M408D Final Exam Review – Day 4 – SOLUTIONS

1) Determine $f_x - f_y$ when $f(x,y) = x^2 + 3xy + y^2$

$$f(x,y) = x^2 + 3xy + y^2$$
$$f_x = 2x + 3y$$
$$f_y = 3x + 2y$$
$$f_x - f_y = 2x + 3y - (3x + 2y) = y - x$$

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2) Determine f_{xy} when $f(x,y) = \sin(xy) + e^{\frac{x}{y}}$

$$f(x,y) = \sin(xy) + e^{\frac{x}{y}}$$
$$f_x = \cos(xy)\cdot y + e^{\frac{x}{y}}\cdot\frac{1}{y}$$
$$= y\cos(xy) + \frac{e^{\frac{x}{y}}}{y}$$
$$f_{xy} = \left\{(1)\cos(xy) + y\left(-\sin(xy)\cdot x\right)\right\} + \frac{y\left(e^{\frac{x}{y}}\cdot\frac{-x}{y^2}\right) - e^{\frac{x}{y}}(1)}{y^2}$$
$$= \cos(xy) - xy\sin(xy) - \frac{e^{\frac{x}{y}}}{y^2}\left(\frac{x}{y}+1\right)$$

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3) Suppose $w = f(x,y,z)$ where f is some function. Suppose that x, y and z are functions of t. Further suppose that $f_x = yz + 2x$, $f_y = ze^{yz} + xz + 2y$, and $f_z = ye^{yz} + xy$. If $x(0) = 2$, $y(0) = 1, z(0) = 1, x'(0) = 3, y'(0) = 4,$ and $z'(0) = 5$, find the derivative of w with respect to t when $t = 0$.

From the multivariable Chain Rule, we have that:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Using the notation provided in the problem, this becomes:

$$\frac{dw}{dt} = f_x\cdot\frac{dx}{dt} + f_y\cdot\frac{dy}{dt} + f_z\cdot\frac{dz}{dt}$$

And evaluating this expression at $t = 0$ might look like the following…

$$\begin{aligned}\left.\frac{dw}{dt}\right|_{t=0} &= \left.\left(f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} + f_z \cdot \frac{dz}{dt} \right)\right|_{t=0} \\ &= \left.\left(f_x \cdot \frac{dx}{dt} \right)\right|_{t=0} + \left.\left(f_y \cdot \frac{dy}{dt} \right)\right|_{t=0} + \left.\left(f_z \cdot \frac{dz}{dt} \right)\right|_{t=0} \\ &= \left(f_x\big|_{t=0} \right)\cdot\left(\left.\frac{dx}{dt}\right|_{t=0} \right) + \left(f_y\big|_{t=0} \right)\cdot\left(\left.\frac{dy}{dt}\right|_{t=0} \right) + \left(f_z\big|_{t=0} \right)\cdot\left(\left.\frac{dz}{dt}\right|_{t=0} \right) \\ &= \left(f_x\big|_{t=0} \right)\cdot\left(x'(0) \right) + \left(f_y\big|_{t=0} \right)\cdot\left(y'(0) \right) + \left(f_z\big|_{t=0} \right)\cdot\left(z'(0) \right)\end{aligned}$$

And now, using the specific functions and values we are given in the problem…

$$\begin{aligned}\left.\frac{dw}{dt}\right|_{t=0} &= \left(f_x\big|_{t=0} \right)\cdot\left(x'(0) \right) + \left(f_y\big|_{t=0} \right)\cdot\left(y'(0) \right) + \left(f_z\big|_{t=0} \right)\cdot\left(z'(0) \right) \\ &= \left((yz+2x)\big|_{t=0} \right)\cdot(3) + \left(\left(ze^{yz}+xz+2y\right)\big|_{t=0} \right)\cdot(4) + \left(\left(ye^{yz}+xy\right)\big|_{t=0} \right)\cdot(5) \\ &= \left(y(0)\cdot z(0) + 2\cdot x(0) \right)\cdot(3) + \left(z(0)\cdot e^{y(0)\cdot z(0)} + x(0)\cdot z(0) + 2\cdot y(0) \right)\cdot(4) + \left(y(0)e^{y(0)\cdot z(0)} + x(0)\cdot y(0) \right)\cdot(5) \\ &= \left((1)\cdot(1) + 2\cdot(2) \right)\cdot(3) + \left((1)\cdot e^{(1)\cdot(1)} + (2)\cdot(1) + 2\cdot(1) \right)\cdot(4) + \left((1)e^{(1)\cdot(1)} + (2)\cdot(1) \right)\cdot(5) \\ &= (1+4)\cdot(3) + (e+2+2)\cdot(4) + (e+2)\cdot(5) \\ &= 15 + 4e + 16 + 5e + 10 \\ &= 9e + 41\end{aligned}$$

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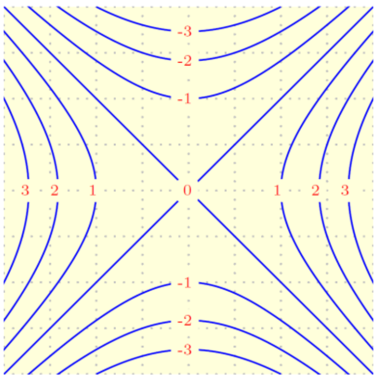
4) Given the following contour map for conic section function $f(x,y)$



A) What is the name for the graph of $f(x,y)$?

Hyperbolic Paraboloid

B) If you were standing at the point $P\left(3,0,f(3,0)\right)$, what would the values of f_x and f_y be (positive, negative or zero)?

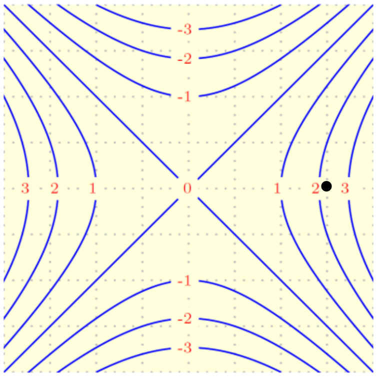
We recall that f_x is the slope of the surface when traveling in the *x*-positive direction.

That is, $f_x\big|_{(a,b)}$ is the slope of the surface at the point $\left(a,b,f(a,b)\right)$ looking "due east" or to the right.

Similarly, f_y is the slope of the surface when traveling in the *y*-positive direction.

The contour map is a look from above, down onto the *xy*-plane.

If we look at the location $(3,0)$ on the *xy*-plane, and imagine being at the point $P(3,0,f(3,0))$ on the surface of f …

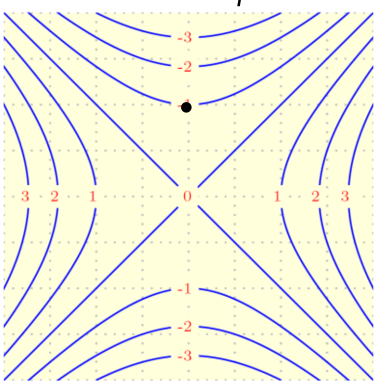


The place we are standing is altitude 2.3-ish and looking to the right, we'd be looking toward altitude 3, in other words, increasing. And so, $f_x > 0$

If we look in the upward direction, we would be looking tangentially to the contour line. The altitude at least very locally would remain at height 2.3, and so the altitude remains constant. That is, $f_y = 0$

C) If you were standing at the point $P(0,2,f(0,2))$, what would the values of f_x and f_y be (positive, negative or zero)?

If we look at the location $(0,2)$ on the *xy*-plane, and imagine being at the point $P(0,2,f(0,2))$ on the surface of f …



The place we are standing is altitude -1 and looking to the right, the altitude at least very locally would remain at height -1, and so the altitude remains constant. That is, $f_x = 0$

If we look in the upward direction, we'd be looking toward altitude -2, in other words, decreasing. And so, $f_y < 0$

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5) The radius of a right circular cone is increasing at a rate of 2 inches per minute while the height is decreasing at a rate of 1 inch per minute. Determine the rate of change of the volume when $r = 3$ and $h = 4$

The volume of a right circular cone is given by: $\frac{1}{3}\pi r^2 h$

For our purposes, we could be even more explicit, stating: $V(r,h) = \frac{1}{3}\pi r^2 h$

Note that the problem implies both *r* and *h* are functions of time, that is, $r = r(t)$ and $h = h(t)$

This problem calls for us to evaluate $\frac{dV}{dt}$ at a specific instance.
Therefore, we'll use the multivariable Chain Rule to take the derivative of *V* with respect to time, *t*

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

Computing the partials…

$$V(r,h) = \frac{1}{3}\pi r^2 h$$

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h$$

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

And so, substituting in our partial derivative expression and rates, we have…

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \left(\frac{2}{3}\pi r h\right) \cdot (2) + \left(\frac{1}{3}\pi r^2\right) \cdot (-1) \quad \text{(radius increasing, but height decreasing)}$$

$$\frac{dV}{dt} = \frac{4}{3}\pi r h - \frac{1}{3}\pi r^2$$

And finally, we evaluate this expressing at the instance when $r = 3$ and $h = 4$…

$$\frac{dV}{dt} = \frac{4}{3}\pi r h - \frac{1}{3}\pi r^2$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=3\\h=4}} = \left.\frac{4}{3}\pi r h - \frac{1}{3}\pi r^2\right|_{\substack{r=3\\h=4}}$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=3\\h=4}} = \frac{4}{3}\pi(3)(4) - \frac{1}{3}\pi(3)^2$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=3\\h=4}} = 13\pi \text{ in}^3/\text{min}$$

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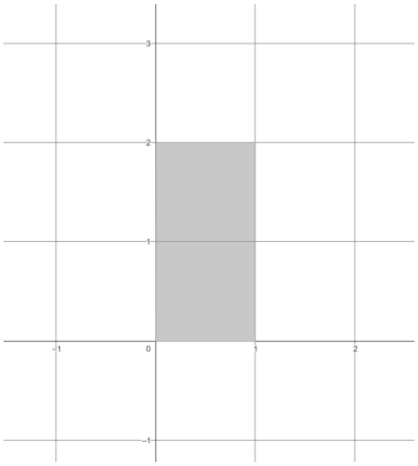
6) $\iint_{\Omega} x^2 + y \; dxdy$

where $\Omega = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2\}$

The area over which we are integrating is a rectangle:



And the integrand doesn't seem particularly difficult to integrate either with respect to *x* or with respect to *y*.
So we can just take the order of integration as given, that is, integrating first with respect to *x*, then with respect to *y*.

$$
\begin{aligned}
\iint_{\Omega}\left(x^2+y\right)dxdy &= \int_0^2\int_0^1\left(x^2+y\right)dxdy \\
&= \int_0^2\left(\frac{1}{3}x^3+xy\right)\Bigg]_0^1 dy \\
&= \int_0^2\left(\frac{1}{3}+y\right)dy \\
&= \left(\frac{1}{3}y+\frac{1}{2}y^2\right)\Bigg]_0^2 \\
&= \frac{1}{3}(2)+\frac{1}{2}(2)^2 \\
&= \frac{8}{3}
\end{aligned}
$$

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7) Evaluate the iterated integral $I=\int_0^1\int_0^3\frac{3y+x^2}{1+y^2}\,dxdy$

Note that in the numerator of the integrand, the *x*-variable occupies only one term, so we should separate the fraction into two terms. Note also the denominator has no *x*'s, so it may be considered as a constant coefficient with respect to *x*. We will rewrite it, then integrate with respect to *x*:

$$\int_0^1\int_0^3 \frac{3y+x^2}{1+y^2}\,dxdy = \int_0^1\int_0^3\left(\frac{3y}{1+y^2}+\frac{x^2}{1+y^2}\right)dxdy$$

$$= \int_0^1\left(\frac{3y}{1+y^2}\cdot x+\frac{1}{1+y^2}\cdot\frac{x^3}{3}\right)\Bigg]_0^3 dy$$

$$= \int_0^1\left(\frac{9y}{1+y^2}+\frac{9}{1+y^2}\right)dy$$

We will separate this into two integrals…

$$9\int_0^1 \frac{y}{1+y^2}\,dy + 9\int_0^1 \frac{1}{1+y^2}\,dy$$

The first of the integrals can be handled via *u*-substitution, with $u = 1+y^2$, and $du = 2ydy$
The corresponding lower bound is $u = 1$ and the corresponding upper bound is $u = 2$

$$I_1 = 9\int_0^1 \frac{y}{1+y^2}\,dy = \frac{9}{2}\int_1^2 \frac{1}{u}\,du = \frac{9}{2}\ln u\Big]_1^2 = \frac{9}{2}(\ln 2 - \ln 1) = \frac{9}{2}\ln 2$$

The second integral can be integrated using arctan:

$$I_2 = 9\int_0^1 \frac{1}{1+y^2}\,dy = 9\tan^{-1} y\Big]_0^1 = 9\left(\tan^{-1}1 - \tan^{-1}0\right) = 9\cdot\frac{\pi}{4}$$

Therefore:

$$\int_0^1\int_0^3 \frac{3y+x^2}{1+y^2}\,dxdy = \frac{9}{2}\ln 2 + 9\cdot\frac{\pi}{4} = \frac{9}{2}\left(\ln 2+\frac{\pi}{2}\right)$$

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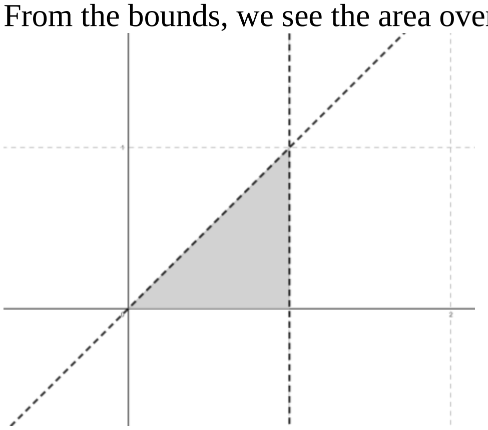
8) $\int\limits_0^1\int\limits_0^x (x+2y)\,dydx$

From the bounds, we see the area over which we are integrating looks like this:



As given, the order of integration is with respect to *y* first. By looking at the integrand, this doesn't appear to be an issue, so we'll keep the order of integration and proceed to compute the double integral.

$$\int_0^1\int_0^x (x+2y)\,dydx = \int_0^1 \left(xy+y^2\right)\Big]_0^x dx$$
$$= \int_0^1 \left(x^2+x^2\right)dx$$
$$= 2\int_0^1 x^2 dx$$
$$= \frac{2}{3}x^3\Big]_0^1$$
$$= \frac{2}{3}$$

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9) Find the volume of the solid bounded by the coordinate planes and the plane $x+y+z=1$

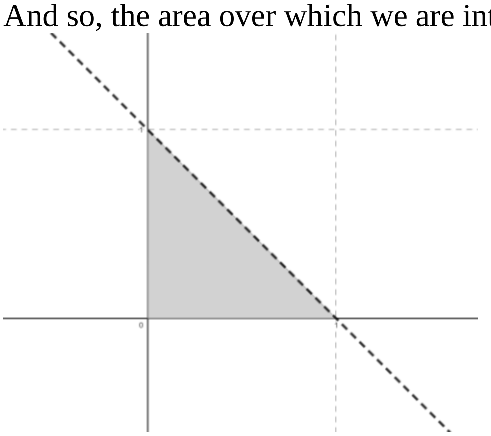
The plane $x+y+z=1$ intersects the *x*-axis at $(1,0,0)$
(we can think of solving for this intersection by letting $y=0$ and $z=0$)
Similarly, the plane $x+y+z=1$ intersects the *y*-axis at $(0,1,0)$ and the *z*-axis at $(0,0,1)$.

And "between" these points, the plane is above the *xy*-plane, which we can see with some made up sample points like: $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Also, we can determine how the plane intersects the *xy*-plane, by letting $z=0$
And we see the intersection forms the line: $y=1-x$

And so, after some investigation we see that the solid lies in the first octant, above the region formed by the lines: $x=0$, $y=0$, and $y=1-x$

And so, the area over which we are integrating would look like this:



And the function existing above this region, forming the top of our solid is given by:
$z=1-x-y$

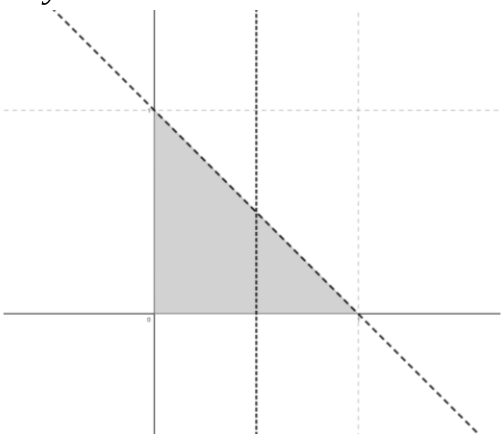
It doesn't appear that this function will be difficult to integrate, either with respect to *x* or with respect to *y* first.
Computing either $\iint(1-x-y)\,dxdy$ or $\iint(1-x-y)\,dydx$ seem equally easy.

So, we'll choose the latter, to compute $\iint (1-x-y)\,dydx$

For the "outer" integral, this one with respect to *x*, in determining the bounds, we can think "what is absolute smallest and absolute largest value that variable takes on over this space?" We might answer *x* ranges from $x=0$ to $x=1$ over this space.

Now, for the "inner" integral, we can ask "for each one of those *x*-values, for instance $x=1/2$, what does the y-value range from and to?"

So, for $x=1/2$ (for example), we can see the *y*-values (in the shaded region) range from $y=0$ to $y=$ the line



And this is true for all sample values of *x*, that is, if we had chosen $x=1/3$ or whatever: the *y*-values (in the shaded region) range from $y=0$ to $y=$ the line

So, now we can be slightly more fancy and say that for each *x* value, *y* ranges from $y=0$ to $y=1-x$

And that allows us to set up our bounds for the double integral: $\int_0^1\int_0^{1-x}(1-x-y)\,dy\;dx$
The only thing left, is to compute the answer.

The volume of the solid is given by…

$$V=\int_0^1\int_0^{1-x}(1-x-y)\,dy\;dx$$

$$=\int_0^1\left(\left(1-x\right)y-\frac{1}{2}y^2\right)\Bigg]_0^{1-x}dx$$

$$=\int_0^1\left(\left(1-x\right)^2-\frac{1}{2}\left(1-x\right)^2\right)dx$$

$$=\frac{1}{2}\int_0^1\left(1-x\right)^2\,dx$$

$$=-\frac{1}{6}\left(1-x\right)^3\Bigg]_0^1$$

$$=-\frac{1}{6}\left\{(0)^3-(1)^3\right\}$$

$$=\frac{1}{6}$$

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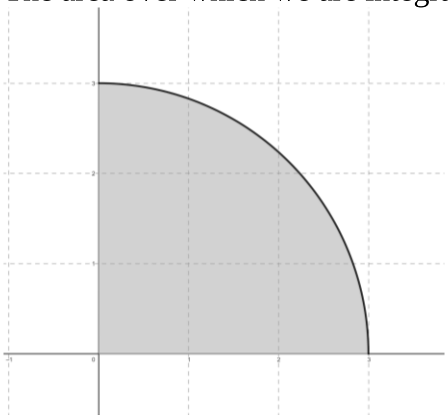
10) Find the volume, V, of the solid under the graph of the function $f(x, y) = 2x + y$ and over the region A in the first octant enclosed by a circle with center at the origin and radius 3.

The area over which we are integrating would look like this:



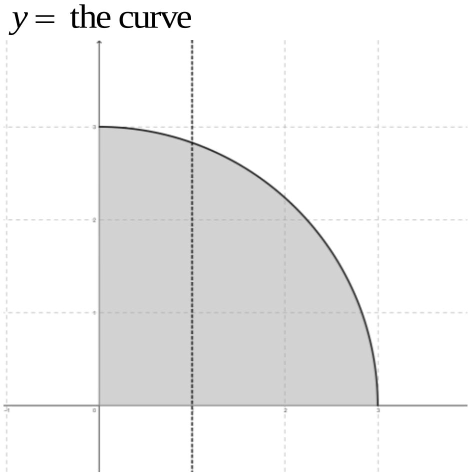
And the function existing above this region, forming the top of our solid is given by:
$f(x, y) = 2x + y$

It doesn't appear that this function will be difficult to integrate, either with respect to x or with respect to y first.
So, we'll choose the latter, to compute: $\iint (2x + y)\, dy dx$

For the "outer" integral, this one with respect to x, in determining the bounds, we can think "what is absolute smallest and absolute largest value that variable takes on over this space?" We might answer x ranges from $x = 0$ to $x = 3$ over this space.

Now, for the "inner" integral, we can ask "for each one of those x-values, for instance $x = 1$, what does the y-value range from and to?"

So, for $x = 1$ (for example), we can see the y-values (in the shaded region) range from $y = 0$ to $y =$ the curve



And this is true for all sample values of x, that is, if we had chosen $x = 2$ or whatever: the y-values (in the shaded region) range from $y = 0$ to $y =$ the curve

So, now we can be slightly more fancy and say that for each x value, y ranges from $y=0$ to $y=\sqrt{9-x^2}$

And that allows us to set up our bounds for the double integral: $\int_0^3\int_0^{\sqrt{9-x^2}}(2x+y)\,dydx$

The only thing left, is to compute the answer.

$$
\begin{aligned}
V &= \int_0^3\int_0^{\sqrt{9-x^2}}(2x+y)\,dydx \\
&= \int_0^3\left(2xy+\frac{1}{2}y^2\right)\Bigg|_0^{\sqrt{9-x^2}}dx \\
&= \int_0^3\left(2x\sqrt{9-x^2}+\frac{1}{2}\left(9-x^2\right)\right)dx \\
&= \int_0^3\left(2x\sqrt{9-x^2}+\frac{9}{2}-\frac{1}{2}x^2\right)dx \\
&= \left(-\frac{2}{3}\left(9-x^2\right)^{3/2}+\frac{9}{2}x-\frac{1}{6}x^3\right)\Bigg|_0^3 \\
&= \left(0+\frac{27}{2}-\frac{27}{6}\right)-\left(-\frac{2}{3}(27)+0-0\right) \\
&= 27\left(\frac{1}{2}-\frac{1}{6}+\frac{2}{3}\right) \\
&= 27(1)=27
\end{aligned}
$$

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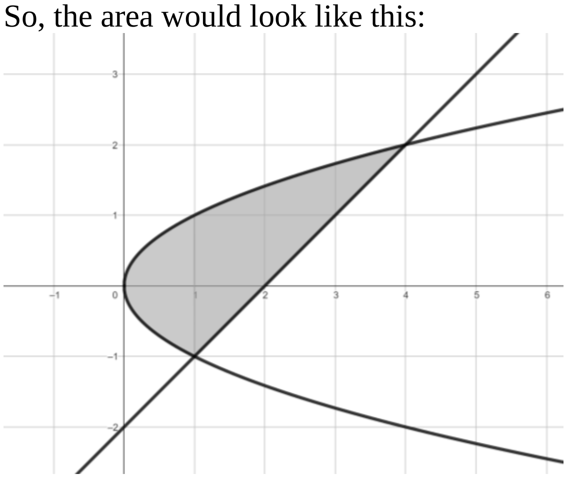
11) Evaluate the double integral $I=\iint_D y\,dA$

when D is the region bounded by $x-y=2$ and $y^2=x$

The region over which we are integrating is bounded by: $y=x-2$ and $y=\pm\sqrt{x}$

So, the area would look like this:



Determining the intersection of the boundaries of our region…

$y = x - 2$ and $y = \pm\sqrt{x}$

$x - 2 = \pm\sqrt{x}$

$(x-2)^2 = \left(\pm\sqrt{x}\right)^2$

$(x-2)^2 = x$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = 1$; $x = 4$

And the corresponding y-values are then:

$y = x - 2$

$y = (1) - 2 = -1$

$y = (4) - 2 = 2$

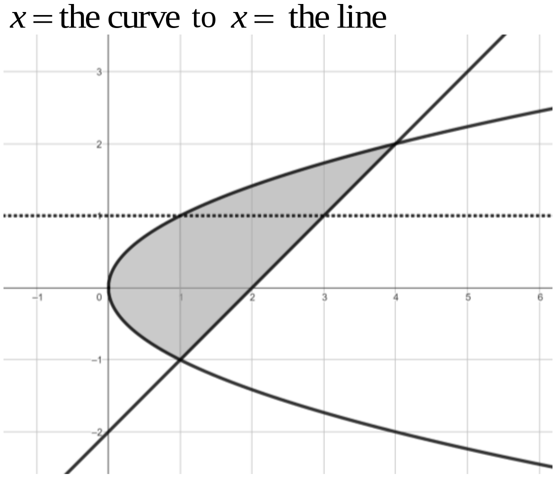
Given the shape of the region over which we are integrating, it is much easier to choose an order of integration with *dy* on the "outside" – that is, integrating with respect to *x* first.
Choosing this order means that we don't have to break up the integral into two integrals as we would need to if integrating with respect to *y* first.

So, we will choose to evaluate: $\iint y \; dxdy$

For the "outer" integral, this one with respect to *y*, in determining the bounds, we can think "what is absolute smallest and absolute largest value that variable takes on over this space?" We might answer *y* ranges from $y = -1$ to $y = 2$ over this space.

Now, for the "inner" integral, we can ask "for each one of those *y*-values, for instance $y = 1$, what does the *x*-value range from and to?"

So, for $y = 1$ (for example), we can see the *x*-values (in the shaded region) range from $x = \text{the curve}$ to $x = \text{ the line}$



And this is true for all sample values of *y*, that is, if we had chosen $y = 1/2$ or whatever, the *y*-values (in the shaded region) range from $x = \text{the curve}$ to $x = \text{ the line}$

So, now we can be slightly more fancy and say that for each *y* value, *x* ranges from $x = y^2$ to $x = y + 2$

And that allows us to set up our bounds for the double integral: $\int_{-1}^{2}\int_{y^2}^{y+2} y\ dxdy$

The only thing left, is to compute the answer.

$$
\begin{aligned}
I &= \iint_D y\ dA \\
&= \int_{-1}^{2}\int_{y^2}^{y+2} y\ dx\ dy \\
&= \int_{-1}^{2} (yx)\Big]_{y^2}^{y+2} dy \\
&= \int_{-1}^{2} \left(y(y+2) - y\left(y^2\right) \right)\ dy \\
&= \int_{-1}^{2} \left(y^2 + 2y - y^3 \right)\ dy \\
&= \left(\frac{1}{3}y^3 + y^2 - \frac{1}{4}y^4 \right)\Bigg]_{-1}^{2} \\
&= \left(\frac{8}{3} + 4 - 4 \right) - \left(-\frac{1}{3} + 1 - \frac{1}{4} \right) \\
&= \frac{9}{4}
\end{aligned}
$$

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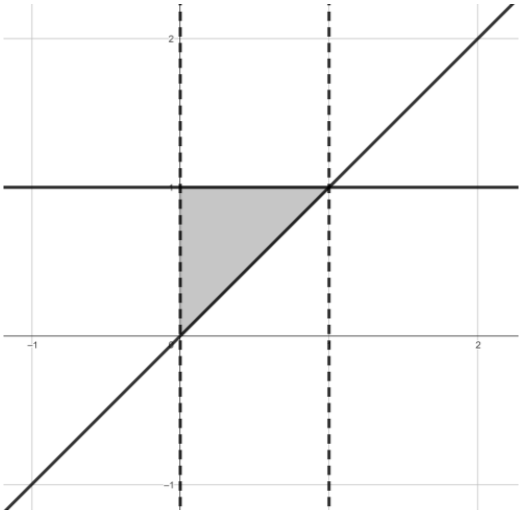
12) $\int_0^1 \int_x^1 x\sin(y^3)dydx$

We notice that attempting to integrate the double-integral in the order given (first with respect to *y*, then with respect to *x*) would be very difficult – in fact, it would be impossible with any techniques we have learned up until now (we simply don't have a y^2 term in the integrand, which would be needed for a *u*-substitution attempt, and no other identities or substitutions seem to alleviate the problem).

Therefore, we must reverse the order of integration to have any success with this problem.

Looking at the region over which we are integrating, we see it is bounded by the extremes:
x ranges from $x=0$ to $x=1$ and
y ranges from $y=x$ to $y=1$

So, the area would look like this:



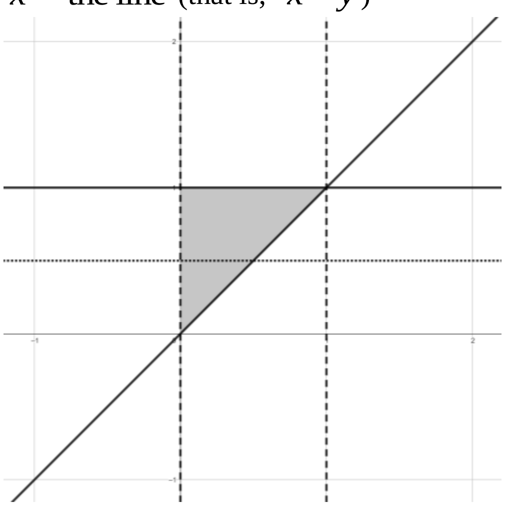
The intersections of the boundaries of our region are: $(0,0),(0,1),(1,1)$

As noted above, we'll need to choose the order of integration: $\iint x\sin\left(y^3\right)dxdy$

For the "outer" integral, this one with respect to *y*, in determining the bounds, we can think "what is absolute smallest and absolute largest value that variable takes on over this space?"
We might answer *y* ranges from $y=0$ to $y=1$ over this space.

Now, for the "inner" integral, we can ask "for each one of those *y*-values, for instance $y=1/2$, what does the *x*-value range from and to?"

So, for $y=1/2$ (for example), we can see the *x*-values (in the shaded region) range from $x=0$ to $x=$ the line (that is, $x=y$)



And this is true for all sample values of *y*, that is, if we had chosen $y=1/3$ or whatever: the *x*-values (in the shaded region) range from $x=0$ to $x=y$

And that allows us to set up our bounds for the double integral: $\int_0^1\int_0^y x\sin\left(y^3\right)dxdy$
The only thing left, is to compute the answer.

$$V = \int_0^1 \int_0^y x\sin\left(y^3\right)dxdy$$

$$= \int_0^1 \left(\frac{1}{2}x^2 \sin\left(y^3\right)\right)\Bigg|_0^y dy$$

$$= \int_0^1 \left(\frac{1}{2}y^2 \sin\left(y^3\right)\right)dy$$

$$= \frac{1}{2}\int_0^1 \left(y^2 \sin\left(y^3\right)\right)dy$$

$$= \frac{1}{2}\left(-\frac{1}{3}\cos\left(y^3\right)\right)\Bigg|_0^1$$

$$= -\frac{1}{6}\left(\cos(1) - \cos(0)\right)$$

$$= \frac{1}{6}\left(\cos(0) - \cos(1)\right)$$

$$= \frac{1}{6}\left(1 - \cos(1)\right)$$

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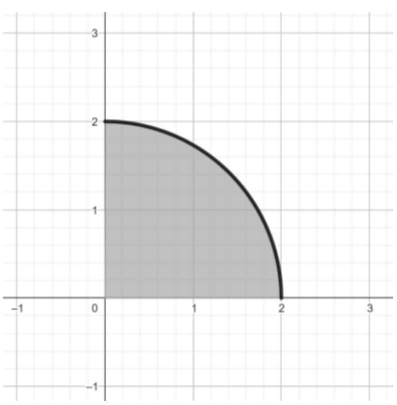
13) Use conversion to polar coordinates to evaluate the integral $\iint_D e^{-x^2-y^2}dA$ where D is the region in the first quadrant of the xy-plane inside the graph of $x = \sqrt{4-y^2}$

From the description of the region D, we have that the region over which we are integrating looks like:



In polar coordinates, this region is described by: $\left\{(r,\theta) \mid 0 \le r \le 2 \;,\; 0 \le \theta \le \frac{\pi}{2}\right\}$

Therefore, using the conversion formula, we find that…

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{\pi/2}\int_0^2 e^{-(r\cos\theta)^2-(r\sin\theta)^2}\,r\;dr\;d\theta$$
$$= \int_0^{\pi/2}\int_0^2 e^{-\left\{r^2\left(\cos^2\theta+\sin^2\theta\right)\right\}}\,r\;dr\;d\theta$$
$$= \int_0^{\pi/2}\int_0^2 e^{-r^2}\,r\;dr\;d\theta$$
$$= \int_0^{\pi/2}\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^2\;d\theta$$
$$= \int_0^{\pi/2} -\frac{1}{2}\left(e^{-4}-1\right)\;d\theta$$
$$= -\frac{1}{2}\left(e^{-4}-1\right)\theta\Big|_0^{\pi/2}$$
$$= -\frac{1}{2}\left(e^{-4}-1\right)\left(\frac{\pi}{2}-0\right)$$
$$= -\frac{\pi}{4}\left(e^{-4}-1\right)$$

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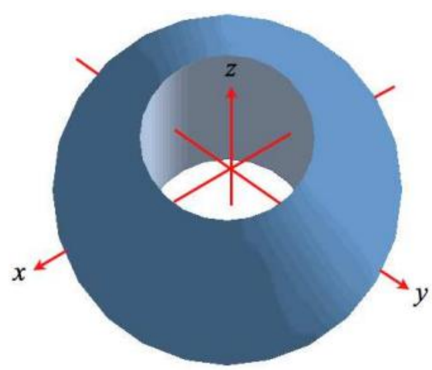
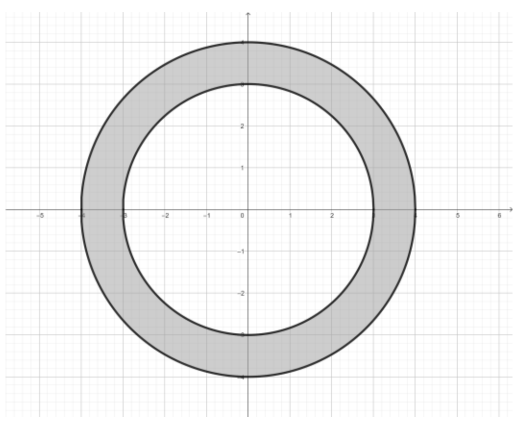
14) The solid shown lies inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=9$

Find the volume of the part of this solid lying above the *xy*-plane.



The region over which we are integrating would look like this:



The inner circle is given by $x^2+y^2=9$

To find the outer circle, we are looking at the intersection of $x^2+y^2+z^2=16$ and the *xy*-plane.

Letting $z=0$, we have $x^2+y^2=16$

This region is described in Cartesian coordinates as: $R = \left\{(x, y) \,|\, 9 \le x^2 + y^2 \le 16\right\}$

The function above this region is given by: $f(x, y) = \sqrt{16 - x^2 - y^2}$

The volume of the solid in a Cartesian double integral is then: $V = \iint_R \sqrt{16 - x^2 - y^2}\, dxdy$

In polar coordinates, the above region is described by: $\left\{(r, \theta) \;\middle|\; 3 \le r \le 4\ ,\ 0 \le \theta \le 2\pi\right\}$

$f(r\cos\theta, r\sin\theta) = \sqrt{16 - (r\cos\theta)^2 - (r\sin\theta)^2} = \sqrt{16 - r^2}$

And so, after conversion to polar coordinates, we have:

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_3^4 \sqrt{16 - (r^2)}\ r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_3^4 r\sqrt{16 - r^2}\ dr\ d\theta \\
&= \int_3^4 \int_0^{2\pi} r\sqrt{16 - r^2}\ d\theta\ dr \\
&= \int_3^4 \left(r\sqrt{16 - r^2}\right)\theta\Big|_0^{2\pi}\ dr \\
&= \int_3^4 2\pi\left(r\sqrt{16 - r^2}\right) dr \\
&= 2\pi \int_3^4 r\sqrt{16 - r^2}\ dr \\
&= 2\pi\left(-\frac{1}{3}\left(16 - r^2\right)^{3/2}\right)\Bigg|_3^4 \\
&= 2\pi\left\{(0) - \left(-\frac{1}{3}(16 - 9)\right)^{3/2}\right\} \\
&= \frac{2\pi}{3} \cdot 7^{3/2}
\end{aligned}
$$

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