EXAMPLE 6 The position of a particle at time t is given by the parametric equations $x = 2t + 3$, $y = 4t^2$, $t \geqslant 0$. Find the speed of the particle when it is at the point $(5, 4)$.

SOLUTION By Equation 8, the speed of the particle at any time t is

$$v(t) = \sqrt{2^2 + (8t)^2} = 2\sqrt{1 + 16t^2}$$

The particle is at the point $(5, 4)$ when $t = 1$, so its speed at that point is $v(1) = 2\sqrt{17} \approx 8.25$. (If distance is measured in meters and time in seconds, then the speed is approximately 8.25 m/s.) ■

■ Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f', g' are continuous, $g(t) \geqslant 0$, and C is traversed exactly once as t increases from α to β. If C is rotated about the x-axis, then the area of the resulting surface is given by

$$\boxed{9} \qquad S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, where ds is given by Formula 6.

EXAMPLE 7 Show that the surface area of a sphere of radius r is $4\pi r^2$.

SOLUTION The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \leqslant t \leqslant \pi$$

about the x-axis. Therefore, from Formula 9, we get

$$\begin{aligned} S &= \int_0^\pi 2\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi\int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi\int_0^\pi r\sin t \cdot r\, dt \\ &= 2\pi r^2\int_0^\pi \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^\pi = 4\pi r^2 \end{aligned}$$ ■

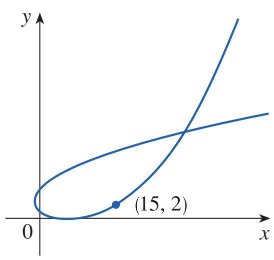
10.2 Exercises

1–4 Find dx/dt, dy/dt, and dy/dx.

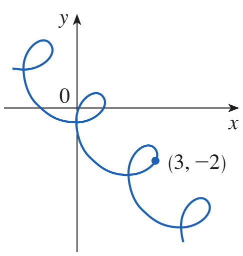
1. $x = 2t^3 + 3t, \quad y = 4t - 5t^2$
2. $x = t - \ln t, \quad y = t^2 - t^{-2}$
3. $x = te^t, \quad y = t + \sin t$
4. $x = t + \sin(t^2 + 2), \quad y = \tan(t^2 + 2)$

5–6 Find the slope of the tangent to the parametric curve at the indicated point.

5. $x = t^2 + 2t, \quad y = 2^t - 2t$



6. $x = t + \cos \pi t, \quad y = -t + \sin \pi t$



7–10 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

7. $x = t^3 + 1, \quad y = t^4 + t; \quad t = -1$

8. $x = \sqrt{t}, \quad y = t^2 - 2t; \quad t = 4$

9. $x = \sin 2t + \cos t, \quad y = \cos 2t - \sin t; \quad t = \pi$

10. $x = e^t \sin \pi t, \quad y = e^{2t}; \quad t = 0$

11–12 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

11. $x = \sin t, \quad y = \cos^2 t; \quad \left(\frac{1}{2}, \frac{3}{4}\right)$

12. $x = \sqrt{t + 4}, \quad y = 1/(t + 4); \quad \left(2, \frac{1}{4}\right)$

13–14 Find an equation of the tangent to the curve at the given point. Then graph the curve and the tangent.

13. $x = t^2 - t, \quad y = t^2 + t + 1; \quad (0, 3)$

14. $x = \sin \pi t, \quad y = t^2 + t; \quad (0, 2)$

15–20 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

15. $x = t^2 + 1, \quad y = t^2 + t$

16. $x = t^3 + 1, \quad y = t^2 - t$

17. $x = e^t, \quad y = te^{-t}$

18. $x = t^2 + 1, \quad y = e^t - 1$

19. $x = t - \ln t, \quad y = t + \ln t$

20. $x = \cos t, \quad y = \sin 2t, \quad 0 < t < \pi$

21–24 Find the points on the curve where the tangent is horizontal or vertical. You may want to use a graph from a calculator or computer to check your work.

21. $x = t^3 - 3t, \quad y = t^2 - 3$

22. $x = t^3 - 3t, \quad y = t^3 - 3t^2$

23. $x = \cos \theta, \quad y = \cos 3\theta$

24. $x = e^{\sin\theta}, \quad y = e^{\cos\theta}$

25. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.

26. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$. Then find the exact coordinates.

27–28 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

27. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

28. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

29. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Graph the curve.

30. Graph the curve $x = -2 \cos t$, $y = \sin t + \sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

31. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d \sin \theta$, $y = r - d \cos \theta$ in terms of θ. (See Exercise 10.1.49.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

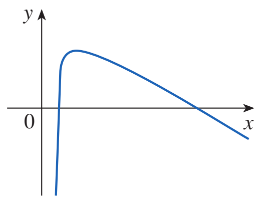
32. (a) Find the slope of the tangent to the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$ in terms of θ. (Astroids are explored in the Discovery Project following Section 10.1.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

33. At what point(s) on the curve $x = 3t^2 + 1$, $y = t^3 - 1$ does the tangent line have slope $\frac{1}{2}$?

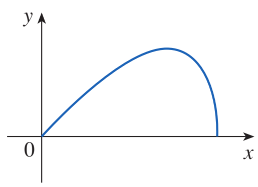
34. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

35–36 Find the area enclosed by the given parametric curve and the x-axis.

35. $x = t^3 + 1, \quad y = 2t - t^2$

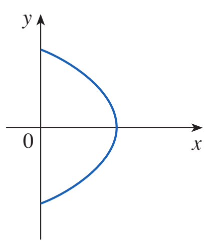


36. $x = \sin t, \quad y = \sin t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

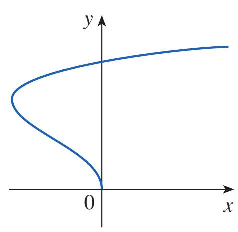


37–38 Find the area enclosed by the given parametric curve and the y-axis.

37. $x = \sin^2 t,$
$y = \cos t$



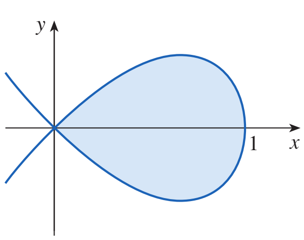
38. $x = t^2 - 2t,$
$y = \sqrt{t}$



39. Use the parametric equations of an ellipse, $x = a \cos \theta$, $y = b \sin \theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

40. Find the area of the region enclosed by the loop of the curve

$$x = 1 - t^2, \quad y = t - t^3$$

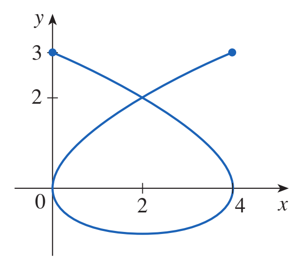


41. Find the area under one arch of the trochoid of Exercise 10.1.49 for the case $d < r$.

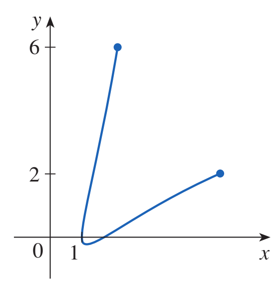
42. Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.
(a) Find the area of $\mathcal{R}$.
(b) If $\mathcal{R}$ is rotated about the x-axis, find the volume of the resulting solid.
(c) Find the centroid of $\mathcal{R}$.

T **43–46** Set up an integral that represents the length of the part of the parametric curve shown in the graph. Then use a calculator (or computer) to find the length correct to four decimal places.

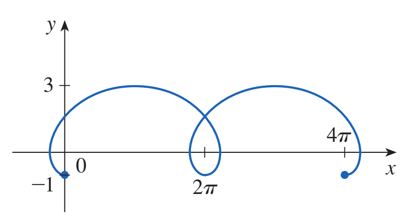
43. $x = 3t^2 - t^3, \quad y = t^2 - 2t$



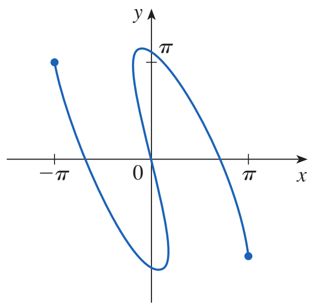
44. $x = t + e^{-t}, \quad y = t^2 + t$



45. $x = t - 2 \sin t, \quad y = 1 - 2 \cos t, \quad 0 \leqslant t \leqslant 4\pi$



46. $x = t\cos t, \quad y = t - 5\sin t$



47–50 Find the exact length of the curve.

47. $x = \frac{2}{3}t^3, \quad y = t^2 - 2, \quad 0 \leqslant t \leqslant 3$

48. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 2$

49. $x = t\sin t, \quad y = t\cos t, \quad 0 \leqslant t \leqslant 1$

50. $x = 3\cos t - \cos 3t, \quad y = 3\sin t - \sin 3t, \quad 0 \leqslant t \leqslant \pi$

51–52 Graph the curve and find its exact length.

51. $x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \leqslant t \leqslant \pi$

52. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \leqslant t \leqslant 3\pi/4$

53. Graph the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ and find its length correct to four decimal places.

54. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

55–56 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

55. $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

56. $x = \cos^2 t, \quad y = \cos t, \quad 0 \leqslant t \leqslant 4\pi$

57–60 The parametric equations give the position (in meters) of a moving particle at time t (in seconds). Find the speed of the particle at the indicated time or point.

57. $x = 2t - 3, \quad y = 2t^2 - 3t + 6; \quad t = 5$

58. $x = 2 + 5\cos\left(\frac{\pi}{3}t\right), \quad y = -2 + 7\sin\left(\frac{\pi}{3}t\right); \quad t = 3$

59. $x = e^t, \quad y = te^t; \quad (e, e)$

60. $x = t^2 + 1, \quad y = t^4 + 2t^2 + 1; \quad (2, 4)$

61. A projectile is fired from the point $(0, 0)$ with an initial velocity of v_0 m/s at an angle α above the horizontal. (See Exercise 10.1.58.) If we assume that air resistance is negligible, then the position (in meters) of the projectile after t seconds is given by the parametric equations

$$x = (v_0\cos\alpha)t \qquad y = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity.

(a) Find the speed of the projectile when it hits the ground.

(b) Find the speed of the projectile at its highest point.

62. Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$$

where e is the eccentricity of the ellipse $(e = c/a$, where $c = \sqrt{a^2 - b^2}\,)$.

63. (a) Graph the **epitrochoid** with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

What parameter interval gives the complete curve?

(b) Use a calculator or computer to find the approximate length of this curve.

64. A curve called **Cornu's spiral** is defined by the parametric equations

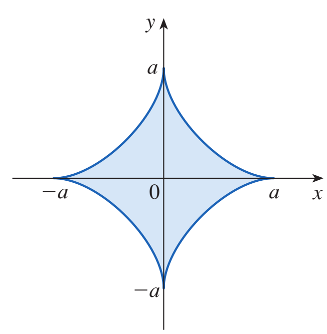
$$x = C(t) = \int_0^t \cos(\pi u^2/2)\,du$$
$$y = S(t) = \int_0^t \sin(\pi u^2/2)\,du$$

where C and S are the Fresnel functions that were introduced in Chapter 5.

(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?

(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

65–66 The curve shown in the figure is the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (Astroids are explored in the Discovery Project following Section 10.1.)



65. Find the area of the region enclosed by the astroid.

66. Find the perimeter of the astroid.

67–70 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use a calculator or computer to find the surface area correct to four decimal places.

67. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

68. $x = \sin t, \quad y = \sin 2t, \quad 0 \leqslant t \leqslant \pi/2$

69. $x = t + e^t, \quad y = e^{-t}, \quad 0 \leqslant t \leqslant 1$

70. $x = t^2 - t^3, \quad y = t + t^4, \quad 0 \leqslant t \leqslant 1$

71–73 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

71. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

72. $x = 2t^2 + 1/t, \quad y = 8\sqrt{t}, \quad 1 \leqslant t \leqslant 3$

73. $x = a \cos^3\theta, \quad y = a \sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

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74. Graph the curve

$$x = 2 \cos\theta - \cos 2\theta$$
$$y = 2 \sin\theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the exact area of the resulting surface. (Use your graph to help find the correct parameter interval.)

75–76 Find the surface area generated by rotating the given curve about the y-axis.

75. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 5$

76. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 1$

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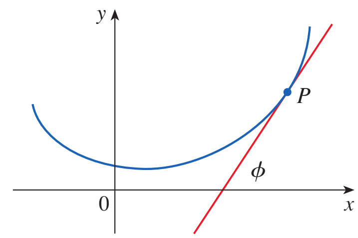
77. If f' is continuous and $f'(t) \neq 0$ for $a \leqslant t \leqslant b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, can be put in the form $y = F(x)$. [*Hint:* Show that f^{-1} exists.]

78. Use Formula 1 to derive Formula 9 from Formula 8.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

79–83 Curvature The *curvature* at a point P of a curve is defined as

$$\kappa = \left| \frac{d\phi}{ds} \right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.



79. For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

80. By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$ with parameter x, show that the formula in Exercise 79 becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$

81. Use the formula in Exercise 79 to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

82. (a) Use the formula in Exercise 80 to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.
(b) At what point does this parabola have maximum curvature?

83. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

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84. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the grazing area available for the cow.