M 408C - Differential and Integral Calculus

Week 7 - 3.5, 3.6, 3.8, 3.9

Quest HW 07 - Due Monday at 11:30p.

Gradescope HW 07 - Due Wednesday at 11:30p on Gradescope

§3.5, #64

§3.6, #85, 86

§3.9, #29, 46, 47, 48

Additional Questions:

#1) Find the x- and y-intercepts of the line tangent to $f(x) = |\ln(0.5x + 0.5e^{-1})|$ when $x = e^{-1}$.

#2) Find equation of the line tangent to $y^3 - 4xy = 8$ that passes through the point $(-3, 0)$. Write your answer in slope-intercept form ($y = mx + b$).

#3) A population grows according to the differential equation $P'(t) = 0.7P(t)$. If the initial population is $P(0) = 10,000$, find the formula for $P(t)$.

Additional Thing: This is usually the point in the semester where students start to feel run down. It's much easier to be successful in the classroom when we feel our best. Here are my 4 tips to feeling better each day:

1) Drink more water. Giant energy drinks and giant coffee drinks feel good, and might be ok in moderation, but nothing is better for yourself than staying hydrated with water.

2) Eat real food. I love pizza, chicken strips, and a coke as much as the everyone else but it's not going to do much to help you feel better. Try eating more healthy foods like vegetables.

3) Sleep more. It's not always easy to find time to get enough sleep, but you should make every effort to fall asleep at a decent time. Its amazing how consistently getting a good nights sleep makes us feel so much better.

4) Exercise a little. You don't need to be training for a marathon but you should get up and move around every day. There are plenty of things to do around campus but I want to focus on what you can do on campus. Everyone is a member of UT RecSports which gives you access to 7 facilities. You can lift weights, walk around the track, play basketball, take a yoga class, take a spin class, swim, use the climbing wall, etc etc. There's something for everyone.

As a graduate student at UT I joined an intramural softball team we one year we **won 2nd place**. This is memorialized in Gregory Gym where you can find a picture of me and our softball team. Look for the 2010-2011 Intramural Champion Frame, Graduate Co-Ed Softball Runner-Up picture. Take a selfie with this picture for an additional 20 points on your HW. Since you are already at Gregory Gym maybe you can find 30 minutes to do something active!

FYI - As a graduate student I much preferred the Rec Center and Bellmont Hall to Gregory Gym. Find a friend and go explore those less-crowded spaces!

49. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

50. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

51. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

52. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

53–56 Orthogonal Trajectories Two curves are *orthogonal* if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are *orthogonal trajectories* of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

53. $x^2 + y^2 = r^2, \quad ax + by = 0$

54. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

55. $y = cx^2, \quad x^2 + 2y^2 = k$

56. $y = ax^3, \quad x^2 + 3y^2 = b$

57. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

58. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

59. The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas.

(a) If T remains constant, use implicit differentiation to find dV/dP.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592\ \text{L}^2\text{-atm/mole}^2$ and $b = 0.04267$ L/mole.

60. (a) Use implicit differentiation to find y' if

$$x^2 + xy + y^2 + 1 = 0$$

(b) Plot the curve in part (a). What do you see? Prove that what you see is correct.

(c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

61. The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

62. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Illustrate part (a) by graphing the ellipse and the normal line.

63. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

64. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

65. Use implicit differentiation to find dy/dx for the equation

$$\frac{x}{y} = y^2 + 1 \qquad y \neq 0$$

and for the equivalent equation

$$x = y^3 + y \qquad y \neq 0$$

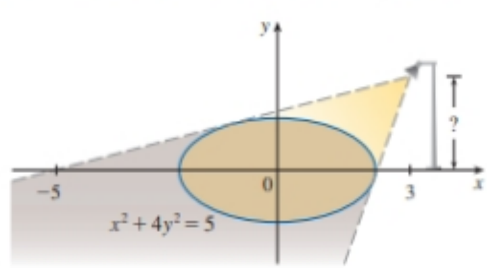
Show that although the expressions you get for dy/dx look different, they agree for all points that satisfy the given equation.

66. The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

67. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



73. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

74. $R(t) = \arcsin(1/t)$

75. $y = x\sin^{-1}x + \sqrt{1 - x^2}$

76. $y = \cos^{-1}(\sin^{-1}t)$

77. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}}$

78. $y = \arctan\sqrt{\dfrac{1-x}{1+x}}$

79–80 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

79. $f(x) = \sqrt{1 - x^2}\arcsin x$ **80.** $f(x) = \arctan(x^2 - x)$

81. Prove the formula for $(d/dx)(\cos^{-1}x)$ by the same method as for $(d/dx)(\sin^{-1}x)$.

82. (a) One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \le y < \pi/2$ or $\pi \le y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

(b) Another way of defining $\sec^{-1}x$ that is sometimes used is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \le y \le \pi$, $y \ne \pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

83. Derivatives of Inverse Functions Suppose that f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

84–86 Use the formula in Exercise 83.

84. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

85. If $f(x) = x + e^x$, find $(f^{-1})'(1)$.

86. If $f(x) = x^3 + 3\sin x + 2\cos x$, find $(f^{-1})'(2)$.

87. Suppose that f and g are differentiable functions and let $h(x) = f(x)^{g(x)}$. Use logarithmic differentiation to derive the formula

$$h' = g \cdot f^{g-1} \cdot f' + (\ln f) \cdot f^g \cdot g'$$

88. Use the formula in Exercise 87 to find the derivative.
(a) $h(x) = x^3$ (b) $h(x) = 3^x$ (c) $h(x) = (\sin x)^x$

3.7 Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

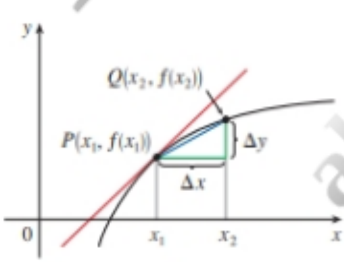
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is **the average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 1. Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line at $P(x_1, f(x_1))$. Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}$$



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

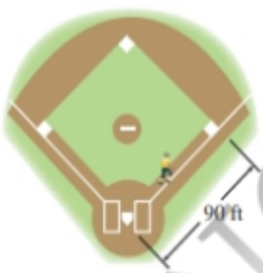
FIGURE 1

17. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

18. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

19. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

20. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
(a) At what rate is his distance from second base decreasing when he is halfway to first base?
(b) At what rate is his distance from third base increasing at the same moment?



21. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

22. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

23–24 Use the fact that the distance (in meters) a dropped stone falls after t seconds is $d = 4.9t^2$.

23. A woman stands near the edge of a cliff and drops a stone over the edge. Exactly one second later she drops another stone. One second after that, how fast is the distance between the two stones changing?

24. Two men stand 10 m apart on level ground near the edge of a cliff. One man drops a stone and one second later the other man drops a stone. One second after that, how fast is the distance between the two stones changing?

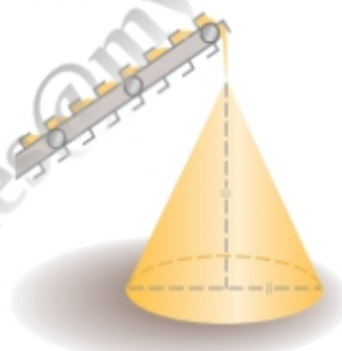
25. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

26. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $(\frac{1}{3}, 1)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

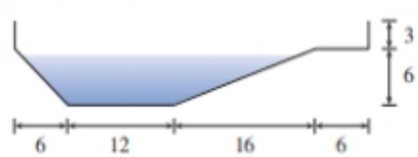
27. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

28. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?

29. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



30. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft^3/min, how fast is the water level rising when the depth at the deepest point is 5 ft?



31. The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long?

32. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

33. A car is traveling north on a straight road at 20 m/s and a drone is flying east at 6 m/s at an elevation of 25 m. At one instant the drone passes directly over the car. How fast is the distance between the drone and the car changing 5 seconds later?

34. If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

35. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 6 ft from the wall?

36. According to the model we used to solve Example 2, what happens as the top of the ladder approaches the ground? Is the model appropriate for small values of y?

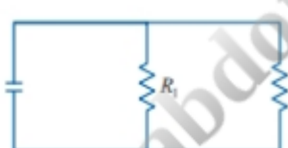
37. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

38. A faucet is filling a hemispherical basin of diameter 60 cm with water at a rate of 2 L/min. Find the rate at which the water is rising in the basin when it is half full. [Use the following facts: 1 L is 1000 cm^3. The volume of the portion of a sphere with radius r from the bottom to a height h is $V = \pi\left(rh^2 - \frac{1}{3}h^3\right)$, as we will show in Chapter 6.]

39. If two resistors with resistances R_1 and R_2 are connected in parallel, as shown in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of 0.3 Ω/s and 0.2 Ω/s, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



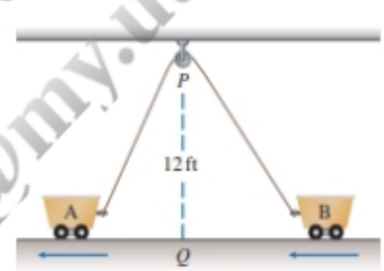
40. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

41. Two straight roads diverge from an intersection at an angle of 60°. Two cars leave the intersection at the same time, the first traveling down one road at 40 mi/h and the second traveling down the other road at 60 mi/h. How fast is the distance between the cars changing after half an hour? [*Hint:* Use the Law of Cosines (Formula 21 in Appendix D).]

42. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when its average length was 18 cm?

43. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°? [*Hint:* Use the Law of Cosines (Formula 21 in Appendix D).]

44. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. (See the figure.) The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



45. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

46. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

47. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?

48. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

49. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

50. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

51. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

52. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

53. Suppose that the volume V of a rolling snowball increases so that dV/dt is proportional to the surface area of the snowball at time t. Show that the radius r increases at a constant rate, that is, dr/dt is constant.

3.10 Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2.7.2.) This observation is the basis for a method of finding approximate values of functions.

Linearization and Approximation

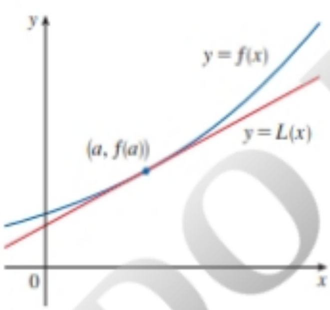


FIGURE 1

It might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

The linear function whose graph is this tangent line, that is,

1 $$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

is called the **linearization** of f at a. The approximation $f(x) \approx L(x)$ or

2 $$\boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

is called the **linear approximation** or **tangent line approximation** of f at a.

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 1, we see