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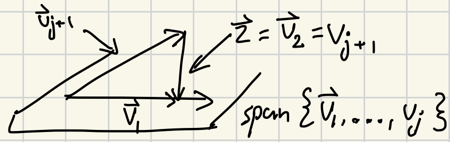
M 340L Matrices and Matrix Calculations

What if we have a basis which is not orthogonal but we need an orthogonal basis?

The Gram-Schmidt Orthogonalization Process

If $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ is a linearly independent set, then $\{\vec{v}_1, \vec{v}_2, \dots, v_k\}$ defined as:



$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$\vdots$$

$$\vec{v}_k = \vec{u}_k - \frac{\vec{u}_k \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}v_1 - \dots - \frac{\vec{u} \cdot \vec{v}_{k-1}}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}}\vec{v}_{k-1}$$

Satisfy span $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ = span $\{\vec{v}_1, \dots, \vec{v}_k\}$ and $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal set.

Example #2

For the vectors $S = \left\{ \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}, \begin{bmatrix}-4\\-7\\-8\\4\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\-7\\7\\0\end{bmatrix} \right\}$, write the vector $\vec{y} = \begin{bmatrix}9\\-4\\-1\\7\\-2\end{bmatrix}$ as a sum of a vector in span S and a vector in $S^{\perp}$. Do Grand-Schmidt to S.

$$\vec{v}_1 = \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}v_1 = \begin{bmatrix}-4\\-7\\8\\9\\1\end{bmatrix} - \frac{\begin{bmatrix}-4\\-7\\8\\9\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}}{\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}} \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} = \begin{bmatrix}-4\\-7\\-8\\4\\1\end{bmatrix} - \frac{\overset{28}{(-4+14+8+8+2)}}{\underset{14}{1+4+1+4+4}} \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}$$

$$= \begin{bmatrix}-4\\-7\\-8\\4\\1\end{bmatrix} - 2\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} = \begin{bmatrix}-6\\-3\\-6\\0\\-3\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v} \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{u}_2 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix}3\\-2\\-7\\7\\0\end{bmatrix} - \frac{\begin{bmatrix}3\\-2\\-7\\7\\0\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}}{\underbrace{\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}}_{14}} \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} - \frac{\begin{bmatrix}3\\-2\\-7\\7\\0\end{bmatrix} \cdot \begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix}}{\begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix} \cdot \begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix}} \begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}3\\-2\\-7\\7\\0\end{bmatrix} - \frac{3+4+7+14+0}{14}\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} - \frac{6-2-14+0+0}{4+1+4+0+1}\begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}3\\-2\\-7\\7\\0\end{bmatrix} - 2\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} + 1\begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix} = \begin{bmatrix}3\\3\\-3\\3\\-3\end{bmatrix} \xrightarrow{1/3} \begin{bmatrix}1\\1\\-1\\1\\-1\end{bmatrix}$$

After Grand Schmidt: $\left\{ \begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\1\\-1\end{bmatrix} \right\}$, $\vec{y} = \begin{bmatrix}9\\-4\\-1\\7\\-2\end{bmatrix}$

$$\hat{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot v_1}\vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{y} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3$$

$$\hat{y} = \begin{bmatrix}7\\0\\-3\\7\\2\end{bmatrix} = \frac{\overset{28}{9+8+1+14-4}}{14}\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} + \frac{\overset{10}{18-4-2+0-2}}{10}\begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix} + \frac{\overset{15}{9-4+1+7+2}}{5}\begin{bmatrix}1\\1\\-1\\1\\-1\end{bmatrix} = 2\begin{bmatrix}1\\-2\\-1\\2\\2\end{bmatrix} + 1\begin{bmatrix}2\\1\\2\\0\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\-1\\1\\-1\end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}9\\-4\\-1\\7\\-2\end{bmatrix} - \begin{bmatrix}7\\0\\-3\\7\\2\end{bmatrix} = \begin{bmatrix}2\\-4\\2\\0\\-4\end{bmatrix} = \vec{z}$$

Orthogonal to each other

Example #1

$\left\{ \underset{\vec{u}_1}{\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix}7\\-1\\0\\1\\5\end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix}} \right\}$ Not an orthogonal set!

Replace this with an orthogonal set

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} \qquad \vec{v}_2 = \vec{u}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \begin{bmatrix}7\\-1\\0\\1\\5\end{bmatrix} - \frac{\begin{bmatrix}7\\-1\\0\\1\\5\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}} \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}7\\-1\\0\\1\\5\end{bmatrix} - \frac{12}{4}\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}7\\-1\\0\\1\\5\end{bmatrix} - 3\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\0\\4\\2\end{bmatrix}$$

$$\begin{bmatrix}4\\2\\0\\4\\2\end{bmatrix} \xrightarrow{1/2} \begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix} =$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \frac{\vec{u}_3 \cdot (\cancel{2}\vec{w})}{(\cancel{2}\vec{w}) \cdot (\cancel{2}\vec{w})}(\cancel{2}\vec{w}) \to \frac{\vec{u}_3 \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \cdot \vec{w}$$

$$= \begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} - \frac{\begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}} \begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} - \boxed{\frac{\begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} \cdot \begin{bmatrix}4\\2\\0\\4\\2\end{bmatrix}}{\begin{bmatrix}4\\2\\0\\4\\2\end{bmatrix} \cdot \begin{bmatrix}4\\2\\0\\4\\2\end{bmatrix}}} \begin{bmatrix}4\\2\\0\\4\\2\end{bmatrix}$$

$$= \begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} - \frac{(-8)}{4}\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} - \frac{\begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} \cdot \begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix}}{\begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix} \cdot \begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix}} \begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix}$$

Don't forget that $\vec{v}_2$ was reduce by $1/2$

$$= \begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} + 2\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} - \frac{30}{10}\begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}2\\6\\1\\8\\4\end{bmatrix} + 2\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix} - 3\begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}-2\\1\\1\\0\\3\end{bmatrix} = \vec{v}_3$$

$$= \begin{bmatrix}-2\\1\\1\\0\\3\end{bmatrix} \qquad \left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\-1\\0\\-1\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}-2\\1\\1\\0\\3\end{bmatrix}} \right\} = \text{our orthogonal set}$$

Check:
$\vec{v}_1 \cdot \vec{v}_2 = 2-1-2+1 = 0$
$\vec{v}_2 \cdot \vec{v}_3 = -2-1+0+0+3 = 0$
$\vec{v}_1 \cdot \vec{v}_3 = -2-1+0+0+3 = 0$
Check for orthogonality

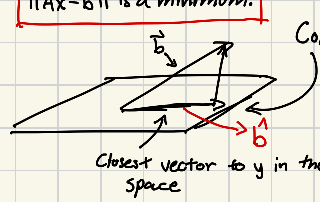
Least Squares Approximations

$A\vec{x} = b$ [an inconsistent system]

What is the "closest" $\vec{x}$ we can get to a solution?

We want to find $\hat{x}$ such that

$\boxed{\|A\hat{x} - \vec{b}\| \text{ is a minimum.}}$



$\hat{b} = \text{proj}_{\text{Col}A}\vec{b}$, so then $\hat{x}$ satisfies $A\hat{x} = \hat{b}$.

$z = \hat{b} - A\hat{x}$ (is in the orthogonal complement in Col A)

$$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$$

$$\vec{a}_i \cdot (\vec{b} - A\hat{x}) = 0 \ \forall i$$

$$a_i^T(\vec{b} - A\hat{x}) = 0$$

$$\left.\begin{array}{l} a_1^T(\vec{b} - A\hat{x}) = 0 \\ a_2^T(\vec{b} - A\hat{x}) = 0 \\ \vdots \\ \vec{a}_n(\vec{b} - A\hat{x}) = 0 \end{array}\right\} \begin{bmatrix}\vec{a}_1^T\\ \vec{a}_2^T\\ \vdots\\ \vec{a}_n^T\end{bmatrix}(\vec{b} - A\hat{x}) = \vec{0}$$

Or $A^T(\vec{b} - A\hat{x}) = \vec{0}$

$$\boxed{A^TA\hat{x} = A^T\vec{b}}$$

the normal equations