M 408C - Differential and Integral Calculus

Week 8 - 3.10, 4.1, 4.2

Quest HW 08 - Due Monday at 11:30p.

Gradescope HW 08 - Due **Monday** at 11:30p on Gradescope.

§4.1, #35

§4.2, #15, 18

Additional Questions:

#1) Use linearization to estimate $\sqrt{3.9}$. *Hint: Let* $f(x) = \sqrt{x}$*, and* a *be an integer close to* 3.9.

#2) Find the absolute minimum and absolute maximum of the function $f(x) = x^{2/3}(6-x)^{1/3}$ on the interval $[0, 6]$.

#3) Find the absolute minimum and absolute maximum of the function $f(x) = \cos^2(x) + \sin(x)$ on the interval $[0, \pi/2]$.

Additional Thing: You have enough to work on this week! Here is my advice: Get it done early and get some good sleep Monday night.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = 3 - 2x, \quad x \geq -1$

16. $f(x) = x^2, \quad -1 \leq x < 2$

17. $f(x) = 1/x, \quad x \geq 1$

18. $f(x) = 1/x, \quad 1 < x < 3$

19. $f(x) = \sin x, \quad 0 \leq x < \pi/2$

20. $f(x) = \sin x, \quad 0 < x \leq \pi/2$

21. $f(x) = \sin x, \quad -\pi/2 \leq x \leq \pi/2$

22. $f(t) = \cos t, \quad -3\pi/2 \leq t \leq 3\pi/2$

23. $f(x) = \ln x, \quad 0 < x \leq 2$

24. $f(x) = |x|$

25. $f(x) = 1 - \sqrt{x}$

26. $f(x) = e^x$

27. $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x \leq 0 \\ 2 - 3x & \text{if } 0 < x \leq 1 \end{cases}$

28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leq x < 1 \\ 4 - 2x & \text{if } 1 \leq x \leq 3 \end{cases}$

29–48 Find the critical numbers of the function.

29. $f(x) = 3x^2 + x - 2$

30. $g(v) = v^3 - 12v + 4$

31. $f(x) = 3x^4 + 8x^3 - 48x^2$

32. $f(x) = 2x^3 + x^2 + 8x$

33. $g(t) = t^5 + 5t^3 + 50t$

34. $A(x) = |3 - 2x|$

35. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

36. $h(p) = \dfrac{p - 1}{p^2 + 4}$

37. $p(x) = \dfrac{x^2 + 2}{2x - 1}$

38. $q(t) = \dfrac{t^2 + 9}{t^2 - 9}$

39. $h(t) = t^{3/4} - 2t^{1/4}$

40. $g(x) = \sqrt[3]{4 - x^2}$

41. $F(x) = x^{4/5}(x - 4)^2$

42. $h(x) = x^{-1/3}(x - 2)$

43. $f(x) = x^{1/3}(4 - x)^{2/3}$

44. $f(\theta) = \theta + \sqrt{2}\cos\theta$

45. $f(\theta) = 2\cos\theta + \sin^2\theta$

46. $p(t) = te^{4t}$

47. $g(x) = x^2 \ln x$

48. $B(u) = 4\tan^{-1}u - u$

49–50 A formula for the *derivative* of a function f is given. How many critical numbers does f have?

49. $f'(x) = 5e^{-0.1|x|}\sin x - 1$

50. $f'(x) = \dfrac{100\cos^2 x}{10 + x^2} - 1$

51–66 Find the absolute maximum and absolute minimum values of f on the given interval.

51. $f(x) = 12 + 4x - x^2, \quad [0, 5]$

52. $f(x) = 5 + 54x - 2x^3, \quad [0, 4]$

53. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

54. $f(x) = x^3 - 6x^2 + 5, \quad [-3, 5]$

55. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$

56. $f(t) = (t^2 - 4)^3, \quad [-2, 3]$

57. $f(x) = x + \dfrac{1}{x}, \quad [0.2, 4]$

58. $f(x) = \dfrac{x}{x^2 - x + 1}, \quad [0, 3]$

59. $f(t) = t - \sqrt[3]{t}, \quad [-1, 4]$

60. $f(x) = \dfrac{e^x}{1 + x^2}, \quad [0, 3]$

61. $f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$

62. $f(\theta) = 1 + \cos^2\theta, \quad [\pi/4, \pi]$

63. $f(x) = x^{-2}\ln x, \quad \left[\frac{1}{2}, 4\right]$

64. $f(x) = xe^{x/2}, \quad [-3, 1]$

65. $f(x) = \ln(x^2 + x + 1), \quad [-1, 1]$

66. $f(x) = x - 2\tan^{-1}x, \quad [0, 4]$

67. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$, $0 \leq x \leq 1$.

68. Use a graph to estimate the critical numbers of $f(x) = |1 + 5x - x^3|$ correct to one decimal place.

69–72
(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.
(b) Use calculus to find the exact maximum and minimum values.

69. $f(x) = x^5 - x^3 + 2, \quad -1 \leq x \leq 1$

70. $f(x) = e^x + e^{-2x}, \quad 0 \leq x \leq 1$

71. $f(x) = x\sqrt{x - x^2}$

72. $f(x) = x - 2\cos x, \quad -2 \leq x \leq 0$

15–18 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

15. $f(x) = 2x^2 - 3x + 1$, $[0, 2]$

16. $f(x) = x^3 - 3x + 2$, $[-2, 2]$

17. $f(x) = \ln x$, $[1, 4]$

18. $f(x) = 1/x$, $[1, 3]$

19–20 Find the number c that satisfies the conclusion of the Mean Value Theorem on the given interval. Graph the function, the secant line through the endpoints, and the tangent line at $(c, f(c))$. Are the secant line and the tangent line parallel?

19. $f(x) = \sqrt{x}$, $[0, 4]$

20. $f(x) = e^{-x}$, $[0, 2]$

21. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

22. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

23–24 Show that the equation has exactly one real solution.

23. $2x + \cos x = 0$

24. $x^3 + e^x = 0$

25. Show that the equation $x^3 - 15x + c = 0$ has at most one solution in the interval $[-2, 2]$.

26. Show that the equation $x^4 + 4x + c = 0$ has at most two real solutions.

27. (a) Show that a polynomial of degree 3 has at most three real zeros.
(b) Show that a polynomial of degree n has at most n real zeros.

28. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two zeros. Show that f' has at least one zero.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three zeros. Show that f'' has at least one real zero.
(c) Can you generalize parts (a) and (b)?

29. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

30. Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

31. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?

32. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint:* Apply the Mean Value Theorem to the function $h = f - g$.]

33. Show that $\sin x < x$ if $0 < x < 2\pi$.

34. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

35. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leq |a - b| \qquad \text{for all } a \text{ and } b$$

36. If $f'(x) = c$ (c a constant) for all x, use Corollary 7 to show that $f(x) = cx + d$ for some constant d.

37. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2mm] 1 + \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

38–39 Use the method of Example 6 to prove the identity.

38. $\arctan x + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$, $x > 0$

39. $2 \sin^{-1}x = \cos^{-1}(1 - 2x^2)$, $x \geq 0$

40. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

41. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

42. **Fixed Points** A number a is called a *fixed point* of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

4.3 What Derivatives Tell Us about the Shape of a Graph

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives. Because $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve