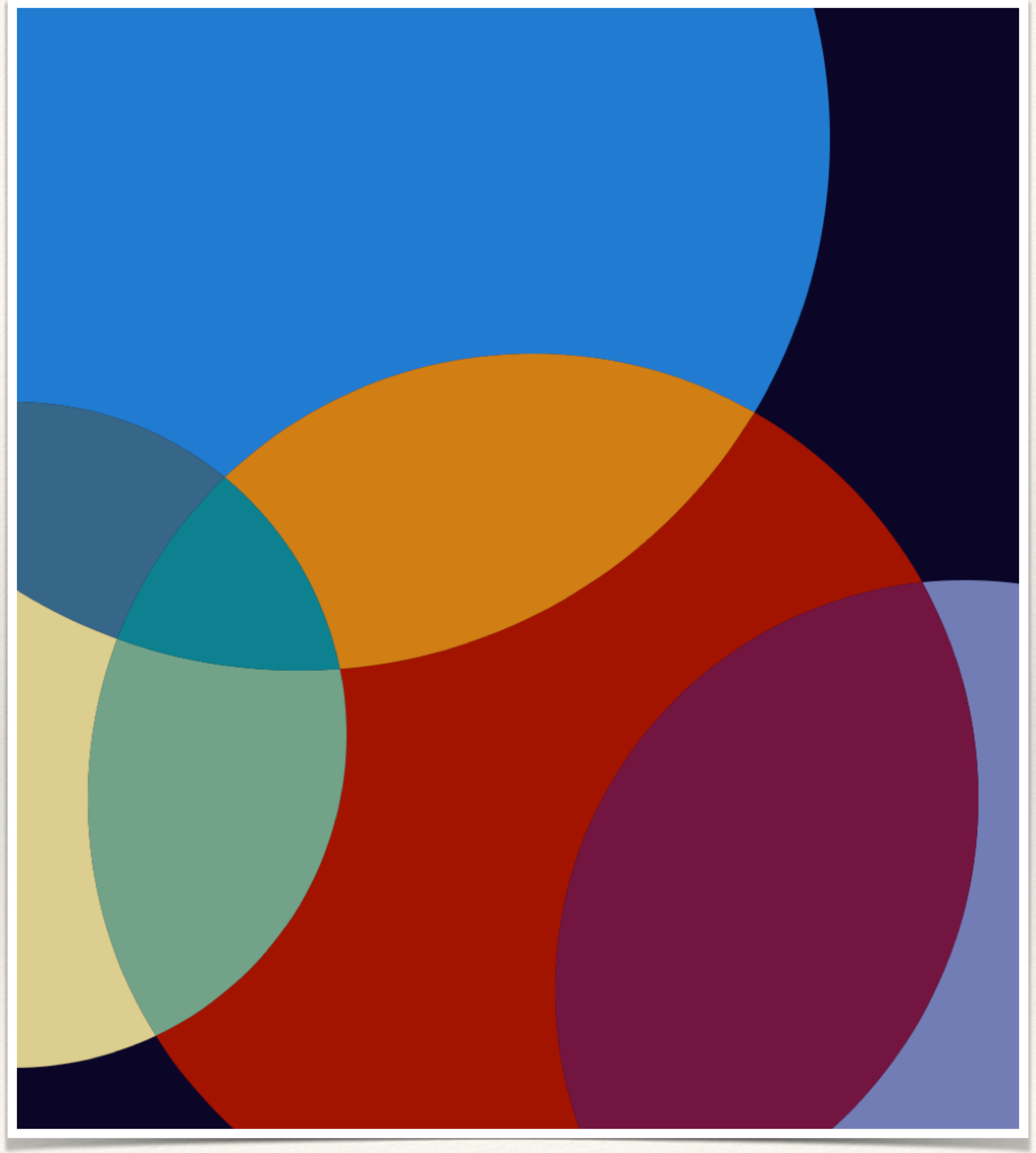
Lecture 9

Combinatorics

Dr. David Zmiaikou



Binomial Coefficients and Pascal's Triangle

Recall Combinations?
$\stackrel{\text{def}}{=}$ What about Binomial Coefficients?

- Let S be an n-element set. A subset of S of cardinality r, where $0 \leq r \leq n$, is called an r-**combination** of S.
- The number of r-combinations is denoted by

$$\boldsymbol{C(n, r)} \quad \text{or} \quad \binom{\boldsymbol{n}}{\boldsymbol{r}}.$$

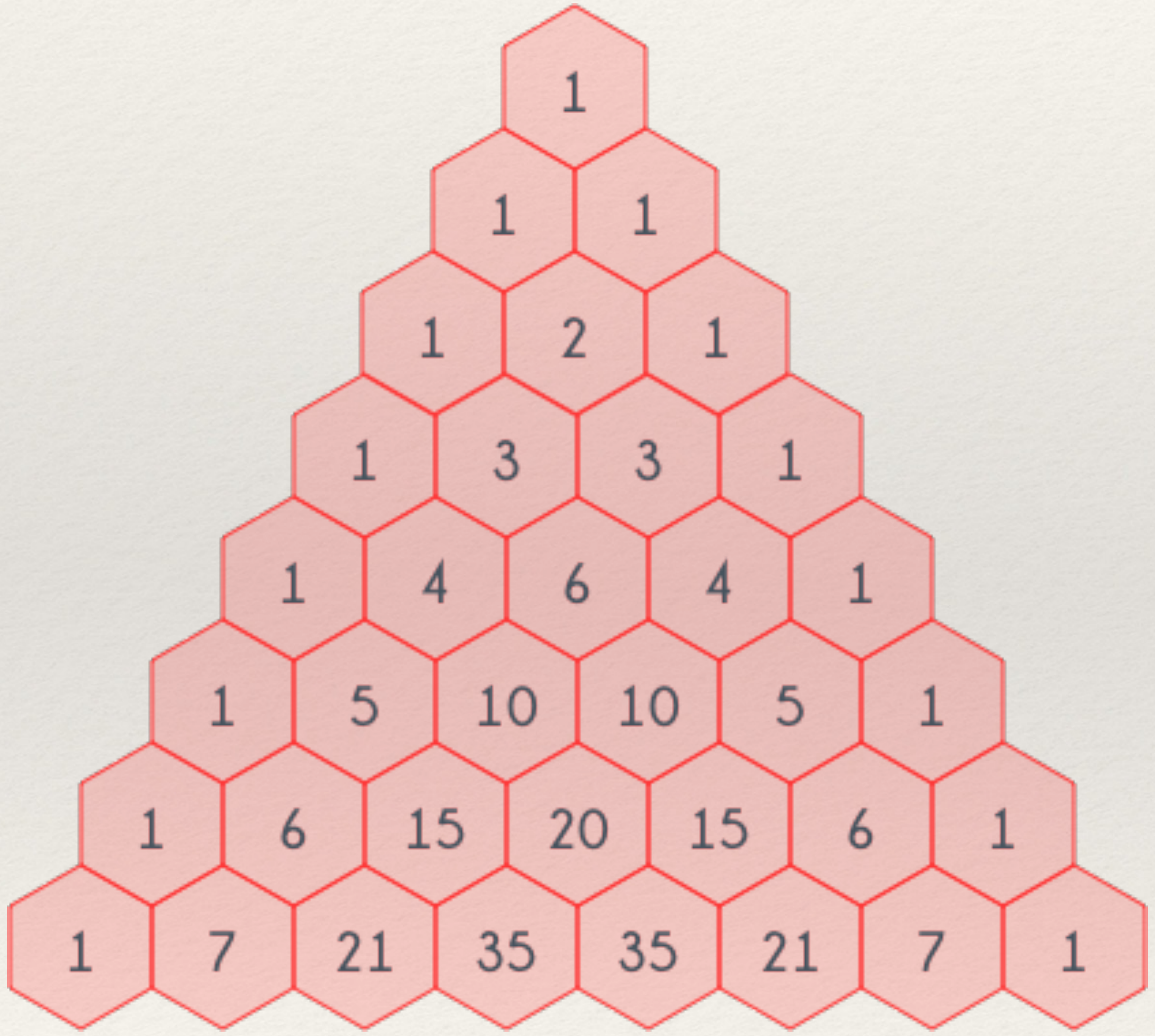
  This number is equal to $\dfrac{n!}{r!(n-r)!}$.
- This number is called a "**binomial coefficient**".

$\stackrel{\text{def}}{=}$ What is Pascal's Triangle?

- **Pascal's triangle** (or Pascal triangle) is a triangular array of the binomial coefficients.

row 0								$\binom{0}{0}$								
row 1							$\binom{1}{0}$		$\binom{1}{1}$							
row 2						$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$						
row 3					$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$					
row 4				$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$				
row 5			$\binom{5}{0}$		$\binom{5}{1}$		$\binom{5}{2}$		$\binom{5}{3}$		$\binom{5}{4}$		$\binom{5}{5}$			
row 6		$\binom{6}{0}$		$\binom{6}{1}$		$\binom{6}{2}$		$\binom{6}{3}$		$\binom{6}{4}$		$\binom{6}{5}$		$\binom{6}{6}$		
row 7	$\binom{7}{0}$		$\binom{7}{1}$		$\binom{7}{2}$		$\binom{7}{3}$		$\binom{7}{4}$		$\binom{7}{5}$		$\binom{7}{6}$		$\binom{7}{7}$	
		$\dots$		$\dots$		$\dots$		$\dots$		$\dots$		$\dots$		$\dots$		

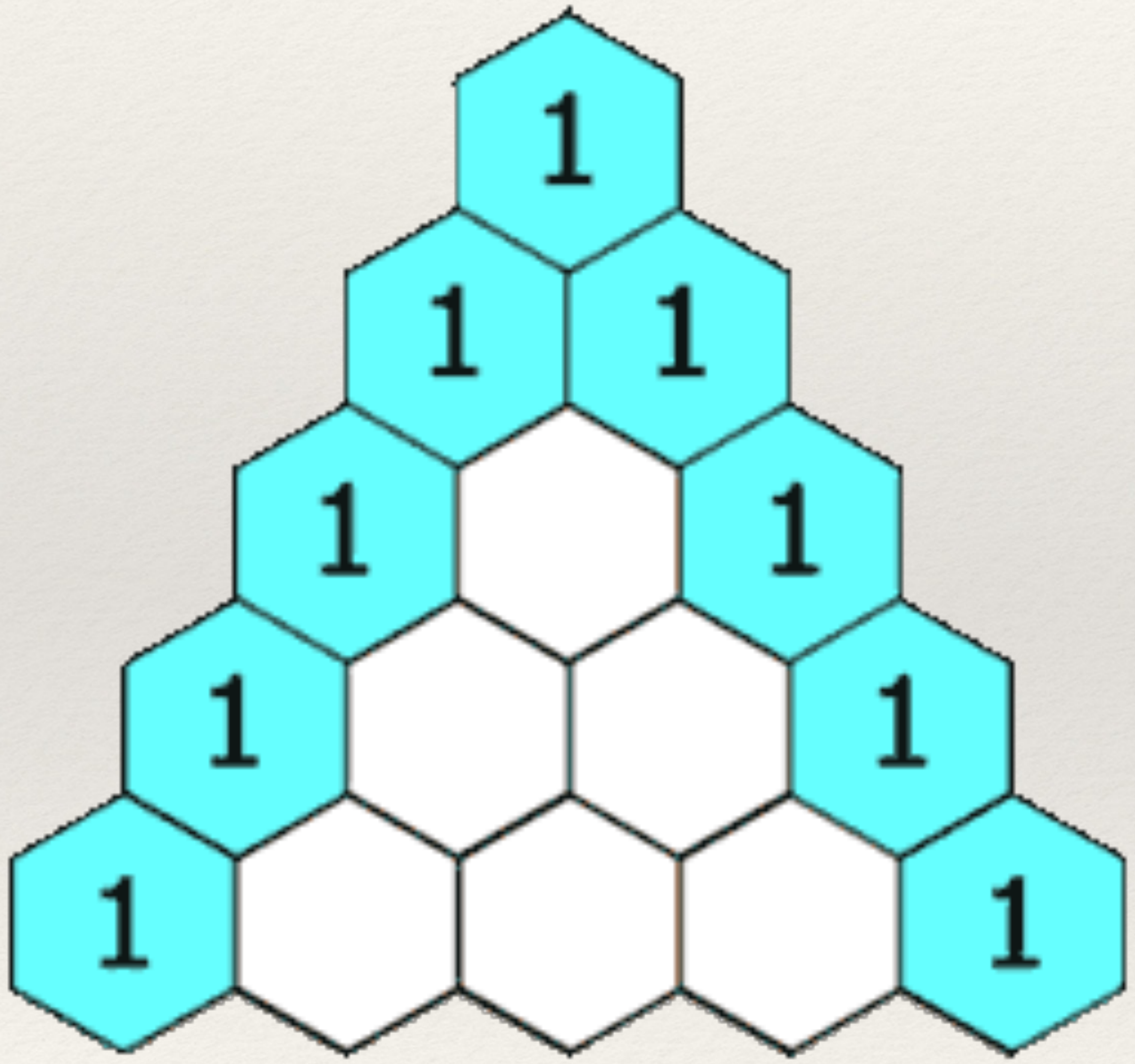
What is Pascal's Triangle?



- **Pascal's triangle** (or Pascal triangle) is a triangular array of the binomial coefficients.
- It is easy to construct it row by row recursively without actually calculating the binomial coefficients (factorials grow exponentially!). This is based on the identity:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

What is Pascal's Triangle?



- It is easy to construct it row by row recursively without actually calculating the binomial coefficients (factorials grow exponentially). This is based on the identity:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

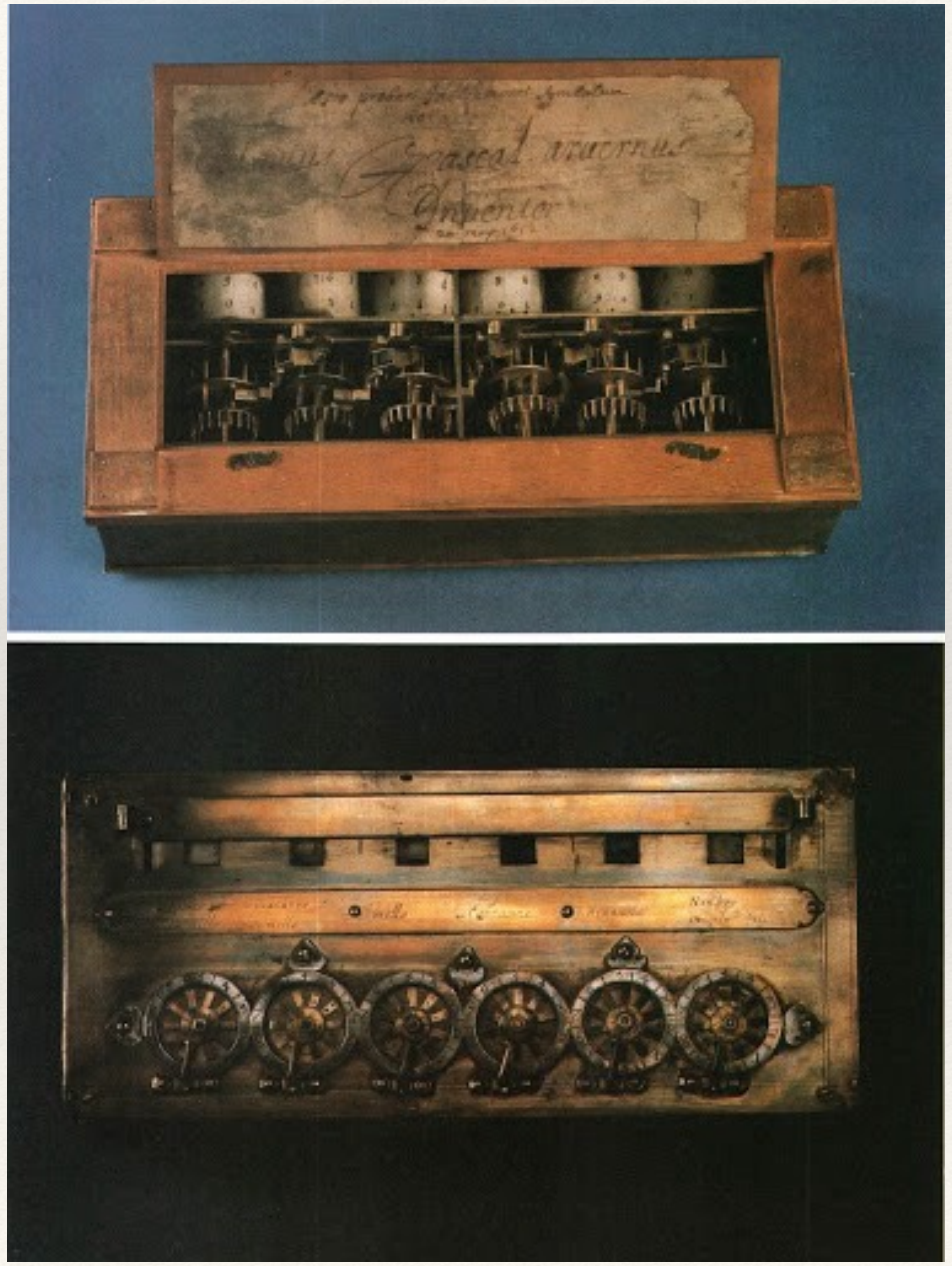
- So, every number inside Pascal's triangle is equal to the sum of the two neighbours above it.

⌛ A bit of history…

- **Blaise Pascal** (19 June 1623 – 19 August 1662) was a French mathematician, physicist, inventor, philosopher, writer, and Catholic theologian.
- Considering that he lived for only 39 years, it is quite remarkable how much the French he accomplished. He was fascinated with geometry during his early teens and discovered several geometric facts on his own.

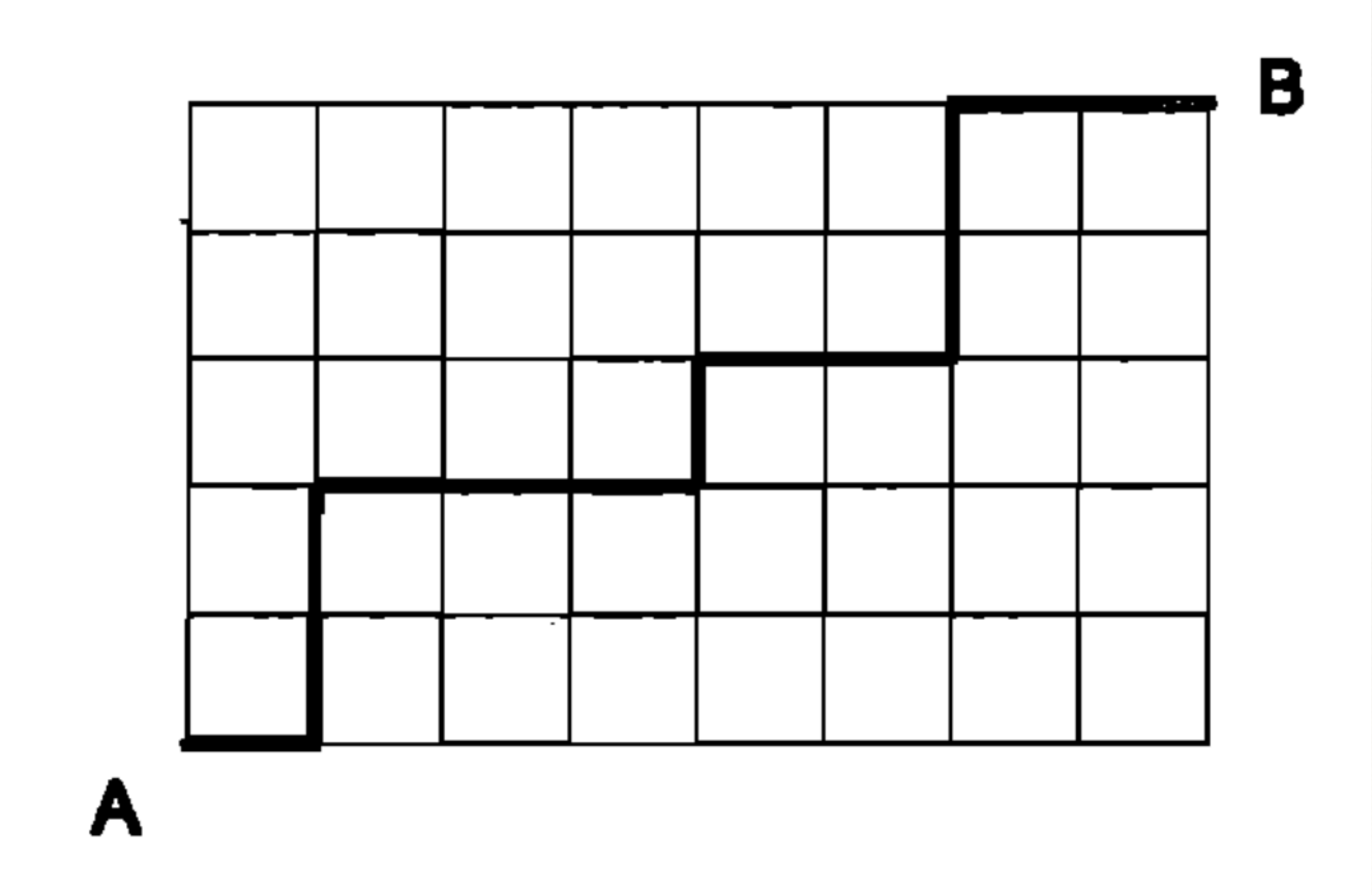
⌛ A bit of history…

- Pascal’s father was employed as a tax collector. In order to help his father, Pascal spent the 3-year period 1642-1645 attempting to invent a digital calculator, which he succeeded in doing. This device, called the Pascaline, became the first digital calculator. He is quite possibly the second person to invent a mechanical calculator.
- In 1968, a programming language (PASCAL) was named for him.



Example 1

The map of a town is depicted in Figure. All its streets are one-way, so that you can drive only "east" or "north" How many different ways are there to reach point *B* starting from *A*?



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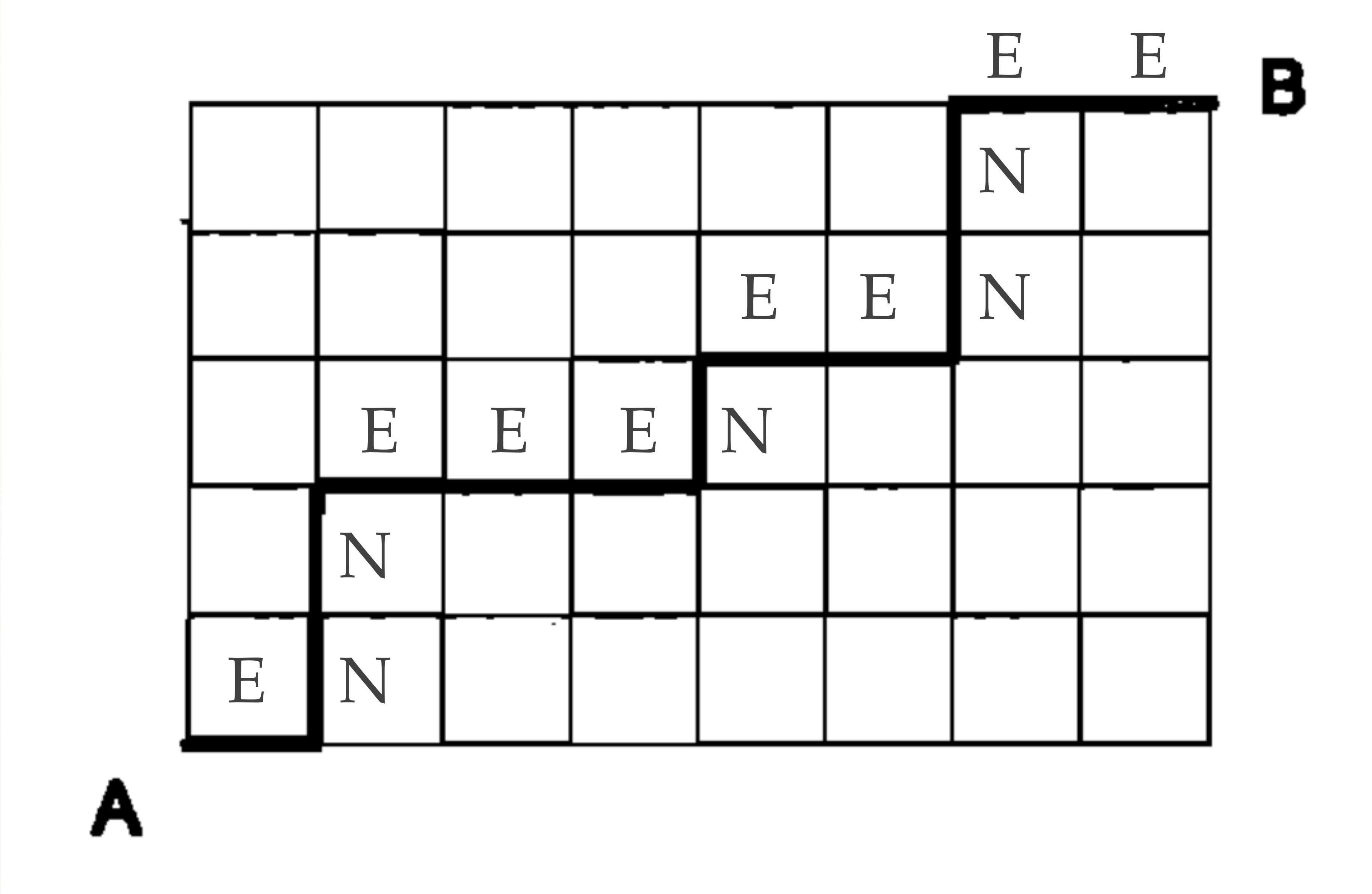
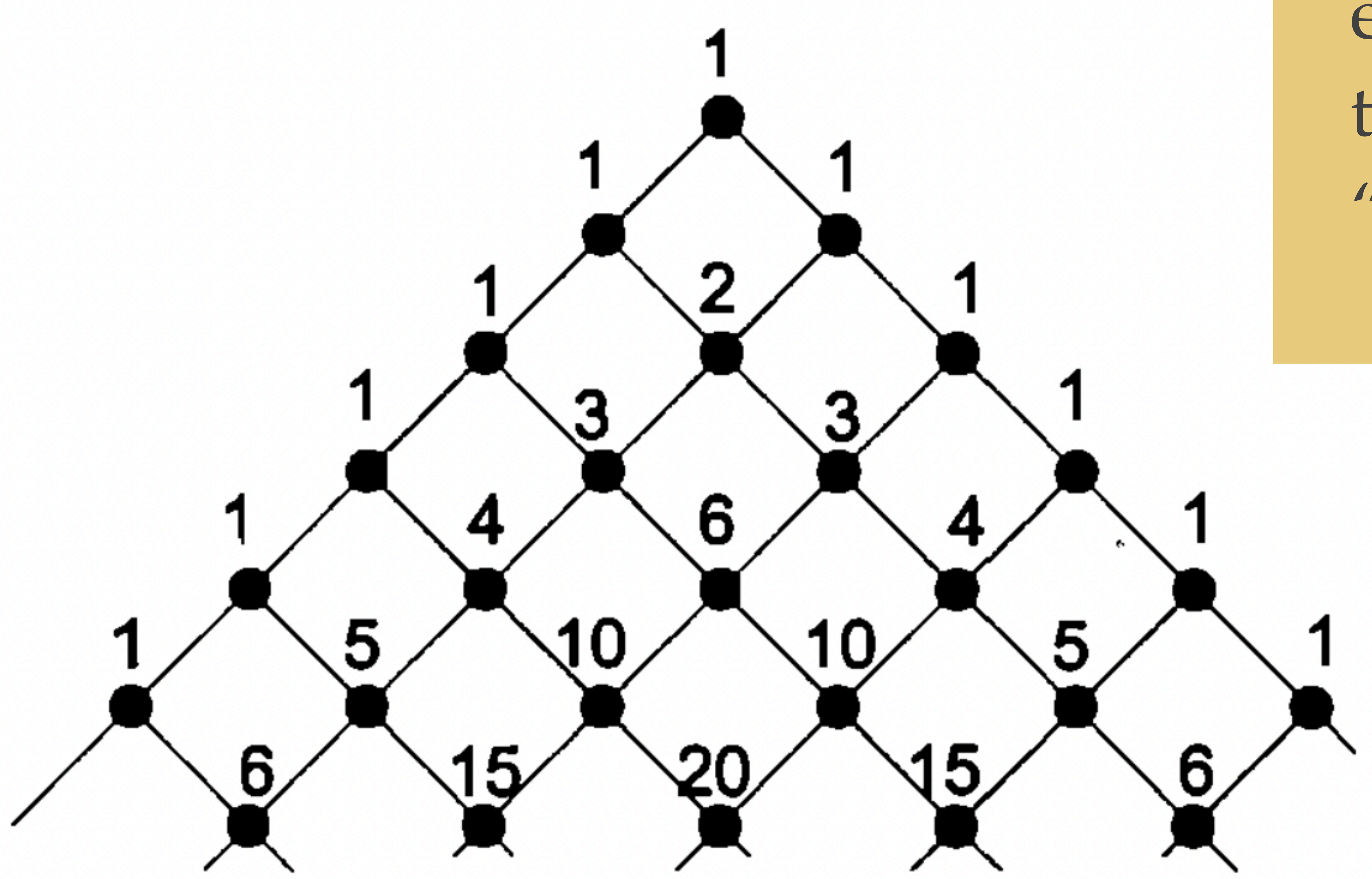
The map of a town is depicted in Figure. All its streets are one-way, so that you can drive only "east" or "north" How many different ways are there to reach point $\boldsymbol{B}$ starting from $\boldsymbol{A}$?



Solution. Let us call any segment of the grid connecting two neighbouring nodes a "street" It is clear that each route from A to B consists of exactly 13 streets, 8 of which are horizontal and 5 are vertical. Given any route, we will construct a sequence of the letters N and E in the following way: when we drive "north", we add the letter N to the sequence, and when we drive "east", we add the letter E to the sequence. For instance, the route in Figure corresponds to the sequence ENNEEENEENNEE. Each sequence constructed in this way contains 13 letters — 8 letters E and 5 letters N. It remains to calculate the number of such sequences.

Any sequence is uniquely determined by the list of the 5 places occupied by the letters N. Five places out of 13 can be chosen in $\binom{13}{5}$ ways. Thus the number of sequences, and, therefore, the number of routes, equals $\binom{13}{5}$. The same reasoning for a $m \times n$ rectangle gives us the result $\binom{m+n}{m}$.

Pascal's Triangle as a Grid

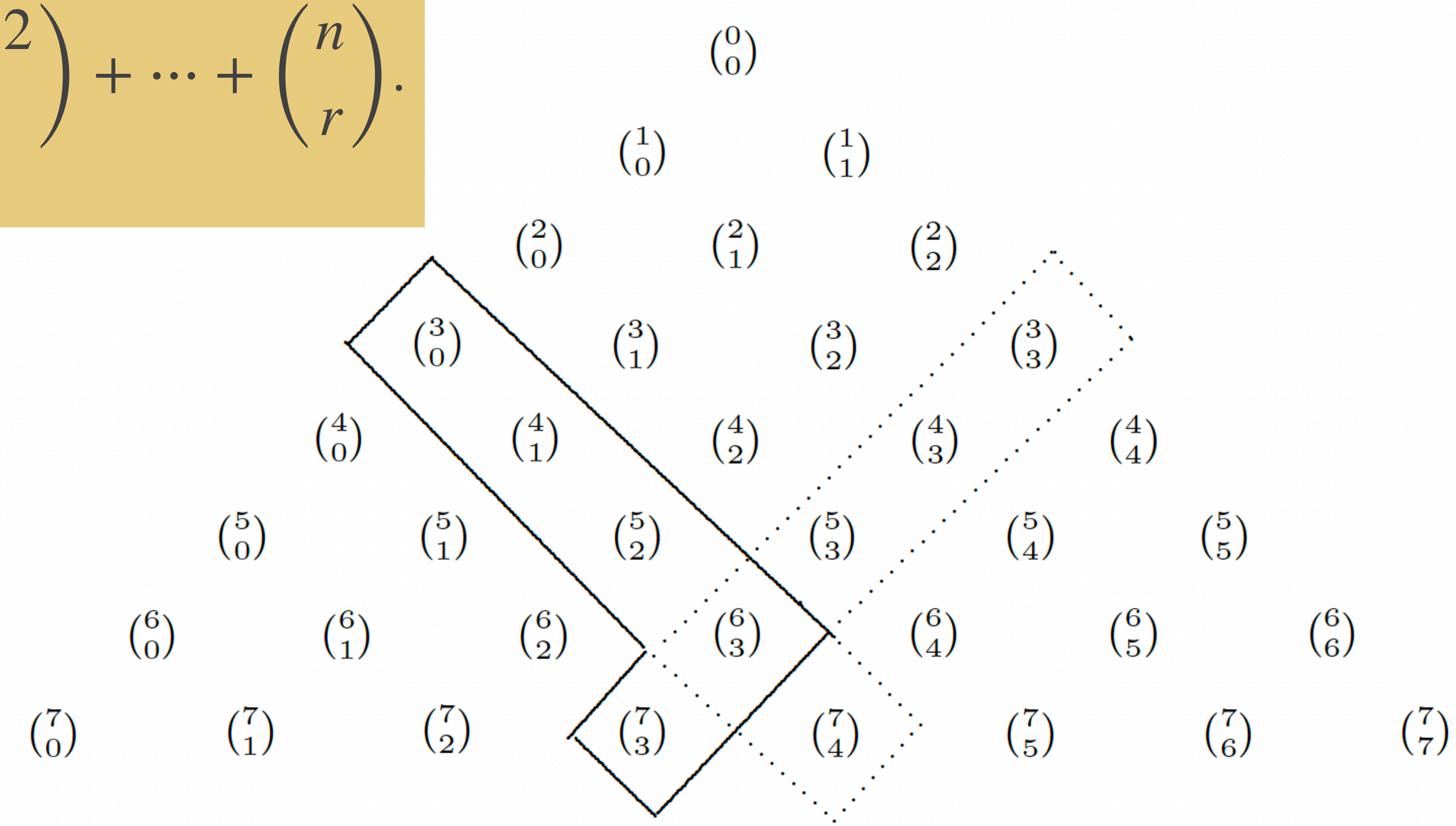


- **Theorem 1**. Let us depict Pascal's triangle as a triangular grid with the binomial coefficients at its nodes (see Figure). Then the label at every node is exactly the number of ways to get to this node from the top node by moving along the grid lines with only "down left" and "down right" directions allowed.

The Hockey Stick Theorem

- **Theorem 2**. For every two integers r and n with $0 \le r \le n$, one has the equality

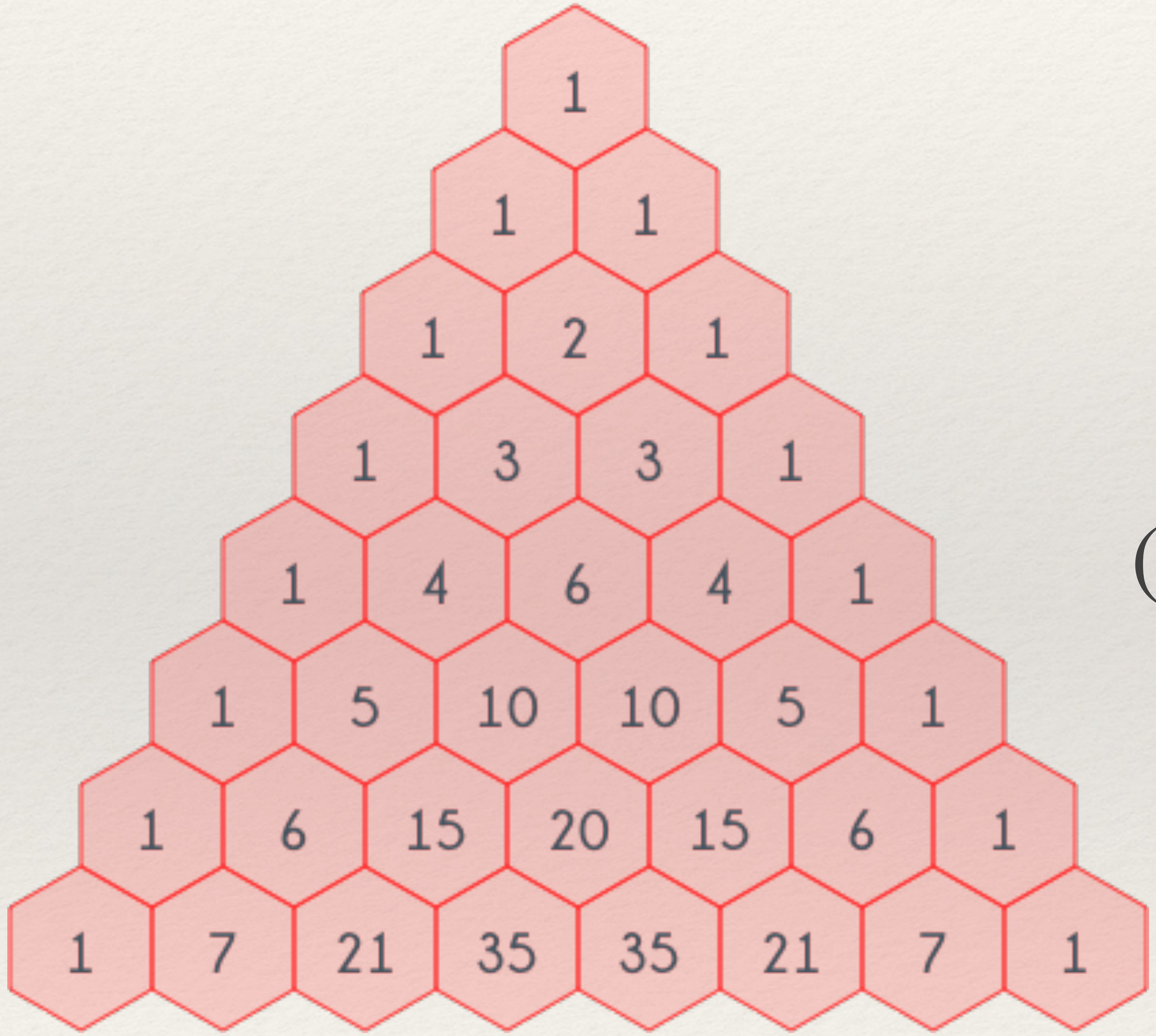
$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}.$$



Newton’s Binomial Theorem

Example 2

The rows in the Pascal triangle may seem familiar for another reason.



$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

Newton's Binomial Theorem

- **Theorem 3**. For every nonnegative integer n, one has the equality

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y^1 + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{r} x^{n-r}y^r + \cdots + \binom{n}{n} y^n.$$

Example 3

- **Theorem 3**. For every nonnegative integer n, one has the equality

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$$\begin{aligned}(x+y)^5 &= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\end{aligned}$$

Example 4

Suppose that n is a positive integer whose last digit is 5, that is, $n = 10a + 5$ for a nonnegative integer a. Then

$$\begin{aligned} n^2 &= (10a+5)^2 = (10a)^2 + 2\cdot(10a)\cdot 5 + 5^2 = 100a^2 + 100a + 25 \\ &= 100a(a+1) + 25 \end{aligned}$$

$$n = \overbrace{\underline{\qquad}\ \ \underline{\qquad}\ \ \cdots\ \ \underline{\qquad}}^{a}\ \ \underline{\ 5\ }. \quad\Rightarrow\quad n^2 = \overbrace{\underline{\qquad}\ \ \underline{\qquad}\ \ \cdots\ \ \underline{\qquad}}^{a(a+1)}\ \ \underline{\ 2\ }\ \ \underline{\ 5\ }.$$

For example, to square 35, we therefore multiply 3 and 3 + 1 (obtaining 12) and follow it by 25, that is, $(35)^2 = 1225$.

Also, $(125)^2$ is $12 \cdot 13 = 156$ followed by 25 or $(125)^2 = 15625$.

Example 5

We mention one other curious property of some of the numbers in the Pascal triangle. Recall that the Fibonacci numbers are defined recursively

$$F_1 = 1,\ F_2 = 1 \ \text{ and } \ F_n = F_{n-1} + F_{n-2} \ \text{ for } n \geq 3.$$

First observe that $\binom{0}{0} = 1 = F_1$ and $\binom{1}{0} = 1 = F_2$. Also

$$\binom{2}{0} + \binom{1}{1} = 2 = F_3$$
$$\binom{3}{0} + \binom{2}{1} = 3 = F_4$$
$$\binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5 = F_5$$
$$\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8 = F_6\,.$$

Fibonacci sequence as sums of binomial coefficients

- **Theorem 4**. For each positive integer *n*, the *n*th Fibonacci number is

$$F_n = \begin{cases} \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k-1} & \textit{if } n = 2k \\ \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k} & \textit{if } n = 2k+1. \end{cases}$$

Counting Idea: Balls and Walls

Example 6

- Six boxes are numbered 1 through 6. How many ways are there to put 20 identical balls into these boxes so that none of them is empty?

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  Solution. Let us arrange the balls in a row. To determine the distribution of the balls in the boxes we must partition this row into six groups of balls using five walls: the first group for the first box, the second group for the second box, et cetera (see Figure).

  

  Thus, the number of ways to distribute our balls in the boxes equals the number of ways to put five walls into gaps between the balls in the row. Any wall can be in any of 19 gaps (there are $19 = 20 - 1$ gaps between 20 balls), and no two of them can be in the same gap (this would mean that one of the groups is empty). Therefore, the number of all possible partitions is $\binom{19}{5}$.

Example 7

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  Solution. Consider a row of 25 objects: 20 identical balls and 5 identical walls, which are arranged in an arbitrary order. Any such row corresponds without ambiguity to some partition of balls: balls located to the left of the first wall, go to the first box; balls between the first and the second wall go to the second box, et cetera (perhaps, some pair of walls are adjacent in the row, resulting in an empty box).

  

  Therefore, the number of partitions is equal to the number of all possible rows of 20 balls and 5 walls; that is, to $\binom{25}{5}$ (the row is completely determined by the 5 places occupied by the walls).

Example 8

- How many ways are there to represent the natural number n as a sum of
  a) k natural numbers?
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  Hint. Represent n as the sum of ones:
  $$n = 1 + 1 + \cdots + 1.$$
  Call these ones "balls", and call the k summands from the statement “boxes".

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  a) k natural numbers?
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  Representations that differ in the order of the summands are different.

  Hint. Represent n as the sum of nones:
  $$n = 1 + 1 + \cdots + 1.$$
  Call these nones "balls", and call the k summands from the statement “boxes".

  Answers. a) $\dbinom{n-1}{k-1}$; b) $\dbinom{n+k-1}{k-1}$.

Generating Functions

Example 9

- Suppose that we were to select two balls from a bowl consisting of one red ball (R), one blue ball (B), one green ball (G) and one yellow ball (Y). How many possible outcomes are there?

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  Solution. Then there are $\binom{4}{2} = 6$ possible outcomes,

  namely: RB, RG, RY, BG, BY, GY.

Example 9

- Suppose that we were to select two balls from a bowl consisting of one red ball (R), one blue ball (B), one green ball (G) and one yellow ball (Y). How many possible outcomes are there?

  Another solution. Take a look at the following product:
  $$(1+x)\cdot(1+x)\cdot(1+x)\cdot(1+x).$$
  Let the first x corresponds to the red ball, the second — to the blue ball, the third — to the green one, and the fourth — to the yellow one. Then the number of number of ways of choosing two balls is equal to coefficient in front of x^2, which we can find by the Binomial Theorem:
  $$(1+x)^4 = 1 + 4x + \mathbf{6}x^2 + 4x^3 + x^4.$$

Example 10

- A bowl consists of 30 balls, namely 10 balls of each of the colours red, blue and green. In how many different ways can 10 balls be selected from the bowl such that
  (1) at least one and at most 7 balls are red,
  (2) at least 2 and at most 4 balls are blue and
  (3) at most 3 balls are green?

Example 10

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  (1) at least one and at most 7 balls are red,
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  (3) at most 3 balls are green?

  Solution. Take a look at the following product:
  $$(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7) \cdot (x^2 + x^3 + x^4) \cdot (1 + x + x^2 + x^3).$$
  Let the first term correspond to the red balls, the second — to the blue balls, and the third — to the green ones. Then the number of number of ways of choosing 10 balls is equal to coefficient in front of x^{10}, which is **11**.

Thank you!