Exercise 1

- derive orthog. proj on range

$$A = \begin{array}{cc} a_1 & a_2 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ is not normalized} \rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P_A = Q \Sigma Q^T, \quad \Sigma = Q^T P Q = \mathrm{diag}(1,1,0)$$

$$Q = \left[\tfrac{1}{\sqrt{2}} a_1 \,\middle|\, a_2 \,\middle|\, 0 \right]$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P = B(B^T B)^{-1} B^T$$

$$(B^TB)^{-1} = \frac{1}{6}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

$$B(B^TB)^{-1}B^T = \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

- QR decomposition
- A matrix: its columns are already orthogonal

$$a_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Q matrix: $\begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}$

R matrix: $Q^TA = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow A = QR = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

- B matrix

Gram-Schmigt orthog.

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad e_1 = \frac{g_1}{\|g_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$g_2 = b_2 - (b_2, e_1) e_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$(b_2, e_1) = \frac{1}{\sqrt{2}}(1 + 0 + 1) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\Rightarrow \quad Q = \begin{bmatrix} 1/\sqrt{2} & 1 \\ 0 & 1 \\ 1/\sqrt{2} & -1 \end{bmatrix}$$

$$R = Q^T B = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad B = \begin{bmatrix} 1/\sqrt{2} & 1 \\ 0 & 1 \\ 1/\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{bmatrix}$$