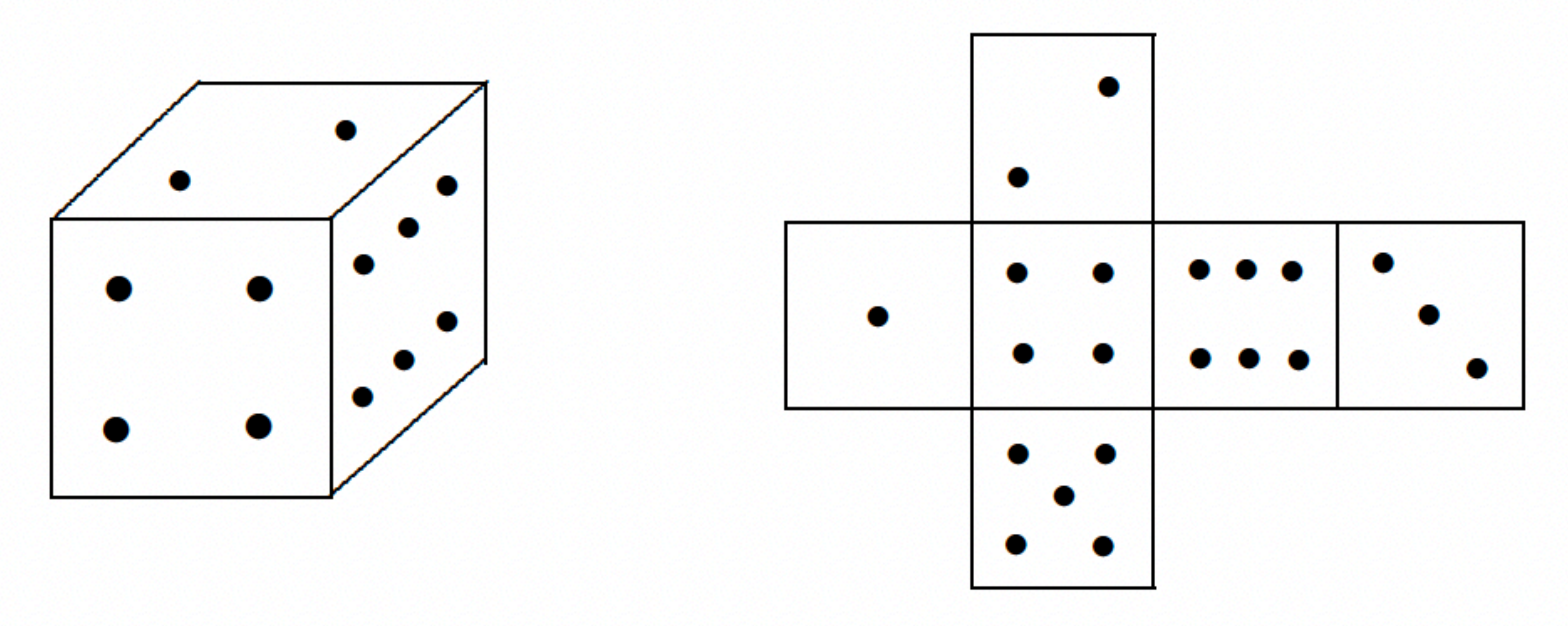
Selecting one ball at random from a bowl containing three red balls, two blue balls and one green ball is an experiment that yields one of three possible outcomes, namely a red ball (R), a blue ball (B) or a green ball (G).

In this case, the **sample space** is $S = \{R, B, G\}$.

Example 4

You probably know that a die is a cube where each of its six faces has a number of spots, ranging from 1 to 6. When a die is tossed, rolled or thrown (on a flat surface), we typically consider the number of spots on the upper face as the outcome of this experiment.

Therefore, the **experiment** of rolling a single die has six possible **outcomes**: 1, 2, 3, 4, 5, 6.
The **sample space** here is $S = \{1, 2, 3, 4, 5, 6\}$.



Example 5

Many games concerning dice (the plural of die), however, involve a pair of dice rather than a single die. When we toss a pair of dice, obtaining as **outcome** (a, b), where a is the number obtained for the first die and b is the number resulting from the second die, then we are typically interested in the number $a + b$, the sum of the numbers of the two dice.

Since there are 6 possible numbers for the first die and 6 for the second die, the number of possible ordered pairs (a, b) from tossing two dice is $6 \cdot 6 = 36$ by the Multiplication Principle. However, there are only 11 distinct sums, namely, $2, 3, \ldots, 12$.

Example 5

It may seem that the sample space in this case should consist of the 11 outcomes 2, 3, . . . , 12 rather than the 36 outcomes mentioned above, but **the choice of the sample space depends on which questions interest us**.

Suppose that we are only interested in the sum of the numbers of two dice when the dice are rolled. Then the appropriate **sample space** may very well be

$$S = \{2, 3, \ldots, 12\}.$$

Example 5

On the other hand, if we are interested in whether the number obtained on at least one of the two dice is 3, then this **event** E can be described more clearly by writing

$$E = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),$$
$$(1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}.$$

An appropriate **sample space** in this case would most likely be

$$S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 5), (6, 6)\},$$

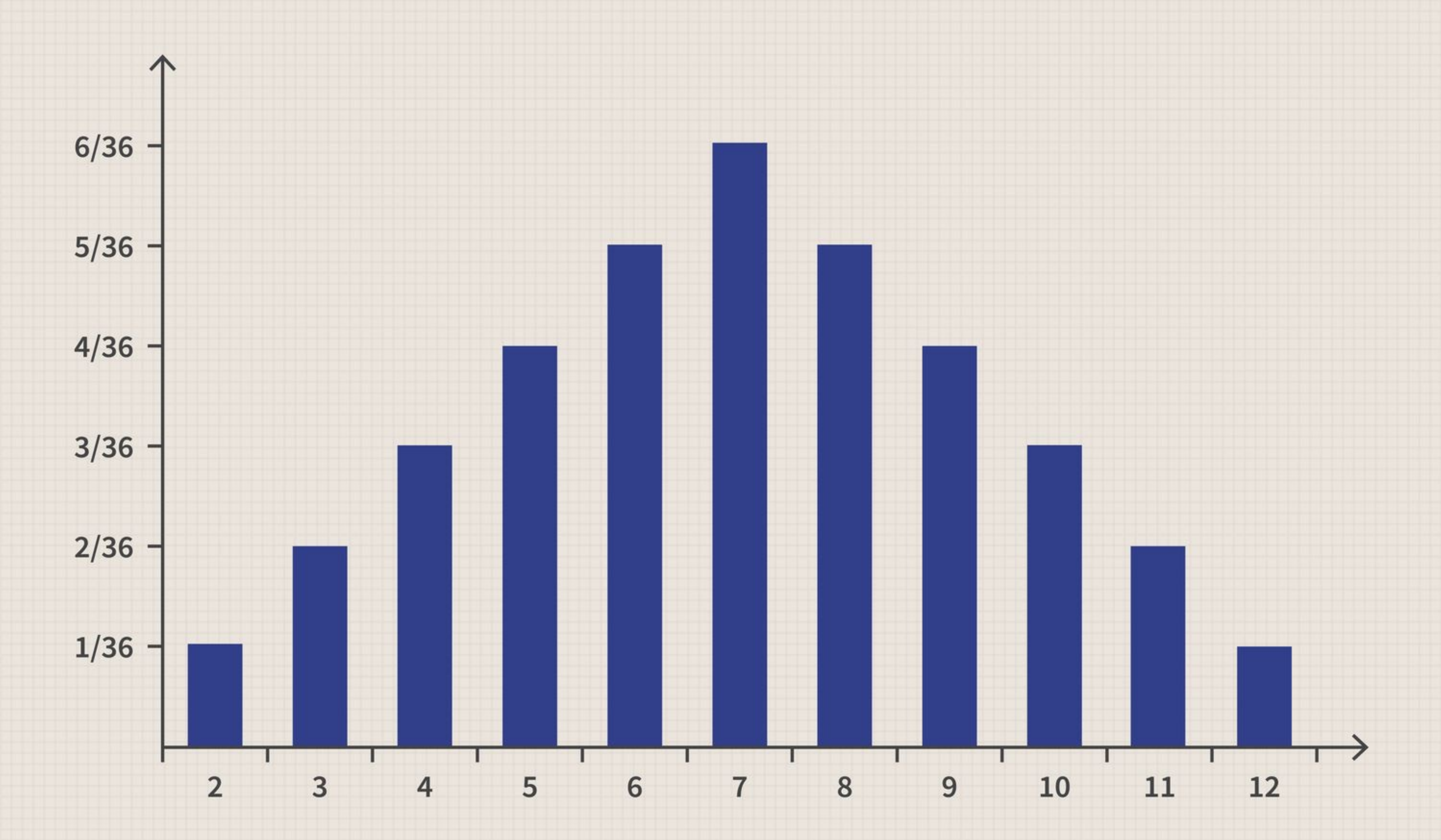
which consists of all 36 possible ordered pairs (a, b) from tossing two dice.

Example 6

Suppose that a casino sells a die, where instead of having a face with a single spot, the logo of the casino is on that face and the remaining five faces have spots as with a standard die. If such a die is rolled and **what interests us is whether the logo occurs**, then one **outcome** is the logo and the other **outcome** is no logo, that is, the **sample space** here is

$$S = \{\text{logo, no logo}\}.$$

$\stackrel{\text{def}}{=}$ Probability Function?

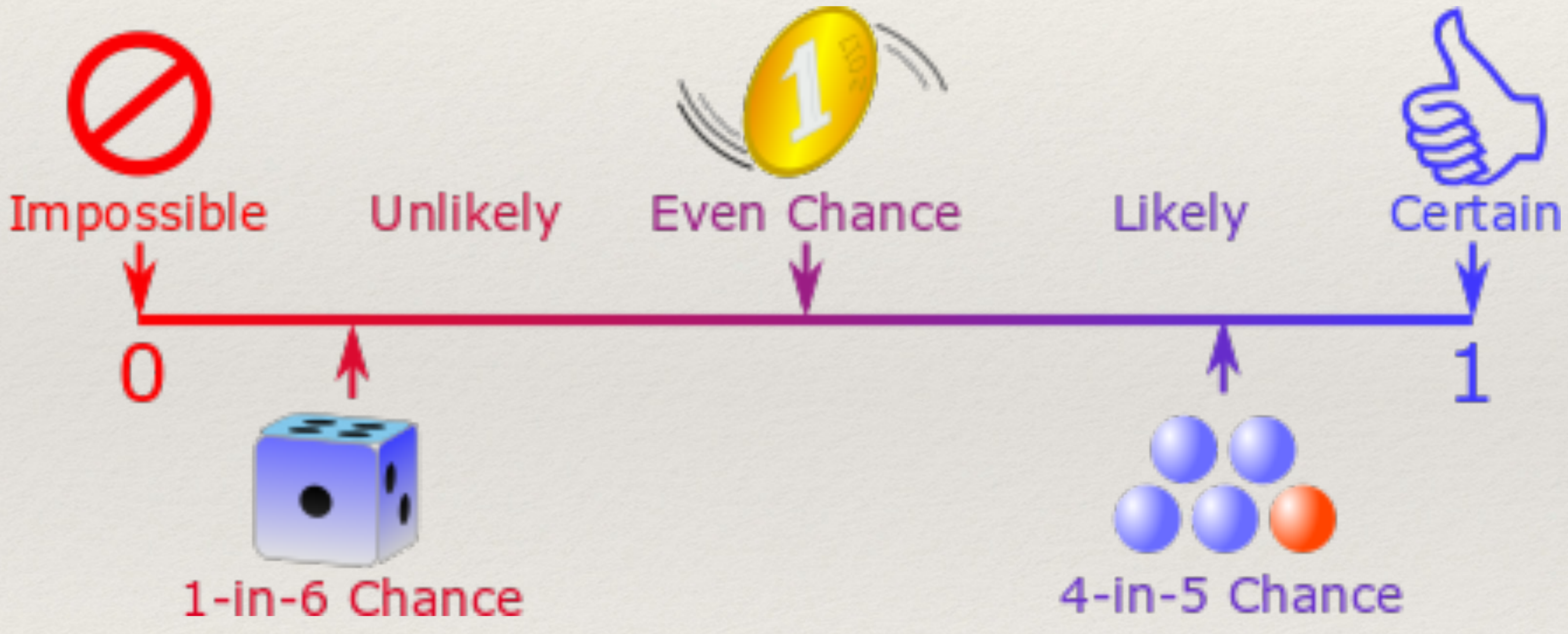


- For a set S, a function $p : S \to [0,1]$ is a **probability function** if

$$\sum_{s \in S} p(s) = 1.$$

- For an element $s \in S$, the number $p(s)$ is the **probability** of s.

$\stackrel{\text{def}}{=}$ Probability of an Event?

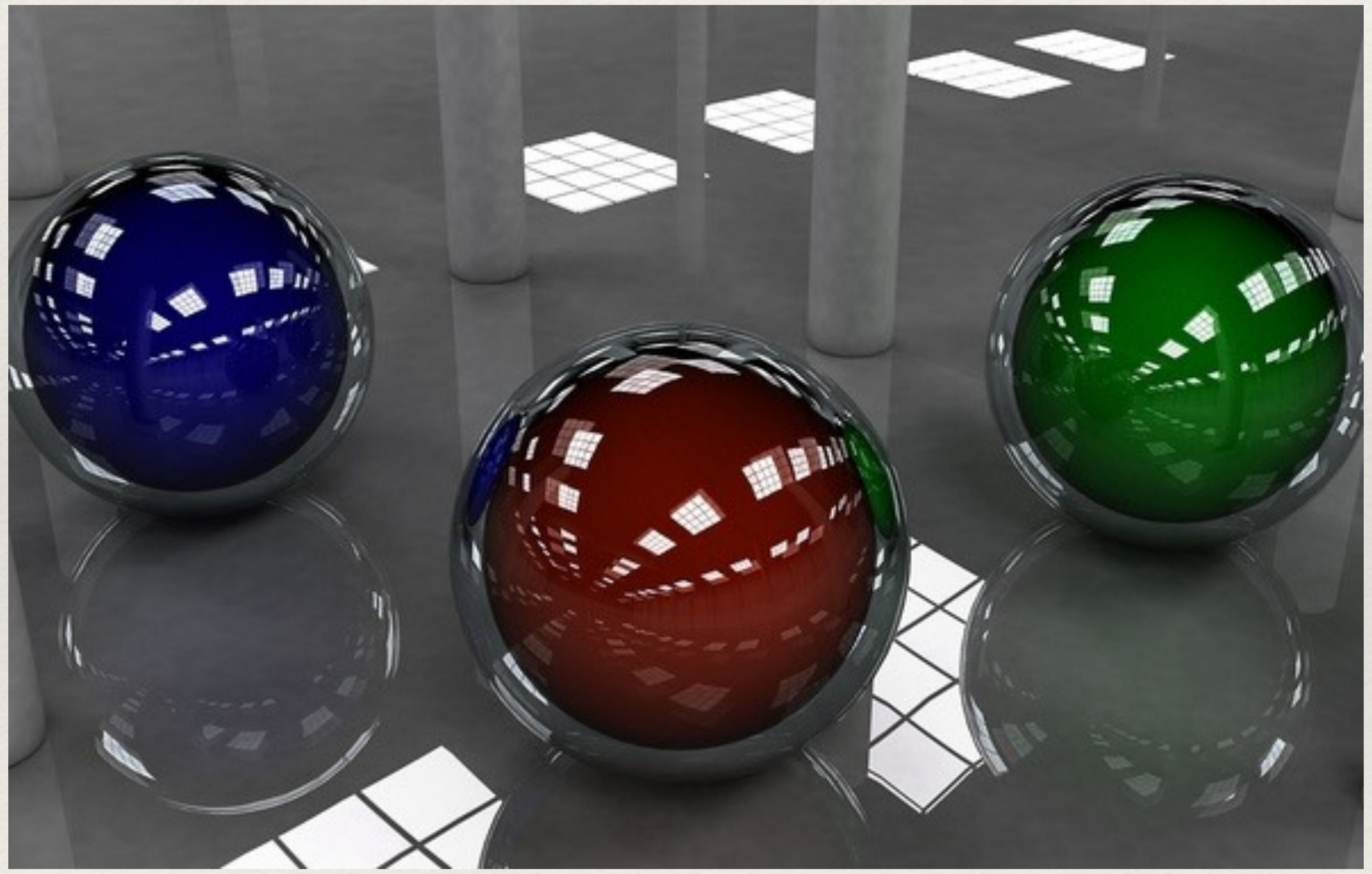


- The **probability** $p(E)$ of an event E in a sample space S is the sum of the probabilities of the outcomes in that event, that is,

$$p(E) = \sum_{s \in E} p(s).$$

- For an element $s \in S$, the number $p(s)$ is the **probability** of s.
- We have $0 \leq p(E) \leq 1$ for every event E. In particular, $p(\varnothing) = 0$ and $p(S) = 1$.

Example 7



Selecting one ball at random from a bowl containing 3 red balls, 2 blue balls and 1 green ball is an experiment that yields one of three possible outcomes, namely a red ball (R), a blue ball (B) or a green ball (G). The **sample space** is $S = \{\text{R, B, G}\}$.

In this case, it is reasonable to assume that selecting any one of the six balls is **equally likely**. It makes sense, therefore, to define a **probability function** p on S by

$$p(R) = \frac{3}{6} = \frac{1}{2}, \quad p(B) = \frac{2}{6} = \frac{1}{3} \quad \text{and} \quad p(G) = \frac{1}{6}.$$

Example 8

Suppose that a casino sells a die, where instead of having a face with a single spot, the logo of the casino is on that face and the remaining five faces have spots as with a standard die. The **sample space** is $S = \{\text{logo, no logo}\}$.

Since one of the six faces has the logo and five of the six faces have no logo, it is logical to define a **probability function** p on S by

$$p(\text{logo}) = \frac{1}{6} \quad \text{and} \quad p(\text{no logo}) = \frac{5}{6}.$$

$\stackrel{\text{def}}{=}$ Uniform Probability Function?

$$P(A) = \frac{\text{Number of favorable outcomes to A}}{\text{Total number of outcomes}}$$

- In general then, if a sample space S is finite, $|S| = n \geq 1$ and each outcome in S is **equally likely** to occur, then it is logical to define

$$p(s) = \frac{1}{|S|} = \frac{1}{n}$$

for each outcome s. Then p is called a **uniform probability function**.

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- If a (finite) sample space S is assigned a uniform probability function and E is an event in S, then in this case,
$$p(E) = \frac{|E|}{|S|}.$$
This is the definition of probability given by Pascal and Fermat.

Example 9

A coin is flipped 2 times. If heads and tails are equally likely to occur, then what is the probability that heads and tails occur once each (in either order)?

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Solution. The **sample space** is

$$S = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}.$$

The event that heads and tails occur once each is $E = \{\mathrm{HT}, \mathrm{TH}\}$.
Since the probability is **uniform** in this case, the probability of E is

$$p(E) = \frac{|E|}{|S|} = \frac{2}{4} = \frac{1}{2}.$$

Example 10

A die is tossed once and each of the six possible outcomes is equally likely to occur. What is the probability that the number obtained is odd?

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A die is tossed once and each of the six possible outcomes is equally likely to occur. What is the probability that the number obtained is odd?

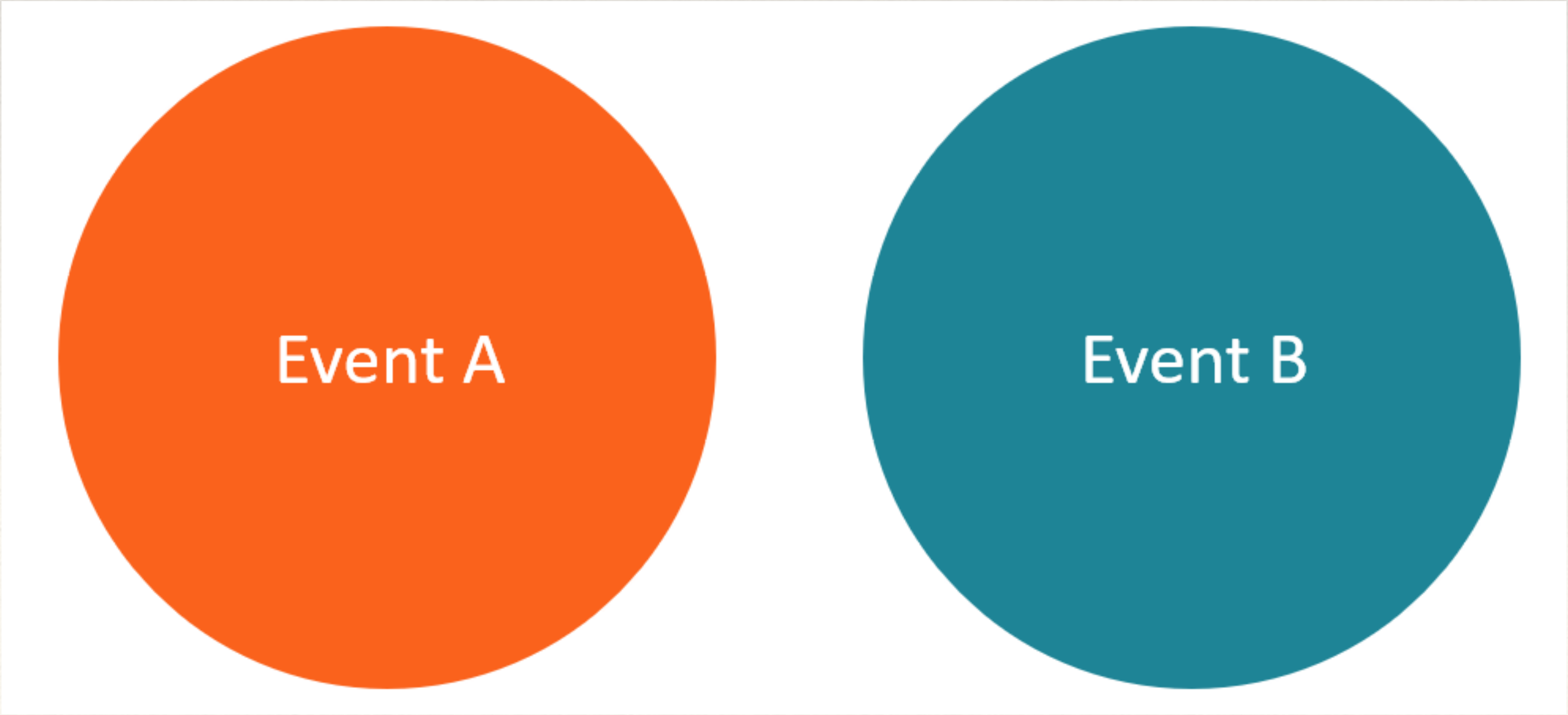
Solution. The **sample space** is $S = \{1, 2, 3, 4, 5, 6\}$ and the **event** of obtaining an odd number on this toss of the die is $E = \{1, 3, 5\}$. Since this probability function is also **uniform**, the probability of E is

$$p(E) = \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}.$$

≝ Mutually Exclusive Event?

- Two events E and F in a sample space S are **mutually exclusive** if the sets E and F are *disjoint*, that is, if E and F cannot occur simultaneously. Then we have

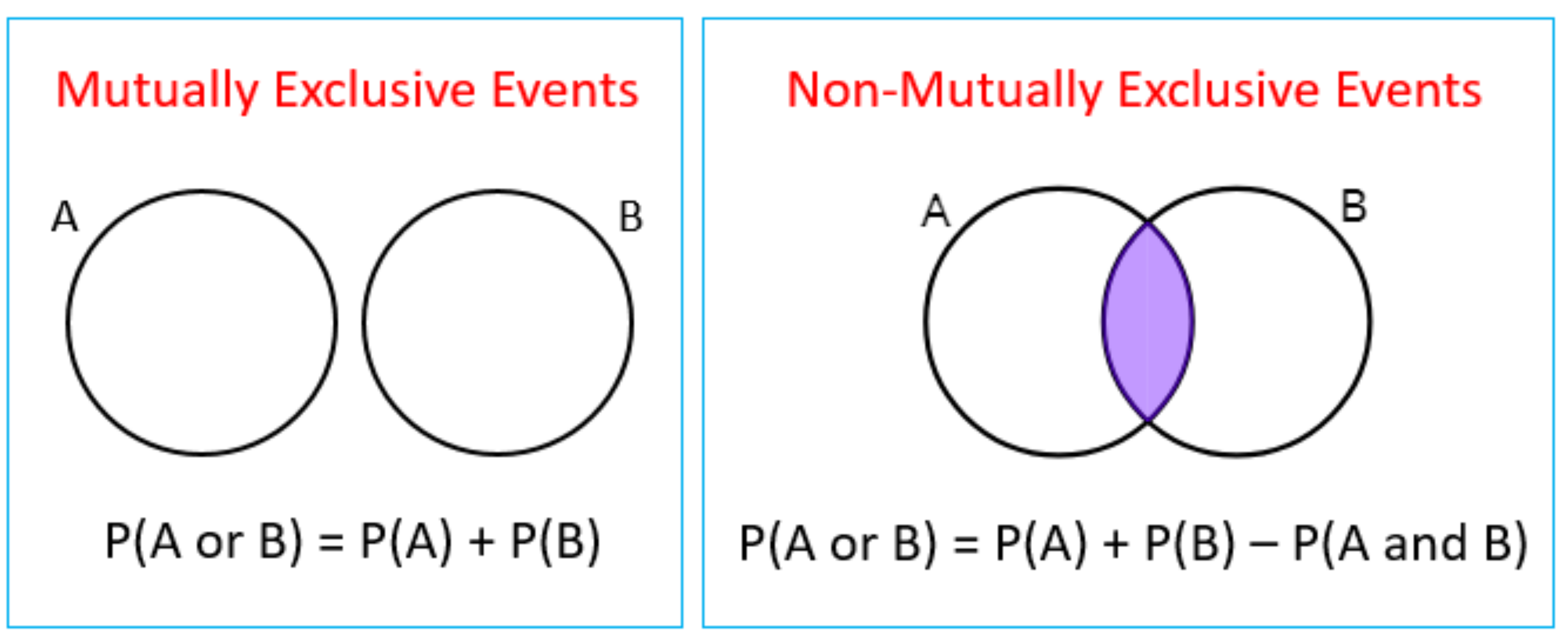
$$p(E \cup F) = \frac{|E \cup F|}{|S|} = \frac{|E| + |F|}{|S|} = \frac{|E|}{|S|} + \frac{|F|}{|S|} = p(E) + p(F).$$



In General

- If E and F are events that are not necessarily mutually exclusive, then it follows from the Principle of Inclusion-Exclusion

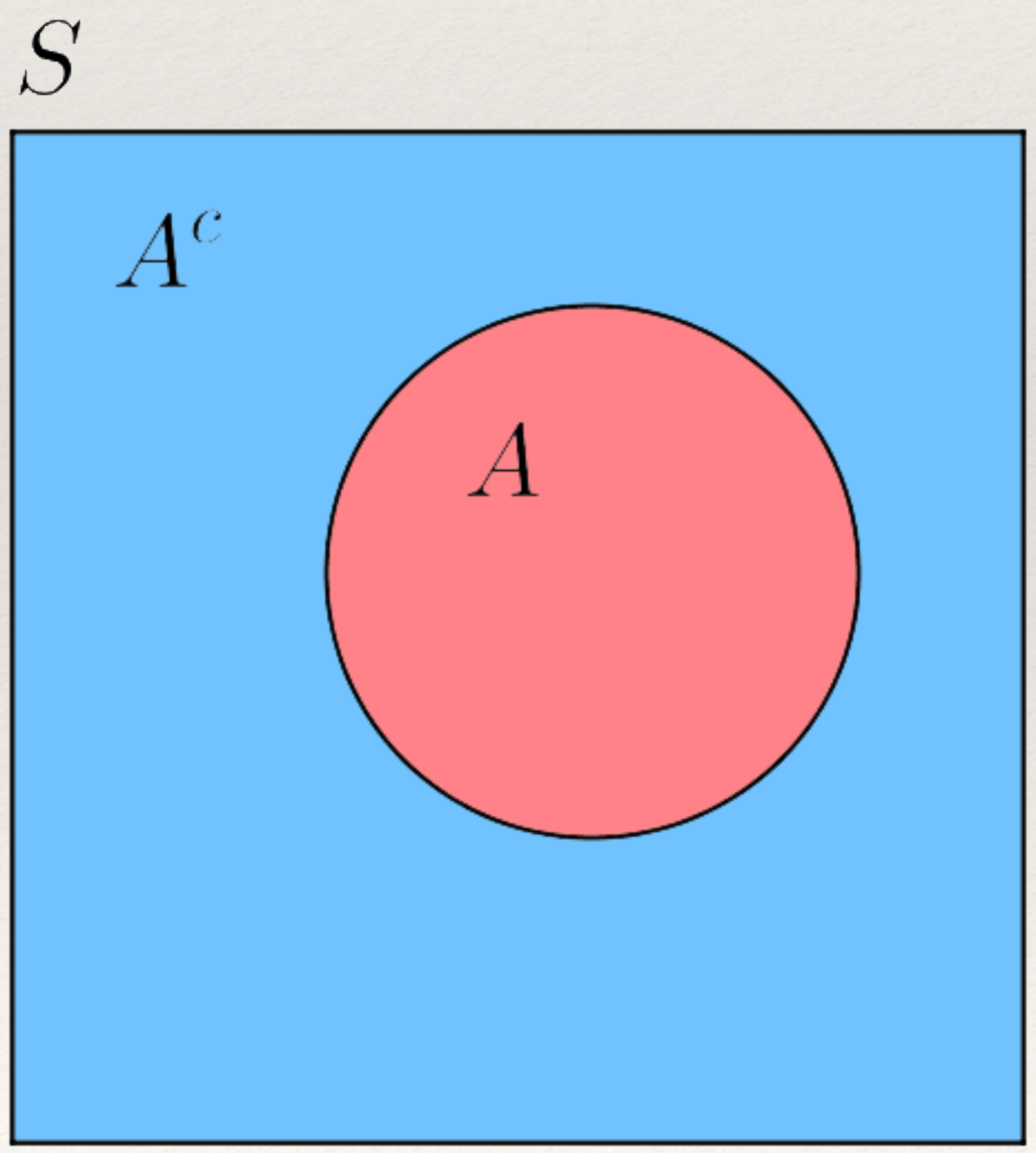
$$p(E \cup F) = p(E) + p(F) - p(E \cap F).$$



$\stackrel{\text{def}}{=}$ Complementary Event?

- For an event E in a sample space S, the **complementary event** $\overline{E}$ (or E^c) of E consists of those outcomes in S not belonging to E, that is, $\overline{E} = S - E$. Therefore,

$$p(\overline{E}) = \frac{|S - E|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - p(E).$$



Sample Space S, event A, and complement A^c

Example 11

A pair of dice is tossed.

(a) What is the probability of obtaining a sum of 7?

(b) What is the probability of obtaining a sum of 7 or a sum of 10?

(c) What is the probability of obtaining 3 on at least one of the two dice?

(d) What is the probability of obtaining a sum that is different from 7?

Example 11

Value of first die \ Value of second die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A pair of dice is tossed.
(a) What is the probability of obtaining a sum of 7?

Solution. (a) The sample space $S = \{2, 3, \ldots, 12\}$ seems like a good choice here and we wish to determine $p(7)$. This will not be a uniform probability function in this case as surely $p(7) \neq p(2)$, for example. Looking at the table, we see that 7 occurs in 6 of the 36 entries in the table, each of which is equally likely to occur. Thus

$$p(7) = 6/36 = 1/6.$$

Example 11

		Value of second die					
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A pair of dice is tossed.
(b) What is the probability of obtaining a sum of 7 or a sum of 10?

Solution. (b) We see that 10 occurs in 3 of the 36 entries in the table. Since obtaining a sum of 7 and obtaining a sum of 10 are **mutually exclusive events**, it follows that

$$p(\{7, 10\}) = p(7) + p(10) = \frac{1}{6} + \frac{3}{36} = \frac{1}{4}.$$

Example 11

(c) What is the probability of obtaining 3 on at least one of the two dice?

Solution. (c) As we saw earlier, the **sample space**

$$S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 5), (6, 6)\}$$

seems more appropriate here, where each of the 36 **outcomes** (ordered pairs) is **equally likely** to occur. The event that interests us is

$E = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$.

Since $|E| = 11$ and the probability function here is **uniform**,

$$p(E) = \frac{|E|}{|S|} = \frac{11}{36}.$$

Another way is by looking at E a bit differently. Let E_1 be the event that 3 is the number obtained on the first die and E_2 the event that 3 is the number obtained on the second die. Then $E = E_1 \cup E_2$ and

$$p(E) = p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}.$$

Value of first die \ Value of second die	1	2	3	4	5	6
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Example 11

Value of second die

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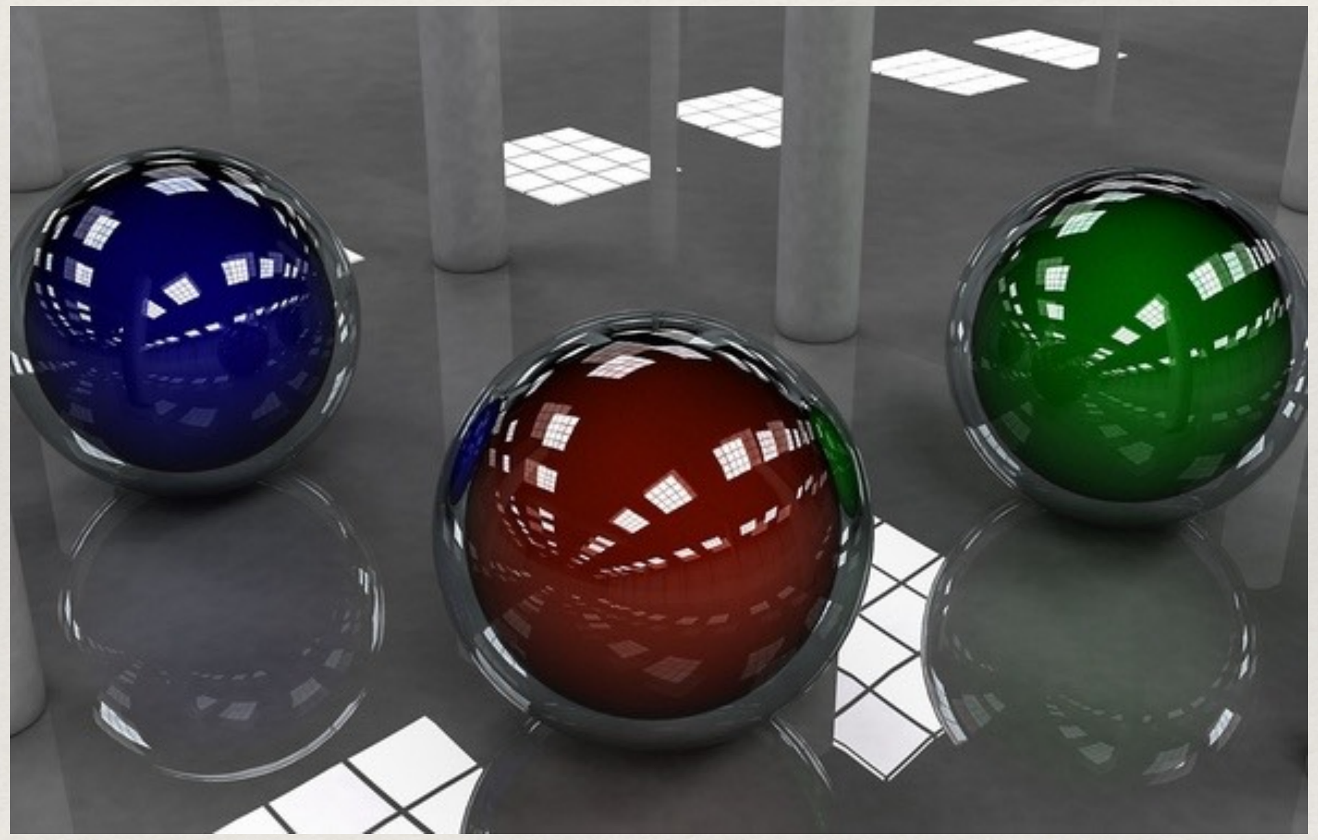
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A pair of dice is tossed.
(d) What is the probability of obtaining a sum that is different from 7?

Solution. (d) Since the event $\overline{E}$ of not obtaining a sum of 7 is the complementary event of the event E of obtaining a sum of 7, it follows that

$$p(\overline{E}) = 1 - p(E) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Example 12

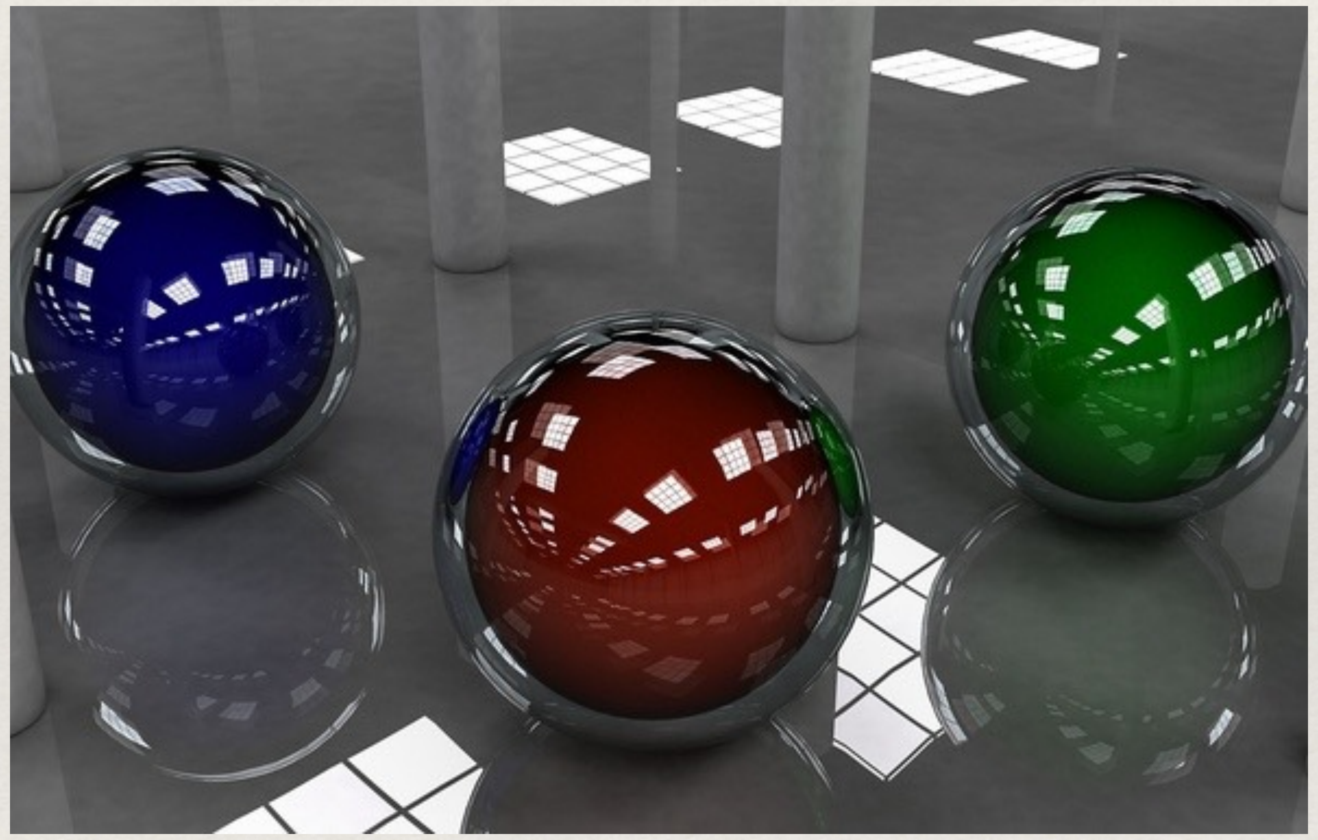


A large box contains 15 balls that are identical except for their colors: 6 are colored red, 5 are colored blue and 4 are colored green.

(a) What is the probability that two balls selected at random from the box have the same color?

(b) What is the probability that two balls selected at random from the box are colored the same if the first ball is returned to the box before the second ball is selected?

Example 12



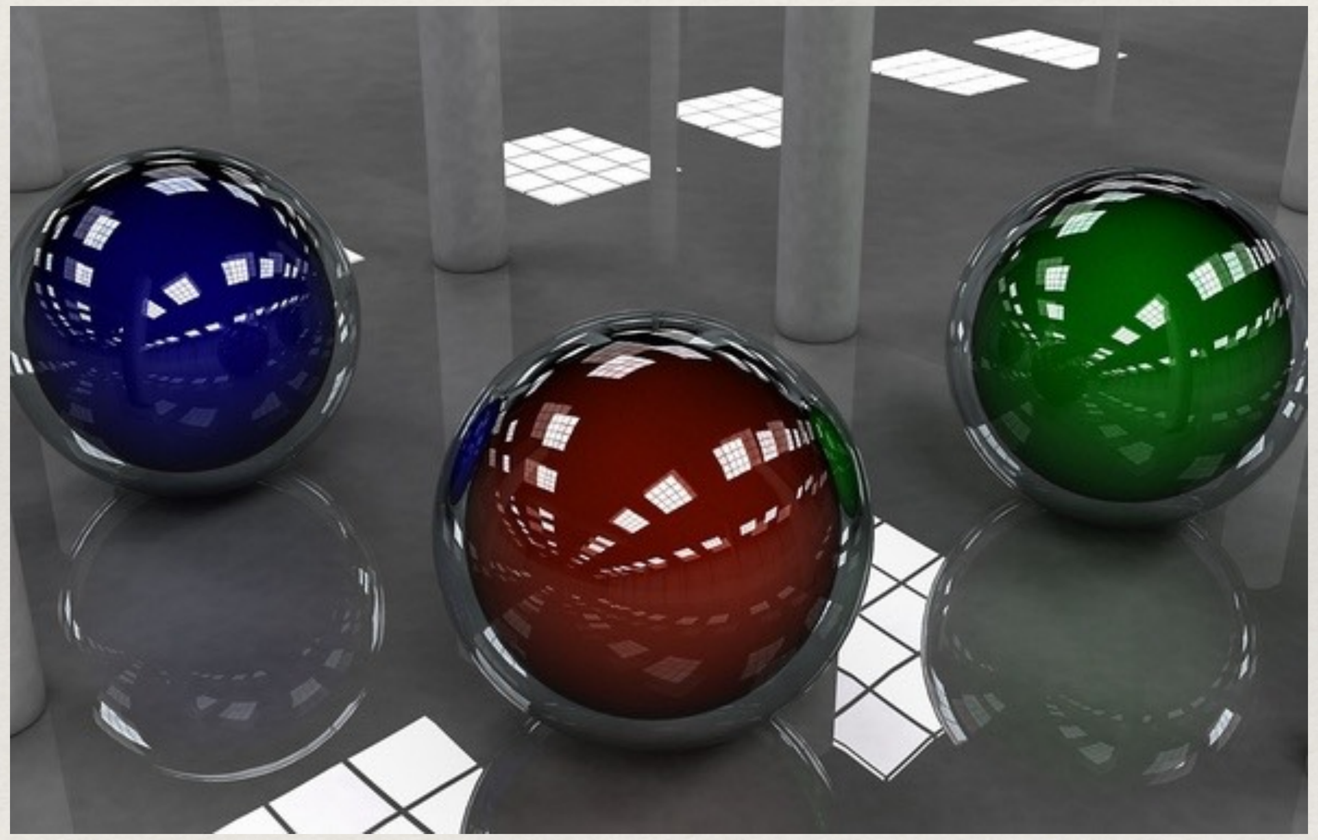
A large box contains 15 balls that are identical except for their colors: 6 are colored red, 5 are colored blue and 4 are colored green. (a) What is the probability that two balls selected at random from the box have the same color?

Solution. (a) There are $\binom{15}{2} = \frac{15 \cdot 14}{2} = 105$ ways to select two balls from the box and so the **sample space** S in this case has 105 elements. Let E be the event that two balls selected at random from the box have the same color. There are $\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ ways to select two red balls, 10 ways to select two blue balls and 6 ways to select two green balls. Then the number of ways to select two balls of the same color is therefore $|E| = 15 + 10 + 6 = 31$. Hence

$$p(E) = |E|/|S| = 31/105 \approx 0.295.$$

Example 12

A large box contains 15 balls that are identical except for their colors: 6 are colored red, 5 are colored blue and 4 are colored green. (b) What is the probability that two balls selected at random from the box are colored the same if the first ball is returned to the box before the second ball is selected?



Solution. (b) Since there are $15 \cdot 15 = 225$ ways to select two balls from the box in this manner, the **sample space** has 225 elements. There are $6 \cdot 6 = 36$ ways to select two red balls, $5 \cdot 5 = 25$ ways to select two blue balls and $4 \cdot 4 = 16$ ways to select two green balls. Hence the number of ways to select two balls of the same color is $36 + 25 + 16 = 77$. So the probability of selecting two balls of the same color in this case is $77/225 \approx 0.342$. Therefore, it is a bit more likely to select two balls of the same color if the first ball is returned to the box before the second ball is selected.

Example 13

A standard deck of playing cards consists of 52 cards. These 52 cards are divided into 4 types of 13 cards each. The types are referred to as suits, called hearts (♥), diamonds (♦), clubs (♣) and spades (♠). In each suit, there is one card of each of the following kinds: 2, 3, 4, 5, 6, 7, 8, 9, 10, jack (J), queen (Q), king (K), ace (A).

A hand that consists of five cards of the same suit is called a **flush**. A hand has **two pairs** if it has a pair of one kind, a second pair of another kind and one card of a third kind.

(a) What is the probability of being dealt a flush?

(b) What is the probability of being dealt a hand of two pairs?

Example 13

(a) What is the probability of being dealt a flush?

(b) What is the probability of being dealt a hand of two pairs?

Solution. (a) The number of ways of being dealt a flush where the cards are of a particular suit is $\binom{13}{5} = 1287$. Since there are four different suits, the number of ways of being dealt a flush is $4 \cdot 1287 = 5148$. Therefore, the probability of being dealt a flush is $5148/2{,}598{,}960 \approx \mathbf{0.00198}$.

Example 13

(a) What is the probability of being dealt a flush?

(b) What is the probability of being dealt a hand of two pairs?

Solution. (b) Since there are 13 different kinds, there are $\binom{13}{2} = 78$ different choices for the kinds of the two pairs. Since there are 4 cards of each kind, there are 6 possibilities for the two cards of the same kind in each pair. There are $52 - 2 \cdot 4 = 44$ possibilities for the card of the third kind. Hence the number of hands consisting of two pairs is $78 \cdot 6 \cdot 6 \cdot 44 = 123{,}552$ by the Multiplication Principle and the probability of being dealt such a hand is $123{,}552/2{,}598{,}960 \approx \mathbf{0.048}$.

Several Experiments

- **Theorem 1**. If an event E of a sample space $S = S_1 \times S_2 \times \cdots \times S_n$ consists of an event E_1 in S_1 followed by an event E_2 in S_2 and so on, then

$$p(E) = p(E_1) \cdot p(E_2) \cdot \cdots \cdot p(E_n).$$

Example 14

A pair of dice is tossed until a sum of 7 is obtained for the two dice.

(a) What is the probability that 7 is obtained on the first toss of the dice?

(b) What is the probability that 7 is obtained for the first time on the second toss of the dice?

(c) What is the probability that 7 is obtained for the first time on the third toss of the dice?

(d) What is the probability that 7 is obtained for the first time on some toss of the dice?

(e) Use (d) to determine the probability of never getting a 7 on successive tosses of the dice.

Thank you!