Vectors in $\mathbb{R}^n$

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Lecture 3

A column vector is an $n \times 1$ matrix

Linear Algebra Physics

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

Vector Addition

If $\vec{u}, \vec{v} \in \mathbb{R}^n$, then $\vec{u} + \vec{v}$ is the vector in $\mathbb{R}^n$ whose ith element is $u_i + v_i$.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \cdots \\ u_n + v_n \end{bmatrix} \tag{1}$$

Example #1

$$\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

The vector addition below cannot be allowed and will be seen as not defined or in other words undefined,

$$\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \textbf{not defined}$$

$$+8 = \textbf{not defined}$$

Scalar Multiplication

If $\vec{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then the scalar multiple of c and $\vec{u}$, written as $c\vec{u}$ is the vector in $\mathbb{R}^n$ whose ith component is cu_i.

$$c\vec{u} = c \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ ... \\ cu_n \end{bmatrix} \tag{2}$$

If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\{$ is a set of vectors in $\mathbb{R}^n$ and $\{c_1, c_2, ..., c_k\}$ is a set of scalars in $\mathbb{R}$, then

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + ... + c_k\vec{v}_k \tag{3}$$

is the linear combination of the $\vec{v}$'s using the c's as weights.

If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the span $s = \text{span}\{\vec{v}_1, ..., \vec{v}_k\}$ is the set of **ALL** linear combination of these vectors.

Example 2

$$\text{span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\} = \{c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} | c_1, c_2 \in \mathbb{R}\} \tag{4}$$

Problem Set 1

$$\begin{aligned} a_{1\,1}x_1 + a_{1\,2}x_2 + ... + a_{1\,n}x_n &= b_1 \\ a_{2\,1}x_1 + a_{2\,2}x_2 + ... + a_{2\,n}x_2 &= b_2 \\ a_{m\,1}x_1 + ... + a_{m\,n}x_n &= b_m \end{aligned}$$

If A is an $m \times x$ matrix and $\vec{x}$ is a vector in $\mathbb{R}^n$ then $A\vec{x}$ is the vector in $\mathbb{R}^n$ which is the linear combination of the columns of A using the components of as weights.

$$A\vec{x} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & ... & \vec{a}_n\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + ... + x_n a_n \tag{5}$$

Problem 1

$$\begin{aligned}
\begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 1 & 0 & -2 \\ 0 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} &= 2\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 7 \\ -1 \end{bmatrix}
\end{aligned}$$

* can be written as $A\vec{x} = b$, can be written as a linear combination of the columns of A?

If $A = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & ... & \vec{a}_2\end{bmatrix}$, is $\vec{b}$ in span$\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$?

Pivot in every row A= will always have a solution.

If A is an $m \times n$ matrix, then the following are equivalent.

- $\forall \vec{b} \in \mathbb{R}^m$, $A\vec{x} = b$ has a solution
- Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A

- The columns of A span $\mathbb{R}^m$A$has a pivot in every row$