Lecture 2

Relations and Functions

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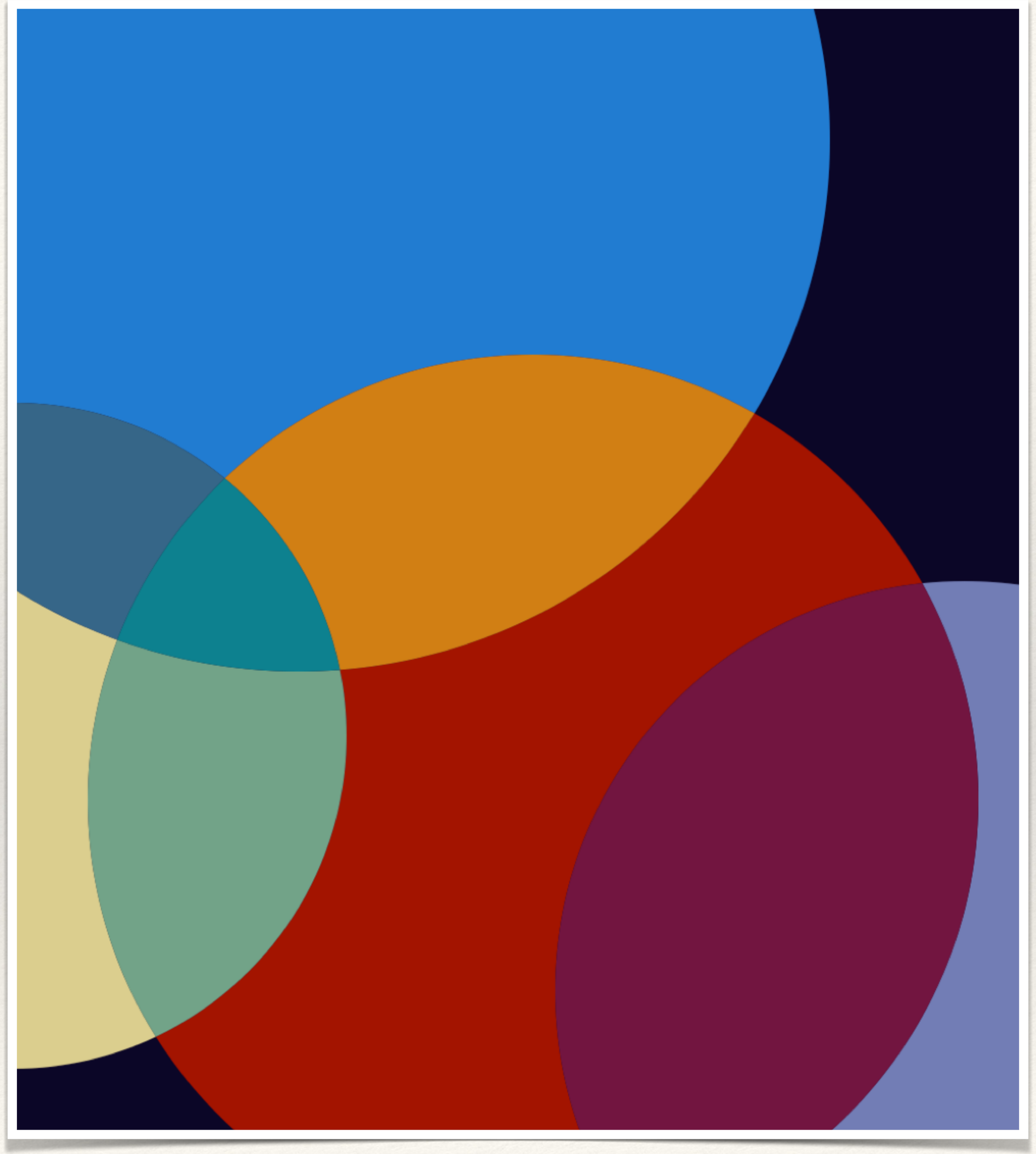


What is a Relation?

- For two sets A and B, the **Cartesian product** $A \times B$ is the set of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. That is,

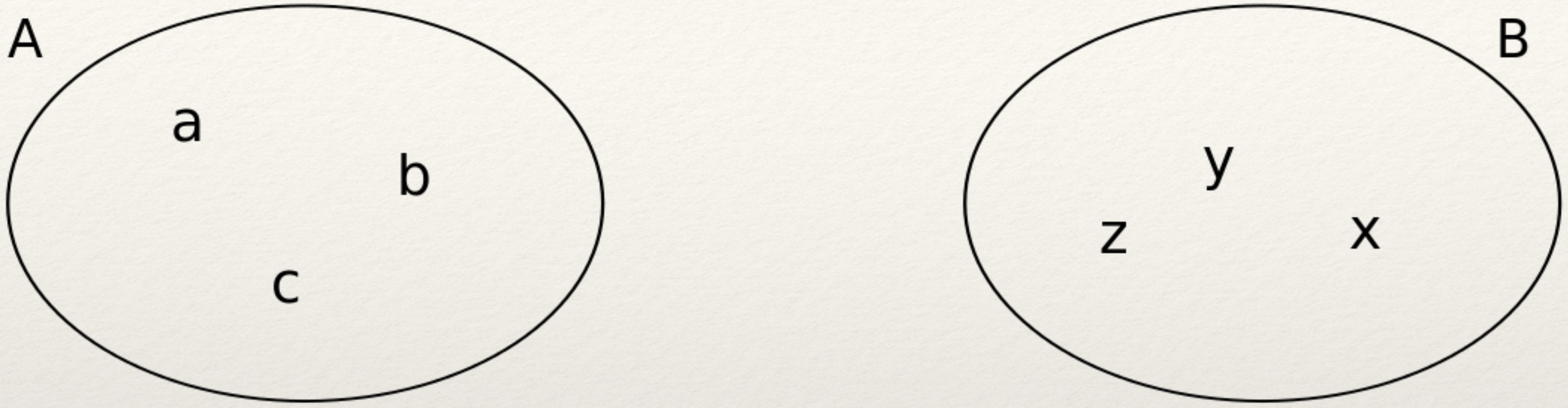
$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

- A **relation** (also called a **binary relation**) R from a set A to a set B is a subset of $A \times B$.

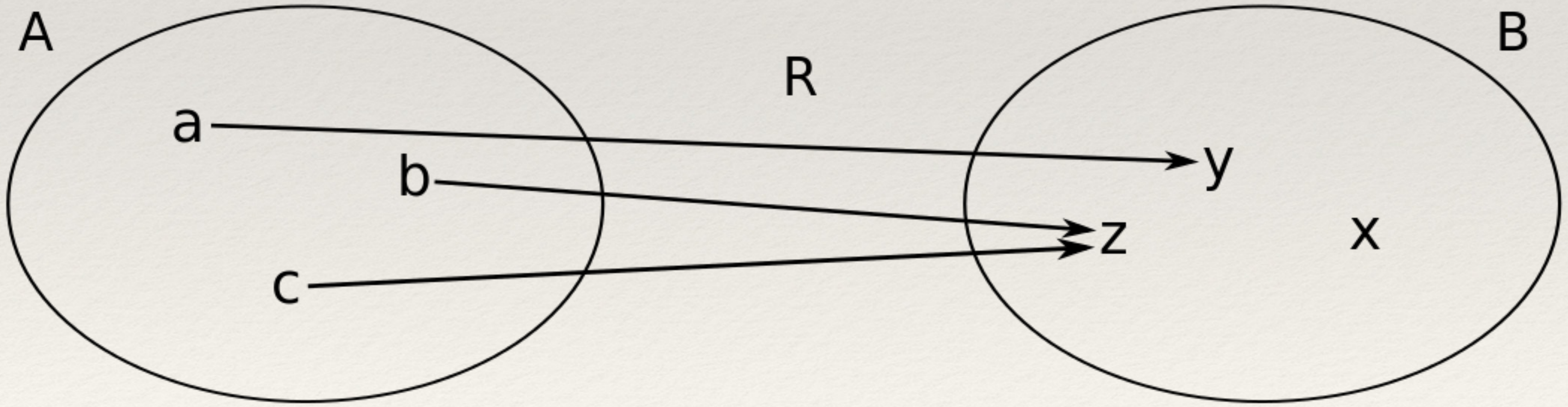
- If $(a, b) \in R$, then a is said to be **related** to b, and we write

$$a \, R \, b$$

Example 1



$$A \times B = \{(a,x), (b,x), (c,x), (a,y), (b,y), (c,y), (a,z), (b,z), (c,z)\}$$



$$R = \{(a,y), (b,z), (c,z)\}$$

Example 2

- For the sets $A = \{0, 1\}$ and $B = \{1, 2, 3\}$, let
  $R = \{(0, 2), (0, 3), (1, 2)\}$ be a relation from A to B.

  Then $0 \, R \, 2$, $0 \, R \, 3$ and $1 \, R \, 2$.

  Since 0 is not related to 1, and 1 is not related to 3, we can also indicate this by writing

  $$0 \not R \, 1 \quad \text{and} \quad 1 \not R \, 3.$$

Example 3

- Let A be the set of positive integers and let B denote the set of negative integers. Define a relation R from A to B:

  $$a \; R \; b \;\; \text{if} \;\; a + b \in \mathbb{N}.$$

  Give examples of some pairs of elements that are related by R and some that are not.

Example 3

- Let A be the set of positive integers and let B denote the set of negative integers. Define a relation R from A to B:
$$a \; R \; b \;\text{ if }\; a+b \in \mathbb{N}.$$
Give examples of some pairs of elements that are related by R and some that are not.

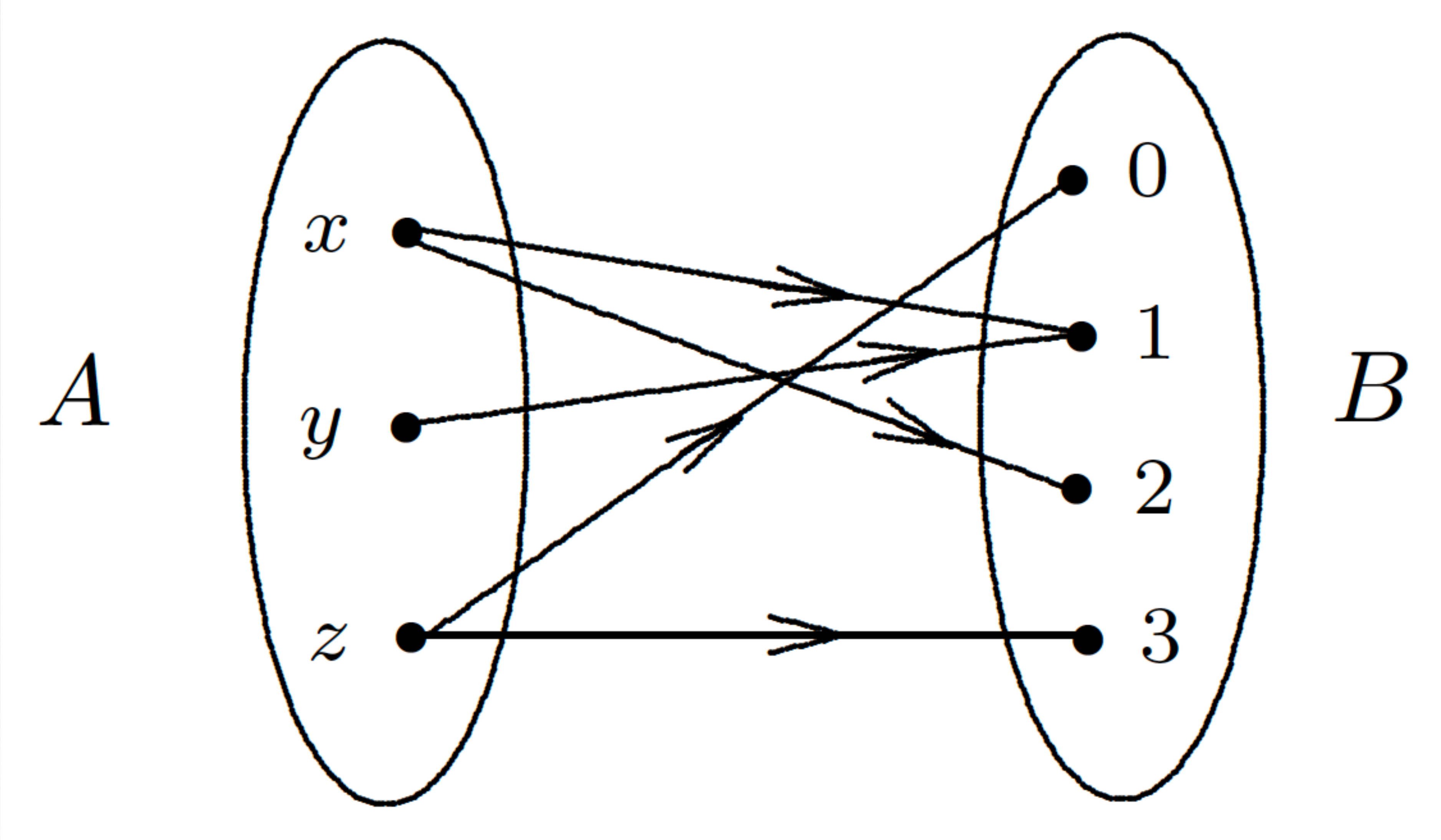
  Solution. We have $5 \; R \; (-2)$, since $5+(-2)=3 \in \mathbb{N}$. Because $2+(-2)=0 \notin \mathbb{N}$, it follows that $2 \not{R} \; (-2)$.

Example 4

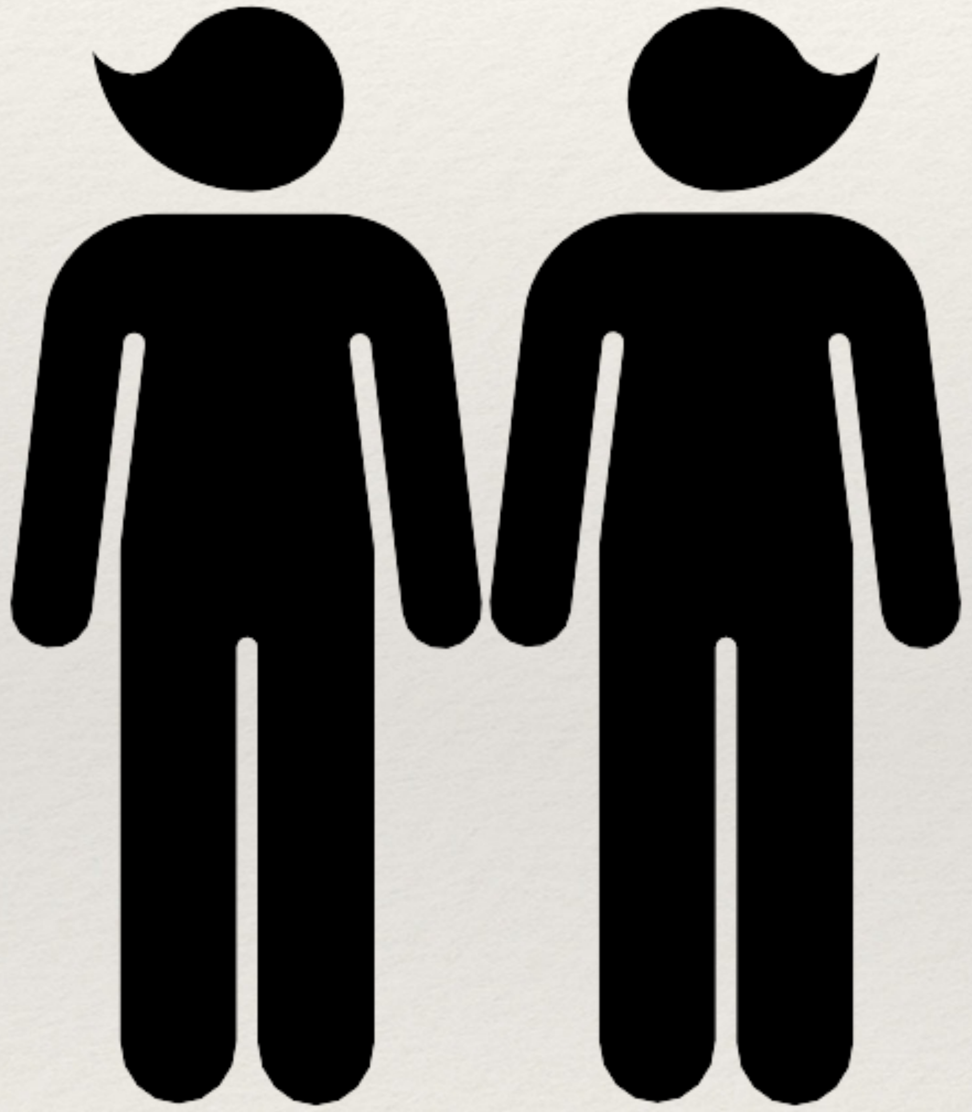
- For the sets $A = \{x, y, z\}$ and $B = \{0,1,2,3\}$,

$$R = \{(x,1), (x,2), (y,1), (z,0), (z,3)\}$$

is a relation from A to B, which can be presented in the diagram below.



Types of Relations on a Set



- A **relation** R on a set S is a relation from S to S. In other words, R is a subset of $S \times S$.

- Let R be a relation defined on a nonempty set S. Then R is

  — **reflexive** if $a \; R \; a$ for all $a \in S$;

  — **symmetric** if whenever $a \; R \; b$, then $b \; R \; a$;

  — **transitive** if whenever $a \; R \; b$ and $b \; R \; c$, then $a \; R \; c$.

Example 5

- Let $S = \{1, 2, 3, 4\}$. Consider the following relation on S:

  $$R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1)\}.$$

  Which of the properties reflexive, symmetric and transitive does R possess?

- Let R be a relation defined on a nonempty set S. Then R is
  — **reflexive** if $a \, R \, a$ for all $a \in S$;
  — **symmetric** if whenever $a \, R \, b$, then $b \, R \, a$;
  — **transitive** if whenever $a \, R \, b$ and $b \, R \, c$, then $a \, R \, c$.

Example 5

- Let $S = \{1, 2, 3, 4\}$. Consider the following relation on S:

  $$R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1)\}.$$

  Which of the properties reflexive, symmetric and transitive does R possess?

  Solution. The relation R is not reflexive since $(4,4) \notin R$.
  The relation R is not symmetric, since $(3,1) \in R$ but $(1,3) \notin R$.
  The relation R is not transitive, since $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$.

Example 6

- A relation R is defined on the set $\mathbb{N}$ of positive integers by $a\,R\,b$ if $a < b$. Which of the properties reflexive, symmetric and transitive does R possess?

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- A relation R is defined on the set $\mathbb{N}$ of positive integers by $a\,R\,b$ if $a < b$. Which of the properties reflexive, symmetric and transitive does R possess?

  Solution. It isn't reflexive ($a \nless a$), it isn't symmetric ($a < b \not\Rightarrow b < a$) but it is transitive ($a < b,\, b < c \;\Rightarrow\; a < c$).

Example 7

- A relation R is defined on the set $\mathbb{Z}$ of integers by $a \, R \, b$ if $ab \geq 0$. Which of the properties reflexive, symmetric and transitive does R possess?

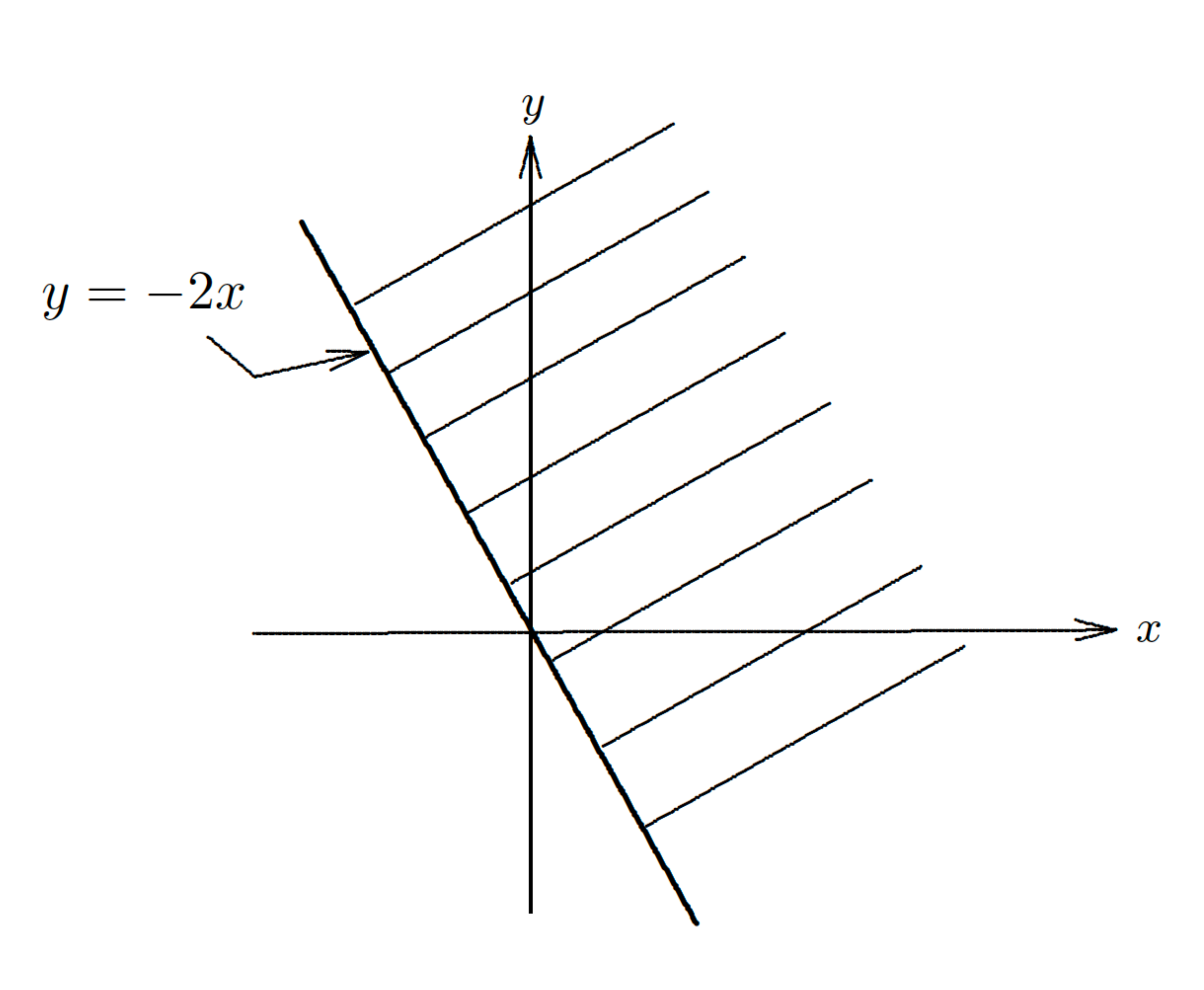
Example 7

- A relation R is defined on the set $\mathbb{Z}$ of integers by $a \, R \, b$ if $ab \geq 0$. Which of the properties reflexive, symmetric and transitive does R possess?

  Answer. Reflexive, symmetric but not transitive.

Example 8

- A relation R is defined on the set $\mathbb{R}$ of real numbers by
$$x \, R \, y \quad \text{if} \quad 2x + y \geq 0.$$
That is, $x \, R \, y$ if (x, y) is a point in the Euclidean plane that lies on or to the right of the line $y = -2x$.



(a) Give an example of two real numbers a and b such that $a \, R \, b$ and two real numbers c and d such that $c \not R \, d$.

(b) Is R reflexive?

(c) Is R symmetric?

(d) Is R transitive?

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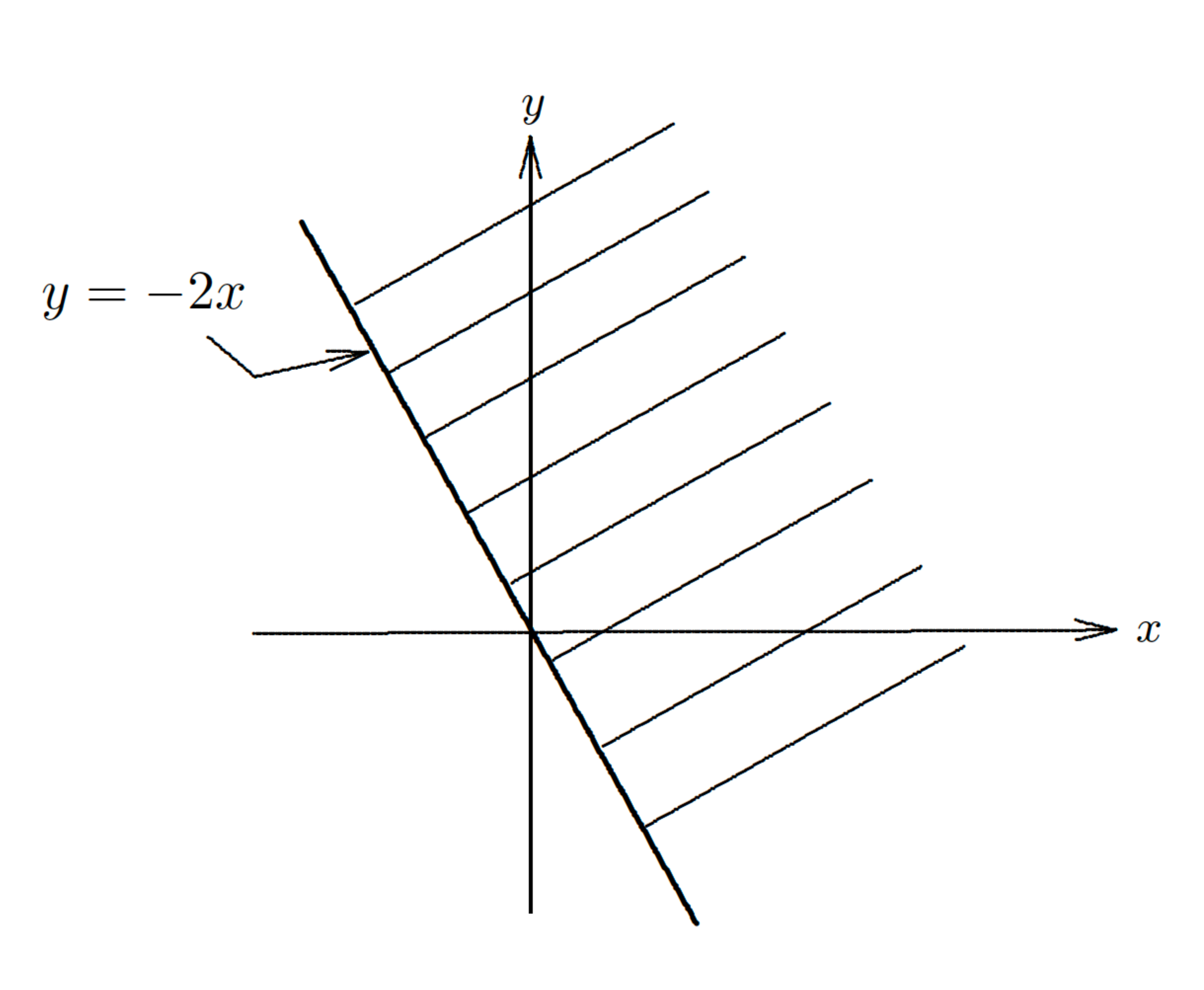
(a) Give an example of two real numbers a and b such that $a \; R \; b$ and two real numbers c and d such that $c \not{R} \; d$.

(b) Is R reflexive?

(c) Is R symmetric?

(d) Is R transitive?

Answer. (a) $3 \; R \; 1$ and $(-3) \not{R} \; (-1)$; (b) no; (c) no; (d) no.



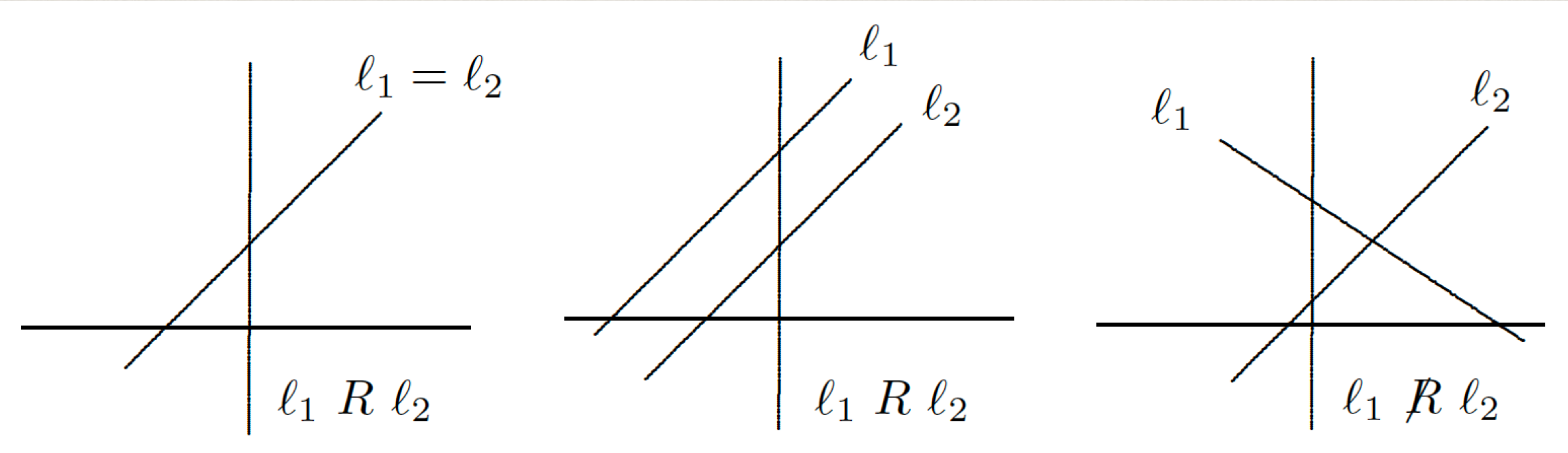
$\stackrel{\text{def}}{=}$ Equivalence Relations

- A relation R on a nonempty set is an **equivalence relation** if R is reflexive, symmetric and transitive.

- Let R be an equivalence relation on a set A. For $a \in A$, the **equivalence class** $[a]$ is defined by
$$[a] = \{x \in A \ : \ x \, R \, a\}.$$
In other words, $[a]$ is the subset of all elements of A that are related to a by R.

Example 9

- A relation R is defined on the set L of straight lines in the Euclidean plane by $l_1 \; R \; l_2$ if two lines l_1 and l_2 coincide or are parallel. Explain why R is an equivalence relation.



Example 10

- A relation R is defined on $\mathbb{N} \times \mathbb{N}$ by
$$(a, b)\ R\ (c, d) \quad \text{if} \quad ad = bc.$$
Is R is an equivalence relation?

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- A relation R is defined on $\mathbb{N} \times \mathbb{N}$ by
$$(a, b)\ R\ (c, d) \quad \text{if} \quad ad = bc.$$
Is R is an equivalence relation?

  Answer. Yes, because R is reflexive, symmetric and transitive.

Example 11

- Let $S = \{1, 2, 3, 4, 5, 6\}$. Consider the following relation on S:

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (2, 3), (2, 6), (3, 2), (3, 6), (4, 1), (6, 2), (6, 3)\}.$$

  (a) Is R an equivalence relation?

  (b) Describe the equivalence classes [1], [2], [3], [4], [5], [6].

- Let R be an equivalence relation on a set A. For $a \in A$, the **equivalence class** $[a]$ is defined by

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- Let $S = \{1, 2, 3, 4, 5, 6\}$. Consider the following relation on S:

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (2, 3), (2, 6), (3, 2), (3, 6), (4, 1), (6, 2), (6, 3)\}.$$

(a) Is R an equivalence relation?

(b) Describe the equivalence classes $[1], [2], [3], [4], [5], [6]$.

Answer. (a) Yes;

(b) $[1] = \{1, 4\}$, $[2] = \{2, 3, 6\}$, $[3] = \{2, 3, 6\}$,
$[4] = \{1, 4\}$, $[5] = \{5\}$, $[6] = \{2, 3, 6\}$.

Example 12

- A relation R is defined on $\mathbb{Z}$ by $a \; R \; b$ if $a + b$ is even.

  (a) Show that R is an equivalence relation.

  (b) Describe the equivalence classes [0], [1], [–3] and [4].

Example 12

- A relation R is defined on $\mathbb{Z}$ by $a\,R\,b$ if $a + b$ is even.

  (a) Show that R is an equivalence relation.

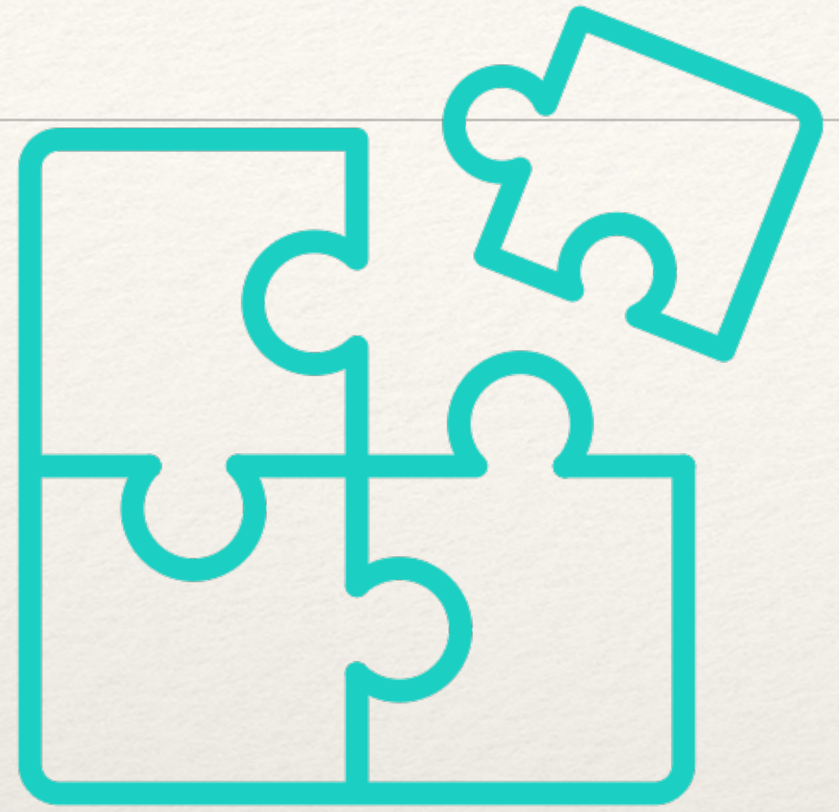
  (b) Describe the equivalence classes [0], [1], [−3] and [4].

  Answer. (a) One has to check that R is is reflexive, symmetric and transitive.
  (b) The equivalence classes are

$$
\begin{aligned}
[0] &= \{x \in \mathbb{Z}: x\,R\,0\} = \{x \in \mathbb{Z} : x + 0 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is even}\} = \{\ldots,-4,-2, 0, 2, 4, \ldots\}\\
[1] &= \{x \in \mathbb{Z} : x\,R\,1\} = \{x \in \mathbb{Z} : x + 1 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is odd}\} = \{\ldots,-5,-3,-1, 1, 3, 5, \ldots\}\\
[-3] &= \{x \in \mathbb{Z} : x\,R\,(-3)\} = \{x \in \mathbb{Z} : x - 3 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is odd}\} = \{\ldots,-5,-3,-1, 1, 3, 5, \ldots\}\\
[4] &= \{x \in \mathbb{Z} : x\,R\,4\} = \{x \in \mathbb{Z} : x + 4 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is even}\} = \{\ldots,-4,-2, 0, 2, 4, \ldots\}.
\end{aligned}
$$

  Consequently, in this case [0] = [4] and [1] = [−3].

Equivalence Relations

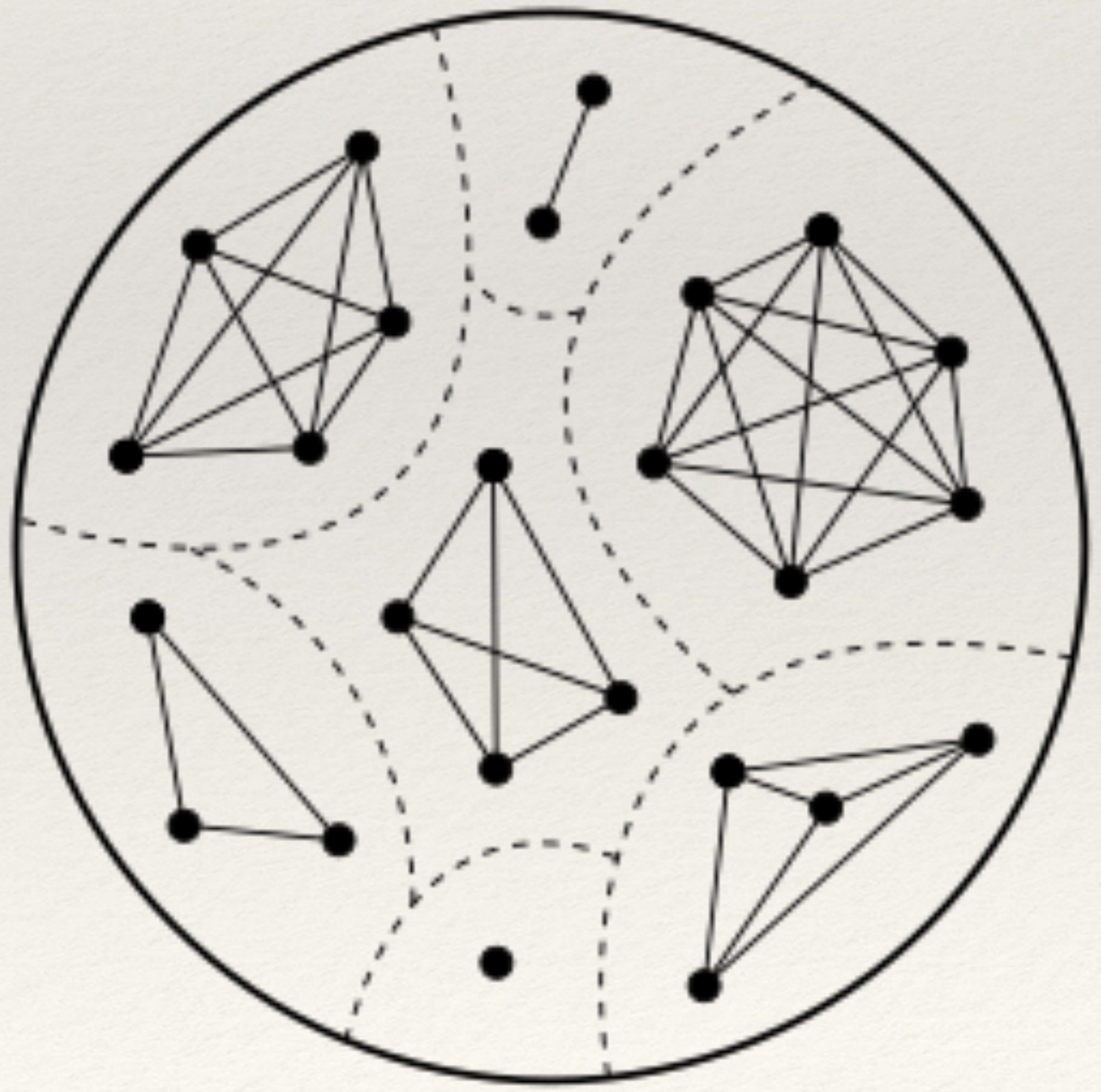


- A **partition** of a nonempty set A is a collection of nonempty subsets of A such that every element of A belongs to exactly one of these subsets.

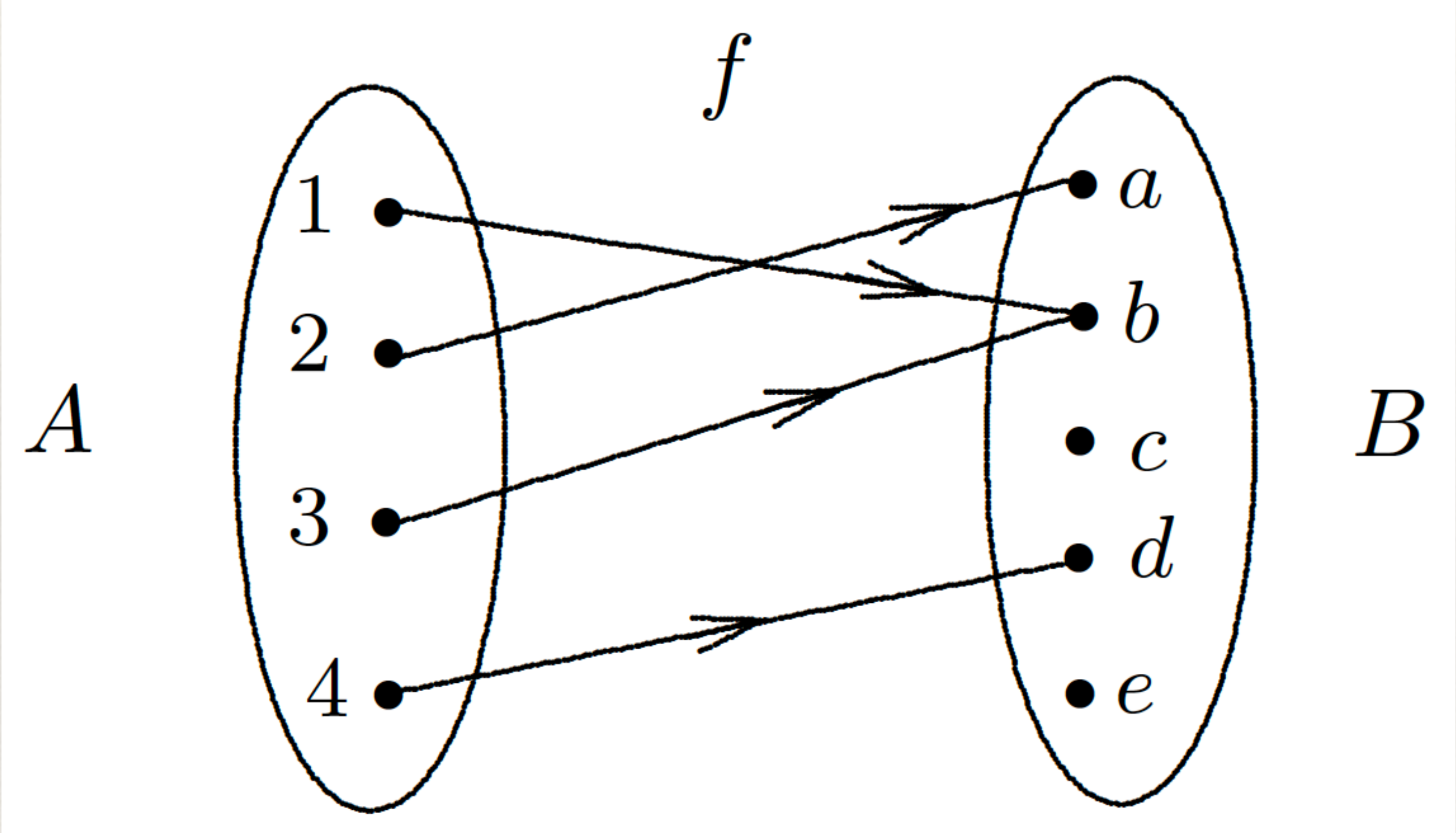
- **Theorem 1.** Let R be an equivalence relation on a nonempty set A and let a and b be elements of A. Then $[a] = [b]$ if and only if $a \; R \; b$.

- **Theorem 2.** Let R be an equivalence relation defined on a nonempty set A. Then the set of all distinct equivalence classes of A resulting from R form a partition of A.

  In particular, if $[a]$ and $[b]$ are two equivalence classes, then either $[a] = [b]$ or $[a] \cap [b] = \varnothing$.



≝ Functions



- Let A and B be two nonempty sets. A **function** f **from** A **to** B is a relation from A to B that associates with each element of A a *unique* element of B. It is denoted by $f : A \to B$.

- The set A is called the **domain** of f, and B is the **codomain** of f.

- If $b \in B$ is the unique element assigned to $a \in A$ by f, then we write $b = f(a)$ and say that b is the **image** of a under f.

- Let X be a subset of A. The **image** of X under f is the set
$$f(X) = \{f(x) : x \in X\}.$$

- The **range** of f is the image of its domain, that is $f(A)$.

Example 13

- Let $A = \{p, q\}$ and $B = \{0, 1, 2, 3, 5\}$. How many functions from A to B are there?

Example 13

- Let $A = \{p, q\}$ and B = $\{0, 1, 2, 3, 4\}$. How many functions from A to B are there?

  Answer. $25 = 5^2$.

$\stackrel{\text{def}}{=}$ One-to-one Functions (Injections)

- For two nonempty sets A and B, a function $f : A \to B$ is said to be **one-to-one** if every two *distinct* elements of A have *distinct* images in B, that is,

$$a \neq b \;\Rightarrow\; f(a) \neq f(b).$$

A one-to-one function is also referred to as an **injective function** or an **injection**.

Example 14

- Let $A = \{a, b, c\}$ and $B = \{w, x, y, z\}$. Consider the functions $f : A \to B$ and $g : A \to B$ defined by

  $f = \{(a, x), (b, z), (c, w)\}$ and $g = \{(a, w), (b, y), (c, w)\}$.

  Are they one-to-one?

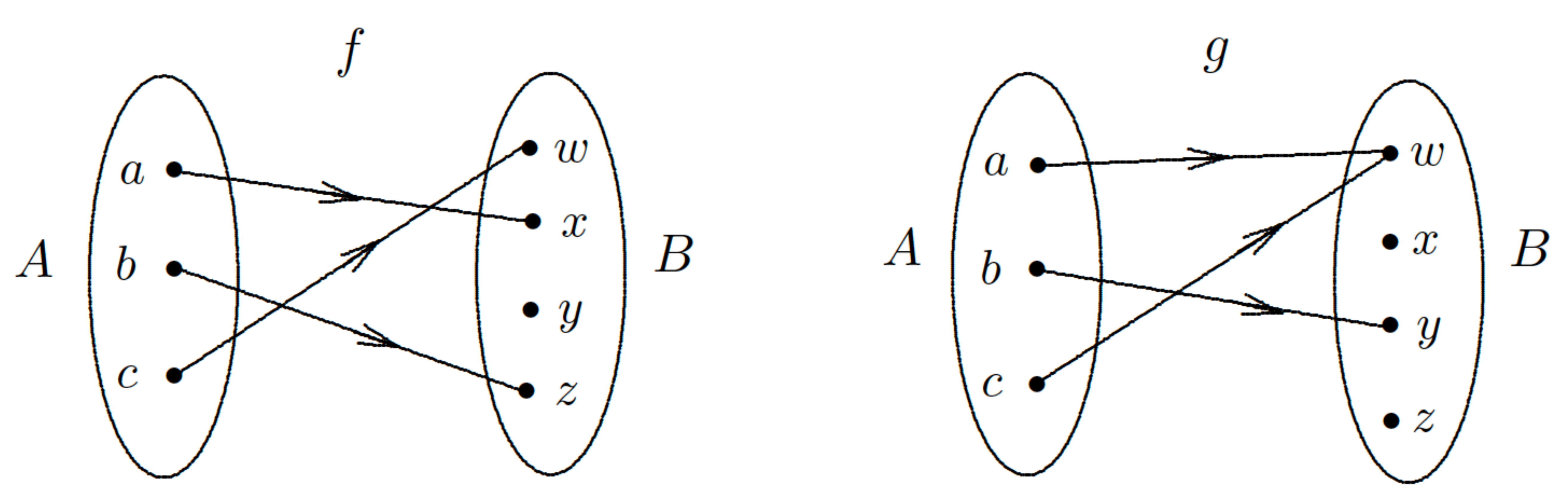
Example 14

- Let $A = \{a, b, c\}$ and $B = \{w, x, y, z\}$. Consider the functions $f : A \to B$ and $g : A \to B$ defined by

  $$f = \{(a, x), (b, z), (c, w)\} \text{ and } g = \{(a, w), (b, y), (c, w)\}.$$

  Are they one-to-one?

  Answer. f is injective but g isn't.



Example 15

- Check whether the following functions are one-to-one:

  (a) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 + 1$ for $x \in \mathbb{R}$.

  (b) $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = \lceil n/2 \rceil$ for $n \in \mathbb{Z}$.

  (c) $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) = x^2 - 3x + 1$ for $x \in \mathbb{R}$.

  (d) $k : \mathbb{R} \to \mathbb{R}$ is defined by $k(x) = 5x - 3$ for $x \in \mathbb{R}$.

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  (d) $k : \mathbb{R} \to \mathbb{R}$ is defined by $k(x) = 5x - 3$ for $x \in \mathbb{R}$.

  Answer. (a) no; (b) no; (c) no; (d) yes.

$\stackrel{\text{def}}{=}$ Onto Functions (Surjections)

- A function $f : A \to B$ is called **onto** if every element of B is the image of some element of A, that is,

$$f(A) = B.$$

An onto function is also called a **surjective function** or a **surjection**.

Example 16

- Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d\}$. Consider the functions $f : A \to B$ and $g : A \to B$ defined by

  $f = \{(1, b), (2, d), (3, d), (4, a), (5, c)\}$ and
  $g = \{(1, a), (2, a), (3, c), (4, c), (5, d)\}$.

  Are they onto?

Example 16

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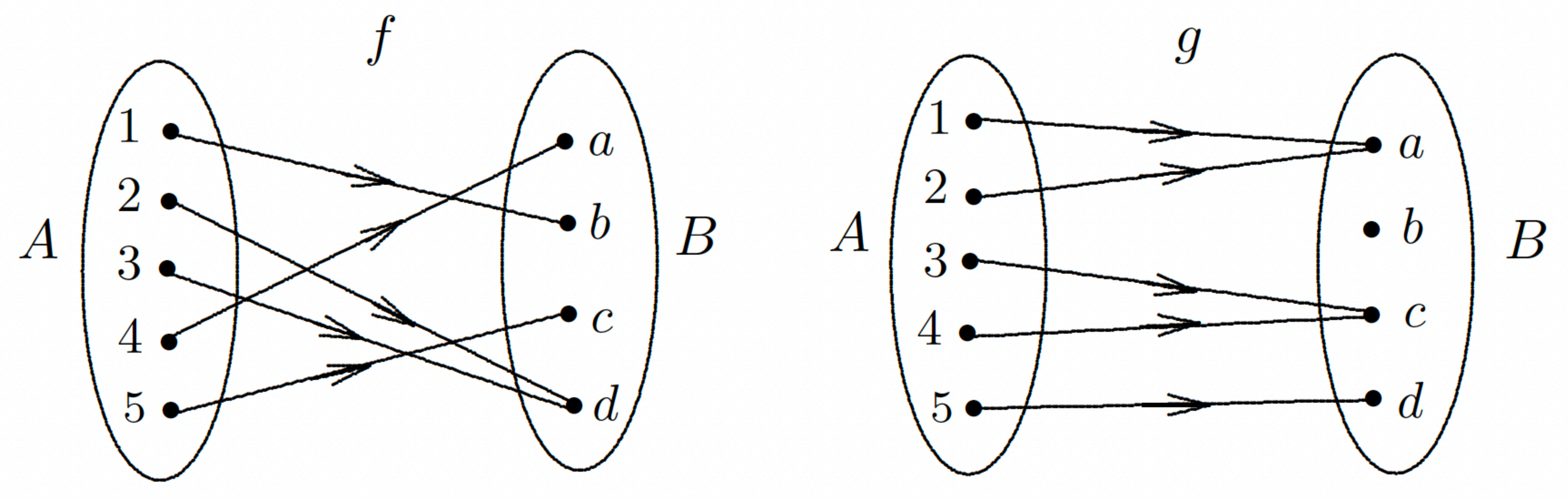
  $f = \{(1, b), (2, d), (3, d), (4, a), (5, c)\}$ and
  $g = \{(1, a), (2, a), (3, c), (4, c), (5, d)\}$.

  Are they onto?

  Answer. f is surjective but g isn't.



Example 17

- Check whether the following functions are onto:

(a) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 4x - 9$ for $x \in \mathbb{R}$.

(b) $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 3n$ for $n \in \mathbb{Z}$.

(c) $y : \mathbb{R} \to \mathbb{R}$ is defined by $y(x) = x^2 - 2x + 5$ for $x \in \mathbb{R}$.

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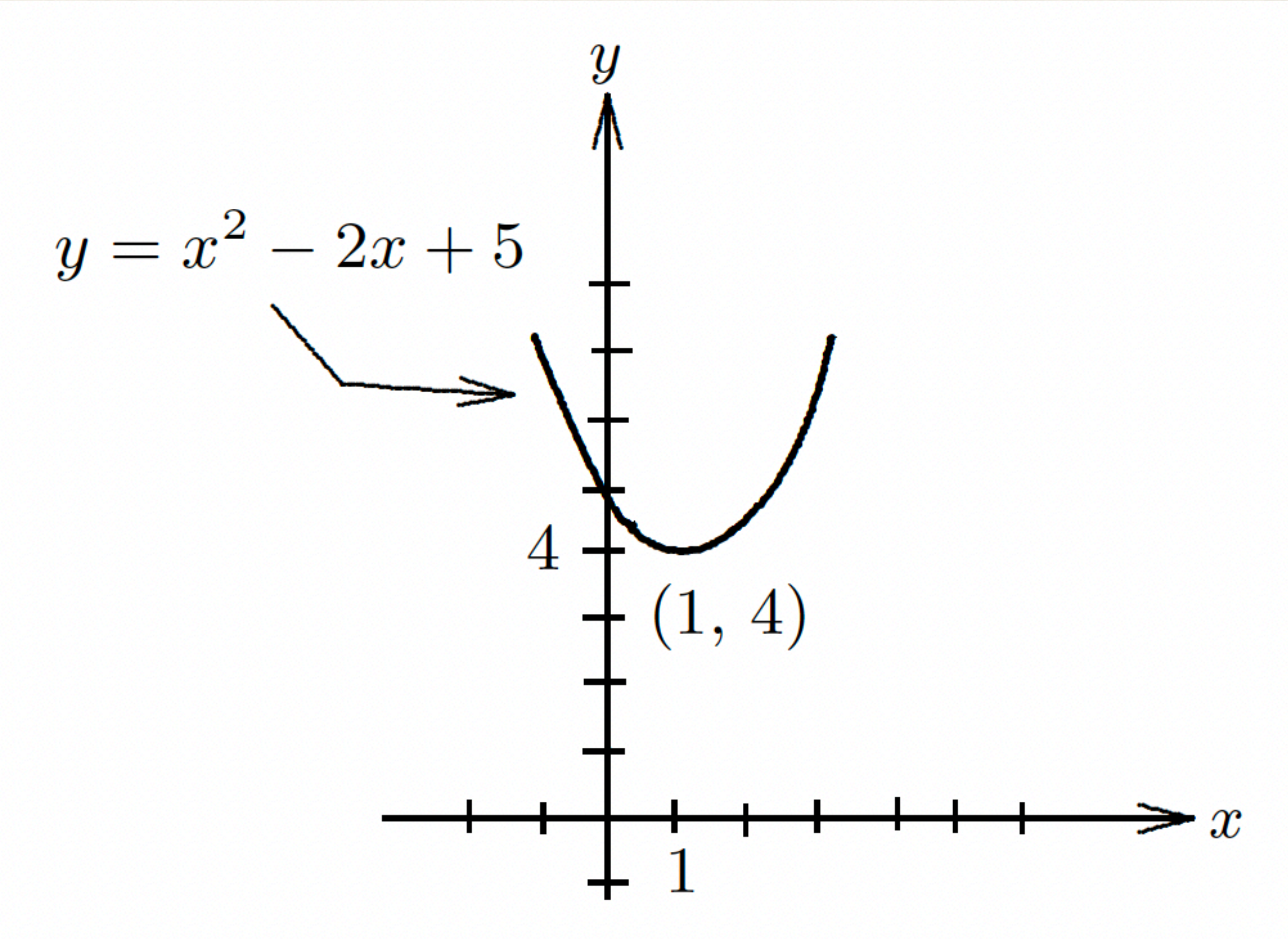
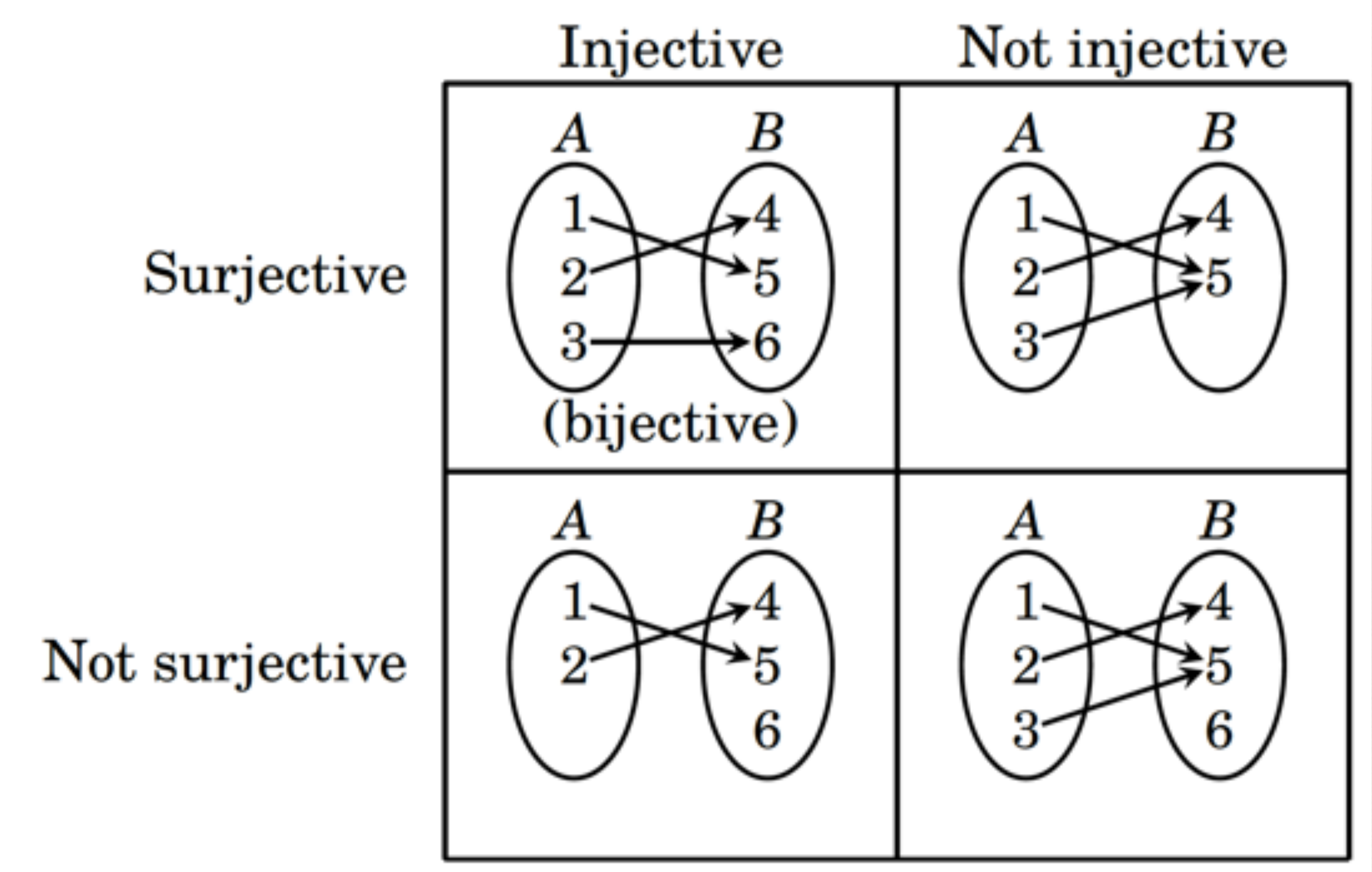
  Answer. (a) yes; (b) no; (c) no.



$\stackrel{\text{def}}{=}$ Bijections



- If a function $f : A \to B$ is one-to-one and onto, then it is called a **bijective function**, a **bijection** or a **one-to-one correspondence**.

bijection = injection + surjection

- A bijective function f from A to A is also called a **permutation** on (or of) A.

Example 18

- Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, w\}$. Consider the function $f : A \to B$ defined by

$$f = \{(1, y), (2, w), (3, z), (4, x)\}.$$

Is it bijective?

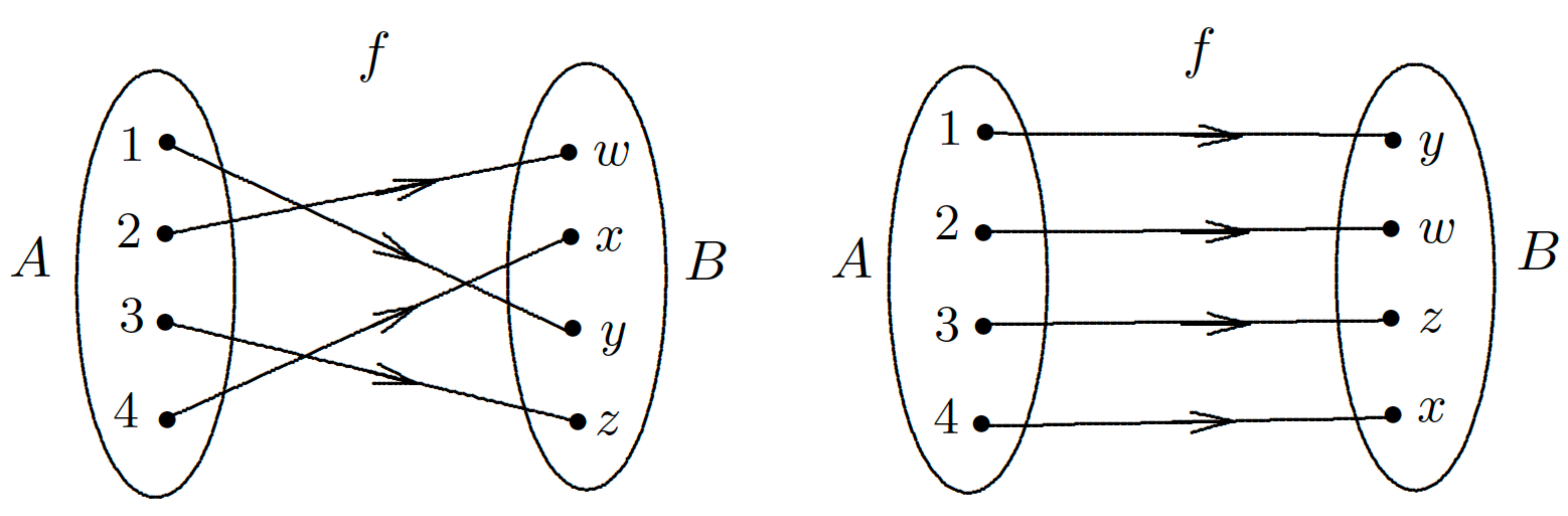
Example 18

- Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, w\}$. Consider the function $f : A \to B$ defined by

$$f = \{(1, y), (2, w), (3, z), (4, x)\}.$$

Is it bijective?

Answer. yes.



Example 19

- Check whether the following functions are bijective:

  (a) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $f(x) = \sqrt{x}$ for $x \in \mathbb{R}^+$.

  (b) $g : \mathbb{R}^+ \to \mathbb{R}$ is defined by $g(x) = \sqrt{x}$ for $x \in \mathbb{R}^+$.

  (c) $h : \mathbb{R}^+ \to \mathbb{R}$ is defined by $h(x) = \sqrt[3]{x}$ for $x \in \mathbb{R}^+$.

  (d) $u : \mathbb{R} \to \mathbb{R}$ is defined by $u(x) = \sqrt[3]{x}$ for $x \in \mathbb{R}$.

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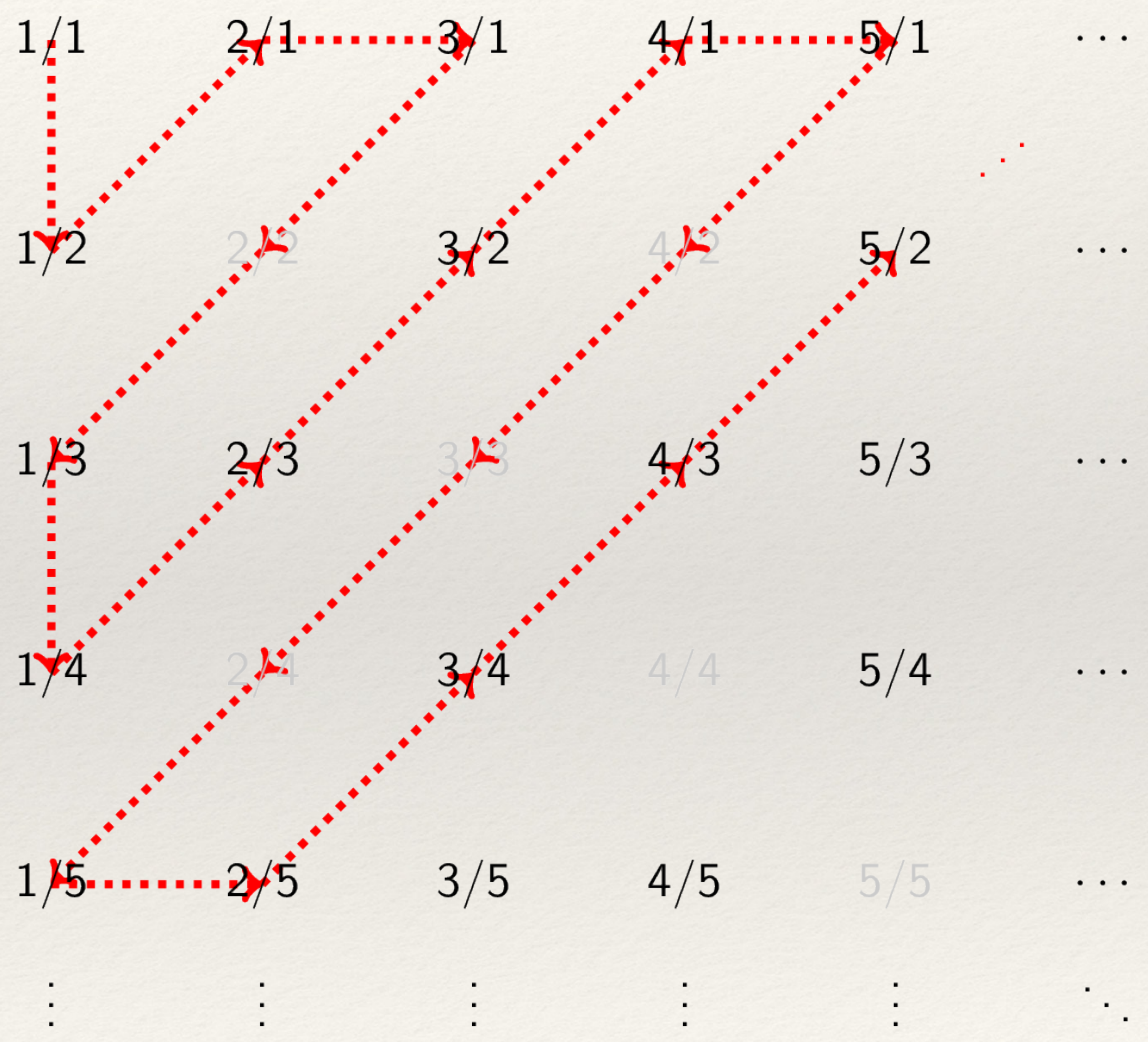
(c) $h : \mathbb{R}^+ \to \mathbb{R}$ is defined by $h(x) = \sqrt[3]{x}$ for $x \in \mathbb{R}^+$.

(d) $u : \mathbb{R} \to \mathbb{R}$ is defined by $u(x) = \sqrt[3]{x}$ for $x \in \mathbb{R}$.

Answer. (a) yes; (b) no; (c) no; (d) yes.

Functions and Cardinalities

- **Theorem 3.** Let $f : A \to B$ be a function.
  - If f is injective, then $|A| \leq |B|$.
  - If f is surjective, then $|A| \geq |B|$.
  - If f is bijective, then $|A| = |B|$.
- That is how one could compare "cardinalities" of infinite sets: if there exists a bijection between them, then have the same "cardinality".
- If there is a bijection from $\mathbb{N}$ to a set A, then the set A is said to be **countable** (or **denumerable**).



Example 20

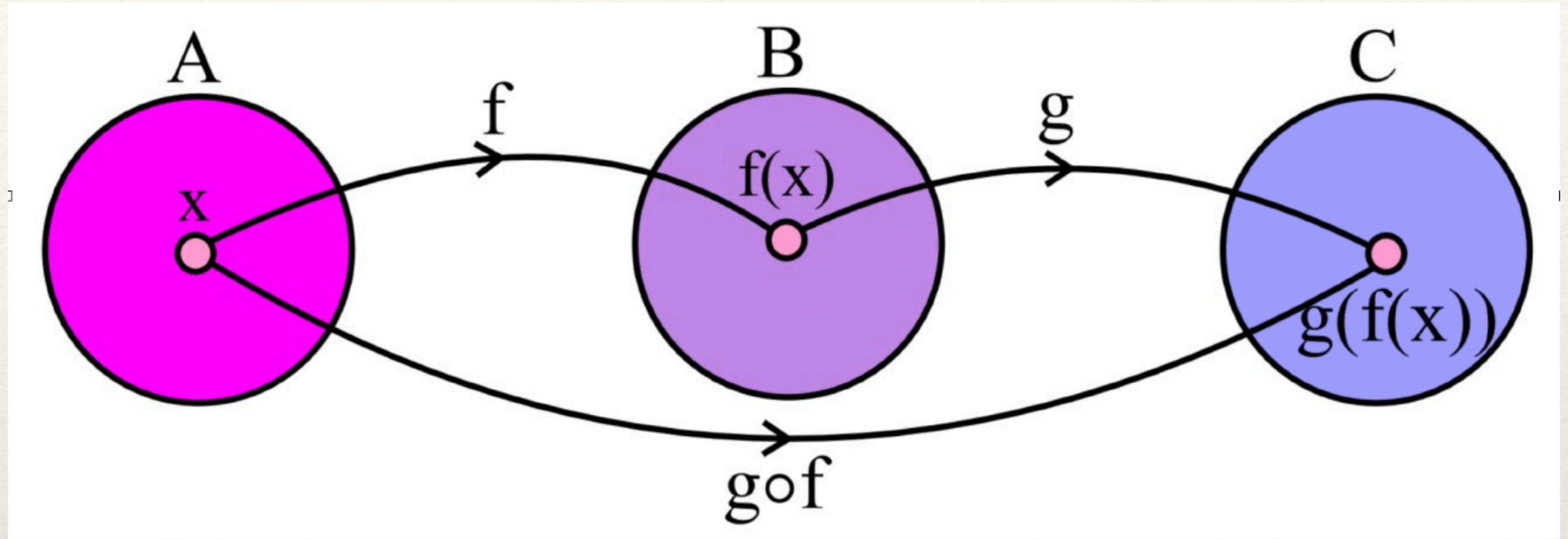
- $\mathbb{Q}$ is countable
- $\mathbb{R}$ isn't countable

$$
\begin{array}{cccccc}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \cdots \\
1/4 & 2/4 & 3/4 & 4/4 & 5/4 & \cdots \\
1/5 & 2/5 & 3/5 & 4/5 & 5/5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

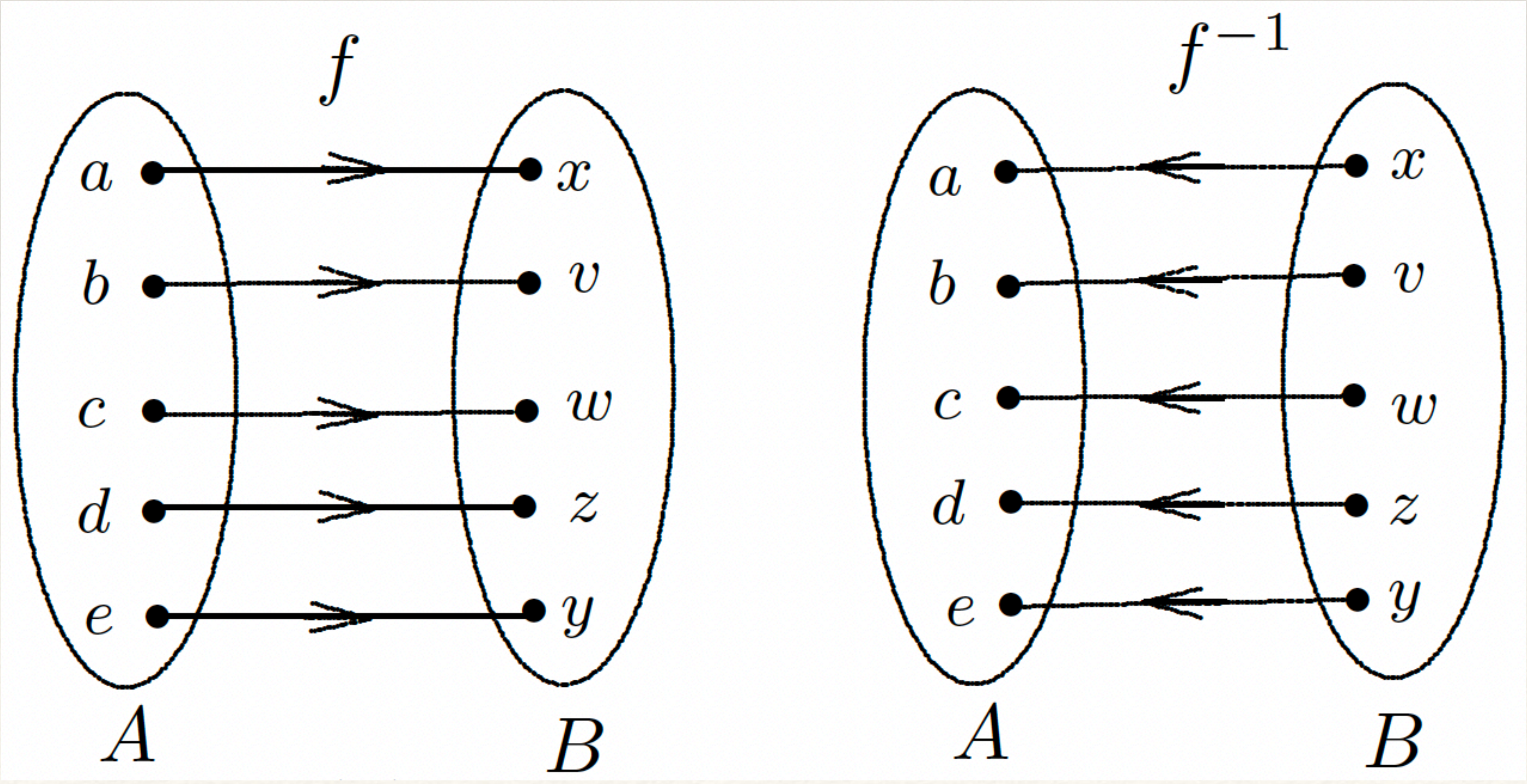
Compositions and Inverse Functions

- **Theorem 4.** Let $f : A \to B$ and $g : B \to C$ be two functions.

  - If f and g are injective, then so is $g \circ f$.
  - If f and g are surjective, then so is $g \circ f$.
  - If f and g are bijective, then so is $g \circ f$.



- **Theorem 5.** A function $f : A \to B$ has an inverse function $f^{-1} : B \to A$ if and only if f is bijective.
  Moreover, if f is bijective, then so is f^{-1}.



Thank you!