CSE416 “Digital Control” Final Exam Model Answer
January, 2021

Question (1):

1. a.

$$\mathcal{C} = [B_c \; A_cB_c] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Controllability matrix has full rank, and the system is controllable.

$$\mathcal{O} = \begin{bmatrix} C \\ CA_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Observability matrix has full rank, and the system is observable.

1. b.

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

$$A = \phi(T) = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix}$$

$$B = \int_0^T \phi(\tau)B_c d\tau = \int_0^T \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & cos\tau \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau = \int_0^T \begin{bmatrix} sin\tau \\ cos\tau \end{bmatrix} d\tau = \begin{bmatrix} -cos\tau \\ sin\tau \end{bmatrix}_0^T$$

$$B = \begin{bmatrix} 1 - cosT \\ sinT \end{bmatrix}$$

Then, the discrete state variable model is:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 - cosT \\ sinT \end{bmatrix} u(k)$$

1. c.

$$\mathcal{C} = [B \; AB] = \begin{bmatrix} 1 - cosT & \cos T - \cos^2 T + \sin^2 T \\ sinT & -\sin T + 2\sin T\cos T \end{bmatrix}$$

$$\Delta(\mathcal{C}) = -\sin T + 2\sin T\cos T + \cos T\sin T - 2\sin T\cos^2 T - \sin T\cos T + \sin T\cos^2 T - \sin^3 T$$

$$\Delta(\mathcal{C}) = 2\sin T\cos T - \sin T\cos^2 T - \sin T\,(1 + \sin^2 T)$$

$$\Delta(\mathcal{C}) = 2\sin T\cos T - \sin T\cos^2 T - 2\sin T + \sin T\cos^2 T$$

$$\Delta(\mathcal{C}) = -2\sin T\,(1 - \cos T)$$

The system loses controllability for $T = n\pi.\; n = 0,1,2, \ldots.$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos T & \sin T \end{bmatrix}$$

$$\Delta(\mathcal{O}) = \sin T$$

The system loses observability for $T = n\pi.\; n = 0,1,2, \ldots.$

1. d.

Transfer function for analog system:

$$G(s) = C(sI - A_c)^{-1}B_c + D$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s}{s^2+1} & \dfrac{1}{s^2+1} \\ \dfrac{-1}{s^2+1} & \dfrac{s}{s^2+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+1}$$

Transfer function for digital system:

$$G(z) = C(zI - A)^{-1}B + D$$

$$G(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z - \cos T & -\sin T \\ \sin T & z - \cos T \end{bmatrix}^{-1} \begin{bmatrix} 1 - \cos T \\ \sin T \end{bmatrix}$$

$$G(z) = \frac{1}{(z - \cos T)^2 \sin^2 T} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z - \cos T & \sin T \\ -\sin T & z - \cos T \end{bmatrix} \begin{bmatrix} 1 - \cos T \\ \sin T \end{bmatrix}$$

$$G(z) = \frac{(z - \cos T)(1 - \cos T) + \sin^2 T}{(z - \cos T)^2 \sin^2 T}$$

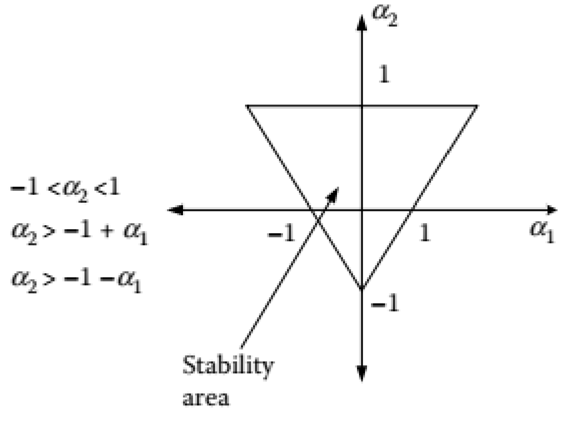
2.

$$Q(z) = z^2 + \alpha_1 z + \alpha_2.$$

❶ $Q(1) = 1 + \alpha_1 + \alpha_2 > 0$

❷ $(-1)^2 Q(-1) = 1 - \alpha_1 + \alpha_2 > 0$

❸ $|\alpha_2| < 1 \rightarrow -1 < \alpha_2 < 1$



Question (2):

1. a.

$$M(z) = E(z) - 0.9z^{-1}E(z) + z^{-1}M(z)$$

$$(1 - z^{-1})M(z) = (1 - 0.9z^{-1})E(z)$$

$$D(z) = \frac{M(z)}{E(z)} = \frac{z - 0.9}{z - 1}$$

1. b.

$$G(z) = \mathcal{Z}\left\{\frac{1 - e^{-Ts}}{s}\frac{1}{s+2}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s(s+2)}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{0.5}{s} - \frac{0.5}{s+2}\right\}$$

$$G(z) = \frac{z-1}{z}\left(\frac{0.5z}{z-1} - \frac{0.5z}{z - e^{-2}}\right)$$

$$G(z) = \frac{z-1}{z}\left(\frac{0.5z(z - e^{-2}) - 0.5z(z-1)}{(z-1)(z-e^{-2})}\right)$$

$$G(z) = \frac{0.4323}{z - 0.1353}$$

The open loop transfer function $D(z)G(z)$ has one pole at $z - 1$

The system is type one, therefore for a unit-step input the steady state error is zero and the steady state response is 1.

1. c.

The closed loop TF is $T(z) = \frac{D(z)G(z)}{1+D(z)G(z)} = \frac{0.4323z - 0.3891}{z^2 - 0.703z - 0.2538}$

Closed loop poles are $0.9658, -0.2628$

The more dominant pole is 0.9658.

Time constant $\tau = -\frac{T}{\ln r} = -\frac{1}{\ln(0.9658)} = 28.74\ seconds$

1. d.

Approximate settling time = $4\tau = 114.95\ seconds$

Also 5τ is correct.

1. e.

Unit step response

$$C(z) = \frac{0.4323z(z - 0.9)}{(z - 0.9658)(z + 0.2628)(z - 1)}$$

$$\frac{C(z)}{z} = \frac{1}{z-1} - \frac{0.6760}{z - 0.9658} - \frac{0.3240}{z + 0.2628}$$

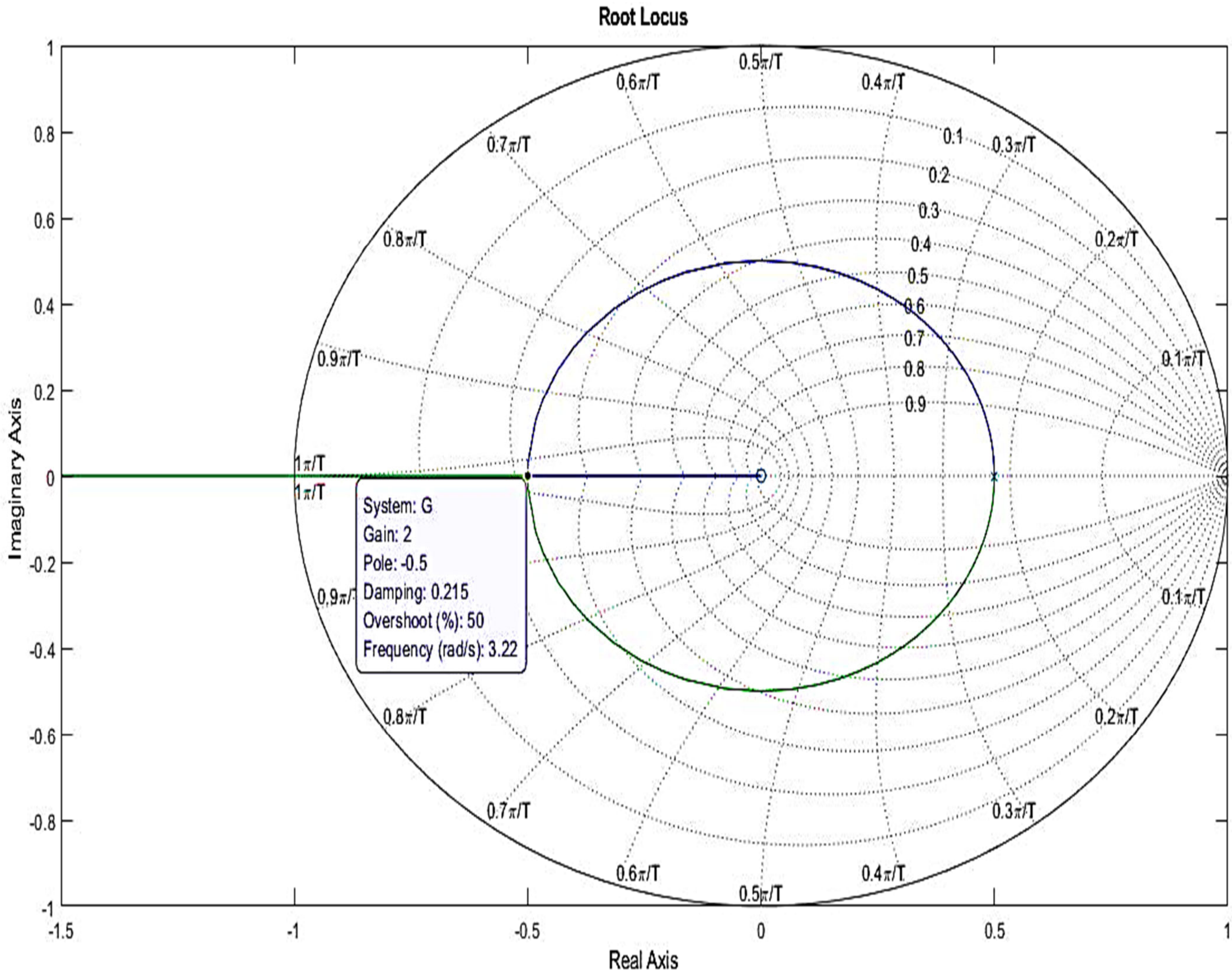
$$C(z) = \frac{z}{z-1} - \frac{0.6760z}{z - 0.9658} - \frac{0.3240z}{z + 0.2628}$$

$$c(k) = 1 - 0.6760(0.9658)^k - 0.3240(-0.2628)^k$$

1. f.

Substituting with $k = 115 \rightarrow c(115) = 0.988$

2.



Question (3):

1.

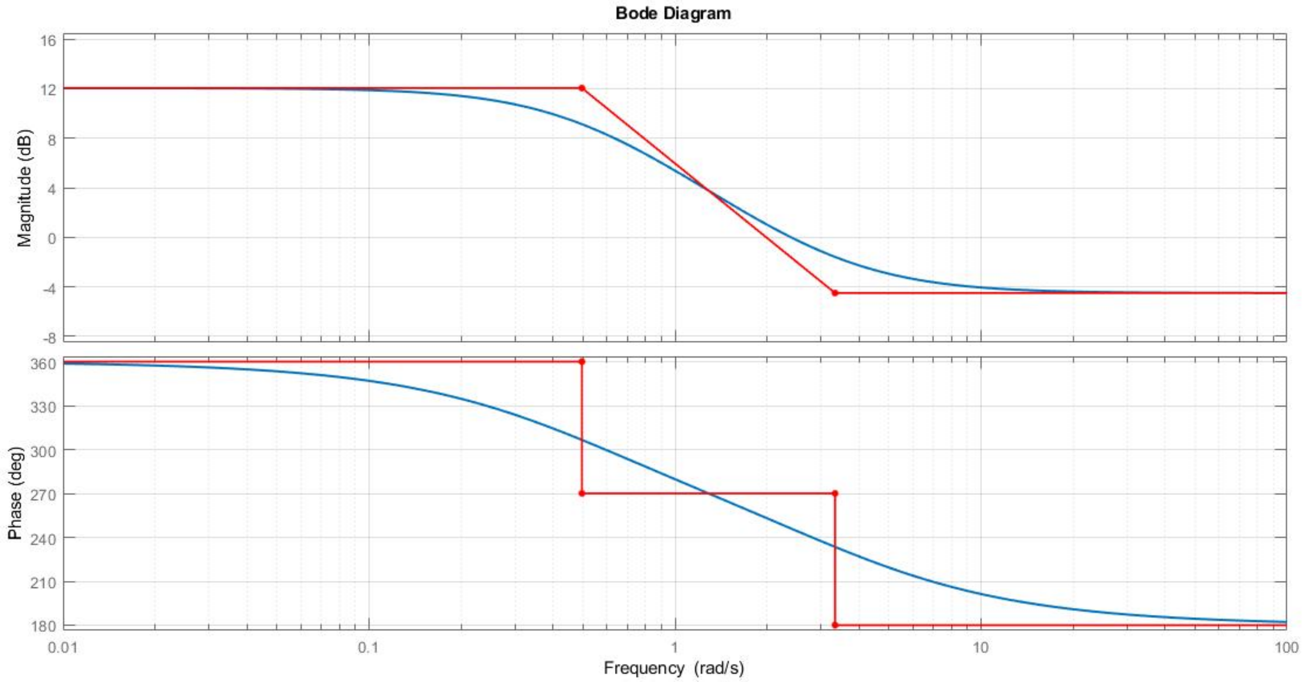
$$G(z) = \frac{1.037}{z - 0.7408}$$

We first apply the bilinear transform:

$$z = \frac{1 + (\frac{T}{2})w}{1 - (\frac{T}{2})w} = \frac{1 + 0.3w}{1 - 0.3w}$$

$$G(w) = -0.5957\frac{w - 3.3333}{w}$$

$$G(j\omega) = 4\frac{1 - \frac{j\omega}{3.3333}}{1 + \frac{j\omega}{0.4963}}$$



2.

$$e_{ss} = \frac{1}{1 + \lim_{z\to 1} G(z)H(z)} = \frac{1}{1 + \lim_{z\to 1}\frac{0.04148}{z - 0.7408}} = \frac{1}{1 + 0.16} = 0.862$$

The steady state error is 0.862 or 86.2%

3.

To achieve a steady state error of 0.05, we will design a phase lag with a dc gain K such that:

$$0.05 = \frac{1}{1 + 0.16K} \to 0.05 + 0.05 \times 0.16K = 1 \to K = 118.75$$

The required phase margin is $\phi_m = 45°$

Choose the phase crossover frequency as:

$$\angle G(j\omega_{w1}) = -180° + \phi_m + 5° = -130°$$

From your sketched bode diagram, $\omega_{w1} \approx 4$

The zero location $\omega_{w0} = 0.1\omega_{w1} = 0.4$

The pole location $\omega_{wp} = \frac{0.1\omega_{w1}}{a_0|G(j\omega_{w1})|\times H} = \frac{0.4}{118.75\times 0.7694\times 0.04} = 0.1094$

The compensator in the w-plane $D(w) = 118.75\frac{1+\frac{w}{0.4}}{1+\frac{w}{0.1094}}$

Back to z-plane using bilinear transform (inverse) $w = \frac{2}{T}\frac{z-1}{z+1}$

$D(z) = \frac{35.22z-27.67}{z-0.9364}$

4.

$$\frac{M(z)}{E(z)} = \frac{35.22z - 27.67}{z - 0.9364} \rightarrow zM(z) - 0.9364M(z) = 35.22zE(z) - 27.67E(z)$$

$$m(k+1) - 0.9364m(k) = 35.22e(k+1) - 27.67e(k)$$

$$m(k) = 35.22e(k) - 27.67e(k-1) + 0.9364m(k-1)$$

Question (4):

1.

$$\tau = 4, \zeta = 0.707$$

$$\ln r = -\frac{T}{\tau} = -\frac{1}{4} \rightarrow r = 0.7788$$

$$\theta^2 = (\ln r)^2 \left(\frac{1}{\zeta^2} - 1\right) = \frac{1}{16} \times \left(\frac{1}{0.707^2} - 1\right) \rightarrow \theta = \pm 0.25$$

The required poles are $z_{1,2} = 0.7788\angle \pm 0.25$

$\alpha_c(z) = z^2 - 2r\cos\theta\ z + r^2$

$\alpha_c(z) = z^2 - 2 \times 0.7788\cos 0.25\ z + (0.7788)^2$

Desired characteristic equation is: $\alpha_c(z) = z^2 - 1.5092z + 0.6065$

$\alpha_c(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 - 1.5092\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 0.6065 = \begin{bmatrix} 0.0973 & 0.4908 \\ 0 & 0.0973 \end{bmatrix}$

Applying Ackermann's formula:

$$K = [0 \quad 1][B \quad AB]^{-1}\alpha_c(A) = [0.3892 \quad 1.7686]$$

2.

$A_{aa} = 1, A_{ab} = 1, A_{ba} = 0, A_{bb} = 1, B_a = 0.125, B_b = 0.25$

The observer dynamics:

$\tau = 2$

$\alpha_e(z) = z - e^{-\frac{T}{\tau}} = z - 0.6065$

$\alpha_e(A_{bb}) = A_{bb} - 0.6065 = 1 - 0.6065 = 0.3935$

$G = \alpha_e(A_{bb})[A_{ab}]^{-1}[1] = 0.3935$

$$\mathbf{q}_b(k+1) = (\mathbf{A}_{bb} - \mathbf{G}\mathbf{A}_{ab})\mathbf{q}_b(k) + \mathbf{G}\mathbf{y}(k+1) + (\mathbf{A}_{ba} - \mathbf{G}\mathbf{A}_{aa})\mathbf{y}(k) + (\mathbf{B}_b - \mathbf{G}\mathbf{B}_a)\mathbf{u}(k)$$

$$\mathbf{q}_b(k) = 0.6065\mathbf{q}_b(k-1) + 0.3935\mathbf{y}(k) - 0.3935\mathbf{y}(k-1) + 0.2008\mathbf{u}(k)$$

3.

$$\boxed{D_{ce}(z) = \frac{-U(z)}{Y(z)} = K_1 + \mathbf{K}_b[z\mathbf{I} - \mathbf{A}_{bb} + \mathbf{G}\mathbf{A}_{ab} + (\mathbf{B}_b - \mathbf{G}\mathbf{B}_a)\mathbf{K}_b]^{-1} \cdot [\mathbf{G}z + \{\mathbf{A}_{ba} - \mathbf{G}\mathbf{A}_{aa} - K_1(\mathbf{B}_b - \mathbf{G}\mathbf{B}_a)\}]}$$

$$D_{ce}(z) = 0.3892 + 1.7686(z - 0.2514)^{-1}.(0.3935z - 0.4717) = \frac{1.085z - 0.9321}{z - 0.2514}$$

4.

$$\frac{M(z)}{E(z)} = \frac{1.085z - 0.9321}{z - 0.2514}$$

$$zM(z) - 0.2514M(z) = 1.085zE(z) - 0.9321E(z)$$

$$m(k) = 1.085e(k) - 0.9321e(k-1) - 0.2514m(k-1)$$

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Define variables error = 0, control_action = 0, previous_error = 0,
previous_control_action = 0;

Start loop:

     read error;
     control_action= .085*error-0.9321*previous_error-
                         0.2514*previous_control_action;

     previous_error = error;
     previous_control_action = control_action;

     output control_action;

end loop.
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