Subject: 401133011 تمرین سری سوم بهینه سازی محدب - امین عبدی پیرامل

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سوال 1) قیدهای نامساوی : $f_i(x)$

$$L(x_i,\lambda_i) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

$$= 4x_1 + 3x_2 + \lambda_1^T(-5x_1 - x_2 + 11) + \lambda_2^T(-2x_1 - x_2 + 8) + \lambda_3^T(-x_1 - 2x_2 + 7)$$

$$+ \lambda_4^T(-x_1) + \lambda_5^T(-x_2)$$

$$g(\lambda) = \inf_x \left(L(x,\lambda)\right) = \inf\left(L(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\right)$$

$$\cancel{\frac{\partial L}{\partial x_1} = 4 - 5\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4 = 0 \xrightarrow{\times 3} 12 - 15\lambda_1 - 6\lambda_2 - 3\lambda_3 - 3\lambda_4 = 0}$$

$$\cancel{\frac{\partial L}{\partial x_2} = 3 - \lambda_1 - \lambda_2 - 2\lambda_3 - \lambda_5 = 0 \xrightarrow{\times 4} 12 + 4\lambda_1 + 4\lambda_2 + 8\lambda_3 + 4\lambda_5 = 0}$$

$$\cancel{(+) \Rightarrow -11\lambda_1}$$

$$g = \inf \Rightarrow \underbrace{(4 - 5\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4)^T x_1 + (3 - \lambda_1 - \lambda_2 - 2\lambda_3 - \lambda_5)^T x_2}_{A} + \underbrace{11\lambda_1 + 8\lambda_2 + 7\lambda_3}_{B}$$

$$= B + \inf A \Rightarrow \text{maximize } B$$

$$\text{s.t.} \begin{cases} 4 - 5\lambda_1 - 2\lambda_2 - \lambda_3 - \lambda_4 = 0 \\ 3 - \lambda_1 - \lambda_2 - 2\lambda_3 - \lambda_5 = 0 \end{cases}$$

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سوال ۲

$p_j \geq 0$

$$\min \sum_{j=1}^{n} P_j$$

$$\text{s.t.} \sum_{j=1}^{n} g_{i,j} \cdot P_j \geq P \quad , i=1,\dots m$$

$$p_j \geq 0 \quad , j=1,\dots,n$$

شروط KKT :

1) Primal constraints: $f_i(x) \leq 0 \quad , h_i(x) = 0$

$$\hookrightarrow \begin{cases} g_{i,j} \cdot P_j \geq P \checkmark & , i=1,\dots,m \, , j=1,\dots,n \\ P_j \geq 0 & , j=1,\dots n \end{cases}$$

2) Dual constraints : $\lambda_{i,j} \geq 0 \quad \checkmark$

3) Complementry slackness: $\lambda_i f_i(x) = 0$

$$\hookrightarrow \lambda_{ij} \left(\sum_{j=1}^{n} g_{i,j} \cdot P_j - P \right) = 0 , \quad i=1,\dots,m , j=1,\dots,n \quad \left(\sum_{j=1}^{n} g_{i,j} \cdot P_j \geq P \right)$$

4) $\nabla L = 0 \quad , L = \sum_{j=1}^{n} P_j + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{i,j} \cdot \left(P - \sum_{j=1}^{n} g_{i,j} \cdot P_j\right)$

$$\rightarrow \frac{\partial L}{\partial p_j} = 1 - \sum_{i=1}^{m} \lambda_{i,j} \cdot g_{ij} = 0 \quad \checkmark$$

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سوال ۳) $P(x_i = k) = \dfrac{e^{-\mu_i}\mu_i^{k}}{k!}$, $y = (y_1, y_2, \dots, y_m)$

$$y_j = \sum_{i=1}^{n} y_{ji}$$

اسلاید ۱۵) $l(a,b) = l(\mu \mid y) = \cancel{\sum_{j=1}^{m}\sum_{i=1}^{n}} \prod_{i=1}^{m} \prod_{i=1}^{n} \log\left(\dfrac{e^{-\mu_i} \cdot \mu_i^{y_{ji}}}{y_{ji}!}\right)$

$$\Rightarrow l(\mu \mid y) = \sum_{i=1}^{m} \sum_{i=1}^{n} \left(-\mu_i + y_{ji}\log \mu_i - \log(y_{ji}!)\right)$$

$$\max\ l(\mu \mid y)$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{m} p_{ji} = 1 \\ p_{ji} \geq 0 \end{cases}$$

سوال ۴) تابع لجستیک با نویز (تمرین اضافه 15)

$$p = \text{Prob}(y=1) = \frac{\exp(a^T u + b + v)}{1+\exp(a^T u + b + v)} \quad , \quad 1-p = \text{Prob}(y=0)$$

$$l(a,b) = \log \prod_{i=1}^{q} p_i \prod_{i=q+1}^{m} (1-p_i)$$

$$= \sum_{i=1}^{q} \log p_i + \sum_{i=q+1}^{m} \log(1-p_i)$$

$$= \sum_{i=1}^{q} \log \frac{\exp(a^T u_i + b + v)}{1+\exp(a^T u_i + b + v)} + \sum_{i=q+1}^{m} \log \frac{1}{1+\exp(a^T u_i + b + v)}$$

$$l(a,b) = \underbrace{\sum_{i=1}^{q} (a^T u_i + b + v)}_{\text{همواره منفی}} - \underbrace{\sum_{i=1}^{m} \log(\underbrace{1+\exp(a^T u_i + b + v)}_{\geq 1})}_{\text{همواره مثبت}}$$

همواره منفی

$$\max \ l(a,b)$$

$$\min \ -l(a,b) \longrightarrow \text{Convex} \checkmark$$

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جواب a)

$$\min \sum_{i=1}^{m} \varphi(r_i) \qquad x \in R^n, A \in R^{m \times n}$$

$$s.t.\ r = Ax - b$$

$$L(x, r, \lambda) = \sum_{i=1}^{m} \varphi(r_i) + \lambda^T (Ax - b - r)$$

$$g(\lambda) = \inf L(x, r, \lambda) \quad , \quad \text{میدانیم:}\ \varphi^*(y) = \sup_x \{x^T y - \varphi(x)\}$$

$$\Rightarrow g(\lambda) = -\sum_{i=1}^{m} \varphi^*(-\lambda_i) - b^T \lambda$$

$$\max g(\lambda)$$
$$s.t\ \lambda \geq 0$$

b) $\varphi(x) = \begin{cases} 0, & |x| \leq w/2 \\ |x| - w/2, & |x| > w/2 \end{cases} \longrightarrow \varphi^*(y) = \begin{cases} 0, & |y| \leq 1/2 \\ |y| - 1/2, & |y| > 1/2 \end{cases}$

$$\Rightarrow g(\lambda) = -\sum_{i=1}^{m} \varphi^*(-\lambda_i) - b^T \lambda$$

$$= -b^T \lambda + -\sum_{i=1}^{m} -\lambda_i + 1/2$$

$$\max -b^T \lambda + \sum \lambda_i - 1/2$$

$$s.t.\ \lambda \geq 0$$

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سوال ۶)

$H = \{ x \in R^n \mid a^T x + b \geq 0 \}$

$$\text{minimize} \quad f(x) = \tfrac{1}{2} \|x\|^2 \longrightarrow QP \quad \text{فرم}$$

$$\text{s.t} \quad : \quad g(x) = a^T x + b \geq 0$$

جواب شروط KKT به فرم دستگاه خطی

$$L(x,\lambda) = \tfrac{1}{2}\|x\|^2 - \lambda(a^T x + b)$$

KKT: شرط ۴ : $A^T \lambda = 0$ ✓

۱ : $A x^* = b$

سوال ۱۰)

$$\min \quad f_0(x)$$

$$\text{s.t.} \quad f_i(x) \leq 0 \quad , i = 1, \dots, m$$

KKT $\leadsto$ شرط سوم ; $\lambda_i f_i(x^*) = 0$

$$\Rightarrow \lambda_i \nabla f_0(x^*)^T (x - x^*) + \lambda_i \sum_{i=1}^{m} \nabla f_i(x^*)^T (x - x^*) = 0 \quad \text{(I)}$$

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x^*) = 0$$

$$\Rightarrow \nabla f_0(x^*)^T (x - x^*) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x^*)^T (x - x^*) = 0 \quad \text{(II)}$$

$$\xrightarrow{(I,II)} \lambda_i \nabla f_0(x^*)^T (x - x^*) + \lambda_i \sum_{i=1}^{m} \nabla f_i(x^*)^T (x - x^*) = \nabla f_0(x^*)^T (x - x^*) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x^*)^T (x - x^*)$$

$$\Rightarrow (x - x^*) \left(\nabla f_0(x^*)^T + \sum_{i=1}^{m} \lambda_i \nabla f_i(x^*)^T \right) + \sum_{i=1}^{m} (\lambda_i \nabla f_i(x^*))^T (x - x^*) = 0$$

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$$\Rightarrow \nabla f(x^*)^T(x-x^*) + \sum_{i=1}^{m} (\lambda_i \nabla f_i(x^*)^T)(x-x^*) = 0$$ ادامه (7

KKT: $f_i(x^*) = 0$ یا $\lambda_i = 0$

$$\rightarrow \nabla f(x^*)^T(x-x^*) + \sum_{i=1}^{m} (\lambda_i \nabla f_i(x^*)^T)(x-x^*) = \nabla f(x^*)^T(x-x^*) \geq 0 \checkmark$$

برای x feasible بودن

سوال (8

$$\varphi(x) = -\sum_{i=1}^{m} \log(-f_i(x))$$

$$\tilde{\varphi}(x) = \sum_{i=1}^{m} t\log(-f_i(x)) - t\log(x^Tx - R^2)$$

$$\rightarrow \nabla^2(tf_0(x) + \tilde{\varphi}(x)) = t\nabla^2 f_0(x) + \nabla^2 \tilde{\varphi}(x) \succeq aI$$

$$\nabla^2 \varphi_i(x) = \frac{1}{(f_i(x))^2} \nabla f_i(x) \nabla f_i(x)^T - \frac{1}{f_i(x)} \nabla^2 f_i(x)$$

چطوری؟