Course: Digital Control | **Mid-Term Exam (1) Solution Manual**
Date: 18 / 11 / 2019 | **Time: 1.30 H**

Question (1): (10 marks):

From the figure, we obtain:

$x(0) = 0$

$x(1) = 0.25$

$x(2) = 0.50$

$x(3) = 0.75$

$x(k) = 1, \qquad k = 4, 5, 6, \ldots$

Then the z transform of $x(k)$ can be given by

$$
\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} x(k) z^{-k} \\
&= 0.25z^{-1} + 0.50z^{-2} + 0.75z^{-3} + z^{-4} + z^{-5} + z^{-6} + \cdots \\
&= 0.25(z^{-1} + 2z^{-2} + 3z^{-3}) + z^{-4}\frac{1}{1 - z^{-1}} \\
&= \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4}}{4(1 - z^{-1})} \\
&= \frac{1}{4}\frac{z^{-1}(1 + z^{-1} + z^{-2} + z^{-3})(1 - z^{-1})}{(1 - z^{-1})^2} \\
&= \frac{1}{4}\frac{z^{-1}(1 - z^{-4})}{(1 - z^{-1})^2}
\end{aligned}
$$

Notice that the curve $x(t)$ can be written as

$$x(t) = \tfrac{1}{4}t - \tfrac{1}{4}(t - 4)1(t - 4)$$

where $1(t - 4)$ is the unit-step function occurring at $t = 4$. Since the sampling period $T = 1$ sec, the z transform of $x(t)$ can also be obtained as follows:

$$
\begin{aligned}
X(z) &= \mathcal{Z}[x(t)] = \mathcal{Z}[\tfrac{1}{4}t] - \mathcal{Z}[\tfrac{1}{4}(t - 4)1(t - 4)] \\
&= \frac{1}{4}\frac{z^{-1}}{(1 - z^{-1})^2} - \frac{1}{4}\frac{z^{-4}z^{-1}}{(1 - z^{-1})^2} \\
&= \frac{1}{4}\frac{z^{-1}(1 - z^{-4})}{(1 - z^{-1})^2}
\end{aligned}
$$

Question (2):

$$G(s)=\frac{V(s)}{U(s)}=\frac{1}{Ms+b}=\frac{K/\tau}{s+1/\tau}$$

where $K=1/b$ and $\tau=M/b$

For a zero-order hold:

$$G(z)=(1-z^{-1})Z\left[\frac{G(s)}{s}\right]$$

$$\frac{G(s)}{s}=\frac{K/\tau}{s(s+1/\tau)}=(K/\tau)\left[\frac{\tau}{s}-\frac{\tau}{s+1/\tau}\right]$$

Thereby

$$G(z)=(1-z^{-1})Z\left[\left(\frac{K}{\tau}\right)\left[\frac{\tau}{s}-\frac{\tau}{s+1/\tau}\right]\right]$$

$$=(1-z^{-1})Z\left[K\left[\frac{1}{s}-\frac{1}{s+1/\tau}\right]\right]$$

$$\Rightarrow G(z)=\frac{z-1}{z}K\left[\frac{z}{z-1}-\frac{z}{z-e^{-T/\tau}}\right]=K\left[1-\frac{z-1}{z-e^{-T/\tau}}\right]$$

or

$$G(z)=K\left[\frac{z-e^{-T/\tau}-z+1}{z-e^{-T/\tau}}\right]=K\left[\frac{1-e^{-T/\tau}}{z-e^{-T/\tau}}\right]$$

Question (3): **(10 marks).**

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -0.3x_1(k) - 1.1x_2(k) + r(k)$$

and

$$y(k) = -0.3x_1(k) - 1.1x_2(k) + r(k)$$

Hence, the state equations can be written in vector form as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.3 & -1.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k)$$
$$y(k) = [-0.3 \quad -1.1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + r(k)$$

It holds that

$$H(z) = C^T(zI - A)^{-1}b + D$$

Calculation of $(zI - A)$

$$(zI - A) = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.3 & -1.1 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0.3 & z+1 \end{bmatrix}$$

Calculation of $(zI - A)^{-1}$

$$(zI - A)^{-1} = \begin{pmatrix} z & 1 \\ 0.3 & z+1.1 \end{pmatrix}^{-1} = \frac{1}{z^2 + 1.1z + 0.3} \begin{bmatrix} z+1.1 & 1 \\ -0.3 & z \end{bmatrix}$$

$$H(z) = C^T(zI - A)^{-1}b + D$$
$$= [-0.3 \quad -1.1] \frac{1}{z^2 + 1.1z + 0.3} \begin{bmatrix} z+1.1 & 1 \\ -0.3 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \Rightarrow$$

After some algebraic manipulations, the transfer function is

$$H(z) = \frac{z^2}{z^2 + z + 0.3}$$