3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

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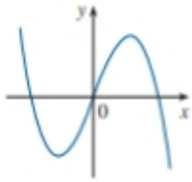
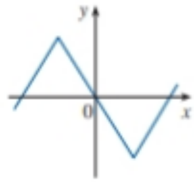

(b)

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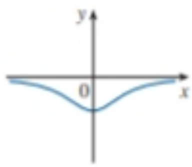
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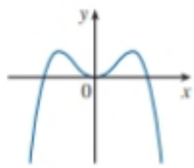
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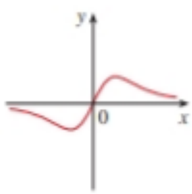
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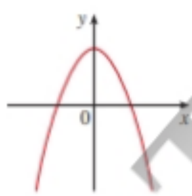
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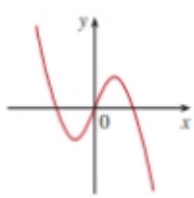
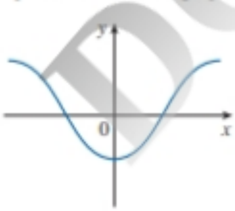

IV


4–11 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

4.

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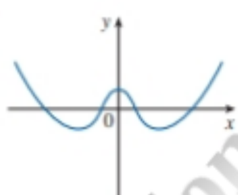
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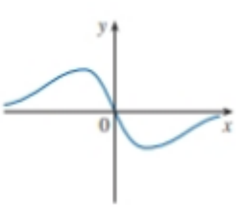
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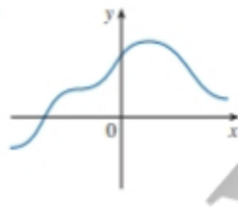
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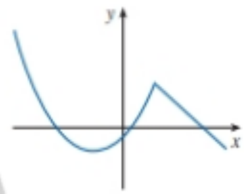
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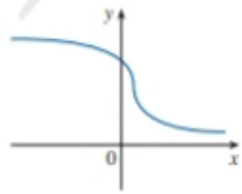
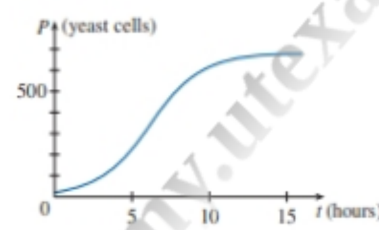
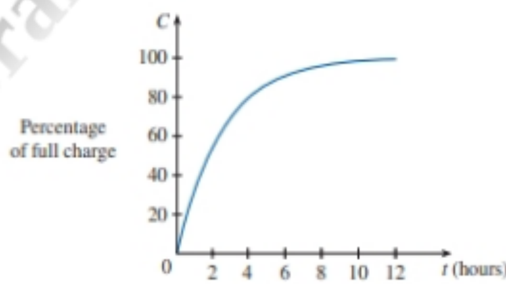


12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).
(a) What is the meaning of the derivative $C'(t)$?
(b) Sketch the graph of $C'(t)$. What does the graph tell you?



14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.
(a) What is the meaning of the derivative $F'(v)$?
(b) Sketch the graph of $F'(v)$.

37. The table gives the height as time passes of a typical pine tree grown for lumber at a managed site.

Tree age (years)	14	21	28	35	42	49
Height (feet)	41	54	64	72	78	83

Source: Arkansas Forestry Commission

If $H(t)$ is the height of the tree after t years, construct a table of estimated values for H' and sketch its graph.

38. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (°C)	15.5	17.7	20.0	22.4	24.4
Weight gained (g)	37.2	31.0	19.8	9.7	−9.8

If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W' and sketch its graph. What are the units for $W'(x)$?

Source: Adapted from J. Chadwick Jr., "Temperature Effects on Growth and Stress Physiology of Brook Trout: Implications for Climate Change Impacts on an Iconic Cold-Water Fish." *Masters Theses*. Paper 897. 2012. scholarworks.umass.edu/theses/897.

39. Let P represent the percentage of a city's electrical power that is produced by solar panels t years after January 1, 2020.

(a) What does dP/dt represent in this context?

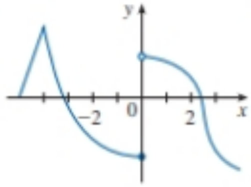
(b) Interpret the statement

$$\left.\frac{dP}{dt}\right|_{t=2} = 3.5$$

40. Suppose N is the number of people in the United States who travel by car to another state for a vacation in a year when the average price of gasoline is p dollars per gallon. Do you expect dN/dp to be positive or negative? Explain.

41–44 The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

41.



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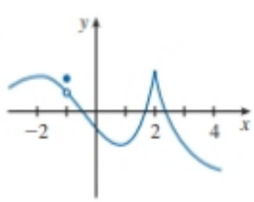

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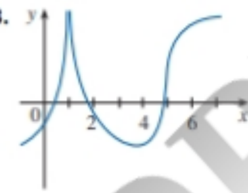

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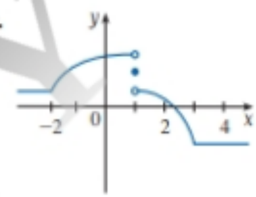
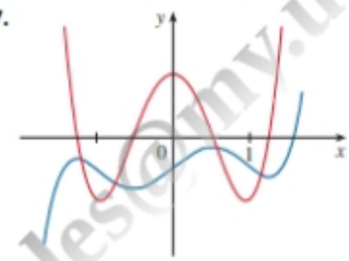


45. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

46. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

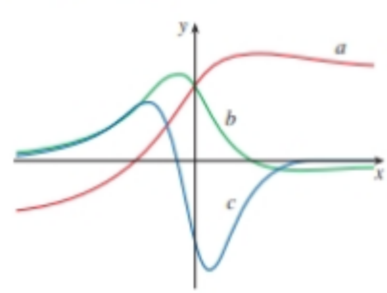
47–48 The graphs of a function f and its derivative f' are shown. Which is bigger, $f'(-1)$ or $f''(1)$?

47.



48.

49. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ at a point (x, e^x) is equal to the y-coordinate of the point (see Figure 7).

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

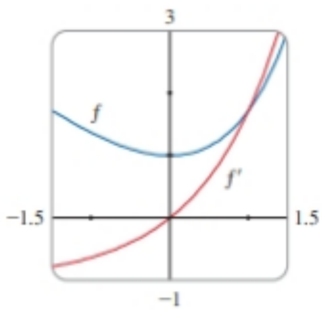


FIGURE 8

In Section 2.8 we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

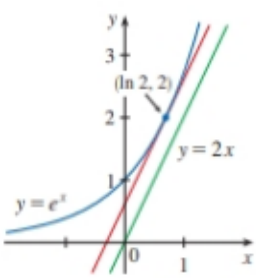


FIGURE 9

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x-coordinate of the point in question be a. Then the slope of the tangent line at that point is e^a. This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \qquad a = \ln 2$$

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.) ■

3.1 Exercises

1. (a) How is the number e defined?
(b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What is the slope of the tangent line at that point?
(b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
(c) Which of the two functions in part (b) grows more rapidly when x is large?

3–34 Differentiate the function.

3. $g(x) = 4x + 7$

4. $g(t) = 5t + 4t^2$

5. $f(x) = x^{75} - x + 3$

6. $g(x) = \frac{7}{4}x^2 - 3x + 12$

7. $f(t) = -2e^t$

8. $F(t) = t^3 + e^3$

9. $W(v) = 1.8v^{-3}$

10. $r(z) = z^{-5} - z^{1/2}$

11. $f(x) = x^{3/2} + x^{-3}$

12. $V(t) = t^{-3/5} + t^4$

13. $s(t) = \dfrac{1}{t} + \dfrac{1}{t^2}$

14. $r(t) = \dfrac{a}{t^2} + \dfrac{b}{t^4}$

15. $y = 2x + \sqrt{x}$

16. $h(w) = \sqrt{2}\,w - \sqrt{2}$

17. $g(x) = \dfrac{1}{\sqrt{x}} + \sqrt[4]{x}$

18. $W(t) = \sqrt{t} - 2e^t$

19. $f(x) = x^3(x + 3)$

20. $F(t) = (2t - 3)^2$

21. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$

22. $S(R) = 4\pi R^2$

23. $f(x) = \dfrac{3x^2 + x^3}{x}$

24. $y = \dfrac{\sqrt{x} + x}{x^2}$

25. $G(r) = \dfrac{3r^{3/2} + r^{5/2}}{r}$

26. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t}$

27. $j(x) = x^{2.4} + e^{2.4}$

28. $k(r) = e^r + r^e$

29. $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$

30. $G(q) = (1 + q^{-1})^2$

31. $D(t) = \dfrac{1 + 16t^2}{(4t)^3}$

32. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v}$

33. $P(w) = \dfrac{2w^2 - w + 4}{\sqrt{w}}$

34. $y = e^{x+1} + 1$

35–36 Find dy/dx and dy/dt.

35. $y = tx^2 + t^3x$

36. $y = \dfrac{t}{x^2} + \dfrac{x}{t}$

37–40 Find an equation of the tangent line to the curve at the given point.

37. $y = 2x^3 - x^2 + 2, \quad (1, 3)$

38. $y = 2e^x + x, \quad (0, 2)$

39. $y = x + \dfrac{2}{x}, \quad (2, 3)$

40. $y = \sqrt[4]{x} - x, \quad (1, 0)$

41–42 Find equations of the tangent line and normal line to the curve at the given point.

41. $y = x^4 + 2e^x, \quad (0, 2)$

42. $y = x^{3/2}, \quad (1, 1)$

43–44 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

43. $y = 3x^2 - x^3, \quad (1, 2)$

44. $y = x - \sqrt{x}, \quad (1, 0)$

45–46 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

45. $f(x) = x^4 - 2x^3 + x^2$

46. $f(x) = x^5 - 2x^3 + x - 1$

47. (a) Graph the function

$$f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$$

in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 2.8.1.)

(c) Calculate $f'(x)$ and use this expression to graph f'. Compare with your sketch in part (b).

48. (a) Graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 2.8.1.)

(c) Calculate $g'(x)$ and use this expression to graph g'. Compare with your sketch in part (b).

49–50 Find the first and second derivatives of the function.

49. $f(x) = 0.001x^5 - 0.02x^3$

50. $G(r) = \sqrt{r} + \sqrt[3]{r}$

51–52 Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f, f', and f''.

51. $f(x) = 2x - 5x^{3/4}$

52. $f(x) = e^x - x^3$

53. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

(a) the velocity and acceleration as functions of t,

(b) the acceleration after 2 s, and

(c) the acceleration when the velocity is 0.

54. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

(a) Find the velocity and acceleration as functions of t.

(b) Find the acceleration after 1 s.

(c) Graph the position, velocity, and acceleration functions on the same screen.

55. Biologists have proposed a cubic polynomial to model the length L of Alaskan rockfish at age A:

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. Calculate

$$\left.\frac{dL}{dA}\right|_{A=12}$$

and interpret your answer.

56. The number of tree species S in a given area A in a forest reserve has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

where A is measured in square meters. Find $S'(100)$ and interpret your answer.

57. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas.
(a) Suppose that the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa. Write V as a function of P.
(b) Calculate dV/dP when $P = 50$ kPa. What is the meaning of the derivative? What are its units?

T **58.** Car tires need to be inflated properly because overinflation or underinflation can cause premature tread wear. The data in the table show tire life L (in thousands of miles) for a certain type of tire at various pressures P (in lb/in^2).

P	26	28	31	35	38	42	45
L	50	66	78	81	74	70	59

(a) Use a calculator or computer to model tire life with a quadratic function of the pressure.
(b) Use the model to estimate dL/dP when $P = 30$ and when $P = 40$. What is the meaning of the derivative? What are the units? What is the significance of the signs of the derivatives?

59. Find the points on the curve $y = x^3 + 3x^2 - 9x + 10$ where the tangent is horizontal.

60. For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?

61. Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

62. Find an equation of the line that is both tangent to the curve $y = x^4 + 1$ and parallel to the line $32x - y = 15$.

63. Find equations for two lines that are both tangent to the curve $y = x^3 - 3x^2 + 3x - 3$ and parallel to the line $3x - y = 15$.

64. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

65. Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

66. Where does the normal line to the parabola $y = x^2 - 1$ at the point $(-1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

67. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

68. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

69. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)

70. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
(a) $f(x) = x^n$ (b) $f(x) = 1/x$

71. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

72. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 9.)

73. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

74. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.

75. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$$

Is f differentiable at 1? Sketch the graphs of f and f'.

76. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geq 2 \end{cases}$$

Give a formula for g' and sketch the graphs of g and g'.

77. (a) For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f'.
(b) Sketch the graphs of f and f'.

78. Where is the function $h(x) = |x - 1| + |x + 2|$ differentiable? Give a formula for h' and sketch the graphs of h and h'.

79. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

80. Suppose the curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$ and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of a, b, c, and d.

81. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

82. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

83. What is the value of c such that the line $y = 2x + 3$ is tangent to the parabola $y = cx^2$?

84. The graph of any quadratic function $f(x) = ax^2 + bx + c$ is a parabola. Prove that the average of the slopes of the tangent lines to the parabola at the endpoints of any interval $[p, q]$ equals the slope of the tangent line at the midpoint of the interval.