Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$M_{yz} = \iiint\limits_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy$$

$$= \frac{2\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \frac{4\rho}{7}$$

$$M_{xy} = \iiint\limits_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2}\right]_{z=0}^{z=x} dx\, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy$$

$$= \frac{\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{7}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14}\right)$$

15.6 Exercises

1. Evaluate the integral in Example 1, integrating first with respect to y, then z, and then x.

2. Evaluate the integral $\iiint_E (xy + z^2)\, dV$, where

$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 3\}$$

using three different orders of integration.

3–8 Evaluate the iterated integral.

3. $\displaystyle\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y)\, dx\, dy\, dz$

4. $\displaystyle\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy\, dz\, dx\, dy$

5. $\displaystyle\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz$

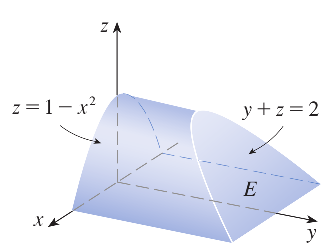
6. $\displaystyle\int_0^{\pi/2} \int_0^{2x} \int_0^{x+z} \cos(x - 2y + z)\, dy\, dz\, dx$

7. $\displaystyle\int_1^3 \int_{-1}^{2} \int_{-y}^{z} \frac{z}{y}\, dx\, dz\, dy$

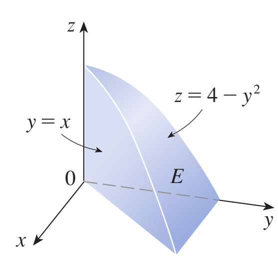
8. $\displaystyle\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z\, dz\, dy\, dx$

9–12
(a) Express the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral for the given function f and solid region E.
(b) Evaluate the iterated integral.

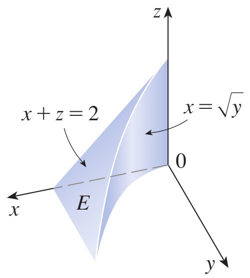
9. $f(x, y, z) = x$



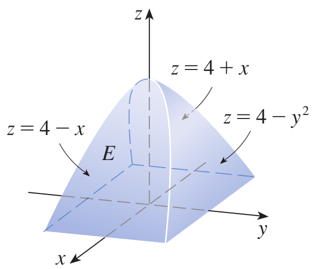
10. $f(x, y, z) = xy$



11. $f(x, y, z) = x + y$



12. $f(x, y, z) = 2$



13–22 Evaluate the triple integral.

13. $\iiint_E y\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant x, x - y \leqslant z \leqslant x + y\}$

14. $\iiint_E e^{z/y}\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1, 0 \leqslant z \leqslant xy\}$

15. $\iiint_E (1/x^3)\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y^2, 1 \leqslant x \leqslant z + 1\}$

16. $\iiint_E \sin y\, dV$, where E lies below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$

17. $\iiint_E 6xy\, dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

18. $\iiint_E (x - y)\, dV$, where E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$

19. $\iiint_T y^2\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

20. $\iiint_T xz\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$

21. $\iiint_E x\, dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

22. $\iiint_E z\, dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

23–26 Use a triple integral to find the volume of the given solid.

23. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

24. The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$

25. The solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$

26. The solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$

27. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.

T (b) Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to find the exact value of the triple integral in part (a).

28–30 Midpoint Rule for Triple Integrals In the *Midpoint Rule for triple integrals* we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate the value of the integral. Divide B into eight sub-boxes of equal size.

28. $\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 4, 0 \leqslant z \leqslant 4\}$

29. $\iiint_B \cos(xyz)\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1\}$

30. $\iiint_B \sqrt{x}\, e^{xyz}\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 2\}$

31–32 Sketch the solid whose volume is given by the iterated integral.

31. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$

32. $\displaystyle\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$

33–36 Express the integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces.

33. $y = 4 - x^2 - 4z^2$, $\quad y = 0$

34. $y^2 + z^2 = 9$, $\quad x = -2$, $\quad x = 2$

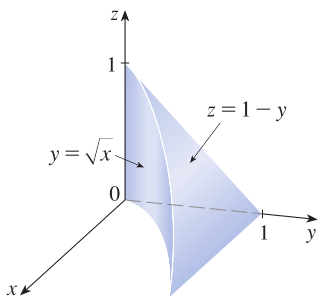
35. $y = x^2$, $\quad z = 0$, $\quad y + 2z = 4$

36. $x = 2$, $\quad y = 2$, $\quad z = 0$, $\quad x + y - 2z = 2$

37. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

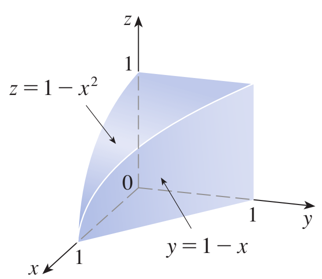
Rewrite this integral as an equivalent iterated integral in the five other orders.



38. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



39–40 Write five other iterated integrals that are equal to the given iterated integral.

39. $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$ **40.** $\int_0^1 \int_y^1 \int_0^z f(x, y, z)\, dx\, dz\, dy$

41–42 Evaluate the triple integral using only geometric interpretation and symmetry.

41. $\iiint_C (4 + 5x^2yz^2)\, dV$, where C is the cylindrical region $x^2 + y^2 \leqslant 4$, $-2 \leqslant z \leqslant 2$

42. $\iiint_B (z^3 + \sin y + 3)\, dV$, where B is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

43–46 Find the mass and center of mass of the solid E with the given density function ρ.

43. E lies above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$; $\rho(x, y, z) = 3$

44. E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$; $\rho(x, y, z) = 4$

45. E is the cube given by $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$; $\rho(x, y, z) = x^2 + y^2 + z^2$

46. E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$; $\rho(x, y, z) = y$

47–50 Assume that the solid has constant density k.

47. Find the moments of inertia for a cube with side length L if one vertex is located at the origin and three edges lie along the coordinate axes.

48. Find the moments of inertia for a rectangular brick with dimensions a, b, and c and mass M if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes.

49. Find the moment of inertia about the z-axis of the solid cylinder $x^2 + y^2 \leqslant a^2$, $0 \leqslant z \leqslant h$.

50. Find the moment of inertia about the z-axis of the solid cone $\sqrt{x^2 + y^2} \leqslant z \leqslant h$.

51–52 Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

51. The solid of Exercise 25; $\rho(x, y, z) = \sqrt{x^2 + y^2}$

52. The hemisphere $x^2 + y^2 + z^2 \leqslant 1$, $z \geqslant 0$; $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

T **53.** Let E be the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ with the density function $\rho(x, y, z) = 1 + x + y + z$. Use a computer algebra system to find the exact values of the following quantities for E.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

T **54.** If E is the solid of Exercise 22 with density function $\rho(x, y, z) = x^2 + y^2$, find the following quantities, correct to three decimal places.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

55. The joint density function for random variables X, Y, and Z is $f(x, y, z) = Cxyz$ if $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 2$, and $f(x, y, z) = 0$ otherwise.
(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.
(c) Find $P(X + Y + Z \leqslant 1)$.