How determinants interact with elementary row operations: Lecture 12

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Class	M 340L
AI summary	This document discusses determinants and their interactions with elementary row operations. It provides examples of swapping rows, multiplying rows by a number, and replacing rows with the sum of another row and a multiple. It also explores the determinant of the transpose of a matrix and the determinant of matrix products. The document concludes with a discussion on bases and coordinates relative to a basis.
Última edición por	Abdon Morales

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 3 & 2 & 3 \\ 4 & 1 & 3 & 1 & 5 & 8 \\ -2 & 2 & 1 & 3 & 5 & 6 \\ 0 & 1 & 0 & 3 & 0 & -1 \end{bmatrix}$$

1. If you swap two rows, the determinant is multiplied by -1.
2. If you multiply a row by a number, you multiply the determinant by that number.

3. If you replace a row by the sum of that row and a multiple of another row, the determinant does not change.

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1 \times 4) - 2 \times 3 = -2$$

Then swap $R_1 \text{ and } R_3$,

$$\det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = (3 \times 2) - (4 \times 1) = 2$$

Then multiply $R_1 \text{ and } R_2$,

$$\det \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} = 2 \times 4 - 4 \times 3 = -4$$

Then replace R_2 by $R_2 - 3R_1$,

$$\det \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = -2$$

We can also get the determinant of A, by getting the determinant of the transpose of A^T, and vice versa can be applied.

$$\det A^T = \det A \tag{1}$$

Here's another example of using determinants for solving the initial matrix in the beginning of these notes.

Example #1

$$\det\begin{bmatrix}2&1&0&2\\0&1&3&-1\\-2&0&6&-2\\4&2&3&7\end{bmatrix}\xrightarrow[R_4-R_2]{R_3-2R_2}\det\begin{bmatrix}2&1&0&2\\0&1&3&-1\\-2&-2&0&0\\4&1&0&8\end{bmatrix}$$

$$=3(-1)\det\begin{bmatrix}2&1&2\\-2&-2&0\\4&1&8\end{bmatrix}=3(-1)(-2)\det\begin{bmatrix}2&1&-2\\1&1&0\\4&1&8\end{bmatrix}$$

$$\longrightarrow 6\det\begin{bmatrix}2&1&2\\1&1&0\\4&1&8\end{bmatrix}\xrightarrow{R_3-4R_1}12\det\begin{bmatrix}2&1&1\\1&1&0\\-4&-3&8\end{bmatrix}$$

$$=12\times 1(+1)\det\begin{bmatrix}1&1\\-4&-3\end{bmatrix}=12$$

Example #2

$$\det\begin{bmatrix}2&1&1&2&1&1\\0&1&1&1&-2&1\\0&1&1&3&2&3\\4&1&3&1&5&8\\-2&2&1&3&5&6\\0&1&0&3&0&-1\end{bmatrix}\xrightarrow[R_5+R_1]{R_4-2R_1}\det\begin{bmatrix}2&1&1&2&1&1\\0&1&1&1&-2&1\\0&1&1&3&2&3\\0&-1&1&-3&3&6\\0&3&2&5&6&7\\0&1&0&3&0&-1\end{bmatrix}=$$

$$2\det\begin{bmatrix}1&1&1&-2&1\\1&1&3&2&3\\-1&1&-3&3&6\\3&2&5&6&7\\1&0&3&0&-1\end{bmatrix}\xrightarrow[R_4-2R_1]{R_2-R_1\text{ and }R_3-R_1}$$

Another important equation for using determinants is with matrix products such as below:

$$\det(AB)=\det A\times\det B$$

Here's an example of using determinants for matrix products:

Examples #3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}; \ AB = \begin{bmatrix} -4 & 7 \\ -8 & 15 \end{bmatrix}$$
$$\det A = -2, \ \det B = 2; \det AB = -4(15) - (-8)7 = -60 + 56 = 4$$

So if A is invertible, $AA^{-1} = I$, $I = \det I = \det(AA^{-1}) = \det A \times \det A^{-1}$.

$$\text{The determinant of the inverse of A matrix is also } \frac{1}{\det A}.$$

A **basis** for a space is a linearly independent spanning set of A.

$$H = \text{space}$$
$$\mathrm{B} = \{\vec{b}_1, \vec{b}_2, ..., \vec{b}_n\} \leftarrow \text{Basis for H and an ordered set}$$
$$\text{If } \vec{x} \in H, \text{ there exists } \beta_1, \beta_2, ..., \beta_n \text{ such that}$$
$$\vec{x} = \beta_1 b_1 + \beta_2 b_2 + ... + \beta_n b_n,$$
$$\text{and these } \beta\text{'s are } \underline{\text{unique}}.$$

The coordinates of $\vec{x}$ are relative to the B basis are:

$$[x]_{\mathrm{B}} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ ... \\ \beta_n \end{bmatrix}$$

Example #4

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= 2\vec{e}_1 - 3\vec{e}_2 \vee 2\hat{i} - 3\hat{j}$$