11/30/2023

Last Time: Exam III

Today: 6.5, Avg value of a function

Future: Final Exam, Friday, Dec 8^{th}, 8AM!

Find average of $1, 7, \sqrt{2}, -3 \Rightarrow \dfrac{1+7+\sqrt{2}-3}{4}$

Find the average of $y_1, y_2, \ldots, y_n = \dfrac{y_1+y_2+\cdots+y_n}{n}$

Q_1: What is the average value of $f(x) = x^2$ from $x=0$ to $x=3$?

Q_n: $\dfrac{f(0)+f(3)}{2} = 4.5$

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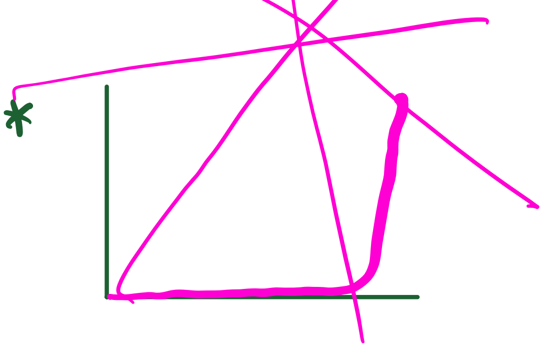

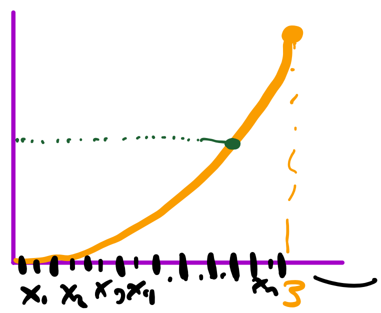

n $x_i \Rightarrow$

n subintervals

Computing Average is tough since there are ∞ y-values on $0 \le x \le 3$

Approx: $\dfrac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$, $\Delta x = \dfrac{b-a}{n} \Leftrightarrow n = \dfrac{b-a}{\Delta x}$

$$= \frac{\sum_{i=1}^{n} f(x_i)}{\frac{b-a}{\Delta x}} = \frac{\sum_{i=1}^{n} f(x_i)\cdot \Delta x}{b-a}$$

$$f_{avg} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}{b-a} = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx$$

Find f_{avg} of $f(x) = x^2$ on $[0,3]$

$$f_{avg} = \frac{1}{3-0} \int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{1}{3}x^3\right]_0^3 = \frac{1}{9}\left[3^3 - 0^3\right] = \boxed{3}$$

⊛ f_{avg} formula works for any function ⊛

If $f(x)$ is continuous on $[a,b]$, then there is a c, $a < c < b$, s.t.

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx = f(c)$$

Find value of c in the problem above.

$\Rightarrow f_{avg} = 3 = f(c) = c^2 \Rightarrow c^2 = 3, \; c = \pm\sqrt{3}$

only $0 < \sqrt{3} < 3 \Rightarrow c = \sqrt{3}$

f is continuous $\Rightarrow f_{avg} = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$\Longrightarrow f(c)\cdot(b-a) = \int_a^b f(x)\,dx \leftarrow$ MVT for integrals

MVT for derivatives $\frac{f(b)-f(a)}{b-a} = f'(c)$

$f'(c)(b-a) = f(b)-f(a)$

$$\underbrace{\underbrace{f(c)}_{\substack{h \text{ in} \\ y\text{-dir}}}\cdot\underbrace{(b-a)}_{\substack{\ell \text{ on} \\ x\text{-axis}}}}_{\text{rectangle}} = \underbrace{\int_a^b f(x)\,dx}_{\text{Area under graph}^*}, \qquad f(c) = f_{avg}$$

Ex: Find the average value of $f(x)$ on $[0, 5]$

$$f(x) = \begin{cases} 2x+1 & 0 \le x \le 3 \\ 4-3x^2 & 3 < x \le 5 \end{cases} \Rightarrow \frac{1}{5}\int_0^5 f(x)\,dx = \frac{1}{5}(12-90) = \frac{-78}{5}$$

$$\int_0^3 2x+1\,dx = [x^2+x]_0^3 = 12, \quad \int_3^5 4-3x^2\,dx = 4x-x^3\Big|_3^5 = -90$$

Ex: Find the average value of $f(x) = \frac{t}{(t^2+1)^2}$ from 0 to 1.

$$f_{avg} = \frac{1}{1-0}\int_0^1 \frac{t}{(t^2+1)^2}\,dt = \int_1^2 \frac{t}{u^2}\cdot\frac{du}{2t} = \frac{1}{2}\int_1^2 u^{-2}\,du$$

$u = t^2+1$

$du = 2t\,dt$

$dt = \frac{du}{2t}$

$t=1,\ u=2$

$t=0,\ u=1$

$$= \frac{1}{2}\cdot\frac{-1}{u} = \frac{1}{2}\left[-\frac{1}{2} - \frac{-1}{1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) = \boxed{1/4}$$

Ex: Avg value of $f(x) = \cos^4(x)\cdot\sin(x)$ from $[0, \pi/2]$.

$$f_{avg} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos^4(x)\cdot\sin(x)\,dx = \frac{2}{\pi}\int_1^0 u^4\cdot\frac{du}{-1}$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$dx = \frac{du}{-\sin(x)}$

$x=0,\ u=1$

$x=\pi/2,\ u=0$

$$= \frac{2}{\pi}\int_0^1 u^4\,du$$

$$= \frac{2}{\pi}\cdot\frac{1}{5}u^5\Big|_0^1$$

$$= \cdots = \boxed{\frac{2}{5\pi}}$$

Final Exam:

Fri, Dec 8^{th}, 8AM - 10AM, This classroom

$\approx$ 25 Multi Choice Qns

- $\approx$ 30% Exam 1 Material
- $\approx$ 30% " 2 "
- $\approx$ 30% " 3 "
- $\approx$ 10% Today