Eigenvectors and Eigenvalues: Lecture 13

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Class	M 340L
AI summary	In this lecture on eigenvectors and eigenvalues, we explore the concepts through examples. We start with a simple example involving basis vectors and determine the coordinates of a vector in different bases. Then, we move on to defining eigenvectors and eigenvalues, showcasing examples and finding eigenvectors for a given matrix. The characteristic polynomial and characteristic equation are introduced as well.
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Here, we continue from the content of yesterday's lecture. Let's start with a simple example...

Example #1

$$\underbrace{\mathrm{B}}_{\text{Each is a basis for } \Re^2} = \{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}\}, C = \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$$

$[\vec{x}]_C = \begin{bmatrix}3\\1\end{bmatrix}$, what are $[\vec{x}]_{\mathrm{B}}$?

$$[\vec{x}]_{\mathrm{B}} = P_{\mathrm{B}\leftarrow C}[\vec{x}]_C;\ [P_{\mathrm{B}}|P_C] \xrightarrow{RREF} [I|P_{\mathrm{B}\leftarrow C}](1)$$

$$\begin{bmatrix}2 & 5 & | & 1 & 3\\1 & 3 & | & -1 & 4\end{bmatrix} \xleftrightarrow{R_1 \text{ swap } R_2} \begin{bmatrix}1 & 3 & | & -1 & 4\\2 & 5 & | & 1 & 3\end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix}1 & 3 & | & -14 & \\0 & -1 & | & 3 & -5\end{bmatrix}$$

Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix, $\vec{v}$ is an **eigenvector**(also know as *characteristic vector(s)*)for A with a corresponding **eigenvalue**(also know as *characteristic value(s)*) λ, if $\vec{v}$ is a non-zero vector where

$$A\vec{v} = \lambda\vec{v} \tag{2}$$

Example #1

$$A = \begin{bmatrix} 5 & 3 \\ 6 & -4 \end{bmatrix} \to \begin{bmatrix} 5 & 3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\underbrace{\lambda = 2}_{\text{eigenvalue}}, \quad \underbrace{\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{eigenvector pairs}}$$

Example #1.1

Let's say $\lambda = -1$ is an eigenvalue for A, find an eigenvector.

$$\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \vec{v} = -1\vec{v} \qquad A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\cancel{(A - \lambda)\vec{v} = \vec{0}} \longrightarrow \text{(Not defined)}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$
$$\text{The eigenvectors are non-zero solutions to this homogenous system.}$$
$$\lambda = -1 : A - (-1)I = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -3 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{6}R_1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \to \begin{matrix} x_2 = s \\ x_1 - \dfrac{1}{2}x_2 = 0 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$(A - \lambda I)\vec{v} = \vec{0}$, λ is an eigenvalue when this equation has non-trivial solutions

$$\iff A - \lambda I \text{ has a free variable} \tag{3}$$
$$\iff A - \lambda I \text{ is singular} \tag{4}$$
$$\iff \det(\underbrace{A - \lambda I}_{\text{chracteristic polynomial}} = 0) \text{ [charateristic equation]} \tag{5}$$

The trace A, $tr\, A = \sum_n^{i=1} a_{ii} \vee \sum_n^{i=1} \lambda_i$ (including multiplicities), $\det A = \prod_n^{i=1} \lambda_i$ (including multiplicities).

Example #1.2

$$A = \begin{bmatrix} 7 & 6 \\ -3 & 2 \end{bmatrix} \quad \begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 7 & -\lambda \\ -3 & -2-\lambda \end{bmatrix} \end{aligned} \quad \begin{aligned} &0 = (7-\lambda)(-2-\lambda) - 6(-3) \\ &0 = \lambda^2 - 7\lambda + 2\lambda - 14 + 18 \\ &0 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) \\ &\lambda = [1, 4] \end{aligned}$$

$$A = \begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix} \; \lambda = 1, 4; \;\; \lambda = 1 : A - 1I = \begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Def:	$\lambda = 4 : A - 4I =$
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} (\vee \begin{bmatrix} -d \\ c \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\lambda = 1 : \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \; \lambda = 4, \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$