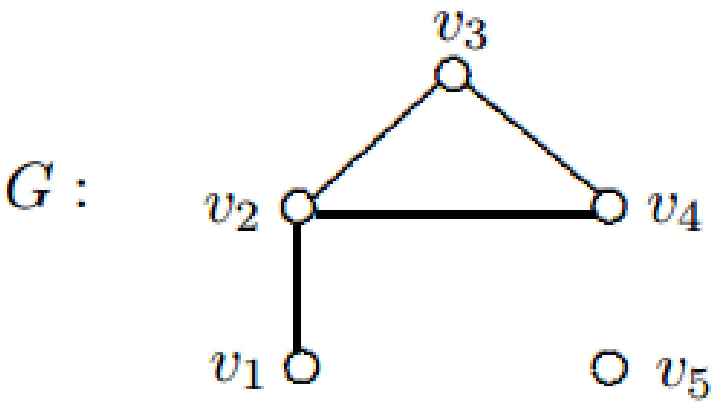
Lecture 11. Graphs

DR. YARASLAU ZADVORNY

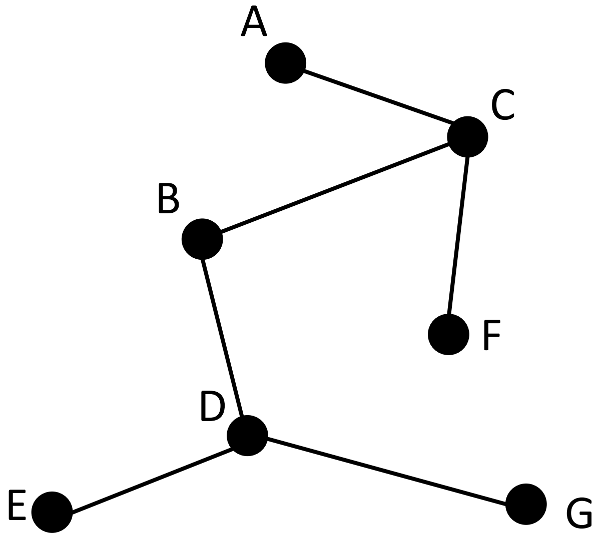
Ways to define a graph: the Adjacency Matrix

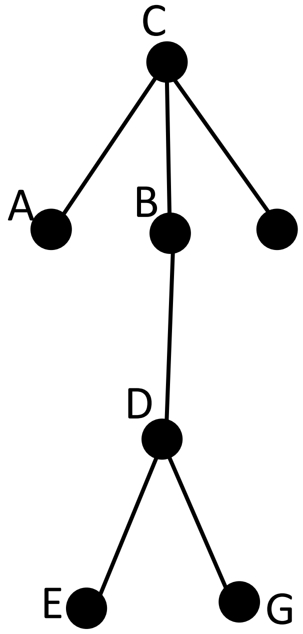


$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix: Disadvantages

	A	B	C	D	E	F	G
A	0	0	1	0	0	0	0
B	0	0	1	1	0	0	0
C	1	1	0	0	0	1	0
D	0	1	0	0	1	0	1
E	0	0	0	1	0	0	0
F	0	1	0	0	0	0	0
G	0	0	0	1	0	0	0





Adjacency Matrix: Disadvantages

$A : \{C\}$

$B : \{C, D\}$

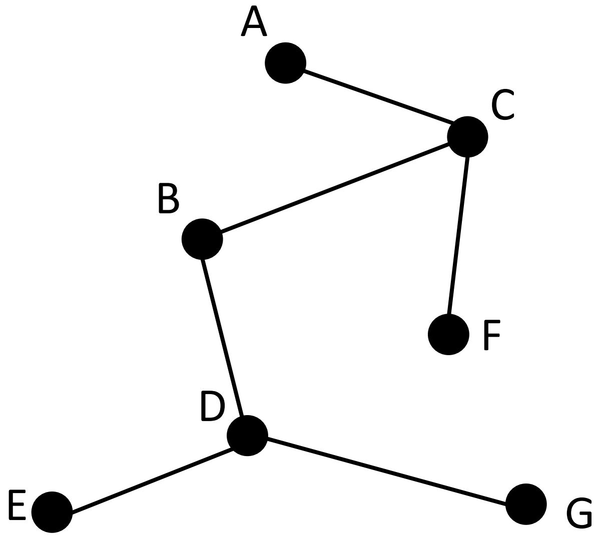
$C : \{A, B, F\}$

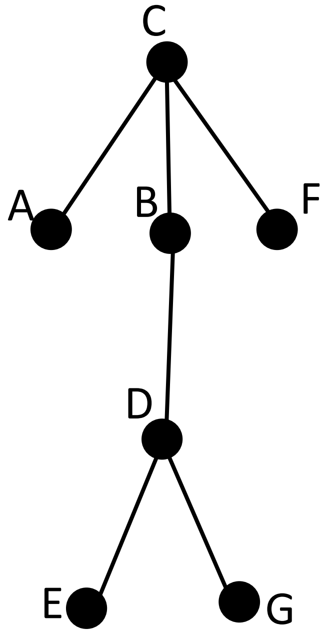
$D : \{B, E, G\}$

$E : \{D\}$

$F : \{C\}$

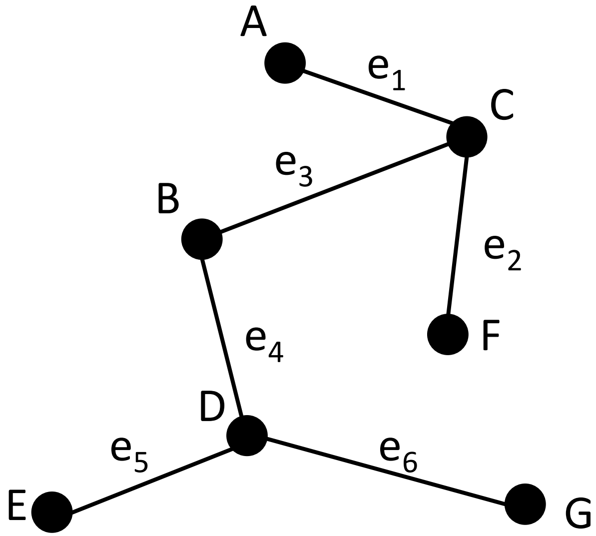
$G : \{D\}$

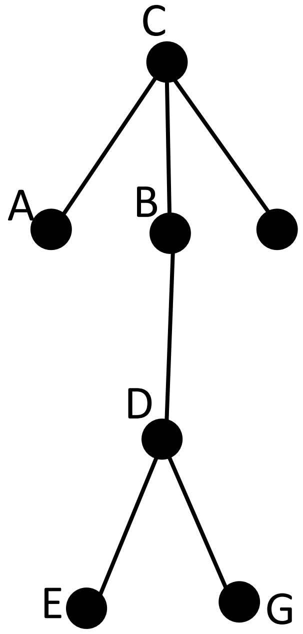




Adjacency Matrix: Disadvantages

$$
\begin{array}{c|ccccccc}
 & A & B & C & D & E & F & G \\
\hline
e_1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
e_2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
e_3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
e_4 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
e_5 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
e_6 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
$$





Adjacency Matrix: Disadvantages

$$e_1 = \{A, C\}$$

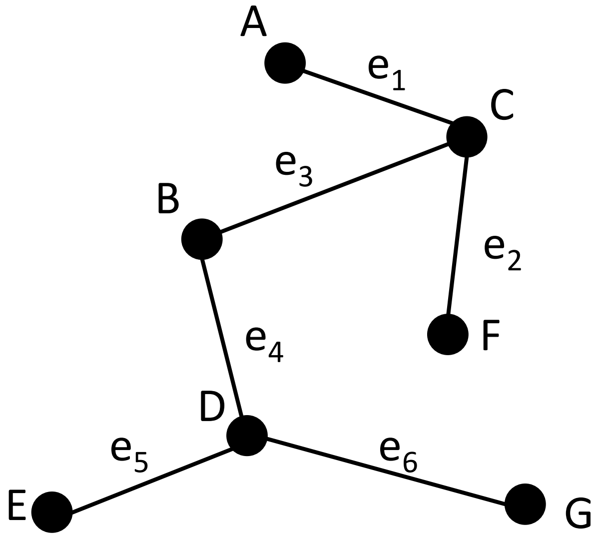
$$e_2 = \{C, F\}$$

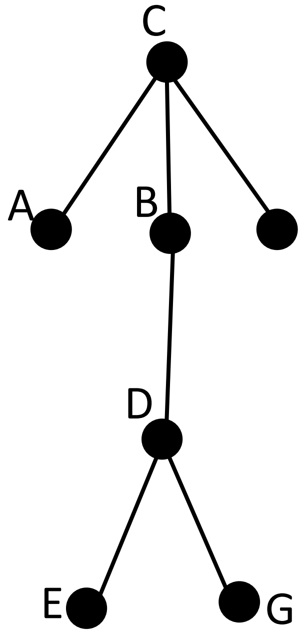
$$e_3 = \{B, C\}$$

$$e_4 = \{B, D\}$$

$$e_5 = \{D, E\}$$

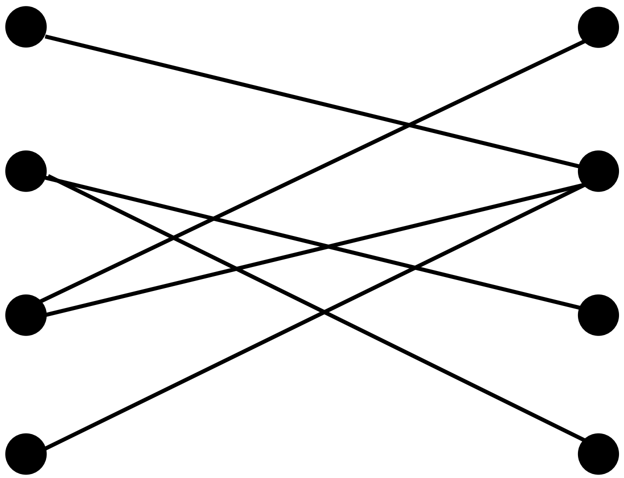
$$e_6 = \{D, G\}$$





Bipartite Graphs

A graph is a **bipartite graph** if the set of its vertices can be dissected into two nonintersecting parts, V_1 and V_2, in such a way that there are no two adjacent vertices in the same part:



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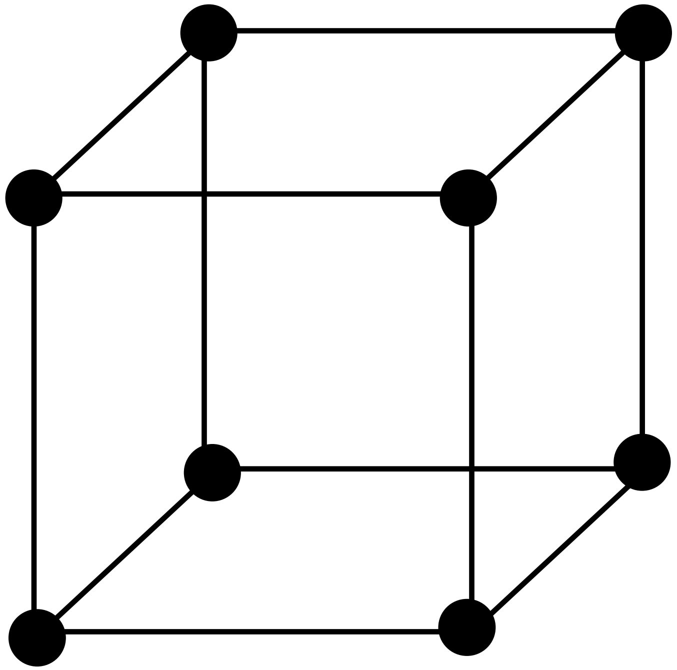

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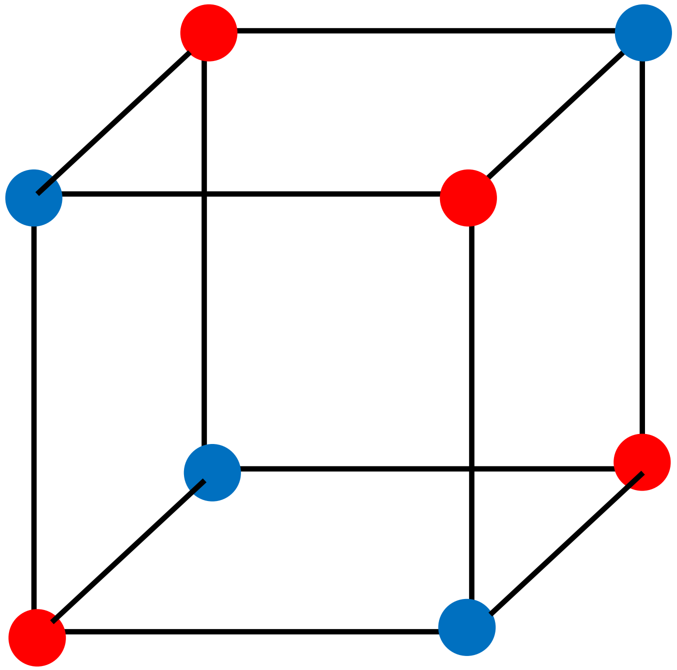
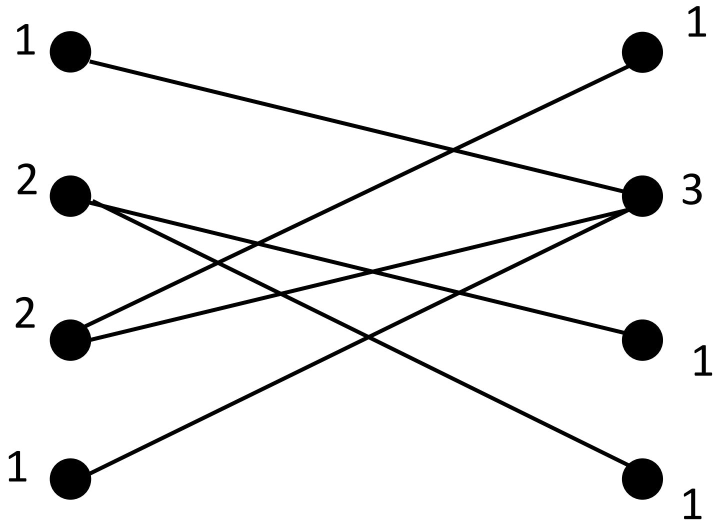


Properties of Bipartite Graphs

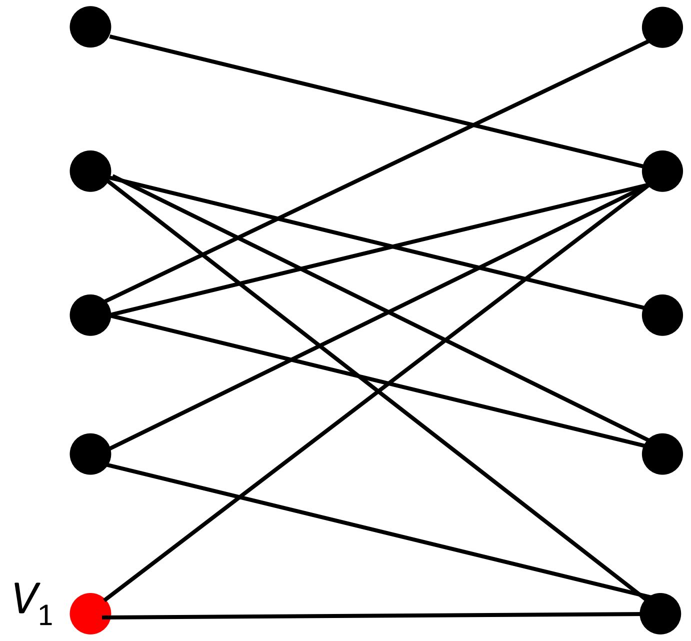
The sum of the degrees of the vertices in each of the parts is the same:



1 + 2 + 2 + 1 = 6 = 1 + 3 + 1 + 1

Properties of Bipartite Graphs

A graph is bipartite if and only if doesn't contain cycles of odd length.



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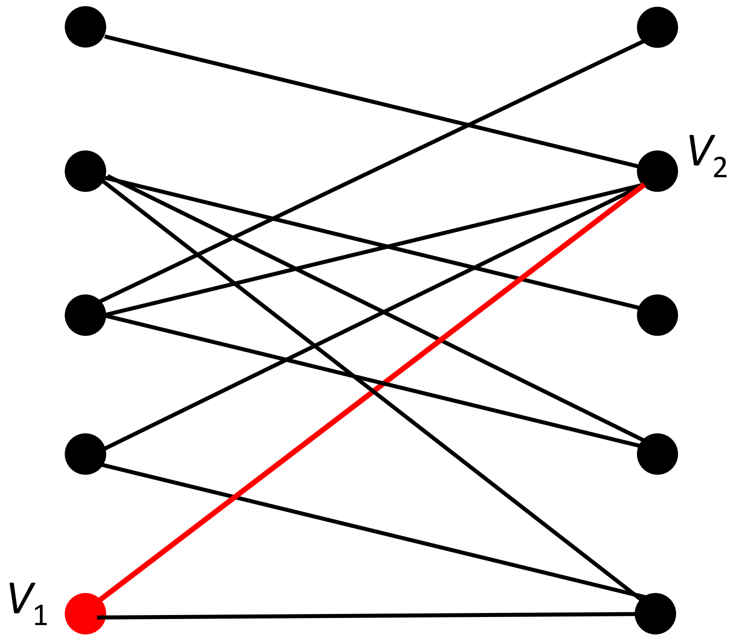

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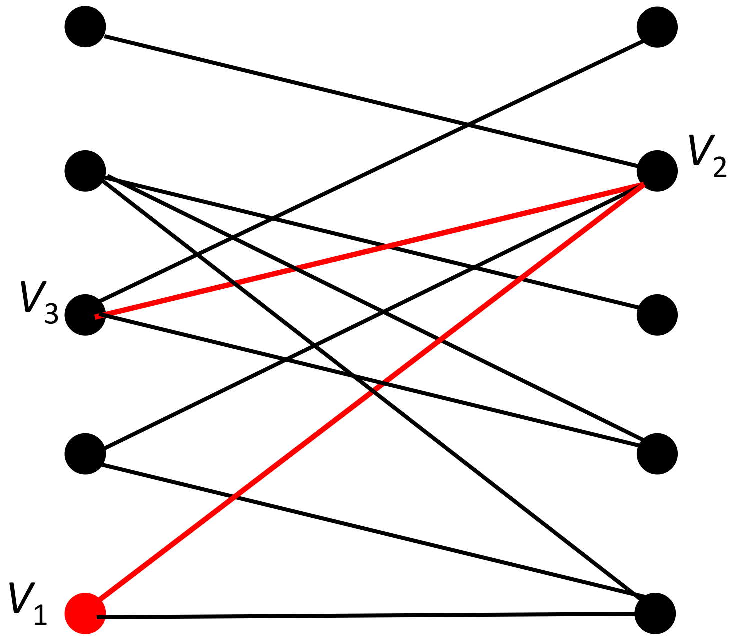

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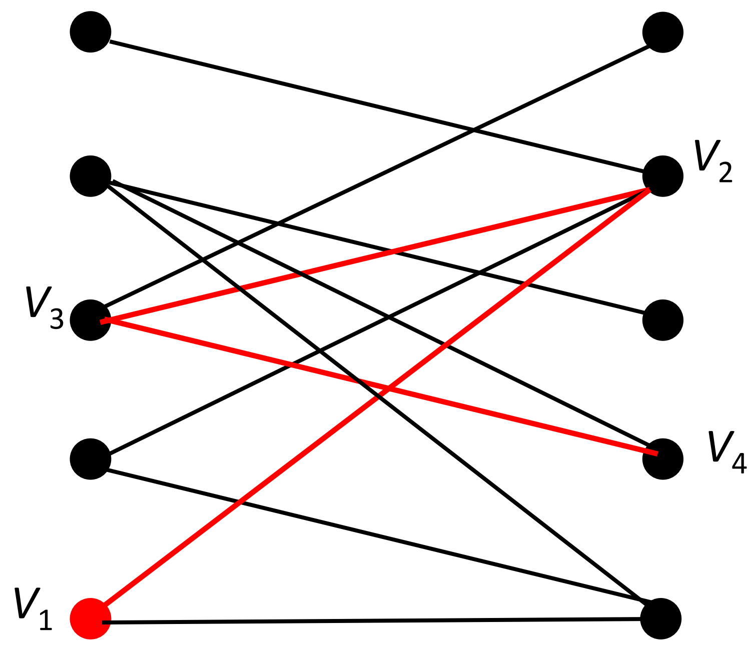

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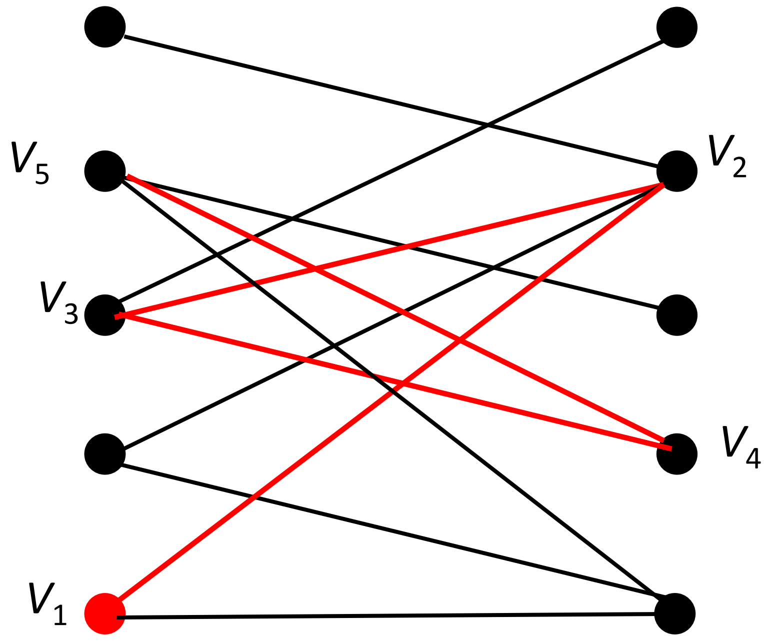

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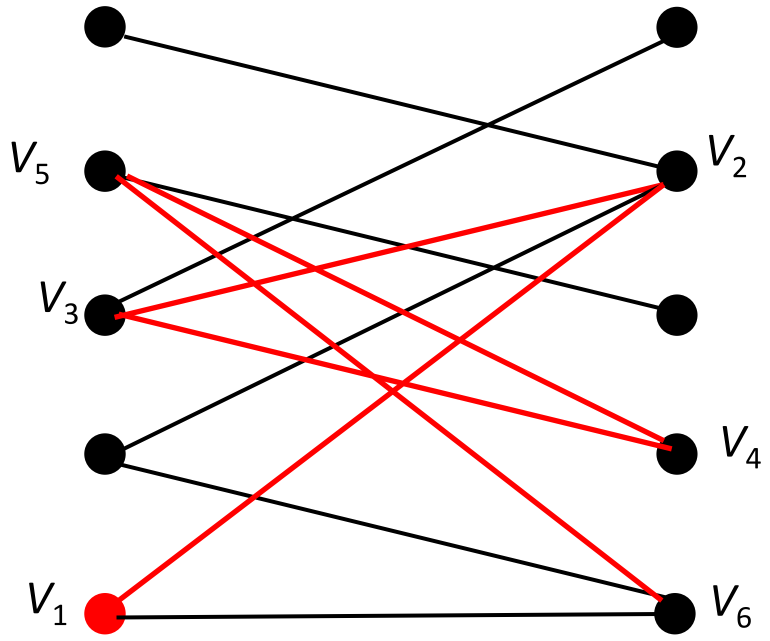

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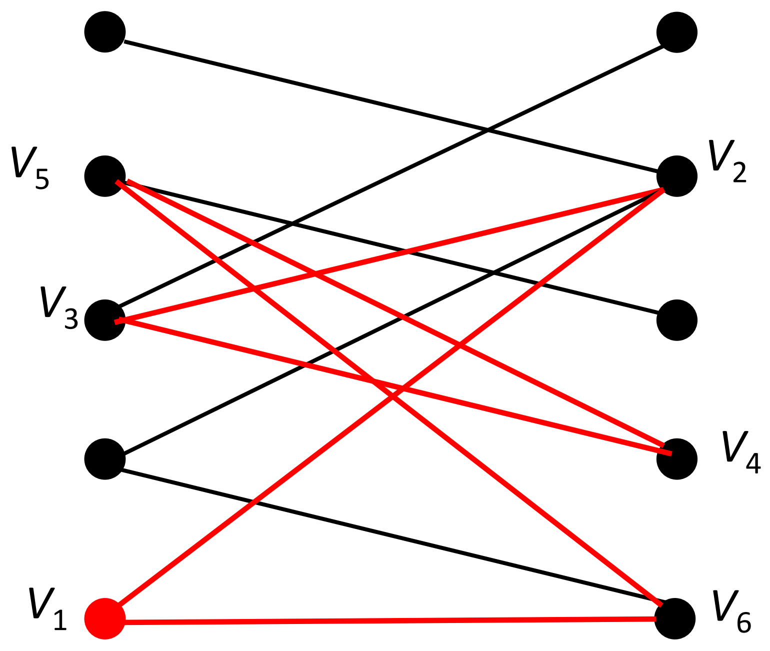

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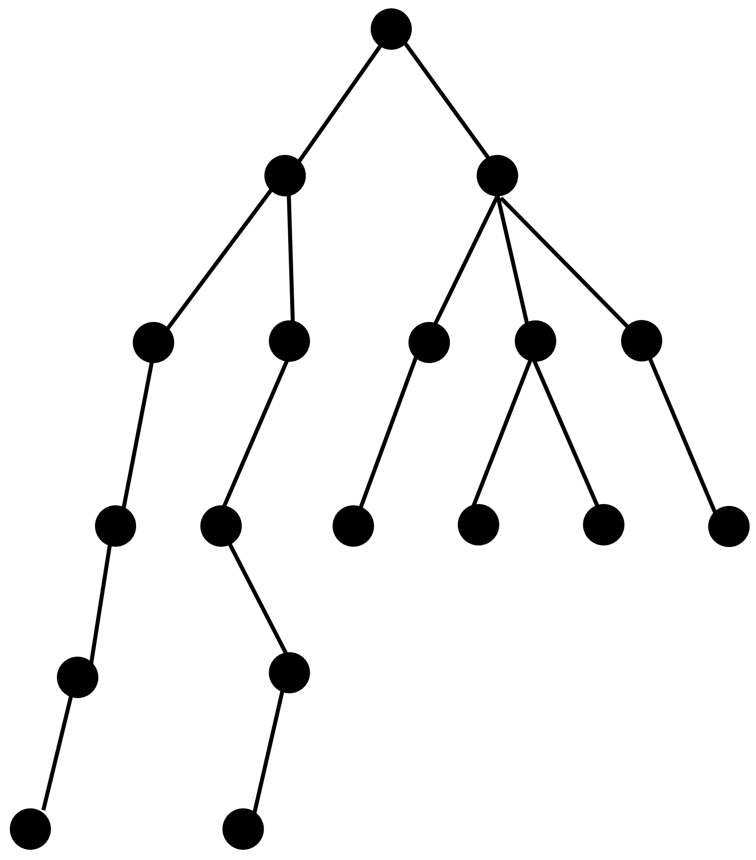

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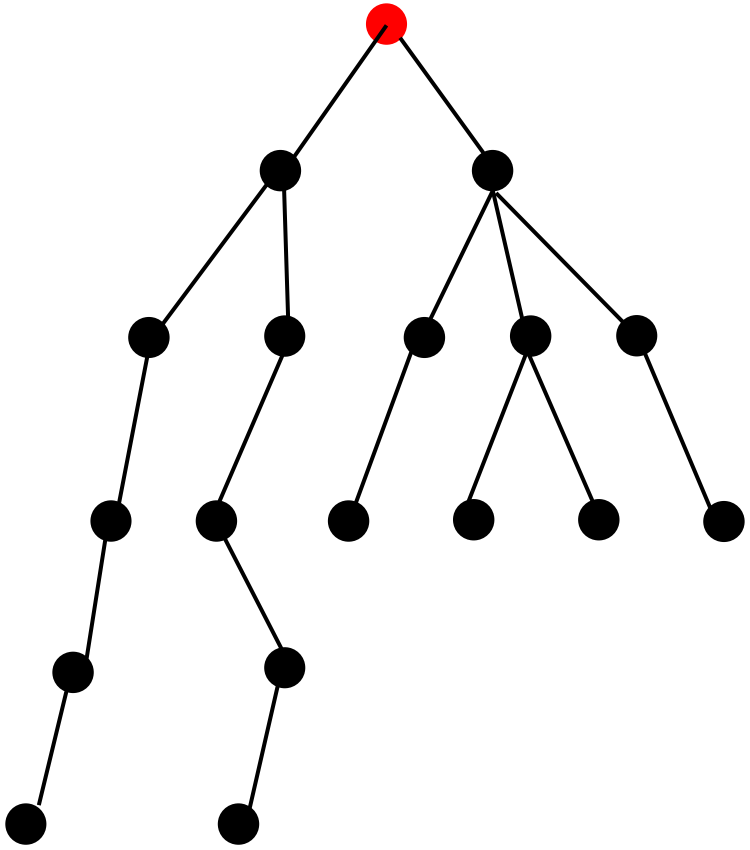

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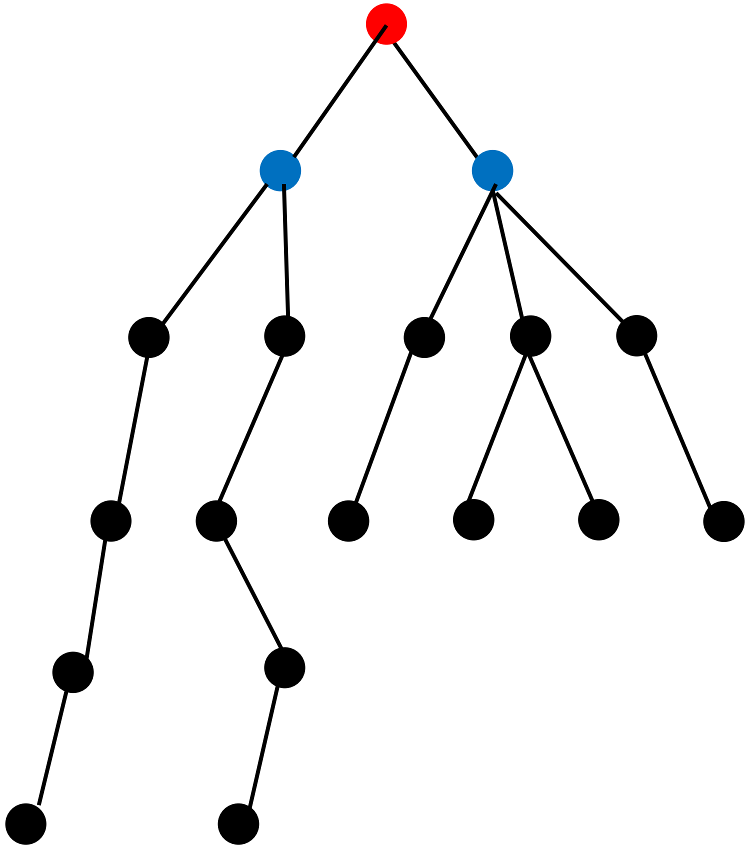

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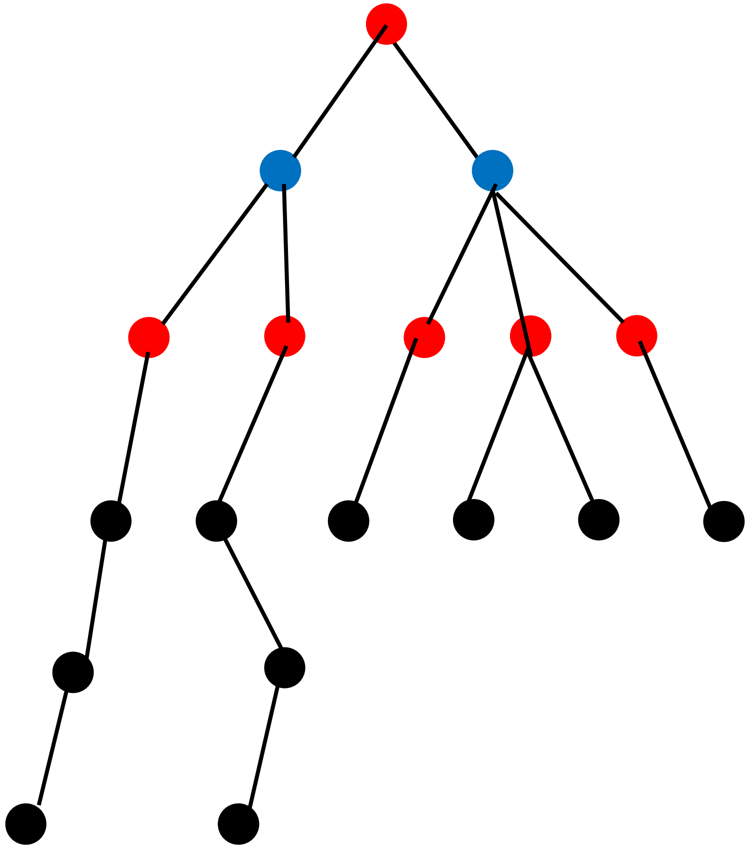

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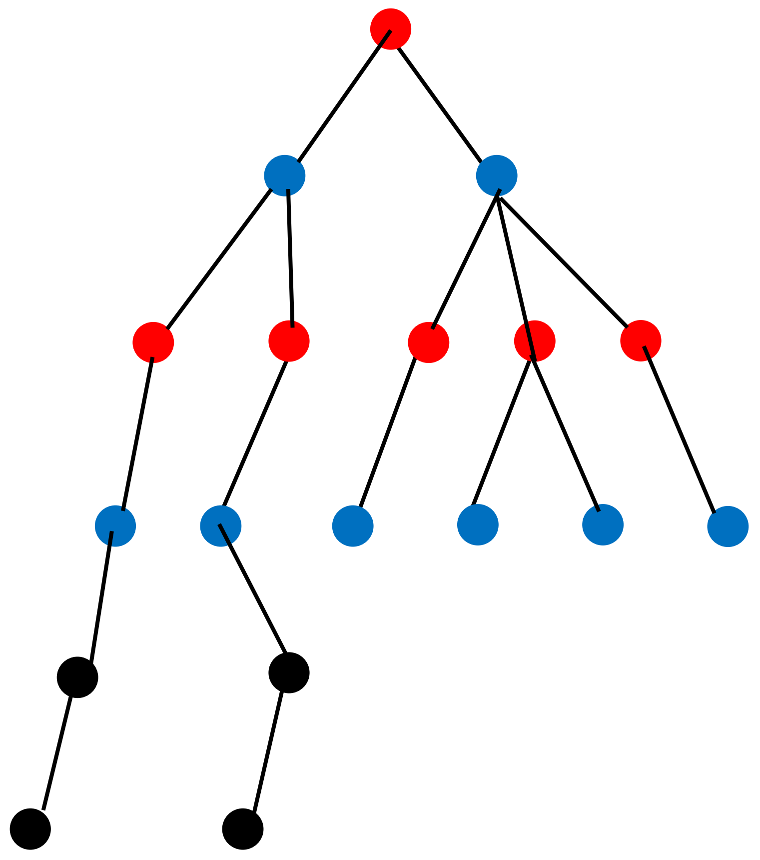

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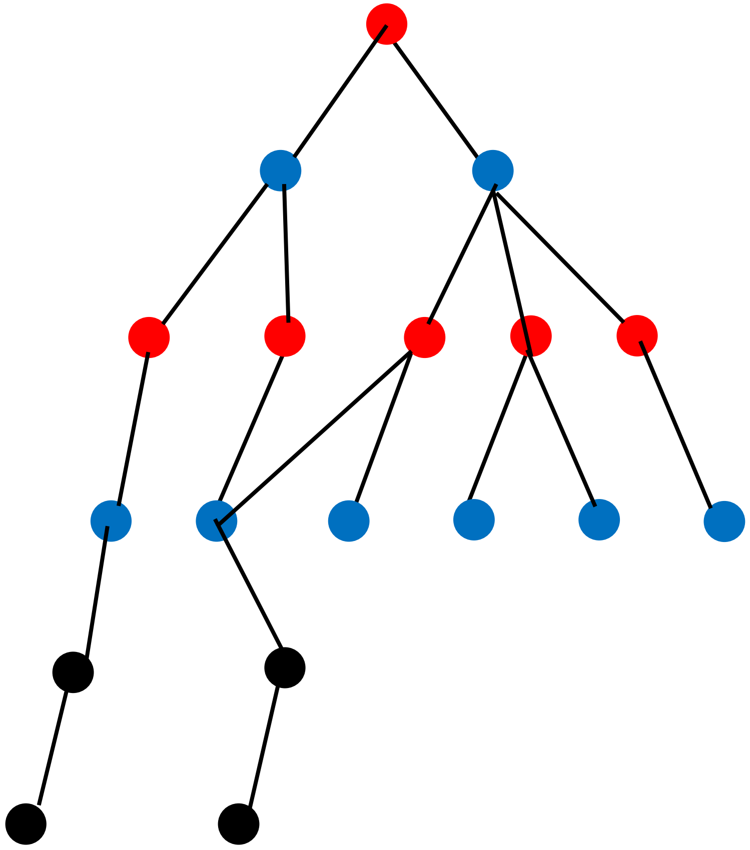

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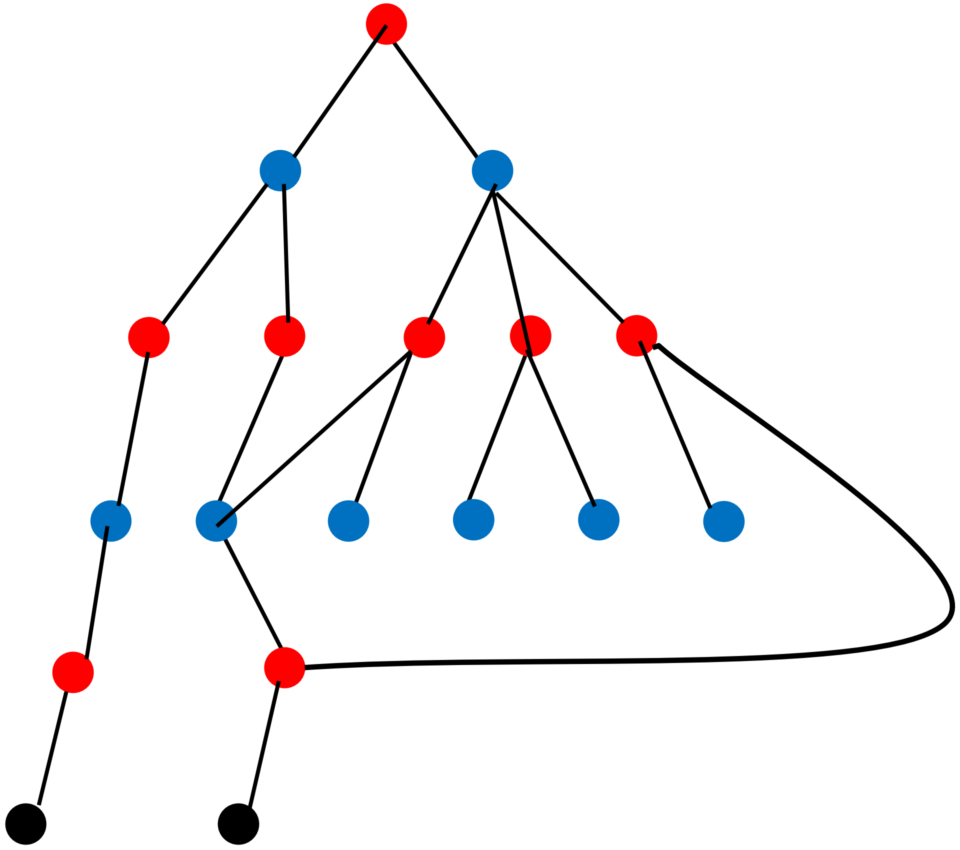

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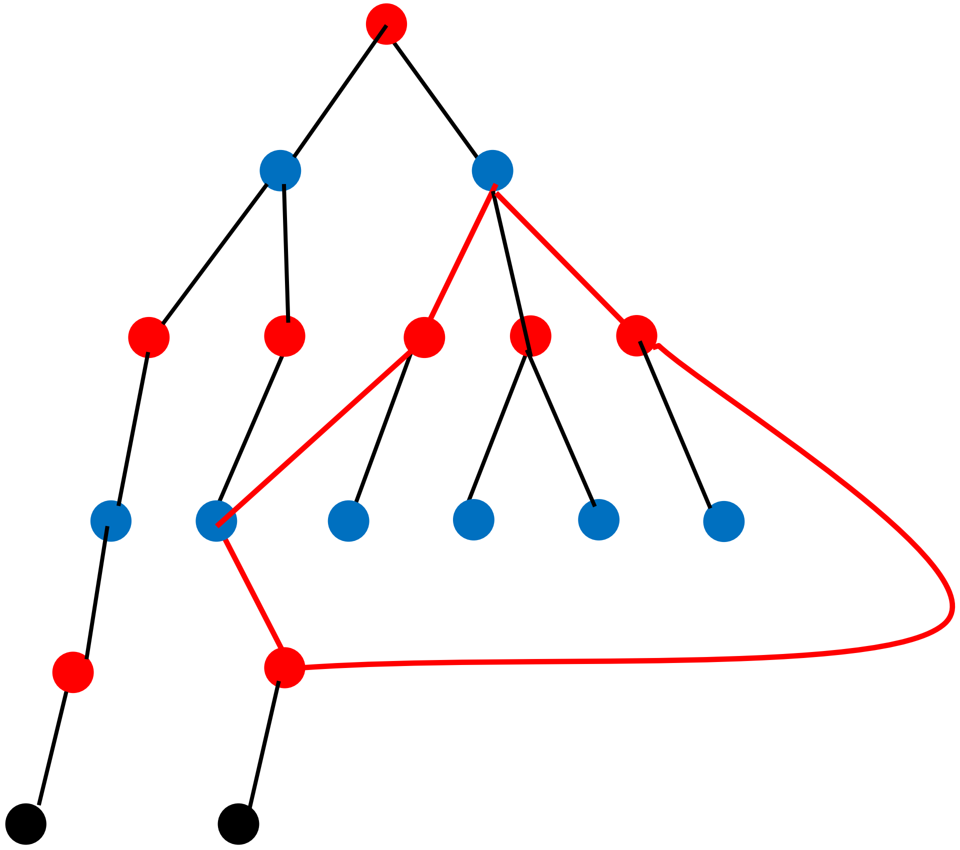
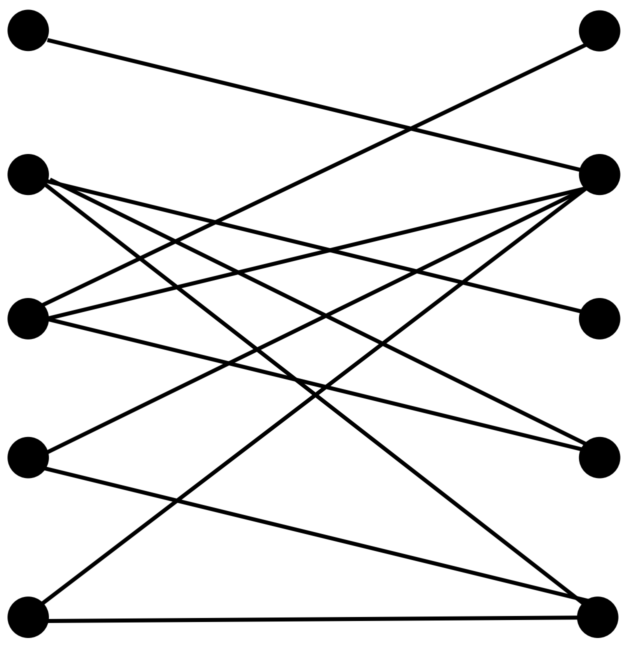


A Problem

In a school grade, each boy is friends with 5 girls and each girl is friends with 6 boys. Who is more in this class - boys or girls, and how many times?



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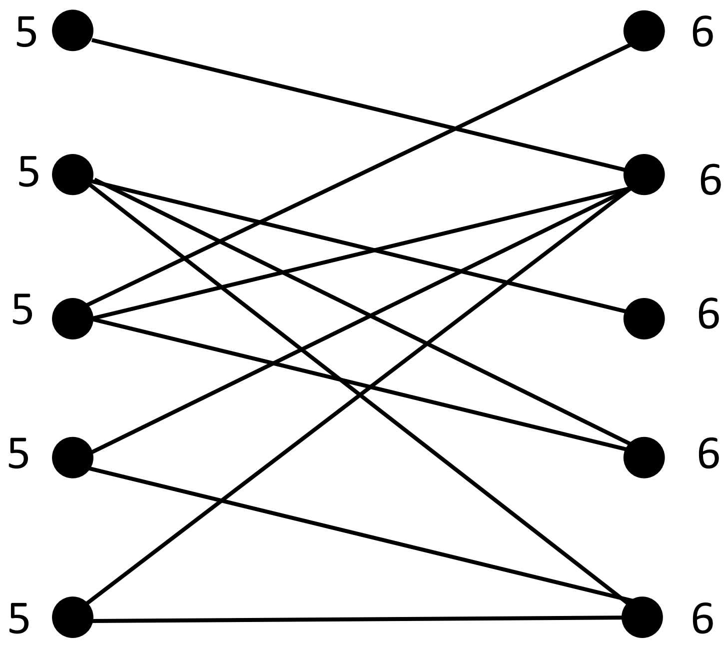


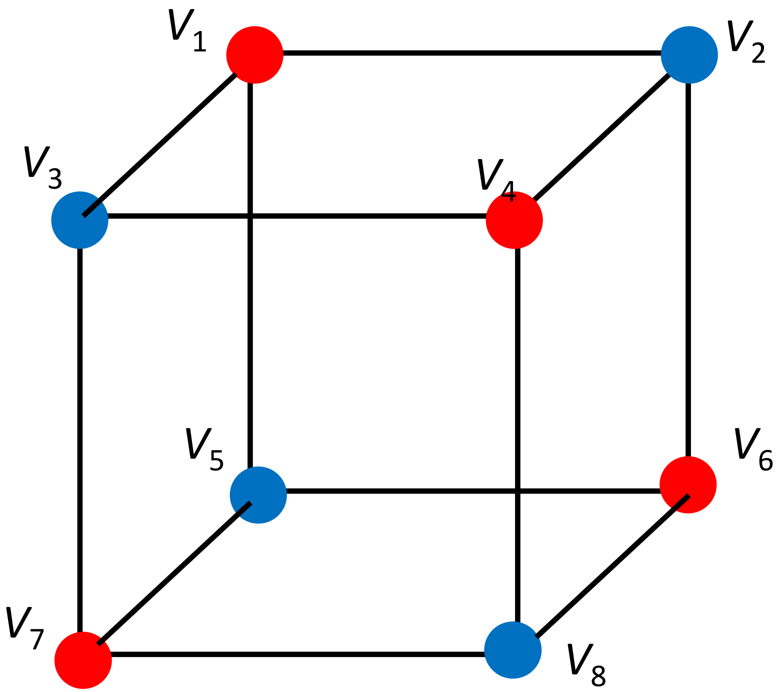
Let there are b boys and g girls. Then

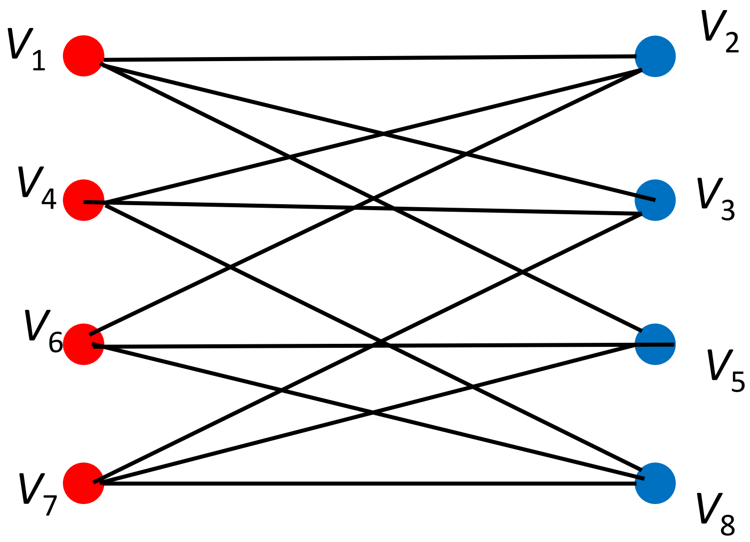
$$5b = 6g,$$

There are more boys in 6/5 times.

Isomorphic Graphs

Two graphs are isomorphic, if the number of vertices in them are equal and the vertices of both graphs can be labelled with labels $\{V_1, V_2, \ldots, V_k\}$ in such a way that the vertices Vi and Vj in the first graph are adjacent if and only if the vertices V_i and V_j are adjacent in the second graph.

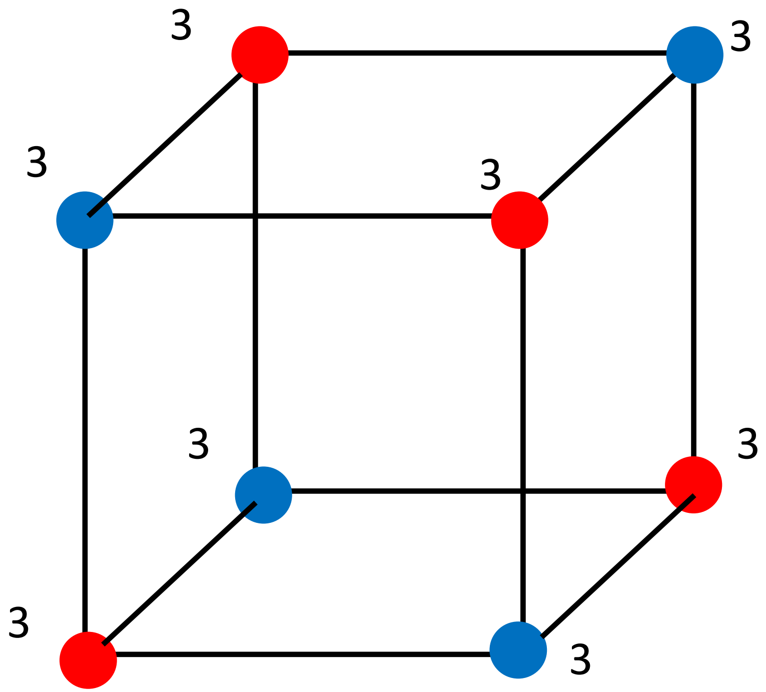


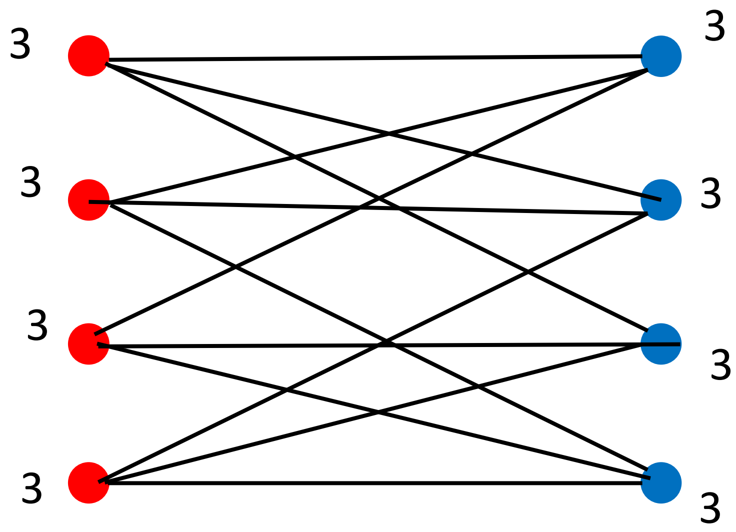


Properties of isomorphic graphs

The number of edges in two isomorphic graphs are equal.

Moreover, the sequences of degrees of vertices are also the same.





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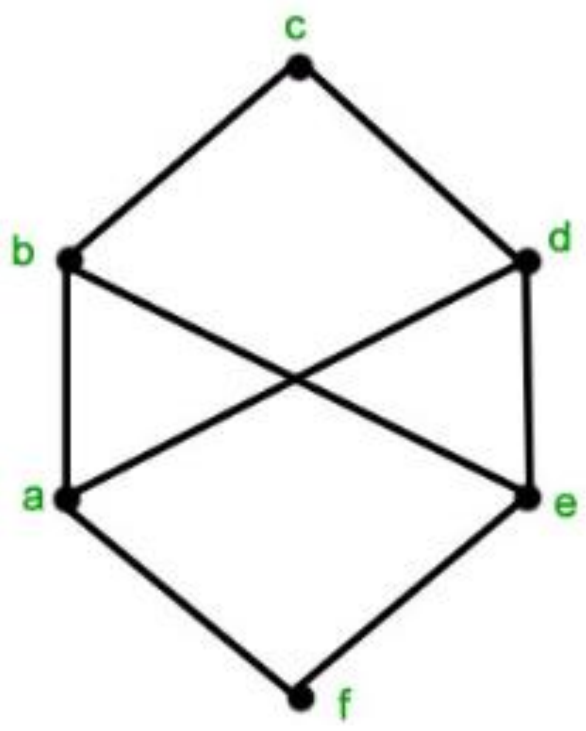

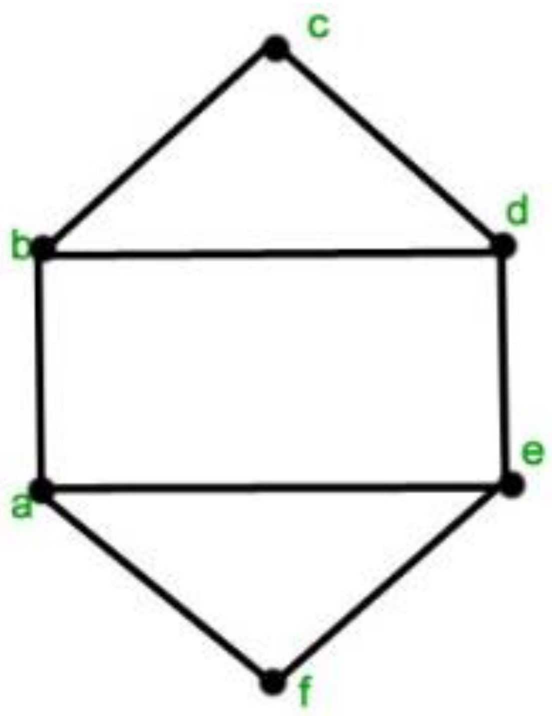

Planar graphs

A graph is planar if it can be drawn in such a way that its edges do not intersect each other.

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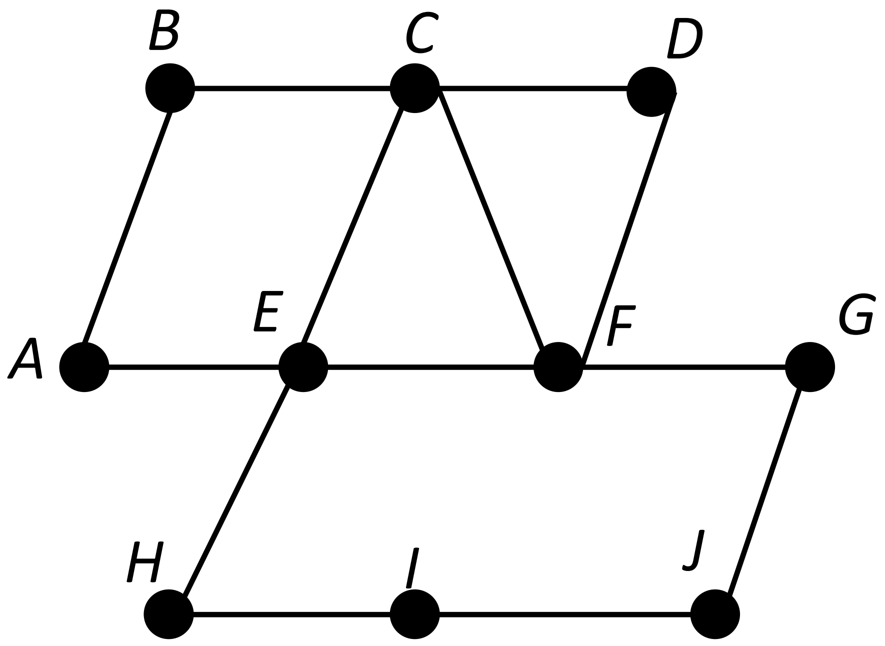

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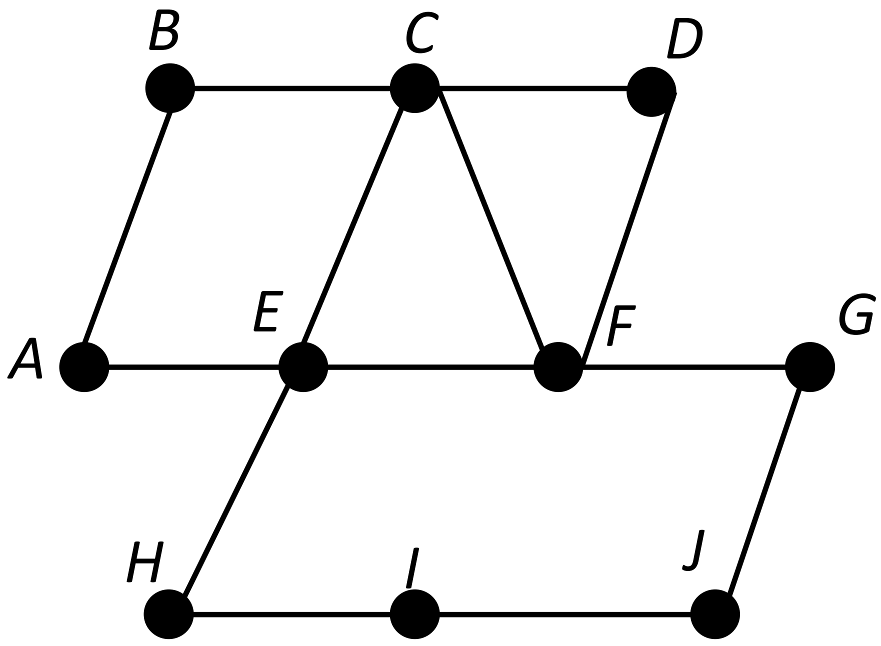

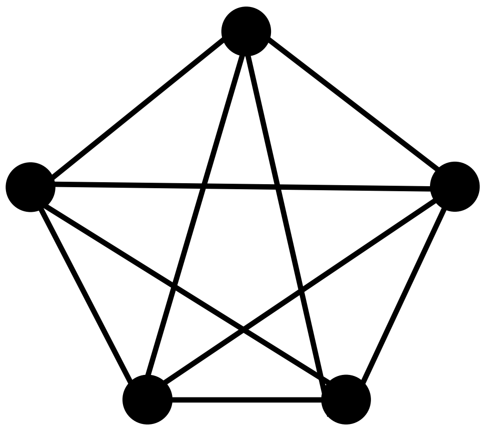

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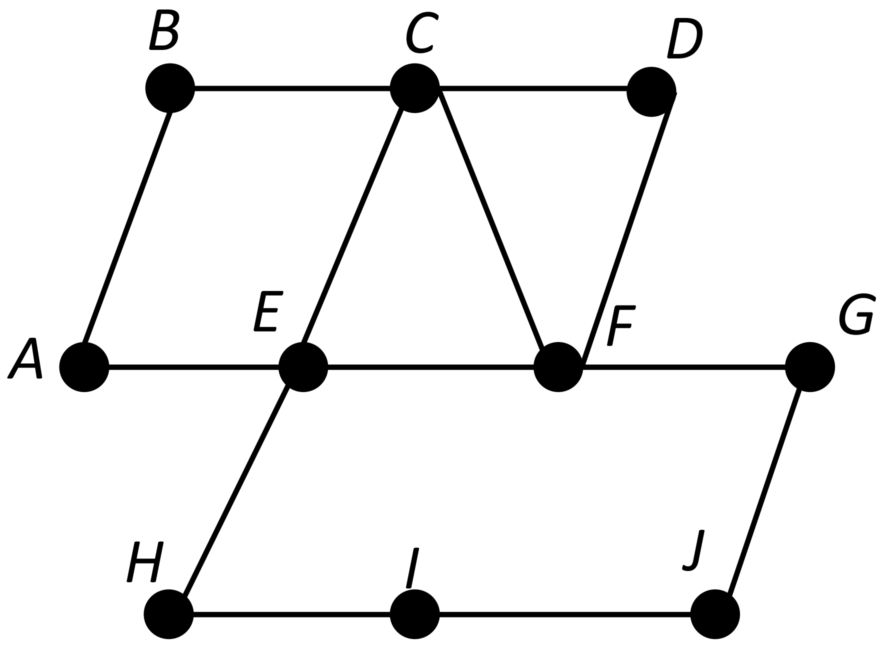

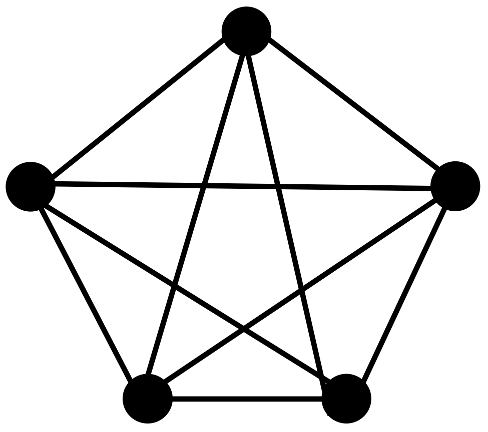

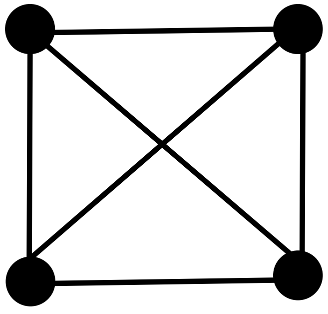

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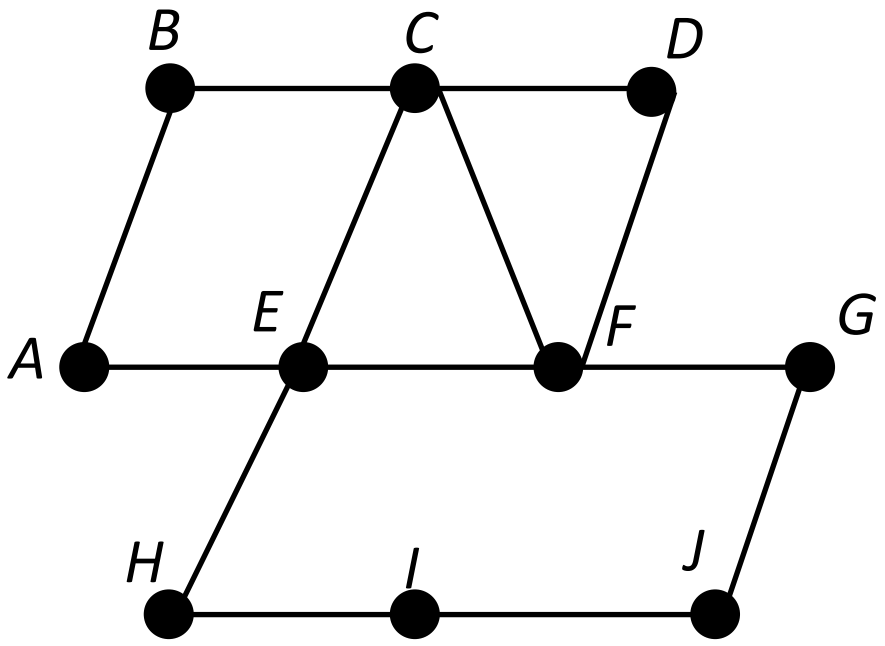

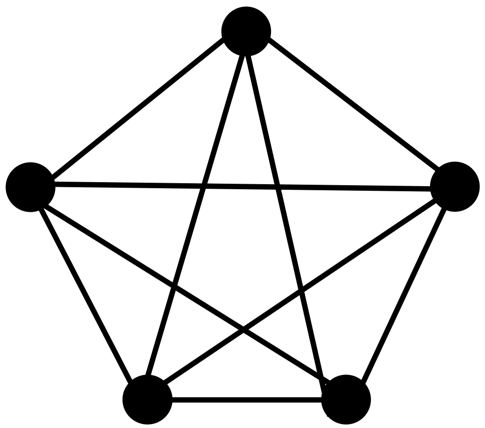

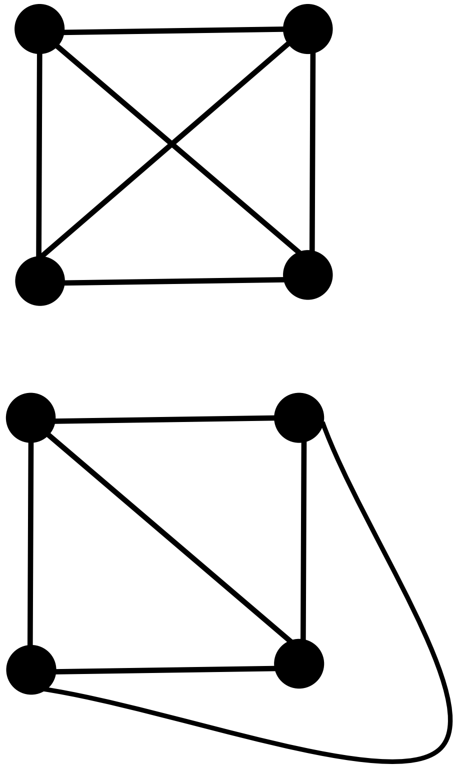

Planar graphs: Euler’s Theorem

For any connected planar graph there holds equality $V – E + F = 2$ (here V is the number of vertices, E is the number of edges, and F is the number of faces, that is, the number of parts into which the plane is dissected when this graph is drawn on it in such a way that its edges do not intersect each other).

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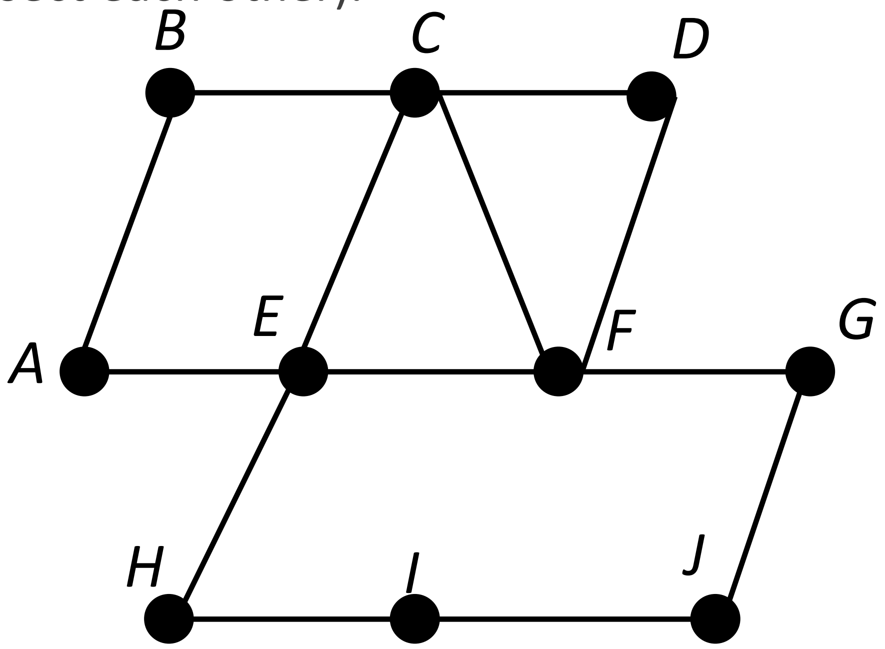

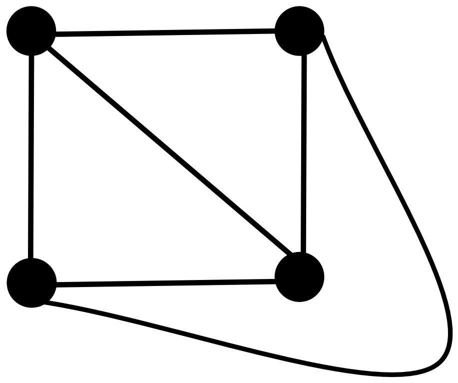

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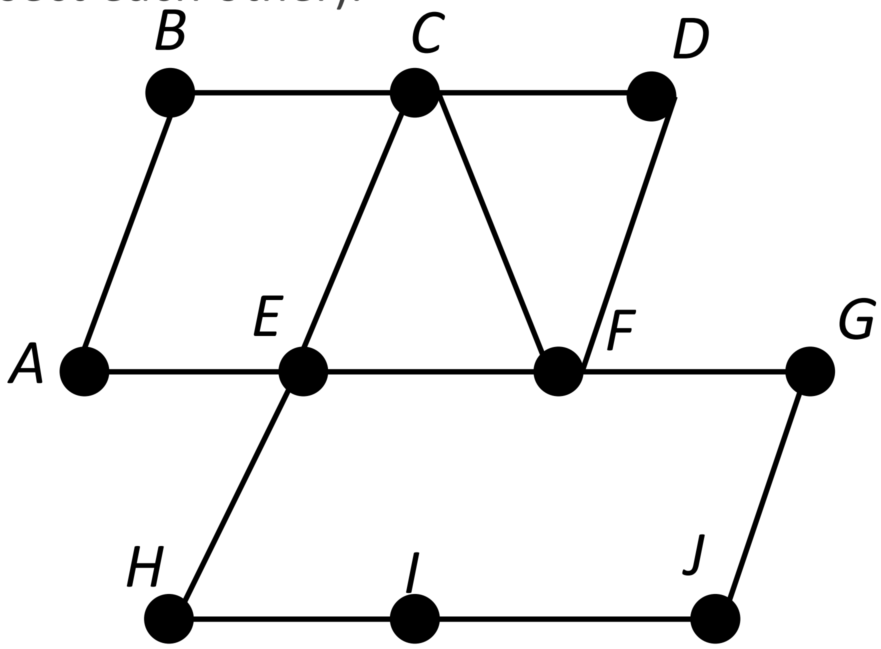

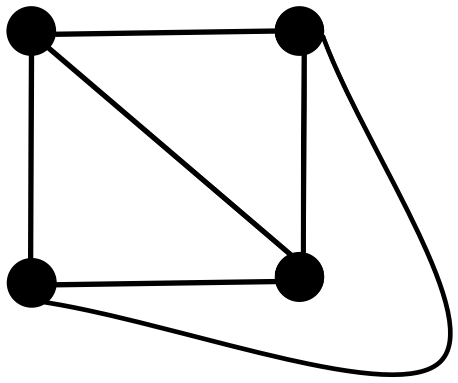

$V = 4$, $E = 6$, $F = 4$

$4 - 6 + 4 = 2$

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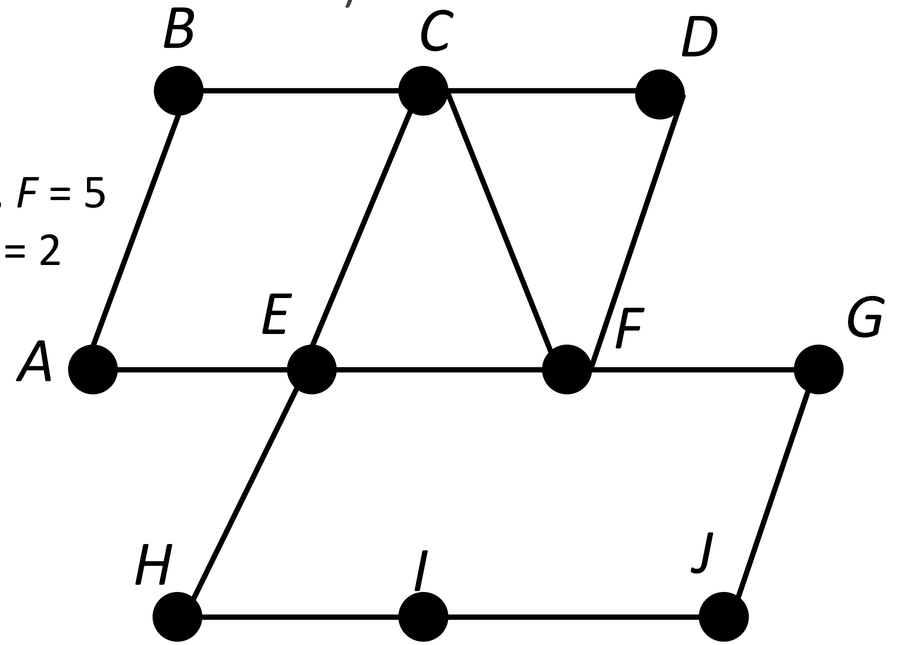
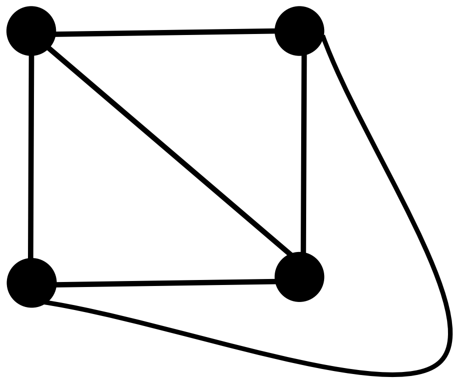
$V = 10$, $E = 13$, $F = 5$
$10 - 13 + 5 = 2$





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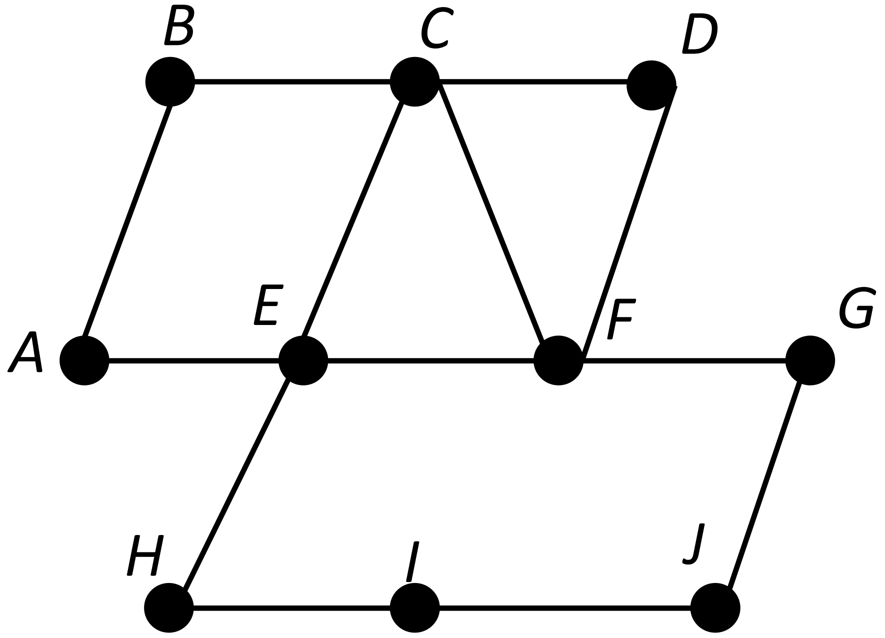

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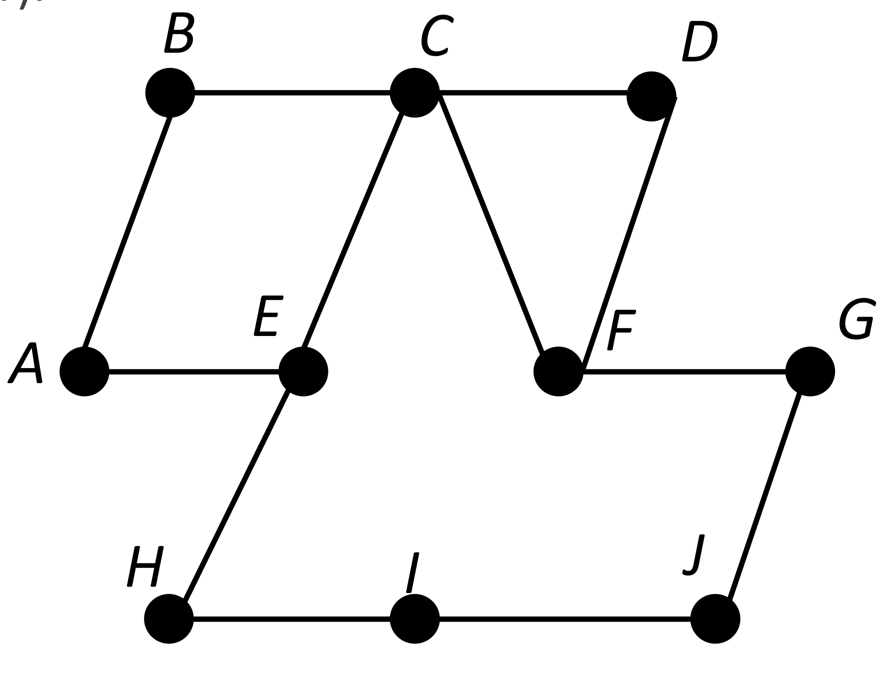

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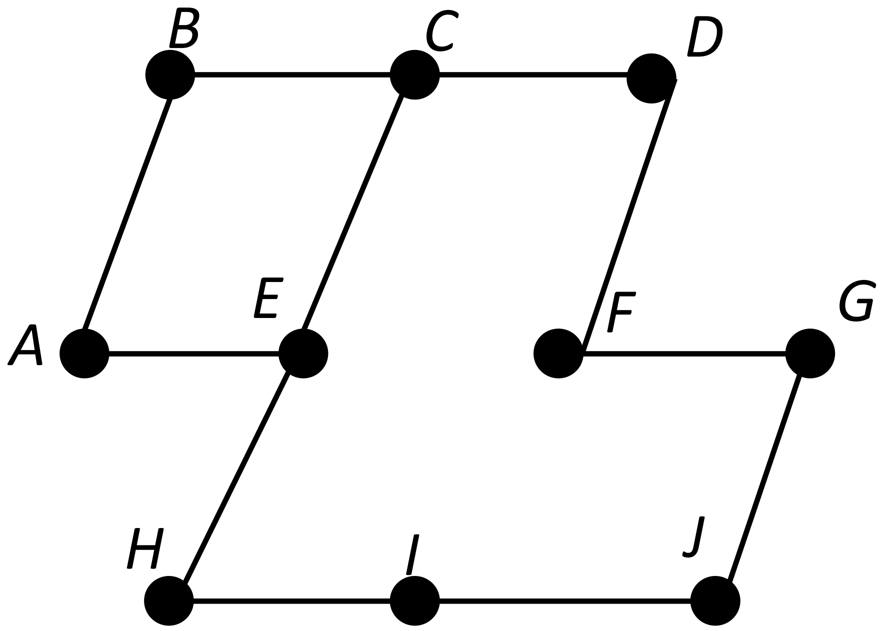

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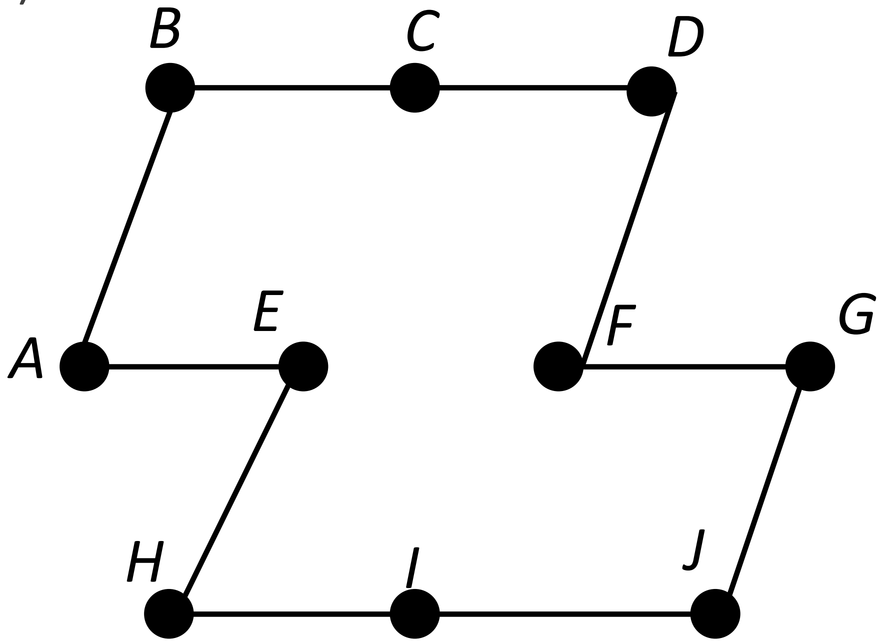

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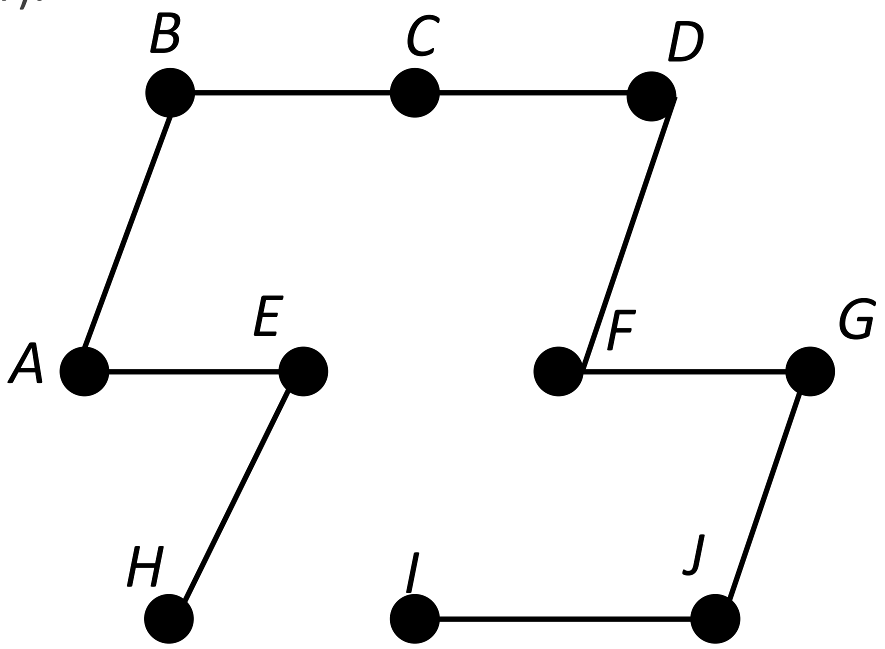
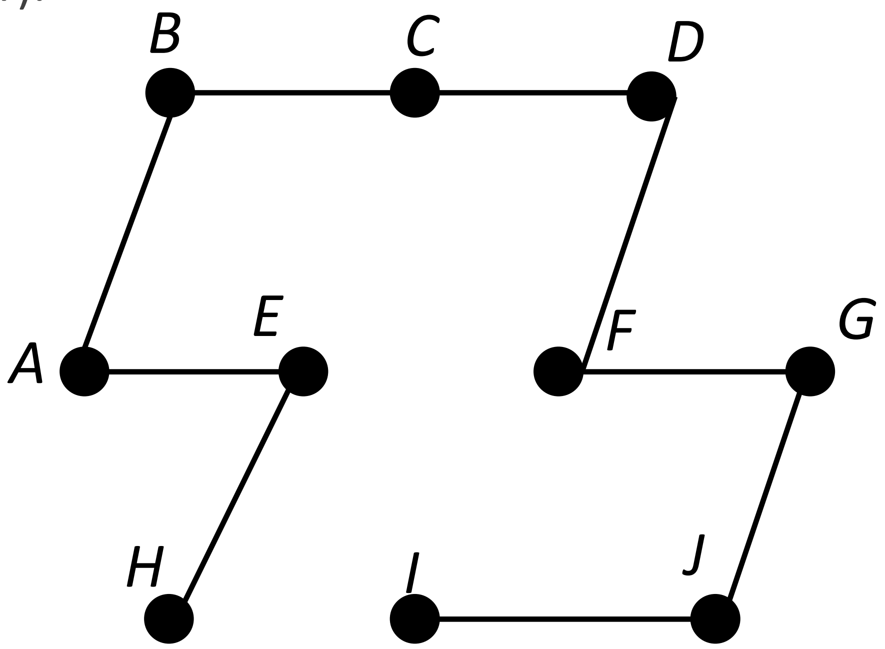


Planar graphs: Euler's Theorem

For any planar graph there holds equality V – E + F = 2 (here V is the number of vertices, E is the number of edges, and F is the number of faces, that is, the number of parts into which the plane is dissected when this graph is drawn on it in such a way that its edges do not intersect each other).



Finally, we get a tree.

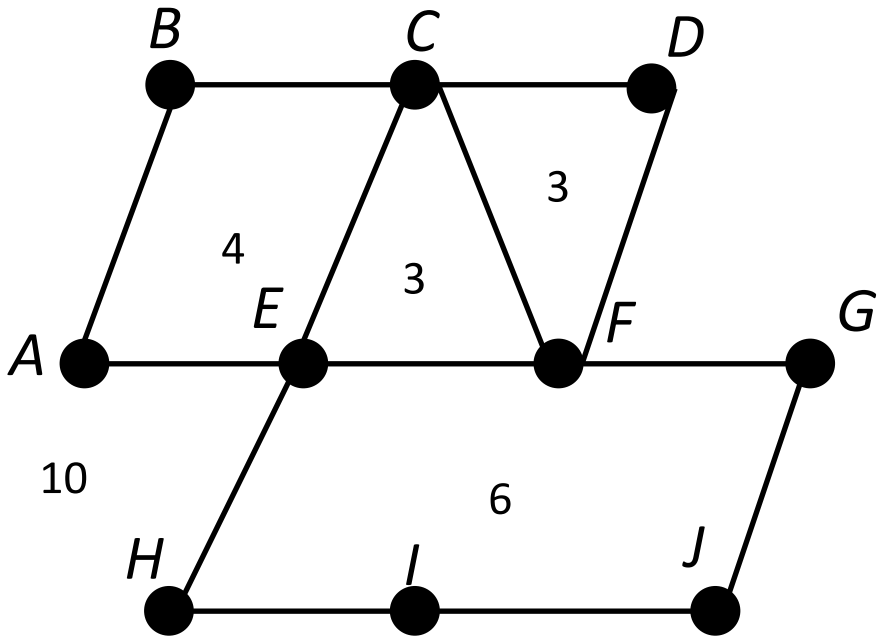
$V = n$, $E = n - 1$, $F = 1$

$n - (n - 1) + 1 = 2$

Handshaking lemma for faces

Let us say that the degree of a face is a number of edges which are adjacent to this face.

The sum of degrees of faces equals to 2E where E is the number of edges.

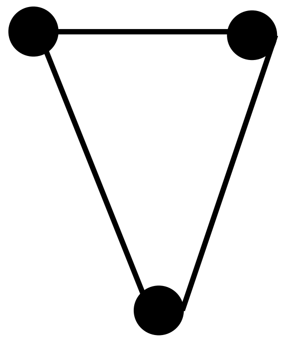


$10 + 4 + 3 + 3 + 6 = 26 = 13 \times 2$

Handshaking lemma for faces

The sum of degrees of faces equals to $2E$ where E is the number of edges.

In particular, $3F \leq 2E$.



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$$3V + 2E \geq 6 + 3E$$

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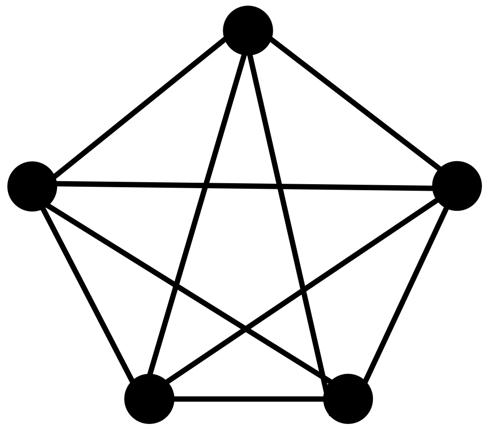
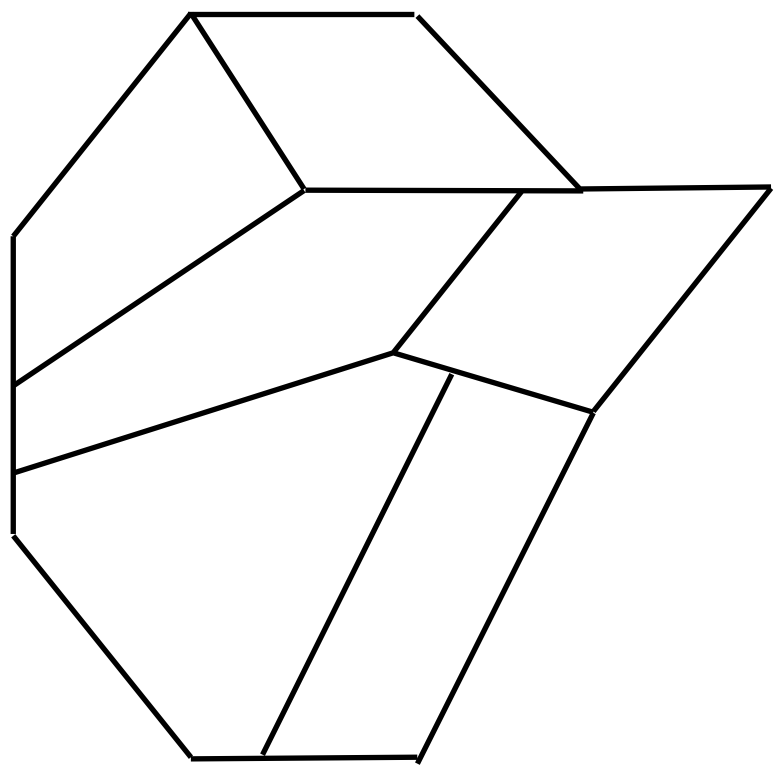


Consequense. The graph K_5 is not a planar graph because for this graph $V = 5$ and $E = 10$.

Four color theorem



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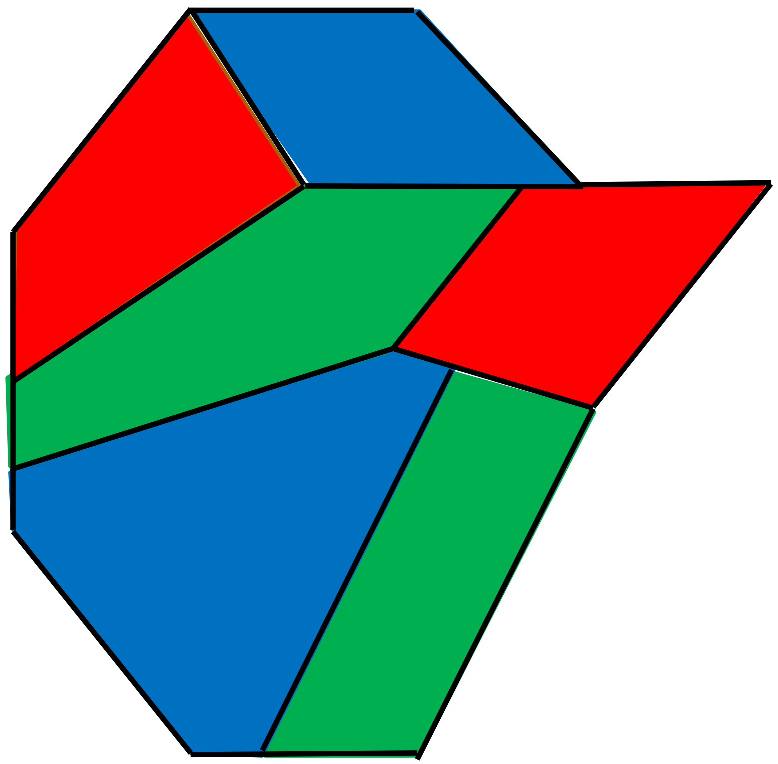

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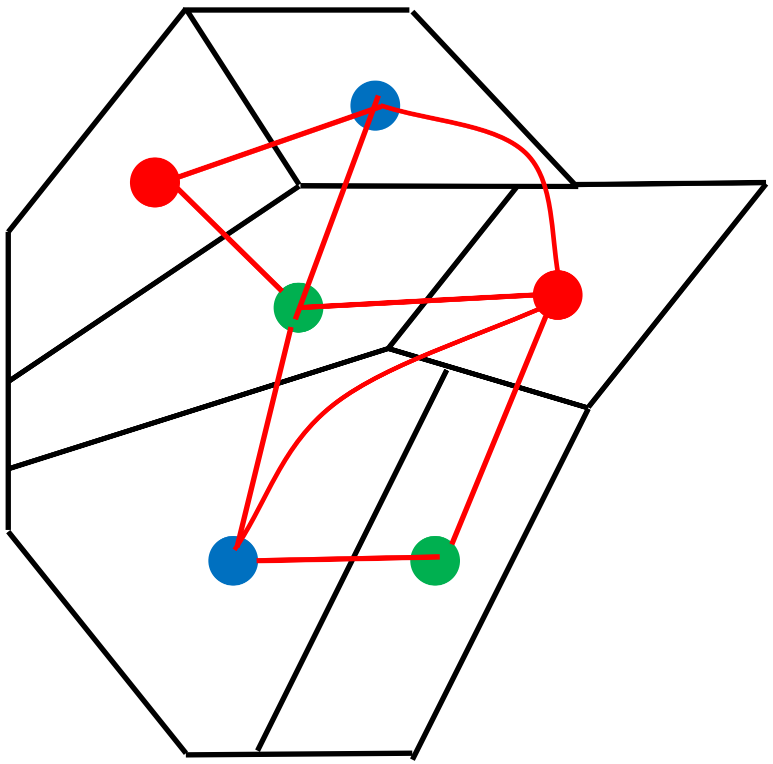
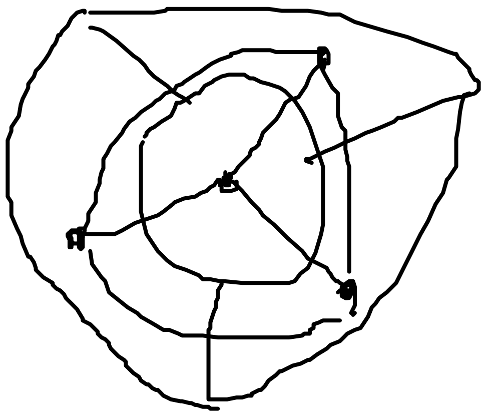
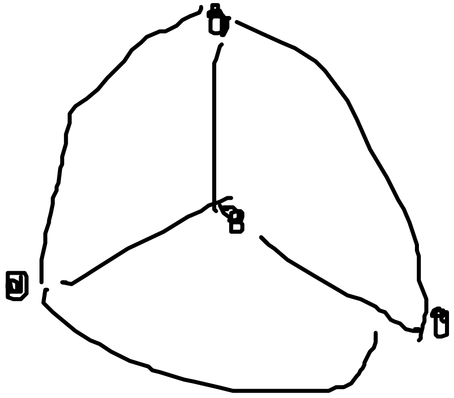


Four color theorem

The four color theorem states that there always exists a regular coloring of a planar graph into 4 colors.





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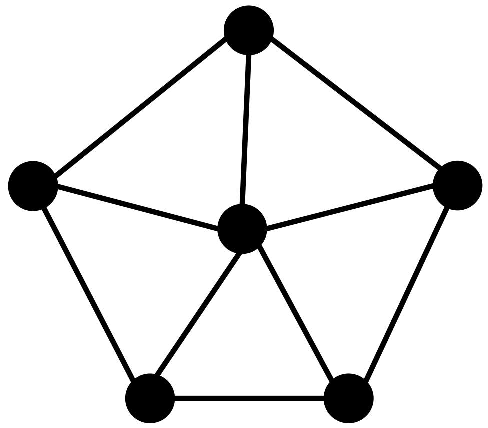

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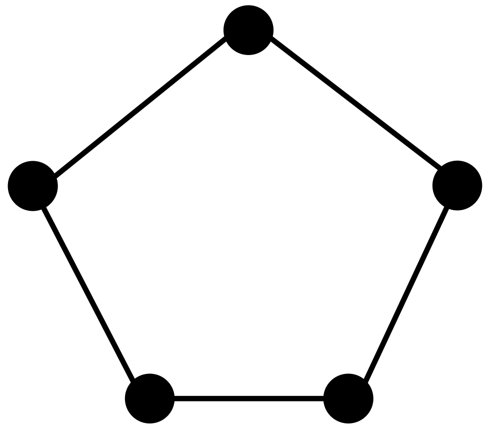

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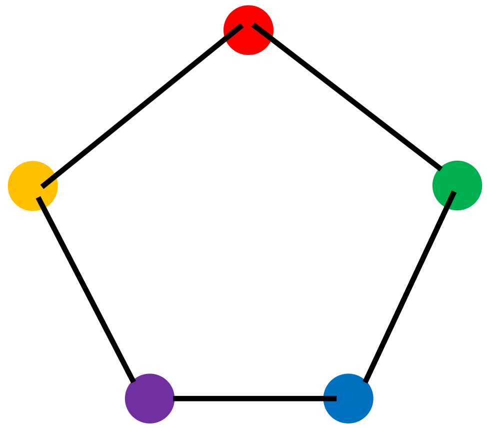

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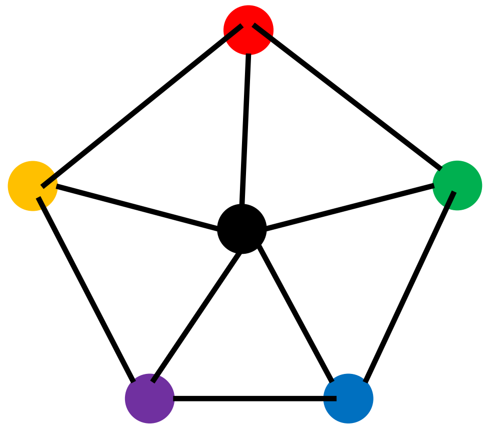

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