Lecture Notes (July 2, 2024)

M 340L Matrices and Matrix Calculations
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Discrete Dynamical Systems

$\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \dots$

$\begin{bmatrix} R_{k+1} \\ W_{k+1} \end{bmatrix} = \begin{bmatrix} 1.2 & -0.2 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} R_k \\ W_k \end{bmatrix}$

$\vec{x}_{k+1} = A\vec{x}_k$, $\vec{x}_0$ is known

$\vec{x}_1 = A\vec{x}_0$

$\vec{x}_2 = A\vec{x}_1 = A^2\vec{x}_0$

$\vec{x}_3 = A\vec{x}_2 = A^3\vec{x}_0 \dots \vec{x}_k = A^k\vec{x}_0$

Pretend A is diagonalizable matrix, with eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ and corresponding eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.

Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$,

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

$$\vec{x}_1 = A\vec{x}_0 = A(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)$$

$$= c_1A\vec{v}_1 + c_2A\vec{v}_2 + \dots + c_nA\vec{v}_n$$

$$= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \dots + c_n\lambda_n\vec{v}_n$$

$$\vec{x}_2 = A\vec{x}_1 = c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2 + \dots + c_n\lambda_n^2\vec{v}_n$$

$$\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \dots + c_n\lambda_n^k\vec{v}_n$$

Example #1

$\vec{x}_{k+1} = \begin{bmatrix} -0.8 & 2.4 \\ -0.2 & -2.2 \end{bmatrix}\vec{x}_k$, $\vec{x}_0 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

$\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$ (The discrete system for this example)

$0 = \det(A - \lambda I) = (-0.8-\lambda)(-2.2-\lambda) - (0.2)(2.4)$

$= \lambda^2 + 2.2\lambda + 0.8\lambda + 1.76 + .48$

$0 = \lambda^2 + 3\lambda + 2.24 \longrightarrow \lambda = \dfrac{-3 \pm \sqrt{9 - 4(2.24)}}{2} = \dfrac{-3 \pm \sqrt{9 - 8.96}}{2}$

$= \dfrac{-3 \pm \sqrt{0.04}}{2} = \dfrac{-3 \pm 0.2}{2} = -1.6, -1.4$

$\lambda = -1.6: A - (-1.6)I = \begin{bmatrix} -0.8+1.6 & 2.4 \\ -0.2 & -2.2+1.6 \end{bmatrix} = \begin{bmatrix} 0.8 & 2.4 \\ -0.2 & -0.6 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2.4 \\ 0.8 \end{bmatrix} \xrightarrow{100} \dots \to \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda = -1.4: A - (-1.4)I = \begin{bmatrix} -0.8+1.4 & 2.4 \\ -0.2 & -2.2+1.4 \end{bmatrix} = \begin{bmatrix} 0.6 & 2.4 \\ -0.2 & -0.8 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

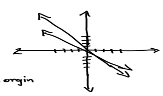
$\left[\lambda_1 = -1.6, \vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}; \lambda_2 = -1.4, \vec{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right]$

$\vec{x}_k = c_1(-1.6)^k\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2(-1.4)^k\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\vec{x}_0 = \begin{bmatrix} 6 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $c\,\{2, -3\}$

$\vec{x}_k = 2(-1.6)^k\begin{bmatrix} -3 \\ 1 \end{bmatrix} - 3(-1.4)^k\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\hookrightarrow$ as $k \to \infty$, it gets larger, as the orgin is a repeller.



Example #2

$\vec{x}_{k+1} = \begin{bmatrix} 1 & -1.4 \\ 0.7 & -1.1 \end{bmatrix}\vec{x}_k$, $\vec{x}_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\lambda = -0.4: A - (-0.4)I = \begin{bmatrix} 1.4 & -1.4 \\ 0.7 & -0.7 \end{bmatrix}$

$\begin{bmatrix} 1.4 & -1.4 \\ 0.7 & -0.7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$

$\lambda = 0.3: A - 0.3I = \begin{bmatrix} 0.7 & -1.4 \\ 0.7 & -1.4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$0 = (1-\lambda)(-1.1-\lambda) - (-1.4)(0.7)$

$= \lambda^2 - \lambda + 1.1\lambda - 1.1 + 0.98$

$0 = \lambda^2 + 0.1\lambda - 0.12$

$0 = (\lambda + 0.4)(\lambda - 0.3)$

$\lambda = -0.4, 0.3$

$\vec{x}_k = c_1(-0.4)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(0.3)^k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $c\,\{5, -1\}$

$\vec{x}_k = 5(-0.4)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} - (0.3)^k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Both λ's < 1 in absolute value, so as $k \to \infty$; both $\longrightarrow 0$.
The origin is an attractor!

Example #3

$\vec{x}_{k+1} = \begin{bmatrix} 0.2 & -0.3 \\ 1.8 & 1.7 \end{bmatrix}\vec{x}_k$, $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0.8: A - 0.8I = \begin{bmatrix} -0.6 & -0.3 \\ 1.8 & 0.9 \end{bmatrix}$

$\begin{bmatrix} -0.6 & -0.3 \\ 1.8 & 0.9 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \vec{0}$

$\lambda = 1.1: A - 1.1I = \begin{bmatrix} -0.9 & -0.3 \\ 1.8 & 0.6 \end{bmatrix}$

$\begin{bmatrix} -0.9 & -0.3 \\ 1.8 & 0.6 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \vec{0}$

$0 = (0.2-\lambda)(1.7-\lambda) - (-0.3)(1.8)$

$= \lambda^2 - 0.2\lambda - 1.7\lambda + 0.34 + 0.54$

$0 = \lambda^2 - 1.9\lambda + 0.88$

$0 = (\lambda - 0.8)(\lambda - 1.1)$

$\lambda = 0.8, 1.1$

$\vec{x}_k = c_1(0.8)^k\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2(1.1)^k\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $c\,\{4, 3\}$

$\vec{x}_k = \underbrace{4(0.8)^k}_{\text{Approaches zero}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \underbrace{3(1.1)^k}_{\text{Approches } \infty}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

λ_1 approaches zero, as $k \to \infty$; and λ_2 approaches ∞, as $k \to \infty$. Thus, the origin is a saddle point

Theorem:

$\vec{x}_{k+1} = A\vec{x}_k$, $\vec{x}_0$ is known

$\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$

$\{|\lambda_1|, |\lambda_2|\} > 1 \longrightarrow$ the origin is a repeller

$\{|\lambda_1|, |\lambda_2|\} < 1 \longrightarrow$ the origin is an attractor

$[|\lambda_1| < 1, |\lambda_2| > 1] \longrightarrow$ the origin is a saddle point

Differential Systems

$\dfrac{d\vec{x}}{dt}(t) = A\vec{x}(t)$, $\vec{x}(0) = \vec{x}_0 \equiv \vec{x}'(t) = A\vec{x}(t) \begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{cases}$

Example #1 (Basic Differential Eqnts)

$y' = \alpha y$, $y^{(0)} = y_0$

$y(t) = y_0e^{\alpha t}$

A is diagonalizable with eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ and eigenvalues $\{\lambda_1, \dots, \lambda_n\}$

Consider $y(t) = e^{\lambda_i t}\vec{v}_i$

$y'(t) = \lambda_i e^{\lambda_i t}\vec{v}_i \equiv e^{\lambda_i t}\lambda_i\vec{v}_i$

$= e^{\lambda_i t}A\vec{v}_i = A(e^{\lambda_i t}\vec{v}_i)$

$\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + \dots + c_ne^{\lambda_n t}\vec{v}_n$ where $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

Example #2

$\vec{x}'(t) = \begin{bmatrix} -0.8 & 2.4 \\ -0.2 & -2.2 \end{bmatrix}\vec{x}(t)$, $\vec{x}(0) = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

$\lambda_1 = -1.6$, $\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$; $\lambda_2 = -1.4$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Sol: $\vec{x}(t) = c_1e^{-1.6t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2e^{-1.4t}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\vec{x}(0) = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $c\,\{-3, 2\}$

As $e^{\lambda_i t}$ approaches zero as $t \to \infty$; the origin is an attractor.

$\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2$

$\text{Re}\,\lambda_1, \text{Re}\,\lambda_2 < 0 \longrightarrow$ It is an attractor (to the origin)

$\text{Re}\,\lambda_1, \text{Re}\,\lambda_2 > 0 \longrightarrow$ It is a repeller (to the origin)

$\text{Re}\,\lambda_1 > 0, \text{Re}\,\lambda_2 < 0 \longrightarrow$ It is a saddle point (the origin)

There's a graph for reference in Freeform