(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)[1 + 2\sin(\pi/3)]}{[1 + \sin(\pi/3)][1 - 2\sin(\pi/3)]} = \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})}$$

$$= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore there are horizontal tangents at the points $(2, \pi/2)$, $\left(\frac{1}{2}, 7\pi/6\right)$, $\left(\frac{1}{2}, 11\pi/6\right)$ and vertical tangents at $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta} = -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry,
$$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$

Thus there is a vertical tangent line at the pole (see Figure 9). ■

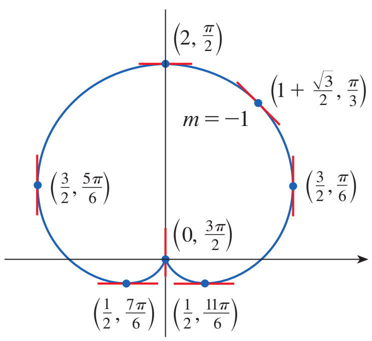


FIGURE 9
Tangent lines for $r = 1 + \sin\theta$

NOTE Instead of having to remember Equation 7, we could employ the method used to derive it. For instance, in Example 5 we could have written parametric equations for the curve as

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

10.4 Exercises

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = \sqrt{2\theta}$, $0 \leqslant \theta \leqslant \pi/2$

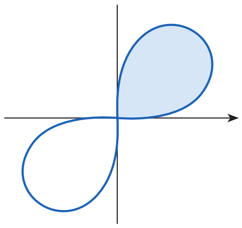
2. $r = e^{\theta}$, $3\pi/4 \leqslant \theta \leqslant 3\pi/2$

3. $r = \sin\theta + \cos\theta$, $0 \leqslant \theta \leqslant \pi$

4. $r = 1/\theta$, $\pi/2 \leqslant \theta \leqslant 2\pi$

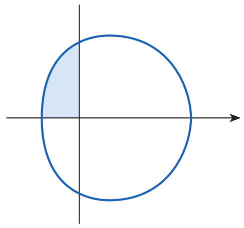
5–8 Find the area of the shaded region.

5.



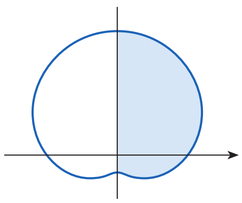
$r^2 = \sin 2\theta$

6.



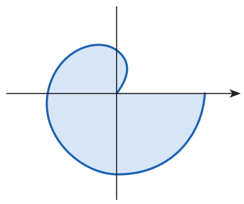
$r = 2 + \cos\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sqrt{\ln\theta},\ 1 \leqslant \theta \leqslant 2\pi$

9–12 Sketch the curve and find the area that it encloses.

9. $r = 4\cos\theta$

10. $r = 2 + 2\cos\theta$

11. $r = 3 - 2\sin\theta$

12. $r = 2\sin 3\theta$

13–16 Graph the curve and find the area that it encloses.

13. $r = 2 + \sin 4\theta$

14. $r = 3 - 2\cos 4\theta$

15. $r = \sqrt{1 + \cos^2(5\theta)}$

16. $r = 1 + 5\sin 6\theta$

17–21 Find the area of the region enclosed by one loop of the curve.

17. $r = 4\cos 3\theta$

18. $r^2 = 4\cos 2\theta$

19. $r = \sin 4\theta$

20. $r = 2\sin 5\theta$

21. $r = 1 + 2\sin\theta$ (inner loop)

22. Find the area enclosed by the loop of the **strophoid** $r = 2\cos\theta - \sec\theta$.

23–28 Find the area of the region that lies inside the first curve and outside the second curve.

23. $r = 4\sin\theta, \quad r = 2$

24. $r = 1 - \sin\theta, \quad r = 1$

25. $r^2 = 8\cos 2\theta, \quad r = 2$

26. $r = 1 + \cos\theta, \quad r = 2 - \cos\theta$

27. $r = 3\cos\theta, \quad r = 1 + \cos\theta$

28. $r = 3\sin\theta, \quad r = 2 - \sin\theta$

29–34 Find the area of the region that lies inside both curves.

29. $r = 3\sin\theta, \quad r = 3\cos\theta$

30. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

31. $r = \sin 2\theta, \quad r = \cos 2\theta$

32. $r = 3 + 2\cos\theta, \quad r = 3 + 2\sin\theta$

33. $r^2 = 2\sin 2\theta, \quad r = 1$

34. $r = a\sin\theta, \quad r = b\cos\theta, \quad a > 0,\ b > 0$

35. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

36. Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

37–42 Find all points of intersection of the given curves.

37. $r = \sin\theta, \quad r = 1 - \sin\theta$

38. $r = 1 + \cos\theta, \quad r = 1 - \sin\theta$

39. $r = 2\sin 2\theta, \quad r = 1$

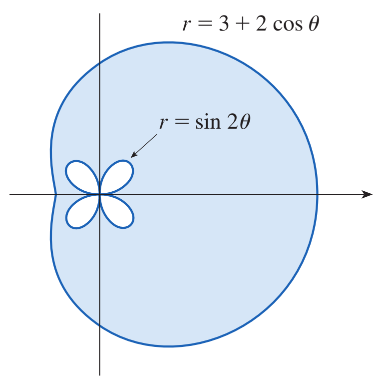
40. $r = \cos\theta, \quad r = \sin 2\theta$

41. $r^2 = 2\cos 2\theta, \quad r = 1$

42. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

43–46 Find the area of the shaded region.

43.



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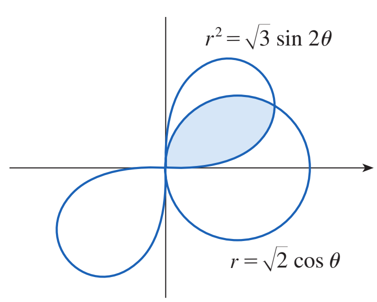

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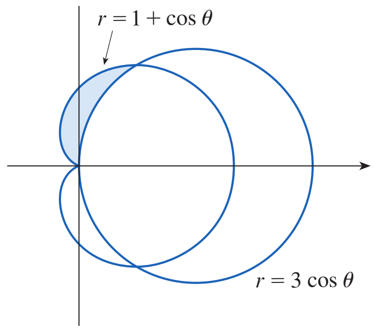

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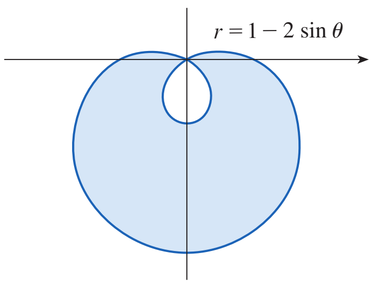
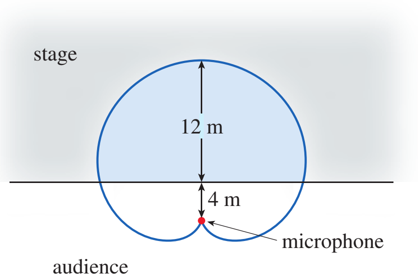


47. The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graph to find the approximate values of θ at which the curves intersect. Then use these values to estimate the area that lies inside both curves.

48. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



49–52 Find the exact length of the polar curve.

49. $r = 2\cos\theta$, $0 \leqslant \theta \leqslant \pi$

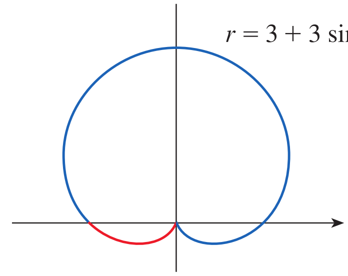
50. $r = e^{\theta/2}$, $0 \leqslant \theta \leqslant \pi/2$

51. $r = \theta^2$, $0 \leqslant \theta \leqslant 2\pi$

52. $r = 2(1 + \cos\theta)$

53–54 Find the exact length of the portion of the curve shown in blue.

53.



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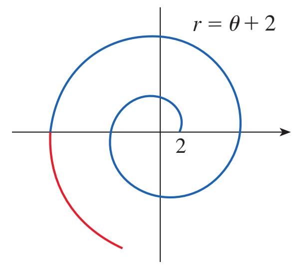


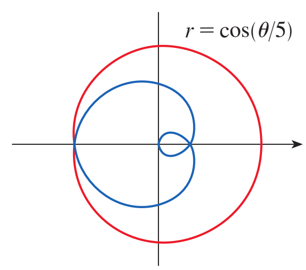
55–56 Find the exact length of the curve. Use a graph to determine the parameter interval.

55. $r = \cos^4(\theta/4)$

56. $r = \cos^2(\theta/2)$

57–58 Set up, but do not evaluate, an integral to find the length of the portion of the curve shown in blue.

57.



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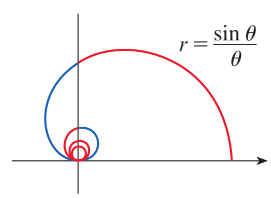


59–62 Use a calculator or computer to find the length of the curve correct to four decimal places. If necessary, graph the curve to determine the parameter interval.

59. One loop of the curve $r = \cos 2\theta$

60. $r = \tan\theta$, $\pi/6 \leqslant \theta \leqslant \pi/3$

61. $r = \sin(6\sin\theta)$

62. $r = \sin(\theta/4)$

63–68 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

63. $r = 2\cos\theta, \quad \theta = \pi/3$

64. $r = 2 + \sin 3\theta, \quad \theta = \pi/4$

65. $r = 1/\theta, \quad \theta = \pi$

66. $r = \sin\theta + 2\cos\theta, \quad \theta = \pi/2$

67. $r = \cos 2\theta, \quad \theta = \pi/4$

68. $r = 1 + 2\cos\theta, \quad \theta = \pi/3$

69–72 Find the points on the given curve where the tangent line is horizontal or vertical.

69. $r = \sin\theta$

70. $r = 1 - \sin\theta$

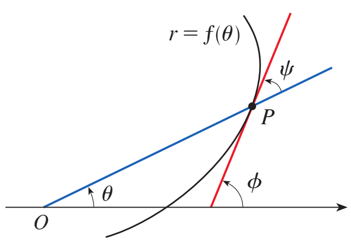
71. $r = 1 + \cos\theta$

72. $r = e^{\theta}$

73. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



74. (a) Use Exercise 73 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.

(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.

(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

75. (a) Use Formula 10.2.9 to show that the area of the surface generated by rotating the polar curve

$$r = f(\theta) \qquad a \leqslant \theta \leqslant b$$

(where f' is continuous and $0 \leqslant a < b \leqslant \pi$) about the polar axis is

$$S = \int_a^b 2\pi r\sin\theta\,\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

76. (a) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$ (where f' is continuous and $0 \leqslant a < b \leqslant \pi$), about the line $\theta = \pi/2$.

(b) Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \pi/2$.

10.5 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

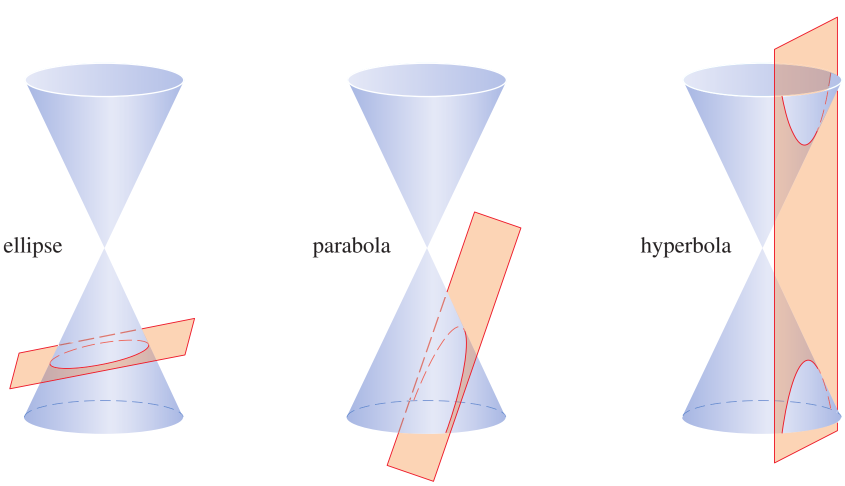


FIGURE 1
Conics