Lecture 16

Discrete Random Variables

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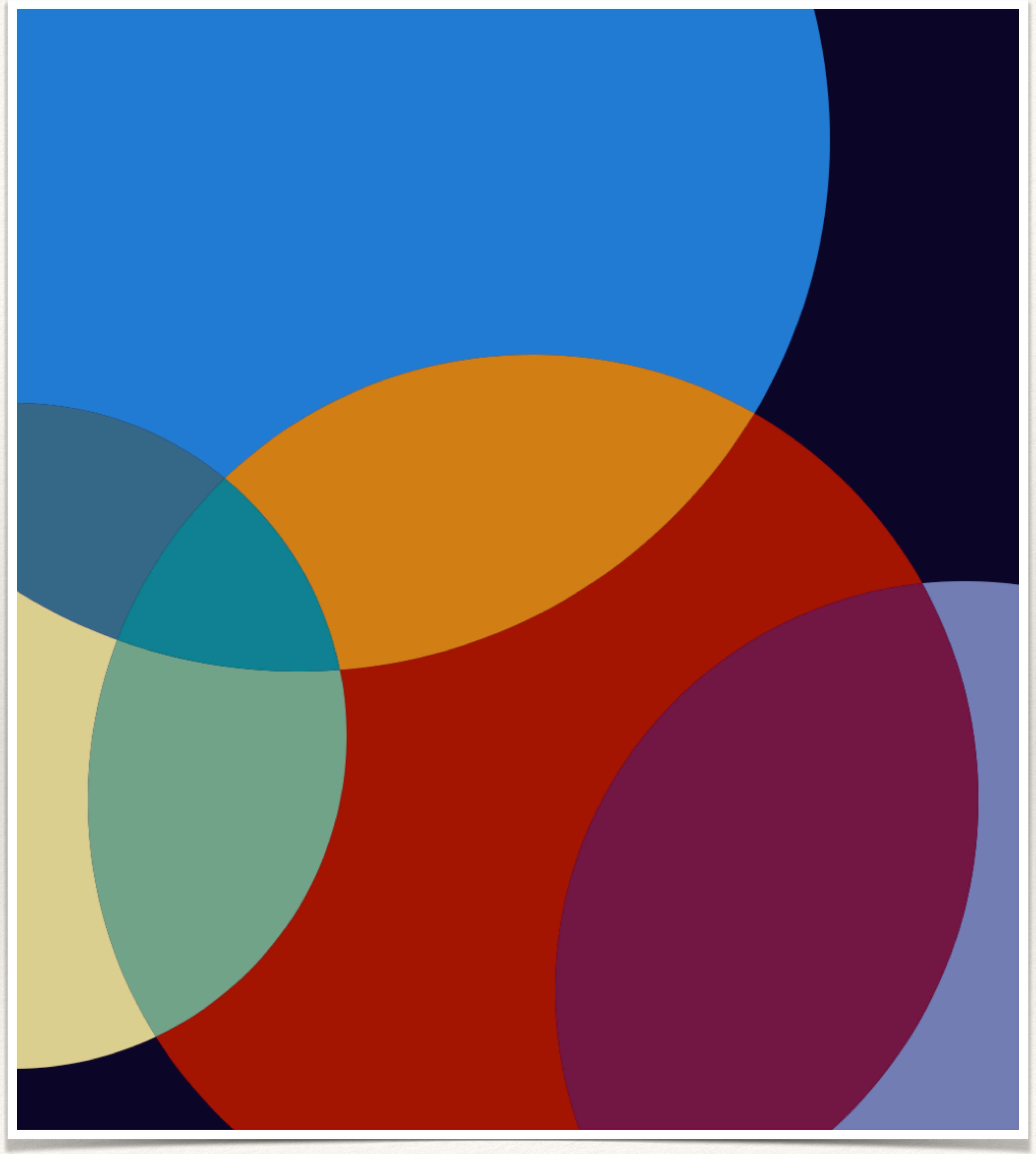
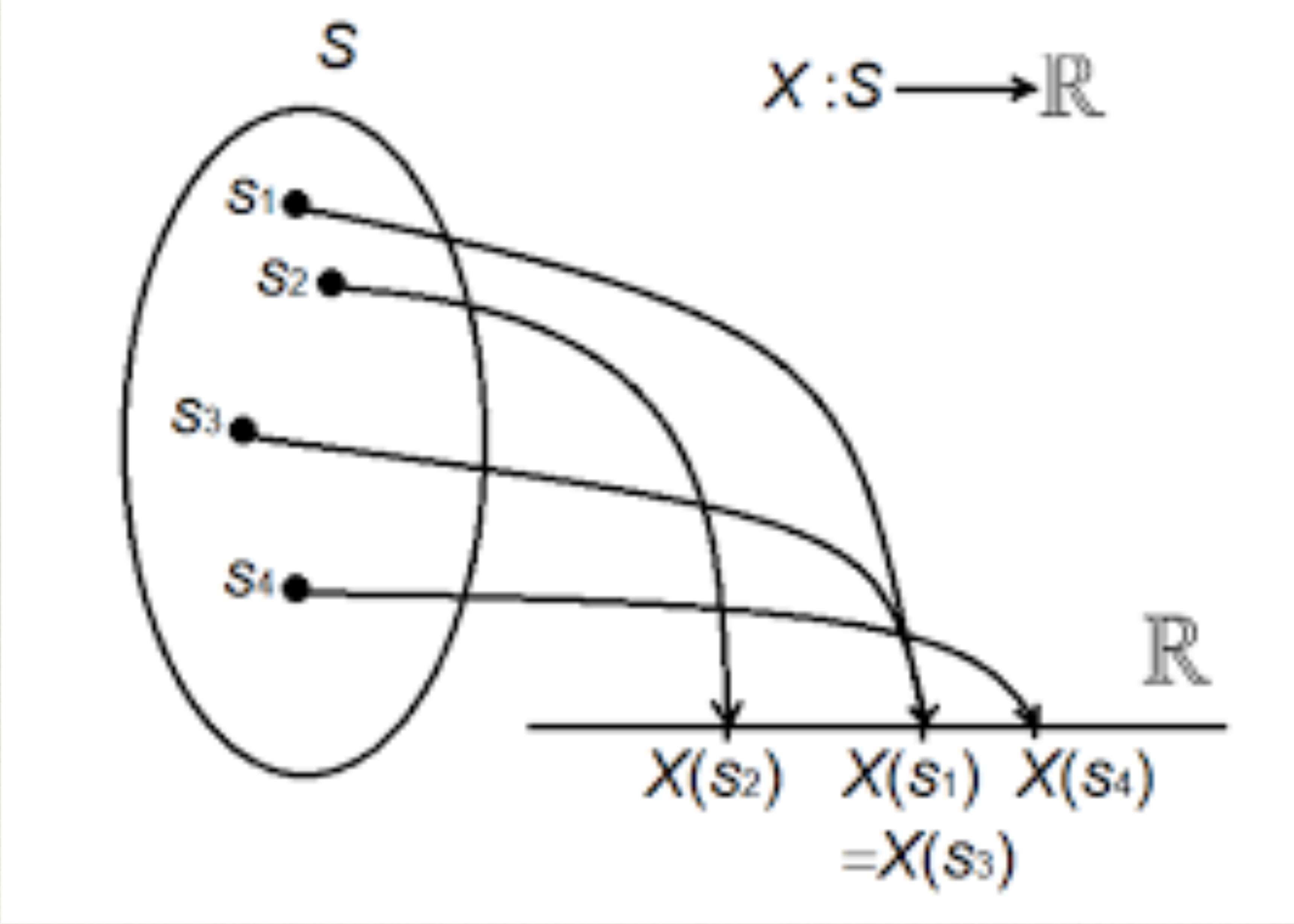


Example 1

You pay \$7.50 to play the following game. A pair of dice is rolled and you receive n dollars if the sum of the numbers on the dice is n. Will you play this game?

$\stackrel{\text{def}}{=}$ What is a Random Variable?



- Let S be a sample space on which is defined a probability function. A **random variable** on S is any function X from S to the set $\mathbb{R}$ of real numbers
$$X : S \to \mathbb{R}.$$
- The **range** $X(S)$ of X is the set of all values of X.

Example 2

A box contains two red balls and three blue balls. Three balls are selected at random from the box.

For each **outcome** $t \in S$ of this experiment, let $X(t)$ denote the number of red balls selected in the outcome t.

Then X is a **random variable** on the sample space of this experiment.

Since each outcome contains at most two red balls, the **range** of X is $\{0, 1, 2\}$.

Example 3

A coin is flipped three times, resulting in the **sample space**

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}.$$

We can define the following **random variable** X on the sample space of this experiment:

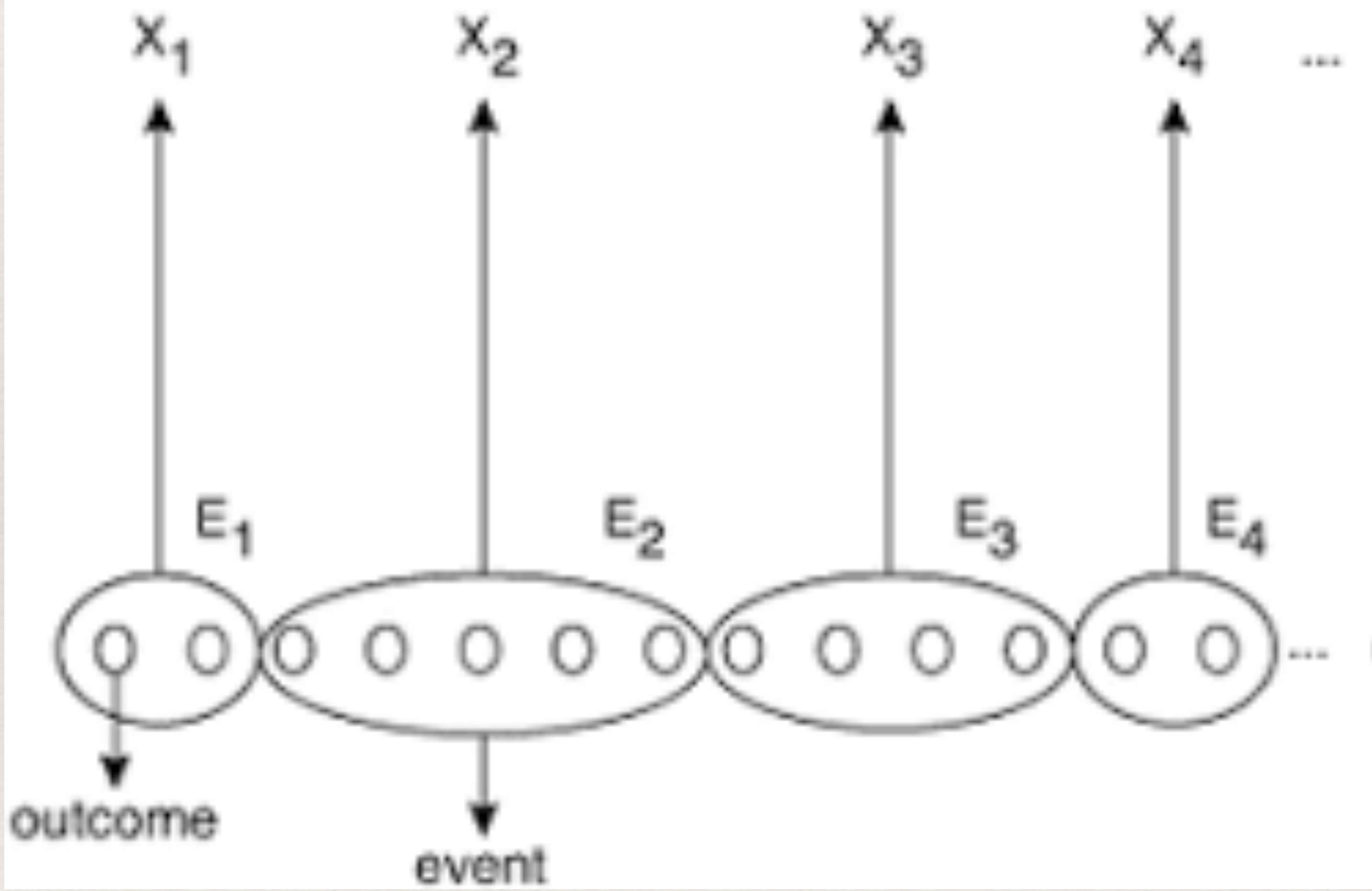
$$X(\text{TTT}) = 0,\ \ X(\text{HHH}) = X(\text{HHT}) = X(\text{HTH}) = X(\text{HTT}) = 1,$$
$$X(\text{THH}) = X(\text{THT}) = 2,\quad X(\text{TTH}) = 3.$$

We can also present this information in the following table:

t	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$X(t)$	1	1	1	1	2	2	3	0

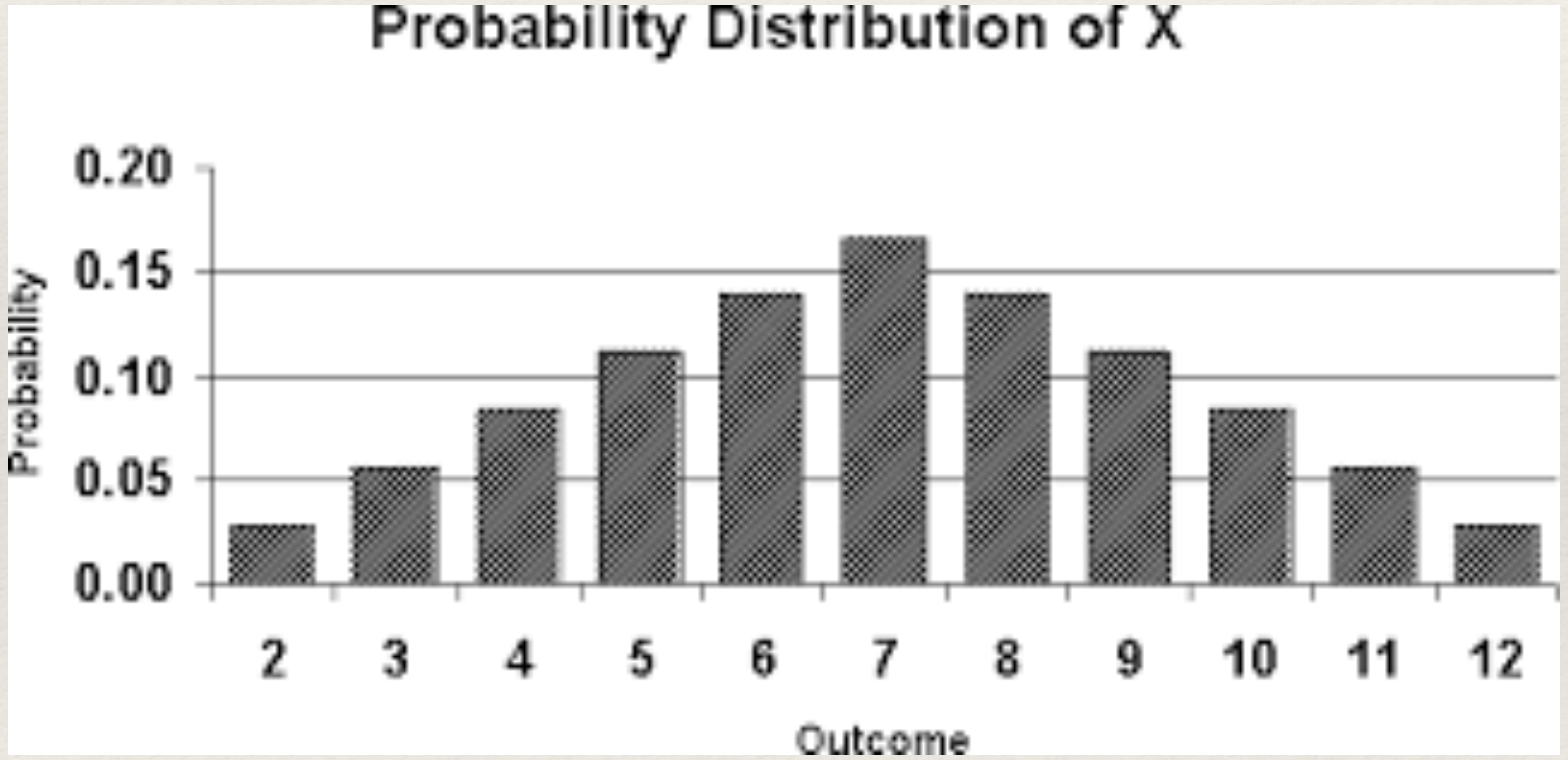
The **range** of X is $\{0, 1, 2, 3\}$.

$\stackrel{\text{def}}{=}$ The Probability of a Random Variable to Take a Value



- Let $X : S \to \mathbb{R}$ be a random variable on the sample space S of some experiment. Let $r \in \mathbb{R}$ be a real number. Then we will denote by $\boldsymbol{X = r}$ the following **event** (subset of S):
$$\{s \in S \mid X(s) = r\}.$$
- We write $\boldsymbol{p(X = r)}$ for the **probability of the event** in which each outcome has the value r.

$\stackrel{\text{def}}{=}$ Distribution of a Random Variable?

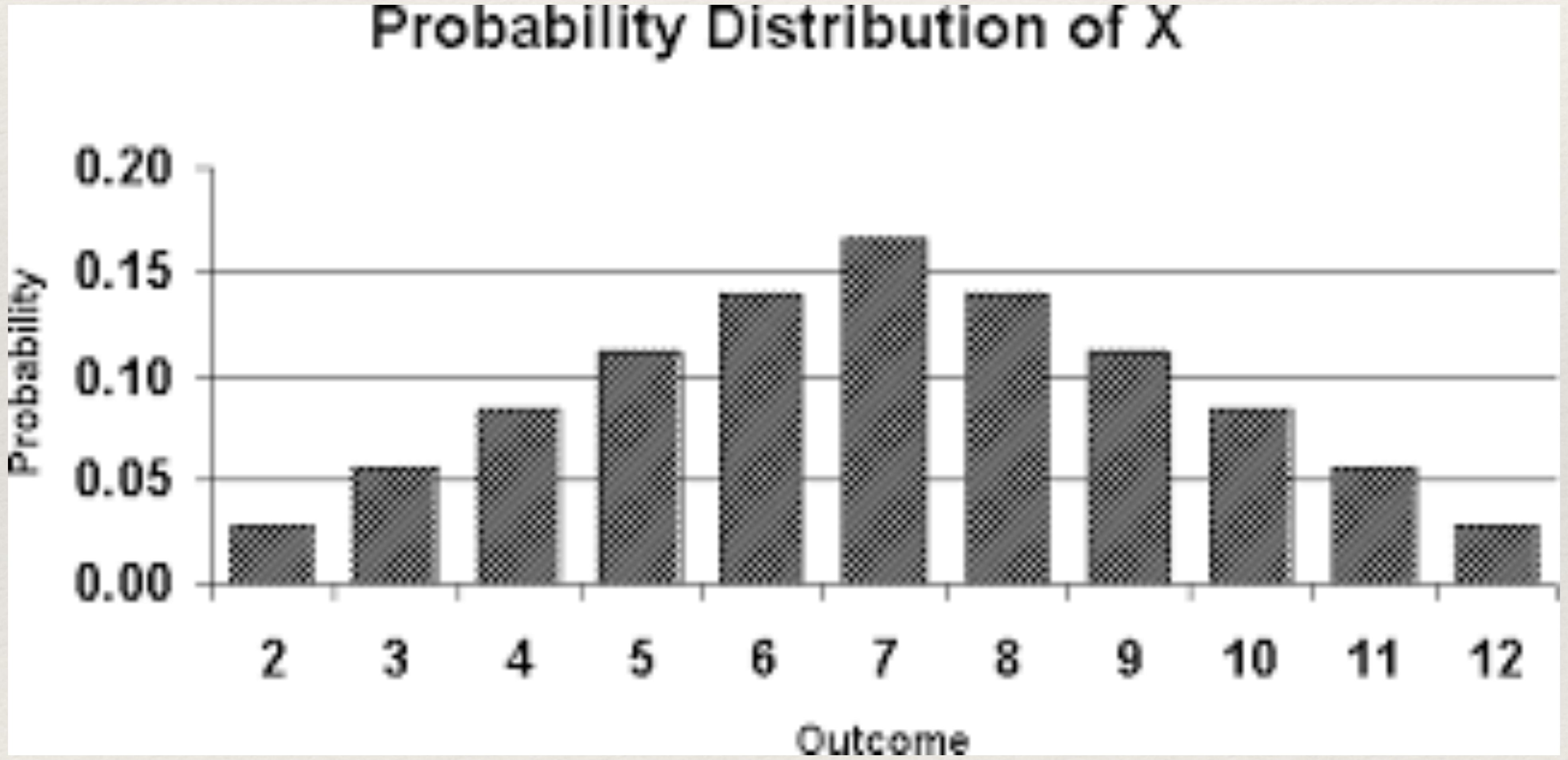


- Let $X : S \to \mathbb{R}$ be a random variable. The **distribution** of X is the set of pairs

$$(r,\ \ p(X = r))$$

for each r in the range of X.

$\stackrel{\text{def}}{=}$ Distribution of a Random Variable?



- Let $X : S \to \mathbb{R}$ be a random variable. The **distribution** of X is the set of pairs
$$(r,\ \ p(X = r))$$
for each r in the range of X.
- The distribution of X is typically expressed by just writing $p(X = r)$ for each $r \in X(S)$.
- We necessarily have that
$$\sum_{r \in X(S)} p(X = r) = 1.$$

Example 4

For the experiment of rolling a pair of dice, the **sample space** is

$$S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 5), (6, 6)\}.$$

Define a **random variable** X on S by

$$X((a, b)) = a + b$$

for each outcome $(a, b) \in S$. What is the **distribution** of X?

Example 4

For the experiment of rolling a pair of dice, the **sample space** is

$$S = \{(1,1), (1,2), (1,3), \ldots, (6,5), (6,6)\}.$$

Define a **random variable** X on S by

$$X((a,b)) = a + b$$

for each outcome $(a,b) \in S$. What is the **distribution** of X?

Value of second die

Value of first die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Solution. The **range** of X is $\{2, 3, \ldots, 12\}$. The **distribution** is given in the table:

r	2	3	4	5	6	7	8	9	10	11	12
$p(X=r)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Example 5

A box contains two red balls and three blue balls. Three balls are selected at random from the box. For each outcome t of this experiment, let $X(t)$ denote the number of red balls selected in the outcome t. What is the **distribution** of X?

Example 5

A box contains two red balls and three blue balls. Three balls are selected at random from the box. For each outcome t of this experiment, let $X(t)$ denote the number of red balls selected in the outcome t. What is distribution of X?

Solution. Since there are $\binom{5}{3} = 10$ ways to choose three balls from the bowl, the sample space S consists of 10 elements. The **range** of X is $X(S) = \{0, 1, 2\}$. The distribution of X is

$$p(X = 0) = \frac{\binom{3}{3}}{10} = \frac{1}{10}, \quad p(X = 1) = \frac{\binom{2}{1} \cdot \binom{3}{2}}{10} = \frac{6}{10},$$

$$p(X = 2) = \frac{\binom{2}{2} \cdot \binom{3}{1}}{10} = \frac{3}{10}.$$

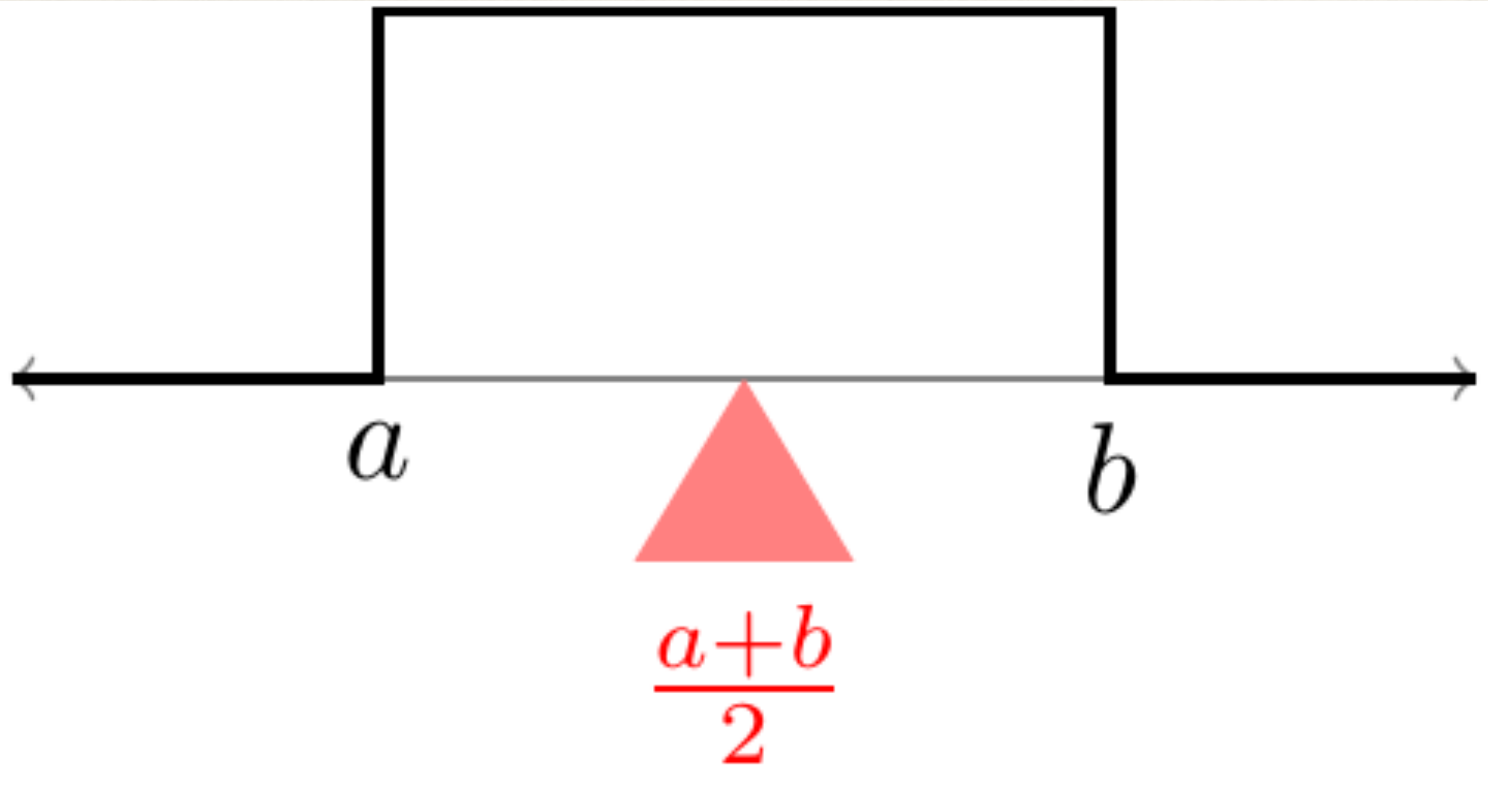
As a table:

r	0	1	2
$p(X = r)$	0.1	0.6	0.3

We can check that $p(X = 0) + p(X = 1) + p(X = 2) = 1$.

Expected Value

$\stackrel{\text{def}}{=}$ Expected Value of a Random Variable?



- Let $X : S \to \mathbb{R}$ be a random variable. The **expected value** $E(X)$ of X is the sum of the products of probabilities of the outcomes s with the values $X(s)$

$$E(X) = \sum_{s \in S} p(s) \cdot X(s).$$

How to Calculate the Expected Value by Knowing the Distribution?

- **Theorem 1**. For a random variable $X : S \to \mathbb{R}$, the expected value is

$$E(X) = \sum_{r \in X(S)} r \cdot p(X = r)$$

Example 6

A box contains two red balls and three blue balls. Three balls are selected at random from the box. For each outcome t of this experiment, let $X(t)$ denote the number of red balls selected in the outcome. What is the expected value of X?

Solution. The **distribution** of X is:

r	0	1	2
$p(X = r)$	0.1	0.6	0.3

Therefore, by the theorem we have

$$E(X) = 0 \cdot p(X = 0) + 1 \cdot p(X = 1) + 2 \cdot p(X = 2) = 0 \cdot 0.1 + 1 \cdot 0.6 + 2 \cdot 0.3 = 1.2$$

Example 7

A fair coin is flipped four times. How many heads would one expect to get?

Solution. Let S be the **sample space** of the $2^4 = 16$ outcomes and let X be the **random variable** that assigns to each outcome the number of heads that occurs. There is 1 outcome with no heads, 4 outcomes with one head, 6 outcomes with two heads, 4 outcomes with three heads and 1 outcome with four heads. So:

r	0	1	2	3	4
$p(X=r)$	$1/16$	$4/16$	$6/16$	$4/16$	$1/36$

Therefore, by the theorem we have

$$E(X) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = 2.$$

Example 8 (aka Example 1)

You pay \$7.50 to play the following game. A pair of dice is rolled and you receive n dollars if the the sum of the numbers on the dice is n. Will you play this game?

Example 8 (aka Example 1)

You pay \$7.50 to play the following game. A pair of dice is rolled and you receive n dollars if the sum of the numbers on the dice is n. Will you play this game?

		Value of second die					
		1	2	3	4	5	6
Value of first die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Solution. For the experiment of rolling a pair of dice, the **sample space** is $S = \{(1,1), (1,2), (1,3), \ldots, (6,5), (6,6)\}$.
Consider the **random variable** X on S defined by

$$X((a,b)) = a + b \quad \text{for each outcome } (a,b) \in S.$$

The **range** of X is $\{2, 3, \ldots, 12\}$. The **distribution** is given in the table:

r	2	3	4	5	6	7	8	9	10	11	12
$p(X=r)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Then, the expected value is

$$E(X) = 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36} = 7$$

which is less than 7.5. So, better not to play this game!

Example 9

At a carnival it costs \$10 to play a game, where three balls are selected from a box consisting of three red balls, three blue balls and three green balls. If all three balls selected have the same color, then you win \$63, while if all balls selected have different colors, you win \$21. If you were to play this game 10 times, how much would you expect to win or lose?

Example 9

At a carnival it costs \$10 to play a game, where three balls are selected from a box consisting of three red balls, three blue balls and three green balls. If all three balls selected have the same color, then you win \$63, while if all balls selected have different colors, you win \$21. If you were to play this game 10 times, how much would you expect to win or lose?

Solution. There are $\binom{9}{3} = 84$ different outcomes and so the **sample space** has 84 elements. Define a **random variable** X by assigning 63 to an outcome in which all 3 balls selected have the same color, 21 to an outcome in which all balls selected have different colors and 0 to all other outcomes. Only one of these outcomes consists of 3 red balls (or 3 blue balls or 3 green balls). Therefore, $p(X = 63) = 3/84 = 1/28$. There are $3 \cdot 3 \cdot 3 = 27$ outcomes where all the balls have different colors. Hence, $p(X = 21) = 27/84 = 9/28$. Hence $p(X = 0) = 1 - 1/28 - 9/28 = 18/28 = 9/14$. The **distribution** of X is:

r	63	21	0
$p(X = r)$	$1/28$	$9/28$	$9/14$

Therefore, we have $E(X) = 63 \cdot \frac{1}{28} + 21 \cdot \frac{9}{28} + 0 \cdot \frac{9}{14} = 9.$

If the game is played 10 times, then you would expect to lose \$10.

Thank you!