M 408C - Differential and Integral Calculus

Week 10 - 4.4, 4.7

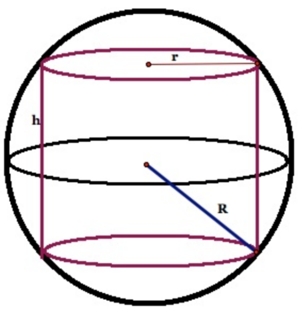
Quest HW 10 - Due Monday at 11:30p.

Gradescope HW 10 - Due Wednesday at 11:30p on Gradescope.

§4.4, #27, 53, 85, 86

§4.7, #14, 26, 42, 55
Additional Questions:

#1) A cylinder is inscribed inside a sphere of radius $R = 10$ as in the image below. What is the maximum possible value of the volume of the cylinder.



#2) You are at one end of a perfectly circular lake with radius 5km. You need to get to the opposite end. You can run on land at a rate of 4km/hr and you can swim at a rate of 2km/hr. You will either run around the lake, swim directly across the lake, or swim and run as in questions #56 on page 345. What is the fastest path and how long does it take? What is the slowest path and how long does it take?

Additional Thing: Try number 7 on page 369. It's a doozy but doable!

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore
$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

4.4 Exercises

1–4 Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$
$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For any limit that is not an indeterminate form, evaluate it where possible.

1. (a) $\lim_{x\to a} \frac{f(x)}{g(x)}$ (b) $\lim_{x\to a} \frac{f(x)}{p(x)}$

(c) $\lim_{x\to a} \frac{h(x)}{p(x)}$ (d) $\lim_{x\to a} \frac{p(x)}{f(x)}$

(e) $\lim_{x\to a} \frac{p(x)}{q(x)}$

2. (a) $\lim_{x\to a} [f(x)p(x)]$ (b) $\lim_{x\to a} [h(x)p(x)]$

(c) $\lim_{x\to a} [p(x)q(x)]$

3. (a) $\lim_{x\to a} [f(x) - p(x)]$ (b) $\lim_{x\to a} [p(x) - q(x)]$

(c) $\lim_{x\to a} [p(x) + q(x)]$

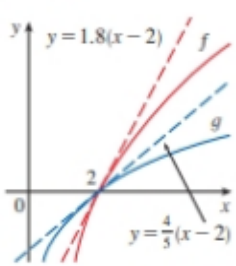
4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ (b) $\lim_{x\to a} [f(x)]^{p(x)}$

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ (d) $\lim_{x\to a} [p(x)]^{f(x)}$

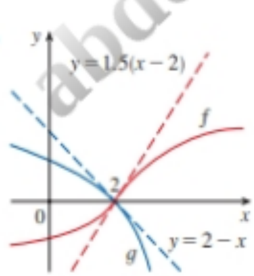
(e) $\lim_{x\to a} [p(x)]^{q(x)}$ (f) $\lim_{x\to a} \sqrt[q(x)]{p(x)}$

5–6 Use the graphs of f and g and their tangent lines at (2, 0) to find $\lim_{x\to 2} \frac{f(x)}{g(x)}$.

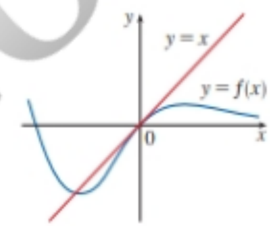
5.



6.



7. The graph of a function f and its tangent line at 0 are shown. What is the value of $\lim_{x\to 0} \frac{f(x)}{e^x - 1}$?



8–70 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

8. $\lim_{x\to 3} \frac{x-3}{x^2-9}$

9. $\lim_{x\to 4} \frac{x^2-2x-8}{x-4}$ **10.** $\lim_{x\to -2} \frac{x^3+8}{x+2}$

11. $\lim_{x\to 1} \frac{x^7-1}{x^3-1}$ **12.** $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

13. $\lim_{x\to \pi/4} \frac{\sin x - \cos x}{\tan x - 1}$ **14.** $\lim_{x\to 0} \frac{\tan 3x}{\sin 2x}$

15. $\lim_{t\to 0} \frac{e^{2t}-1}{\sin t}$ **16.** $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

17. $\lim_{x\to 1} \frac{\sin(x-1)}{x^3+x-2}$ **18.** $\lim_{\theta\to \pi} \frac{1+\cos\theta}{1-\cos\theta}$

19. $\lim_{x\to \infty} \frac{\sqrt{x}}{1+e^x}$ **20.** $\lim_{x\to \infty} \frac{x+x^2}{1-2x^2}$

21. $\lim_{x\to 0^+} \frac{\ln x}{x}$ **22.** $\lim_{x\to \infty} \frac{\ln\sqrt{x}}{x^2}$

23. $\lim_{x\to 3} \frac{\ln(x/3)}{3-x}$ **24.** $\lim_{t\to 0} \frac{8^t-5^t}{t}$

25. $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$ **26.** $\lim_{u\to \infty} \frac{e^{u/10}}{u^3}$

$\boxed{\textbf{27.}\ \lim_{x\to 0} \frac{e^x+e^{-x}-2}{e^x-x-1}}$ **28.** $\lim_{x\to 0} \frac{\sinh x - x}{x^3}$

29. $\displaystyle\lim_{x\to 0} \frac{\tanh x}{\tan x}$

30. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$

31. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1}x}{x}$

32. $\displaystyle\lim_{x\to \infty} \frac{(\ln x)^2}{x}$

33. $\displaystyle\lim_{x\to 0} \frac{x3^x}{3^x - 1}$

34. $\displaystyle\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}$

35. $\displaystyle\lim_{x\to 0} \frac{\ln(1 + x)}{\cos x + e^x - 1}$

36. $\displaystyle\lim_{x\to 1} \frac{x \sin(x - 1)}{2x^2 - x - 1}$

37. $\displaystyle\lim_{x\to 0^+} \frac{\arctan 2x}{\ln x}$

38. $\displaystyle\lim_{x\to 0} \frac{x^2 \sin x}{\sin x - x}$

39. $\displaystyle\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$, $b \neq 0$

40. $\displaystyle\lim_{x\to \infty} \frac{e^{-x}}{(\pi/2) - \tan^{-1}x}$

41. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

42. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x \sin(x^2)}$

43. $\displaystyle\lim_{x\to \infty} x \sin(\pi/x)$

44. $\displaystyle\lim_{x\to \infty} \sqrt{x}\, e^{-x/2}$

45. $\displaystyle\lim_{x\to 0} \sin 5x \csc 3x$

46. $\displaystyle\lim_{x\to -\infty} x \ln\left(1 - \frac{1}{x}\right)$

47. $\displaystyle\lim_{x\to \infty} x^3 e^{-x^2}$

48. $\displaystyle\lim_{x\to \infty} x^{3/2} \sin(1/x)$

49. $\displaystyle\lim_{x\to 1^+} \ln x \tan(\pi x/2)$

50. $\displaystyle\lim_{x\to (\pi/2)^-} \cos x \sec 5x$

51. $\displaystyle\lim_{x\to 1} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right)$

52. $\displaystyle\lim_{x\to 0} (\csc x - \cot x)$

$\boxed{\textbf{53.}\ \displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)}$

54. $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\tan^{-1}x}\right)$

55. $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\tan x}\right)$

56. $\displaystyle\lim_{x\to \infty} (x - \ln x)$

57. $\displaystyle\lim_{x\to 0^+} x^{\sqrt{x}}$

58. $\displaystyle\lim_{x\to 0^+} (\tan 2x)^x$

59. $\displaystyle\lim_{x\to 0} (1 - 2x)^{1/x}$

60. $\displaystyle\lim_{x\to \infty} \left(1 + \frac{a}{x}\right)^{bx}$

61. $\displaystyle\lim_{x\to 1^+} x^{1/(1-x)}$

62. $\displaystyle\lim_{x\to \infty} (e^x + 10x)^{1/x}$

63. $\displaystyle\lim_{x\to \infty} x^{1/x}$

64. $\displaystyle\lim_{x\to \infty} x^{e^{-x}}$

65. $\displaystyle\lim_{x\to 0^+} (4x + 1)^{\cot x}$

66. $\displaystyle\lim_{x\to 0^+} (1 - \cos x)^{\sin x}$

67. $\displaystyle\lim_{x\to 0^+} (1 + \sin 3x)^{1/x}$

68. $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$

69. $\displaystyle\lim_{x\to 0^+} \frac{x^x - 1}{\ln x + x - 1}$

70. $\displaystyle\lim_{x\to \infty} \left(\frac{2x - 3}{2x + 5}\right)^{2x+1}$

71–72 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

71. $\displaystyle\lim_{x\to \infty} \left(1 + \frac{2}{x}\right)^x$

72. $\displaystyle\lim_{x\to 0} \frac{5^x - 4^x}{3^x - 2^x}$

73–74 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Also, calculate the exact value of the limit.

73. $f(x) = e^x - 1$, $g(x) = x^3 + 4x$

74. $f(x) = 2x \sin x$, $g(x) = \sec x - 1$

75. Prove that

$$\lim_{x\to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

76. Prove that

$$\lim_{x\to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches infinity more slowly than any power of x.

77–78 What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

77. $\displaystyle\lim_{x\to \infty} \frac{x}{\sqrt{x^2 + 1}}$

78. $\displaystyle\lim_{x\to (\pi/2)^-} \frac{\sec x}{\tan x}$

79. Investigate the family of curves $f(x) = e^x - cx$. In particular, find the limits as $x \to \pm\infty$ and determine the values of c for which f has an absolute minimum. What happens to the minimum points as c increases?

80. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}\left(1 - e^{-ct/m}\right)$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t\to\infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c\to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

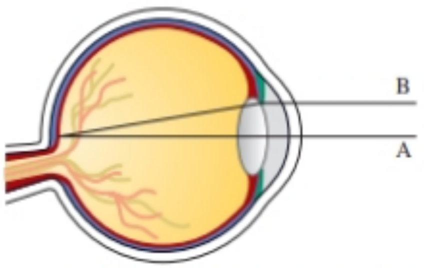
81. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

82. Light enters the eye through the pupil and strikes the retina, where photoreceptor cells sense light and color. W. Stanley Stiles and B. H. Crawford studied the phenomenon in which measured brightness decreases as light enters farther from the center of the pupil (see the figure).



A light beam A that enters through the center of the pupil measures brighter than a beam B entering near the edge of the pupil.

They detailed their findings of this phenomenon, known as the *Stiles–Crawford effect of the first kind*, in an important paper published in 1933. In particular, they observed that the amount of luminance sensed was *not* proportional to the area of the pupil as they expected. The percentage P of the total luminance entering a pupil of radius r mm that is sensed at the retina can be described by

$$P = \frac{1 - 10^{-\rho r^2}}{\rho r^2 \ln 10}$$

where ρ is an experimentally determined constant, typically about 0.05.

(a) What is the percentage of luminance sensed by a pupil of radius 3 mm? Use $\rho = 0.05$.

(b) Compute the percentage of luminance sensed by a pupil of radius 2 mm. Does it make sense that it is larger than the answer to part (a)?

(c) Compute $\lim_{r \to 0^+} P$. Is the result what you would expect? Is this result physically possible?

Source: Adapted from W. Stiles and B. Crawford, "The Luminous Efficiency of Rays Entering the Eye Pupil at Different Points." *Proceedings of the Royal Society of London, Series B: Biological Sciences* 112 (1933): 428–50.

83. Logistic Equations Some populations initially grow exponentially but eventually level off. Equations of the form

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

where M, A, and k are positive constants, are called *logistic equations* and are often used to model such populations. (We will investigate these in detail in Chapter 9.) Here M is called the *carrying capacity* and represents the maximum population size that can be supported, and

$$A = \frac{M - P_0}{P_0}$$

where P_0 is the initial population.

(a) Compute $\lim_{t \to \infty} P(t)$. Explain why your answer is to be expected.

(b) Compute $\lim_{M \to \infty} P(t)$. (Note that A is defined in terms of M.) What kind of function is your result?

84. A metal cable has radius r and is covered by insulation so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

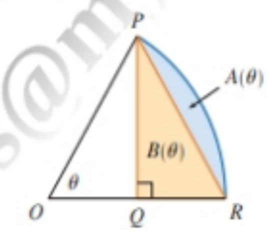
(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

85. The first appearance in print of l'Hospital's Rule was in the book *Analyse des infiniment petits* published by the Marquis de l'Hospital in 1696. This was the first calculus textbook ever published. The example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

86. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



87. Evaluate

$$\lim_{x \to \infty}\left[x - x^2 \ln\left(\frac{1 + x}{x}\right)\right]$$

88. Suppose f is a positive function. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, show that

$$\lim_{x \to a} [f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the volume.

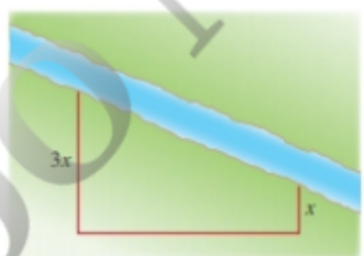
(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the volume as a function of one variable.

(f) Finish solving the problem and compare the answer with your estimate in part (a).

13. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

14. A farmer has 1200 ft of fencing for enclosing a trapezoidal field along a river as shown. One of the parallel sides is three times longer than the other. No fencing is needed along the river. Find the largest area the farmer can enclose.



15. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs $20 per linear foot to install and the farmer is not willing to spend more than $5000, find the dimensions for the plot that would enclose the most area.

16. If the farmer in Exercise 15 wants to enclose 8000 square feet of land, what dimensions will minimize the cost of the fence?

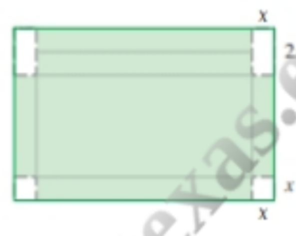
17. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.

(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

18. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

19. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

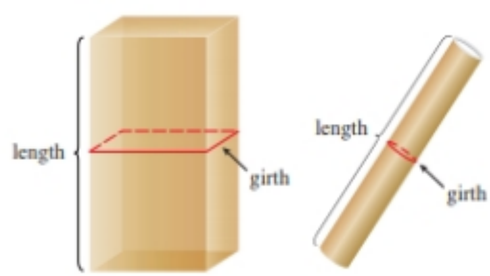
20. A box with an open top is to be constructed from a 4 ft by 3 ft rectangular piece of cardboard by cutting out squares or rectangles from each of the four corners, as shown in the figure, and bending up the sides. One of the longer sides of the box is to have a double layer of cardboard, which is obtained by folding the side twice. Find the largest volume that such a box can have.



21. A rectangular storage container without a lid is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the least expensive such container.

22. Rework Exercise 21 assuming the container has a lid that is made from the same material as the sides.

23. A package to be mailed using the US postal service may not measure more than 108 inches in length plus girth. (Length is the longest dimension and girth is the largest distance around the package, perpendicular to the length.) Find the dimensions of the rectangular box with square base of greatest volume that may be mailed.



24. Refer to Exercise 23. Find the dimensions of the cylindrical mailing tube of greatest volume that may be mailed using the US postal service.

25. Find the point on the line $y = 2x + 3$ that is closest to the origin.

26. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

27. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

28. Find, correct to two decimal places, the coordinates of the point on the curve $y = \sin x$ that is closest to the point $(4, 2)$.

29. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

30. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

31. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

32. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.

33. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

34. If the two equal sides of an isosceles triangle have length a, find the length of the third side that maximizes the area of the triangle.

35. If one side of a triangle has length a and another has length $2a$, show that the largest possible area of the triangle is a^2.

36. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 4 - x^2$. What is the largest possible area of the rectangle?

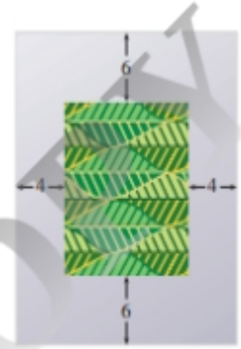
37. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

38. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

39. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

40. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 1.1.72.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

41. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.



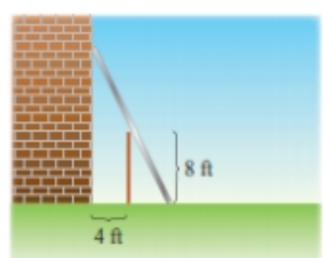
42. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

43. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

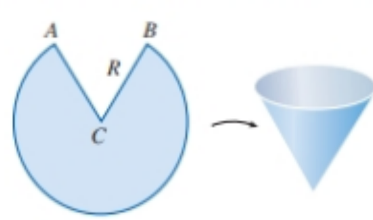
44. Answer Exercise 43 if one piece is bent into a square and the other into a circle.

45. If you are offered one slice from a round pizza (in other words, a sector of a circle) and the slice must have a perimeter of 32 inches, what diameter pizza will reward you with the largest slice?

46. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



47. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



48. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

49. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

50. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

51. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

52. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.

(a) Determine the value of v that minimizes E.

(b) Sketch the graph of E.

Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

53. In a beehive, each cell is a regular hexagonal prism, open at one end; the other end is capped by three congruent rhombi forming a trihedral angle at the apex, as in the figure. Let θ be the angle at which each rhombus meets the altitude, s the side length of the hexagon, and h the length of the longer base of the trapezoids on the sides of the cell. It can be shown that if s and h are held fixed, then the volume of the cell is constant (independent of θ), and for a given value of θ the surface area S of the cell is

$$S = 6sh - \tfrac{3}{2}s^2 \cot \theta + \tfrac{3}{2}\sqrt{3}s^2 \csc \theta$$

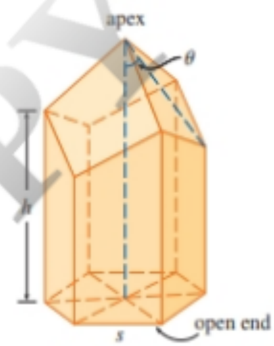
It is believed that bees form their cells in such a way as to minimize surface area, thus using the least amount of wax in cell construction.

(a) Calculate $dS/d\theta$.

(b) What angle θ should the bees prefer?

(c) Determine the minimum surface area of the cell in terms of s and h.

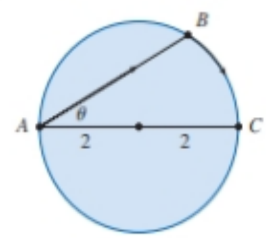
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than $2°$.



54. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

55. Solve the problem in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

56. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?



57. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

T **58.** Suppose the refinery in Exercise 57 is located 1 km north of the river. Where should P be located?

59. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If