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Task: Numerical Integration (Trapezoidal Rule)

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1. Introduction and Theoretical Foundation

The rigorous evaluation of definite integrals is a paramount operation in applied computational engineering. While the First Fundamental Theorem of Calculus relies on finding elementary algebrai1c antiderivatives, this is frequently inapplicable in real-world scenarios.
The Trapezoidal Rule, classified formally as a closed Newton-Cotes formula of degree $n = 1$, provides a highly adaptable algorithmic solution. It circumvents intractable integrands—such as the Gaussian distributions critical to probability and AI models—or discrete arrays of empirical telemetry by approximating the region beneath the curve as a sequence of geometric trapezoids.

2. Mathematical Formulation

Instead of utilizing a single linear secant which introduces catastrophic truncation error over wide intervals, computational scientists deploy the Composite Trapezoidal Rule.

- The global integration domain $[a, b]$ is subdivided into a fine-grained mesh of n contiguous subintervals.
- The uniform width of each subinterval (step size) is calculated as $h = \frac{b-a}{n}$.
- The overall integral is efficiently computed using the factorized algebraic summation:
$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$
- This specific composite formulation represents the absolute algorithmic standard implemented in modern quantitative analysis platforms.

3. Error Analysis and Asymptotic Efficiency

A numerical approximation is only computationally viable if its intrinsic errors can be rigorously quantified.

- Global Truncation Error (E_T): The error is bounded by $|E_T| \leq \frac{(b-a)^3}{12n^2}M$, where $M = max_{x\in[a,b]} |f''(x)|$.
- Algorithmic Scaling: The error is inversely proportional to the square of the number of subintervals (n^2), formally classifying it as a second-order methodology with $O(h^2)$ asymptotic convergence.
- Geometric Concavity: If the target function is strictly concave up ($f''(x) > 0$), the rule is mathematically guaranteed to systematically overestimate the true analytic value.

4. Advanced Computational Methodologies

While traditionally viewed as an $O(h^2)$ algorithm, the Trapezoidal Rule possesses advanced characteristics when deployed under specific algorithmic constraints:

- Exponential Convergence: When applied to analytic, smoothly periodic functions over exactly one full period, the truncation error bypasses polynomial constraints and plummets geometrically toward absolute zero. The boundary derivative differences systematically annihilate to zero, elevating the convergence to $O(e^{-cn})$.
- Romberg Integration: To drastically elevate precision limits without increasing computational load, algorithms utilize Richardson Extrapolation. By recursively combining lower-quality $O(h^2)$ estimates computed at sequentially halved step sizes, the algorithm mathematically nullifies dominant error terms, successfully synthesizing highly refined estimations governed strictly by $O(h^4)$ or $O(h^6)$ error limits.

5. Practical Empirical Demonstration

In data-scarce empirical regimes where continuous analytical models are absent, this algorithm excels.
Consider empirical telemetry recording an object's instantaneous forward velocity uniformly every 1 second ($h = 1$):

- Sampled Velocity (m/s): $\{2, 4, 8, 16\}$ evaluated at temporal nodes $\{0, 1, 2, 3\}$.
- Applying the Composite Formula: $Area \approx \frac{1}{2}[2 + 2(4 + 8) + 16]$.
- Result: $\frac{1}{2}[2 + 24 + 16] = 21$ meters.