M408D Final Exam Review – Day 1 – SOLUTIONS

1) Determine which integration technique would be used to solve the following integration problems. Do not work the problem completely.

A) $\int \frac{dx}{x^2 - 4x + 13}$	denominator – completing the square ; then if not an inv trig integral, maybe try trig sub
B) $\int \frac{x^2}{2x^3 - 1} dx$	u-sub
C) $\int (\cos\theta + 1)^2 \, d\theta$	expand expression ; then maybe power-reducing identity for cosine
D) $\int x^3 \ln x \, dx$	integration by parts
E) $\int \frac{\csc^2 x}{\cot^3 x} dx$	maybe u-sub or converting everything to sines and cosines, then see where it goes…
F) $\int e^{2x} \cos 3x \, dx$	integration by parts
G) $\int \cos^3\theta \sin^4\theta \, d\theta$	keep one of the cosines to the side and convert everything else to sines using Pyth ident: $\sin^2\theta + \cos^2\theta = 1$
H) $\int \frac{x^2}{\sqrt{2x - x^2}} dx$	completing the square on the radicand ; then maybe trig sub
I) $\int \frac{2s}{\sqrt[3]{6 - 5s^2}} ds$	u-sub

J) $\int \frac{x}{\sqrt{x^2 + 4x + 8}} dx$	completing the square on the radicand ; then maybe trig sub
K) $\int \frac{2x^2 - 9x}{(x-2)^3} dx$	partial fraction decomposition
L) $\int_1^e \frac{\ln x}{x} dx$	u-sub
M) $\int \tan^4\theta \, d\theta$	trig Pyth ident: $\tan^2\theta + 1 = \sec^2\theta$ to start…
N) $\int \frac{e^{\frac{1}{x}}}{x^2} dx$	u-sub
O) $\int_0^{\frac{\pi}{4}} x \cos x dx$	integration by parts
P) $\int \frac{x^2 - 4x + 7}{(x+1)(x^2 - 2x + 3)} dx$	partial fraction decomposition
Q) $\int x^2 e^{2x} \, dx$	integration by parts

Integration by Substitution

2)

$$\int \frac{2s}{\sqrt[3]{6-5s^2}}ds \qquad \text{Let } u=6-5s^2 \;\; ; \;\; du=-10s\ ds$$

$$=2\int \frac{(s\ ds)}{\sqrt[3]{6-5s^2}} \qquad -\frac{du}{10}=s\ ds$$

$$=2\int \frac{\left(-\frac{du}{10}\right)}{\sqrt[3]{u}}$$

$$=-\frac{1}{5}\int u^{-1/3}du$$

$$=-\frac{1}{5}\cdot\frac{3}{2}u^{2/3}+C$$

$$=-\frac{3}{10}\left(6-5s^2\right)^{2/3}+C$$

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3)

$$\int \frac{x^2}{2x^3-1}dx \qquad \text{Let } u=2x^3-1 \;\; ; \;\; du=6x^2dx$$

$$=\int \frac{(x^2dx)}{2x^3-1} \qquad \frac{du}{6}=x^2dx$$

$$=\int \frac{\left(\frac{du}{6}\right)}{u}$$

$$=\frac{1}{6}\int \frac{1}{u}du$$

$$=\frac{1}{6}\ln|u|+C$$

$$=\frac{1}{6}\ln\left|2x^3-1\right|+C$$

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4)

$$\int_1^e \frac{\ln x}{x}\,dx$$ Let $u=\ln x$; $du=\frac{1}{x}dx$

$$=\int_0^1 u\,du$$ $x=1 \rightarrow u=\ln(1)=0$

$$=\frac{1}{2}u^2\Big]_0^1$$ $x=e \rightarrow u=\ln(e)=1$

$$=\frac{1}{2}$$

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5)

$$\int \frac{\csc^2 x}{\cot^3 x}\,dx$$

$$=\int \frac{\frac{1}{\sin^2 x}}{\frac{\cos^3 x}{\sin^3 x}}\,dx$$

$$=\int \frac{\sin x}{\cos^3 x}\,dx$$ Let $u=\cos x$; $du=-\sin x\,dx$

$$=\int u^{-3}(-du)$$

$$=\frac{1}{2}u^{-2}+C$$

$$=\frac{1}{2}\cdot\frac{1}{\cos^2 x}+C$$

$$=\frac{1}{2}\sec^2 x+C$$

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6)

$$\int \frac{dx}{x^2-4x+13}$$

$$= \int \frac{dx}{(x-2)^2+9}$$

$$= \int \frac{dx}{9\left(\frac{(x-2)^2}{9}+1\right)}$$

$$= \frac{1}{9}\int \frac{dx}{\left(\frac{x-2}{3}\right)^2+1} \qquad \text{Let } u = \frac{x-2}{3} \quad ; \quad du = \frac{1}{3}dx$$

$$= \frac{1}{9}\int \frac{3du}{u^2+1} \qquad 3du = dx$$

$$= \frac{1}{3}\int \frac{1}{u^2+1}\, du$$

$$= \frac{1}{3}\tan^{-1} u + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$$

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7)

$$\int \frac{e^{\frac{1}{x}}}{x^2}dx \qquad \text{Let } u = \frac{1}{x} \quad ; \quad du = -\frac{1}{x^2}dx$$

$$= \int e^{\frac{1}{x}} \cdot \frac{1}{x^2}dx$$

$$= \int e^u(-du)$$

$$= -e^u + C$$

$$= -e^{\frac{1}{x}} + C$$

Trigonometric Integrals

8)

$$\int \cos^3\theta \sin^4\theta \, d\theta$$
$$= \int \cos\theta\left(\cos^2\theta\right)\sin^4\theta \, d\theta$$
$$= \int \cos\theta\left(1-\sin^2\theta\right)\sin^4\theta \, d\theta$$
$$= \int \cos\theta\left(\sin^4\theta - \sin^6\theta\right) d\theta \qquad \text{Let } u = \sin\theta \quad ; \quad du = \cos\theta \, d\theta$$
$$= \int \left(u^4 - u^6\right) du$$
$$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$$
$$= \frac{1}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta + C$$

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9)

$$\int \left(\cos\theta + 1\right)^2 d\theta$$
$$= \int \left(\cos^2\theta + 2\cos\theta + 1\right) d\theta$$
$$= \int \cos^2\theta \, d\theta + \int 2\cos\theta \, d\theta + \int d\theta$$
$$= \int \cos^2\theta \, d\theta + 2\sin\theta + \theta + C$$
$$= 2\sin\theta + \theta + C + \int \cos^2\theta \, d\theta$$
$$= 2\sin\theta + \theta + C + \int \frac{1}{2}\left(1+\cos 2\theta\right) d\theta \qquad \text{Recall the identity: } \cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$$
$$= 2\sin\theta + \theta + C + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$
$$= \frac{1}{4}\sin 2\theta + 2\sin\theta + \frac{3}{2}\theta + C$$

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10)

$\int \tan^4 \theta \, d\theta$

$= \int \tan^2 \theta \cdot \tan^2 \theta \, d\theta$

$= \int \left(\sec^2 \theta - 1\right) \tan^2 \theta \, d\theta$ Recall the identity: $\tan^2 \theta + 1 = \sec^2 \theta$

$= \int \left(\tan^2 \theta \sec^2 \theta - \tan^2 \theta\right) \, d\theta$

$= \int \tan^2 \theta \sec^2 \theta \, d\theta - \int \tan^2 \theta \, d\theta$ Let $u = \tan \theta$; $du = \sec^2 \theta \, d\theta$

$= \int u^2 du - \int \tan^2 \theta \, d\theta$

$= \dfrac{1}{3} u^3 + C - \int \tan^2 \theta \, d\theta$

$= \dfrac{1}{3} \tan^3 \theta + C - \int \tan^2 \theta \, d\theta$

$= \dfrac{1}{3} \tan^3 \theta + C - \int \left(\sec^2 \theta - 1\right) \, d\theta$

$= \dfrac{1}{3} \tan^3 \theta + C - \left(\tan \theta - \theta\right)$

$= \dfrac{1}{3} \tan^3 \theta - \tan \theta + \theta + C$

Integration by Parts Reminder: $\int u \, dv = uv - \int v \, du$

11)

$\int x^3 \ln x \, dx$ Recall the IBP formula: $\int u dv = uv - \int v du$

$= \int \ln x \left(x^3 \, dx\right)$ Let $u = \ln x$; $dv = x^3 dx$

$= \left(\ln x\right)\left(\dfrac{1}{4} x^4\right) - \int \left(\dfrac{1}{4} x^4\right)\left(\dfrac{1}{x} dx\right)$ $du = \dfrac{1}{x} dx$; $v = \dfrac{1}{4} x^4$

$= \dfrac{1}{4} x^4 \ln x - \dfrac{1}{4} \int x^3 dx$

$= \dfrac{1}{4} x^4 \ln x - \dfrac{1}{4}\left(\dfrac{1}{4} x^4\right) + C$

$= \dfrac{1}{4} x^4 \left(\ln x - \dfrac{1}{4}\right) + C$

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12)

$\int_0^{\pi/4} x\cos x\, dx$ Recall the IBP formula: $\int u dv = uv - \int v du$

$= \int_0^{\pi/4} x\left(\cos x\, dx\right)$ Let $u = x$; $dv = \cos x\, dx$

$= \left((x)(\sin x)\right)\Big]_0^{\pi/4} - \int_0^{\pi/4} (\sin x)(dx)$ $du = dx$; $v = \sin x$

$= x\sin x\Big]_0^{\pi/4} + (\cos x)\Big]_0^{\pi/4}$

$= \left\{\frac{\pi}{4}\sin\frac{\pi}{4} - 0\sin 0\right\} + \left\{\cos\frac{\pi}{4} - \cos 0\right\}$

$= \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} - 0 + \frac{\sqrt{2}}{2} - 1$

$= \frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + 1\right) - 1$

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13)

$\int e^{2x}\cos 3x\, dx$ Recall the IBP formula: $\int u dv = uv - \int v du$

$= \int \cos 3x\left(e^{2x}\, dx\right)$ Let $u = \cos 3x$; $dv = e^{2x} dx$

$= (\cos 3x)\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)(-3\sin 3x\, dx)$ $du = -3\sin 3x\, dx$; $v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\, dx$ Let $u = \sin 3x$; $dv = e^{2x} dx$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int \sin 3x\left(e^{2x}\, dx\right)$ $du = 3\cos 3x\, dx$; $v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left\{(\sin 3x)\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)(3\cos 3x\, dx)\right\}$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left\{\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\, dx\right\}$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\, dx$

And so, we have that for $I = \int e^{2x}\cos 3x\, dx$

$I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}(I)$

Solving for *I* we get…

$$I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}(I)$$

$$I+\frac{9}{4}I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x$$

$$\frac{13}{4}I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x$$

$$I=\frac{4}{13}\left(\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x\right)$$

$$I=\frac{2}{13}e^{2x}\cos 3x+\frac{3}{13}e^{2x}\sin 3x$$

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14)

$\int x^2e^{2x}\,dx$ Recall the IBP formula: $\int udv=uv-\int vdu$

$=\int x^2\left(e^{2x}\,dx\right)$ Let $u=x^2$; $dv=e^{2x}dx$

$=\left(x^2\right)\left(\frac{1}{2}e^{2x}\right)-\int\left(\frac{1}{2}e^{2x}\right)(2xdx)$ $du=2xdx$; $v=\frac{1}{2}e^{2x}$

$=\frac{1}{2}x^2e^{2x}-\int x\left(e^{2x}dx\right)$ Let $u=x$; $dv=e^{2x}dx$

$=\frac{1}{2}x^2e^{2x}-\left\{(x)\left(\frac{1}{2}e^{2x}\right)-\int\left(\frac{1}{2}e^{2x}\right)(dx)\right\}$ $du=dx$; $v=\frac{1}{2}e^{2x}$

$$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{2}\int e^{2x}dx$$

$$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{2}\left(\frac{1}{2}e^{2x}\right)+C$$

$$=\frac{1}{2}\left(x^2e^{2x}-xe^{2x}+\frac{1}{2}e^{2x}\right)+C$$

Integration by Trigonometric Substitution

15)

$$\int \frac{x}{\sqrt{x^2+4x+8}}\,dx$$

$$= \int \frac{x}{\sqrt{\left(x^2+4x+4\right)+4}}\,dx$$

$$= \int \frac{x}{\sqrt{\left(x+2\right)^2+4}}\,dx$$

Let $x+2=2\tan\theta$; $dx=2\sec^2\theta\ d\theta$

$$= \int \frac{\left(2\tan\theta-2\right)}{\sqrt{\left(2\tan\theta\right)^2+4}}\left(2\sec^2\theta\ d\theta\right)$$

$x=2\tan\theta-2$

$$= 4\int \frac{\left(\tan\theta-1\right)}{\sqrt{4\tan^2\theta+4}}\left(\sec^2\theta\ d\theta\right)$$

$$= 4\cdot\frac{1}{2}\int \frac{\left(\tan\theta-1\right)\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,d\theta$$

$$= 2\int \frac{\left(\tan\theta-1\right)\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta$$

Recall the identity: $\tan^2\theta+1=\sec^2\theta$

$$= 2\int \left(\tan\theta-1\right)\sec\theta\ d\theta$$

$$= 2\int \left(\sec\theta\tan\theta-\sec\theta\right)d\theta$$

$$= 2\int \sec\theta\tan\theta\ d\theta-2\int\sec\theta\ d\theta$$

$$= 2\left(\sec\theta\right)-2\left(\ln\left|\sec\theta+\tan\theta\right|\right)+C$$

$$= 2\left(\sec\theta-\ln\left|\sec\theta+\tan\theta\right|\right)+C$$

back-subbing using the declaration $x+2=2\tan\theta$

$$= 2\left(\left(\frac{\sqrt{\left(x+2\right)^2+4}}{2}\right)-\ln\left|\left(\frac{\sqrt{\left(x+2\right)^2+4}}{2}\right)+\left(\frac{x+2}{2}\right)\right|\right)+C$$

and right-triangle consequences

$$= \sqrt{\left(x+2\right)^2+4}-2\ln\left|\frac{1}{2}\left(\sqrt{\left(x+2\right)^2+4}+x+2\right)\right|+C$$

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16)

$$\int \frac{x^2}{\sqrt{2x-x^2}}\,dx$$

$$= \int \frac{x^2}{\sqrt{-\left(x^2-2x\right)}}\,dx$$

$$= \int \frac{x^2}{\sqrt{-\left(x^2-2x+1-1\right)}}\,dx$$

$$= \int \frac{x^2}{\sqrt{-\left(\left(x-1\right)^2-1\right)}}\,dx$$

$$= \int \frac{x^2}{\sqrt{1-\left(x-1\right)^2}}\,dx \qquad \text{Let } x-1=\sin\theta \quad ; \quad dx=\cos\theta\; d\theta$$

$$= \int \frac{\left(\sin\theta+1\right)^2}{\sqrt{1-\left(\sin\theta\right)^2}}\left(\cos\theta\; d\theta\right) \qquad x=\sin\theta+1$$

$$= \int \left(\sin\theta+1\right)^2\, d\theta$$

$$= \int \left(\sin^2\theta+2\sin\theta+1\right)\, d\theta \qquad \text{Recall the identity: } \sin^2\theta=\frac{1}{2}\left(1-\cos 2\theta\right)$$

$$= \int \left(\frac{1}{2}\left(1-\cos 2\theta\right)+2\sin\theta+1\right)\, d\theta$$

$$= \int \left(-\frac{1}{2}\cos 2\theta+2\sin\theta+\frac{3}{2}\right)\, d\theta$$

$$= -\frac{1}{2}\left(\frac{1}{2}\sin 2\theta\right)+2\left(-\cos\theta\right)+\frac{3}{2}\theta+C$$

$$= -\frac{1}{4}\sin 2\theta-2\cos\theta+\frac{3}{2}\theta+C$$

$$= -\frac{1}{4}\left(2\sin\theta\cos\theta\right)-2\cos\theta+\frac{3}{2}\theta+C$$

$$= -\frac{1}{2}\sin\theta\cos\theta-2\cos\theta+\frac{3}{2}\theta+C \qquad \text{back-subbing using the declaration } x-1=\sin\theta$$

$$= -\frac{1}{2}\left(x-1\right)\left(\sqrt{1-\left(1-x\right)^2}\right)-2\left(\sqrt{1-\left(1-x\right)^2}\right)+\frac{3}{2}\left(\sin^{-1}\left(x-1\right)\right)+C \qquad \text{and right-triangle consequences}$$

$$= -\frac{1}{2}\left(x-1\right)\sqrt{1-\left(1-x\right)^2}-2\sqrt{1-\left(1-x\right)^2}+\frac{3}{2}\sin^{-1}\left(x-1\right)+C$$

Integration Using Partial Fractions

17) $\displaystyle\int \frac{x^2-4x+7}{(x+1)(x^2-2x+3)}\,dx$

Since the denominator has a linear term and an irreducible quadratic, we have the partial fraction decomposition:

$$\frac{x^2-4x+7}{(x+1)(x^2-2x+3)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-2x+3}$$

$$x^2-4x+7=A\left(x^2-2x+3\right)+\left(Bx+C\right)\left(x+1\right)$$

$$x^2-4x+7=Ax^2-2Ax+3A+Bx^2+Cx+Bx+C$$

$$x^2-4x+7=\left(A+B\right)x^2+\left(-2A+C+B\right)x+\left(3A+C\right)$$

$$A+B=1$$

$$-2A+C+B=-4$$

$$3A+C=7$$

Putting these together...

$A=2 \quad ; \quad B=-1 \quad ; \quad C=1$

Therefore…

$$\frac{x^2-4x+7}{(x+1)(x^2-2x+3)}=\frac{\left(2\right)}{x+1}+\frac{\left(-1\right)x+\left(1\right)}{x^2-2x+3}$$

$$\int\frac{x^2-4x+7}{(x+1)(x^2-2x+3)}\,dx=\int\left(\frac{2}{x+1}+\frac{-x+1}{x^2-2x+3}\right)dx$$

$$=\int\frac{2}{x+1}dx-\int\frac{x-1}{x^2-2x+3}dx$$

$$=2\ln\left|x+1\right|-\int\frac{x-1}{x^2-2x+3}dx$$

Let $u=x^2-2x+3$; $du=\left(2x-2\right)dx$

$$=2\ln\left|x+1\right|-\int\frac{\left(x-1\right)dx}{\left(x^2-2x+3\right)}$$

$$\frac{du}{2}=\left(x-1\right)dx$$

$$=2\ln\left|x+1\right|-\int\frac{\frac{du}{2}}{u}$$

$$=2\ln\left|x+1\right|-\frac{1}{2}\int\frac{1}{u}du$$

$$=2\ln\left|x+1\right|-\frac{1}{2}\ln\left|u\right|+C$$

$$=2\ln\left|x+1\right|-\frac{1}{2}\ln\left|x^2-2x+3\right|+C$$

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18) $\displaystyle\int\frac{2x^2-9x}{(x-2)^3}dx$

Since the denominator has repeated linear terms, we have the partial fraction decomposition:

$$\frac{2x^2-9x}{(x-2)^3}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}$$

$$2x^2-9x=A(x-2)^2+B(x-2)+C$$

$$2x^2-9x=Ax^2-4Ax+4A+Bx-2B+C$$

$$2x^2-9x=(A)x^2+(B-4A)x+(4A-2B+C)$$

$$A=2$$

$$B-4A=-9$$

$$4A-2B+C=0$$

Putting these together...

$A=2 \quad ; \quad B=-1 \quad ; \quad C=-10$

Therefore…

$$\frac{2x^2-9x}{(x-2)^3}=\frac{2}{x-2}-\frac{1}{(x-2)^2}-\frac{10}{(x-2)^3}$$

$$\int\frac{2x^2-9x}{(x-2)^3}dx=\int\left(\frac{2}{x-2}-\frac{1}{(x-2)^2}-\frac{10}{(x-2)^3}\right)dx$$

$$=2\ln|x-2|+\frac{1}{x-2}+\frac{5}{(x-2)^2}+C$$

Improper Integrals

19) $\int_0^\infty xe^{-2x}dx$

We see that the integral is improper, and so the initial rewrite is:

$$\int_0^\infty xe^{-2x}dx=\lim_{t\to\infty}\int_0^t xe^{-2x}dx$$

Now, we notice that $\int xe^{-2x}dx$ requires integration by parts: $\int udv=vu-\int vdu$

Let: $\quad u=x \qquad dv=e^{-2x}dx$

$\qquad du=dx \qquad v=-\frac{1}{2}e^{-2x}$

So,

$$\int xe^{-2x}dx=\left(-\frac{1}{2}e^{-2x}\right)(x)-\int\left(-\frac{1}{2}e^{-2x}\right)(dx)$$

$$=-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}dx$$

$$=-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}$$

(and since we'll immediately be using this result in a definite integral, we'll leave the " $+C$ " off.

Now, since $\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$

We have that:

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t xe^{-2x}dx &= \lim_{t\to\infty}\left\{\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)\Bigg]_0^t\right\} \\
&= \lim_{t\to\infty}\left\{\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) - \left(0 - \frac{1}{4}\right)\right\} \\
&= \lim_{t\to\infty}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) + \frac{1}{4} \\
&= \lim_{t\to\infty}\left(-\frac{t}{2e^{2t}} - \frac{1}{4e^{2t}}\right) + \frac{1}{4} \\
&= \lim_{t\to\infty}\left(-\frac{t}{2e^{2t}}\right) - \lim_{t\to\infty}\left(\frac{1}{4e^{2t}}\right) + \frac{1}{4} \qquad \text{(so long as both limits exist)}
\end{aligned}$$

Now,

$$\lim_{t\to\infty}\left(-\frac{t}{2e^{2t}}\right) \to -\frac{\infty}{\infty} \quad \left(L'H \ \ \text{form}\right)$$

So,

$$\lim_{t\to\infty}\left(-\frac{t}{2e^{2t}}\right) \overset{L'H}{=} \lim_{t\to\infty}\left(-\frac{1}{4e^{2t}}\right) = 0$$

And

$$\lim_{t\to\infty}\left(\frac{1}{4e^{2t}}\right) = 0$$

And so, in total:

$$\begin{aligned}
\int_0^\infty xe^{-2x}dx &= \lim_{t\to\infty}\int_0^t xe^{-2x}dx \\
&= \lim_{t\to\infty}\left(-\frac{t}{2e^{2t}}\right) - \lim_{t\to\infty}\left(\frac{1}{4e^{2t}}\right) + \frac{1}{4} \\
&= (0) - (0) + \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

(or is said to converge)

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20) $I = \int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\, dx$

$$
\begin{aligned}
I &= \int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\, dx \\
&= \lim_{t\to -2^+} \int_t^{14} \frac{1}{\sqrt[4]{x+2}}\, dx \\
&= \lim_{t\to -2^+} \int_t^{14} (x+2)^{-1/4}\, dx \\
&= \lim_{t\to -2^+} \left(\frac{4}{3}(x+2)^{3/4}\right)\Bigg]_t^{14} \\
&= \lim_{t\to -2^+} \left\{\left(\frac{4}{3}(14+2)^{3/4}\right)-\left(\frac{4}{3}(t+2)^{3/4}\right)\right\} \\
&= \lim_{t\to -2^+} \left\{\frac{32}{3}-\frac{4}{3}(t+2)^{3/4}\right\} \\
&= \frac{32}{3}-\frac{4}{3}\lim_{t\to -2^+}(t+2)^{3/4} \\
&= \frac{32}{3}-\frac{4}{3}(0) \\
&= \frac{32}{3}
\end{aligned}
$$

(or is said to converge)

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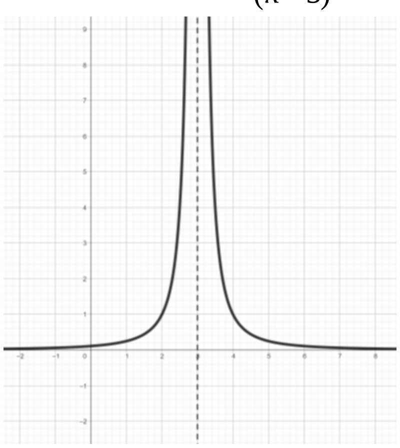
21) $\int_1^5 \frac{dx}{(x-3)^2}$

We notice that $y=\frac{1}{(x-3)^2}$ has a vertical asymptote at $x=3$



We would like to break up the integral $\int_1^5 \frac{dx}{(x-3)^2} = \int_1^3 \frac{dx}{(x-3)^2} + \int_3^5 \frac{dx}{(x-3)^2}$

and evaluate the two parts however we quickly notice that won't work since $\int \frac{dx}{(x-3)^2} = -\frac{1}{x-3}$

cannot be evaluated at $x=3$

So instead, we write:

$$\int_1^5 \frac{dx}{(x-3)^2} = \lim_{t\to 3^-}\int_1^t \frac{dx}{(x-3)^2} + \lim_{s\to 3^+}\int_s^5 \frac{dx}{(x-3)^2}$$

Now,

$$\lim_{t\to 3^-}\int_1^t \frac{dx}{(x-3)^2} = \lim_{t\to 3^-}\left\{ -\frac{1}{x-3}\Big]_1^t \right\}$$

$$= \lim_{t\to 3^-}\left\{ -\frac{1}{t-3} + \frac{1}{1-3} \right\}$$

$$= \lim_{t\to 3^-}\left(-\frac{1}{t-3} \right) - \frac{1}{2} \to +\infty$$

And

$$\lim_{s\to 3^+}\int_s^5 \frac{dx}{(x-3)^2} = \lim_{s\to 3^+}\left\{ -\frac{1}{x-3}\Big]_s^5 \right\}$$

$$= \lim_{s\to 3^+}\left\{ -\frac{1}{5-3} + \frac{1}{s-3} \right\}$$

$$= \lim_{s\to 3^+}\left(\frac{1}{s-3} \right) - \frac{1}{2} \to +\infty$$

Therefore,

$$\int_1^5 \frac{dx}{(x-3)^2} \to \infty \quad \text{(or is said to be divergent)}$$
