Lecture Notes (July 3rd, 2024)

M 340L Matrices and Matrix Calculations
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Dot Products (also known as Inner Products)

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$,

$$\underset{n\times 1}{\vec{u}} \cdot \underset{n\times 1}{\vec{v}} = \underset{1\times n}{\vec{u}^T} \underset{n\times 1}{\vec{v}} \equiv u_1v_1 + u_2v_2 + \dots + u_nv_n$$

Properties of the dot product

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
3. $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$
4. $\vec{u}\cdot\vec{u} \geq 0$, and $\vec{u}\cdot\vec{u} = 0 \iff \vec{u} = \vec{0}$

The length/magnitude/norm of a vector is

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}.$$

A unit vector is a vector whose length is 1.

If $\vec{v}$ is not $\vec{0}$, then you can normalize $\vec{v}$ by replacing it with $\frac{\vec{v}}{\|\vec{v}\|}$, the unit vector in the direction of $\vec{v}$.

$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \left.\frac{\vec{v}}{\|\vec{v}\|}\right\} \text{direction}$$

Also $\|\vec{v}\| \equiv |\vec{v}|$.

Example #1

$$\vec{v} = \begin{bmatrix}2\\1\\-1\end{bmatrix},\ \|\vec{v}\| = \sqrt{4+1+1} = \sqrt{6}$$

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}2/\sqrt{6}\\1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}$$

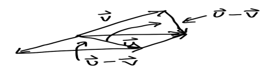
Constant Property

$$\|c\vec{v}\| = \sqrt{(c\vec{v}_1)^2 + (c\vec{v}_2)^2 + \dots + (c\vec{v}_n)^2}$$
$$= \sqrt{c^2(\vec{v}_1^2 + \vec{v}_2^2 + \dots + \vec{v}_n^2)}$$
$$= \sqrt{c^2}\sqrt{\vec{v}_1^2 + \vec{v}_2^2 + \dots + \vec{v}_n^2}$$
$$= |c|\,\|\vec{v}\|$$

Example #2

Distance between $\vec{u}$ and $\vec{v}$

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$



Orthogonal Vectors

Two vectors are orthogonal if $\vec{u}\cdot\vec{v} = 0$.

Example #3

$$\vec{u} = \begin{bmatrix}1\\2\end{bmatrix},\ \vec{v} = \begin{bmatrix}-2\\1\end{bmatrix} \qquad \vec{u}\cdot\vec{v} = 1(-2) + 2(1) = 0$$

In general,

$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta \iff 0 \leq \theta \leq \pi}$$

angle between $\vec{u}$ and $\vec{v}$

If S is a set in $\mathbb{R}^n$, the orthogonal complement to S, denoted as $S^\perp$ (read as "s-perp") is the set of vectors in $\mathbb{R}^n$ orthogonal to every vector in S.

Example # 3.1

$$S = \left\{\begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}$$

I want to find $S^\perp$

$\vec{v}$ is in $S^\perp$ if $\begin{bmatrix}1\\2\\1\\2\end{bmatrix}\cdot\vec{v} = 0$ and $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\cdot\vec{v} = 0$

$$\begin{bmatrix}1 & 2 & 1 & 2\end{bmatrix}\vec{v} = \vec{0}$$
$$\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}\vec{v} = \vec{0} \longrightarrow \begin{bmatrix}1 & 2 & 1 & 2\\1 & 1 & 0 & 0\end{bmatrix}\vec{v} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\xrightarrow{R_2-R_1} \begin{bmatrix}1 & 2 & 1 & 2\\0 & -1 & -1 & -2\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 1 & 2\end{bmatrix};\ \vec{v} = s\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}; \quad s,t \in \mathbb{R}$$

Sol: $S^\perp = \text{span}\left\{\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-2\\0\\1\end{bmatrix}\right\}$

$\hookrightarrow$ $S^\perp$ is always a subspace of $\mathbb{R}^n$

Orthogonal Set

An orthogonal set is a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ where $\vec{v}_i\cdot\vec{v}_j = 0$ when $i \neq j$

Example #4

$$\left\{\underset{\vec{v}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\0\\-1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\0\\0\end{bmatrix}}, \underset{\vec{v}_4}{\begin{bmatrix}1\\-2\\1\end{bmatrix}}\right\}$$

This set is orthogonal

Linear Combinations

$\vec{v}_1\cdot\vec{v}_2 = 1+0-1 = 0$
$\vec{v}_1\cdot\vec{v}_3 = 0+0+0 = 0$
$\vec{v}_1\cdot\vec{v}_4 = 1-2+1 = 0$
$\vec{v}_2\cdot\vec{v}_3 = 0+0+0 = 0$
$\vec{v}_2\cdot\vec{v}_4 = 1+0-1 = 0$
$\vec{v}_3\cdot\vec{v}_4 = 0+0+0 = 0$

Theorem

If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal set of non-zero vectors, then the set is *linearly independent*.

Proof

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$

$$\vec{v}_i\cdot(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) = \vec{v}_i\cdot\vec{0}$$

$$c_1\underbrace{\cancel{\vec{v}_i\cdot\vec{v}_1}}_{0} + \underbrace{\cancel{\dots}}_{0} + c_i\vec{v}_i\cdot\vec{v}_i + \underbrace{\cancel{\dots}}_{0} + c_k\underbrace{\cancel{\vec{v}_i\cdot\vec{v}_k}}_{0} = \vec{0}$$

$$c_i\vec{v}_i\cdot\vec{v}_i = \vec{0}.$$

Since $\vec{v}_i \neq 0$, $\vec{v}_i\cdot\vec{v}_i \neq 0 \Longrightarrow c_i = 0$ so all c's $= 0$, and the set is *linearly independent* ∎

Orthogonal Basis

An orthogonal basis is a basis which is also an orthogonal set.

Say $\{\vec{v}_1 \dots \vec{v}_k\}$ is an orthogonal basis for W.

If $\vec{x} \in W$

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$
$$\vec{x}\cdot\vec{v}_i = (c_1\vec{v}_1 + \dots + c_k\vec{v}_k)\cdot\vec{v}_i$$
$$\vec{x}\cdot\vec{v}_i = c_i\,\vec{v}_i\cdot\vec{v}_i,\quad c_i = \frac{\vec{x}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}$$

Example #5

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}\right\} \longleftarrow \text{Orthogonal basis for } \mathbb{R}^3$$

$$\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3v_3,\quad c_1 = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}} = \frac{1+2+3}{1+0+1} = 2$$

$$c_2 = \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix}}{\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix}} = \frac{1+0-3}{1+0+1} = -1$$

$$c_3 = \frac{\vec{x}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}}{\begin{bmatrix}1\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}} = \frac{1-4+3}{1+4+1} = 0$$

$$\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + (-1)\begin{bmatrix}1\\0\\-1\end{bmatrix} + 0\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

Example #6 (What if your basis isn't orthogonal)

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\},\ \vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$c_1 = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = 1;\ c_2 = \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{-1+2}{1+1} = 1/2$$

$$c_3 = \frac{\vec{x}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{2+2+3}{4+1+1} = \frac{7}{6}$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \longrightarrow 1\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} + 7/6\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\boxed{\text{If your basis isn't orthogonal, you must do row reduction.}}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \neq \begin{bmatrix}1-1/2+7/3\\1/2+7/6\\7/6\end{bmatrix} = \begin{bmatrix}17/6\\10/6\\7/6\end{bmatrix}$$