Least Squares Approximations

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M 340L Matrices and Matrix Calculations

$A\vec{x} = \vec{b} \longleftarrow$ inconsistent system

$A^T A\hat{x} = A^T\vec{b} \longleftarrow$ normal equations

$\|A\hat{x} - \vec{b}\| \longrightarrow$ Least squares error

Example #1

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & -1 \end{bmatrix}}_{A}\vec{x} = \underbrace{\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}_{b}$$

$$A^T A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}\hat{x} = \begin{bmatrix} 7 \\ -7 \end{bmatrix},\ \hat{x} = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix} \longrightarrow \|A\hat{x} - b\| = \left\| \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\|$$

$$\|A\hat{x} - b\| = \cancel{\frac{1}{3}\sqrt{16+1+4}}\ \frac{\sqrt{21}}{3} = \left\| \begin{bmatrix} -4/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -5/3 \\ -4/3 \\ 8/3 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\|$$

Try: $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

$$\left\| A\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{3} \approx 1.73$$

$A^T A\hat{x} = A^T\vec{b}$; has a unique solution $\iff$ the columns of A are linearly independent.

Example #2

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 2 \end{bmatrix}\hat{x} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix},\ A^T\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & 12 & 2 \\ 12 & 24 & 4 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1/3 - 2s \\ s \end{bmatrix},\ s \in \mathbb{R}$$

$$A\hat{x} = \begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1/3 - 2s \\ s \end{bmatrix} = \begin{bmatrix} 1/3 - 2s + 2s \\ -2/3 + 4s - 4s \\ 1/3 - 2s + 2s \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} \longrightarrow \|A\hat{x} - \vec{b}\| = \left\| \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 2/3 \\ -5/3 \\ 8/3 \end{bmatrix} \right\| = \frac{\sqrt{93}}{3}$$

$$\|A\hat{x} - \vec{b}\| = 4.82182538$$

Example #3 (Use computer)

Collection of points: $(1,1)$, $(2,1.5)$, $(3,3)$, $(4,4.5)$, $(5,7)$

$y = \alpha_0 + \alpha_1 x$

$\alpha_0 + \alpha_1(1) = 1$

$\alpha_0 + \alpha_1(2) = 1.5$

$\alpha_0 + \alpha_1(3) = 3$

$\alpha_0 + \alpha_1(4) = 4.5$

$\alpha_0 + \alpha_1(5) = 7$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}}_{\hat{x}} = \begin{bmatrix} 1 \\ 1.5 \\ 3 \\ 4.5 \\ 7 \end{bmatrix}$$

$$A^T A\hat{x} = A^T\vec{b} \rightarrow A^T A = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix},\ A^T\vec{b} = \begin{bmatrix} 17 \\ 66 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -1.1 \\ 1.5 \end{bmatrix} \longrightarrow \|A\hat{x} - b\| = 1.095$$

$$y = -1.1 + 1.5x$$

What he's looking for in the test!

What if we fit these points to a parabola?

$$y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

$$\begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \\ 1 & 5 & 5^2 \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 3 \\ 4.5 \\ 7 \end{bmatrix} \quad \text{(use computer)}$$

Example #4

Coll. of Points: $(1,1.5)$, $(1.5,3)$, $(2,3.5)$, $(2.5,3)$, $(3,2)$, $(3.5,1)$, $(4,-1)$, $(4.5,-2.5)$, $(5,-3.5)$, $(5.5,-3.5)$

$$y = \alpha_0\cos x + \alpha_1\sin x,\ A = \overbrace{\begin{bmatrix} \cos(1.0) & \sin(1.0) \\ \cos(1.5) & \sin(1.5) \\ \cos(2.0) & \sin(2.0) \\ \cdots & \cdots \\ \cos(5.5) & \sin(5.5) \end{bmatrix}}^{\text{Everything in radians}},\ \text{and } \vec{b} = \begin{bmatrix} 1.5 \\ 3 \\ \vdots \\ -3.5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4.0233 & -0.2151 \\ -0.2151 & 5.9767 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -8.0463 \\ 18.1904 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} -1.8407 \\ 2.9773 \end{bmatrix}$$

$$\|A\hat{x} - b\| \approx 0.7810 \longrightarrow \boxed{y = -1.8407\cos x + 2.9773\sin x}$$

The linear regression