CSE416 “Digital Control” Final Exam Model Answer

January, 2022

Question (1):

1.

If a transfer function contains a transcendental term e^{-Ts}, then it can be replaced by an infinite series that may contain a zero in the vicinity of the integrator (the pole at the origin).

Transcendental means a function that is not a polynomial or not in an algebraic form.

Using Taylor’s series: $e^{-Ts} = 1 - Ts + \frac{(Ts)^2}{2!} - \frac{(Ts)^3}{3!} + \frac{(Ts)^4}{4!} - \frac{(Ts)^5}{5!} + \cdots$

Then,

$$G_{ho}(s) = \frac{1-e^{-Ts}}{s} = \frac{1 - \left(1 - Ts + \frac{(Ts)^2}{2!} - \frac{(Ts)^3}{3!} + \frac{(Ts)^4}{4!} - \frac{(Ts)^5}{5!} + \cdots\right)}{s}$$

$$= \frac{Ts - \frac{(Ts)^2}{2!} + \frac{(Ts)^3}{3!} - \frac{(Ts)^4}{4!} + \frac{(Ts)^5}{5!} + \cdots}{s}$$

$$G_{ho}(s) = T - \frac{T^2 s}{2!} + \frac{T^3 s^2}{3!} - \frac{T^4 s^3}{4!} + \frac{T^5 s^4}{5!} + \cdots$$

Therefore, this equivalent series of the zero-order hold transfer function does not contain an integrator or a pole at the origin.

2. a.

$T = 0.001,\ a = 40$

$$G_p(s) = \frac{1600}{s^2 - 1600}$$

$$G(s) = G_{ho}(s)G_p(s) = \frac{1-e^{-Ts}}{s} \cdot \frac{1600}{s^2-1600} = (1-e^{-Ts})\left(\frac{1600}{s(s^2-1600)}\right)$$

$$\frac{1600}{s(s^2-1600)} = \frac{A}{s} + \frac{B}{s+40} + \frac{C}{s-40}$$

$$A(s+40)(s-40) + Bs(s-40) + Cs(s+40) = 1600$$

$$s = 40 \rightarrow C = 0.5$$

$$s = 0 \rightarrow A = -1$$

$$A + B + C = 0 \rightarrow B = 0.5$$

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\left(\frac{1600}{s(s^2 - 1600)}\right)\right\}$$

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{-\frac{1}{s} + \frac{0.5}{s + 40} + \frac{0.5}{s - 40}\right\}$$

$$G(z) = (1 - z^{-1})\left(-\frac{z}{z-1} + 0.5\frac{z}{z - e^{-40\times 0.001}} + 0.5\frac{z}{z - e^{40\times 0.001}}\right)$$

$$G(z) = (1 - z^{-1})\left(-\frac{z}{z-1} + 0.5\frac{z}{z - 0.9608} + 0.5\frac{z}{z - 1.0408}\right)$$

$$G(z) = 8 \times 10^{-4}\frac{z + 1}{z^2 - 2.002z + 1}$$

2. b.

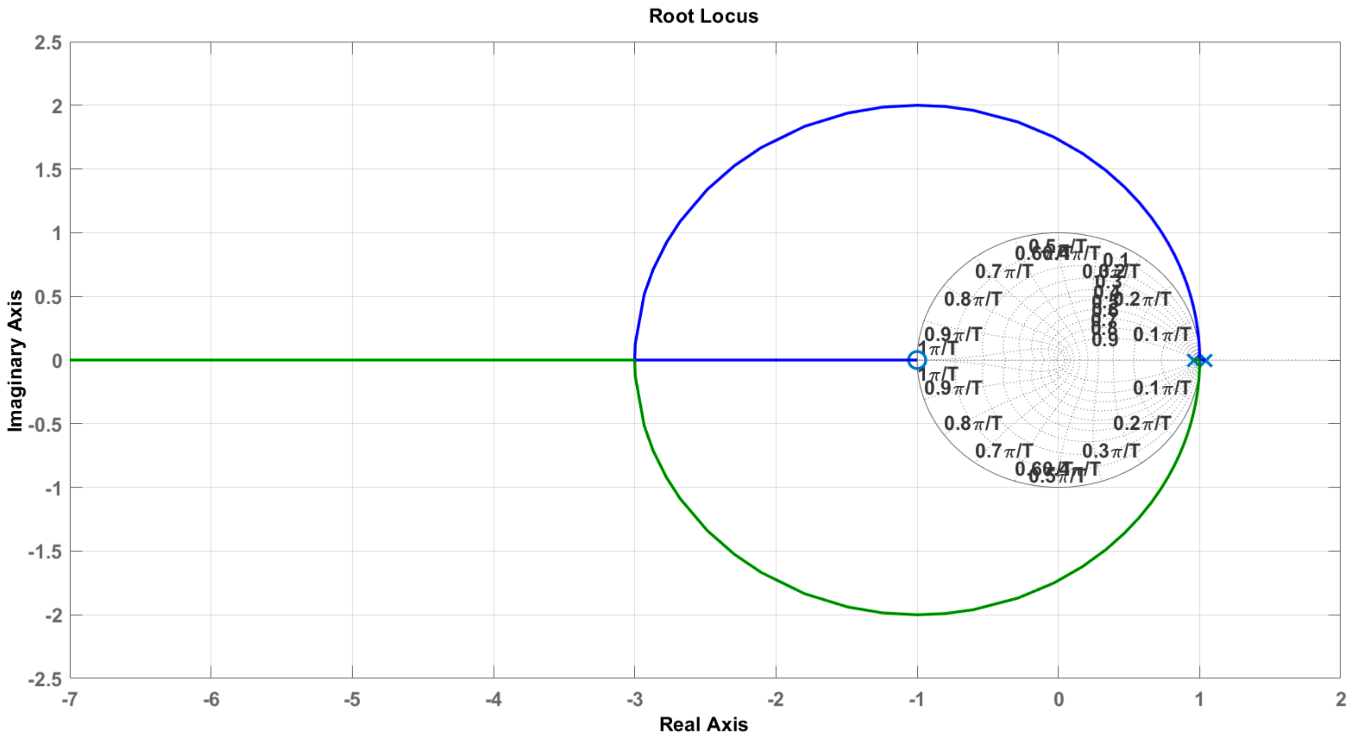
Continuous-time poles are $s_{1,2} = \pm 40$

$z_{1,} = e^{40*0.001} = 1.0457$, $z_2 = e^{-40*0.001} = 0.9608$ which are the poles of the discrete-time transfer function.

Therefore, the discrete-time poles map to continuous-time poles by $z = e^{sT}$

This relationship holds here as the sampling frequency is much higher than the frequency of the poles.

2. c.



2. d.

As seen in the root locus. All the values of the proportional controller will drive the poles outside the unit circle. Therefore, the proportional controller cannot stabilize this system.

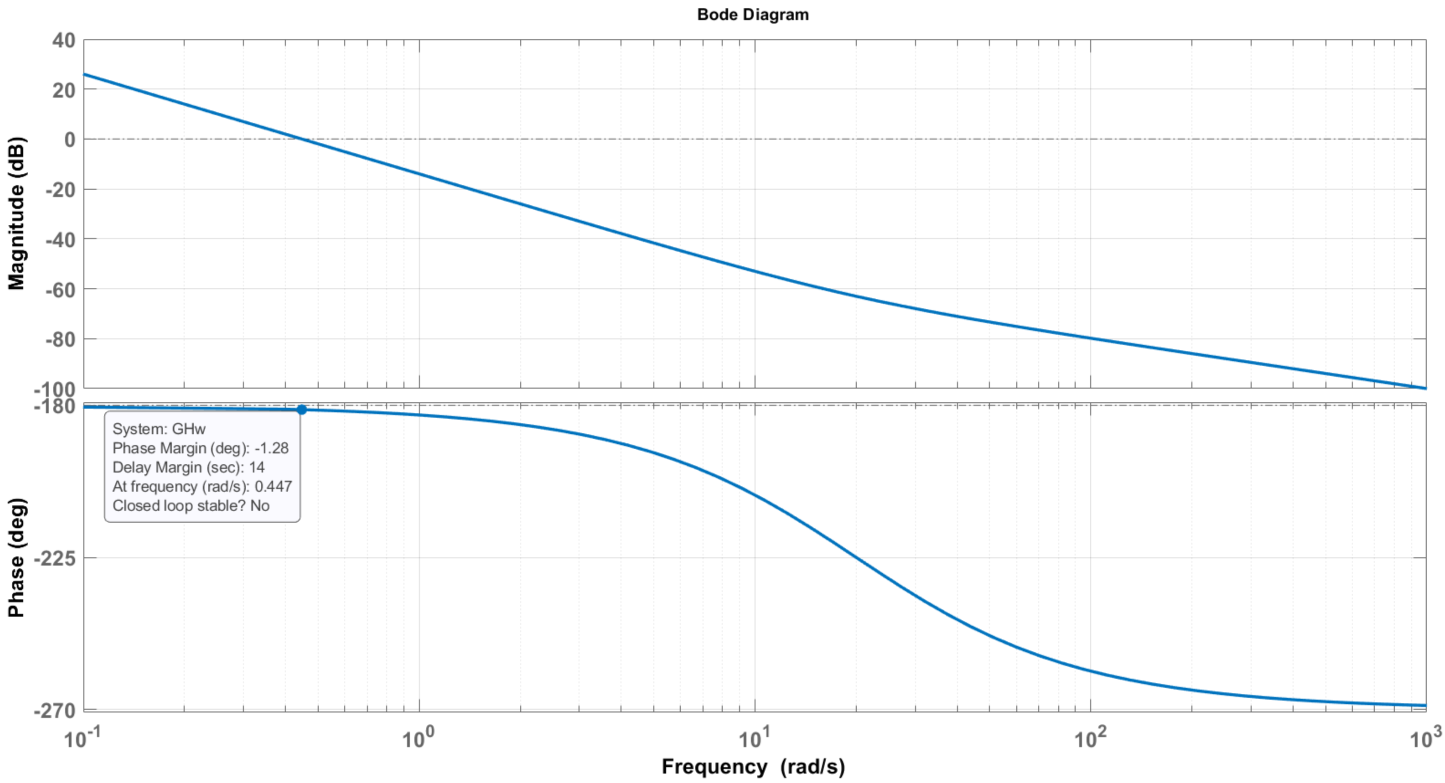
Question (2):

1.

Substitute with bilinear transform $z = \frac{1+\left(\frac{T}{2}\right)w}{1-\left(\frac{T}{2}\right)w}$ to get the w-domain transfer function.

$$G(w) = \frac{0.05(z+1)}{(z-1)^2}\Bigg|_{\frac{0.05(z+1)}{(z-1)^2}} = -\frac{w-20}{2w^2}$$

We should plot the bode diagram for $G(w)H_k = -0.02\,\frac{w-20}{2w^2}$



2. From the bode diagram, the phase margin is $-1.28°$. Therefore, the system is unstable.

3. The PI controller is a phase-lag compensator. From the bode diagram, this system cannot be stabilized by a phase-lag and require a phase-lead compensator.

4.

$$D(w) = K_p + K_D\frac{w}{1 + w\left(\frac{T}{2}\right)}$$

$\phi_m = 45°$ at $\omega_{w1} = 1\,rad/sec$

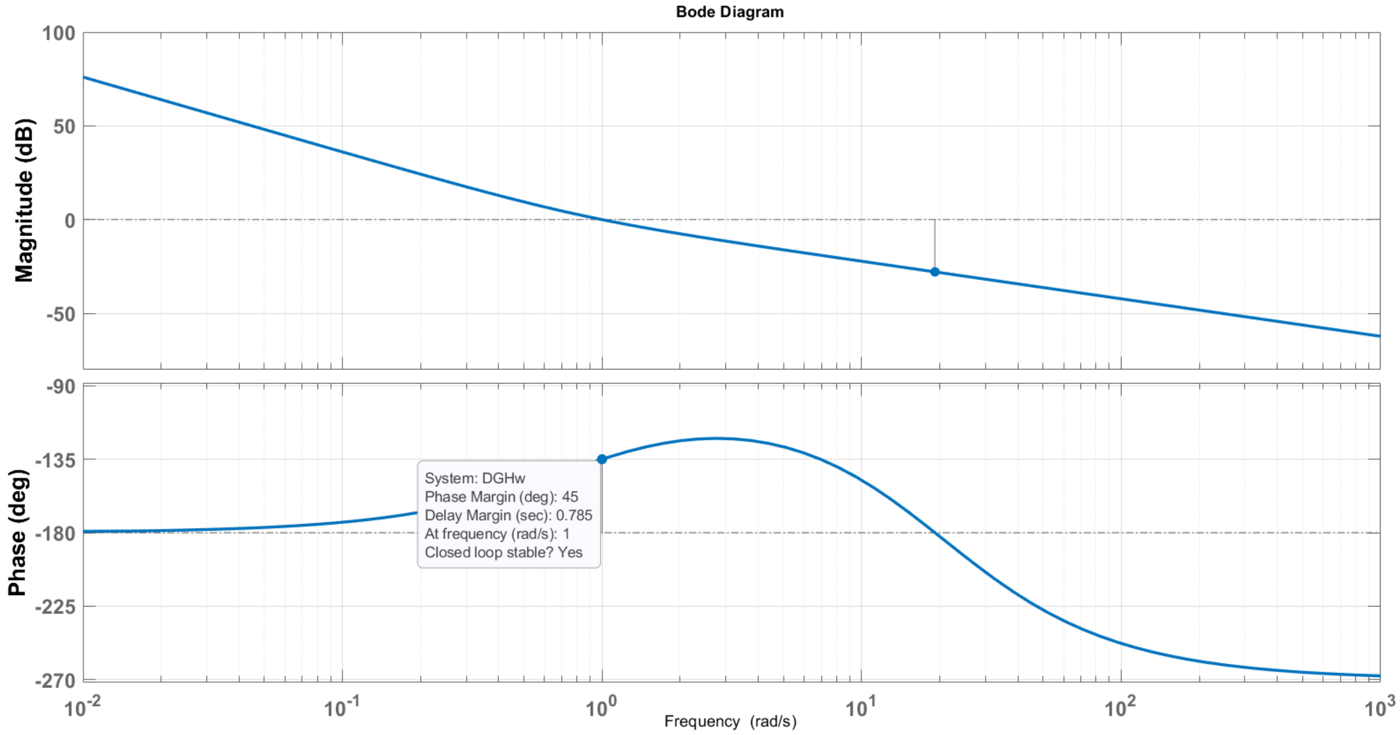
$|GH(j1)| = 0.2002$

$$\theta = \angle D(j\omega_{w1}) = 180° + \phi_m - \angle GH(j\omega_{w1}) = 180° + 45° - (-182.8624°) = 407.8624° = 47.8624°$$

$$K_D = \left(\frac{\sin\theta}{|GH(j\omega_{w1})|}\right)\left(\frac{\left(\frac{2}{T}\right)^2 + \omega_{w1}^2}{\omega_{w1}\left(\frac{2}{T}\right)^2}\right) = \left(\frac{\sin(47.8624°)}{0.2002}\right)\left(\frac{(20)^2+1}{1\times(20)^2}\right) = 3.7123$$

$$K_P = \frac{\cos\theta}{|GH(j\omega_{w1})|} - \frac{K_D\omega_{w1}^2\left(\frac{2}{T}\right)}{\left(\frac{2}{T}\right)^2 + \omega_{w1}^2} = \frac{\cos(47.8624°)}{0.2002} - \frac{3.7123\times 1\times 20)}{(20)^2+1} = 3.1652$$

$$D(w) = 3.1652 + 3.7123\frac{w}{1+0.05w}$$



5.

$$D(z) = K_P + K_D\frac{z-1}{Tz} = \frac{4.029z - 3.712}{0.1z}$$

Let $E(z)$ be the error and $U(z)$ the controller output or the control action.
Then,

$$D(z) = \frac{U(z)}{E(z)} = \frac{4.029z - 3.712}{0.1z} = \frac{4.029 - 3.712z^{-1}}{0.1}$$

$$0.1U(z) = 4.029E(z) - 3.712z^{-1}E(z)$$

$$u(k) = 40.29e(k) - 37.12e(k-1)$$

Question (3):

$$A = \begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.125 \\ 0 \end{bmatrix}, C = [1 \quad 1], D = 0, T = 0.1$$

1.
To check the stability, we first obtain the transfer function
$G(z) = C(z\mathbf{I} - A)^{-1}B$

$z\mathbf{I} - A = \begin{bmatrix} z - 2.01 & 1 \\ -1 & z \end{bmatrix}$

$(z\mathbf{I} - A)^{-1} = \dfrac{1}{z^2 - 2.01z + 1}\begin{bmatrix} z & -1 \\ 1 & z - 2.01 \end{bmatrix}$

$G(z) = \dfrac{1}{z^2 - 2.01z + 1}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} z & -1 \\ 1 & z - 2.01 \end{bmatrix}\begin{bmatrix} 0.125 \\ 0 \end{bmatrix}$

$G(z) = \dfrac{1}{z^2 - 2.01z + 1}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0.125z \\ 0.125 \end{bmatrix}$

$G(z) = \dfrac{0.125z + 0.125}{z^2 - 2.01z + 1}$

The characteristic equation of $G(z)$ is $z^2 - 2.01z + 1 = 0$
Poles are 1.1052 and 0.9048
There is one pole outside the unit circle, therefore the system is unstable.
The poles also can be directly obtained by the eigenvalues of A, but we didn't cover it in our course.

2.

$$AB = \begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.125 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2512 \\ 0.125 \end{bmatrix}$$
$$CA = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3.01 & -1 \end{bmatrix}$$

The controllability matrix is
$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0.125 & 0.2512 \\ 0 & 0.125 \end{bmatrix}$

The controllability matrix is
$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3.01 & -1 \end{bmatrix}$
Both matrices are full rank, therefore the system is completely controllable and observable.

3.
τ=0.5, $\zeta = 0.707$
To get the locations of the desired closed-loop poles:
$\ln r = -\dfrac{0.1}{0.5} = -0.2 \rightarrow r = 0.8187$

$\theta^2 = (\ln r)^2\left(\dfrac{1}{\zeta^2} - 1\right) = (-0.2)^2\left(\dfrac{1}{0.707^2} - 1\right) \rightarrow \theta = \pm 0.2$

$z_{1,2} = 0.8187\angle \pm 0.2$

$\alpha_c = \big(z - r(\cos\theta + j\sin\theta)\big)\big(z - r(\cos\theta - j\sin\theta)\big)$

$\alpha_c = z^2 - 2r\cos\theta\, z + r^2$
$\alpha_c = z^2 - 2 \times 0.8187 \times \cos(0.2)\, z + (0.8187)^2$
$\alpha_c = z^2 - 1.6048z + 0.6703$

$K = [0 \quad 1][B \quad AB]^{-1}\alpha_c(A)$

$[B \quad AB] = \begin{bmatrix} 0.125 & 0.2512 \\ 0 & 0.125 \end{bmatrix}$

$[B \quad AB]^{-1} = \begin{bmatrix} 8 & -16.0801 \\ 0 & 8 \end{bmatrix}$

$\alpha_c(A) = \begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix}^2 - 1.6048\begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix} + 0.6703\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.4847 & -0.4052 \\ 0.4052 & -0.3297 \end{bmatrix}$

$K = [0 \quad 1]\begin{bmatrix} 8 & -16.0801 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 0.4847 & -0.4052 \\ 0.4052 & -0.3297 \end{bmatrix} = \begin{bmatrix} 3.2417 \\ -2.6375 \end{bmatrix}$

The feedback control law is:

$$u(k) = 3.2417x_1(k) - 2.6375x_2(k)$$

4.
For the predictor observer:

$\tau = 0.25, \zeta = 1$

The poles location for the estimator:

$\ln r = -\dfrac{0.1}{0.25} = -0.4 \rightarrow r = 0.6703$

$\zeta = 1 \rightarrow \theta = 0$

$\alpha_e = (z - 0.6703)^2 = z^2 - 1.3406z + 0.4493$

$G = \alpha_e(A)\begin{bmatrix} C \\ CA \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3.01 & -1 \end{bmatrix}$
$\begin{bmatrix} C \\ CA \end{bmatrix}^{-1} = \begin{bmatrix} 0.2494 & 0.2494 \\ 0.7506 & -0.2494 \end{bmatrix}$
$\alpha_e(A) = \begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix}^2 - 1.3406\begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix} + 0.4493\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7948 & -0.6694 \\ 0.6694 & -0.5507 \end{bmatrix}$
$G = \begin{bmatrix} 0.7948 & -0.6694 \\ 0.6694 & -0.5507 \end{bmatrix}\begin{bmatrix} 0.2494 & 0.2494 \\ 0.7506 & -0.2494 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3651 \\ 0.3042 \end{bmatrix}$

The observer equation is:

$$\mathbf{q}(k+1) = \begin{bmatrix} 1.6449 & -1.3651 \\ 0.6958 & -0.3042 \end{bmatrix}\mathbf{q}(k) + \begin{bmatrix} 0.3651 \\ 0.3042 \end{bmatrix}y(k) + \begin{bmatrix} 0.125 \\ 0 \end{bmatrix}u(k)$$

5.
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Define variables:
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q1 = 0; q2 = 0; y = 0; u = 0;

Start loop:
     read y;

     // compute control law
     u = 3.2417*q1-2.6375*q2;

     // update states estimation for next loop (prediction)
     q1 = 1.6449*q1 - 1.3651*q2 + 0.3651*y+0.0125*u;
     q2 = 0.6958*q1 - 0.3042*q2 + 0.3042*y;

     output u;
end loop.
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You can also get the controller-estimator transfer function. Then, implement the equivalent difference equation. However, this would waste your time in exam!!!

Double Checking:

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{e}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{GC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix}$$

$$\begin{bmatrix} A-BK & BK \\ 0 & A-GC \end{bmatrix} = \begin{bmatrix} 1.6048 & -0.6703 & 0.4052 & -0.3297 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1.6449 & -1.3651 \\ 0 & 0 & 0.6958 & -0.3042 \end{bmatrix}$$

The poles are

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0.8024 + 0.1627i
0.8024 - 0.1627i
0.6703 + 0.0000i
0.6703 + 0.0000i
```

which are the poles for the closed-loop control in addition to the observer poles.