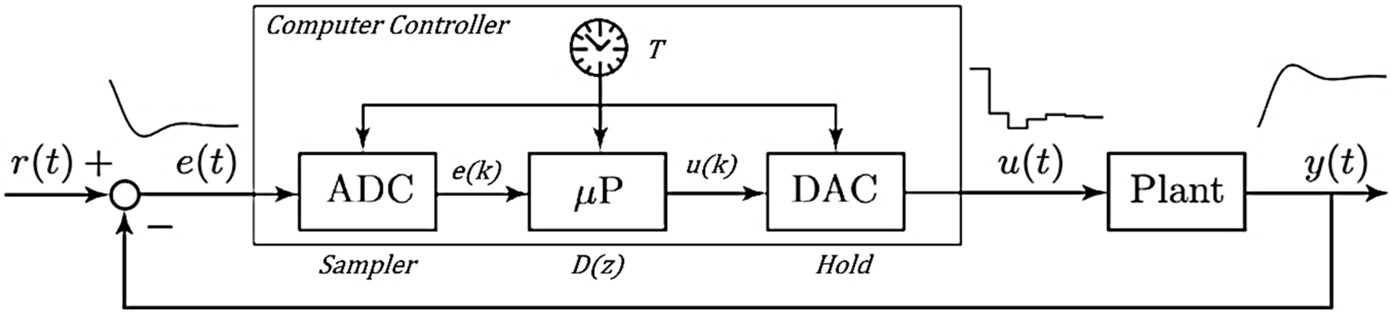
CSE416 “Digital Control” Final Exam Model Answer
January, 2020

Question (1):

1.

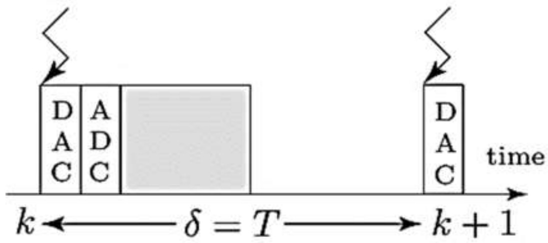


The signals are:
$r(t)$ continuous-time reference signal.
$e(t)$ continuous-time error signal.
$e(k)$ sampled error signal (in discrete time).
$u(k)$ computed control action in discrete time.
$u(t)$ control action in continuous time after hold.
$y(t)$ continuous-time output signal.

2.

Due to computation, additional delays arise. This delay should be added to plant during analysis $e^{-\delta s}P(s)$.
No RTOS can guarantee δ to be constant to achieve synchronization.
Solution can be one of the two simple approaches:

1. The controller should have a robustness margin (Phase margin).
2. Avoid varying computation delays by artificially delaying the output until the next interrupt stimulus arrives as in case C. This event will be strictly synchronous such that in this case constant delays result. The analysis can be described by the augmented plant $e^{-Ts}P(s)$.



3. a.

The Euler backward emulation technique is applied by using the substitution:

$$s = \frac{z-1}{Tz}$$

Inserting this equation into the controller $C(s)$ yields:

$$C(z) = \frac{2\frac{z-1}{Tz}+1}{\frac{z-1}{Tz}+\alpha} = \frac{z(2+T)-2}{z(1+\alpha T)-1}$$

3.b.

The controller $C(z)$ is stable if its pole z_p fulfills the condition $|z_p| < 1$.

The pole z_p is obtained from the equation $z(1 + \alpha T) - 1 = 0$.

$$z_p = \frac{1}{1+\alpha T}$$

From the condition above, α must satisfy:

$$\left|\frac{1}{1+\alpha T}\right| < 1$$

$$|1+\alpha T| > 1$$

$$1+\alpha T > 1 \qquad \text{or} \qquad 1+\alpha T < -1$$

These inequalities constraints lead to the solutions $\alpha > 0$ or $\alpha < -\frac{2}{T}$.

3.c.

For $C(s)$ being stable, its pole $s_p = -\alpha$ must lie in the left half of the complex plane, i.e. $Re\{s_p\} < 0 \Rightarrow \alpha > 0$.

For both continuous and digital controllers, the condition on α is that $\alpha > 0$.

<u>Question (2):</u>

1. a.

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\left(\frac{1}{s+1}\right)\right\} = (1-z^{-1})\mathcal{Z}\left\{\frac{1}{s(s+1)}\right\}$$

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{1}{s}-\frac{1}{s+1}\right\} = \left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}-\frac{z}{z-e^{-T}}\right)$$

$$G(z) = \frac{1-e^{-T}}{z-e^{-T}}$$

The characteristic equation is given by:

$$1+K.G(z) = 0$$

$$1+K\frac{1-e^{-T}}{z-e^{-T}} = 0$$

$$z - e^{-T} + K(1 - e^{-T}) = 0$$

1. b.

The condition for stability is $|z_p| < 1$.

For $T = 1$, the characteristic equation is: $z - 0.3679 + 0.6321K = 0$

$$|-0.3679 + 0.6321K| < 1$$

$$-1 < -0.3679 + 0.6321K < 1$$

$$-0.6321 < 0.6321K < 1.3679$$

Since K is assumed in question as $K > 0$, the range for stability is $K > 2.1641$

Similarly, for $T = 0.1$, range for stability is $K > 20.0167$

Similarly, for $T = 0.01$, range for stability is $K > 200.0017$

1. c.

Reducing sampling time increases the margin of stability, that is obvious from increasing the ranges of K for stability.

2.

The closed-loop characteristic equation is: $1 + G(z)H(z) = 0$

$$1 + \frac{K(z+0.995)}{(z-1)(z-0.905)} = 0$$

We plot the poles and the zeros, then determine the parts of the locus in the real axis as shown in the plot.

No need to compute asymptotic lines, it is obvious from the plot.

To compute breakaway and break-in points:

$$K = -\frac{(z-1)(z-0.905)}{z+0.995} = -\frac{z^2 - 1.905z + 0.905}{z+0.995}$$

$$\frac{dK}{dz} = 0 \Rightarrow \begin{array}{l} z^2 + 1.99z - 2.76 = 0 \\ z_1 = 0.954, z_2 = -2.934 \end{array}$$

$$K|_{z=0.954} = 0.001 > 0 \quad K|_{z=-2.934} = 7.789 > 0$$

One of them is a breakaway point and the other is a break-in.

To compute intersection points with the unit circle, we apply the Jury test:

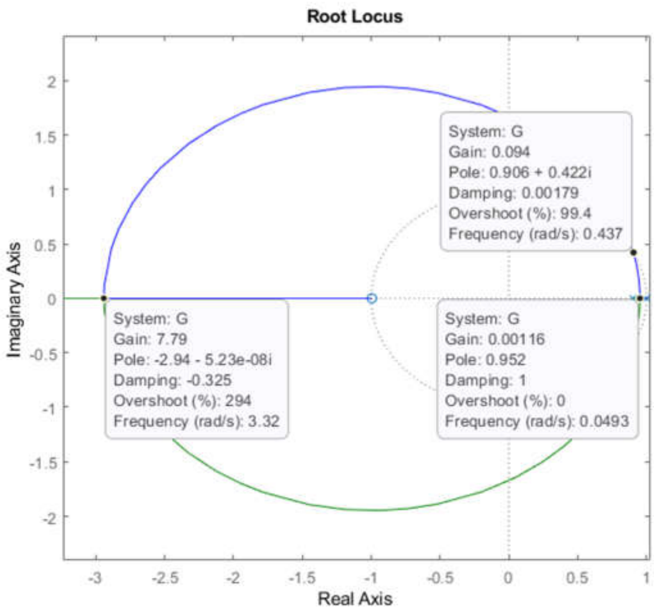
$$z^2 + (K - 1.905)z + 0.905 + 0.995K = 0$$

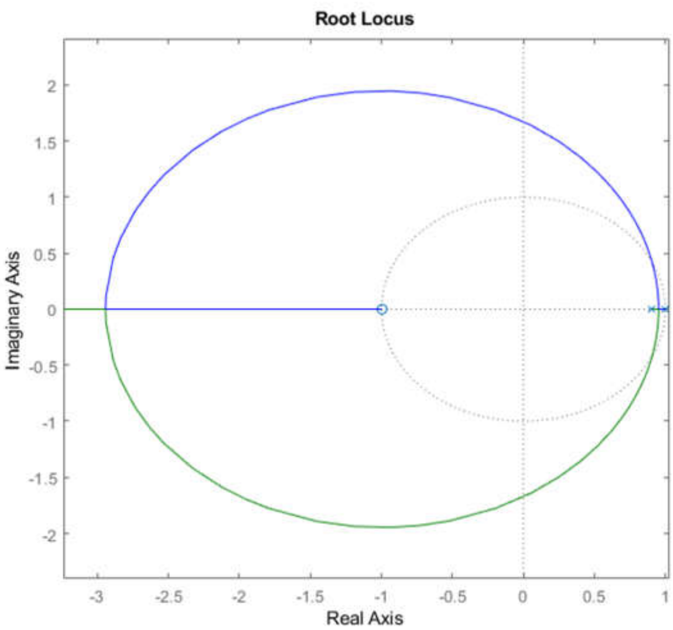
Conditions from stability for Jury test are:

$$\left.\begin{array}{r} Q(1)>0 \\ Q(-1)>0 \\ |Q(0)|<1 \end{array}\right\} \Rightarrow 0<K<0.095$$

Critical value for K for marginal stability is:

$$K_{cr}=0.095 \Rightarrow \begin{array}{l} z^2-1.81z+1=0 \\ z_1=0.905 \pm j0.425 \end{array}$$





Question (3):

1. a.

To apply Routh-Hurwitz, we first apply the bilinear transform:

$$z = \frac{1 + (\frac{T}{2})w}{1 - (\frac{T}{2})w}$$

For $T = 2$, $z = \frac{1+w}{1-w}$

$$G(w) = \frac{K\left(\frac{1+w}{1-w} + 0.8\right)}{\left(\frac{1+w}{1-w} - 1\right)\left(\frac{1+w}{1-w} - 0.6\right)}$$

$$= \frac{K\left(\frac{1+w+0.8(1-w)}{1-w}\right)}{\left(\frac{1+w-1(1-w)}{1-w}\right)\left(\frac{1+w-0.6(1-w)}{1-w}\right)}$$

$$= \frac{K(1+w+0.8(1-w))(1-w)}{(1+w-1(1-w))(1+w-0.6(1-w))}$$

$$= \frac{K(1.8+0.2w)(1-w)}{(1+w-1+w)(0.4+1.6w)}$$

$$G(w) = \frac{K(-0.1w^2 - 0.8w + 0.9)}{(0.4w + 1.6w^2)}$$

The characteristic equation is $1 + G(w) = 0$

$$1 + \frac{K(-0.1w^2 - 0.8w + 0.9)}{(0.4w + 1.6w^2)} = 0$$

$$(0.4w + 1.6w^2) + K(-0.1w^2 - 0.8w + 0.9) = 0$$

$$(1.6 - 0.1K)w^2 + (0.4 - 0.8K)w + 0.9K = 0$$

Applying Routh array:

$$\begin{array}{c|cc} w^2 & (1.6-0.1K) & 0.9K \\ w^1 & (0.4-0.8K) & 0 \\ w^0 & 0.9K & \end{array}$$

The following conditions must hold for stability:

$(1.6-0.1K)>0$	$(0.4-0.8K)>0$	$0.9K > 0$
$1.6>0.1K$	$0.4>0.8K$	$K > 0$
$K<\dfrac{1.6}{0.1}$	$K<\dfrac{0.4}{0.8}$	
$K < 16$	$K < 0.5$	

Then, the aggregate condition inequality is:

$$0 < K < 0.5$$

1. b.

The characteristic equation is:

$$1+G(z)=0$$

$$1+\frac{K(z+0.8)}{(z-1)(z-0.6)}=0$$

$$(z-1)(z-0.6)+K(z+0.8)=0$$

$$(z^2-1.6z+0.6)+K(z+0.8)=0$$

$$z^2+(K-1.6)z+(0.6+0.8K)=0$$

$$Q(z)=z^2+(K-1.6)z+(0.6+0.8K)$$

Applying Jury method:

z^0	z^1	z^2
$0.6+0.8K$	$(K-1.6)$	1

Necessary conditions are:

$Q(1)>0$	$(-1)^2 Q(-1)>0$	$\lvert a_0\rvert < a_2$
$(1)^2+(K-1.6)1+(0.6+0.8K)>0$	$Q(-1)>0$	$\lvert 0.6+0.8K\rvert<1$
$1+(K-1.6)+(0.6+0.8K)>0$	$(-1)^2+(K-1.6)(-1)+(0.6+0.8K)>0$	$\lvert 0.8K\rvert<0.4$
$1.8K>0$	$1-K+1.6+(0.6+0.8K)>0$	$K<\dfrac{0.4}{0.8}$
$K>0$	$-0.2K+3.2>0$	<0.5
	$3.2>0.2K$	
	$K<\dfrac{3.2}{0.2}$	
	<16	

Then, the aggregate condition for stability is:

$$0 < K < 0.5$$

2.

The maximum frequency can be determined from Nyquist frequency:

$$f < \frac{f_s}{2}$$

Therefore, $\omega < \frac{2\pi}{2T} = \frac{\pi}{0.1} = 31.4159\, rad/sec$

$$G(z) = Z\left\{\frac{1-e^{-sT}}{s}\frac{5}{s+5}\right\} = \frac{1-e^{-0.5}}{z-e^{-0.5}},$$

$$G(z) = \frac{0.393}{z-0.606}.$$

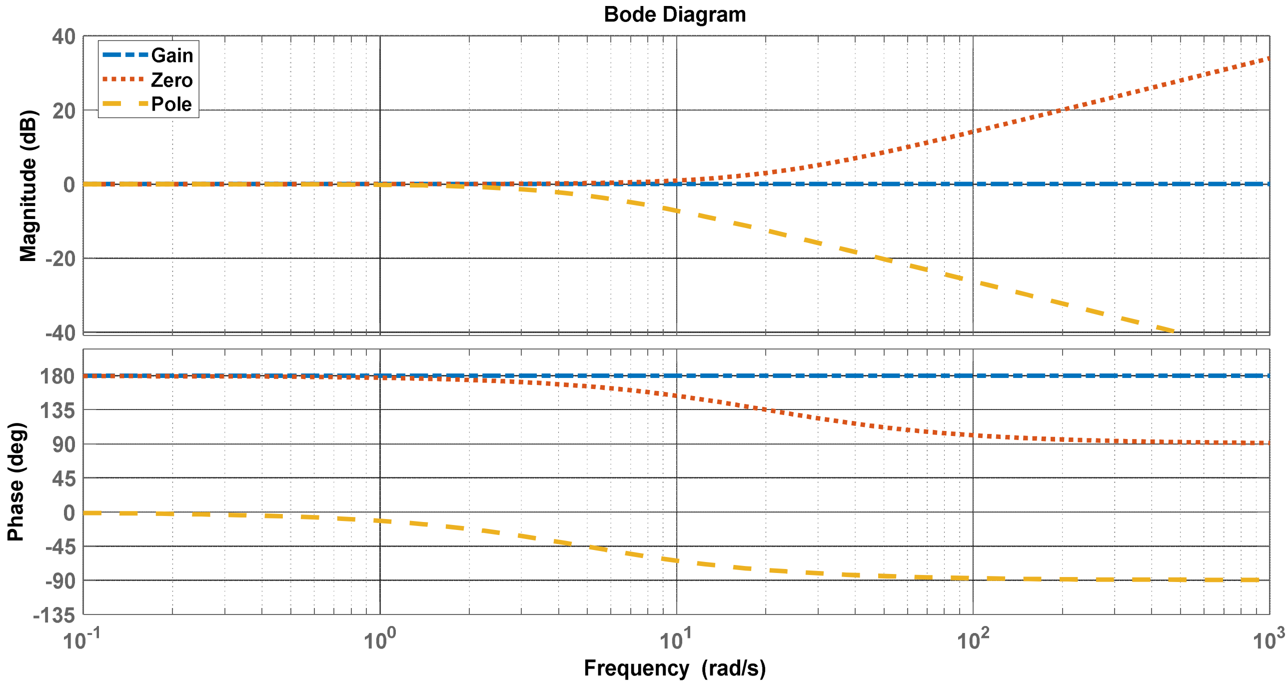
Applying bilinear transform:

$$z = \frac{1+\left(\frac{T}{2}\right)w}{1-\left(\frac{T}{2}\right)w} = \frac{1+0.05w}{1-0.05w}$$

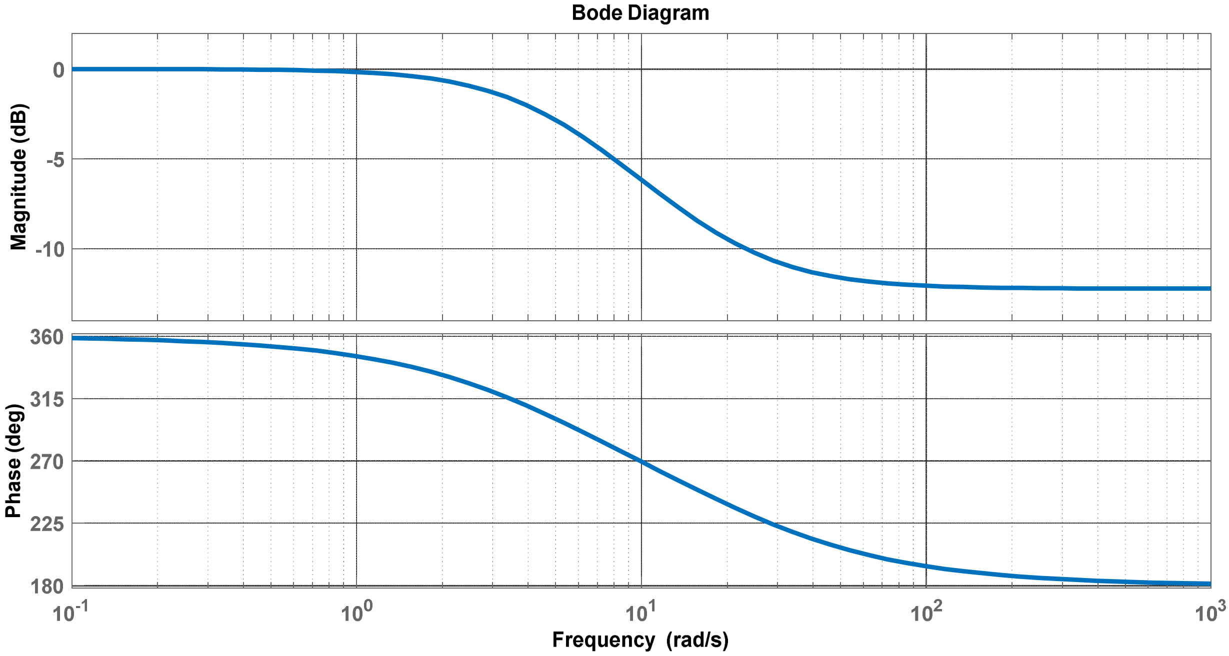
$$G(w) = \frac{0.3935}{\frac{1+0.05w}{1-0.05w} - 0.6065} = \frac{-0.01968w + 0.3935}{0.08033w + 0.3935}$$

$$G(w) = \frac{-0.24494(w-20)}{(w+4.899)} = \frac{-1.0020\left(\frac{j\omega}{20} - 1\right)}{\left(\frac{j\omega}{4.899} + 1\right)}$$

Bode diagram of each component



Complete bode diagram:



Question (4):

1. a.

We calculate the state transition matrix $\phi_c(t)$:

$$\phi_c(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

$$A_d = \phi_c(T) = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} 1 & 0.4323 \\ 0 & 0.1353 \end{bmatrix}$$

$$B_d = \left[\int_0^{\tau} \phi_c(\tau) d\tau\right] B = \begin{bmatrix} \frac{1}{2}\left(T + \frac{1}{2}(e^{-2T} - 1)\right) \\ \frac{1}{2}(1 - e^{-2T}) \end{bmatrix} = \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix}$$

$$C_d = C = [1 \quad 0]$$

Thus, the discrete-time state variable model is:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.4323 \\ 0 & 0.1353 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

1. b.

The discrete transfer function can be obtained from the discrete-time state space model by:

$$G(z) = C_d(zI - A_d)^{-1}B_d$$

$$G(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & -0.4323 \\ 0 & z-0.1353 \end{bmatrix}^{-1} \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix}$$

$$G(z) = \frac{1}{(z-1)(z-0.1353)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-0.1353 & 0.4323 \\ 0 & z-0.1353 \end{bmatrix} \begin{bmatrix} 0.2838 \\ 0.4323 \end{bmatrix}$$

$$G(z) = \frac{0.2838z + 0.1485}{z^2 - 1.1353z + 0.1353}$$

2. a.

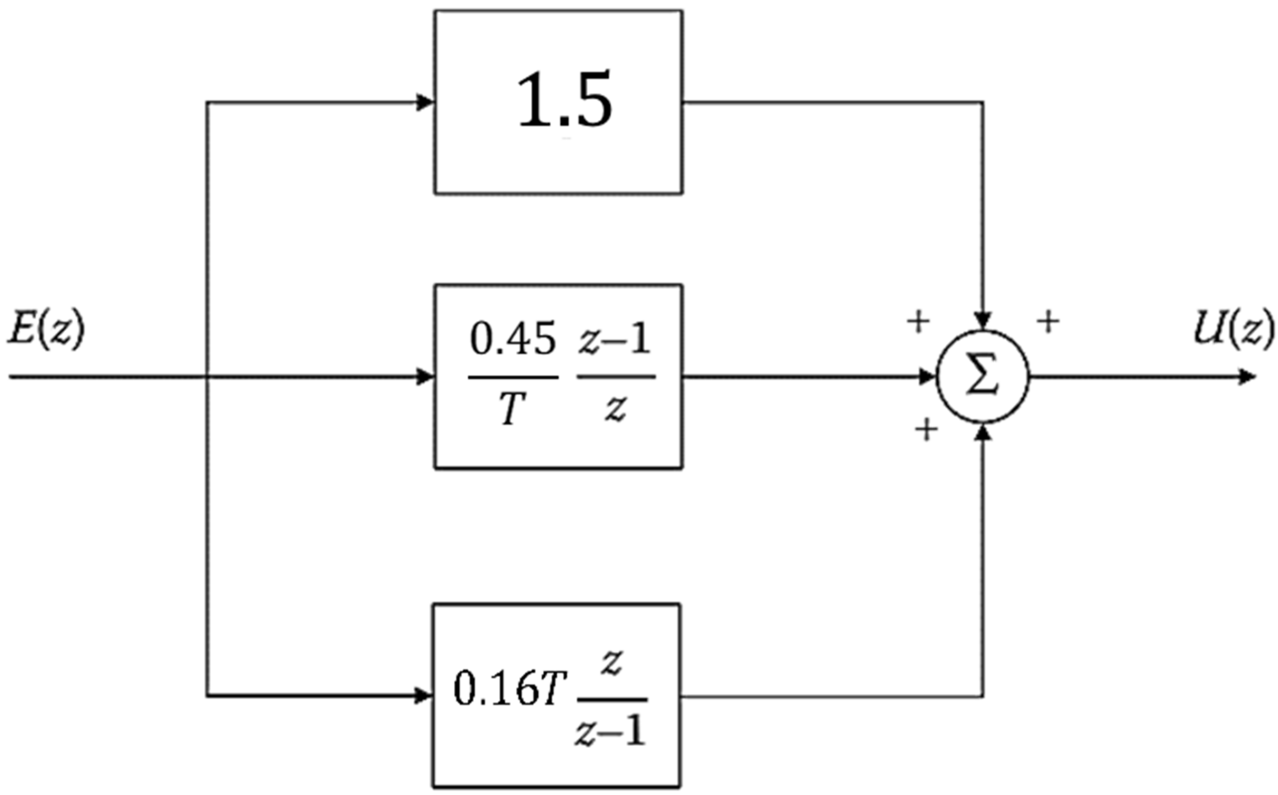
The continuous time differential equation for PID is:

$$u(t) = 1.5e(t) + 0.16\int_0^{\tau} e(\tau)d\tau + 0.45\frac{de(t)}{dt}$$

By emulation:

$$u(k) = 1.5e(k) + 0.16T\sum_{j=1}^{k} e(j) + \frac{0.45}{T}\big(e(k) - e(k-1)\big)$$

2. b.



2. c.

```
static double   Ts = 1,
                Kp = 1.5,
                Ki = 0.16,
                Kd = 0.45,
                Isat = 3;


double PID_Controller(double Y, double Ref)
{
   static double U;
   static double err, err_prev = 0;

    /* Calculate error of the ball's position */
   err = Ref - Y;

   /* Calculate the I part */
   if ( Ki == 0.0 )
     I_u = 0.0;  /* Set the I part to zero if Ki == 0 */
   else
     I_u += Ki * err * Ts;

   /* Saturation of the I part */
   if ( Isat > 0.0 )
   {
     if ( I_u >  Isat )   I_u =  Isat;
     if ( I_u < -Isat )   I_u = -Isat;
   }

   /* Add vertical PID parts: PID = P + I + D */
   U = Kp*err + I_u + Kd*(err-err_prev)/Ts ;

   err_prev = err;


    /* Check control limits */
    if( U > +5.0 ) U = +5.0;
    if( U < -5.0 ) U = -5.0;

   return U;
}
```