This print-out should have 7 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 10 & 3 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. $A^{-1} = \begin{bmatrix} 13 & -3 & -3 \\ -4 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

2. $A^{-1} = \begin{bmatrix} 3 & -3 & -3 \\ -4 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

3. $A^{-1} = \begin{bmatrix} 3 & -4 & -1 \\ -3 & 1 & 0 \\ -3 & 1 & -1 \end{bmatrix}$

4. $A^{-1} = \begin{bmatrix} 13 & -4 & -1 \\ -3 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$ **correct**

Explanation:

The inverse matrix A^{-1} can be computed by reducing the augmented matrix

$$[\,A \quad I_3\,] = \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 3 & 10 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

to row-reduced echelon form $[\,I_3 \quad A^{-1}\,]$.

Now after row reduction downwards we see that

$$[\,A \quad I_3\,] \sim \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{bmatrix}.$$

But then after row reduction upwards,

$$[\,A \quad I_3\,] \sim \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 10 & -3 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 13 & -4 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently,

$$A^{-1} = \begin{bmatrix} 13 & -4 & -1 \\ -3 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}.$$

002 10.0 points

Determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & -1 \\ -2 & -2 & -3 \end{bmatrix}.$$

1. $A^{-1} = \begin{bmatrix} 7 & -4 & -2 \\ 8 & -5 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

2. $A^{-1} = \begin{bmatrix} 3 & 8 & -5 \\ -4 & 1 & 3 \\ -2 & -2 & -1 \end{bmatrix}$

3. $A^{-1} = \begin{bmatrix} 7 & 8 & -5 \\ -4 & -5 & 3 \\ -2 & -2 & 1 \end{bmatrix}$ **correct**

4. $A^{-1} = \begin{bmatrix} 3 & -4 & -2 \\ 8 & 1 & -2 \\ -5 & 3 & -1 \end{bmatrix}$

Explanation:

The inverse matrix A^{-1} can be computed by reducing the augmented matrix

$$[\,A \quad I_3\,] = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & -3 & -1 & 0 & 1 & 0 \\ -2 & -2 & -3 & 0 & 0 & 1 \end{bmatrix}$$

to row-reduced echelon form $[\,I_3 \quad A^{-1}\,]$.

Now after row reduction downwards we see that

$$[\,A \quad I_3\,] \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & 0 \\ 0 & 2 & -5 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix}.$$

But then after row reduction upwards,

$$[\,A \quad I_3\,] \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & -5 & 3 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 9 & 10 & -6 \\ 0 & 1 & 0 & -4 & -5 & 3 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 7 & 8 & -5 \\ 0 & 1 & 0 & -4 & -5 & 3 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{bmatrix}.$$

Consequently,

$$\boxed{A^{-1} = \begin{bmatrix} 7 & 8 & -5 \\ -4 & -5 & 3 \\ -2 & -2 & 1 \end{bmatrix}.}$$

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003 10.0 points

Solve for X when $AX + B = C$,

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 2 & 2 \\ 1 & 5 \end{bmatrix}.$$

1. $X = \begin{bmatrix} -6 & -2 \\ 10 & 4 \end{bmatrix}$ **correct**

2. $X = \begin{bmatrix} -2 & -2 \\ 10 & -2 \end{bmatrix}$

3. $X = \begin{bmatrix} -6 & -2 \\ -5 & -4 \end{bmatrix}$

4. $X = \begin{bmatrix} -2 & -2 \\ -5 & 4 \end{bmatrix}$

5. $X = \begin{bmatrix} -6 & -2 \\ 2 & -2 \end{bmatrix}$

Explanation:

By the algebra of matrices,

$$X = A^{-1}(C - B).$$

But the inverse of any 2×2 matrix

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

is given by

$$D^{-1} = \begin{bmatrix} \dfrac{d_{22}}{\Delta} & -\dfrac{d_{12}}{\Delta} \\ -\dfrac{d_{21}}{\Delta} & \dfrac{d_{11}}{\Delta} \end{bmatrix}$$

with $\Delta = d_{11}d_{22} - d_{12}d_{21}$.

Thus

$$X = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\left(\begin{bmatrix} 2 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 2 & 2 \end{bmatrix}.$$

Consequently,

$$\boxed{X = \begin{bmatrix} -6 & -2 \\ 10 & 4 \end{bmatrix}}.$$

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004 10.0 points

Determine the unique solution x_1 of the matrix equation

$$A\mathbf{x} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \\ -9 \end{bmatrix}$$

when A has an LU-decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

1. $x_1 = -4$

2. $x_1 = -6$

3. $x_1 = -5$

4. $x_1 = -2$ **correct**

5. $x_1 = -3$

Explanation:

Set $\mathbf{y} = U\mathbf{x}$. Then $A\mathbf{x} = L\mathbf{y} = \mathbf{b}$, and so $\mathbf{y} = L^{-1}\mathbf{b}$. Now

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix},$$

in which case $A\mathbf{x} = \mathbf{b}$ reduces to

$$U\mathbf{x} = L^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ -11 \\ -9 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \\ 2 \end{bmatrix}.$$

But then,

$$U\mathbf{x} = \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \\ 2 \end{bmatrix}.$$

which is equivalent to the system

$$2x_3 = 2, \qquad 4x_2 - x_3 = 11,$$

and

$$-3x_1 + x_2 + 2x_3 = 11.$$

So by back substitution, $x_3 = 1,\ x_2 = 3$ and

$$\boxed{x_1 = -2}.$$

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005 10.0 points

Find U in an LU decomposition of

$$A = \begin{bmatrix} -3 & -3 & 2 & -3 \\ -15 & -15 & 14 & -17 \\ 15 & 15 & 6 & 6 \end{bmatrix}.$$

1. $U = \begin{bmatrix} -3 & 1 & 4 & -2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

2. $U = \begin{bmatrix} -3 & -3 & 2 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ **correct**

3. $U = \begin{bmatrix} 1 & 1 & 4 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. $U = \begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. $U = \begin{bmatrix} 1 & 3 & -2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. $U = \begin{bmatrix} -3 & 3 & -2 & 3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Explanation:

Recall that in a factorization $A = LU$ of an $m \times n$ matrix A, then L is an $m \times m$ lower triangular matrix with ones on the diagonal and U is an $m \times n$ echelon form of A.

We begin by computing U. Now $U = M_0 A$ where j is the number of row operations on A needed to transform A into its echelon form U and M_i is a product of $j - i$ elementary matrices that represent these row operations.

$$\begin{aligned} U &= M_0A = M_1E_1A \\ &= M_1 \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 & 2 & -3 \\ -15 & -15 & 14 & -17 \\ 15 & 15 & 6 & 6 \end{bmatrix} \\ &= M_2E_2(E_1A) \\ &= M_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 & 2 & -3 \\ 0 & 0 & 4 & -2 \\ 15 & 15 & 6 & 6 \end{bmatrix} \\ &= E_3(E_2E_1A) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 & 2 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 16 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -3 & 2 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

Next recall that all elementary matrices are invertible, as is the product of elementary matrices. Thus we can change $U = M_0 A$ to $M_0^{-1} U = A$. This shows that $M_0^{-1} = L$. Hence

$$\begin{aligned} L = M_0^{-1} &= E_1^{-1} E_2^{-1} E_3^{-1} \\ &= E_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -5 & 4 & 1 \end{bmatrix} \end{aligned}$$

Consequently,

$$\boxed{U = \begin{bmatrix} -3 & -3 & 2 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}}.$$

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006 10.0 points

Determine the Unit Lower Triangular matrix L in the unique LU-decomposition of the matrix

$$A = \begin{bmatrix} -2 & 1 & -5 \\ 8 & 0 & 17 \\ -8 & -8 & -6 \end{bmatrix}.$$

1. $L = \begin{bmatrix} 1 & -4 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

2. $L = \begin{bmatrix} -2 & 1 & -5 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$

3. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -5 & -3 & 1 \end{bmatrix}$

4. $L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$ **correct**

5. $L = \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & 17 \\ 0 & 0 & 1 \end{bmatrix}$

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}$

Explanation:

The matrix L can be computed from the elementary matrices reducing A to echelon form.

Set

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{aligned} E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & -5 \\ 8 & 0 & 17 \\ -8 & -8 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & -5 \\ 0 & 4 & -3 \\ -8 & -8 & -6 \end{bmatrix} = A_1, \end{aligned}$$

say. Next set

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{aligned} E_2 A_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & -5 \\ 0 & 4 & -3 \\ -8 & -8 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & -5 \\ 0 & 4 & -3 \\ 0 & -12 & 14 \end{bmatrix} = A_2, \end{aligned}$$

say. Finally, set

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

so that the product

$$\begin{aligned} E_3 A_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & -5 \\ 0 & 4 & -3 \\ 0 & -12 & 14 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & -5 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} = U, \end{aligned}$$

and

$$E_3E_2E_1A = E_3E_2A_1 = E_3A_2 = U$$

is an echelon form of A. But then $A = LU$, setting

$$L = E_1^{-1}E_2^{-1}E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}.$$

Since L is lower triangular with 1's on the diagonal , this provides the LU-decomposition of A. Consequently,

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}}.$$

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007 10.0 points

Determine the lower triangular matrix L in an LU-decomposition of

$$A = \begin{bmatrix} 2 & 1 & -4 \\ 4 & -3 & -4 \\ -8 & -9 & 18 \end{bmatrix}.$$

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$

2. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -8 & -5 & 1 \end{bmatrix}$

3. $L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -5 & 0 \\ -4 & 1 & -2 \end{bmatrix}$

4. $L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -5 & 0 \\ 4 & -1 & -2 \end{bmatrix}$

5. $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}$ **correct**

6. $L = \begin{bmatrix} 2 & 0 & 0 \\ -4 & -5 & 0 \\ 8 & 5 & -2 \end{bmatrix}$

Explanation:

We first determine the elementary matrices reducing A to an echelon form U by row reductions *downwards*.

$$A \sim E_1A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -4 \\ 4 & -3 & -4 \\ -8 & -9 & 18 \end{bmatrix}$$

$$= E_2(E_1A)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -4 \\ 0 & -5 & 4 \\ -8 & -9 & 18 \end{bmatrix}$$

$$= E_3(E_2E_1A)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -4 \\ 0 & -5 & 4 \\ 0 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -4 \\ 0 & -5 & 4 \\ 0 & 0 & -2 \end{bmatrix} = U.$$

But an elementary matrix is always invertible. Thus

$$L = E_1^{-1}E_2^{-1}E_3^{-1}$$

$$= E_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}.$$

Consequently, $A = LU$ with

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}}.$$