Thus, from Equation 5, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Since $y(0) = 20$, we have $-\ln 130 = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Therefore $$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is continuous and $y(0) = 20$, and the right side is never 0, we deduce that $150 - y(t)$ is always positive. Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$ ■

Figure 10 shows the graph of the function $y(t)$ of Example 6. Notice that, as time goes by, the amount of salt approaches 150 kg.

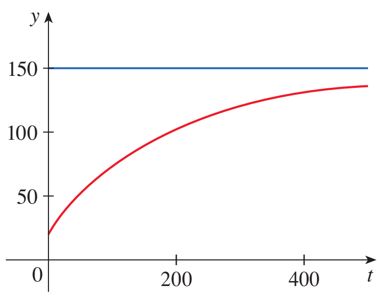


FIGURE 10

9.3 Exercises

1–12 Solve the differential equation.

1. $\dfrac{dy}{dx} = 3x^2y^2$

2. $\dfrac{dy}{dx} = \dfrac{x}{y^4}$

3. $\dfrac{dy}{dx} = x\sqrt{y}$

4. $xy' = y + 3$

5. $xyy' = x^2 + 1$

6. $y' + xe^y = 0$

7. $(e^y - 1)y' = 2 + \cos x$

8. $\dfrac{dy}{dx} = 2x(y^2 + 1)$

9. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1$

10. $\dfrac{dz}{dt} + e^{t+z} = 0$

11. $\dfrac{d\theta}{dt} = \dfrac{t \sec\theta}{\theta e^{t^2}}$

12. $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1 + R^2}}{\ln H}$

13–20 Find the solution of the differential equation that satisfies the given initial condition.

13. $\dfrac{dy}{dx} = xe^y, \quad y(0) = 0$

14. $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

15. $\dfrac{dA}{dr} = Ab^2 \cos br, \quad A(0) = b^3$

16. $x^2y' = k \sec y, \quad y(1) = \pi/6$

17. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

18. $x + 3y^2\sqrt{x^2 + 1}\,\dfrac{dy}{dx} = 0, \quad y(0) = 1$

19. $x \ln x = y\left(1 + \sqrt{3 + y^2}\right) y', \quad y(1) = 1$

20. $\dfrac{dy}{dx} = \dfrac{x \sin x}{y}, \quad y(0) = -1$

21. Find an equation of the curve that passes through the point $(0, 2)$ and whose slope at (x, y) is x/y.

22. Find the function f such that $f'(x) = xf(x) - x$ and $f(0) = 2$.

23. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$.

24. Solve the differential equation $xy' = y + xe^{y/x}$ by making the change of variable $v = y/x$.

25. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.

(b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.

(c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain.

26. Solve the differential equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant C varies?

27. Solve the initial-value problem $y' = (\sin x)/\sin y$, $y(0) = \pi/2$, and graph the (implicitly defined) solution.

28. Solve the differential equation $y' = x\sqrt{x^2 + 1}/(ye^y)$ and graph several members of the family of (implicitly defined) solutions. How does the solution curve change as the constant C varies?

29–30

(a) Use a computer to draw a direction field for the differential equation. Get a printout and use it to sketch some solution curves without solving the differential equation.
(b) Solve the differential equation.
(c) Graph several members of the family of solutions obtained in part (b). Compare with the curves from part (a).

29. $y' = y^2$ **30.** $y' = xy$

31–34 Find the orthogonal trajectories of the family of curves. Graph several members of each family on a common screen.

31. $x^2 + 2y^2 = k^2$ **32.** $y^2 = kx^3$

33. $y = \dfrac{k}{x}$ **34.** $y = \dfrac{1}{x + k}$

35–37 Integral Equations An *integral equation* is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation. [*Hint:* Use an initial condition obtained from the integral equation.]

35. $y(x) = 2 + \displaystyle\int_2^x [t - ty(t)]\,dt$

36. $y(x) = 2 + \displaystyle\int_1^x \frac{dt}{ty(t)}, \quad x > 0$

37. $y(x) = 4 + \displaystyle\int_0^x 2t\sqrt{y(t)}\,dt$

38. Find a function f such that $f(3) = 2$ and

$$(t^2 + 1)f'(t) + [f(t)]^2 + 1 = 0 \qquad t \neq 1$$

[*Hint:* Use the addition formula for $\tan(x + y)$ on Reference Page 2.]

39. Solve the initial-value problem in Exercise 9.2.27 to find an expression for the charge at time t. Find the limiting value of the charge.

40. In Exercise 9.2.28 we discussed a differential equation that models the temperature of a 95°C cup of coffee in a 20°C room. Solve the differential equation to find an expression for the temperature of the coffee at time t.

41. In Exercise 9.1.27 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

42. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: $A + B \rightarrow C$. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[C]}{dt} = k[A][B]$$

(See Example 3.7.4.) Thus, if the initial concentrations are $[A] = a$ moles/L and $[B] = b$ moles/L and we write $x = [C]$, then we have

$$\frac{dx}{dt} = k(a - x)(b - x)$$

(a) Assuming that $a \neq b$, find x as a function of t. Use the fact that the initial concentration of C is 0.
(b) Find $x(t)$ assuming that $a = b$. How does this expression for $x(t)$ simplify if it is known that $[C] = \frac{1}{2}a$ after 20 seconds?

43. In contrast to the situation of Exercise 42, experiments show that the reaction $H_2 + Br_2 \rightarrow 2HBr$ satisfies the rate law

$$\frac{d[HBr]}{dt} = k[H_2][Br_2]^{1/2}$$

and so for this reaction the differential equation becomes

$$\frac{dx}{dt} = k(a - x)(b - x)^{1/2}$$

where $x = [HBr]$ and a and b are the initial concentrations of hydrogen and bromine.
(a) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$.
(b) If $a > b$, find t as a function of x.
[*Hint:* In performing the integration, make the substitution $u = \sqrt{b - x}$.]

44. A sphere with radius 1 m has temperature 15°C. It lies inside a concentric sphere with radius 2 m and temperature 25°C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

If we let $S = dT/dr$, then S satisfies a first-order differential equation. Solve it to find an expression for the temperature $T(r)$ between the spheres.

45. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < r/k$, find $\lim_{t\to\infty} C(t)$ and interpret your answer.

46. A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t, with $x(0) = 0$.

(a) Formulate a mathematical model in the form of an initial-value problem that represents the "flow" of the new currency into circulation.

(b) Solve the initial-value problem found in part (a).

(c) How long will it take for the new bills to account for 90% of the currency in circulation?

47. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

48. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

49. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

50. A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after one hour?

51. Terminal Velocity When a raindrop falls, it increases in size and so its mass at time t is a function of t, namely, $m(t)$. The rate of growth of the mass is $km(t)$ for some positive constant k. When we apply Newton's Law of Motion to the raindrop, we get $(mv)' = gm$, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The *terminal velocity* of the raindrop is $\lim_{t\to\infty} v(t)$. Find an expression for the terminal velocity in terms of g and k.

52. An object of mass m is moving horizontally through a medium which resists the motion with a force that is a function of the velocity; that is,

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = f(v)$$

where $v = v(t)$ and $s = s(t)$ represent the velocity and position of the object at time t, respectively. For example, think of a boat moving through the water.

(a) Suppose that the resisting force is proportional to the velocity, that is, $f(v) = -kv$, k a positive constant. (This model is appropriate for small values of v.) Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels from time $t = 0$?

(b) For larger values of v a better model is obtained by supposing that the resisting force is proportional to the square of the velocity, that is, $f(v) = -kv^2$, $k > 0$. (This model was first proposed by Newton.) Let v_0 and s_0 be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels in this case?

53. Allometric Growth In biology, *allometric growth* refers to relationships between sizes of parts of an organism (skull length and body length, for instance). If $L_1(t)$ and $L_2(t)$ are the sizes of two organs in an organism of age t, then L_1 and L_2 satisfy an allometric law if their specific growth rates are proportional:

$$\frac{1}{L_1}\frac{dL_1}{dt} = k\frac{1}{L_2}\frac{dL_2}{dt}$$

where k is a constant.

(a) Use the allometric law to write a differential equation relating L_1 and L_2 and solve it to express L_1 as a function of L_2.

(b) In a study of several species of unicellular algae, the proportionality constant in the allometric law relating B (cell biomass) and V (cell volume) was found to be $k = 0.0794$. Write B as a function of V.

54. A model for tumor growth is given by the Gompertz equation

$$\frac{dV}{dt} = a(\ln b - \ln V)V$$

where a and b are positive constants and V is the volume of the tumor measured in mm^3.

(a) Find a family of solutions for tumor volume as a function of time.

(b) Find the solution that has an initial tumor volume of $V(0) = 1 \text{ mm}^3$.

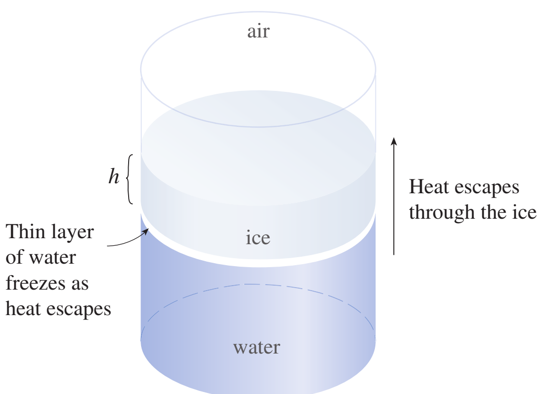
55. Let $A(t)$ be the area of a tissue culture at time t and let M be the final area of the tissue when growth is complete. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A(t)}$. So a reasonable model for the growth of tissue is obtained by assuming that the rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$.

(a) Formulate a differential equation and use it to show that the tissue grows fastest when $A(t) = \frac{1}{3}M$.

T (b) Solve the differential equation to find an expression for $A(t)$. Use a computer to perform the integration.

56. Sea Ice Many factors influence the formation and growth of sea ice. In this exercise we develop a simplified model that describes how the thickness of sea ice is affected over time by the temperatures of the air and ocean water. As we commented in Section 1.2, a good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

Consider a column of air/ice/water as shown in the figure. Let's assume that the temperature T_a (in °C) at the ice/air interface is constant (with T_a below the freezing point of the ocean water) and that the temperature T_w at the ice/water interface also remains constant (where T_w is greater than the water's freezing point).



Energy transfers upward through the ice from the warmer seawater to the colder air in the form of heat Q, measured in joules (J). By Fourier's law of heat conduction, the rate of heat transfer dQ/dt satisfies the differential equation

$$\frac{dQ}{dt} = \frac{kA}{h}(T_w - T_a)$$

where k is a constant called the *thermal conductivity* of the ice, A is the (horizontal) cross-sectional area (in m^2) of the column, and h is the ice thickness (in m).

(a) The loss of a small amount of heat ΔQ from the seawater causes a thin layer of thickness Δh of water at the ice/water interface to freeze. The mass density D (measured in kg/m^3) of seawater varies with temperature, but at the interface we can assume that the temperature is constant (near 0°C) and hence D is constant. Let L be the *latent heat* of seawater, defined as the amount of heat loss required to freeze 1 kg of the water. Show that $\Delta h \approx (1/LAD)\Delta Q$ and hence

$$\frac{dh}{dQ} = \frac{1}{LAD}$$

(b) Use the Chain Rule to write the differential equation

$$\frac{dh}{dt} = \frac{k}{LDh}(T_w - T_a)$$

and explain why this equation predicts the fact that thin ice grows more rapidly than thick ice, and thus a crack in ice tends to "heal" and the thickness of an ice field tends to become uniform over time.

(c) If the thickness of the ice at time $t = 0$ is h_0, find a model for the ice thickness at any time t by solving the differential equation in part (b).

Source: Adapted from M. Freiberger, "Maths and Climate Change: The Melting Arctic," *Plus* (2008): http://plus.maths.org/content/maths-and-climate-change-melting-arctic. Accessed March 9, 2019.

57. Escape Velocity According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m(dv/dt)$ and so

$$m\frac{dv}{dt} = -\frac{mgR^2}{(x + R)^2}$$

(a) Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that

$$v_0 = \sqrt{\frac{2gRh}{R + h}}$$

[*Hint:* By the Chain Rule, $m(dv/dt) = mv(dv/dx)$.]

(b) Calculate $v_e = \lim_{h\to\infty} v_0$. This limit is called the *escape velocity* for the earth. (Another method of finding escape velocity is given in Exercise 7.8.77.)

(c) Use $R = 3960$ mi and $g = 32 \text{ ft/s}^2$ to calculate v_e in feet per second and in miles per second.