The functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^n), exponential functions (b^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.5 Exercises

1–8 Three integrals are given that, although they look similar, may require different techniques of integration. Evaluate the integrals.

1. (a) $\displaystyle\int \frac{x}{1 + x^2}\,dx$ (b) $\displaystyle\int \frac{1}{1 + x^2}\,dx$

(c) $\displaystyle\int \frac{1}{1 - x^2}\,dx$

2. (a) $\displaystyle\int x\sqrt{x^2 - 1}\,dx$ (b) $\displaystyle\int \frac{1}{x\sqrt{x^2 - 1}}\,dx$

(c) $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x}\,dx$

3. (a) $\displaystyle\int \frac{\ln x}{x}\,dx$ (b) $\displaystyle\int \ln(2x)\,dx$

(c) $\displaystyle\int x \ln x\,dx$

4. (a) $\displaystyle\int \sin^2 x\,dx$ (b) $\displaystyle\int \sin^3 x\,dx$

(c) $\displaystyle\int \sin 2x\,dx$

5. (a) $\displaystyle\int \frac{1}{x^2 - 4x + 3}\,dx$ (b) $\displaystyle\int \frac{1}{x^2 - 4x + 4}\,dx$

(c) $\displaystyle\int \frac{1}{x^2 - 4x + 5}\,dx$

6. (a) $\int x \cos x^2\,dx$ (b) $\int x \cos^2 x\,dx$

(c) $\int x^2 \cos x\,dx$

7. (a) $\int x^2 e^{x^3}\,dx$ (b) $\int x^2 e^x\,dx$

(c) $\int x^3 e^{x^2}\,dx$

8. (a) $\int e^x \sqrt{e^x - 1}\,dx$ (b) $\int \dfrac{e^x}{\sqrt{1 - e^{2x}}}\,dx$

(c) $\int \dfrac{1}{\sqrt{e^x - 1}}\,dx$

9–93 Evaluate the integral.

9. $\int \dfrac{\cos x}{1 - \sin x}\,dx$

10. $\int_0^1 (3x + 1)^{\sqrt{2}}\,dx$

11. $\int_1^4 \sqrt{y}\,\ln y\,dy$

12. $\int \dfrac{e^{\arcsin x}}{\sqrt{1 - x^2}}\,dx$

13. $\int \dfrac{\ln(\ln y)}{y}\,dy$

14. $\int_0^1 \dfrac{x}{(2x + 1)^3}\,dx$

15. $\int \dfrac{x}{x^4 + 9}\,dx$

16. $\int t \sin t \cos t\,dt$

17. $\int_2^4 \dfrac{x + 2}{x^2 + 3x - 4}\,dx$

18. $\int \dfrac{\cos(1/x)}{x^3}\,dx$

19. $\int \dfrac{1}{x^3\sqrt{x^2 - 1}}\,dx$

20. $\int \dfrac{2x - 3}{x^3 + 3x}\,dx$

21. $\int \dfrac{\cos^3 x}{\csc x}\,dx$

22. $\int \ln(1 + x^2)\,dx$

23. $\int x \sec x \tan x\,dx$

24. $\int_0^{\sqrt{2}/2} \dfrac{x^2}{\sqrt{1 - x^2}}\,dx$

25. $\int_0^{\pi} t \cos^2 t\,dt$

26. $\int_1^4 \dfrac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

27. $\int e^{x + e^x}\,dx$

28. $\int \dfrac{e^x}{1 + e^{2x}}\,dx$

29. $\int \arctan\sqrt{x}\,dx$

30. $\int \dfrac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx$

31. $\int_0^1 \left(1 + \sqrt{x}\right)^8 dx$

32. $\int (1 + \tan x)^2 \sec x\,dx$

33. $\int_0^1 \dfrac{1 + 12t}{1 + 3t}\,dt$

34. $\int_0^1 \dfrac{3x^2 + 1}{x^3 + x^2 + x + 1}\,dx$

35. $\int \dfrac{dx}{1 + e^x}$

36. $\int \sin\sqrt{at}\,dt$

37. $\int \ln\left(x + \sqrt{x^2 - 1}\right) dx$

38. $\int_{-1}^2 |e^x - 1|\,dx$

39. $\int \sqrt{\dfrac{1 + x}{1 - x}}\,dx$

40. $\int_1^3 \dfrac{e^{3/x}}{x^2}\,dx$

41. $\int \sqrt{3 - 2x - x^2}\,dx$

42. $\int_{\pi/4}^{\pi/2} \dfrac{1 + 4\cot x}{4 - \cot x}\,dx$

43. $\int_{-\pi/2}^{\pi/2} \dfrac{x}{1 + \cos^2 x}\,dx$

44. $\int \dfrac{1 + \sin x}{1 + \cos x}\,dx$

45. $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\,d\theta$

46. $\int_{\pi/6}^{\pi/3} \dfrac{\sin\theta \cot\theta}{\sec\theta}\,d\theta$

47. $\int \dfrac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta}\,d\theta$

48. $\int_0^{\pi} \sin 6x \cos 3x\,dx$

49. $\int \theta \tan^2\theta\,d\theta$

50. $\int \dfrac{1}{x\sqrt{x - 1}}\,dx$

51. $\int \dfrac{\sqrt{x}}{1 + x^3}\,dx$

52. $\int \sqrt{1 + e^x}\,dx$

53. $\int \dfrac{x}{1 + \sqrt{x}}\,dx$

54. $\int \dfrac{(x - 1)e^x}{x^2}\,dx$

55. $\int x^3 (x - 1)^{-4}\,dx$

56. $\int_0^1 x\sqrt{2 - \sqrt{1 - x^2}}\,dx$

57. $\int \dfrac{1}{x\sqrt{4x + 1}}\,dx$

58. $\int \dfrac{1}{x^2\sqrt{4x + 1}}\,dx$

59. $\int \dfrac{1}{x\sqrt{4x^2 + 1}}\,dx$

60. $\int \dfrac{dx}{x(x^4 + 1)}$

61. $\int x^2 \sinh mx\,dx$

62. $\int (x + \sin x)^2\,dx$

63. $\int \dfrac{dx}{x + x\sqrt{x}}$

64. $\int \dfrac{dx}{\sqrt{x} + x\sqrt{x}}$

65. $\int x\sqrt[3]{x + c}\,dx$

66. $\int \dfrac{x \ln x}{\sqrt{x^2 - 1}}\,dx$

67. $\int \dfrac{dx}{x^4 - 16}$

68. $\int \dfrac{dx}{x^2\sqrt{4x^2 - 1}}$

69. $\int \dfrac{d\theta}{1 + \cos\theta}$

70. $\int \dfrac{d\theta}{1 + \cos^2\theta}$

71. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

72. $\int \dfrac{1}{\sqrt{\sqrt{x} + 1}}\,dx$

73. $\displaystyle\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

74. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}\,dx$

75. $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

76. $\displaystyle\int \frac{x^2}{x^6+3x^3+2}\,dx$

77. $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

78. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$

79. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

80. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$

81. $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx$

82. $\displaystyle\int \frac{4^x+10^x}{2^x}\,dx$

83. $\displaystyle\int \frac{dx}{x\ln x - x}$

84. $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

85. $\displaystyle\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

86. $\displaystyle\int \frac{1+\sin x}{1-\sin x}\,dx$

87. $\displaystyle\int x\sin^2 x\cos x\,dx$

88. $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx$

89. $\displaystyle\int \sqrt{1-\sin x}\,dx$

90. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

91. $\displaystyle\int_1^3 \left(\sqrt{\frac{9-x}{x}}-\sqrt{\frac{x}{9-x}}\right)dx$

92. $\displaystyle\int \frac{1}{(\sin x+\cos x)^2}\,dx$

93. $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin 2\theta}\,d\theta$

94. We know that $F(x)=\int_0^x e^{e^t}\,dt$ is a continuous function by FTC1, though it is not an elementary function. The functions

$$\int \frac{e^x}{x}\,dx \qquad \text{and} \qquad \int \frac{1}{\ln x}\,dx$$

are not elementary either, but they can be expressed in terms of F. Evaluate the following integrals in terms of F.

(a) $\displaystyle\int_1^2 \frac{e^x}{x}\,dx$ (b) $\displaystyle\int_2^3 \frac{1}{\ln x}\,dx$

95. The functions $y=e^{x^2}$ and $y=x^2e^{x^2}$ don't have elementary antiderivatives, but $y=(2x^2+1)e^{x^2}$ does. Evaluate $\int (2x^2+1)e^{x^2}\,dx$.

7.6 Integration Using Tables and Technology

In this section we describe how to use tables and mathematical software to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even the most powerful computer software can't find explicit formulas for the antiderivatives of functions like e^{x^2} or the other functions described at the end of Section 7.5.

Tables of Integrals

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand. In some cases, the results obtained are of a simpler form than those given by a computer. A relatively brief table of 120 integrals, categorized by form, is provided on Reference Pages 6–10 at the back of the book. More extensive tables, containing hundreds or thousands of entries, are available in separate publications or on the Internet. When using such tables, remember that integrals do not often occur in exactly the form listed. Usually we need to use the Substitution Rule or algebraic manipulation to transform a given integral into one of the forms in the table.

EXAMPLE 1 The region bounded by the curves $y=\arctan x$, $y=0$, and $x=1$ is rotated about the y-axis. Find the volume of the resulting solid.

SOLUTION Using the method of cylindrical shells, we see that the volume is

$$V=\int_0^1 2\pi x\arctan x\,dx$$