23. Show that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

24. Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

25. Show that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

26. Prove, using the definition of a derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

27–30 Find an equation of the tangent line to the curve at the given point.

27. $y = \sin x + \cos x$, $(0, 1)$

28. $y = x + \sin x$, (π, π)

29. $y = e^x \cos x + \sin x$, $(0, 1)$

30. $y = \dfrac{1 + \sin x}{\cos x}$, $(\pi, -1)$

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31. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

32. (a) Find an equation of the tangent line to the curve $y = 3x + 6 \cos x$ at the point $(\pi/3, \pi + 3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

33. (a) If $f(x) = \sec x - x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.

34. (a) If $f(x) = e^x \cos x$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by graphing f, f', and f''.

35. If $g(\theta) = \dfrac{\sin \theta}{\theta}$, find $g'(\theta)$ and $g''(\theta)$.

36. If $f(t) = \sec t$, find $f''(\pi/4)$.

37. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.
(c) Show that your answers to parts (a) and (b) are equivalent.

38. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

39–40 For what values of x does the graph of f have a horizontal tangent?

39. $f(x) = x + 2 \sin x$ **40.** $f(x) = e^x \cos x$

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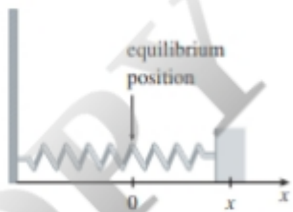
41. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
(a) Find the velocity and acceleration at time t.
(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



42. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)
(a) Find the velocity and acceleration at time t.
(b) Graph the velocity and acceleration functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

43. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

44. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.
(a) Find the rate of change of F with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

45–60 Find the limit.

45. $\displaystyle\lim_{x \to 0} \frac{\sin 5x}{3x}$ **46.** $\displaystyle\lim_{x \to 0} \frac{\sin x}{\sin \pi x}$

47. $\displaystyle\lim_{t \to 0} \frac{\sin 3t}{\sin t}$ **48.** $\displaystyle\lim_{x \to 0} \frac{\sin^2 3x}{x}$

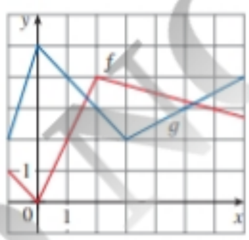
69. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

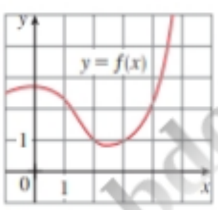
(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

70. Let f and g be the functions in Exercise 69.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

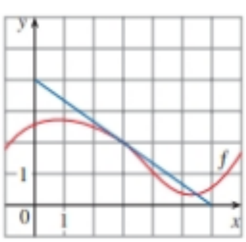
71. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



72. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
(a) $h'(2)$ (b) $g'(2)$



73. If $g(x) = \sqrt{f(x)}$, where the graph of f is shown, evaluate $g'(3)$.



74. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

75. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

76. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

77. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

78. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

79. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

80. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

81. Show that the function $y = e^{2x}(A\cos 3x + B\sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

82. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?

83. Find the 50th derivative of $y = \cos 2x$.

84. Find the 1000th derivative of $f(x) = xe^{-x}$.

85. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \tfrac{1}{4}\sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

86. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

87. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35. In view of these data, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35\sin\left(\frac{2\pi t}{5.4}\right)$$

(a) Find the rate of change of the brightness after t days.
(b) Find, correct to two decimal places, the rate of increase after one day.

25–26 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

25. $x^4y^2 - x^3y + 2xy^3 = 0$ **26.** $y \sec x = x \tan y$

27–36 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

27. $ye^{\sin x} = x \cos y$, $(0, 0)$

28. $\tan(x + y) + \sec(x - y) = 2$, $(\pi/8, \pi/8)$

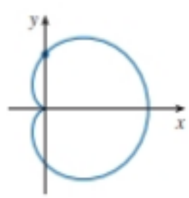
29. $x^{2/3} + y^{2/3} = 4$, $(-3\sqrt{3}, 1)$ (astroid)

30. $y^2(6 - x) = x^3$, $(2, \sqrt{2})$ (cissoid of Diocles)

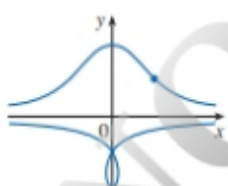
31. $x^2 - xy - y^2 = 1$, $(2, 1)$ (hyperbola)

32. $x^2 + 2xy + 4y^2 = 12$, $(2, 1)$ (ellipse)

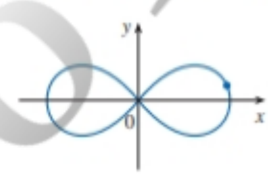
33. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $(0, \frac{1}{2})$ (cardioid)



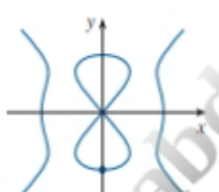
34. $x^2y^2 = (y + 1)^2(4 - y^2)$, $(2\sqrt{3}, 1)$ (conchoid of Nicomedes)



35. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$ (lemniscate)



36. $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$ (devil's curve)



37. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (Graph the implicitly defined curve if possible, or you can graph the upper and lower halves separately.)

38. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

39–42 Find y'' by implicit differentiation. Simplify where possible.

39. $x^2 + 4y^2 = 4$ **40.** $x^2 + xy + y^2 = 3$

41. $\sin y + \cos x = 1$ **42.** $x^3 - y^3 = 7$

43. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

44. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

45. Fanciful shapes can be created by using software that can graph implicitly defined curves.

(a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.

(b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.

(c) Find the exact x-coordinates of the points in part (a).

(d) Create even more fanciful curves by modifying the equation in part (a).

46. (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Graph this curve and discover why.

(b) At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.

47. Find the points on the lemniscate in Exercise 35 where the tangent is horizontal.

48. Show by implicit differentiation that the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has equation

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

49. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

50. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

51. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

52. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

53–56 Orthogonal Trajectories Two curves are *orthogonal* if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are *orthogonal trajectories* of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

53. $x^2 + y^2 = r^2, \quad ax + by = 0$

54. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

55. $y = cx^2, \quad x^2 + 2y^2 = k$

56. $y = ax^3, \quad x^2 + 3y^2 = b$

57. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

58. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

59. The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas.

(a) If T remains constant, use implicit differentiation to find dV/dP.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592\ \text{L}^2\text{-atm/mole}^2$ and $b = 0.04267$ L/mole.

60. (a) Use implicit differentiation to find y' if

$$x^2 + xy + y^2 + 1 = 0$$

(b) Plot the curve in part (a). What do you see? Prove that what you see is correct.

(c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

61. The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

62. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Illustrate part (a) by graphing the ellipse and the normal line.

63. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

64. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

65. Use implicit differentiation to find dy/dx for the equation

$$\frac{x}{y} = y^2 + 1 \qquad y \neq 0$$

and for the equivalent equation

$$x = y^3 + y \qquad y \neq 0$$

Show that although the expressions you get for dy/dx look different, they agree for all points that satisfy the given equation.

66. The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

67. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?