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M340L
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Orthonormal Sets and Basis, and Projections

Orthonormal Sets

An orthogonal set of unit vectors

Example #1

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ An orthogonal basis for $\mathbb{R}^3$

$\downarrow$ normalize

$\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}\right\}$ An orthonormal basis for $\mathbb{R}^3$

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

$c_i = \dfrac{\vec{x}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}$ If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a orthogonal basis.

Orthonormal Basis

An orthogonal basis of unit vectors.

$\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\} \leftarrow$ Orthonormal basis

$c_i = \vec{x}\cdot\vec{u}_i$

$U = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k]$

$\{\vec{u}_1, \dots, \vec{u}_k\}$ is an orthonormal subset of $\mathbb{R}^n$.

$$U^TU = \begin{bmatrix}\vec{u}_1^T\\ \vec{u}_2^T\\ \vdots\\ \vec{u}_k^T\end{bmatrix}[\vec{u}_1, \vec{u}_2, \dots \vec{u}_k]$$

$$= \begin{bmatrix}\vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \cdots & \vec{u}_1^T\vec{u}_k\\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 & \cdots & \vec{u}_2^T\vec{u}_k\\ \vdots & & \ddots & \vdots\\ \vec{u}_k^T\vec{u}_1 & \cdots & \cdots & \vec{u}_k^T\vec{u}_k\end{bmatrix} = \begin{bmatrix}\vec{u}_1\cdot\vec{u}_1 & \vec{u}_1\cdot\vec{u}_2 & \cdots & \vec{u}_1\cdot\vec{u}_k\\ \vdots & \vec{u}_2\cdot\vec{u}_2 & & \vdots\\ \vdots & & \ddots & \vdots\\ \vec{u}_k\cdot\vec{u}_1 & \cdots & \cdots & \vec{u}_k\cdot\vec{u}_k\end{bmatrix},$$

$$= \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & \cdots & 1\end{bmatrix} = I$$

Properties of $\vec{U}$:

1. $\|U\vec{x}\| = \|\vec{x}\|$
2. $(U\vec{x})(U\vec{y}) = \vec{x}\cdot\vec{y}$
3. $(U\vec{x})(U\vec{y}) = 0 \iff \vec{x}\cdot\vec{y} = 0$

$(U\vec{x})\cdot(U\vec{y})$
$= (U\vec{x})^T(U\vec{y})$
$= \vec{x}^TU^TU\vec{y}$
$= \vec{x}^TI\vec{y}$
$= \vec{x}^T\vec{y}$
$= \vec{x}\cdot\vec{y}$

If U is an $n\times n$ matrix with orthonormal columns, it is called orthogonal matrix.

$U^TU = I$

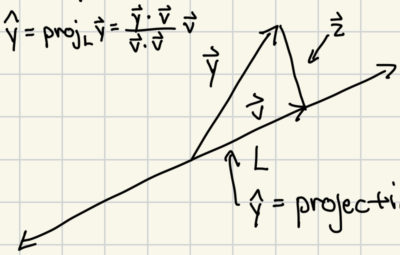
$\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{6}\\1/\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{3}\\-2/\sqrt{6}\\0\end{bmatrix} = \frac{1}{3} - \frac{2}{6} + 0 = 0$ $\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2}\\ 1/\sqrt{3} & -2/\sqrt{6} & 0\\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2}\end{bmatrix}$

$\left\|\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{6}\\1/\sqrt{2}\end{bmatrix}\right\| = \sqrt{1/3 + 1/6 + 1/2} = 1$

Orthogonal Projections

Example #1

$\hat{y} = \text{proj}_L\,\vec{y} = \dfrac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$



$\hat{y}$ = projection of $\vec{y}$ onto L (onto $\vec{v}$)

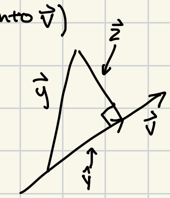
$\vec{y} = \hat{y} + \vec{z} = \begin{bmatrix}3\\3\\3\end{bmatrix} + \begin{bmatrix}-1\\2\\-1\end{bmatrix}$

$\hat{y}\cdot\vec{z} = \begin{bmatrix}3\\3\\3\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\-1\end{bmatrix} = -3+6-3 = 0$

$\vec{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{y} = \begin{bmatrix}2\\5\\2\end{bmatrix}$

$= \text{proj}_L\,\vec{y} = \text{proj}_{\vec{v}}\,\vec{y}$

$\vec{z} = \vec{y} - \hat{y}$

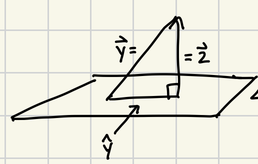


$\hat{y} = \dfrac{\begin{bmatrix}2\\5\\2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}} = \dfrac{2+5+2}{1+1+1}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix}$

$\vec{y} = \hat{y} + \vec{z} = \begin{bmatrix}3\\3\\3\end{bmatrix} + \begin{bmatrix}-1\\2\\-1\end{bmatrix}$

$\hat{y}\cdot\vec{z} = \begin{bmatrix}3\\3\\3\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\-1\end{bmatrix} = -3+6-3 = 0$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}2\\5\\2\end{bmatrix} - \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$



$\mathcal{W}$, a subspace of $\mathbb{R}^n$

Theorem: If $\vec{y}$ is a vector in $\mathbb{R}^n$ and $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\vec{y}$ can be written uniquely as $\vec{y} = \hat{y} + \vec{z}$, where $\hat{y}\in\mathcal{W}$ and $\vec{z}\in\mathcal{W}^\perp$.

Furthermore, if $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for $\mathcal{W}$, then

$$\hat{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \dots + \frac{\vec{y}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k \quad *$$

(existance)

Proof: If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal basis for $\mathcal{W}$, then since $\hat{y}$ is constructed by $*$, it is a linear combination of vectors in $\mathcal{W}$, $\hat{y}$ is itself in $\mathcal{W}$.

Want to show: $\vec{z}\in\mathcal{W}^\perp$; consider $\vec{z}\cdot\vec{v}_i = (\vec{y} - \hat{y})\cdot\vec{v}_i$

$$= \left(\vec{y} - \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 - \dots - \frac{\vec{y}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k\right)\cdot\vec{v}_i$$

(continues up on the right)

(Proof continues here)

$$= \vec{y}\cdot\vec{v}_i - \cancel{\frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\cdot\vec{v}_1\cdot\vec{v}_i}^{\,0} - \cancel{\frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2\cdot\vec{v}_i}^{\,0} - \dots - \frac{\vec{y}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}\vec{v}_i\cdot\vec{v}_i - \dots \cancel{\frac{\vec{y}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k\cdot\vec{v}_i}_{\,0}$$

$$= \vec{y}\cdot\vec{v}_i - \frac{\vec{y}\cdot\vec{v}_i}{\cancel{\vec{v}_i\cdot\vec{v}_i}}\cancel{\vec{v}_i\cdot\vec{v}_i}$$

$$= \vec{y}\cdot\vec{v}_i - \vec{y}\cdot\vec{v}_i = 0$$

Since $\vec{z}\perp\vec{v}_i$ for every i, it must be ortnogonal to every vector in $\mathcal{W}$.

Thus, $\vec{z}\in\mathcal{W}^\perp$.

Uniqueness: Let's pretend $\vec{y} = \hat{y} + \vec{z}$ and $\vec{y} = \vec{\alpha} + \vec{\beta}$, where $\hat{y}, \vec{\alpha}\in\mathcal{W}$ and $\vec{z}, \vec{\beta}\in\mathcal{W}^\perp$.

$\hat{y} + \vec{z} = \vec{\alpha} + \vec{\beta}$

$(\hat{y} - \vec{\alpha})\cdot(\underbrace{\hat{y} - \vec{\alpha}}_{\in\mathcal{W}}) = (\underbrace{\vec{\beta} - \vec{z}}_{\in\mathcal{W}^\perp})\cdot(\hat{y} - \vec{\alpha}) = 0$

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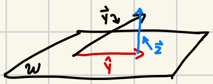
so $\hat{y} - \vec{\alpha} = \vec{0}$ or $\hat{y} = \vec{\alpha}$. Similarly, $\vec{\beta} = \vec{z}$. ■

Example #2

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}}_{\text{Orthogonal basis for } W} = W, \text{ a subspace of } \mathbb{R}^5$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 4 \\ 9 \\ -7 \\ -8 \\ 13 \end{bmatrix} - \begin{bmatrix} 9 \\ 8 \\ -5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -2 \\ -2 \\ -9 \\ 9 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 4 \\ 9 \\ -7 \\ -8 \\ 13 \end{bmatrix}, \hat{y} = \begin{bmatrix} 9 \\ 8 \\ -5 \\ 1 \\ 4 \end{bmatrix}, \vec{z} = \begin{bmatrix} -5 \\ 1 \\ -2 \\ -2 \\ -9 \\ 9 \end{bmatrix}$$



Check: $\hat{y} \cdot \vec{z} = 9(-5) + 8 \cdot 1 + (-5)(-2) + 1(-9) + 4(9)$
$= -18 + 8 + 10$
$= 0$

$\hat{y}$ is the closest vector in W to $\vec{y}$.
$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$, where $\vec{v} \in W$ and $\vec{v} \neq \hat{y}$.
$\|\vec{z}\|$ = the distance from $\vec{y}$ to W.

Write $\vec{y}$ as the sum of a vector in W and a vector in $W^\perp$. (or orthogonally decompose $\vec{y}$ into a sum of a vector in W and a vector in $W^\perp$)

$$\hat{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 + \frac{\vec{y} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3$$

$$\hat{y} = \frac{\begin{bmatrix} 4 \\ 9 \\ -7 \\ -8 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 9 \\ -7 \\ -8 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 9 \\ -7 \\ -8 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{4 + 18 - 21 - 8 + 26}{1 + 4 + 9 + 1 + 4} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \frac{8 + 9 + 0 + 16 - 13}{4 + 1 + 0 + 4 + 1} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \frac{4 + 9 + 14 - 8 + 13}{1 + 1 + 4 + 1 + 1} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ -5 \\ 1 \\ 4 \end{bmatrix} = \hat{y}$$

What if you want to do this projection, and you have a basis for W, but *it's not* orthogonal?