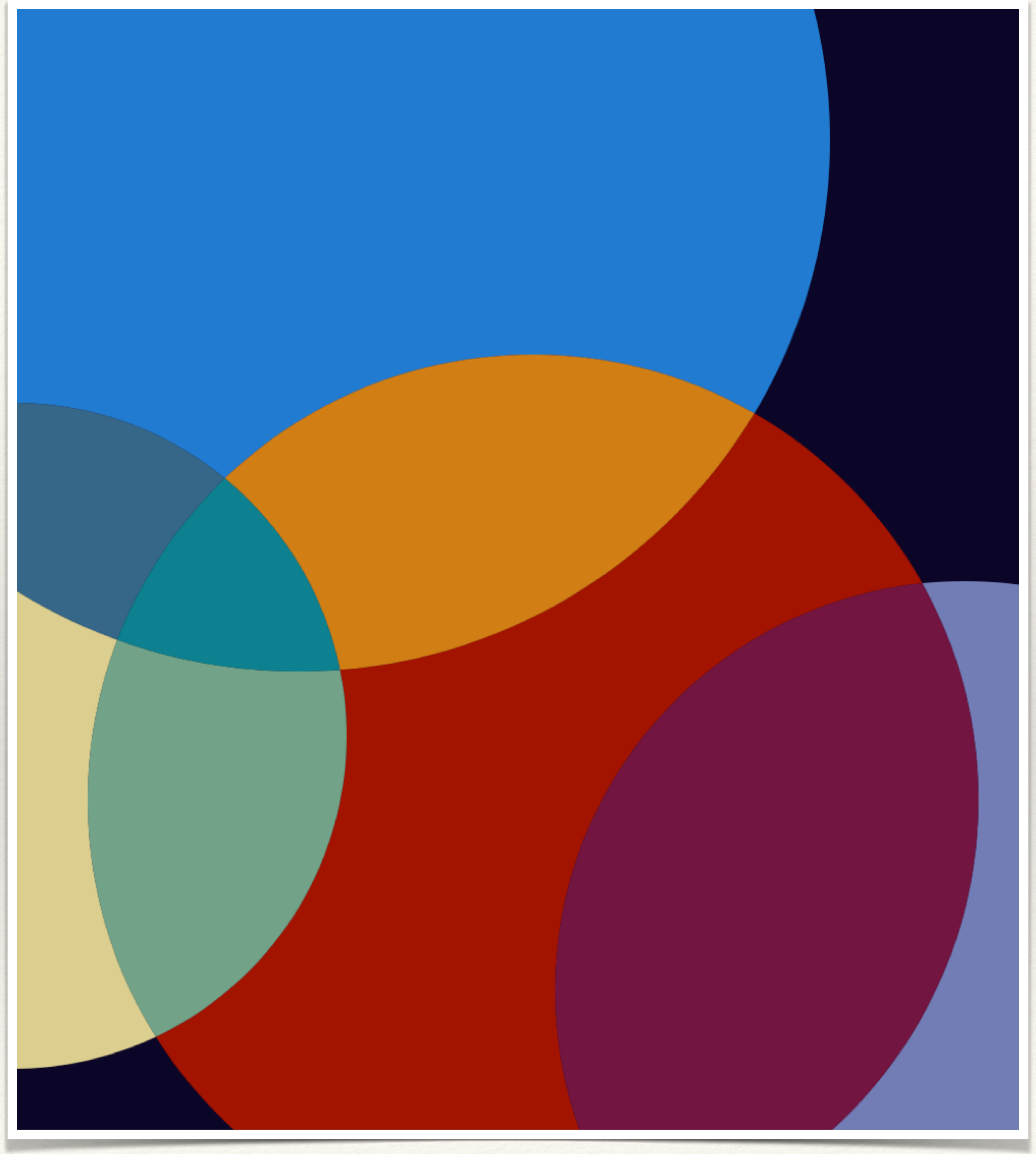
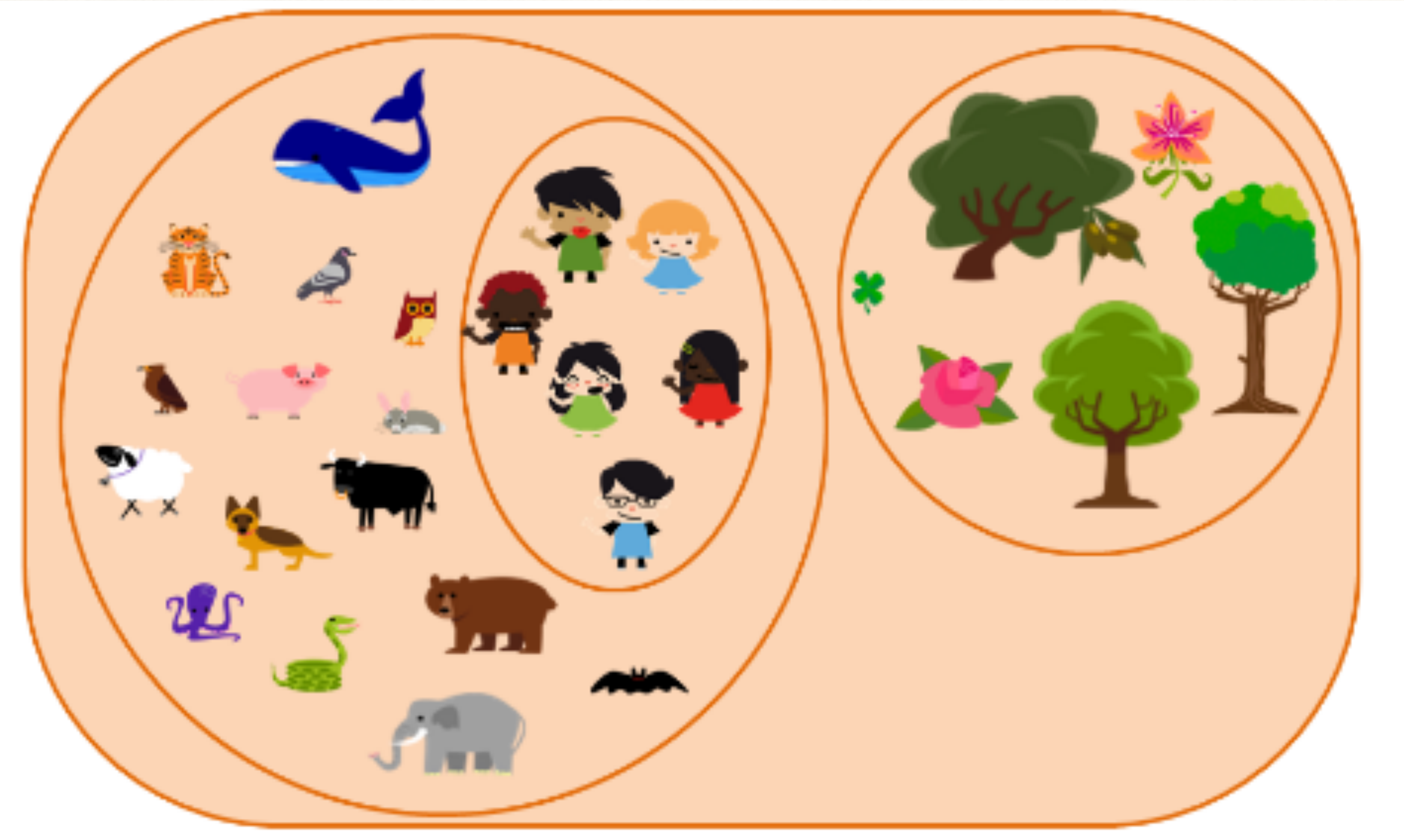
Lecture 1

Set Theory

Dr. David Zmiaikou



What is a set?

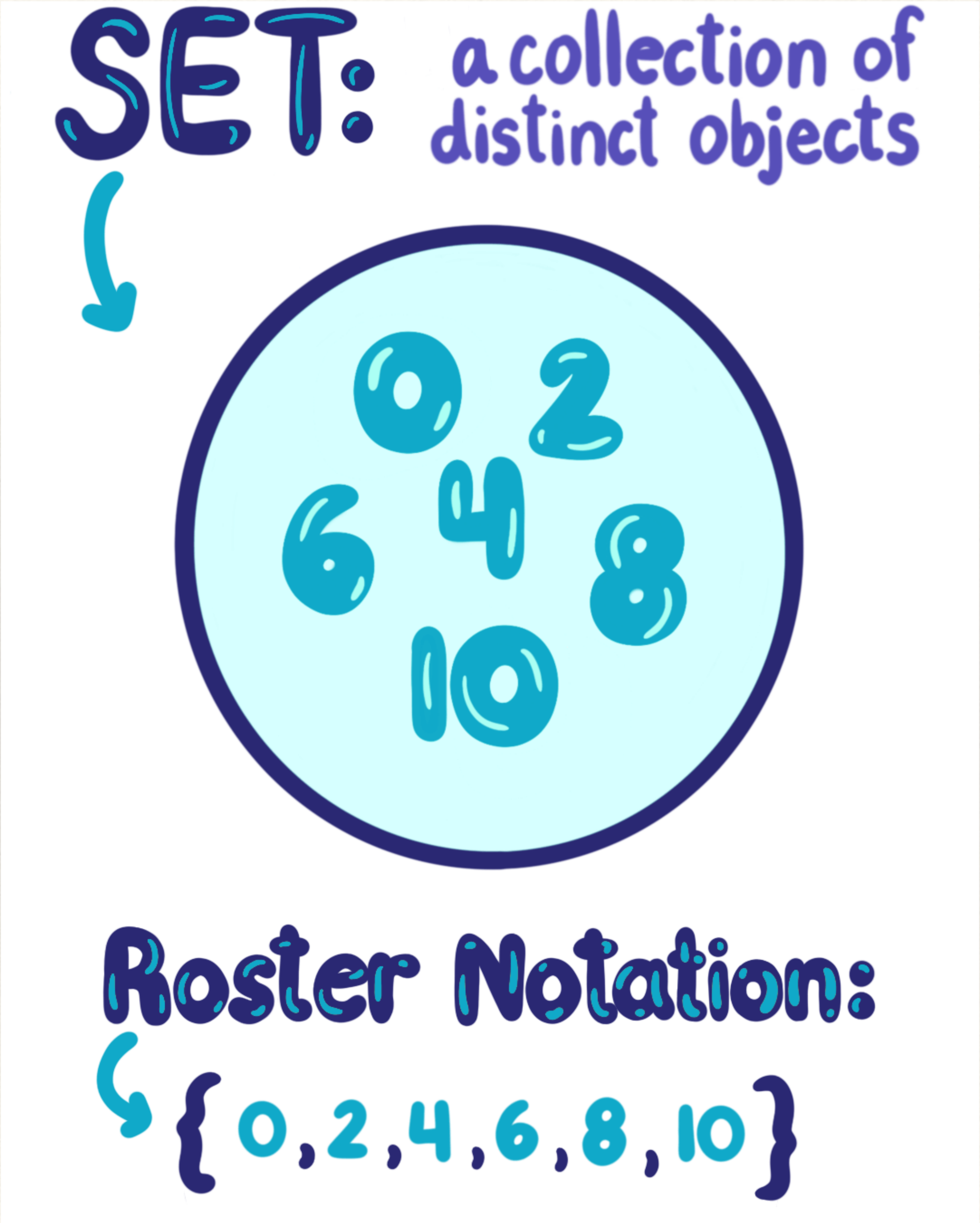


A **set** may be regarded as a collection of different objects.

The objects themselves are called '**elements**' or '**members**' of the set.

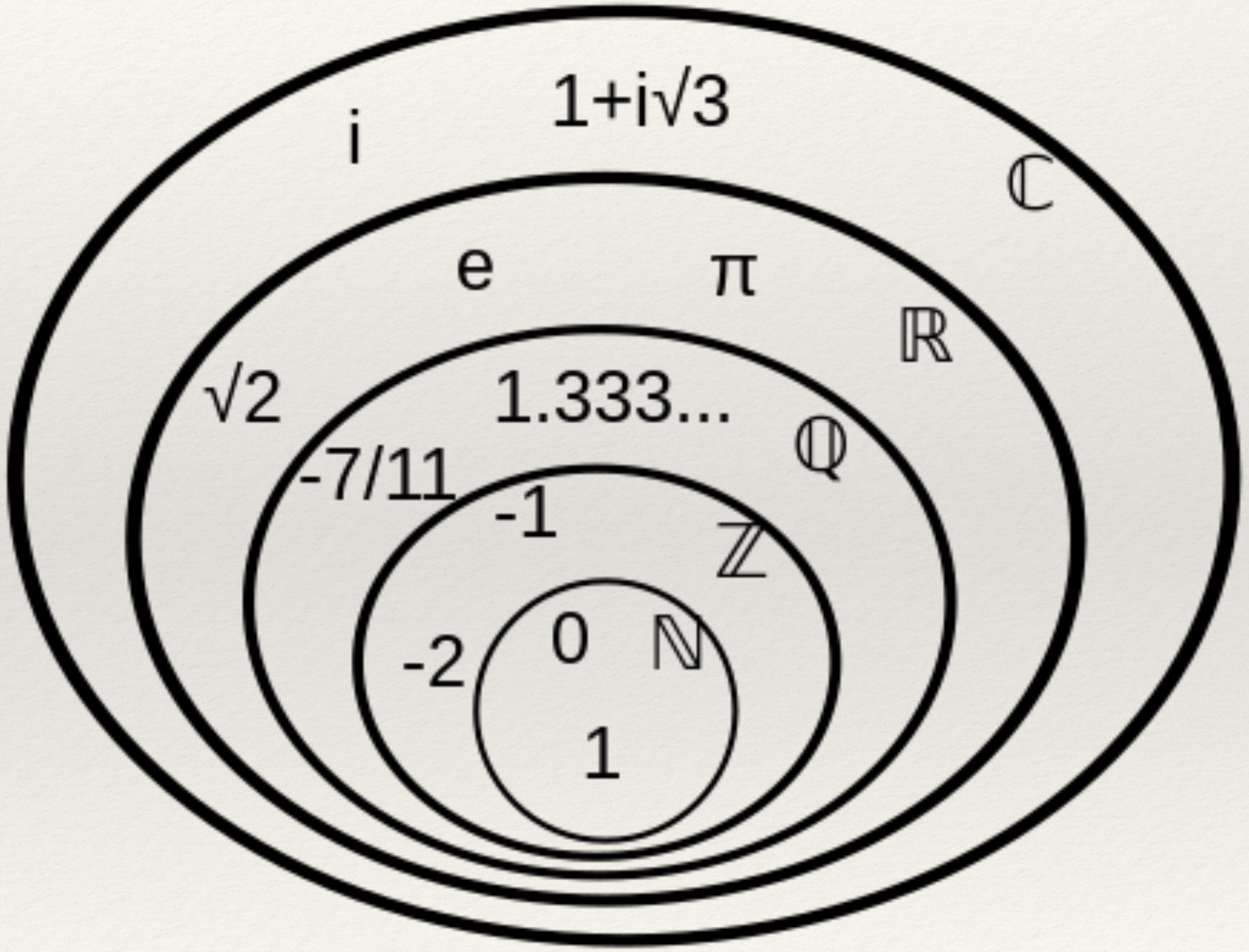
Notation

- **Roster notation**
  $A = \{0, 3, 7, 1\}$
  $B = \{\text{zebra, white, black}\}$
  ☝️ *the order in which the elements are listed does not matter*
- **Semantic notation**
  Let A be the set of the first three natural number.
  Let B be the set of things on my table.
- **Set-builder notation**
  $A = \{n \mid n \text{ is an integer and } 0 < n < 4\}$
  $A = \{n \ : \ n \text{ is an integer and } 0 < n < 4\}$
- **Membership**
  If x is an element of a set A, then we write $x \in A$.



Well-known sets of numbers

- set of all natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$
- set of all integers
  $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- set of all rational numbers
  $\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}$
- set of all real numbers $\mathbb{R}$ including all rational numbers and all irrational numbers
- set of all complex numbers
  $\mathbb{C} = \left\{ a + b\sqrt{-1} \,\middle|\, a, b \in \mathbb{R} \right\}$



⏳ A bit of history

Set theory was founded by a single paper in 1874 by the German mathematician **Georg Cantor** "On a Property of the Collection of All Real Algebraic Numbers”.

He gave the following definition:
“A set is a gathering together into a whole of definite, distinct objects of our perception or our thought—which are called elements of the set.”

Georg Cantor created set theory out of his desire to put the theory of real numbers on a sound basis, and also to understand “the infinity” better.

His ground-breaking work is nevertheless now known as naive set theory because of paradoxical flaws later found in it.

$\stackrel{\text{def}}{=}$ A few definitions

- Two sets A and B are **equal**, written $A = B$, if they consist of exactly the same elements.
- The set containing no elements is called the **empty set**. It is denoted by $\varnothing$.
- For a finite set A (a set with a finite number of elements), we write $|A|$ (or #A) to denote the number of elements in A. This is called the **cardinality** of A.

Examples 1

- If A is the set of letters in the English alphabet, then
  $|A| = ?$

- If C is the set of playing cards in a standard deck of cards,
  $|C| = ?$

Example 1

- If A is the set of letters in the English alphabet, then
  $$|A| = \mathbf{26}$$
- If C is the set of playing cards in a standard deck of cards,
  $$|C| = \mathbf{52}$$

Example 2

- $|\varnothing| = ?$

- $|\{\varnothing\}| = ?$

Example 2

- $\varnothing$ has no elements and so $|\varnothing| = \mathbf{0}$.
- Since the sets $\{\varnothing\}$ and $\varnothing$ do not consist of the same elements, $\{\varnothing\}$ isn't equal to $\varnothing$. (An empty box and a box containing an empty box are not the same thing.) The set $\{\varnothing\}$ has one element, namely the empty set $\varnothing$ and so $|\{\varnothing\}| = \mathbf{1}$.

Example 3

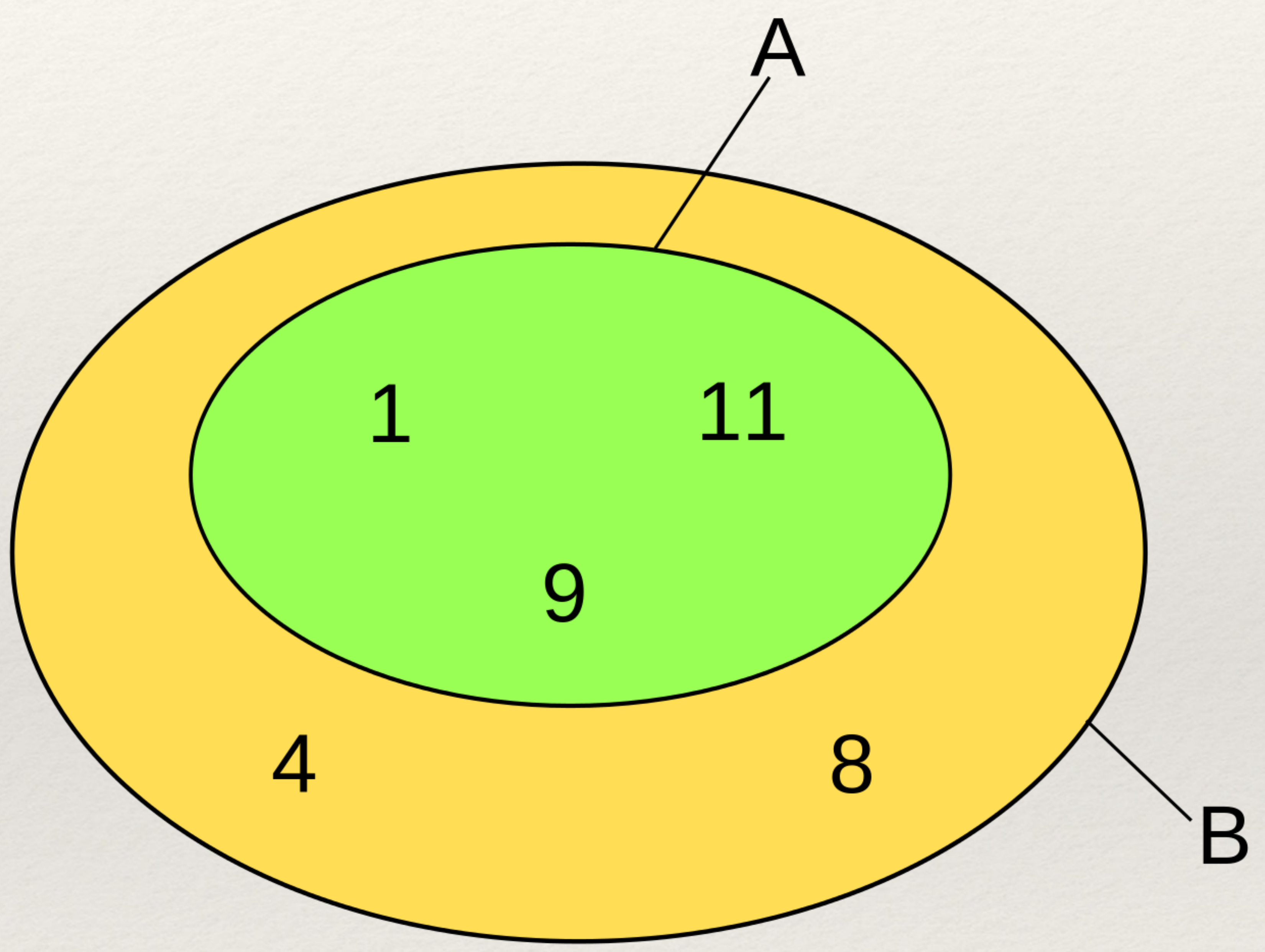
- If $A = \{1, \{1, 3\}, \varnothing, x\}$, then $|A| = ?$

Example 3

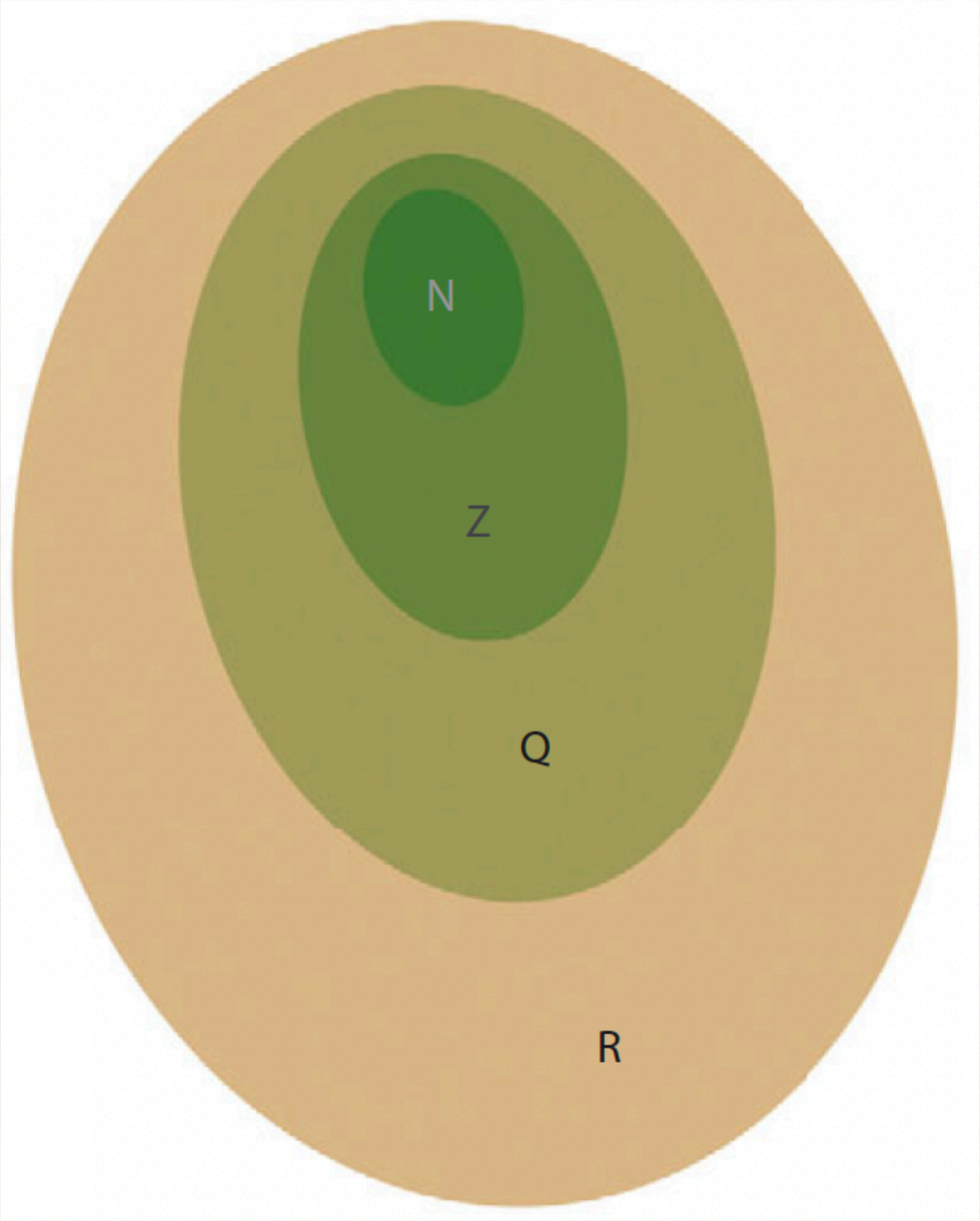
- The set $A = \{1, \{1, 3\}, \varnothing, x\}$ has four elements, two of which are sets, namely $\{1, 3\}$ and $\varnothing$. Since A has four elements, its cardinality $|A|$ equals **4**.

Subsets

- A set A is called a **subset** of a set B, written $A \subseteq B$, if every element of A also belongs to B.
- A set A is a **proper subset** of a set B, written $A \subset B$, if $A \subseteq B$ but $A \neq B$.
- Often when there is a discussion concerning sets, the sets involved are all subsets of some specified set, called the **universal set**, which is usually denoted by U.



Example 4



- $\mathbb{N} \subset \mathbb{Z}$
- $\mathbb{Z} \subset \mathbb{Q}$
- $\mathbb{Q} \subset \mathbb{R}$

Example 5

- If $A = \{a, b, c\}$ and $B = \{a, b, c, d, e\}$, then $A \subseteq B$.

- If $C = \{1, 2\}$ and $D = \{1, 3\}$, then $C \nsubseteq D$ since $2 \in C$ but $2 \notin D$.

Example 6

- If $A = \{a, b, c\}$ and $C = \{a, b, c, d, e\}$, find all subsets B such that $A \subset B \subset C$.

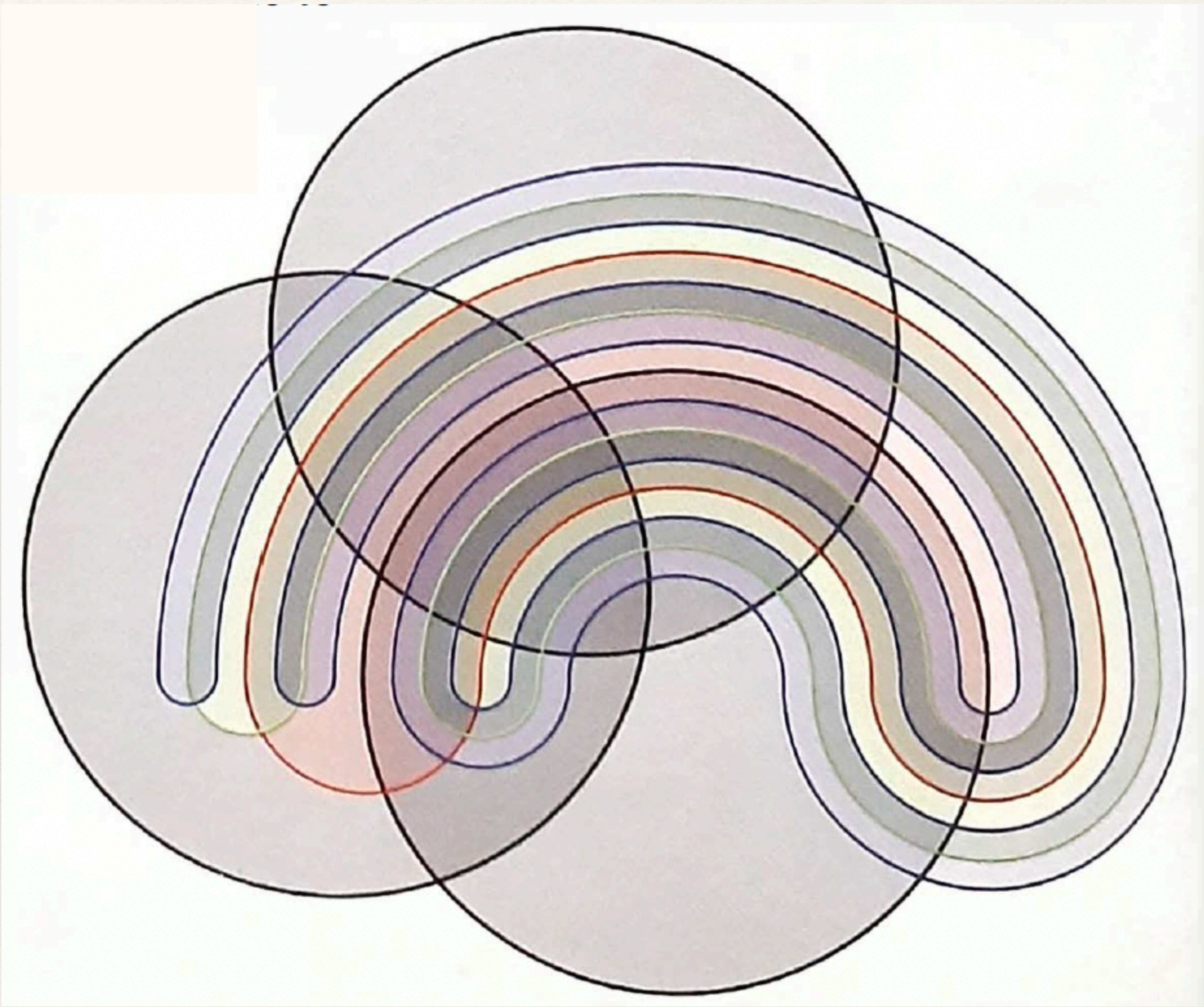
Example 6

- If $A = \{a, b, c\}$ and $C = \{a, b, c, d, e\}$, find all subsets B such that $A \subset B \subset C$.

  Answer. The only such sets B are $\{a, b, c, d\}$ and $\{a, b, c, e\}$.

Venn Diagrams

- A **Venn diagram** is a diagram in the plane that represents pictorially a set or sets under consideration.
- A *rectangle* is often drawn to represent the universal set.
- Within the rectangle, a *closed curve* (often a *circle* or *ellipse*) is drawn for each set being discussed.



Example 6

- Suppose that $U = \{1, 2, \ldots, 6\}$ is the universal set. Draw a Venn diagram for each of the following:

  (a) the set $A = \{1, 2, 5, 6\}$.

  (b) the set $B = \{1, 3, 6\}$.

  (c) the two sets $C = \{1, 2, 4\}$ and $D = \{1, 2, 4, 5, 6\}$.

  (d) the two sets $E = \{1, 2, 3\}$ and $F = \{3, 5, 6\}$.

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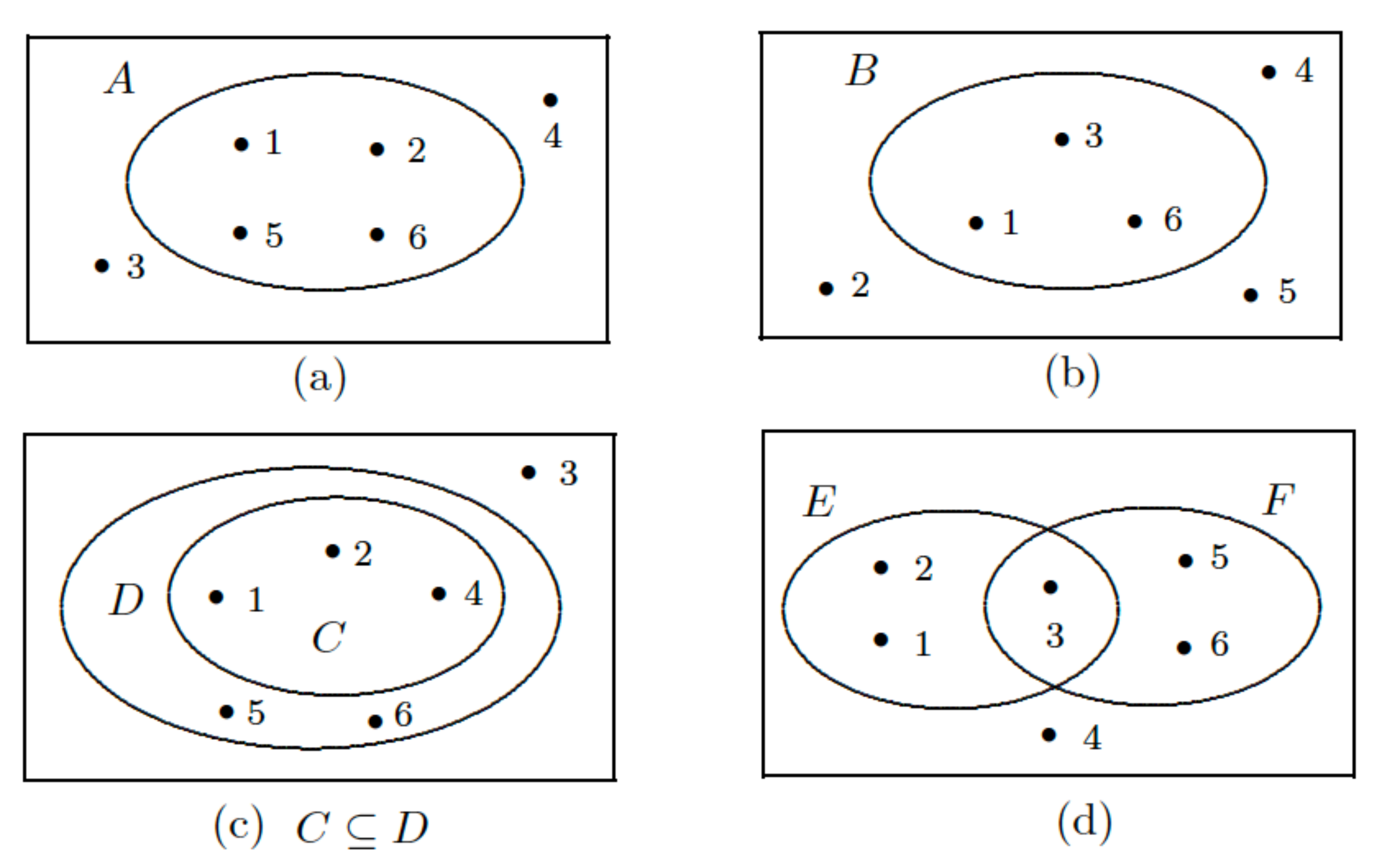
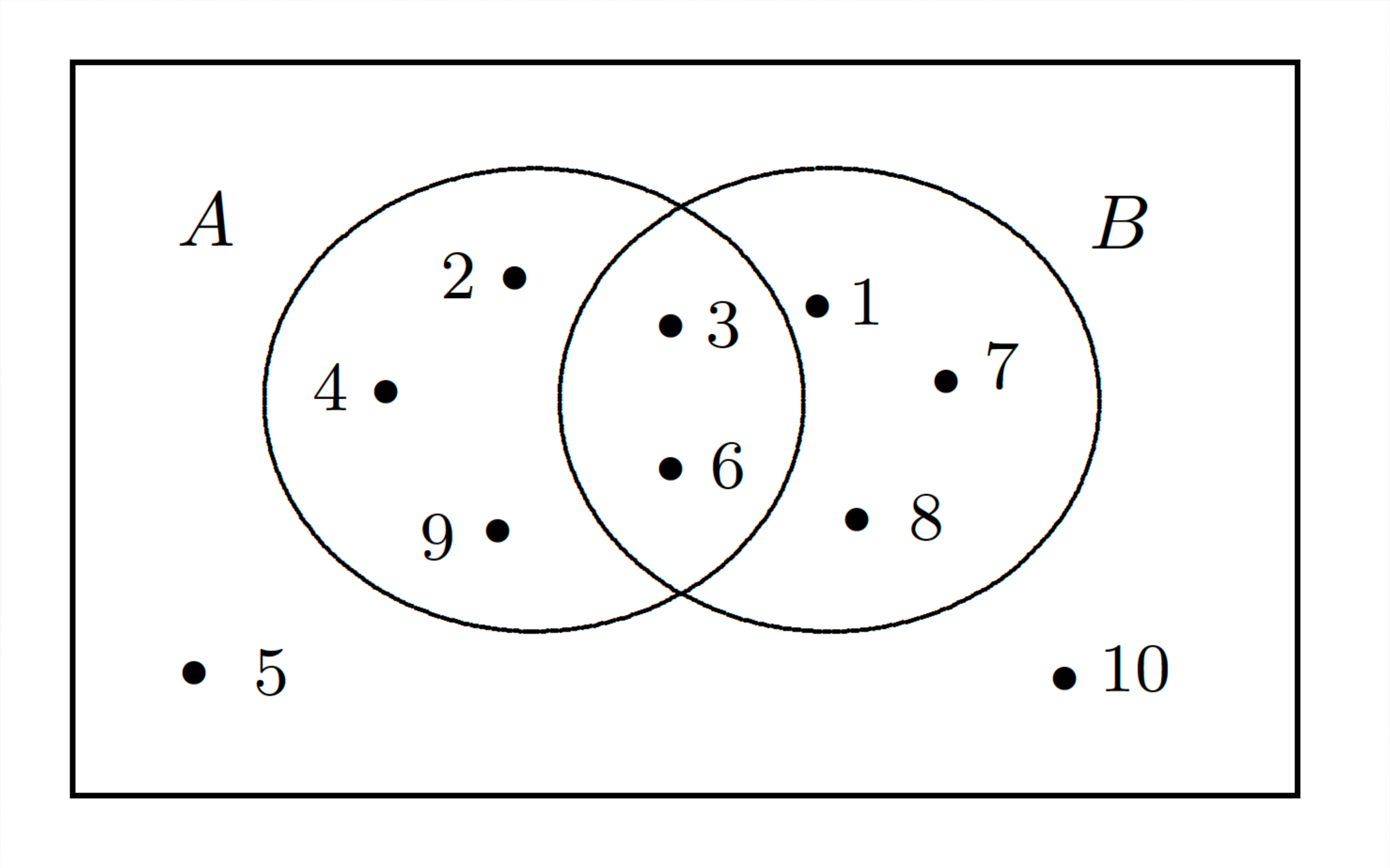


Example 7

- Suppose we have a diagram on the right. We can see that

  - 3 and 6 belong to both sets A and B,
  - 2, 4 and 9 belong to A but not to B,
  - 1, 7 and 8 belong to B but not to A,
  - 5 and 10 belong to neither A nor B.

  Therefore, $A = \{2, 3, 4, 6, 9\}$, $B = \{1, 3, 6, 7, 8\}$ and the universal set is $U = \{1, 2, \ldots, 10\}$.



Power Sets

- The set of all subsets of a set A is called the **power set** of A and is denoted by $\mathscr{P}(A)$.

- **Theorem 1.** If A is a set with $|A| = n$, where n is a nonnegative integer, then
$$|\mathscr{P}(A)| = 2^n.$$

Example 8

- Determine the power set of each of the sets $A = \{a, b\},\ B = \{x, y, z\}, C = \{0, \{\emptyset\}\}$ and $D = \emptyset$.

Example 8

- Determine the power set of each of the sets
  $A = \{a, b\}$, $B = \{x, y, z\}$, $C = \{0, \{\varnothing\}\}$ and $D = \varnothing$.

  Answer.
  $\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, A\}$

  $\mathscr{P}(B) = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, B\}$

  $\mathscr{P}(C) = \{\varnothing, \{0\}, \{\{\varnothing\}\}, C\}$

  $\mathscr{P}(D) = \{\varnothing\}$

Example 9

- For $A = \{x \in \mathbb{Z} : |x| \leq 3\}$, how many elements are in $\mathscr{P}(A)$?

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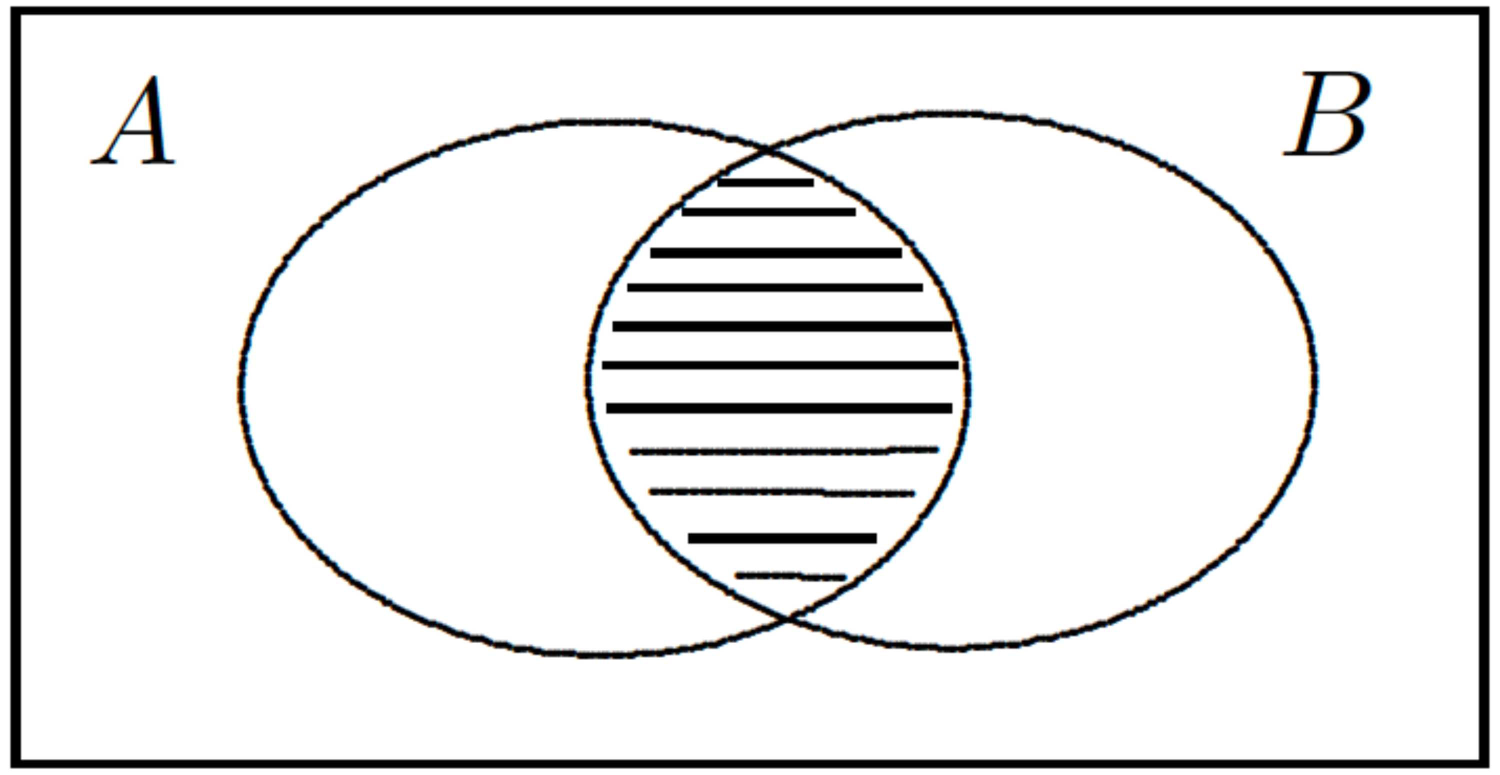
- For $A = \{x \in \mathbb{Z} : |x| \leq 3\}$, how many elements are in $\mathscr{P}(A)$?

  Solution. Since $A = \{-3, -2, -1, 0, 1, 2, 3\}$, it follows that $|A| = 7$. So, by the theorem, we have
  $$|\mathscr{P}(A)| = 2^7 = 128.$$

Set Operations

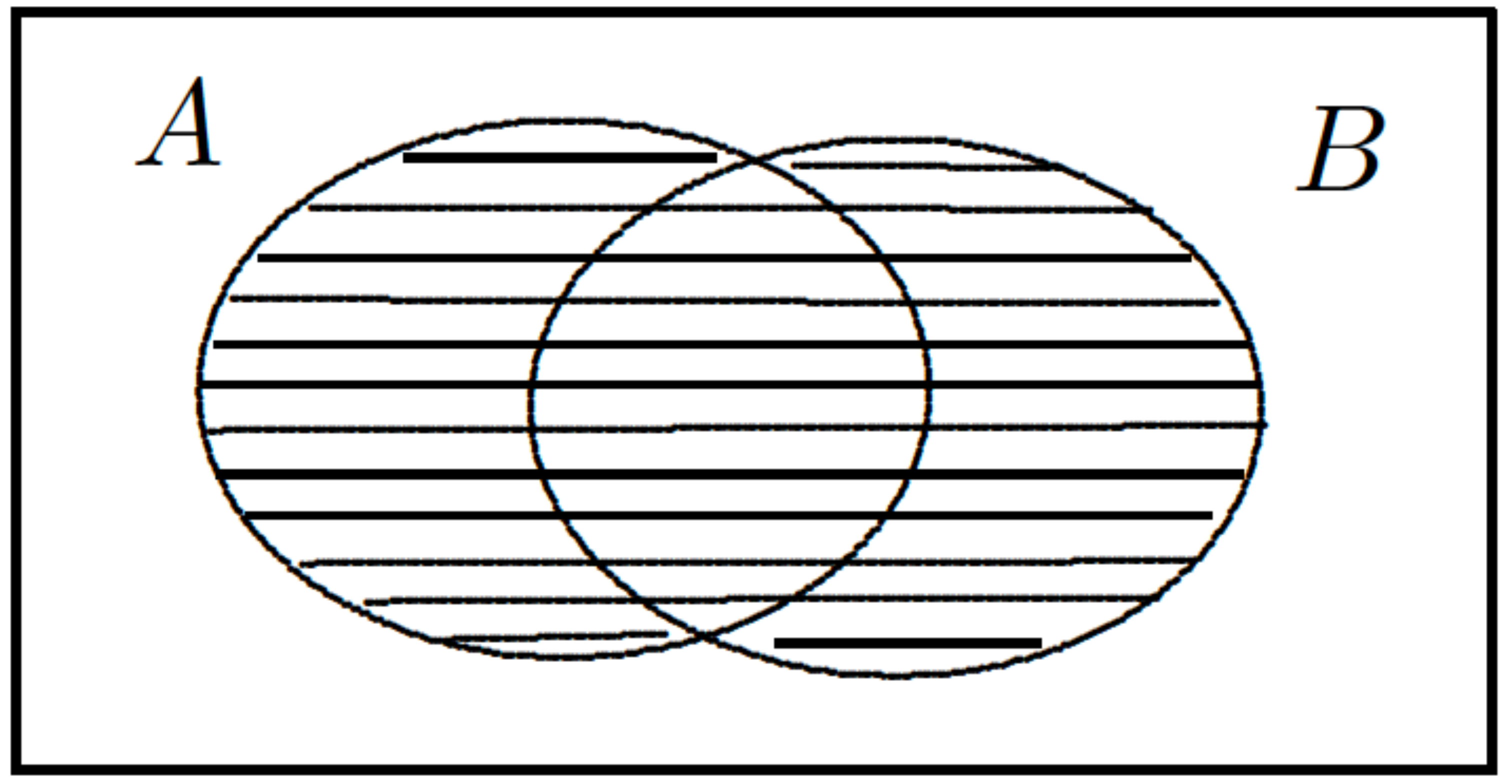
Intersections and Unions

- Let A and B be two sets. The **intersection** $A \cap B$ of A and B is the set of elements belonging to *both* A and B. Thus
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$



$A \cap B$

- Let A and B be two sets. The **union** $A \cup B$ of A and B is the set of elements belonging to *at least one* of A and B. Thus
$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

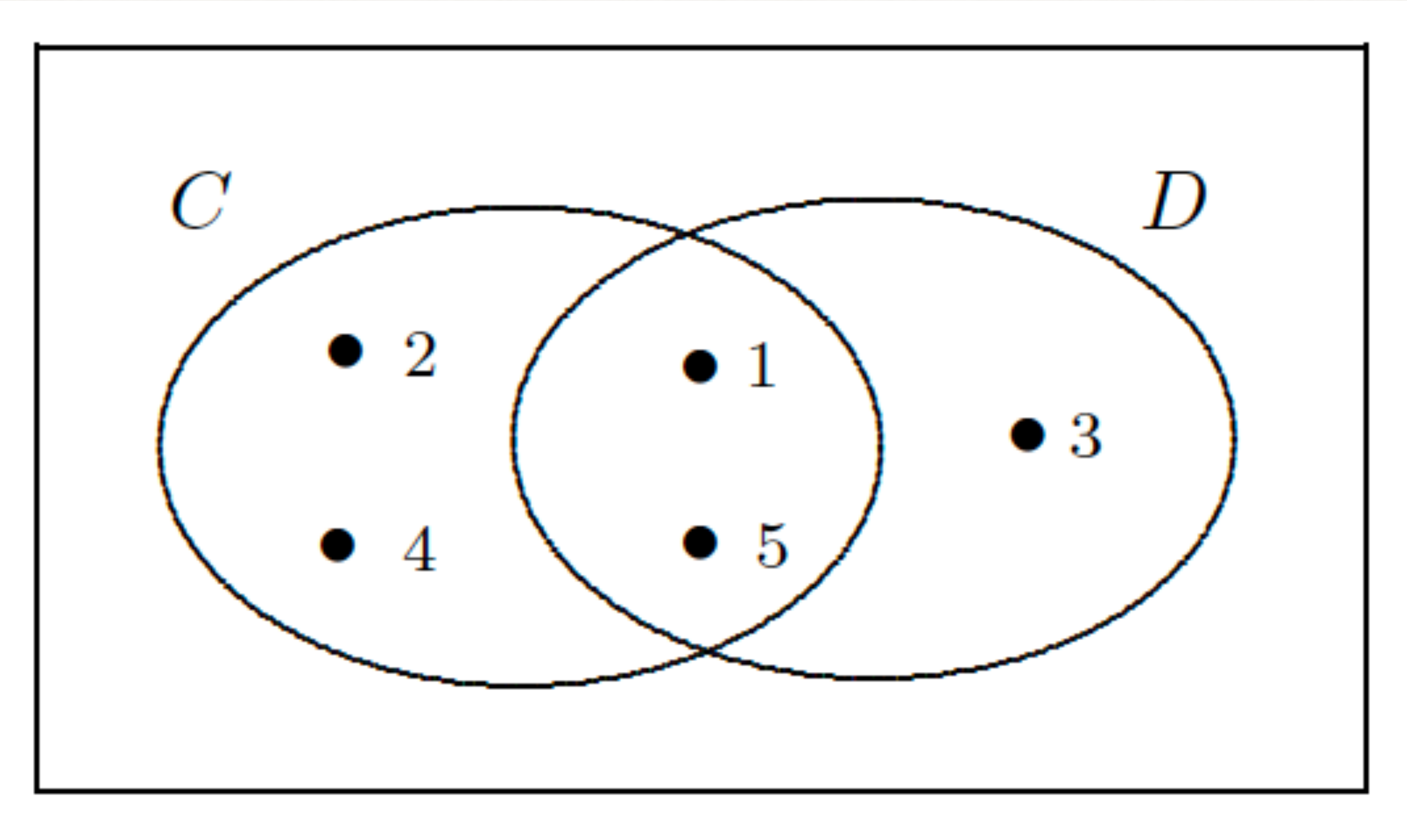


$A \cup B$

Example 10

- For the sets $C = \{1, 2, 4, 5\}$ and $D = \{1, 3, 5\}$, we have

$$C \cap D = \{1, 5\} \text{ and } C \cup D = \{1, 2, 3, 4, 5\}.$$



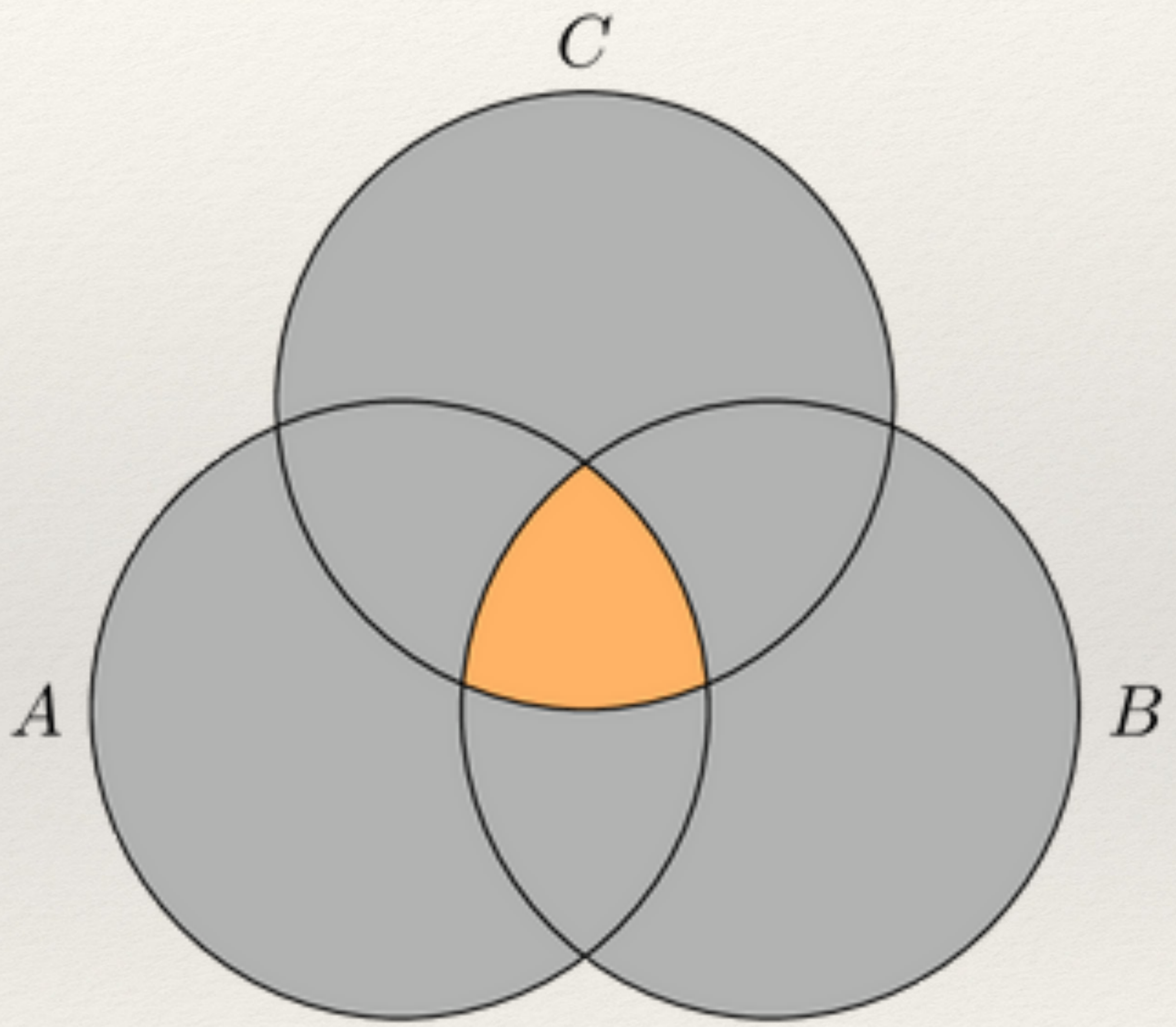
Intersections and Unions

- The **intersection** $A \cap B \cap C$

  $A \cap B \cap C = \{x : x \in A \text{ and } x \in B \text{ and } x \in C\}.$

- The **union** $A \cup B \cup C$

  $A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$



Example 11

- For sets $A = \{1, 3, 4\}$, $B = \{3, 4, 6\}$ and $C = \{2, 3, 5\}$,

$$A \cap B \cap C = \{3\} \text{ and } A \cup B \cup C = \{1, 2, \ldots, 6\}.$$

Intersections and Unions

- **Theorem 2.** For every three sets A, B and C, we have:

  Commutative Laws
  $A \cap B = B \cap A$ and $A \cup B = B \cup A$

  Associative Laws
  $(A \cap B) \cap C = A \cap (B \cap C)$
  $(A \cup B) \cup C = A \cup (B \cup C)$

  Distributive Laws
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Intersections and Unions

- **Theorem 2.** For every three sets A, B and C, we have:

  Commutative Laws
  $A \cap B = B \cap A$ and
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  Associative Laws
  $(A \cap B) \cap C = A \cap (B \cap C)$
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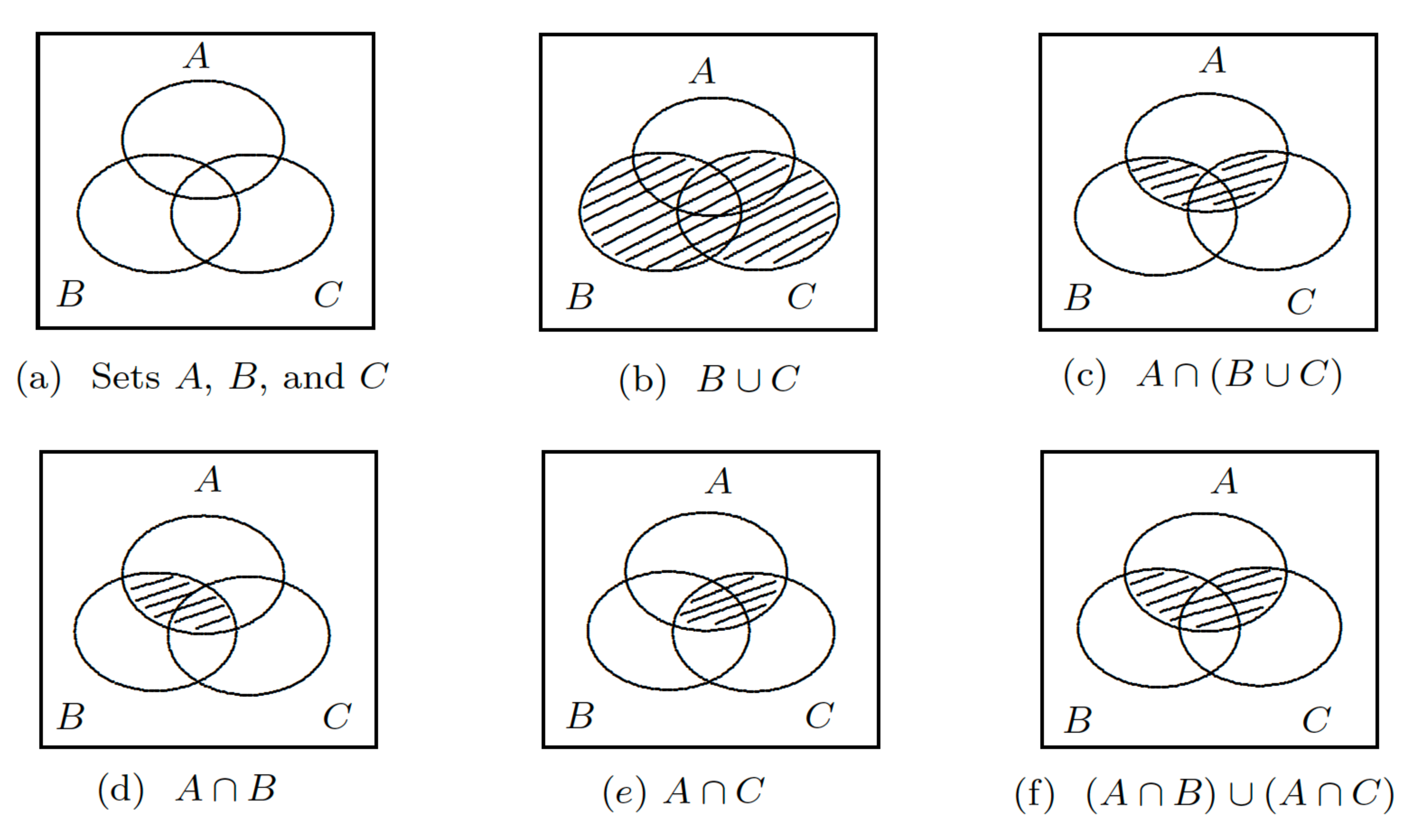
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



Disjoint Sets

- Two sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \varnothing$.

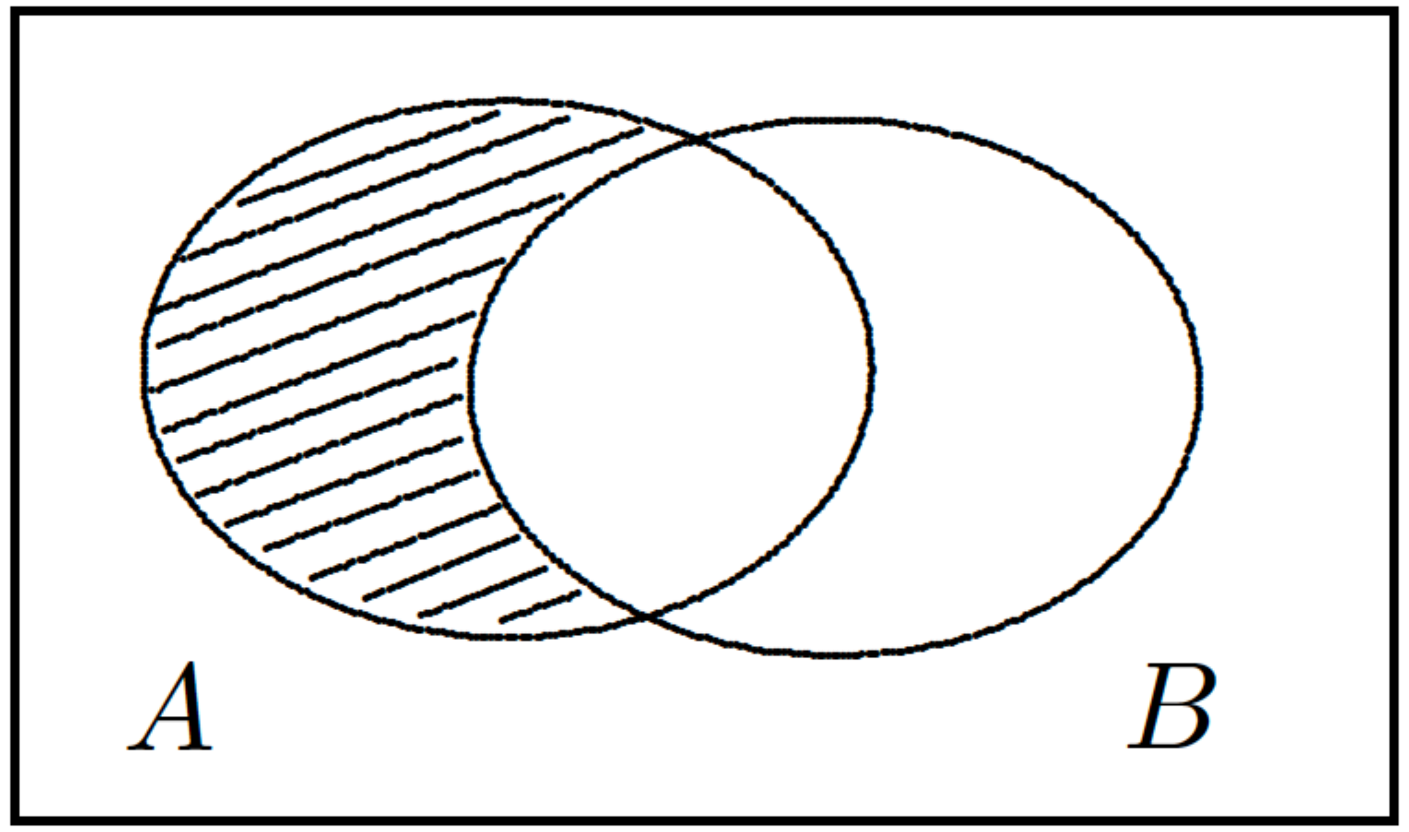
- A collection of sets is said to be **pairwise disjoint** if every two distinct sets in the collection are disjoint.

Example 12

- The sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ are disjoint.
- The set of rational numbers and the set of irrational numbers are disjoint.
- The set of nonnegative integers and the set of nonpositive integers are not disjoint, however, since 0 belongs to both sets.
- The sets $A = \{1, 2, 3\}$, $B = \{4, 5\}$ and $C = \{6, 7, 8\}$ are pairwise disjoint since every two of the sets A, B and C are disjoint.

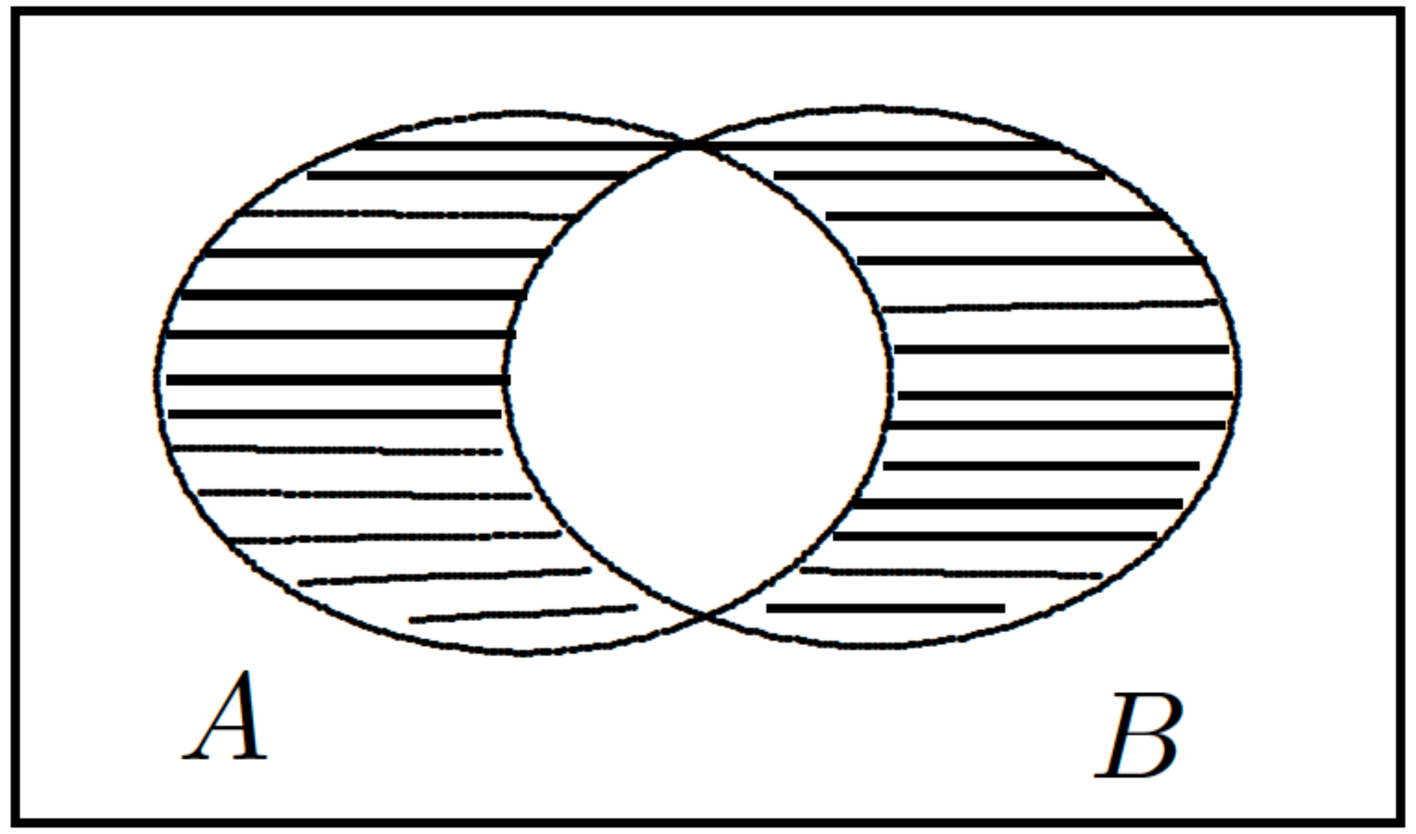
Difference and Symmetric Difference

- The **difference** $A - B$ (or $A \backslash B$) of two sets A and B is defined as
$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$



$A - B$

- The **symmetric difference** $A \oplus B$ (or $A \Delta B$) of two sets A and B is defined as
$$A \oplus B = (A - B) \cup (B - A).$$



$A \oplus B$

Difference and Symmetric Difference

- **Theorem 3.** We also have

$$A \oplus B = (A \cup B) - (A \cap B).$$

Example 13

- For the sets $A = \{i \in \mathbb{Z} : 1 \leq i \leq 10\}$ and $B = \{i \in \mathbb{Z} : 5 \leq i \leq 15\}$, we obtain

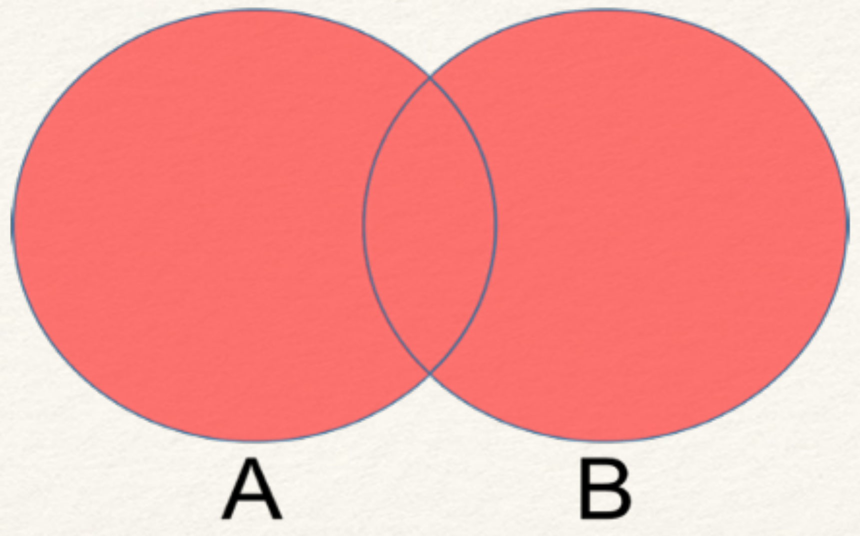
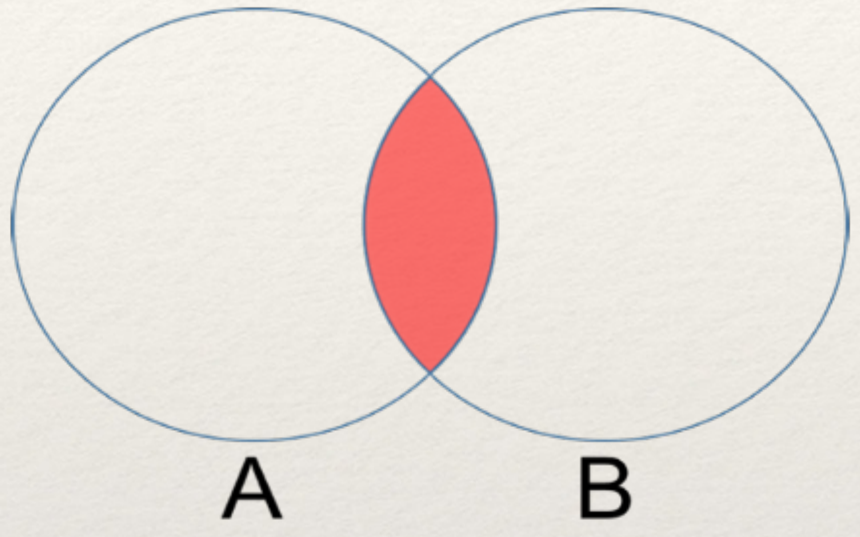
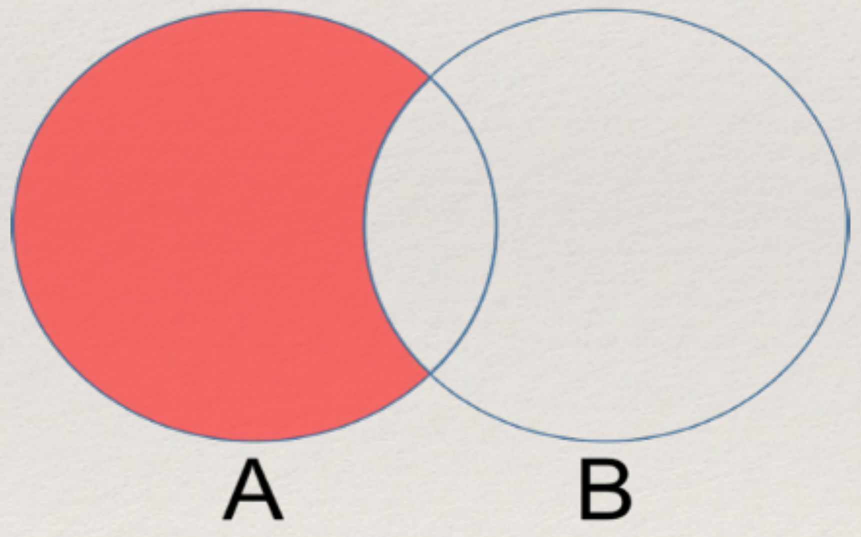
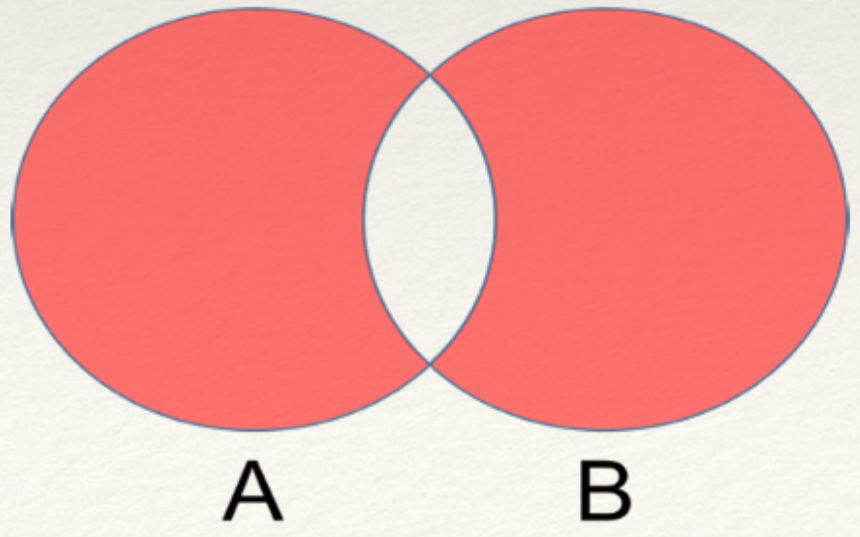
$$A - B = \{i \in \mathbb{Z} : 1 \leq i \leq 4\} = \{1,2,3,4\}$$

$$B - A = \{i \in \mathbb{Z} : 11 \leq i \leq 15\} = \{11,12,13,14,15\}$$

$$A \cup B = \{i \in \mathbb{Z} : 1 \leq i \leq 15\} = \{1,2,\ldots,15\}$$

$$A \cap B = \{i \in \mathbb{Z} : 5 \leq i \leq 10\} = \{5,6,7,8,9,10\}$$

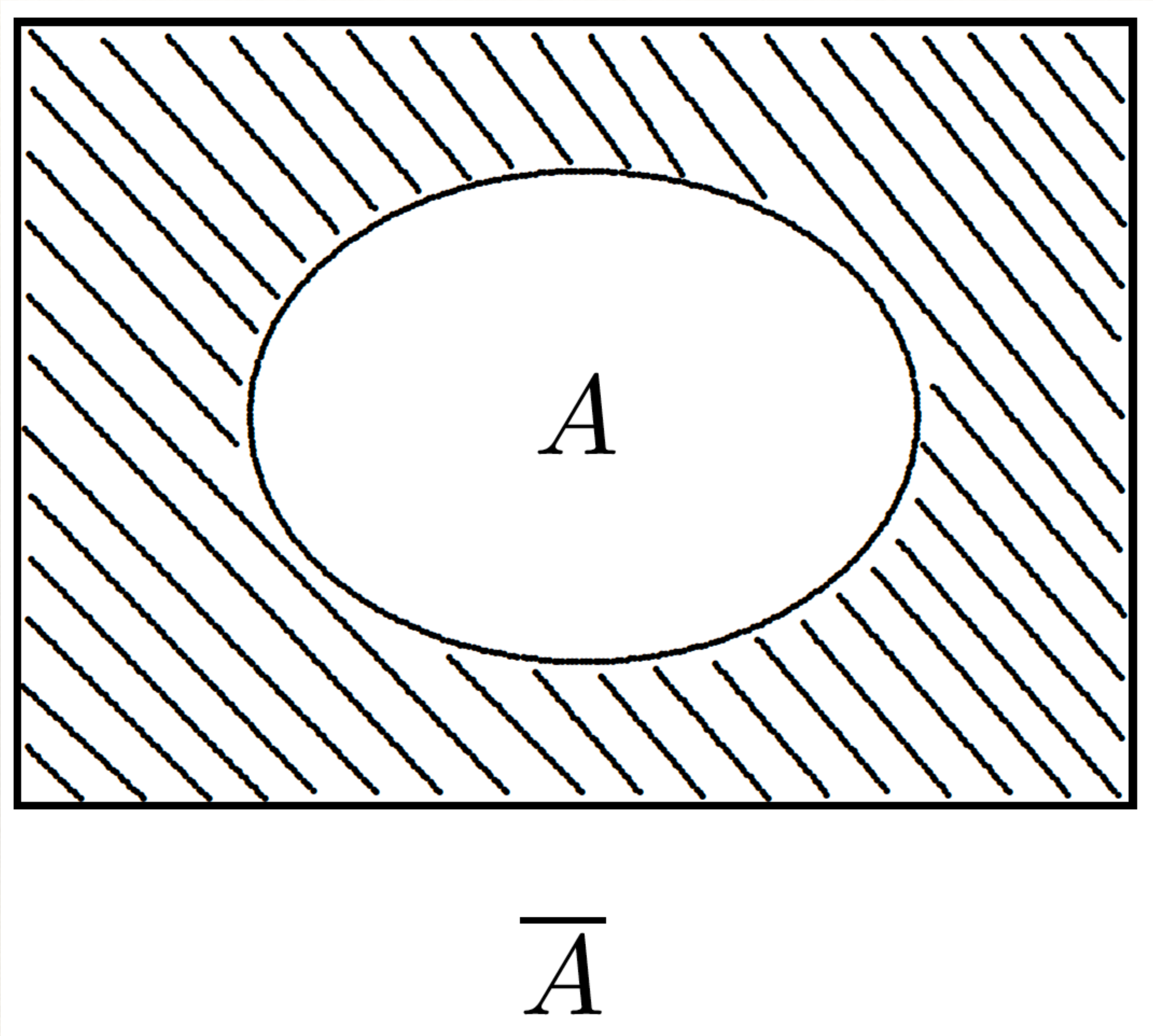
$$A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B) = \{1,2,3,4,11,12,13,14,15\}$$

Set Operation	Venn Diagram	Interpretation
Union		$A \cup B$, is the set of all values that are a member of A, or B, or both.
Intersection		$A \cap B$, is the set of all values that are members of both A and B.
Difference		$A \setminus B$, is the set of all values of A that are not members of B
Symmetric Difference		$A \triangle B$, is the set of all values which are in one of the sets, but not both.

Complement of a Set

- For a set A (which is therefore a subset of the universal set U being considered), the **complement** $\overline{A}$ (or A^c) is the set of elements (in the universal set) not belonging to A. That is,

$$\overline{A} = \{x \in U : x \notin A\}.$$



Examples 14

- Let $U = \mathbb{Z}$ be the universal set and let E be the set of even integers. Then the complement $\overline{E}$ is the set of odd integers.

- $\overline{\overline{A}} = A$ for any set A

Complement of a Set

- **Theorem 4.** For every two sets A and B, we have:

  De Morgan's Laws
  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Example 15

- For the sets $C = \{1, 2, 4, 5\}$ and $D = \{1, 3, 5\}$ and the universal set $U = \{1, 2, 3, 4, 5\}$, it follows that

$$
\begin{aligned}
&C - D = \{2, 4\} \\
&D - C = \{3\} \\
&C \oplus D = \{2, 3, 4\} \\
&\overline{C} = \{3\} \\
&\overline{D} = \{2, 4\} \\
&\overline{C} \cup \overline{D} = \{2, 3, 4\} \\
&C \cap D = \{1,5\} \\
&\overline{C \cap D} = \{2, 3, 4\}
\end{aligned}
$$

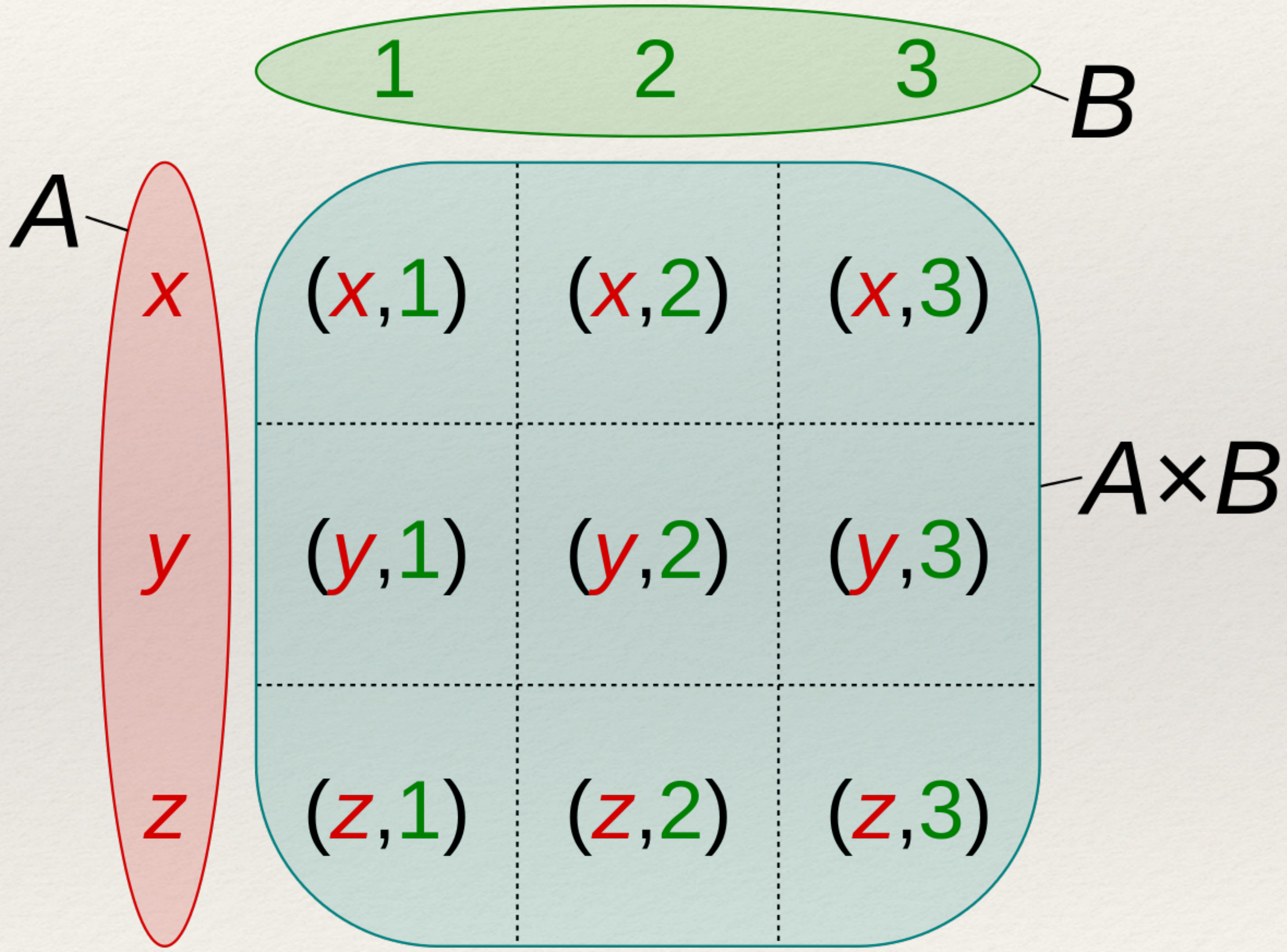
Cartesian Products of Sets

- For two elements a and b, we write (a, b) for the **ordered pair** in which a is the first element (or first coordinate) of the pair and b is the second element (second coordinate) of the pair.

  ☝ *the order matters*: $(1, 2) \neq (2, 1)$

- For two sets A and B, the **Cartesian product** $A \times B$ is the set of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. That is,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$



Examples 15

- For the sets $A = \{1, 2\}$ and $B = \{x, y, z\}$, we have

  $A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$

  $B \times A = \{(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)\}$

  $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

  $B \times B = \{(x, x), (x, y), (x, z), (y, x), (y, y), (y, z), (z, x), (z, y), (z, z)\}$

Thank you!