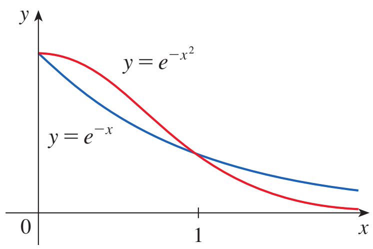

FIGURE 13

and observe that the first integral on the right-hand side is just an ordinary definite integral with a finite value. In the second integral we use the fact that for $x \geqslant 1$ we have $x^2 \geqslant x$, so $-x^2 \leqslant -x$ and therefore $e^{-x^2} \leqslant e^{-x}$. (See Figure 13.) The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-1}$$

Therefore, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}dx$ is convergent. It follows that $\int_0^\infty e^{-x^2}dx$ is convergent also. ■

Table 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

In Example 9 we showed that $\int_0^\infty e^{-x^2}dx$ is convergent without computing its value. In Exercise 84 we indicate how to show that its value is approximately 0.8862. In probability theory it is important to know the exact value of this improper integral, as we will see in Section 8.5; using the methods of multivariable calculus it can be shown that the exact value is $\sqrt{\pi}/2$. Table 1 illustrates the definition of a convergent improper integral by showing how the (computer-generated) values of $\int_0^t e^{-x^2}dx$ approach $\sqrt{\pi}/2$ as t becomes large. In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

Table 2

t	$\int_1^t [(1 + e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. ■

Table 2 illustrates the divergence of the integral in Example 10. It appears that the values are not approaching any fixed number.

7.8 Exercises

1. Explain why each of the following integrals is improper.

(a) $\displaystyle\int_1^4 \frac{dx}{x-3}$ (b) $\displaystyle\int_3^\infty \frac{dx}{x^2-4}$

(c) $\displaystyle\int_0^1 \tan \pi x\,dx$ (d) $\displaystyle\int_{-\infty}^{-1} \frac{e^x}{x}\,dx$

2. Which of the following integrals are improper? Why?

(a) $\displaystyle\int_0^\pi \sec x\,dx$ (b) $\displaystyle\int_0^4 \frac{dx}{x-5}$

(c) $\displaystyle\int_{-1}^3 \frac{dx}{x+x^3}$ (d) $\displaystyle\int_1^\infty \frac{dx}{x+x^3}$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geqslant 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in both the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.
(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.
(c) Find the total area under each curve for $x \geqslant 1$, if it exists.

5–48 Determine whether the integral is convergent or divergent. Evaluate integrals that are convergent.

5. $\displaystyle\int_1^\infty 2x^{-3}\,dx$ **6.** $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,dx$

7. $\displaystyle\int_0^\infty e^{-2x}\,dx$ **8.** $\displaystyle\int_1^\infty \left(\tfrac{1}{3}\right)^x dx$

9. $\displaystyle\int_{-2}^\infty \frac{1}{x+4}\,dx$ **10.** $\displaystyle\int_1^\infty \frac{1}{x^2+4}\,dx$

11. $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$

12. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

13. $\displaystyle\int_{-\infty}^0 \frac{x}{(x^2+1)^3}\,dx$

14. $\displaystyle\int_{-\infty}^{-3} \frac{x}{4-x^2}\,dx$

15. $\displaystyle\int_1^\infty \frac{x^2+x+1}{x^4}\,dx$

16. $\displaystyle\int_2^\infty \frac{x}{\sqrt{x^2-1}}\,dx$

17. $\displaystyle\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$

18. $\displaystyle\int_{-\infty}^{-1} \frac{x^2+x}{x^3}\,dx$

19. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

20. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

21. $\displaystyle\int_{-\infty}^\infty \cos 2t\,dt$

22. $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$

23. $\displaystyle\int_0^\infty \sin^2\alpha\,d\alpha$

24. $\displaystyle\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta$

25. $\displaystyle\int_1^\infty \frac{1}{x^2+x}\,dx$

26. $\displaystyle\int_2^\infty \frac{dv}{v^2+2v-3}$

27. $\displaystyle\int_{-\infty}^0 ze^{2z}\,dz$

28. $\displaystyle\int_2^\infty ye^{-3y}\,dy$

29. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

30. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

31. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4+4}\,dz$

32. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

33. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy$

34. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$

35. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

36. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx$

37. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

38. $\displaystyle\int_{-1}^2 \frac{x}{(x+1)^2}\,dx$

39. $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx$

40. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

41. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

42. $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$

43. $\displaystyle\int_0^{\pi/2} \tan^2\theta\,d\theta$

44. $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$

45. $\displaystyle\int_0^1 r\ln r\,dr$

46. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$

47. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

48. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

49–54 Sketch the region and find its area (if the area is finite).

49. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant e^{-x}\}$

50. $S = \{(x, y) \mid x \leqslant 0,\ 0 \leqslant y \leqslant e^{x}\}$

51. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/(x^3+x)\}$

52. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant xe^{-x}\}$

53. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

54. $S = \{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\,\}$

55. (a) If $g(x) = (\sin^2 x)/x^2$, use a calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2, 5,$ $10, 100, 1000,$ and $10{,}000$. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?
(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

56. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use a calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5, 10, 100, 1000,$ and $10{,}000$. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?
(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

57–64 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

57. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

58. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

59. $\displaystyle\int_2^\infty \frac{1}{x-\ln x}\,dx$

60. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

61. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

62. $\displaystyle\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx$

63. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

64. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

65–68 Improper Integrals that Are Both Type 1 and Type 2
The integral $\int_a^\infty f(x)\,dx$ is improper because the interval $[a, \infty)$ is infinite. If f has an infinite discontinuity at a, then the integral is improper for a second reason. In this case we evaluate the integral by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx \qquad c > a$$

Evaluate the given integral if it is convergent.

65. $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$

66. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx$

67. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$

68. $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$

69–71 Find the values of p for which the integral converges and evaluate the integral for those values of p.

69. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

70. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

71. $\displaystyle\int_0^1 x^p \ln x\,dx$

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72. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.
(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.
(c) Prove your guess using mathematical induction.

73. The *Cauchy principal value* of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$$

Show that $\int_{-\infty}^{\infty} x\,dx$ diverges but the Cauchy principal value of this integral is 0.

74. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

75. We know from Example 1 that the region

$$\mathcal{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$$

has infinite area. Show that by rotating $\mathcal{R}$ about the x-axis we obtain a solid (called *Gabriel's horn*) with finite volume.

76. Use the information and data in Exercise 6.4.35 to find the work required to propel a 1000-kg space vehicle out of the earth's gravitational field.

77. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use Newton's Law of Gravitation (see Exercise 6.4.35) and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

78. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}}\,x(r)\,dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R - r)^2$, find the perceived density $y(s)$.

79. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.
(a) Make a rough sketch of what you think the graph of F might look like.
(b) What is the meaning of the derivative $r(t) = F'(t)$?
(c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

80. As we saw in Section 3.8, a radioactive substance decays exponentially: The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life* M of an atom in the substance is

$$M = -k\int_0^\infty te^{kt}\,dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

81. In a study of the spread of illicit drug use from an enthusiastic user to a population of N users, the authors model the number of expected new users by the equation

$$\gamma = \int_0^\infty \frac{cN(1 - e^{-kt})}{k}\,e^{-\lambda t}\,dt$$

where c, k, and λ are positive constants. Evaluate this integral to express γ in terms of c, N, k, and λ.

Source: F. Hoppensteadt et al., "Threshold Analysis of a Drug Use Epidemic Model," *Mathematical Biosciences* 53 (1981): 79–87.

82. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V}C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^\infty u(t)$ and interpret it.

83. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2+1}\,dx < 0.001$$

84. Estimate the numerical value of $\int_0^\infty e^{-x^2}\,dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\,dx$ and $\int_4^\infty e^{-x^2}\,dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}\,dx$, which is less than 0.0000001.

85–87 The Laplace Transform If $f(t)$ is continuous for $t \geqslant 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

and the domain of F is the set consisting of all numbers s for which the integral converges.

85. Find the Laplace transform of each of the following functions.
(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

86. Show that if $0 \leqslant f(t) \leqslant Me^{at}$ for $t \geqslant 0$, where M and a are constants, then the Laplace transform $F(s)$ exists for $s > a$.

87. Suppose that $0 \leqslant f(t) \leqslant Me^{at}$ and $0 \leqslant f'(t) \leqslant Ke^{at}$ for $t \geqslant 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$ and the Laplace transform of $f'(t)$ is $G(s)$, show that

$$G(s) = sF(s) - f(0) \qquad s > a$$

88. If $\int_{-\infty}^\infty f(x)\,dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

89. Show that $\int_0^\infty x^2 e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$.

90. Show that $\int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln y}\,dy$ by interpreting the integrals as areas.

91. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right) dx$$

converges. Evaluate the integral for this value of C.

92. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$$

converges. Evaluate the integral for this value of C.

93. Suppose f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)\,dx$ is convergent?

94. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx$$

7 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x\,dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
(a) $\int_a^\infty f(x)\,dx$ (b) $\int_{-\infty}^b f(x)\,dx$ (c) $\int_{-\infty}^\infty f(x)\,dx$

7. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.
(a) f has an infinite discontinuity at a.
(b) f has an infinite discontinuity at b.
(c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.