10/31/2023

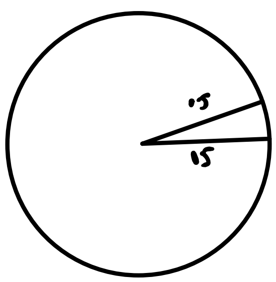
Last Time: Optimization

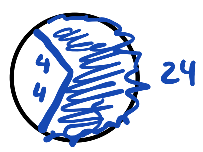
Today: Anti-Derivatives - 4.9

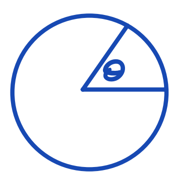
Area + Distance - 5.1

Future: HWs

You are offered a slice of pizza (a sector). The Perimeter of a slice must be 32in. What diameter of pizza maximizes the area of the slice.







Area of S: $\frac{1}{2}r^2\theta$

Arc length: $r\theta$

Optimize: $\boxed{A = \frac{1}{2}r^2\theta}$

Perimeter: $32 = 2r + r\theta$

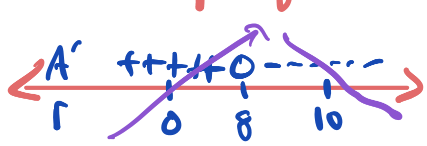
$\therefore \theta = \dfrac{32-2r}{r} = \boxed{32r^{-1} - 2}$

$A = \frac{1}{2} \cdot r^2 \cdot (32r^{-1} - 2)$

$= 16r - r^2$

$A' = 16 - 2r = 0$

$r = 8$



Diameter is 16, we get max size slice.

Antiderivative: An antiderivative of $f(x)$ is a function $F(x)$ s.t. $F'(x) = f(x)$.

Ex: Let $f(x) = 2x^4 - \sin(x)$, find $F(x)$.

$f(x)$	$F(x)$
$c \cdot f(x)$	$c \cdot F(x)$
$f(x) \pm g(x)$	$F(x) \pm G(x)$
x^n	$\frac{1}{n+1} x^{n+1} + C$, $n \neq -1$
$1/x$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$

$\Rightarrow \left[\frac{1}{n+1} x^{n+1}\right]' = \frac{1}{n+1}(n+1)x^n = \underline{x^n}$

$f(x) = 2x^4 - \sin(x)$

$F(x) = 2 \cdot \frac{1}{5}x^5 - (-\cos(x))$

$= 2/5\, x^5 + \cos(x)$

.... $= 2/5\, x^5 + \cos(x) + 1$

$= 2/5\, x^5 + \cos(x) - 5$

(*) If F, G are two antiderivatives of $f(x)$, then $F(x) - G(x) = C$

$\Leftrightarrow$ $F(x) = G(x) + C$

$f''(x) = \frac{1}{3\sqrt{x}}$, $f'(4) = 3$, $f(4) = 5$, find $f(x)$.

$$f''(x) = \tfrac{1}{3}x^{-1/2} \Rightarrow f'(x) = \tfrac{1}{3} \cdot \frac{1}{1/2} x^{1/2} + C = \tfrac{2}{3}\sqrt{x} + C$$

$$\Rightarrow f'(4) = \tfrac{2}{3}\sqrt{4} + C = \tfrac{4}{3} + C = 3 \Rightarrow C = \tfrac{5}{3}$$

$$f'(x) = \tfrac{2}{3}x^{1/2} + \tfrac{5}{3} \Rightarrow f(x) = \tfrac{2}{3} \cdot \frac{1}{3/2} x^{3/2} + \tfrac{5}{3}x + D = \tfrac{4}{9}x^{3/2} + \tfrac{5}{3}x + D$$

$\therefore f(4) = \frac{32}{9} + \frac{20}{3} + D \Rightarrow \frac{92}{9} + D = \frac{45}{9} \Rightarrow D = -\frac{47}{9}$ $\therefore f(x) = \frac{4}{9}x^{3/2} + \frac{5}{3}x - \frac{47}{9}$

$f''(x) = \sin(x)$, $f'(0) = 2$, $f(0) = 3$, find $f(\pi)$.

$f'(x) = -\cos(x) + C$, $f'(0) = -\cos(0) + C = -1 + C = 2 \therefore C = 3$,

$f'(x) = -\cos(x) + 3 \Rightarrow f(x) = -\sin(x) + 3x + D$, $f(0) = -\sin(0) + 3(0) + D = 3$

$f(x) = -\sin(x) + 3x + 3$

$f(\pi) = -\sin(\pi) + 3\pi + 3$

$= 3\pi + 3$

$f'(x) = \tan^2(x) + e^x$, $f(0) = 3$, find $f(\pi/4)$

$$\circledast \quad \sin^2\theta + \cos^2\theta = 1 \xRightarrow{\div \cos^2\theta} \tan^2\theta + 1 = \sec^2\theta$$

$$\Rightarrow \tan^2\theta = \sec^2\theta - 1$$

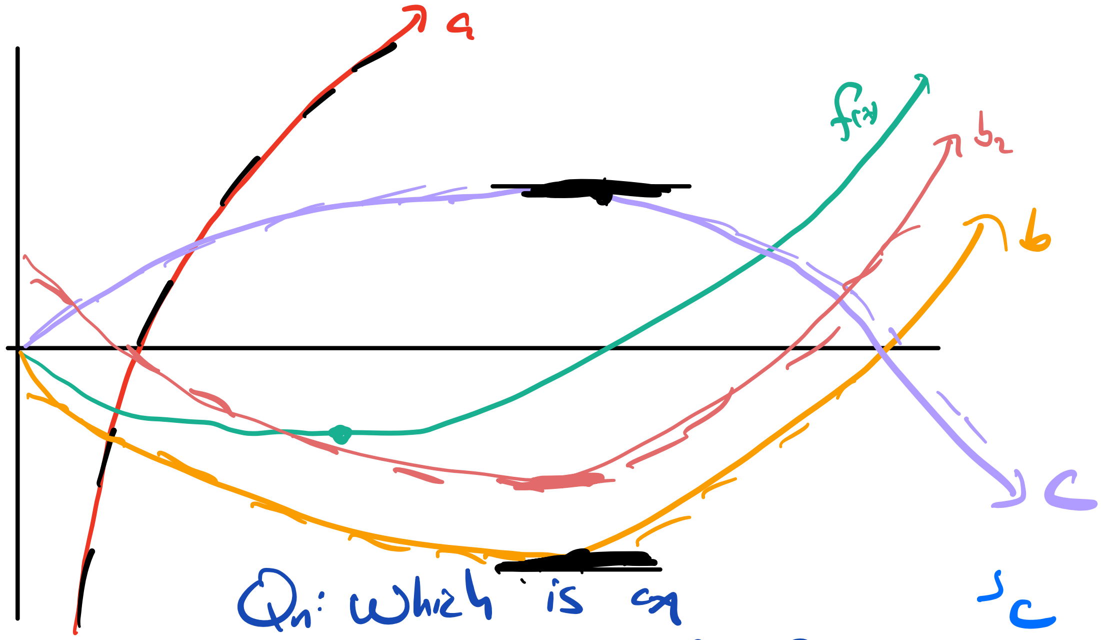
$$\therefore f'(x) = \sec^2\theta - 1 + e^x \Rightarrow f(x) = \tan\theta - x + e^x + C$$

$$f(0) = 0 - 0 + e^0 + C = 1 + C = 3 \quad \therefore C = 2$$

$$f(x) = \tan(\theta) - x + e^x + 2$$

$$f(\pi/4) = \tan(\pi/4) - (\pi/4) + e^{\pi/4} + 2$$

$$= 3 - \pi/4 + e^{\pi/4}$$

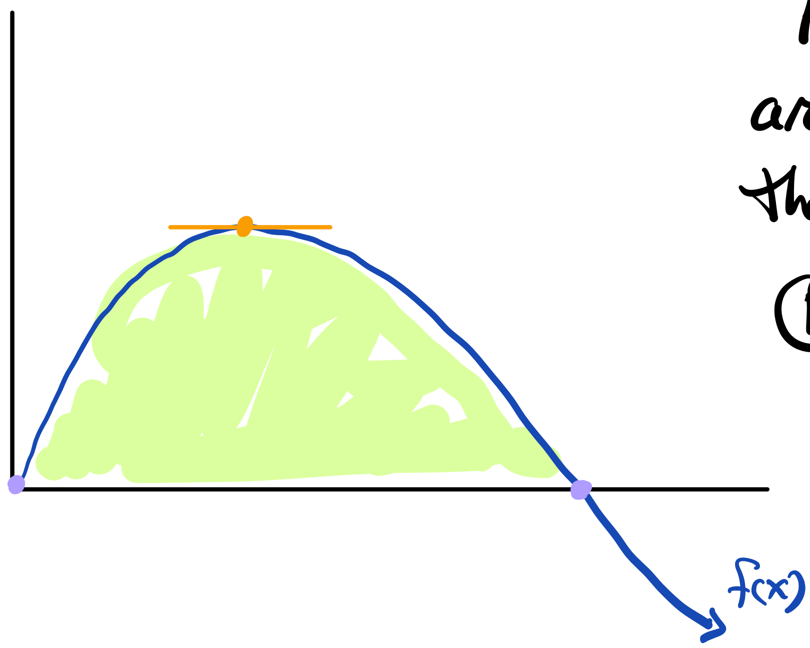


Qn: Which is an antiderivative of f?

$\Longleftrightarrow$ $\cancel{a' = f}$

$b' = f$ $\Longleftrightarrow$ b is an antiderivative.

or $c' = f$

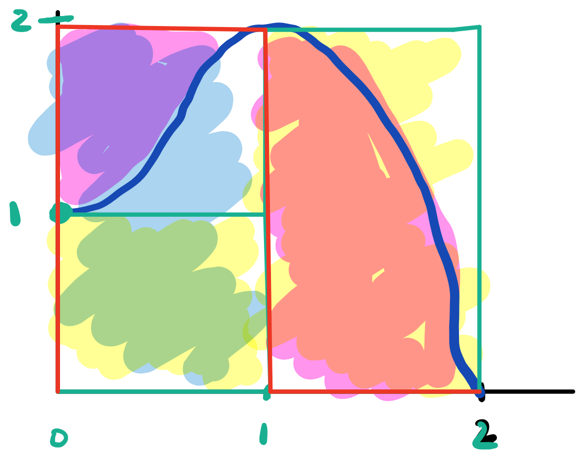


How do we find the area between f(x) and the x-axis?

① Approximate.

$\left[V_{avg} \overset{?}{\Longrightarrow} V(t) \right]$

Start w/ approximations:



between $x=0$ + $x=2$

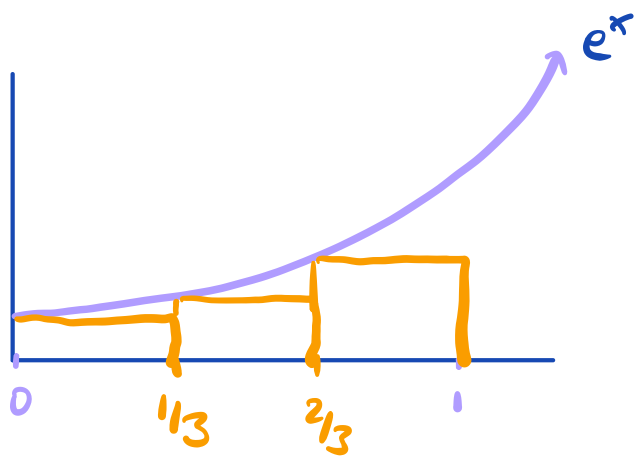
$n=2$, left endpoints:

$A \approx b_1 h_1 + b_2 h_2 = 1(1) + 1(2) = 3$

$n=2$, right endpoint

$A \approx b_1 h_1 + b_2 h_2 = (1)(2) + 1(0) = 2$

Observation: more rectangles ⟺ smaller bases should lead to a better approximation.



approx area under e^x between $x=0, 1$ using $n=3$ and left endpoints.

$$A = b_1 h_1 + b_2 h_2 + b_3 h_3$$

$$= (h_1 + h_2 + h_3) \cdot b$$

$$= \left(e^0 + e^{1/3} + e^{2/3}\right) \tfrac{1}{3}$$

$$= \frac{e^0 + e^{1/3} + e^{2/3}}{3}$$