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FSTE
Cycle d'ingénieurs
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Ex. 2 (AFC)

$$X = \begin{pmatrix} 10 & 10 \\ 5 & 15 \\ 15 & 5 \end{pmatrix} \begin{matrix} 20 \\ 20 \\ 20 \end{matrix}$$

$$\begin{matrix} 30 & 30 & 60 \end{matrix}$$

1°)

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{12} \end{pmatrix} \begin{matrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{matrix}$$

$$\begin{matrix} \frac{1}{2} & \frac{1}{2} & 1 \end{matrix}$$

2°) a) Nuage $\mathcal{B}(I)$:

$M_1 = (\beta_{11}, \beta_{12})$; $M_2 = (\beta_{21}, \beta_{22})$;

et $M_3 = (\beta_{31}, \beta_{32})$

avec $\beta_{ij} = \dfrac{P_{ij}}{P_{i\cdot}\sqrt{P_{\cdot j}}}$

$$M_1 = \left(\frac{P_{11}}{P_{1\cdot}\sqrt{P_{\cdot 1}}} ; \frac{P_{12}}{P_{1\cdot}\sqrt{P_{\cdot 2}}} \right) = \left(\frac{1/2}{\sqrt{1/2}} ; \frac{1/2}{\sqrt{1/2}} \right)$$

$$M_1 = \left(\frac{\sqrt{2}}{2} ; \frac{\sqrt{2}}{2} \right)$$

$$M_2 = \left(\frac{P_{21}}{P_{2\cdot}\sqrt{P_{\cdot 1}}} ; \frac{P_{22}}{P_{2\cdot}\sqrt{P_{\cdot 2}}} \right) = \left(\frac{1/4}{\sqrt{1/2}} ; \frac{3/4}{\sqrt{1/2}} \right)$$

$$M_2 = \left(\frac{\sqrt{2}}{4} ; \frac{3\sqrt{2}}{4} \right)$$

$$M_3 = \left(\frac{P_{31}}{P_{3\cdot}\sqrt{P_{\cdot 1}}} ; \frac{P_{32}}{P_{3\cdot}\sqrt{P_{\cdot 2}}} \right) = \left(\frac{3/4}{\sqrt{1/2}} ; \frac{1/4}{\sqrt{1/2}} \right)$$

$$M_3 = \left(\frac{3\sqrt{2}}{4} ; \frac{\sqrt{2}}{4} \right)$$

2) b) Distances χ^2 dans $\mathcal{B}(I)$:

entre M_i et $M_{i'}$; $d^2(M_i, M_{i'}) = \sum_j (\beta_{ij} - \beta_{i'j})^2$

$$d^2(M_1, M_2) = (\beta_{11} - \beta_{21})^2 + (\beta_{12} - \beta_{22})^2$$
$$= \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{4}\right)^2 = \left(\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{-\sqrt{2}}{4}\right)^2 = \frac{1}{4}$$

$$d^2(M_1, M_3) = (\beta_{11} - \beta_{31})^2 + (\beta_{12} - \beta_{32})^2$$
$$= \left(\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right)^2 = \left(-\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4}$$

$$d^2(M_2, M_3) = (\beta_{21} - \beta_{31})^2 + (\beta_{22} - \beta_{32})^2$$
$$= \left(\frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\right)^2 = \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$$

3) a) Matrice des variances-covariances W du nuage $\mathcal{B}(I)$

$$W = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$$v_{11} = \sum_i p_{i.}\left(\beta_{i1} - \sqrt{p_{.1}}\right)^2 = p_{1.}\left(\beta_{11} - \sqrt{p_{.1}}\right)^2 + p_{2.}\left(\beta_{21} - \sqrt{p_{.1}}\right)^2$$
$$+ p_{3.}\left(\beta_{31} - \sqrt{p_{.1}}\right)^2$$

$$V_{11} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\right)^2 + \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2$$

$$V_{11} = \frac{1}{3}\left(\frac{2}{16}\right) + \frac{1}{3}\left(\frac{2}{16}\right) = \frac{1}{12}.$$

$$V_{22} = p_{1\cdot}\left(\beta_{12} - \sqrt{p_{\cdot 2}}\right)^2 + p_{2\cdot}\left(\beta_{22} - \sqrt{p_{\cdot 2}}\right)^2 + p_{3\cdot}\left(\beta_{32} - \sqrt{p_{\cdot 2}}\right)^2$$

$$= \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)^2$$

$$= \frac{1}{3} \times 0^2 + \frac{1}{3} \times \frac{2}{16} + \frac{1}{3} \times \frac{2}{16} = \frac{1}{12}$$

$$V_{12} = V_{21} = p_{\cdot 1}\left(\beta_{11} - \sqrt{p_{\cdot 1}}\right)\left(\beta_{12} - \sqrt{p_{\cdot 2}}\right) + p_{2\cdot}\left(\beta_{21} - \sqrt{p_{\cdot 1}}\right)\left(\beta_{22} - \sqrt{p_{\cdot 2}}\right) +$$

$$+ p_{3\cdot}\left(\beta_{31} - \sqrt{p_{\cdot 1}}\right)\left(\beta_{32} - \sqrt{p_{\cdot 2}}\right)$$

$$V_{12} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) + \frac{1}{3}\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right) +$$

$$+ \frac{1}{3}\left(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\right) = -\frac{1}{12} = V_{21}$$

D'où

$$W = \begin{pmatrix} \frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{12} \end{pmatrix}$$

b) Valeurs propres de W :

$$\det(W-\lambda I) = \begin{vmatrix} \frac{1}{12}-\lambda & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{12}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{12}-\lambda\right)^2 - \left(\frac{1}{12}\right)^2 = 0$$

$$\Rightarrow \left(\frac{1}{12}\right)^2 - \frac{\lambda}{6} + \lambda^2 - \left(\frac{1}{12}\right)^2 = 0$$

$$\Rightarrow \lambda\left(\lambda - \frac{1}{6}\right) = 0 \Rightarrow \lambda = 0 \ ; \ \lambda = \frac{1}{6}$$

$$\Rightarrow \lambda_{max} = \frac{1}{6}$$

c) Variabilité totale de $\mathcal{B}(I)$:

$$V_B = tr(W) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

4°) Variabilité totale de $\mathcal{C}(I)$:

$$V_C = \lambda_{max} = \frac{1}{6}$$

5°) Variabilité expliquée est $\delta = \frac{V_C}{V_B} = \frac{\lambda_{max}}{tr(W)} = 1$