M408D Final Exam Review – Day 3 – SOLUTIONS

1) Christo, the installation artist, is putting up his new piece called "Very Long Fence." The first fence post he puts up is 3m tall and each subsequent fence post is 60% of the height of the previous one.

a) Assuming he can go on forever (and that we don't have to worry about atomic physics), what is the limit of the heights of these fence posts going to?

$$a_1 = 3 \quad ; \quad a_2 = 3\left(\frac{3}{5}\right) \quad ; \quad a_3 = 3\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = 3\left(\frac{3}{5}\right)^2 \quad ; \quad \ldots \quad ; \quad a_n = 3\left(\frac{3}{5}\right)^{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(3\left(\frac{3}{5}\right)^{n-1}\right) = 3\lim_{n\to\infty}\left(\frac{3}{5}\right)^{n-1}$$

Since $\frac{3}{5} < 1$, larger and larger powers of $\frac{3}{5}$ will tend toward zero.

So, $\lim_{n\to\infty} a_n = 3\lim_{n\to\infty}\left(\frac{3}{5}\right)^{n-1} = 3(0) = 0$

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b) What is the total length of fencing material he used assuming the base of each post starts at the ground?

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 3\left(\frac{3}{5}\right)^{n-1}$$

Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ when $|r| < 1$

So, $\sum_{n=1}^{\infty} 3\left(\frac{3}{5}\right)^{n-1} = \frac{3}{1-\frac{3}{5}} = \frac{15}{2}$ meters

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Determine whether the following series converge or diverge, justify your answer.

If applicable, say if it converges conditionally. (Unless otherwise stated, assume $\sum = \sum_{n=1}^{\infty}$)

2) $\sum \frac{\cos n\pi}{n}$

Notice that $\cos n\pi$ behaves like $(-1)^n$

So, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Which invites us to use the Alternating Series test:

1) $\lim_{n\to\infty} a_n \stackrel{?}{=} 0$

2) $|a_{n+1}| \stackrel{?}{<} |a_n|$

Using this test on our series…

1) Yes ; $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

2) Yes ; $\left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right| \quad\rightarrow\quad \frac{1}{n+1} < \frac{1}{n}$ true for all $n \in \mathbb{Z}^+$

So, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ converges by the Alternating Series Test.

Test for absolute convergence:

$$\sum_{n=1}^{\infty}\left|\frac{\cos n\pi}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

and $\sum_{n=1}^{\infty}\frac{1}{n}$ is the Harmonic series which diverges by the p-series test (p = 1)

So, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ is conditionally convergent.

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3) $\sum \frac{n!}{(n+2)!}$

We could use the RAT!O test, however a little algebra might work well first, too…

$$\sum \frac{n!}{(n+2)!} = \sum \frac{n!}{(n+2)(n+1)n!} = \sum \frac{1}{(n+2)(n+1)} = \sum \frac{1}{n^2+3n+2}$$

We notice that: $\frac{1}{n^2+3n+2} < \frac{1}{n^2}$ (for all positive integers n)

and $\sum \frac{1}{n^2}$ converges by p-series (p = 2)

So, $\sum \frac{n!}{(n+2)!}$ converges by Direct Comparison with p-series $\sum \frac{1}{n^2}$

Since $\sum \frac{n!}{(n+2)!} = \sum \left|\frac{n!}{(n+2)!}\right|$, we have that $\sum \frac{n!}{(n+2)!}$ is absolutely convergent.

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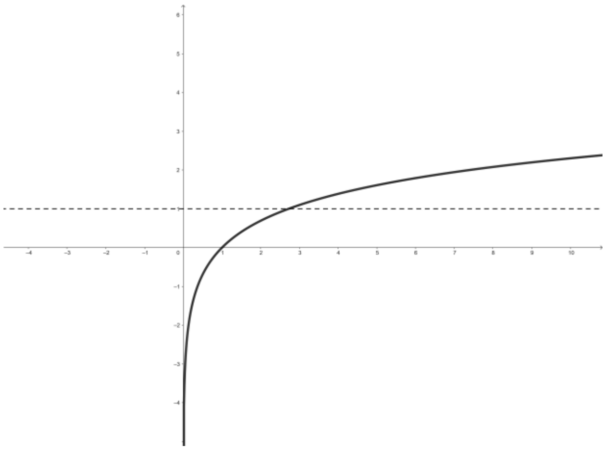
4) $\sum \frac{\ln n}{n}$

This problem can be done with Integral Test, however also notice:

$\ln n > 1 \quad \text{for all } n \geq 3$



So, $\frac{\ln n}{n} > \frac{1}{n} \quad$ for all $n \geq 3$

Therefore, $\sum \frac{\ln n}{n}$ diverges by Direct Comparison with the p-series $\sum \frac{1}{n}$

(Since $\frac{\ln n}{n} > 0 \quad$ for all $n \geq 3$, there is no test need for conditional convergence.)

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5) $\sum \frac{1}{\sqrt{2n^2 + n}}$

This series is not solved by the N-th term test (as the terms go to zero), is not of the Geometric series form (or the Telescoping series form), and is not alternating (so no Alternating Series test).

So, we do some algebra to get a feeling for what the terms "are like" so that we might use the Basic Comparison or the Limit Comparison tests…

Notice that:

$$\frac{1}{\sqrt{2n^2+n}} = \frac{1}{\sqrt{n^2\left(2+\frac{1}{n}\right)}} = \frac{1}{\sqrt{n^2}\sqrt{2+\frac{1}{n}}} = \frac{1}{n\sqrt{2+\frac{1}{n}}}$$

(breaking up the radical is fine since both terms in the radicand are always positive)

And so, we get the feeling that $\frac{1}{\sqrt{2n^2+n}} \approx \frac{1}{n\sqrt{2}}$

And, as far as the series go, we could say $\frac{1}{\sqrt{2n^2+n}} \approx \frac{1}{n}$

I don't want to worry with finding a rigid inequality that holds for all *n*, for instance $\frac{1}{\sqrt{2n^2+n}} < \frac{1}{n}$ is true for all positive integers *n*, but might be a pain to mess with…

So instead, we'll use the Limit Comparison test…

$$\begin{aligned}
\lim_{n\to\infty} \frac{b_n}{a_n} &= \lim_{n\to\infty} \frac{1/n}{1/\sqrt{2n^2+n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2n^2+n}}{n} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^2\left(2+\frac{1}{n}\right)}}{n} \\
&= \lim_{n\to\infty} \frac{\sqrt{n^2}\sqrt{2+\frac{1}{n}}}{n} \\
&= \lim_{n\to\infty} \frac{n\sqrt{2+\frac{1}{n}}}{n} \\
&= \lim_{n\to\infty} \sqrt{2+\frac{1}{n}} \\
&= \sqrt{2}
\end{aligned}$$

Since $0 < \sqrt{2} < \infty$, we have that…

$\sum \frac{1}{\sqrt{2n^2+n}}$ diverges by Limit Comparison with the p-series $\sum \frac{1}{n}$

(Since $\frac{1}{\sqrt{2n^2+n}} > 0$ for all *n*, there is no test need for conditional convergence.)

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6) $\sum \frac{(n^3-1)^{1/2}}{n^3}$

This series is not solved by the N-th term test (as the terms go to zero), is not of the Geometric series form (or the Telescoping series form), and is not alternating (so no Alternating Series test).

So, we do some algebra to get a feeling for what the terms "are like" so that we might use the Basic Comparison or the Limit Comparison tests…

Notice that:

$$\frac{(n^3-1)^{1/2}}{n^3} = \frac{\sqrt{n^3-1}}{n^3} = \frac{\sqrt{n^3\left(1-\frac{1}{n^3}\right)}}{n^3} = \frac{\sqrt{n^3}\sqrt{1-\frac{1}{n^3}}}{n^3} = \frac{\sqrt{1-\frac{1}{n^3}}}{n^{3/2}}$$

So, $\sum \frac{(n^3-1)^{1/2}}{n^3} = \sum \frac{\sqrt{1-\frac{1}{n^3}}}{n^{3/2}}$

Since $\sqrt{1-\frac{1}{n^3}} < 1$ for all positive integers n, we have that:

$\frac{\sqrt{1-\frac{1}{n^3}}}{n^{3/2}} < \frac{1}{n^{3/2}}$ for all positive integers n

And $\sum \frac{1}{n^{3/2}}$ converges by the p-series test (p = 3/2)

So,

$\sum \frac{(n^3-1)^{1/2}}{n^3}$ converges by Direct Comparison with convergent p-series $\sum \frac{1}{n^{3/2}}$

(Since $\frac{(n^3-1)^{1/2}}{n^3} > 0$ for all positive integers n, there is no test need for conditional convergence.)

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7) $\sum \frac{(-1)^n n}{(n+1)^3}$

This is an alternating series.
However, we will test for absolute convergence first…

$$\begin{aligned}\sum \left|\frac{(-1)^n n}{(n+1)^3}\right| &= \sum \frac{n}{(n+1)^3} \\ &= \sum \frac{n}{n^3+3n^2+3n+1} \\ &= \sum \frac{n}{n\left(n^2+3n+3+1/n\right)} \\ &= \sum \frac{1}{n^2+3n+3+1/n}\end{aligned}$$

We notice that $\frac{1}{n^2+3n+3+1/n} < \frac{1}{n^2}$ for all positive integers n

And that $\sum \frac{1}{n^2}$ converges by the p-series test (p = 2)

So, $\sum \left| \frac{(-1)^n n}{(n+1)^3} \right|$ converges by Direct Comparison with the p-series $\sum \frac{1}{n^2}$

Therefore, $\sum \frac{(-1)^n n}{(n+1)^3}$ converges.

And so, $\sum \frac{(-1)^n n}{(n+1)^3}$ is absolutely convergent.

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8) $\sum \frac{3^n n^n}{n!}$

Seeing the factorial, we will try the RAT!O test…

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}(n+1)^{n+1}}{(n+1)!}}{\frac{3^n n^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left(\frac{3^{n+1}(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n n^n} \right) \\
&= \lim_{n\to\infty} \left(\frac{n!}{(n+1)!} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{(n+1)^{n+1}}{n^n} \right) \\
&= \lim_{n\to\infty} \left(\frac{n!}{(n+1)n!} \cdot 3 \cdot \frac{(n+1)^{n+1}}{n^n} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{(n+1)} \cdot 3 \cdot \frac{(n+1)^{n+1}}{n^n} \right) \\
&= \lim_{n\to\infty} \left(3 \cdot \frac{(n+1)^n}{n^n} \right) \\
&= 3 \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n \\
&= 3 \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n \\
&= 3 \cdot e
\end{aligned}$$

Since $3e > 1$, by the RAT!O test, we have that:

$\sum \frac{3^n n^n}{n!}$ diverges

(Since $\frac{3^n n^n}{n!} > 0$ for all positive integers *n*, there is no test need for conditional convergence.)

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9) Find the sum of the series: $\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n}$

We might notice through a little bit of algebra that:

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2^n}{5^n} + \frac{3^n}{5^n} \right)$$

And so, our given series look very much like a sum of two Geometric Series forms:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{when } |r| < 1$$

To proceed in this direction, we need to make one technical note:

$\sum^{\infty} (a_n + b_n) = \sum^{\infty} a_n + \sum^{\infty} b_n$ is allowable when both of $\sum^{\infty} a_n$ and $\sum^{\infty} b_n$ converge absolutely.

(we will have to take this for granted during our work, but will be validated at the end if both of our series $\sum^{\infty} a_n$ and $\sum^{\infty} b_n$ are determined to converge absolutely)

And so, we proceed in this direction of "breaking up" the sum…

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} &= \sum_{n=0}^{\infty} \left(\frac{2^n}{5^n} + \frac{3^n}{5^n} \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{2^n}{5^n} \right) + \sum_{n=0}^{\infty} \left(\frac{3^n}{5^n} \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{2}{5} \right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5} \right)^n\end{aligned}$$

Using the sum of Geometric Series formulas for these two series…

$$\sum_{n=0}^{\infty} \left(\frac{2}{5} \right)^n = \frac{1}{1 - \frac{2}{5}} = \frac{5}{3} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \left(\frac{3}{5} \right)^n = \frac{1}{1 - \frac{3}{5}} = \frac{5}{2}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} = \frac{5}{3} + \frac{5}{2} = \frac{25}{6}$$

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10) True or False.

A) The basic comparison test can be used to show that the series $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+6}$ diverges by comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = \frac{1}{2}$

In part A, the claim is that the <u>basic comparison test</u> can be used to show that the series $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+6}$ diverges by comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = \frac{1}{2}$

From practice, the intuition we've built tells us that $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+6}$ behaves like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ since we expect the constants to be insignificant (which is correct in this case). And we could use the Limit Comparison test to show that our hunch is right:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}+6}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+6} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+6} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{2+\frac{6}{\sqrt{n}}} = \frac{1}{2}$$

So we have that $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+6}$ does in fact diverge by limit comparison to p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = \frac{1}{2}$

However, part A states that this conclusion can also be reached using the <u>basic comparison test</u>, which is false.

Notice in the basic comparison test that we need to use a strict inequality. We would need to show that:

$\frac{1}{2\sqrt{n}+6} > \frac{1}{\sqrt{n}}$ and by the p-series test the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, therefore $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+6}$ is divergent.

However, we cannot establish the initial inequality $\frac{1}{2\sqrt{n}+6} > \frac{1}{\sqrt{n}}$

Instead, we see that $\frac{1}{2\sqrt{n}+6} < \frac{1}{\sqrt{n}}$

So the claim is very close, but not true as stated.

B) The limit comparison test can be used to show that the series $\sum_{n=1}^{\infty} \frac{5n}{2n+n^3}$ converges by comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 2$

In part B, the claim is that the limit comparison test can be used to show that the series $\sum_{n=1}^{\infty} \frac{5n}{2n+n^3}$ converges by comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p = 2$

Notice that $\frac{5n}{2n+n^3} = \frac{5n}{n\left(2+n^2\right)} = \frac{5}{2+n^2} \quad \left(\text{for } n \neq 0\right)$

And so, we could use the limit comparison test to compare $\sum_{n=1}^{\infty} \frac{5}{2+n^2}$ with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{5}{2+n^2}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{5n^2}{2+n^2}=\lim_{n\to\infty}\frac{5n^2}{2+n^2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{5}{\frac{2}{n^2}+1}=5$$

Therefore, the limit comparison test says that $\sum_{n=1}^{\infty}\frac{5n}{2n+n^3}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ behave the same way.

And so, since the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we have that $\sum_{n=1}^{\infty}\frac{5n}{2n+n^3}$ converges, and the claim is true.

C) The integral test can be used to show that the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges to the value $\frac{1}{\ln 2}$ by comparing it to the improper integral $\int_2^{\infty}\frac{1}{x(\ln x)^2}dx$

In part C, in using the integral test to determine the convergence/divergence properties of $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ we would have the improper integral $\int_2^{\infty}\frac{1}{x(\ln x)^2}dx$ to evaluate…

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\int_2^{t}\frac{1}{x(\ln x)^2}dx \qquad \text{Let } u=\ln x\ ;\ \ du=\frac{1}{x}dx$$

$$=\lim_{t\to\infty}\int_{\#_1}^{\#_2}\frac{1}{(u)^2}du$$

$$=\lim_{t\to\infty}\left(-\frac{1}{u}\right)\Bigg]_{\#_1}^{\#_2}$$

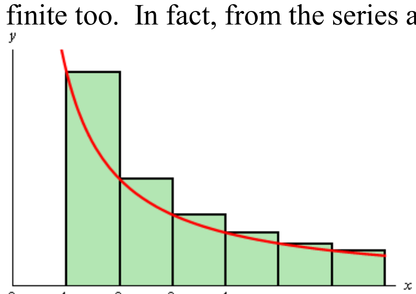
$$=\lim_{t\to\infty}\left(-\frac{1}{\ln x}\right)\Bigg]_{2}^{t}$$

$$=\lim_{t\to\infty}\left(-\frac{1}{\ln t}-\left(-\frac{1}{\ln 2}\right)\right)$$

$$=\frac{1}{\ln 2}$$

Therefore, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges by the Integral test.

However, the statement claims that the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges to the value $\frac{1}{\ln 2}$ which is incorrect. The integral test just says that if the area below the curve is finite, then the sum of the series is finite too. In fact, from the series and area below the curve overlap graphs…



…we have that for certain whatever the sum of the series is, it is greater than the value $\frac{1}{\ln 2}$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = K > \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln 2}$$

And so, part C is false.

D) The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ can be shown to diverge using the p-series test, the integral test, and the geometric series test

Part D claims that the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ can be shown to diverge using the p-series test, the integral test, and the geometric series test.

The Harmonic series fits the template of a p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p=1$

And the p-series test includes diverges of the series for $p=1$

The Harmonic series also can be shown to diverge by the integral test…

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \left(\ln x\right)\Big]_1^t = \lim_{t\to\infty} \left(\ln t\right) = \infty$$

Since the integral diverges, so must the series $\sum_{n=1}^{\infty} \frac{1}{n}$

However, the Harmonic series does not fit the template for the Geometric Series test.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{when } |r|<1 \quad \text{or an equivalent formula} \quad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{when } |r|<1$$

And with no "ratio to a power" in the Harmonic expression, we cannot use the Geometric Series test to show that the Harmonic series diverges.

E) If an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ fails to satisfy either of the two criteria of the alternating series test, 1) $\lim_{n\to\infty} |a_n| = 0$ or 2) $|a_{n+1}| < |a_n|$ then it is known to diverge.

Finally part E claims that if an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ fails to satisfy either of the two criteria of the alternating series test, 1) $\lim_{n\to\infty} |a_n| = 0$ or 2) $|a_{n+1}| < |a_n|$ then it is known to diverge.

This is a common misconception. Instead we should recall that if either of the criteria are not met, the test is inconclusive and another test should used. In a large number of cases, a series that fails the Alternating Series test, fails the first criteria. In which case, we would look to the Basic Divergence test to show that the series diverges:

If a sequence $\{a_n\}$ converges to a nonzero number, diverges, or fails to converge, then $\sum_{n=1}^{\infty} a_n$ diverges.

Compute the following Taylor Polynomials, centered at zero:

11) $T_2(x)$ for $f(x)=\dfrac{x}{x^2+1}$

Provided that a power series representation for a function $f\left(x\right)$ about $x=a$ exists, the Taylor Series for $f\left(x\right)$ about $x=a$ is given by:

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^n$$

$$= f\left(a\right)+f'\left(a\right)\left(x-a\right)+\frac{f''\left(a\right)}{2!}\left(x-a\right)^2+\frac{f'''\left(a\right)}{3!}\left(x-a\right)^3+\cdots$$

And the nth degree Taylor polynomial is given by:

$$T_n\left(x\right)=\sum_{i=0}^{n}\frac{f^{(i)}\left(a\right)}{i!}\left(x-a\right)^i$$

And so, the degree-2 Taylor polynomial centered at zero is then:

$$T_2\left(x\right)=\sum_{i=0}^{2}\frac{f^{(i)}\left(0\right)}{i!}\left(x-0\right)^i$$

$$= f\left(0\right)+f'\left(0\right)\left(x-0\right)+\frac{f''\left(0\right)}{2!}\left(x-0\right)^2$$

$$= f\left(0\right)+f'\left(0\right)\cdot x+\frac{1}{2}f''\left(0\right)\cdot x^2$$

For the function $f\left(x\right)=\dfrac{x}{x^2+1}\ldots$

$$f\left(0\right)=1$$

$$f'\left(x\right)=\frac{\left(x^2+1\right)\left(1\right)-\left(x\right)\left(2x\right)}{\left(x^2+1\right)^2}=\frac{-x^2+1}{\left(x^2+1\right)^2}\quad\rightarrow\quad f'\left(0\right)=1$$

$$f''\left(x\right)=\frac{\left(x^2+1\right)^2\left(-2x\right)-\left(-x^2+1\right)\left(2\left(x^2+1\right)\left(2x\right)\right)}{\left(x^2+1\right)^4}\quad\rightarrow\quad f''\left(0\right)=0$$

So, the second degree Taylor polynomial centered at zero for $f\left(x\right)=\dfrac{x}{x^2+1}$ is:

$$T_2\left(x\right)= f\left(0\right)+f'\left(0\right)\cdot x+\frac{1}{2}f''\left(0\right)\cdot x^2$$

$$=\left(0\right)+\left(1\right)\cdot x+\left(0\right)\cdot x^2$$

$$= x$$

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12) $T_3(x)$ for $f(x)=e^{2x}$

The nth degree Taylor polynomial, $T_n(x)$ for a function $f(x)$ is given by:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

And so, the degree-3 Taylor polynomial centered at zero is then:

$$T_3(x) = \sum_{i=0}^{3} \frac{f^{(i)}(0)}{i!}(x-0)^i$$
$$= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= f(0) + f'(0)\cdot x + \frac{1}{2} f''(0)\cdot x^2 + \frac{1}{6} f'''(0)\cdot x^3$$

For the function $f(x) = e^{2x}\ldots$

$$f(0) = e^0 = 1$$
$$f'(x) = 2e^{2x} \quad \rightarrow \quad f'(0) = 2$$
$$f''(x) = 4e^{2x} \quad \rightarrow \quad f''(0) = 4$$
$$f'''(x) = 8e^{2x} \quad \rightarrow \quad f'''(0) = 8$$

And so, the degree-3 Taylor polynomial centered at zero for $f(x) = e^{2x}$ is:

$$T_3(x) = f(0) + f'(0)\cdot x + \frac{1}{2} f''(0)\cdot x^2 + \frac{1}{6} f'''(0)\cdot x^3$$
$$= (1) + (2)\cdot x + \frac{1}{2}(4)\cdot x^2 + \frac{1}{6}(8)\cdot x^3$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Find the interval of convergence for the following:

13) $\sum \frac{2^k}{(2k)!}(x+1)^k$

Our two main tools for Radius of Convergence (ROC) / Interval of Convergence (IOC) problems are the Root Test and the Ratio Test.

In this problem, we'll use the Ratio Test…

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{2^{k+1}}{(2(k+1))!}(x+1)^{k+1}}{\frac{2^k}{(2k)!}(x+1)^k}\right|$$

$$= \lim_{k\to\infty}\frac{(2k)!}{(2(k+1))!}\cdot\frac{2^{k+1}}{2^k}\cdot\left|\frac{(x+1)^{k+1}}{(x+1)^k}\right| \qquad \text{since all k-terms are positive}$$

$$= \lim_{k\to\infty}\frac{(2k)!}{(2k+2)!}\cdot 2\cdot|x+1|$$

$$= 2|x+1|\cdot\lim_{k\to\infty}\frac{(2k)!}{(2k+2)(2k+1)(2k)!}$$

$$= 2|x+1|\cdot\lim_{k\to\infty}\frac{1}{(2k+2)(2k+1)}$$

$$= 2|x+1|\cdot(0)$$

$$= 0$$

And so,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 0 < 1 \qquad \text{for all values of } x$$

Therefore, $\sum \frac{2^k}{(2k)!}(x+1)^k$ converges for all values of x

And so, the IOC for $\sum \frac{2^k}{(2k)!}(x+1)^k$ is $(-\infty,\infty)$

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14) $\sum \frac{k+1}{k^2 3^k}(x-3)^k$

Our two main tools for Radius of Convergence (ROC) / Interval of Convergence (IOC) problems are the Root Test and the Ratio Test.

In this problem, we'll use the Ratio Test…

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{(k+1)+1}{(k+1)^2 3^{k+1}}(x-3)^{k+1}}{\frac{k+1}{k^2 3^k}(x-3)^k}\right|$$

$$= \lim_{k\to\infty}\frac{(k+2)k^2}{(k+1)^2(k+1)}\cdot\frac{3^k}{3^{k+1}}\cdot\left|\frac{(x-3)^{k+1}}{(x-3)^k}\right|$$ since all k-terms are positive

$$= \lim_{k\to\infty}\frac{k^3+2k^2}{(k+1)^3}\cdot\frac{1}{3}\cdot|x-3|$$

$$= \frac{1}{3}|x-3|\cdot\lim_{k\to\infty}\frac{k^3+2k^2}{k^3+3k^2+3k+1}$$

$$= \frac{1}{3}|x-3|\cdot(1)$$

$$= \frac{1}{3}|x-3|$$

And so, we ask ourselves, "for what values of *x* is the result of the Ratio Test less than one"…

$$\frac{1}{3}|x-3|<1$$

$$|x-3|<3$$

$$-3<x-3<3$$

$$0<x<6$$

And so, temporarily, our IOC is at least $(0,6)$

Now we test the endpoints of that interval, individually…

When $x=0$…

$$\sum\frac{k+1}{k^2 3^k}(x-3)^k = \sum\frac{k+1}{k^2 3^k}(-3)^k$$

$$= \sum\frac{(k+1)3^k(-1)^k}{k^2 3^k}$$

$$= \sum\frac{(k+1)(-1)^k}{k^2}$$

Which invites us to use the Alternating Series Test…

1) $\lim_{n\to\infty} a_n \overset{?}{=} 0$; Yes, since $\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(\frac{k+1}{k^2}\right) = 0$

2) $|a_{n+1}| \overset{?}{<} |a_n|$; Yes, since…

$$|a_{k+1}| \overset{?}{<} |a_k| \quad \rightarrow \quad \left|\frac{k+2}{(k+1)^2}\right| \overset{?}{<} \left|\frac{k+1}{k^2}\right|$$

$\dfrac{k+2}{(k+1)^2} \overset{?}{<} \dfrac{k+1}{k^2}$ since all k-terms are positive

$$k^2(k+2) \overset{?}{<} (k+1)^3$$

$$k^3 + 2k^2 \overset{?}{<} k^3 + 3k^2 + 3k + 1$$

$0 \overset{true}{<} k^2 + 3k + 1$ true for all positive integers k

(here, we could have also showed the second criteria of the AST was satisfied by showing that the function $f(x) = \dfrac{x+1}{x^2}$ is monotonically decreasing – by showing that its derivative is always negative, for x in the positive integers)

And so, $x = 0$ is in the IOC for $\sum \dfrac{k+1}{k^2 3^k}(x-3)^k$

Testing the other endpoint of the interval, when $x = 6\ldots$

$$\begin{aligned}\sum \frac{k+1}{k^2 3^k}(x-3)^k &= \sum \frac{k+1}{k^2 3^k}(3)^k \\ &= \sum \frac{(k+1)3^k}{k^2 3^k} \\ &= \sum \frac{k+1}{k^2}\end{aligned}$$

Ignoring the "+ 1", we might have a hunch that $\sum \dfrac{k+1}{k^2}$ does the same thing as $\sum \dfrac{1}{k}$

Using the Limit Comparison Test to verify…

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{k+1}{k^2}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

And since $0 < 1 < \infty$, the Limit Comparison Test tells us that $\sum \dfrac{k+1}{k^2}$ and $\sum \dfrac{1}{k}$ behave the same way.

Since $\sum \dfrac{1}{k}$ diverges (p-series with p = 1), we then have that $\sum \dfrac{k+1}{k^2}$ diverges.

And so, $x = 6$ is NOT in the IOC for $\sum \dfrac{k+1}{k^2 3^k}(x-3)^k$

Putting all of this together, we have that the IOC for $\sum \dfrac{k+1}{k^2 3^k}(x-3)^k$ is: $[0,6)$

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15) $\sum \frac{(-1)^k}{5^{k+1}}(3x)^k$

Our two main tools for Radius of Convergence (ROC) / Interval of Convergence (IOC) problems are the Root Test and the Ratio Test.

In this problem, we'll kick it Root down…

$$\lim_{k\to\infty}\sqrt[k]{|a_k|}=\lim_{k\to\infty}\sqrt[k]{\left|\frac{(-1)^k}{5^{k+1}}(3x)^k\right|}$$

$$=\lim_{k\to\infty}\sqrt[k]{\left|\frac{(3x)^k}{5^{k+1}}\right|}$$

$$=\lim_{k\to\infty}\sqrt[k]{\frac{1}{5}\cdot\frac{3^k|x|^k}{5^k}}$$

$$=\lim_{k\to\infty}\sqrt[k]{\frac{1}{5}}\sqrt[k]{\frac{3^k|x|^k}{5^k}}$$

$$=\lim_{k\to\infty}\sqrt[k]{\frac{1}{5}}\cdot\frac{3|x|}{5}$$

$$=\frac{3|x|}{5}\cdot\lim_{k\to\infty}\sqrt[k]{\frac{1}{5}}$$

$$=\frac{3|x|}{5}\cdot(1)$$

$$=\frac{3|x|}{5}$$

And so, we ask ourselves, "for what values of *x* is the result of the Root Test less than one"…

$$\frac{3}{5}|x|<1$$

$$|x|<\frac{5}{3}$$

$$-\frac{5}{3}<x<\frac{5}{3}$$

And so, temporarily, our IOC is at least $\left(-\frac{5}{3},\frac{5}{3}\right)$

Now we test the endpoints of that interval, individually…

When $x=-\frac{5}{3}\dots$

$$\sum \frac{(-1)^k}{5^{k+1}}(3x)^k = \sum \frac{(-1)^k}{5^{k+1}}\left(3\cdot\left(-\frac{5}{3}\right)\right)^k$$
$$= \sum \frac{(-1)^k}{5^{k+1}}(-1)^k 5^k$$
$$= \sum \frac{1}{5}$$

Which diverges.

So, $x = -\frac{5}{3}$ is NOT in the IOC for $\sum \frac{(-1)^k}{5^{k+1}}(3x)^k$

Testing the other endpoint, when $x = \frac{5}{3}\dots$

$$\sum \frac{(-1)^k}{5^{k+1}}(3x)^k = \sum \frac{(-1)^k}{5^{k+1}}\left(3\cdot\left(\frac{5}{3}\right)\right)^k$$
$$= \sum \frac{(-1)^k}{5^{k+1}} 5^k$$
$$= \sum \frac{(-1)^k}{5}$$

We might be tempted to use the Alternating Series test here, but would notice quickly that the first criteria is not satisfied. That is, the terms to not go to zero. Which means the AST is inconclusive.

So, we fall back to the Nth-term Test (aka Basic Divergence Test):

If $\lim\limits_{n\to\infty} a_n$ does not exist or $\lim\limits_{n\to\infty} a_n \neq 0$, Then $\sum a_n$ diverges

Here, we see $\lim\limits_{k\to\infty} \frac{(-1)^k}{5}$ does not exist, as it alternates between two values forever.

Therefore, $\sum \frac{(-1)^k}{5}$ diverges and so $x = \frac{5}{3}$ is NOT in the IOC for $\sum \frac{(-1)^k}{5^{k+1}}(3x)^k$

Putting all of this together, we have that the IOC for $\sum \frac{(-1)^k}{5^{k+1}}(3x)^k$ is: $\left(-\frac{5}{3}, \frac{5}{3}\right)$

Determine the power series representations for the following functions.

16) $f(y) = \dfrac{5}{4+y^2}$

The main power series representation that we start from (and hope to tie the given function to), is an adaptation of the sum of a Geometric Series formula: $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ when $|x| < 1$

Therefore, in my scratch work, I hope to "deconstruct" the given function expression $\frac{5}{4+y^2}$ in such a way that I can relate it to the expression $\frac{1}{1-x}$ through a series of multiplications, divisions, substitutions, derivatives, integrations, etc.

Then, I'll repeat that process, making the same modifications to the series expressions…

$$\frac{5}{4+y^2}=5\left(\frac{1}{4+y^2}\right)$$

$$=5\left(\frac{1}{4\left(1+\frac{y^2}{4}\right)}\right)$$

$$=\frac{5}{4}\left(\frac{1}{1+\frac{y^2}{4}}\right)$$

$$=\frac{5}{4}\left(\frac{1}{1-\left(-\frac{y^2}{4}\right)}\right)$$

So, now I see that through a substution of $-\frac{y^2}{4}$ and a multiplication by $\frac{5}{4}$, I can "build up" a series expression for the given function $\frac{5}{4+y^2}$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\frac{1}{1-\left(-\frac{y^2}{4}\right)}=\sum_{n=0}^{\infty}\left(-\frac{y^2}{4}\right)^n$$

$$\frac{1}{1+\frac{y^2}{4}}=\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{4^n}$$

$$\frac{5}{4}\cdot\frac{1}{1+\frac{y^2}{4}}=\frac{5}{4}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{4^n}$$

$$\frac{5}{4\left(1+\frac{y^2}{4}\right)}=\sum_{n=0}^{\infty}\frac{5(-1)^n y^{2n}}{4^{n+1}}$$

$$\frac{5}{4+y^2}=\sum_{n=0}^{\infty}\frac{5(-1)^n y^{2n}}{4^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{5(-1)^n y^{2n}}{\left(2^2\right)^{n+1}}$$

$$=\sum_{n=0}^{\infty}5(-1)^n\frac{y^{2n}}{2^{2(n+1)}}$$

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17) $f(x)=\ln\left(1-x^2\right)$

Again, we'll start with the main power series representation: $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ when $|x|<1$

Therefore, in my scratch work, I hope to "deconstruct" the given function expression $\ln\left(1-x^2\right)$ in such a way that I can relate it to the expression $\frac{1}{1-x}$ through a series of multiplications, divisions, substitutions, derivatives, integrations, etc.

Then, I'll repeat that process, making the same modifications to the series expressions…

Notice the relationship:

$$\int\frac{1}{1-x}\,dx=-\ln(1-x)+C$$

And so, after having an expression for $-\ln(1-x)$, then we can multiply by a negative one to get rid of the negative sign and substitute x^2 for x to arrive at the desired expression…

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = \int \left(\sum_{n=0}^{\infty} x^n \right) dx$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \int x^n dx$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

Here, we pause a moment to determine the value of C for our current expression.
We want our power series representation for the function to be accurate for all x near zero, therefore we can use values of x near or equal to zero to determine the value of C

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\ln(1-x) = -\left\{ x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots \right\} + C$$

Let $x = 0$

$$\ln(1-(0)) = -\left\{ (0) + \frac{1}{2}(0)^2 + \frac{1}{3}(0)^3 + \cdots \right\} + C$$

$$0 = -\{0\} + C$$

$$C = 0$$

And so, we have: $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

And so, after a substitution of x^2 for x we have…

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln\left(1-\left(x^2\right)\right) = -\sum_{n=0}^{\infty} \frac{\left(x^2\right)^{n+1}}{n+1}$$

$$\ln\left(1-x^2\right) = -\sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1}$$

$$\ln\left(1-x^2\right) = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{2(n+1)}$$

And so, we have a power series for our function: $\ln\left(1-x^2\right) = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{2(n+1)}$

While it is not necessary here (but none the less, good practice) we could also advance the index in our series, if we wanted to match answer choices that begin at $n=1$, for instance…

$$\ln\left(1-x^2\right) = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{2(n+1)} = -\left\{\frac{1}{1}x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \cdots\right\}$$
$$= -\sum_{n=1}^{\infty} \frac{1}{n} x^{2n} = -\left\{\frac{1}{1}x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \cdots\right\}$$

And so, we have another, equally valid power series representation, this one with a series that starts at $n=1$

$$\ln\left(1-x^2\right) = \sum_{n=1}^{\infty} -\frac{1}{n} x^{2n}$$

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18) $f(t) = \left(\frac{2t}{4-t}\right)^2$

As before, we do some "deconstructing" of our given function expression to try to find a connection to $\frac{1}{1-x}$ through a series of multiplications, divisions, substitutions, derivatives, integrations, etc.

$$\left(\frac{2t}{4-t}\right)^2 = \frac{4t^2}{(4-t)^2}$$
$$= 4t^2 \cdot \frac{1}{(4-t)^2}$$
$$= 4t^2 \cdot \frac{1}{\left(4\left(1-\frac{t}{4}\right)\right)^2}$$
$$= 4t^2 \cdot \frac{1}{4^2\left(1-\frac{t}{4}\right)^2}$$
$$= \frac{t^2}{4} \cdot \frac{1}{\left(1-\frac{t}{4}\right)^2}$$

This conclusion tells us that if we can find a series expression for $\frac{1}{(1-x)^2}$ then we can simply substitute $\frac{t}{4}$ in for x and then multiply the result by $\frac{t^2}{4}$ and we'll arrive at the desired conclusion.

So, now we're looking for a connection between $\frac{1}{(1-x)^2}$ and $\frac{1}{1-x}$….

Notice that the derivative of $\frac{1}{1-x}$ is very close to $\frac{1}{(1-x)^2}$ …in fact it matches it exactly:

$$\frac{d}{dx}\left[\frac{1}{1-x}\right]=\frac{d}{dx}\left[(1-x)^{-1}\right]=(-1)(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

And so, we have enough information to start "building" our series representation for the function…

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right]=\frac{d}{dx}\left[\sum_{n=0}^{\infty}x^n\right]$$

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}\frac{d}{dx}\left[x^n\right]$$

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}nx^{n-1}$$

$$\frac{1}{\left(1-\left(\frac{t}{4}\right)\right)^2}=\sum_{n=0}^{\infty}n\left(\frac{t}{4}\right)^{n-1}$$

$$\frac{1}{\left(1-\frac{t}{4}\right)^2}=\sum_{n=0}^{\infty}n\frac{t^{n-1}}{4^{n-1}}$$

$$\frac{t^2}{4}\cdot\frac{1}{\left(1-\frac{t}{4}\right)^2}=\frac{t^2}{4}\cdot\sum_{n=0}^{\infty}n\frac{t^{n-1}}{4^{n-1}}$$

$$\frac{t^2}{4\left(1-\frac{t}{4}\right)^2}=\sum_{n=0}^{\infty}n\frac{t^{n-1+2}}{4^{n-1+1}}$$

$$\frac{4}{4}\cdot\frac{t^2}{4\left(1-\frac{t}{4}\right)^2}=\sum_{n=0}^{\infty}n\frac{t^{n+1}}{4^n}$$

$$\frac{4t^2}{4^2\left(1-\frac{t}{4}\right)^2}=\sum_{n=0}^{\infty}n\frac{t^{n+1}}{4^n}$$

$$\frac{4t^2}{\left(4\left(1-\frac{t}{4}\right)\right)^2}=\sum_{n=0}^{\infty}n\frac{t^{n+1}}{4^n}$$

$$\frac{4t^2}{(4-t)^2}=\sum_{n=0}^{\infty}n\frac{t^{n+1}}{4^n}$$

$$\left(\frac{2t}{4-t}\right)^2=\sum_{n=0}^{\infty}n\frac{t^{n+1}}{4^n}$$

And so, we have a power series for our function: $\left(\frac{2t}{4-t}\right)^2 = \sum_{n=0}^{\infty} n\frac{t^{n+1}}{4^n}$

Once again, while it is not necessary in this problem (but none the less, good practice) we could also advance the index in our series, if we wanted to match answer choices that begin at $n=2$, for instance…

Notice that since our first term is zero, we can simply advance the index one step without making any changes to the summand…

$$\begin{aligned}\left(\frac{2t}{4-t}\right)^2 &= \sum_{n=0}^{\infty} n\frac{t^{n+1}}{4^n} = \left\{0\cdot\frac{t^1}{4^0}+1\cdot\frac{t^2}{4^1}+2\cdot\frac{t^3}{4^2}+\cdots\right\} \\ &= \sum_{n=0}^{\infty} n\frac{t^{n+1}}{4^n} = \left\{0+1\cdot\frac{t^2}{4^1}+2\cdot\frac{t^3}{4^2}+\cdots\right\} \\ &= \sum_{n=1}^{\infty} n\frac{t^{n+1}}{4^n} = \left\{1\cdot\frac{t^2}{4^1}+2\cdot\frac{t^3}{4^2}+\cdots\right\}\end{aligned}$$

And so, we have without too much work: $\left(\frac{2t}{4-t}\right)^2 = \sum_{n=1}^{\infty} n\frac{t^{n+1}}{4^n}$

Let's advance the index one more step, to begin at $n=2$

$$\begin{aligned}\left(\frac{2t}{4-t}\right)^2 &= \sum_{n=1}^{\infty} n\frac{t^{n+1}}{4^n} = \left\{1\cdot\frac{t^2}{4^1}+2\cdot\frac{t^3}{4^2}+\cdots\right\} \\ &= \sum_{n=2}^{\infty} (n-1)\frac{t^n}{4^{n-1}} = \left\{1\cdot\frac{t^2}{4^1}+2\cdot\frac{t^3}{4^2}+\cdots\right\}\end{aligned}$$

And so, we have another, equally valid power series representation, this one with a series that starts at $n=2$

$$\left(\frac{2t}{4-t}\right)^2 = \sum_{n=2}^{\infty} (n-1)\frac{t^n}{4^{n-1}}$$
