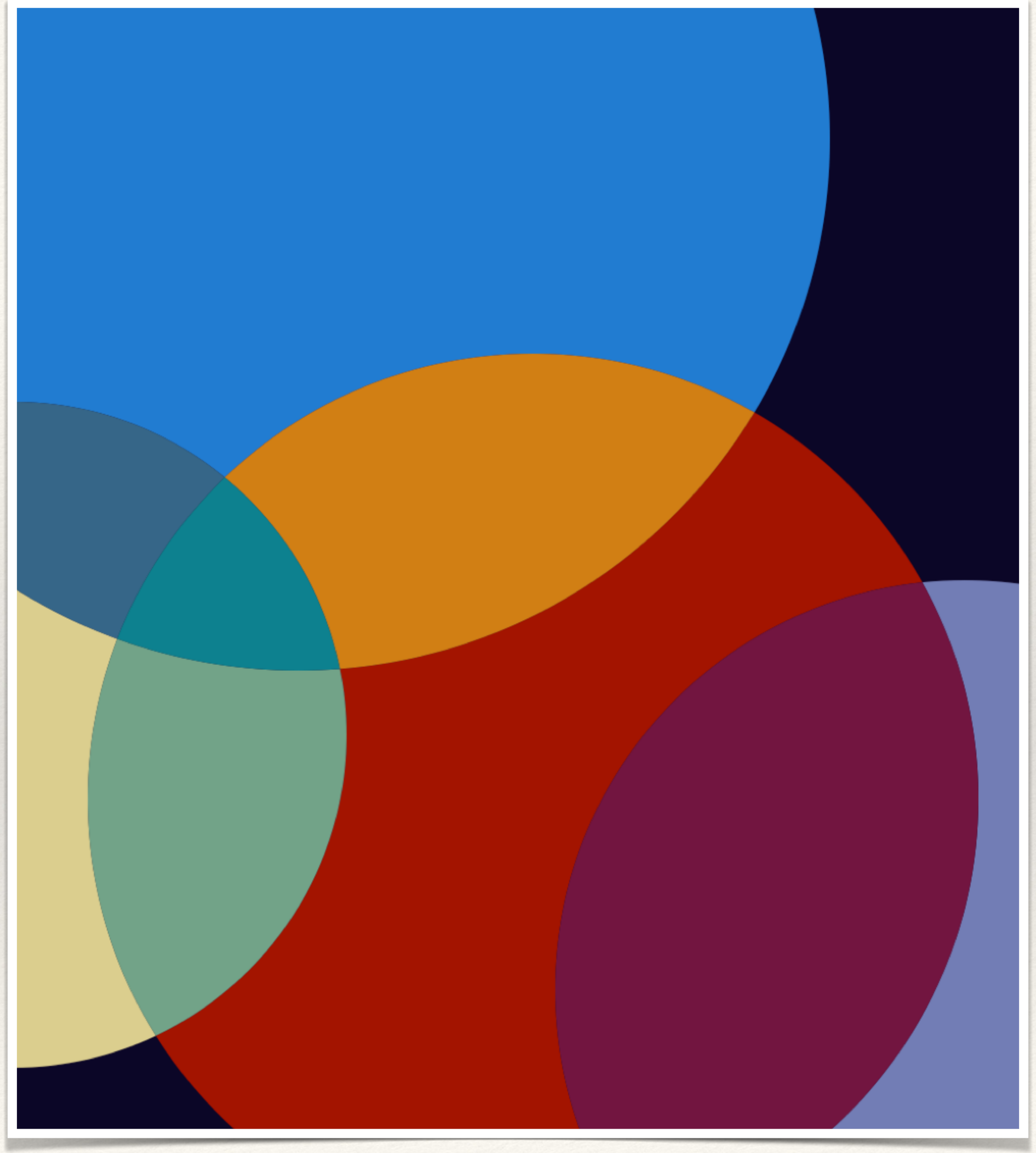
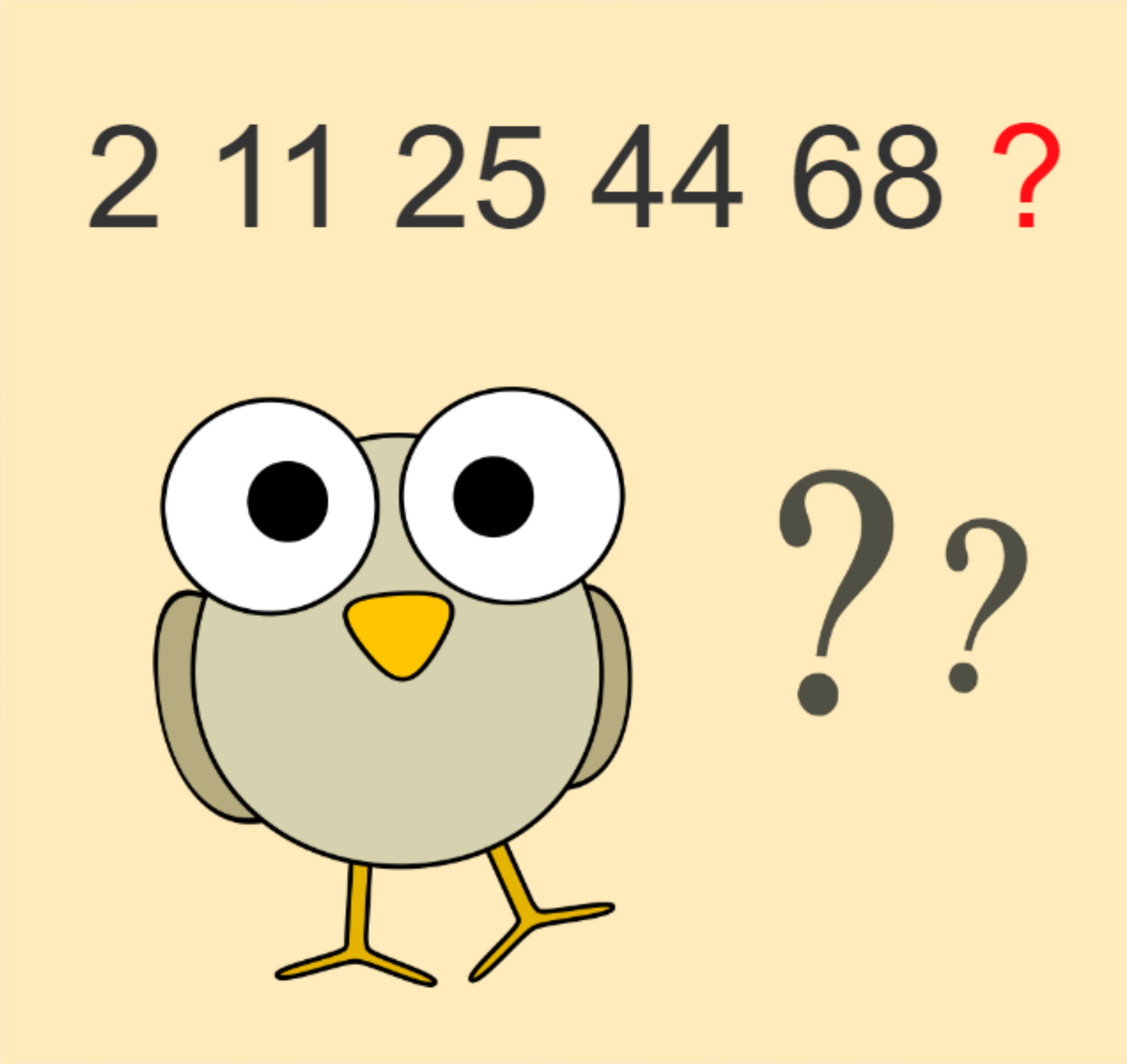
Lecture 5

Sequences

Dr. David Zmiaikou



What is a Sequence?



- A **sequence** is a listing (possibly with repetitions) of elements of some set.
- A sequence can be **infinite** or **finite**.
- An *infinite* sequence is often denoted by
$$a_1, a_2, a_3, \ \dots \quad \text{or} \quad \{a_n\}.$$
- The element a_1 is called the **first term** of the sequence, a_2 the **second term** and so on. The ***n*th term** of the sequence is therefore a_n.

Example 1

The sequence whose nth term is

$$a_n = \frac{(-1)^n}{n} \text{ for } n \in \mathbb{N}$$

can also be expressed as

$$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots$$

or as

$$\left\{ \frac{(-1)^n}{n} \right\}.$$

Example 2

- Find the first three terms of the sequence $\{a_n\}$, where

$$a_n = n^3 - 6n^2 + 12n - 6 \text{ for } n \in \mathbb{N}.$$

Example 2

- Find the first three terms of the sequence $\{a_n\}$, where

$$a_n = n^3 - 6n^2 + 12n - 6 \text{ for } n \in \mathbb{N}.$$

Answer. $a_1 = 1,\ a_2 = 2,\ a_3 = 3.$

Example 3

- $1, 2, 3, 4, \ldots$ is a sequence whose nth term is n.
- $1, 4, 9, 16, \ldots$ is a sequence whose nth term is n^2.
- $1, 8, 27, 64, \ldots$ is a sequence whose nth term is n^3.
- $2, 4, 8, 16, \ldots$ is a sequence whose nth term is 2^n.
- $4, 7, 10, 13, \ldots$ is a sequence whose nth term is $3n + 1$.

Example 4

- Consider the sequence $a_0, a_1, a_2, \ \ldots,$ where

$$a_0 = -\frac{1}{3}, a_1 = \frac{2}{5}, a_2 = -\frac{4}{7} \ \text{ and } \ a_3 = \frac{8}{9}.$$

Determine the nth term of a sequence $\{a_n\}$ whose first four terms are those given above.

Example 4

- Consider the sequence $a_0, a_1, a_2, \ldots$, where

$$a_0 = -\frac{1}{3}, a_1 = \frac{2}{5}, a_2 = -\frac{4}{7} \text{ and } a_3 = \frac{8}{9}.$$

Determine the nth term of a sequence $\{a_n\}$ whose first four terms are those given above.

Answer. $a_n = (-1)^{n+1}\dfrac{2^n}{2n+3}$.

Example 5

- What is the *n*th term of this sequence?

Example 5

- What is the nth term of this sequence?

Answer. $a_n = n! = 1 \cdot 2 \cdot \ldots \cdot n$.

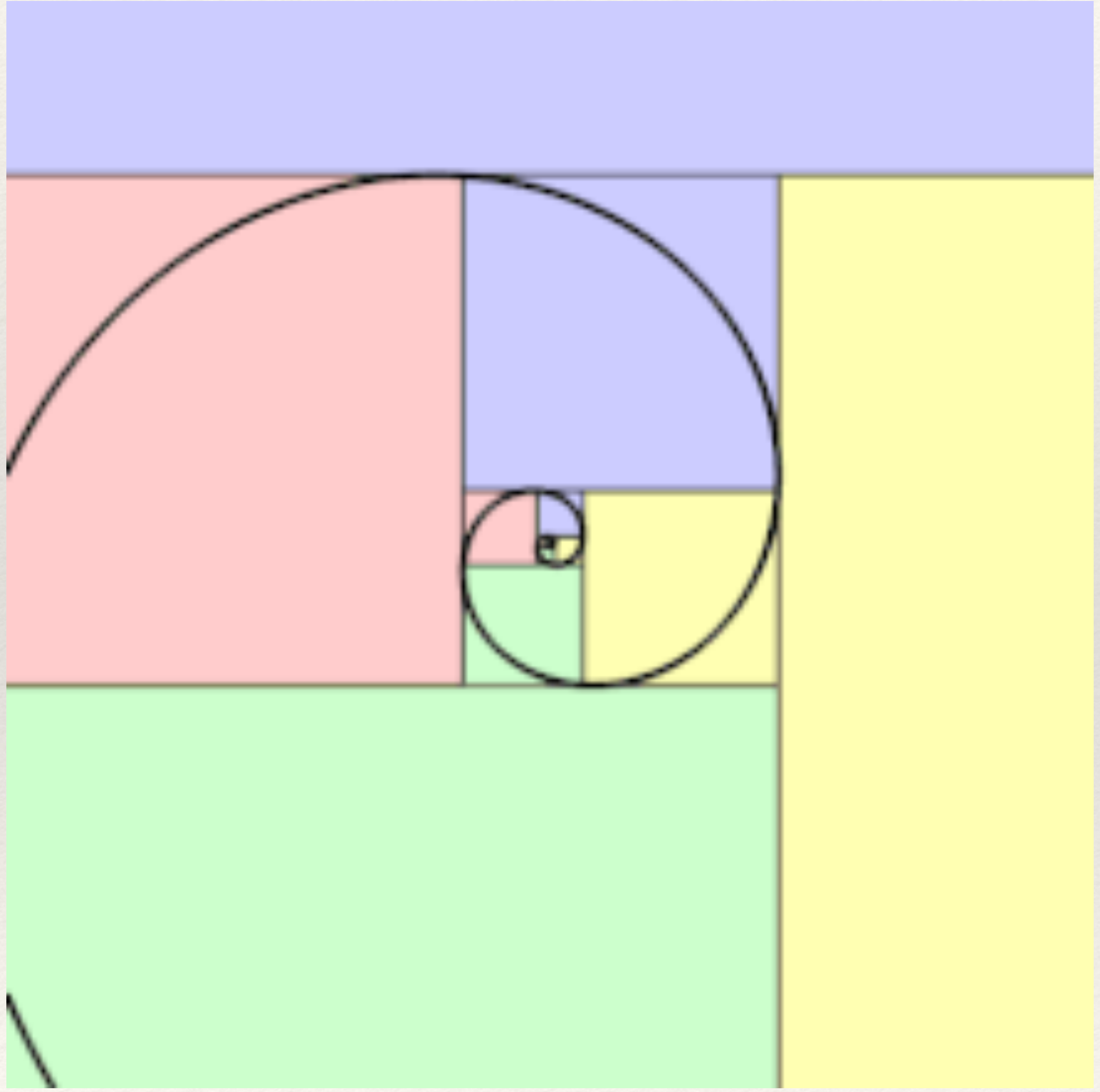
Explicitly Defined Sequences

SEQUENCE	IMPLICIT FORMULA	EXPLICIT FORMULA
$1, 2, 4, 8, 16, \ldots$		
	$a_1 = 10$ $a_n = 5 + a_{n-1}$	
		$a_n = n^2$

- In the examples above the sequences are **explicitly defined**, since we have an explicit formula (a closed-form expression) for the *n*th term.

  That is, the general term of the sequence can be determined simply by evaluating a given expression for an appropriate integer.

Recursively Defined Sequences



- A sequence $a_1, a_2, a_3, \ldots$ is **recursively defined** if:

  (1) for some fixed positive integer t, the terms $a_1, a_2, \ldots, a_t$ are given;

  (2) for each integer $n > t$, the term a_n is defined in terms of one or more of $a_1, a_2, \ldots, a_{n-1}$.

- Here, $a_1, a_2, \ldots, a_t$ are called the **initial values** of $\{a_n\}$.

- The relation that defines a_n in terms of $a_1, a_2, \ldots, a_{n-1}$ is called the **recurrence relation** for $\{a_n\}$.

Example 6

- A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 1 \text{ and } a_n = \frac{n}{n-1}a_{n-1} \text{ for } n \geq 2.$$

Determine a_2, a_3, a_4 and find an explicit formula for a_n.

Example 6

- A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 1 \text{ and } a_n = \frac{n}{n-1} a_{n-1} \text{ for } n \geq 2.$$

Determine a_2, a_3, a_4 and find an explicit formula for a_n.

Answer. $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and in general $a_n = n$.

Example 7

- A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 2,\ a_2 = 3 \ \text{ and } \ a_n = 2a_{n-1} - a_{n-2} \ \text{ for } \ n \geq 3.$$

Determine a_3, a_4, a_5 and find an explicit formula for a_n.

Example 7

- A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 2,\ a_2 = 3 \ \text{ and } \ a_n = 2a_{n-1} - a_{n-2} \ \text{ for } \ n \geq 3.$$

Determine a_3, a_4, a_5 and find an explicit formula for a_n.

Answer. $a_3 = 4$, $a_4 = 5$, $a_5 = 6$ and in general $a_n = n + 1$.

Example 7

- Find a recurrence relation and initial conditions for the sequence 1, 5, 17, 53, 161, 485, … .

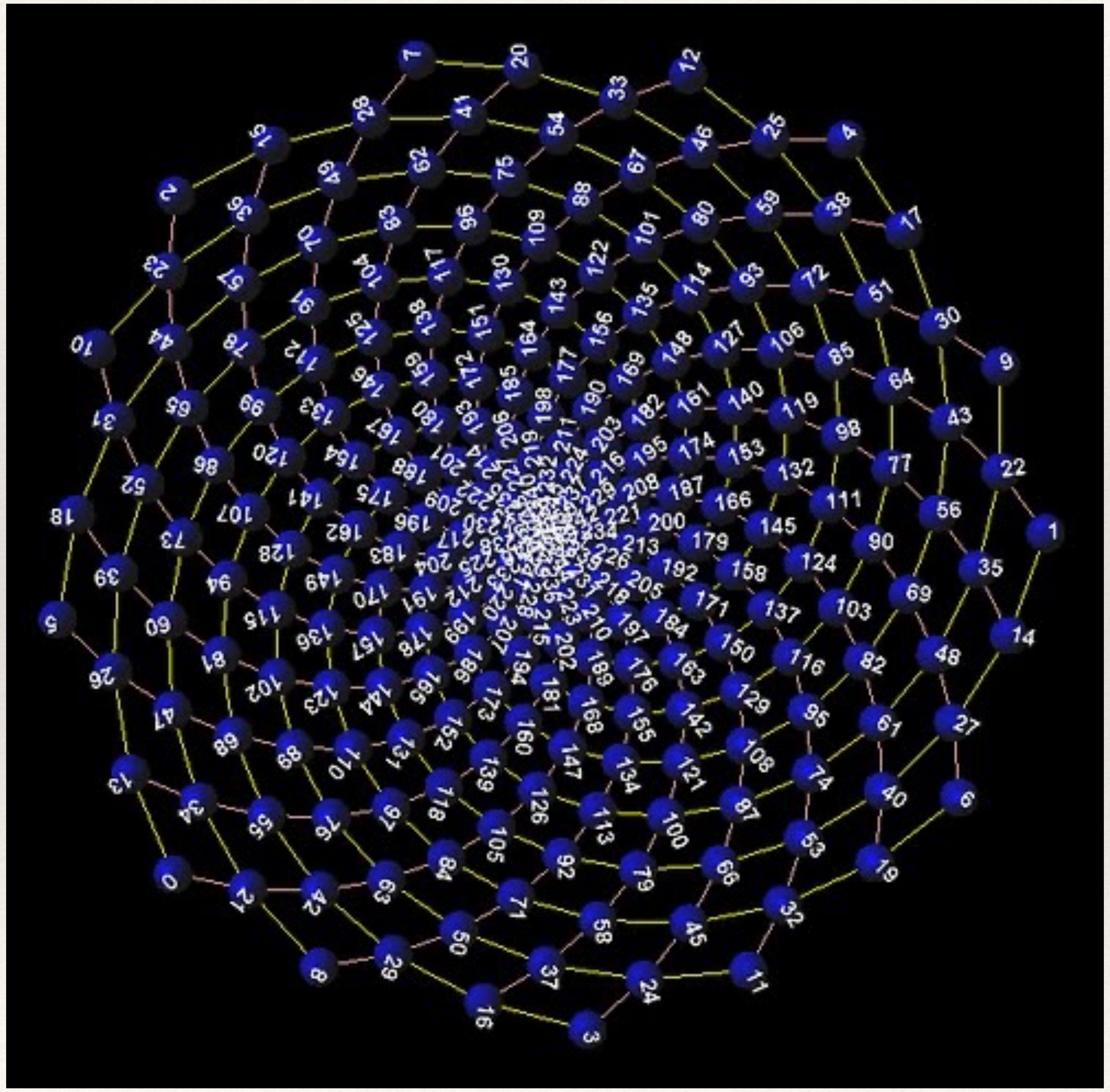
Example 7

- Find a recurrence relation and initial conditions for the sequence 1 , 5 , 17 , 53 , 161 , 485 … .

  Solution. We could look at the differences between terms: 4 , 12 , 36 , 108 , … . Notice that these are growing by a factor of 3. Is the original sequence as well? $1 \cdot 3 = 3$, $5 \cdot 3 = 15$, $17 \cdot 3 = 51$ and so on. It appears that we always end up with 2 less than the next term. Aha!

  So, $a_n = 3a_{n-1} + 2$ for $n \geq 2$ is our recurrence relation and the initial condition is $a_1 = 1$.

Fibonacci Numbers



- The **Fibonacci sequence** $\{F_n\}$ is defined recursively by

$$F_1 = 1,\ F_2 = 1 \ \text{ and}$$

$$F_n = F_{n-1} + F_{n-2} \ \text{ for } \ n \geq 3.$$

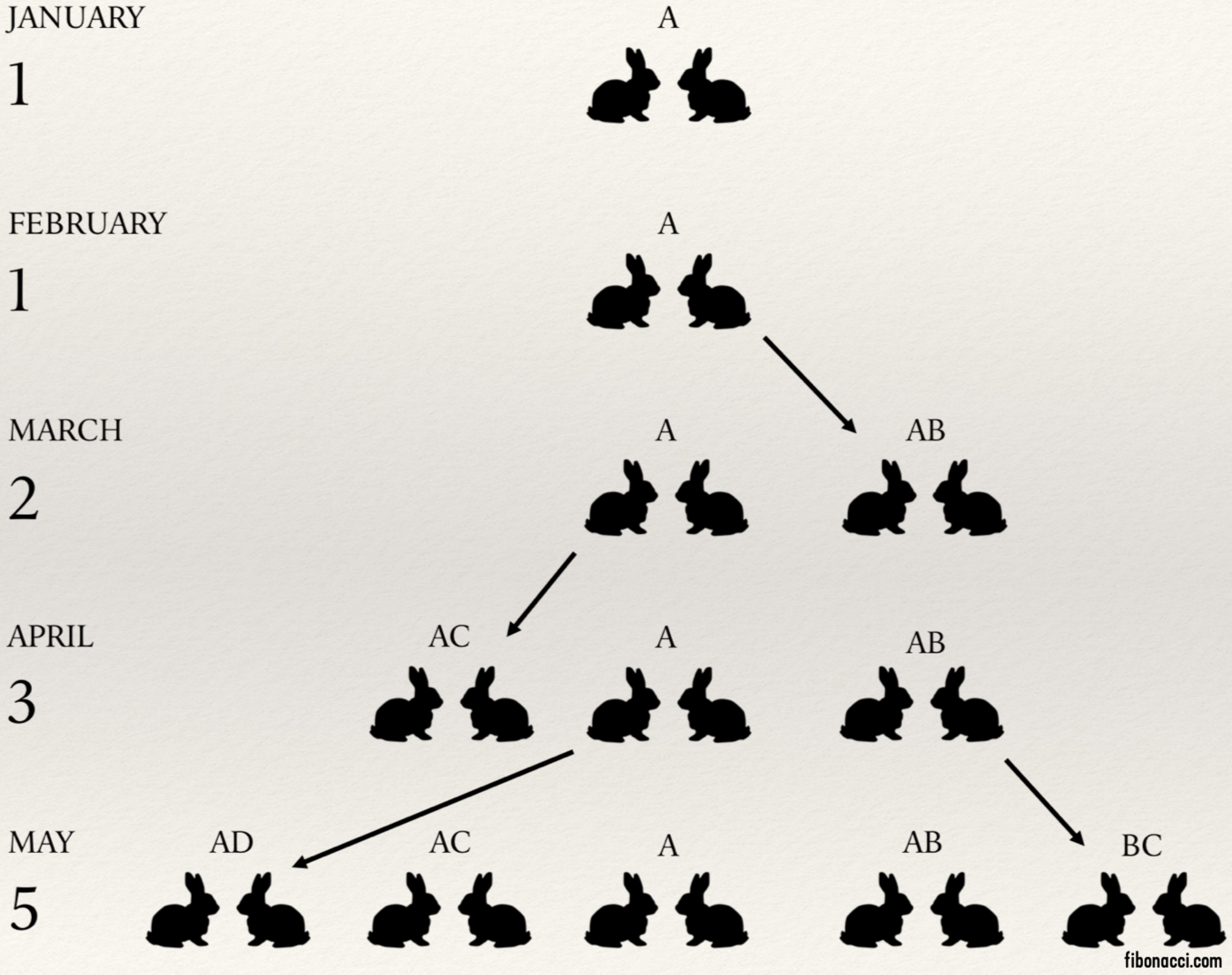
- The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

⌛ A bit of history…

- One of the great European mathematicians of the middle ages was **Leonardo da Pisa**, born in 1175 in Pisa, Italy, known for its famous Leaning Tower. **Leonardo** called himself **Fibonacci**. The name Fibonacci is a shortening of "filius Bonacci", which means "son of Bonaccio". His father's name was Guglielmo Bonaccio. (Bonacci is the plural of Bonaccio.)

  Fibonacci traveled a great deal in his early years about the Mediterranean coast and returned to Pisa in 1200.

⏳ A bit of history…



- With the knowledge he had acquired during his travels, Fibonacci wrote the book *Liber Abaci*, in which he introduced the decimal number system to the Latin-speaking world. There, he also stated the following problem:

  How Many Pairs of Rabbits Are Created by One Pair in One Year?
  A certain man had one pair of rabbits together in a certain enclosed place and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear a single pair and in the second month those born to bear also.

Example 8

- For a positive integer n, let s_n be the number of n-bit strings having no two consecutive 0s.

  (a) Determine s_1, s_2 and s_3.
  (b) Give a recursive definition of s_n for $n \geq 1$.
  (c) Use (b) to determine s_i for $1 \leq i \leq 6$.

Example 8

- For a positive integer n, let s_n be the number of n-bit strings having no two consecutive 0s.

  (a) Determine s_1, s_2 and s_3.
  (b) Give a recursive definition of s_n for $n \geq 1$.
  (c) Use (b) to determine s_i for $1 \leq i \leq 6$.

Solution. (a) Certainly, the two 1-bit strings 0 and 1 do not have two consecutive 0s and so $s_1 = 2$. The 2-bit string 00 is the only one of the four 2-bit strings with two consecutive 0s. Thus $s_2 = 3$. The eight 3-bit strings are

$$000\ 001\ 010\ 011\ 100\ 101\ 110\ 111.$$

Since 5 of these do not have two consecutive 0s, it follows that $s_3 = 5$.

Example 8

- For a positive integer n, let s_n be the number of n-bit strings having no two consecutive 0s.

  (a) Determine s_1, s_2 and s_3.
  (b) Give a recursive definition of s_n for $n \geq 1$.
  (c) Use (b) to determine s_i for $1 \leq i \leq 6$.

(b) We have noted in (a) that $s_1 = 2$ and $s_2 = 3$. For $n \geq 3$, an n-bit string with no two consecutive 0s either

— has 1 as the last bit and so the first $n - 1$ bits give an $(n - 1)$-bit string having no two consecutive 0s or

— has 10 as the last two bits and so the first $n - 2$ bits give an $(n - 2)$-bit string having no two consecutive 0s.

Consequently, for $n \geq 3$, we have $s_n = s_{n-1} + s_{n-2}$ and so the recursive relation is:

$$s_1 = 2,\ s_2 = 3 \ \text{ and } \ s_n = s_{n-1} + s_{n-2} \ \text{ for } \ n \geq 3.$$

Example 8

- For a positive integer n, let s_n be the number of n-bit strings having no two consecutive 0s.

  (a) Determine s_1, s_2 and s_3.
  (b) Give a recursive definition of s_n for $n \geq 1$.
  (c) Use (b) to determine s_i for $1 \leq i \leq 6$.

(c) This gives us $s_1 = 2$, $s_2 = 3$,
$s_3 = s_2 + s_1 = 3 + 2 = 5$, $s_4 = s_3 + s_2 = 5 + 3 = 8$,
$s_5 = s_4 + s_3 = 8 + 5 = 13$, $s_6 = s_5 + s_4 = 13 + 8 = 21$,
all of which, of course, are Fibonacci numbers.

Example 9

- Prove that, for every integer $n \geq 2$,

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n .$$

Example 9

- Prove that, for every integer $n \geq 2$,

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n.$$

Proof. We proceed by induction. When $n = 2$, we have

$$F_1F_3 = 1 \cdot 2 = 1^2 + 1 = F_2^2 + (-1)^2.$$

Thus the formula holds for $n = 2$. Assume that the equality is true for some $n = k$, that is,

$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k \quad \text{or else} \quad F_k^2 = F_{k-1}F_{k+1} - (-1)^k.$$

Let us then show that it is also true for $n = k + 1$:

$$F_kF_{k+2} = F_{k+1}^2 + (-1)^{k+1}.$$

Using the recursion relation for the Fibonacci numbers, we obtain

$$\begin{aligned} F_kF_{k+2} &= F_k(F_k + F_{k+1}) = F_k^2 + F_kF_{k+1} \\ &= [F_{k-1}F_{k+1} - (-1)^k] + F_kF_{k+1} \\ &= F_{k-1}F_{k+1} + F_kF_{k+1} + (-1)^{k+1} \\ &= (F_{k-1} + F_k)F_{k+1} + (-1)^{k+1} \\ &= F_{k+1}F_{k+1} + (-1)^{k+1} = F_{k+1}^2 + (-1)^{k+1}. \end{aligned}$$

By the Principle of Mathematical Induction, we conclude that

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n \quad \text{for every integer } n \geq 2.$$

🙋 A question arises…

- Given a recursively defined sequence, how to find an explicit formula for it?

🙋 A question arises…

- Given a recursively defined sequence, how to find an explicit formula for it?

  Answer. For linear recursive sequences, there is a general method.

The Characteristic Root Technique

- **Theorem 1.** Given a recurrence relation
$$a_n + \alpha a_{n-1} + \beta a_{n-2} = 0,$$
the **characteristic polynomial** is
$$x^2 + \alpha x + \beta$$
giving the **characteristic equation**:
$$x^2 + \alpha x + \beta = 0.$$
If r_1 and r_2 are two distinct roots of the characteristic polynomial (*i.e.*, solutions to the characteristic equation), then the **solution** to the recurrence relation is
$$a_n = ar_1^n + br_2^n,$$
where a and b are constants determined by the initial conditions.

Example 10

- Solve the recurrence relation

$$a_n = 7a_{n-1} - 10a_{n-2}$$

with $a_0 = 2$ and $a_1 = 3$.

Example 10

- Solve the recurrence relation

  $a_n = 7a_{n-1} - 10a_{n-2}$

  with $a_0 = 2$ and $a_1 = 3$.

Solution. Rewrite the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 0.$$

Now form the characteristic equation:

$$x^2 - 7x + 10 = 0$$

and solve for x:

$$(x-2)(x-5) = 0$$

so $x = 2$ and $x = 5$ are the characteristic roots. We therefore know that the solution to the recurrence relation will have the form

$$a_n = a2^n + b5^n.$$

To find a and b, plug in $n = 0$ and $n = 1$ to get a system of two equations with two unknowns:

$$2 = a2^0 + b5^0 = a + b,$$
$$3 = a2^1 + b5^1 = 2a + 5b.$$

Solving this system gives $a = \dfrac{7}{3}$ and $b = -\dfrac{1}{3}$, so the solution to the recurrence relation is

$$a_n = \frac{7}{3}2^n - \frac{1}{3}5^n.$$

Characteristic Root Technique for Repeated Roots

- **Theorem 2.** If the recurrence relation
$$a_n + \alpha a_{n-1} + \beta a_{n-2} = 0,$$
has a characteristic polynomial with only one root r, then the solution to the recurrence relation is
$$a_n = ar^n + bnr^n,$$
where a and b are constants determined by the initial conditions.

Example 11

- Solve the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 4$.

Example 11

- Solve the recurrence relation

  $$a_n = 6a_{n-1} - 9a_{n-2}$$

  with $a_0 = 1$ and $a_1 = 4$.

Solution. The characteristic polynomial is $x^2 - 6x + 9$. We solve the characteristic equation

$$x^2 - 6x + 9 = 0$$

by factoring:

$$(x - 3)^2 = 0$$

so $x = 3$ is the only characteristic root. Therefore we know that the solution to the recurrence relation has the form

$$a_n = a3^n + bn3^n$$

for some constants a and b. Now use the initial conditions:

$$a_0 = 1 = a3^0 + b \cdot 0 \cdot 3^0 = a,$$
$$a_1 = 4 = a \cdot 3^1 + b \cdot 1 \cdot 3^1 = 3a + 3b.$$

Since $a = 1$, we find that $b = \frac{1}{3}$. So, the solution to the recurrence relation is

$$a_n = 3^n + \frac{1}{3}n3^n = 3^{n-1}(3 + n).$$

Example 12

- For the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2}$$

with $F_1 = 1$ and $F_2 = 1$, find a closed-form expression.

Example 12

- For the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2}$$

with $F_1 = 1$ and $F_2 = 1$, find the characteristic equation and a closed-form expression.

Answer. The characteristic equation is $x^2 - x - 1 = 0$ and

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Thank you!