Lecture Notes (June 28, 2024)

M 340L Matrices and Matrix Calculations
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Example #1

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ – Invertiable – Diagonalizable $\quad$ $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ – Invertiable, – Diagnonalizable

$\begin{bmatrix}1&0\\0&0\end{bmatrix}$ – Not invertable – Diagonalizable $\quad$ $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ Not invertiable or diagonalizable

Example #2

A is a diagonalizable matrix

$$A = \begin{bmatrix}-1&2&-6\\3&-2&9\\2&2&7\end{bmatrix},\ \lambda = 2,\ \vec{v} = \begin{bmatrix}-2\\3\\2\end{bmatrix};\ \lambda = 1,\ \vec{v} = \begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix}-1&2&-6\\3&-2&9\\2&2&7\end{bmatrix}\begin{bmatrix}-1&2&-6\\3&-2&9\\2&2&7\end{bmatrix} = \begin{bmatrix}-5&-18&-18\\9&28&39\\18&14&55\end{bmatrix}$$

$A^8 = (AAAAAAAA)$ [sadism] $\qquad A = PDP^{-1}$

$$A^k,\ (N^+ = k) = \underbrace{AAA\ldots}_{k \text{ times}}$$

$$= \underbrace{(PDP^{-1})(PDP^{-1})\ldots}_{k \text{ times}} \longrightarrow PD(P^{-1}P)D(P^{-1}P)\ldots$$

$$= PDIDI\ldots IDIDP^{-1}$$

$$= P\underbrace{DD\ldots DD}_{k \text{ times}}P^{-1}$$

$$\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix} = \begin{bmatrix}a^2&0&0\\0&b^2&0\\0&0&c^2\end{bmatrix};\ \text{if } D = \begin{bmatrix}\lambda_1&0&0\\0&\lambda\ldots&0\\0&0&\lambda_n\end{bmatrix} = D^k = \begin{bmatrix}\lambda_1^k&0&0\\0&\lambda^k\ldots&0\\0&0&\lambda_n^k\end{bmatrix}$$

Def of A^k:

$$A^k = PD^kP^{-1} = P\begin{bmatrix}\lambda_1^k&0&0\\0&\lambda^k\ldots&0\\0&0&\lambda_n^k\end{bmatrix} \blacksquare$$

For A^8

$$= PD^8P^{-1} = \begin{bmatrix}-2&1&-3\\3&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}2^8&0&0\\0&1^8&0\\0&0&1^8\end{bmatrix}P^{-1};\ \text{when } A = \begin{bmatrix}4&4&2\\-2&-1&-2\\-2&0&3\end{bmatrix},\ \text{then } \lambda = 0,$$

$$\vec{v} = \begin{bmatrix}-3\\2\\2\end{bmatrix};\ \lambda = 1,\ \vec{v} = \begin{bmatrix}-2\\1\\1\end{bmatrix};\ \lambda = -1,\ \vec{v} = \begin{bmatrix}1\\3\\4\end{bmatrix}$$

Proof: When A^3

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$$

$$= PIP^{-1} + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \ldots$$

$$= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \ldots\right)P^{-1}$$

$$e^A = \begin{bmatrix}e^{\lambda}&0&0\\0&e^{\lambda_2}&0\\0&0&e^{\lambda_n}\end{bmatrix}P^{-1}$$

P

$$P\begin{bmatrix}1+\lambda_1+\frac{1}{2!}\lambda^2+\frac{1}{3!}\lambda_1^3\ldots&&\\0&1+\lambda_2+\frac{1}{2!}\lambda_2^2+\frac{1}{3!}\lambda_2^3+\ldots&\\0&0&1+\lambda_3+\frac{1}{2!}\lambda_3^2+\frac{1}{3!}\lambda_3^3+\ldots\end{bmatrix}$$

Theorem:

$A = PDP^{-1}$, where $P = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n]$ and $\vec{x} = [\vec{x}]_\varepsilon$; when $[\vec{x}]$ is the coordinates of $\vec{x}$ relative to the eigenvector basis.

$$= PDP^{-1}\vec{x}$$

$$T(x) = PD[\vec{x}]_\varepsilon$$

$$P^{-1}T(x) = D[\vec{x}]_\varepsilon$$

$$[T(x)]_\varepsilon = D[\vec{x}]_\varepsilon$$