4. Ex: Smallest number of terms of $\displaystyle\sum_{m=1}^{\infty} \frac{8}{(m+1)(\ln(m+1))^2} = s$

Rephrase the question: to add for the partial sum $s_n$ to be within $\frac{1}{5}$ of $s$

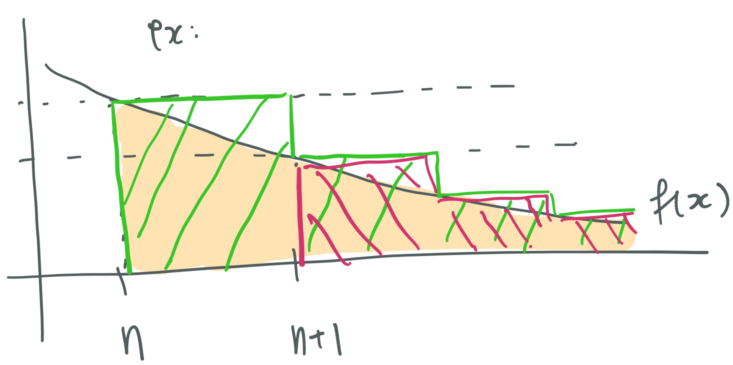
- The partial sum $\displaystyle s_n = \sum_{m=1}^{n} \overbrace{\frac{8}{(m+1)(\ln(m+1))^2}}^{f(m)}$

  is the sum we're looking for.

  We want the value of $n$ so that $s - s_n \le \frac{1}{5}$.

- Notice the squeeze inequality

$$\star \quad \boxed{\sum_{m=n+1}^{\infty} f(m)} \le \int_n^{\infty} f(x)\,dx \le \boxed{\sum_{m=n}^{\infty} f(m)}$$




Now we define $\displaystyle s = \sum_{m=1}^{\infty} f(m)$.

The previous inequality becomes:

$\star \quad s - s_n \leq \int_n^{\infty} f(x)\,dx \leq s - s_{n+1}$

(write out $s - s_n$ using the definition of partial sums to convince yourself of this).

The question wants us to find the smallest $n$ that $s - s_n \leq \frac{1}{5}$.

From the inequality, it is sufficient to find the smallest $n$ such that $\int_n^{\infty} f(x)\,dx \leq \frac{1}{5}$.

Now we solve $\displaystyle\int_n^{\infty} \frac{8}{(x+1)(\ln x+1)^2}\,dx$

$u = \ln(x+1)$

$du = \frac{1}{x+1}\,dx$

$$= \lim_{t\to\infty} \int_n^t 8\,du \cdot \frac{1}{u^2}$$

$$= \lim_{t\to\infty} \left[-\frac{8}{u}\right]_n^t$$

$$= \lim_{t\to\infty} \left[\frac{-8}{\ln(x+1)}\right]_n^t$$

$$= \frac{+8}{\ln(n+1)}.$$

So $\int_n^{\infty} f(x)\,dx \leq \frac{1}{5}$

turns into $\frac{8}{\ln(n+1)} \leq \frac{1}{5}$

$$40 \leq \ln(n+1)$$
$$e^{40} \leq e^{\ln(n+1)}$$
$$e^{40} \leq n+1$$

So $n$ must be at least $e^{40}$.

6. Ex: Determine if $\sum_{k=2}^{\infty} \frac{k^2-4}{k^2+2k}$ conv/ div.

- Notice that the long term behaviour of $\frac{k^2-4}{k^2+2k}$ is $\frac{k^2}{k^2} = 1$. So we intuitively think this series will diverge.

- pf: $\lim_{k\to\infty} \frac{k^2-4}{k^2+2k} = \lim_{k\to\infty} \frac{k^2-4}{k^2+2k} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}}$

$$= \lim_{k\to\infty} \frac{1 - 4/k^2}{1 + 2/k}$$

$= 1.$

By $n^{th}$ term test / divergence test,

$\sum \frac{k^2-4}{k^2+2k}$ diverges.