sometimes use integration by parts to reduce the power. In this way we can find a *reduction formula* as in the next example.

Equation 7 is called a *reduction formula* because the exponent n has been *reduced* to $n - 1$ and $n - 2$.

EXAMPLE 6 Prove the reduction formula

$$\boxed{7} \qquad \int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let

$$u = \sin^{n-1}x \qquad\qquad dv = \sin x\, dx$$

Then

$$du = (n-1)\sin^{n-2}x \cos x\, dx \qquad\qquad v = -\cos x$$

and integration by parts gives

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \cos^2 x\, dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx - (n-1)\int \sin^n x\, dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx$$

or

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\, dx \qquad \blacksquare$$

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\, dx$ in terms of $\int \sin x\, dx$ (if n is odd) or $\int (\sin x)^0\, dx = \int dx$ (if n is even).

7.1 Exercises

1–4 Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int xe^{2x}\, dx$; $\quad u = x,\ dv = e^{2x}\, dx$

2. $\int \sqrt{x}\ln x\, dx$; $\quad u = \ln x,\ dv = \sqrt{x}\, dx$

3. $\int x\cos 4x\, dx$; $\quad u = x,\ dv = \cos 4x\, dx$

4. $\int \sin^{-1}x\, dx$; $\quad u = \sin^{-1}x,\ dv = dx$

5–42 Evaluate the integral.

5. $\int te^{2t}\, dt$

6. $\int ye^{-y}\, dy$

7. $\int x\sin 10x\, dx$

8. $\int (\pi - x)\cos \pi x\, dx$

9. $\int w\ln w\, dw$

10. $\int \frac{\ln x}{x^2}\, dx$

11. $\int (x^2 + 2x)\cos x\, dx$

12. $\int t^2 \sin \beta t\, dt$

13. $\int \cos^{-1}x\, dx$

14. $\int \ln\sqrt{x}\, dx$

15. $\int t^4 \ln t\, dt$

16. $\int \tan^{-1}(2y)\, dy$

17. $\int t\csc^2 t\, dt$

18. $\int x\cosh ax\, dx$

19. $\int (\ln x)^2\, dx$

20. $\int \frac{z}{10^z}\, dz$

21. $\int e^{3x}\cos x\, dx$

22. $\int e^x \sin \pi x\, dx$

23. $\int e^{2\theta}\sin 3\theta\, d\theta$

24. $\int e^{-\theta}\cos 2\theta\, d\theta$

25. $\displaystyle\int z^3 e^z \, dz$

26. $\displaystyle\int (\arcsin x)^2 \, dx$

27. $\displaystyle\int (1 + x^2)\, e^{3x} \, dx$

28. $\displaystyle\int_0^{1/2} \theta \sin 3\pi\theta \, d\theta$

29. $\displaystyle\int_0^1 x\, 3^x \, dx$

30. $\displaystyle\int_0^1 \frac{xe^x}{(1 + x)^2} \, dx$

31. $\displaystyle\int_0^2 y \sinh y \, dy$

32. $\displaystyle\int_1^2 w^2 \ln w \, dw$

33. $\displaystyle\int_1^5 \frac{\ln R}{R^2} \, dR$

34. $\displaystyle\int_0^{2\pi} t^2 \sin 2t \, dt$

35. $\displaystyle\int_0^{\pi} x \sin x \cos x \, dx$

36. $\displaystyle\int_1^{\sqrt{3}} \arctan(1/x) \, dx$

37. $\displaystyle\int_1^5 \frac{M}{e^M} \, dM$

38. $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3} \, dx$

39. $\displaystyle\int_0^{\pi/3} \sin x \ln(\cos x) \, dx$

40. $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4 + r^2}} \, dr$

41. $\displaystyle\int_0^{\pi} \cos x \sinh x \, dx$

42. $\displaystyle\int_0^t e^s \sin(t - s) \, ds$

43–48 First make a substitution and then use integration by parts to evaluate the integral.

43. $\displaystyle\int e^{\sqrt{x}} \, dx$

44. $\displaystyle\int \cos(\ln x) \, dx$

45. $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) \, d\theta$

46. $\displaystyle\int_0^{\pi} e^{\cos t} \sin 2t \, dt$

47. $\displaystyle\int x \ln(1 + x) \, dx$

48. $\displaystyle\int \frac{\arcsin(\ln x)}{x} \, dx$

49–52 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

49. $\displaystyle\int xe^{-2x} \, dx$

50. $\displaystyle\int x^{3/2} \ln x \, dx$

51. $\displaystyle\int x^3 \sqrt{1 + x^2} \, dx$

52. $\displaystyle\int x^2 \sin 2x \, dx$

53. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x \, dx$.

54. (a) Prove the reduction formula

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n - 1}{n} \int \cos^{n-2} x \, dx$$

(b) Use part (a) to evaluate $\int \cos^2 x \, dx$.

(c) Use parts (a) and (b) to evaluate $\int \cos^4 x \, dx$.

55. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n - 1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx$$

where $n \geqslant 2$ is an integer.

(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x \, dx$ and $\int_0^{\pi/2} \sin^5 x \, dx$.

(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n + 1)}$$

56. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \frac{\pi}{2}$$

57–60 Use integration by parts to prove the reduction formula.

57. $\displaystyle\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$

58. $\displaystyle\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

59. $\displaystyle\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n - 1} - \int \tan^{n-2} x \, dx \quad (n \neq 1)$

60. $\displaystyle\int \sec^n x \, dx = \frac{\tan x \sec^{n-2} x}{n - 1} + \frac{n - 2}{n - 1} \int \sec^{n-2} x \, dx \quad (n \neq 1)$

61. Use Exercise 57 to find $\int (\ln x)^3 \, dx$.

62. Use Exercise 58 to find $\int x^4 e^x \, dx$.

63–64 Find the area of the region bounded by the given curves.

63. $y = x^2 \ln x, \quad y = 4 \ln x$

64. $y = x^2 e^{-x}, \quad y = xe^{-x}$

65–66 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

65. $y = \arcsin\left(\frac{1}{2}x\right), \quad y = 2 - x^2$

66. $y = x \ln(x + 1), \quad y = 3x - x^2$

67–70 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves about the given axis.

67. $y = \cos(\pi x/2),\ y = 0,\ 0 \leqslant x \leqslant 1;$ about the y-axis

68. $y = e^x,\ y = e^{-x},\ x = 1;$ about the y-axis

69. $y = e^{-x},\ y = 0,\ x = -1,\ x = 0;$ about $x = 1$

70. $y = e^x,\ x = 0,\ y = 3;$ about the x-axis

71. Calculate the volume generated by rotating the region bounded by the curves $y = \ln x$, $y = 0$, and $x = 2$ about each axis.
(a) The y-axis (b) The x-axis

72. Calculate the average value of $f(x) = x \sec^2 x$ on the interval $[0, \pi/4]$.

73. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was discussed in Example 5.3.3 and is used extensively in the theory of optics. Find $\int S(x)\, dx$. [Your answer will involve $S(x)$.]

74. A Rocket Equation A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8 \text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket (a) one minute after liftoff and (b) after it has consumed 6000 kg of fuel.

75. A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

76. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x) g''(x)\, dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x) g(x)\, dx$$

77. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 x f''(x)\, dx$.

78. (a) Use integration by parts to show that

$$\int f(x)\, dx = x f(x) - \int x f'(x)\, dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\, dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]
(c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).
(d) Use part (b) to evaluate $\int_1^e \ln x\, dx$.

79. (a) Recall that the formula for integration by parts is obtained from the Product Rule. Use similar reasoning to obtain the following integration formula from the Quotient Rule.

$$\int \frac{u}{v^2}\, dv = -\frac{u}{v} + \int \frac{1}{v}\, du$$

(b) Use the formula in part (a) to evaluate $\displaystyle\int \frac{\ln x}{x^2}\, dx$.

80. The Wallis Product Formula for π Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$.
(a) Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$.
(b) Use Exercise 56 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n + 1}{2n + 2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n + 1}{2n + 2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

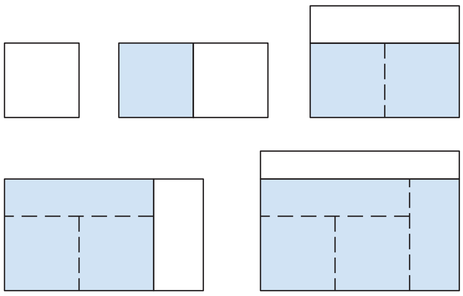
and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.
(d) Use part (c) and Exercises 55 and 56 to show that

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

and is called the *Wallis product*.
(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.



81. We arrived at Formula 6.3.2, $V = \int_a^b 2\pi x f(x)\, dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where f is one-to-one and therefore has an inverse function g. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2\, dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\, dx$$