21. $f(x) = \begin{cases} x+3 & \text{if } x \leqslant -1 \\ 2^x & \text{if } x > -1 \end{cases}$ $\qquad a = -1$

22. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $\qquad a = 1$

23. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $\qquad a = 0$

24. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $\qquad a = 3$

25–26
(a) Show that f has a removable discontinuity at $x = 3$.
(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed").

25. $f(x) = \dfrac{x - 3}{x^2 - 9}$ $\qquad$ **26.** $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$

27–34 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

27. $f(x) = \dfrac{x^2}{\sqrt{x^4 + 2}}$ $\qquad$ **28.** $g(v) = \dfrac{3v - 1}{v^2 + 2v - 15}$

29. $h(t) = \dfrac{\cos(t^2)}{1 - e^t}$

30. $B(u) = \sqrt{3u - 2} + \sqrt[3]{2u - 3}$

31. $L(v) = v \ln(1 - v^2)$ $\qquad$ **32.** $f(t) = e^{-t^2} \ln(1 + t^2)$

33. $M(x) = \sqrt{1 + \dfrac{1}{x}}$ $\qquad$ **34.** $g(t) = \cos^{-1}(e^t - 1)$

35–38 Use continuity to evaluate the limit.

35. $\displaystyle\lim_{x \to 2} x\sqrt{20 - x^2}$ $\qquad$ **36.** $\displaystyle\lim_{\theta \to \pi/2} \sin(\tan(\cos \theta))$

37. $\displaystyle\lim_{x \to 1} \ln\left(\frac{5 - x^2}{1 + x}\right)$ $\qquad$ **38.** $\displaystyle\lim_{x \to 4} 3^{\sqrt{x^2 - 2x - 4}}$

39–40 Locate the discontinuities of the function and illustrate by graphing.

39. $f(x) = \dfrac{1}{\sqrt{1 - \sin x}}$ $\qquad$ **40.** $y = \arctan \dfrac{1}{x}$

41–42 Show that f is continuous on $(-\infty, \infty)$.

41. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 1 \\ \ln x & \text{if } x > 1 \end{cases}$

42. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

43–45 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

43. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$

44. $f(x) = \begin{cases} 2^x & \text{if } x \leqslant 1 \\ 3 - x & \text{if } 1 < x \leqslant 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

45. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

46. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

47. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

49. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

50. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.
(a) Find $(f \circ g)(x)$.
(b) Is $f \circ g$ continuous everywhere? Explain.

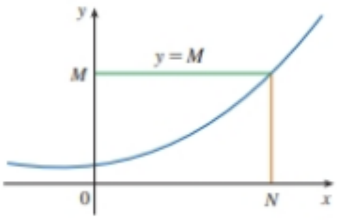


FIGURE 19
$\lim_{x \to \infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

9 Precise Definition of an Infinite Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

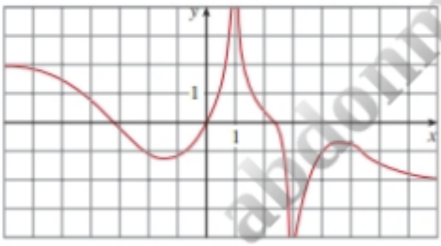
Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 80.)

2.6 Exercises

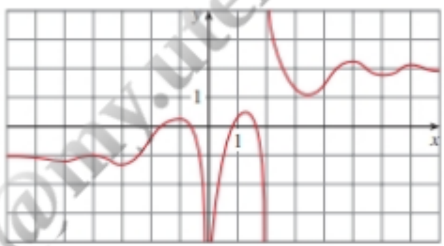
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$
(c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 3} f(x)$
(e) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
(c) $\lim_{x \to 0} g(x)$ (d) $\lim_{x \to 2^-} g(x)$
(e) $\lim_{x \to 2^+} g(x)$ (f) The equations of the asymptotes



5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(2) = 4$, $f(-2) = -4$, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = 2$

6. $f(0) = 0$, $\lim_{x \to 1^-} f(x) = \infty$, $\lim_{x \to 1^+} f(x) = -\infty$,
$\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = -2$

7. $\lim_{x \to 0} f(x) = \infty$, $\lim_{x \to 3^-} f(x) = -\infty$, $\lim_{x \to 3^+} f(x) = \infty$,
$\lim_{x \to -\infty} f(x) = 1$, $\lim_{x \to \infty} f(x) = -1$

8. $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to -2^-} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = -\infty$,
$\lim_{x \to 2} f(x) = \infty$, $\lim_{x \to \infty} f(x) = \infty$

9. $f(0) = 0$, $\lim_{x \to 1} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = -\infty$, f is odd

10. $\lim_{x \to -\infty} f(x) = -1$, $\lim_{x \to 0^-} f(x) = \infty$, $\lim_{x \to 0^+} f(x) = -\infty$,
$\lim_{x \to 3^-} f(x) = 1$, $f(3) = 4$, $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to \infty} f(x) = 1$

11. Guess the value of the limit

$$\lim_{x\to\infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50,$ and 100. Then use a graph of f to support your guess.

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x\to\infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3}$

14. $\displaystyle\lim_{x\to\infty} \sqrt{\frac{9x^3 + 8x - 4}{3 - 5x + x^3}}$

15–42 Find the limit or show that it does not exist.

15. $\displaystyle\lim_{x\to\infty} \frac{4x + 3}{5x - 1}$

16. $\displaystyle\lim_{x\to\infty} \frac{-2}{3x + 7}$

17. $\displaystyle\lim_{t\to-\infty} \frac{3t^2 + t}{t^3 - 4t + 1}$

18. $\displaystyle\lim_{t\to-\infty} \frac{6t^2 + t - 5}{9 - 2t^2}$

19. $\displaystyle\lim_{r\to\infty} \frac{r - r^3}{2 - r^2 + 3r^3}$

20. $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 8x + 2}{4x^3 - 5x^2 - 2}$

21. $\displaystyle\lim_{x\to\infty} \frac{4 - \sqrt{x}}{2 + \sqrt{x}}$

22. $\displaystyle\lim_{u\to-\infty} \frac{(u^2 + 1)(2u^2 - 1)}{(u^2 + 2)^2}$

23. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x + 3x^2}}{4x - 1}$

24. $\displaystyle\lim_{t\to\infty} \frac{t + 3}{\sqrt{2t^2 - 1}}$

25. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$

26. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$

27. $\displaystyle\lim_{x\to-\infty} \frac{2x^5 - x}{x^4 + 3}$

28. $\displaystyle\lim_{q\to\infty} \frac{q^3 + 6q - 4}{4q^2 - 3q + 3}$

29. $\displaystyle\lim_{t\to\infty} \left(\sqrt{25t^2 + 2} - 5t\right)$

30. $\displaystyle\lim_{x\to-\infty} \left(\sqrt{4x^2 + 3x} + 2x\right)$

31. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

32. $\displaystyle\lim_{x\to\infty} \left(x - \sqrt{x}\right)$

33. $\displaystyle\lim_{x\to-\infty} (x^2 + 2x^7)$

34. $\displaystyle\lim_{x\to\infty} (e^{-x} + 2\cos 3x)$

35. $\displaystyle\lim_{x\to\infty} (e^{-2x}\cos x)$

36. $\displaystyle\lim_{x\to\infty} \frac{\sin^2 x}{x^2 + 1}$

37. $\displaystyle\lim_{x\to\infty} \frac{1 - e^x}{1 + 2e^x}$

38. $\displaystyle\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

39. $\displaystyle\lim_{x\to(\pi/2)^+} e^{\sec x}$

40. $\displaystyle\lim_{x\to0^+} \tan^{-1}(\ln x)$

41. $\displaystyle\lim_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)]$

42. $\displaystyle\lim_{x\to\infty} [\ln(2 + x) - \ln(1 + x)]$

43. (a) For $f(x) = \dfrac{x}{\ln x}$ find each of the following limits.

(i) $\displaystyle\lim_{x\to0^+} f(x)$ (ii) $\displaystyle\lim_{x\to1^-} f(x)$ (iii) $\displaystyle\lim_{x\to1^+} f(x)$

(b) Use a table of values to estimate $\displaystyle\lim_{x\to\infty} f(x)$.

(c) Use the information from parts (a) and (b) to make a rough sketch of the graph of f.

44. (a) For $f(x) = \dfrac{2}{x} - \dfrac{1}{\ln x}$ find each of the following limits.

(i) $\displaystyle\lim_{x\to\infty} f(x)$ (ii) $\displaystyle\lim_{x\to0^+} f(x)$

(iii) $\displaystyle\lim_{x\to1^-} f(x)$ (iv) $\displaystyle\lim_{x\to1^+} f(x)$

(b) Use the information from part (a) to make a rough sketch of the graph of f.

45. (a) Estimate the value of

$$\lim_{x\to-\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

46. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x\to\infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

47–52 Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

47. $y = \dfrac{5 + 4x}{x + 3}$

48. $y = \dfrac{2x^2 + 1}{3x^2 + 2x - 1}$

49. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

50. $y = \dfrac{1 + x^4}{x^2 - x^4}$

51. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$

52. $y = \dfrac{2e^x}{e^x - 5}$

40. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to-\infty} g(x) = -\infty$.

41. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x\to-5^+} g(x) = \infty$, and $\lim_{x\to5^-} g(x) = 3$.

42. Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x\to2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ±1, and f is odd.

43–48 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

43. $\displaystyle\lim_{h\to0} \frac{\sqrt{9+h} - 3}{h}$

44. $\displaystyle\lim_{h\to0} \frac{e^{-2+h} - e^{-2}}{h}$

45. $\displaystyle\lim_{x\to2} \frac{x^6 - 64}{x - 2}$

46. $\displaystyle\lim_{x\to1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

47. $\displaystyle\lim_{h\to0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h}$

48. $\displaystyle\lim_{\theta\to\pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

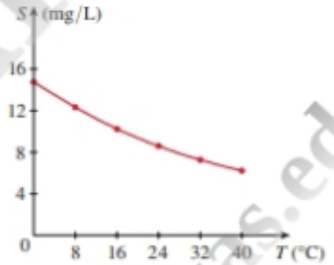
49. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
 (i) from $x = 100$ to $x = 105$
 (ii) from $x = 100$ to $x = 101$
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

50. Let $H(t)$ be the daily cost (in dollars) to heat an office building when the outside temperature is t degrees Fahrenheit.
(a) What is the meaning of $H'(58)$? What are its units?
(b) Would you expect $H'(58)$ to be positive or negative? Explain.

51. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

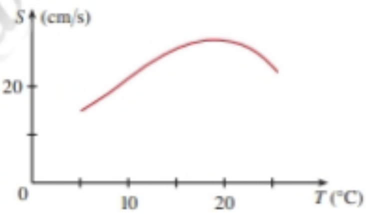
52. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

53. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Source: C. Kupchella et al., *Environmental Science: Living Within the System of Nature*, 2d ed. (Boston: Allyn and Bacon, 1989).

54. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



55. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).

t (hours)	1.0	1.5	2.0	2.5	3.0
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

(a) Find the average rate of change of C with respect to t over each time interval:
 (i) [1.0, 2.0] (ii) [1.5, 2.0]
 (iii) [2.0, 2.5] (iv) [2.0, 3.0]
In each case, include the units.
(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.