Lecture Notes (July 1st, 2024)

M 340L Matrices and Matrix Calculations
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Complex eigenvalues and eigenvectors

Example #1

$$A=\begin{bmatrix}0&-1\\1&0\end{bmatrix},\quad 0=\det(A-\lambda I)$$
$$0=(-\lambda)(-\lambda)-(-1)(1)$$
$$0=\lambda^2+1$$
$$\lambda=\pm i$$

$\lambda=i: A-iI$
$$=\begin{bmatrix}-i&-1\\1&-i\end{bmatrix}\begin{bmatrix}i\\1\end{bmatrix}\qquad \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}-b\\a\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$i\begin{bmatrix}1\\-i\end{bmatrix}=\begin{bmatrix}i\\1\end{bmatrix}\qquad \text{or } \begin{bmatrix}-d\\c\end{bmatrix}$$

$$\lambda=-i: A-(-i)I=\begin{bmatrix}i&-1\\1&i\end{bmatrix}\begin{bmatrix}-i\\1\end{bmatrix};\quad \lambda=\pm i,\ \vec{v}=\begin{bmatrix}\pm i\\1\end{bmatrix}$$

$$\boxed{C=\begin{bmatrix}a&-b\\b&a\end{bmatrix}}$$

$$0=\det(C-\lambda I)=(a-\lambda)(a-\lambda)$$
$$0=(a-\lambda)^2+b^2$$
$$0=(\lambda-a)^2+b^2$$
$$-b^2=(\lambda-a)^2$$

$$\pm ib=\lambda-a$$
$$a\pm ib=\lambda$$

$$\begin{bmatrix}a&-b\\b&a\end{bmatrix}=\underbrace{\begin{bmatrix}r&0\\0&r\end{bmatrix}}_{(r>0)}\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$$
counterclockwise rotation through an angle of θ

$a=r\cos\theta$
$b=r\sin\theta$

contraction $[0<r<1]$
dialation $[r>1]$

Example #2

$$A=\begin{bmatrix}1&-6\\3&-5\end{bmatrix}\quad 0=(1-\lambda)(-5-\lambda)-(-6)(3)$$
$$0=\lambda^2+5\lambda-\lambda-5+18$$
$$0=\lambda^2+4\lambda+13$$

$0=\lambda+4\lambda+4+9$
$-9=(\lambda+2)^2$
$\pm 3i=\lambda+2$
$\underline{-2\pm 3i=\lambda} \longleftrightarrow a\pm ib$

$a=-2$
$b=3$

$$A=PCP^{-1}=\begin{bmatrix}?&?\end{bmatrix}\begin{bmatrix}a&-b\\b&a\end{bmatrix}P^{-1}$$

$\lambda=a-ib$
$\vec{v}=\operatorname{Re}\vec{v}+i\operatorname{Im}\vec{v}$ — real part $\vec{v}$, imaginary part of $\vec{v}$
$P=\begin{bmatrix}\operatorname{Re}\vec{v}&\operatorname{Im}\vec{v}\end{bmatrix}$

$$A=\begin{bmatrix}1&-6\\3&-5\end{bmatrix};\ \lambda=-2-3i$$
$$A-(-2-3i)I=\begin{bmatrix}1-(-2-3i)&-6\\3&-5-(-2-3i)\end{bmatrix}$$
$$=\begin{bmatrix}3+3i&-6\\3&-3+3i\end{bmatrix}\begin{bmatrix}6\\3+3i\end{bmatrix},\begin{bmatrix}3-3i\\3\end{bmatrix}$$
$$\begin{bmatrix}2\\1+i\end{bmatrix}\qquad\begin{bmatrix}-1-i\\1\end{bmatrix}$$
$$\begin{bmatrix}1-i+i-i^2\\1+i\end{bmatrix}=\begin{bmatrix}(1-i)(1+i)\\1+i\end{bmatrix}$$
$$\underbrace{\begin{bmatrix}2\\1\end{bmatrix}}_{\operatorname{Re}\vec{v}}+i\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\operatorname{Im}\vec{v}}=\begin{bmatrix}2+0i\\1+1i\end{bmatrix}\longleftarrow\begin{bmatrix}2\\1+i\end{bmatrix}$$

$$\begin{bmatrix}a&-b\\b&a\end{bmatrix}P^{-1}$$
$$\begin{bmatrix}1&-6\\3&-5\end{bmatrix}=PCP^{-1}=\begin{bmatrix}\operatorname{Re}\vec{v}&\operatorname{Im}\vec{v}\end{bmatrix}$$
$$=\begin{bmatrix}2&0\\1&-1\end{bmatrix}\begin{bmatrix}-2&-3\\3&-2\end{bmatrix}$$
$$=\begin{bmatrix}2&0\\1&-1\end{bmatrix}\begin{bmatrix}-2&-3\\3&-2\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&0\\-1&2\end{bmatrix}$$
$$=\begin{bmatrix}-4&-6\\1&-5\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&0\\-1&2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2&-12\\6&-10\end{bmatrix}=\begin{bmatrix}1&-6\\3&5\end{bmatrix}=\begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}-2&-3\\3&-2\end{bmatrix}P^{-1}$$

Example #3

$$A=\begin{bmatrix}4&3\\-4&-2\end{bmatrix}\quad 0=(4-\lambda)(-2-\lambda)-(-4)(3)$$
$$=\lambda^2+2\lambda-4\lambda-8+12$$
$$0=\lambda^2-2\lambda+4$$

$$\lambda=\frac{2\pm\sqrt{(-2)^2-4\cdot1\cdot4}}{2}=\frac{2\pm\sqrt{4-16}}{2}\quad\frac{2\pm\sqrt{-12}}{2}\quad\frac{2\pm2\sqrt{3}\,i}{2}=1\pm\sqrt{3}i$$

$$\overset{a-bi}{\boxed{\lambda=1-\sqrt{3}i}}: A-(1-\sqrt{3}i)I=\begin{bmatrix}4-(1-\sqrt{3}i)&3\\-4&-2-(1-\sqrt{3}i)\end{bmatrix}$$
$$=\begin{bmatrix}3+\sqrt{3}i&3\\-4&-3+\sqrt{3}i\end{bmatrix}$$

$$A=\begin{bmatrix}-3&0\\3&\sqrt{3}\end{bmatrix}\begin{bmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{bmatrix}P^{-1}$$

Important!
We have to find P and C as we are able to find P^{-1}

$$=\begin{bmatrix}-3&0\\3&\sqrt{3}\end{bmatrix}\begin{bmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{bmatrix}\left(\frac{1}{-3\sqrt{3}}\right)\begin{bmatrix}\sqrt{3}&0\\-3&-3\end{bmatrix}=\frac{1}{3\sqrt{3}}\begin{bmatrix}-3&0\\3&\sqrt{3}\end{bmatrix}\begin{bmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{bmatrix}\begin{bmatrix}-\sqrt{3}&0\\3&3\end{bmatrix}$$
$$=\frac{1}{3\sqrt{3}}\begin{bmatrix}-3&3\sqrt{3}\\6&-2\sqrt{3}\end{bmatrix}\begin{bmatrix}-3\sqrt{3}&0\\3&3\end{bmatrix}=\frac{1}{3\sqrt{3}}\begin{bmatrix}12\sqrt{3}&9\sqrt{3}\\-12\sqrt{3}&-6\sqrt{3}\end{bmatrix}=\begin{bmatrix}4&3\\-4&-2\end{bmatrix}$$