Abdon Morales
July 10th, 2024

M 340L Matrices and Matrix Calculations

Symmetric Matrices

$A^T = A \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

All real symmetric matrices have the following properties:

1. All eigenvalues are real
2. Eigenvectors for different eigenvalues are orthogonal
3. The matrix is diagonalizable

→ If A is symmetric, it is orthogonally diagonalizable,

$$A = PDP^T$$

where D is real & diagonal P is an orthogonal matrix

Proof of Prop. #2: $A^T = A$

Let $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ be the eigenvalue, eigenvector pairs where $\lambda_1 \neq \lambda_2$.

Consider. $\vec{v}_1 \cdot (A\vec{v}_2) = \vec{v}_1 \cdot (\lambda_2 \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$

$= \vec{v}_1^T (A\vec{v}_2) = (\vec{v}_1^T A)\vec{v}_2$

$= (A^T \vec{v}_1)^T \vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2$

$= (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1 \vec{v}_1 \cdot \vec{v}_2$

so $\lambda_1 \vec{v}_1 \cdot \vec{v}_2 = \lambda_2 \vec{v}_1 \cdot \vec{v}_2$ or $(\lambda_1 - \lambda_2)\vec{v}_1 \cdot \vec{v}_2 = 0$

Since $\lambda_1 \neq \lambda_2$, $\vec{v}_1 \cdot \vec{v}_2 = 0$

Example #1

$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ Find eigenvalues and eigenvectors

$0 = (2-\lambda)(-3-\lambda) - 1$

$= \lambda^2 + 3\lambda - 2\lambda - 6 - 1$

$0 = \lambda^2 + \lambda - 7$

$\lambda = \dfrac{-1 \pm \sqrt{1^2 - 4 \cdot 1(-7)}}{2} = \dfrac{-1 \pm \sqrt{29}}{2}$

$\lambda = \dfrac{-1+\sqrt{29}}{2}$, $A - \left(\dfrac{-1+\sqrt{29}}{2}\right)I = \begin{bmatrix} 2 - \left(\frac{-1+\sqrt{29}}{2}\right) & 1 \\ 1 & -3 - \left(\frac{-1+\sqrt{29}}{2}\right) \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2} - \frac{\sqrt{29}}{2} & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{29}}{2} \end{bmatrix} \begin{bmatrix} 1+\sqrt{29} \\ 2 \end{bmatrix} = \vec{v}_1$

$\lambda = \dfrac{-1-\sqrt{29}}{2}$; $A - \left(\dfrac{-1-\sqrt{29}}{2}\right)I = \begin{bmatrix} 2 - \left(\frac{-1-\sqrt{29}}{2}\right) & 1 \\ 1 & -3 - \left(\frac{-1-\sqrt{29}}{2}\right) \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2} + \frac{\sqrt{29}}{2} & 1 \\ 1 & -\frac{5}{2} + \frac{\sqrt{29}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 5+\sqrt{29} \end{bmatrix} = \vec{v}$

$\vec{v}_1 \cdot \vec{v}_2 = -2(5+\sqrt{29}) + 2(5+\sqrt{29}) = 0$

Example #2

$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix}$ Orthogonally diagonalize this matrix.

$\lambda = \{\boxed{0}, 4, -6\}$

→ Matrix is singular if $\lambda = 0$!

$\lambda = 0: A - 0I = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} -4 & 4 & 0 \\ -3 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -4 & 4 & 0 \\ -3 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ normalize

$\lambda = 4: A - 4I = \begin{bmatrix} -3 & -3 & 2 \\ -3 & -3 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$\lambda = 6: A - 6I = \begin{bmatrix} 7 & -3 & 2 \\ -3 & 7 & 2 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{R_1 - R_3} \begin{bmatrix} 5 & -5 & 0 \\ -3 & 7 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$

$A = PDP^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{bmatrix} P^T$

Example #3

$A = \begin{bmatrix} 3 & 4 & 2 \\ 4 & 9 & 4 \\ 2 & 4 & 3 \end{bmatrix}$ Orthogonally diagonalize this matrix.

$\lambda = 13, 1$

$\lambda = 13, A - 13I = \begin{bmatrix} -10 & -4 & 2 \\ 4 & -4 & 4 \\ 2 & 4 & 10 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} -6 & 0 & 6 \\ 4 & -4 & 4 \\ 2 & 4 & -10 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$\lambda_1 = 1: A - 1I = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_2$ $\qquad \vec{u}_1 = \vec{v}_1$

$\vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\vec{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{norm.}} \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix} \xrightarrow{\text{norm.}} \begin{bmatrix} -1/\sqrt{30} \\ -2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}$ $\quad \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix}$

$A = PDP^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{5} & -1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 13 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^T$

The Spectral Decomposition

If A is a symmetric matrix with eigenvalues $\{\lambda_1, \lambda_2, \dots \lambda_n\}$ with corresponding eigenvectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ then:

$$A = \lambda_1 \boxed{\vec{u}_1 \vec{u}_1^T} + \lambda_2 \vec{u}_2 \vec{u}_2^T + \dots + \lambda_n \vec{u}_n \vec{u}_n^T$$

→ Projection matrix

Example #1

$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix}$; $\lambda_1 = 0, \vec{v}_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$; $\lambda_2 = 4, \vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$;

$\lambda_3 = -6, \vec{v}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$

$A = \cancel{\lambda_1 \vec{v}_1 \vec{v}_1^T} \rightarrow 0 + \lambda_2 \vec{v}_2 \vec{v}_2^T + \lambda_3 \vec{v}_3 \vec{v}_3^T$

$= 0 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} + 4 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} - 6 \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$

$= 4 \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 6 \begin{bmatrix} 1/6 & 1/6 & -2/6 \\ 1/6 & 1/6 & -2/6 \\ -2/6 & -2/6 & 4/6 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & -4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & 2 \\ 2 & 2 & -4 \end{bmatrix}$