Lecture 8

Combinatorics

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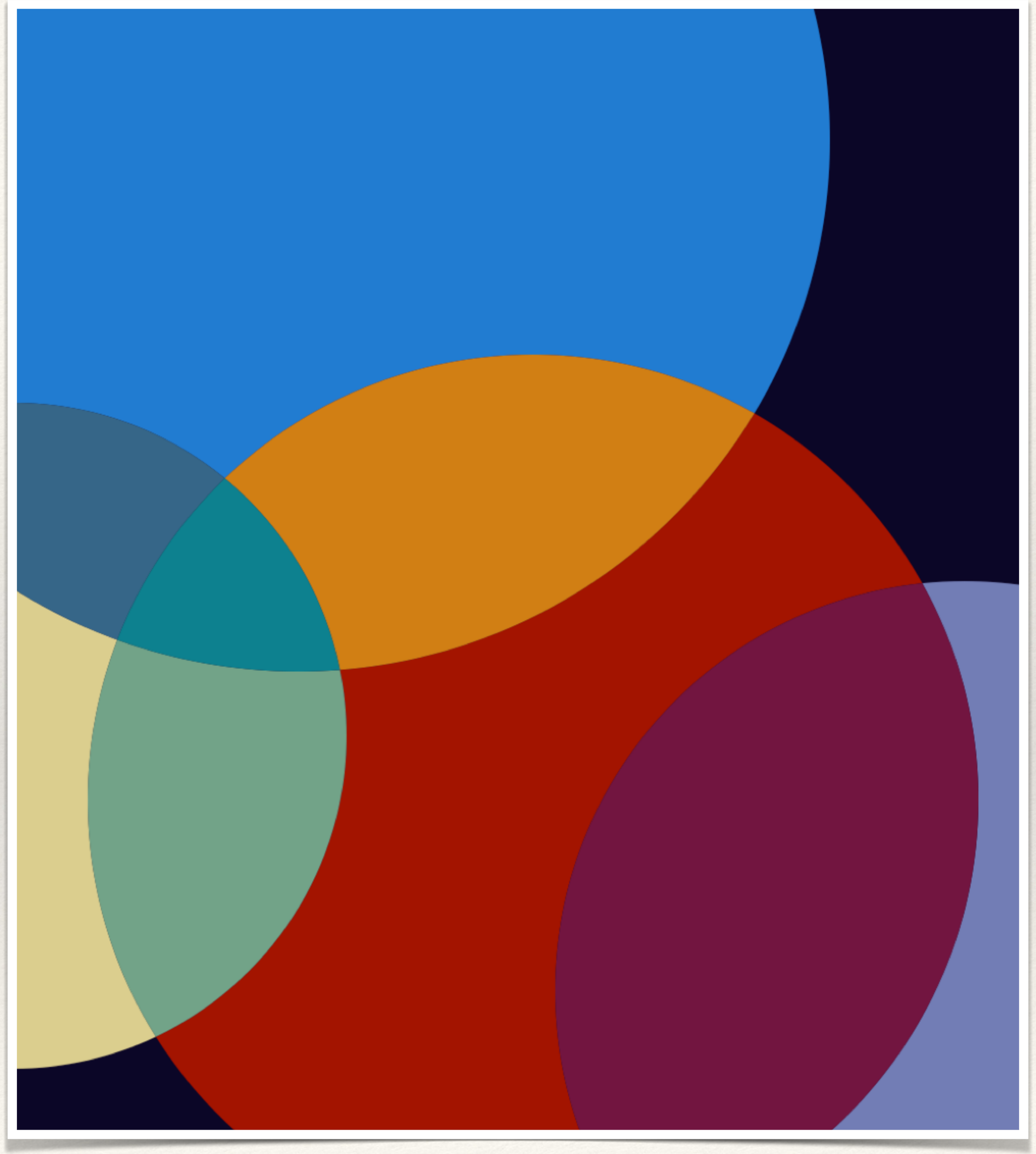
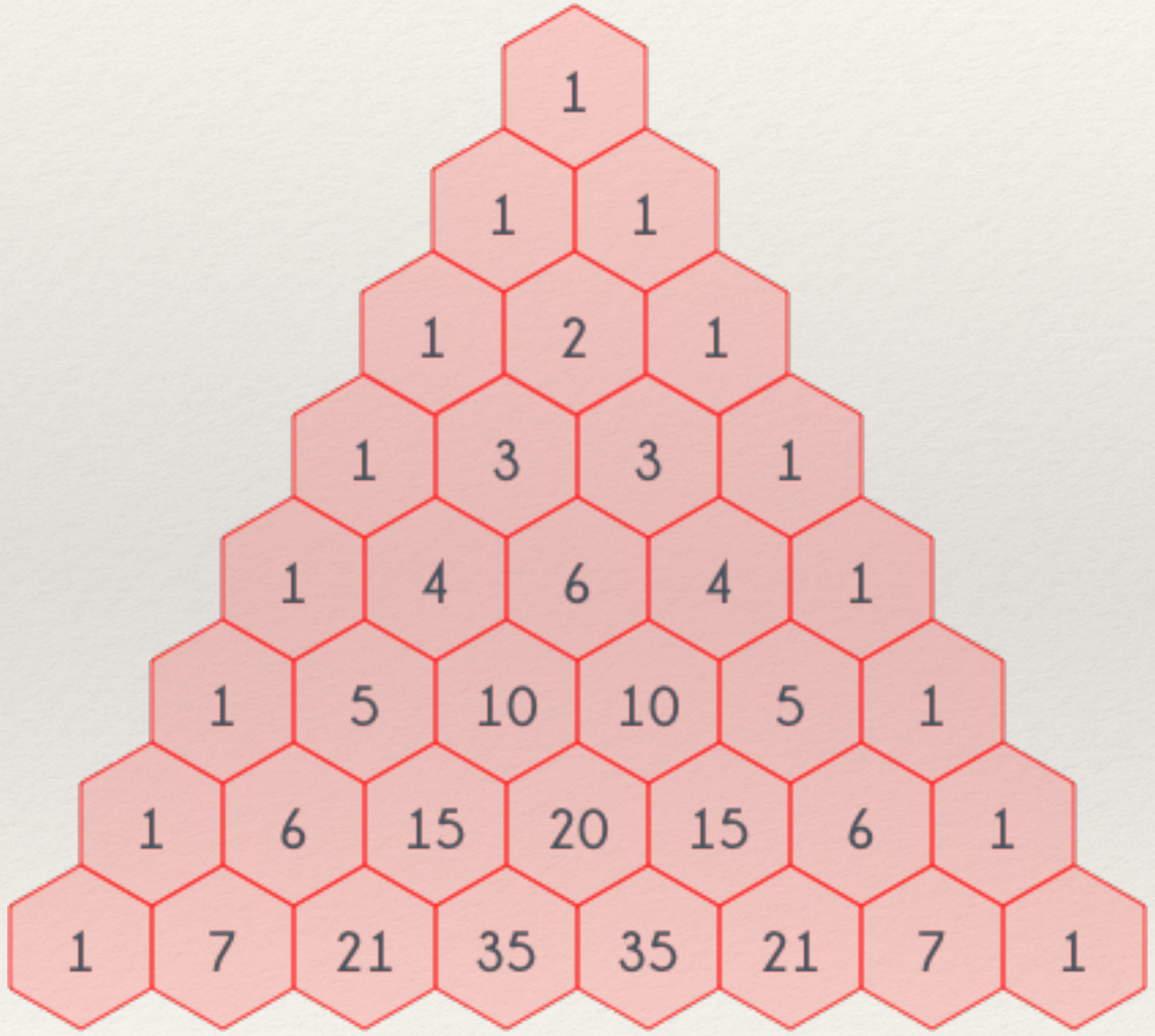


What is Combinatorics?



- Discrete mathematics consists of many areas of mathematics. One of these areas is combinatorics.
- **Combinatorics** is the branch of mathematics primarily concerned with **counting** and deals with the study of configurations or arrangements of objects.

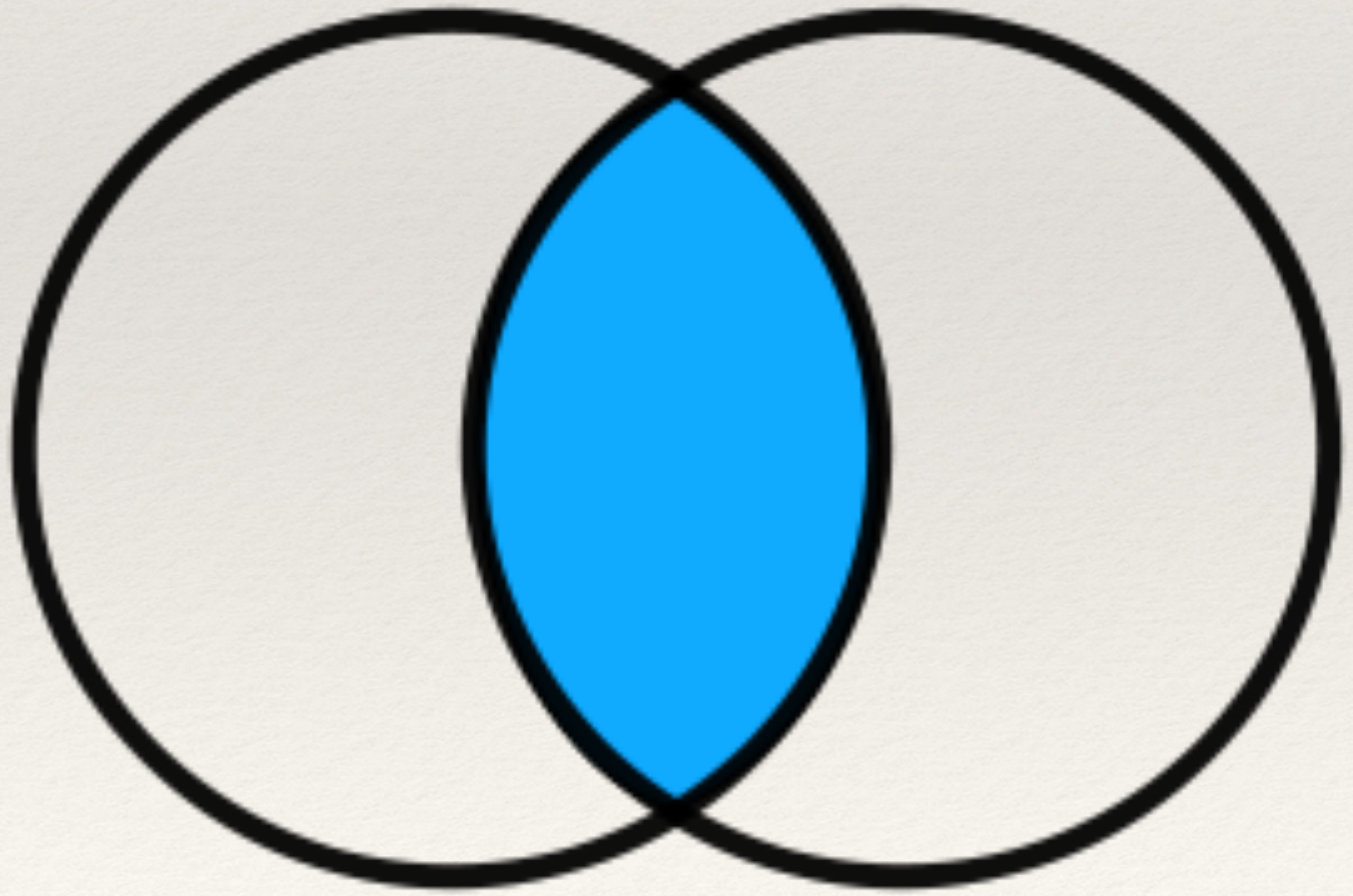
⏳ A bit of history…

- Combinatorics evidently came into prominence during 1915-1916 only after the publication of the classic two-volume treatise Combinatory Analysis by **Percy Alexander MacMahon**.
- Basic combinatorial concepts and enumerative results appeared throughout the ancient world. In the 6th century BCE, ancient Indian physician Sushruta asserts in Sushruta Samhita that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc., thus computing all $2^6 - 1$ possibilities.

Counting Ideas

The Principle of Inclusion-Exclusion for Two Sets

- **Theorem 1.** If A and B are finite sets, then
$$|A \cup B| = |A| + |B| - |A \cap B|.$$



Example 1

In a survey of 100 college students, 35 were registered in Algebra, 52 were registered in Computer Science, and 18 were registered in both courses.

(a) How many students were registered in Algebra or Computer Science?

(b) How many were registered in neither course?

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(a) How many students were registered in Algebra or Computer Science?
(b) How many were registered in neither course?

Solution. (a) Let A = set of students in Algebra, B = set of students in Computer Science. Then the given information tells us that

$$|A| = 35, |B| = 52. |A \cap B| = 18.$$

So, $|A \cup B| = 35 + 52 - 18 = 69.$

(b) Since 100 students were surveyed, it follows that $100 - 69 = 31$ were registered in neither course.

Example 2

- How many 8-bit sequences begin with 110 or end with 1100?

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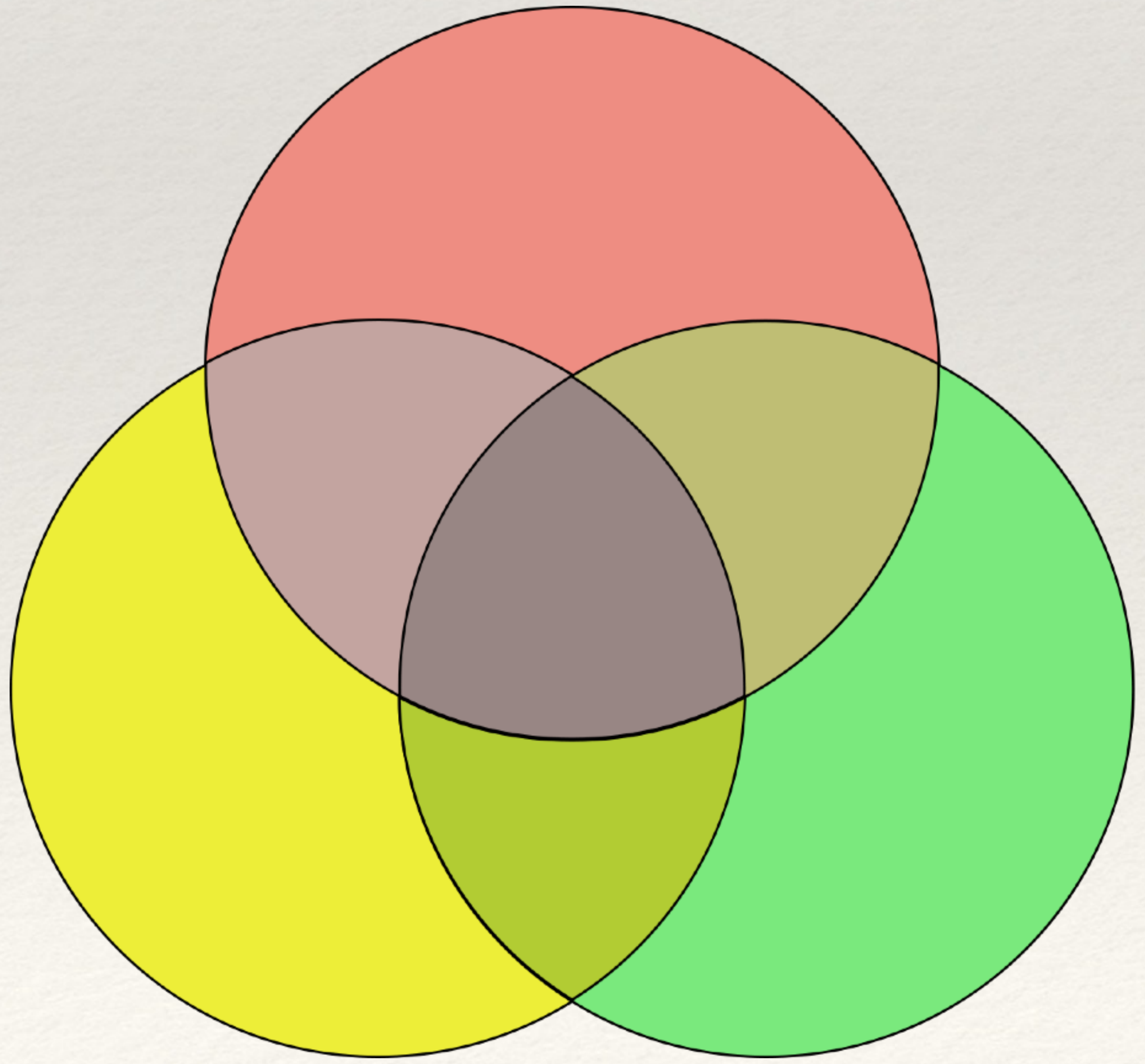
  Solution. We are considering 8-bit sequences of one of the types
  1 1 0 _ _ _ _ _ or _ _ _ _ 1 1 0 0.
  The number of 8-bit sequences that begin with 110 is 2 . 2 . 2 . 2 . 2 = 32 and the number of 8-bit sequences that end with 1100 is 2 . 2 . 2 . 2 = 16. Observe that an 8-bit sequence could begin with 110 and end with 1100. Such a sequence has the following appearance:
  1 1 0 _ 1 1 0 0.
  Therefore, the number of 8-bit sequences that begin with 110 and end with 1100 is 2. It then follows by the Principle of Inclusion-Exclusion that the number of 8-bit sequences with the desired property is $32 + 16 - 2 = 46$

The Principle of Inclusion-Exclusion for Three Sets

- **Theorem 2.** If A, B and C are finite sets, then
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$



Example 3

- The director of a dormitory cafeteria interviews 60 students to determine their likes and dislikes among certain foods. Here is what she learned:
  31 students like beef.
  32 students like chicken.
  31 students like fish.
  15 students like beef and chicken.
  12 students like beef and fish.
  19 students like chicken and fish.
  8 students like all three.

  How many like none of these?

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  Answer. $60 - 56 = 4$.

The Addition Principle

- **Theorem 3**. If two sets A and B have no elements in common, then
$$|A \cup B| = |A| + |B|.$$

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- **Theorem 3**. If two sets A and B have no elements in common (that is $A \cap B = \varnothing$), then
$$|A \cup B| = |A| + |B|.$$

  In general, if, for n sets $A_1, A_2, \ldots, A_n$, no two have elements in common, then
$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|.$$

Example 4

- After a student graduates from college, he wants to work on a Master's degree in computer science. He is considering two universities in South Carolina, three universities in Georgia and five universities in Florida.

  By the Addition Principle, the total number of choices he has for his graduate studies is $2 + 3 + 5 = 10$.

Example 5

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  Answer. $2^3 + 2^4 = 24$.

The Multiplication Principle

- **Theorem 4**. If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, the task of making these selections can be done in

$$p \cdot q \cdot r \cdot \ldots$$

different ways.

Example 6

- The fixed-price dinner at Mabenka Restaurant provides the following choices:

  Appetizer: soup or salad
  Entrée: baked chicken, broiled beef patty, beef liver, or roast beef au jus
  Dessert: ice cream or cheese cake

  How many different meals can be ordered?

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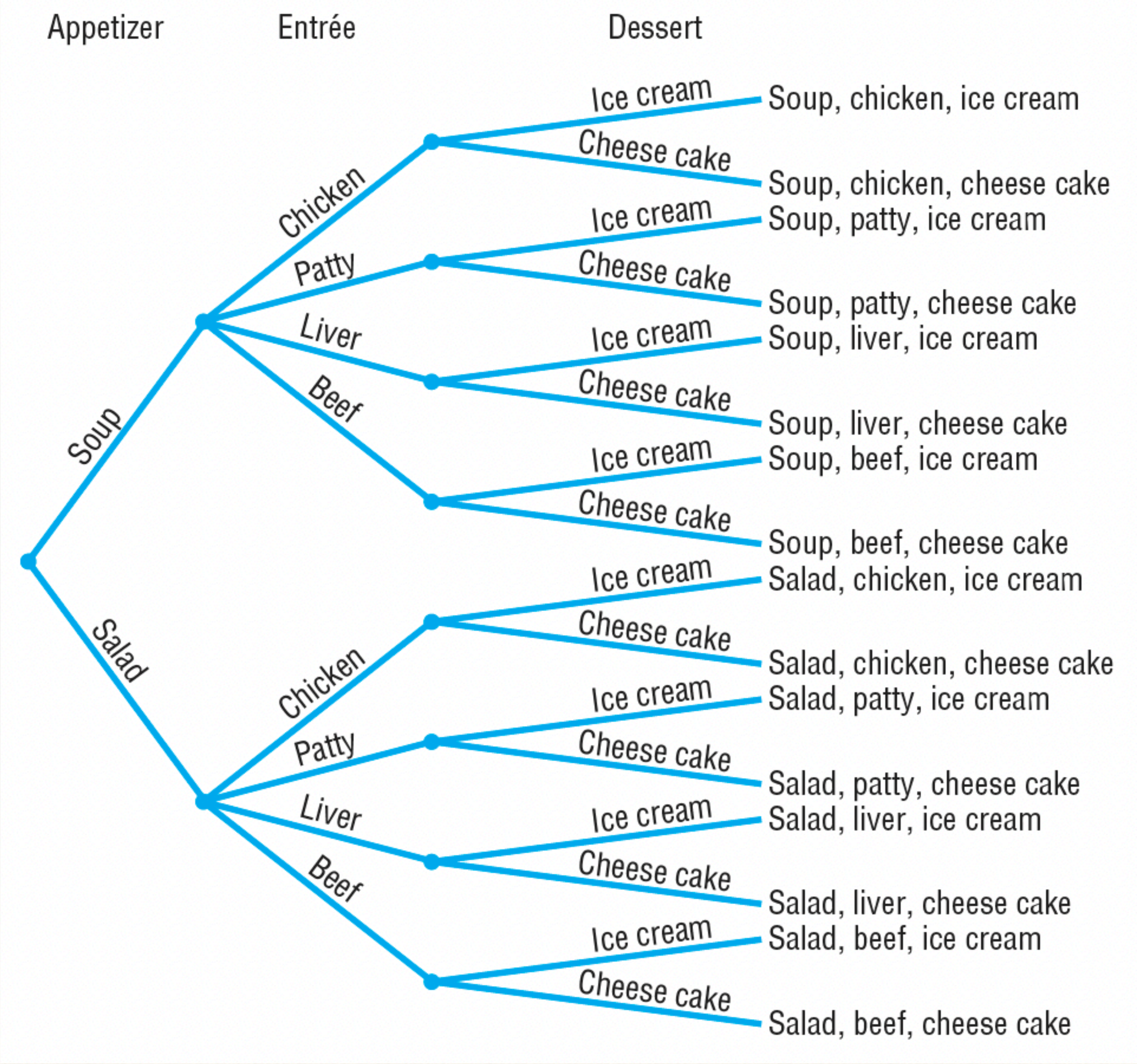
Solution. Ordering such a meal requires three separate decisions:

Choose an Appetizer	**Choose an Entrée**	**Choose a Dessert**
2 choices	4 choices	2 choices

Look at the tree diagram. Note that for each choice of appetizer, there are 4 choices of entrées. And for each of these 2 # 4 = 8 choices, there are 2 choices for dessert. A total of

$$2 \cdot 4 \cdot 2 = 16$$

different meals can be ordered.



Example 7

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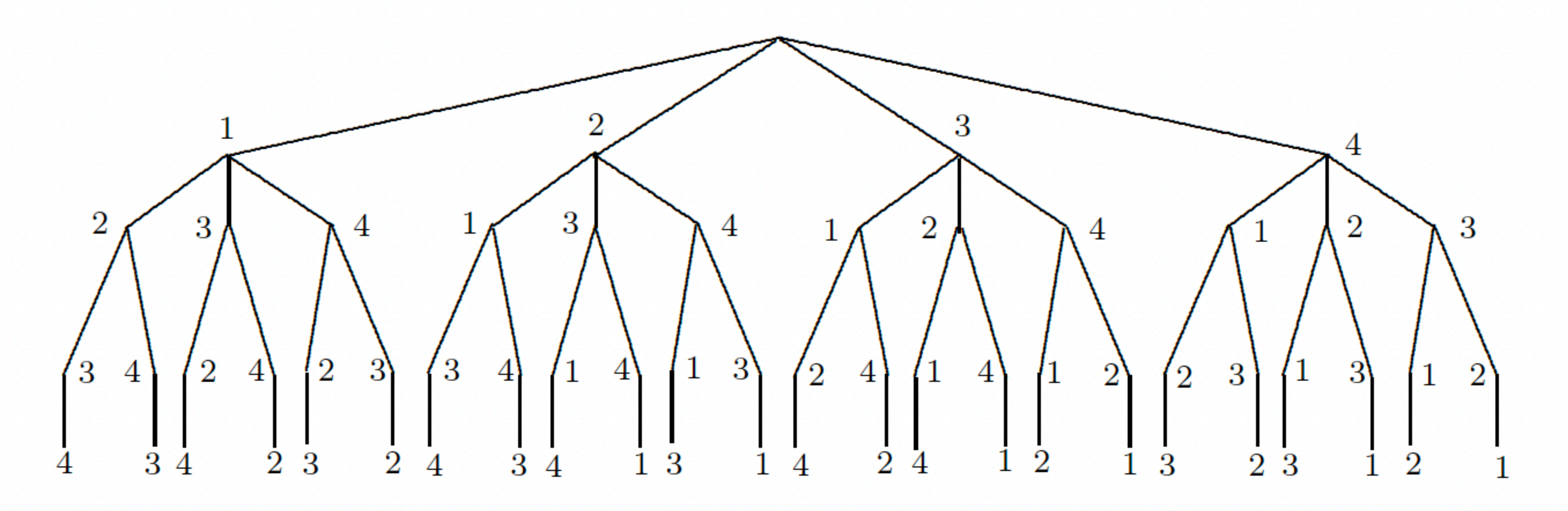
  Solution. It sometimes helps to begin by listing some of the possibilities. The code consists of an uppercase letter followed by a digit, so some possibilities are A1, A2, B3, X0, and so on. The task consists of making two selections: The first selection requires choosing an uppercase letter (26 choices), and the second task requires choosing a digit (10 choices). By the Multiplication Principle, there are $26 \cdot 10 = 260$ different code words of the type described.

Permutations

$\stackrel{\text{def}}{=}$ What is a Permutation?

- A **permutation** is an ordered arrangement of *r* objects chosen from *n* objects.



Three Types of Permutations

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them.
[Distinct, with repetition]

2. The n objects are distinct (different), and repetition is not allowed in the selection of r of them, where $r \leq n$.
[Distinct, without repetition]

3. The n objects are not distinct, and all of them are used in the arrangement.
[Not distinct]

Example 8

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them.
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Counting Airport Codes [Permutation: Distinct, with Repetition]

The International Airline Transportation Association (IATA) assigns three-letter codes to represent airport locations. For example, the airport code for Ft. Lauderdale, Florida, is FLL. Notice that repetition is allowed in forming this code. How many airport codes are possible?

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Solution. An airport code is formed by choosing 3 letters from 26 letters and arranging them in order. In the ordered arrangement, a letter may be repeated. This is an example of a permutation with repetition in which 3 objects are chosen from 26 distinct objects. The task of counting the number of such arrangements consists of making three selections. Each selection requires choosing a letter of the alphabet (26 choices). By the Multiplication Principle, there are

$$26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$$

possible airport codes.

Permutations: Distinct Objects with Repetition

- **Theorem 5**. The number of ordered arrangements of r objects chosen from n objects, in which the n objects are distinct and repetition is allowed, is n^r.

Example 9

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them.
[Distinct, with repetition]

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[Distinct, without repetition]

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[Not distinct]

Forming Codes [Permutation: Distinct, without Repetition]

Suppose that a three-letter code is to be formed using any of the 26 uppercase letters of the alphabet, but no letter is to be used more than once. How many different three-letter codes are there?

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1. The n objects are distinct (different), and repetition is allowed in the selection of r of them.
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Forming Codes [Permutation: Distinct, without Repetition]

Suppose that a three-letter code is to be formed using any of the 26 uppercase letters of the alphabet, but no letter is to be used more than once. How many different three-letter codes are there?

Solution. Some of the possibilities are ABC, ABD, ABZ, ACB, CBA, and so on. The task consists of making three selections. The first selection requires choosing from 26 letters. Since no letter can be used more than once, the second selection requires choosing from 25 letters. The third selection requires choosing from 24 letters. (Do you see why?) By the Multiplication Principle, there are

$$26 \cdot 25 \cdot 24 = 15{,}600$$

different three-letter codes with no letter repeated.

Notation

- The notation $\boldsymbol{P(n,r)}$ represents the number of ordered arrangements of r objects chosen from n distinct objects, where $r \leq n$ and repetition is not allowed.

Example 10

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them.
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Lining People Up [Permutation: Distinct, without Repetition]

In how many ways can 5 people be lined up?

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[Not distinct]

Lining People Up [Permutation: Distinct, without Repetition]

In how many ways can 5 people be lined up?

Solution. The 5 people are distinct. Once a person is in line, that person will not be repeated elsewhere in the line; and, in lining people up, order is important. This is a permutation of 5 objects taken 5 at a time, so 5 people can be lined up in

$$P(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$$

ways.

Permutations of r Objects Chosen from n Distinct Objects without Repetition

- **Theorem 6**. The number of arrangements of n objects using $r \leq n$ of them, in which
  1. the n objects are distinct,
  2. once an object is used it cannot be repeated, and
  3. order is important,
  is given by the formula
$$P(n, r) = \frac{n!}{(n-r)!}.$$

Permutations Involving *n* Objects That Are Not Distinct

- **Theorem 7**. The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, … and n_k are of a kth kind is given by

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$$

where $n = n_1 + n_2 + \cdots + n_k$.

Example 11

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them.
[Distinct, with repetition]

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[Distinct, without repetition]

3. The n objects are not distinct, and all of them are used in the arrangement.
[Not distinct]

Arranging Flags [Permutation: Not Distinct]

How many different vertical arrangements are there of 8 flags if 4 are white, 3 are blue, and 1 is red?

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Arranging Flags [Permutation: Not Distinct]

How many different vertical arrangements are there of 8 flags if 4 are white, 3 are blue, and 1 is red?

Solution. We seek the number of permutations of 8 objects, of which 4 are of one kind, 3 are of a second kind, and 1 is of a third kind. We find that there are

$$\frac{8!}{4! \cdot 3! \cdot 1!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3! \cdot 1!} = 280$$

different arrangements

Example 12

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[Distinct, with repetition]

2. The n objects are distinct (different), and repetition is not allowed in the selection of r of them, where $r \leq n$.
[Distinct, without repetition]

3. The n objects are not distinct, and all of them are used in the arrangement.
[Not distinct]

Forming Different Words [Permutation: Not Distinct]

How many different words (real or imaginary) can be formed using all the letters in the word REARRANGE?

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3. The n objects are not distinct, and all of them are used in the arrangement.
[Not distinct]

Forming Different Words [Permutation: Not Distinct]

How many different words (real or imaginary) can be formed using all the letters in the word REARRANGE?

Solution. Each word formed will have 9 letters: 3 R's, 2 A's, 2 E's, 1 N and 1 G. We find that there are

$$\frac{9!}{3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} = 15{,}120$$

different arrangements

Combinations

What is a Combination?

- Let S be an n-element set. A subset of S of cardinality r, where $0 \leq r \leq n$, is called an r-**combination** of S.
- The number of r-combinations is denoted by $\boldsymbol{C(n,r)}$.

Number of Combinations of n Distinct Objects Taken r at a Time

- **Theorem 8**. The number of arrangements of n objects using $r \leq n$ of them, in which
  1. the n objects are distinct,
  2. once an object is used it cannot be repeated, and
  3. order is **not** important,
  is given by the formula

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Example 13

Forming Committees I

How many different committees of 3 people can be formed from a pool of 7 people?

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Forming Committees I
How many different committees of 3 people can be formed from a pool of 7 people?

Solution. The 7 people are distinct. More important, though, is the observation that the order of being selected for a committee is not significant. The problem asks for the number of combinations of 7 objects taken 3 at a time.

$$C(7,3) = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{6} = 35$$

Thirty-five different committees can be formed.

Example 14

Forming Committees II

In how many ways can a committee consisting of 2 faculty members and 3 students be formed if 6 faculty members and 10 students are eligible to serve on the committee?

Example 14

Forming Committees II

In how many ways can a committee consisting of 2 faculty members and 3 students be formed if 6 faculty members and 10 students are eligible to serve on the committee?

Solution. The problem can be separated into two parts: the number of ways in which the faculty members can be chosen, $C(6,2)$, and the number of ways in which the student members can be chosen, $C(10,3)$. By the Multiplication Principle, the committee can be formed in

$$C(6,2)\cdot C(10,3)=\frac{6!}{2!4!}\cdot\frac{10!}{3!7!}=1800$$

ways.

Thank you!