83. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2+1}\,dx < 0.001$$

84. Estimate the numerical value of $\int_0^\infty e^{-x^2}\,dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\,dx$ and $\int_4^\infty e^{-x^2}\,dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}\,dx$, which is less than 0.0000001.

85–87 The Laplace Transform If $f(t)$ is continuous for $t \geqslant 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

and the domain of F is the set consisting of all numbers s for which the integral converges.

85. Find the Laplace transform of each of the following functions.
(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

86. Show that if $0 \leqslant f(t) \leqslant Me^{at}$ for $t \geqslant 0$, where M and a are constants, then the Laplace transform $F(s)$ exists for $s > a$.

87. Suppose that $0 \leqslant f(t) \leqslant Me^{at}$ and $0 \leqslant f'(t) \leqslant Ke^{at}$ for $t \geqslant 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$ and the Laplace transform of $f'(t)$ is $G(s)$, show that

$$G(s) = sF(s) - f(0) \qquad s > a$$

88. If $\int_{-\infty}^\infty f(x)\,dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

89. Show that $\int_0^\infty x^2e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$.

90. Show that $\int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln y}\;dy$ by interpreting the integrals as areas.

91. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx$$

converges. Evaluate the integral for this value of C.

92. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx$$

converges. Evaluate the integral for this value of C.

93. Suppose f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)\,dx$ is convergent?

94. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx$$

7 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x\,dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
(a) $\int_a^\infty f(x)\,dx$ (b) $\int_{-\infty}^b f(x)\,dx$ (c) $\int_{-\infty}^\infty f(x)\,dx$

7. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.
(a) f has an infinite discontinuity at a.
(b) f has an infinite discontinuity at b.
(c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\int \tan^{-1}x\,dx$ can be evaluated using integration by parts.

2. $\int x^5 e^x\,dx$ can be evaluated by applying integration by parts five times.

3. To evaluate $\displaystyle\int \frac{dx}{\sqrt{25 + x^2}}$ an appropriate trigonometric substitution is $x = 5\sin\theta$.

4. To evaluate $\displaystyle\int \frac{dx}{\sqrt{9 + e^{2x}}}$ we can use the formula in entry 25 of the Table of Integrals to obtain $\ln\left(e^x + \sqrt{9 + e^{2x}}\right) + C$.

5. $\dfrac{x(x^2 + 4)}{x^2 - 4}$ can be put in the form $\dfrac{A}{x + 2} + \dfrac{B}{x - 2}$.

6. $\dfrac{x^2 + 4}{x(x^2 - 4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x + 2} + \dfrac{C}{x - 2}$.

7. $\dfrac{x^2 + 4}{x^2(x - 4)}$ can be put in the form $\dfrac{A}{x^2} + \dfrac{B}{x - 4}$.

8. $\dfrac{x^2 - 4}{x(x^2 + 4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2 + 4}$.

9. $\displaystyle\int_0^4 \frac{x}{x^2 - 1}\,dx = \tfrac{1}{2}\ln 15$

10. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$ is convergent.

11. If $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent, then $\int_0^\infty f(x)\,dx$ is convergent.

12. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

13. (a) Every elementary function has an elementary derivative.
(b) Every elementary function has an elementary antiderivative.

14. If f is continuous on $[0, \infty)$ and $\int_1^\infty f(x)\,dx$ is convergent, then $\int_0^\infty f(x)\,dx$ is convergent.

15. If f is a continuous, decreasing function on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ is convergent.

16. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both convergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is convergent.

17. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both divergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is divergent.

18. If $f(x) \leqslant g(x)$ and $\int_0^\infty g(x)\,dx$ diverges, then $\int_0^\infty f(x)\,dx$ also diverges.

EXERCISES

Note: Additional practice in techniques of integration is provided in Exercises 7.5.

1–50 Evaluate the integral.

1. $\displaystyle\int_1^2 \frac{(x + 1)^2}{x}\,dx$

2. $\displaystyle\int_1^2 \frac{x}{(x + 1)^2}\,dx$

3. $\displaystyle\int \frac{e^{\sin x}}{\sec x}\,dx$

4. $\displaystyle\int_0^{\pi/6} t\sin 2t\,dt$

5. $\displaystyle\int \frac{dt}{2t^2 + 3t + 1}$

6. $\displaystyle\int_1^2 x^5 \ln x\,dx$

7. $\displaystyle\int_0^{\pi/2} \sin^3\theta\,\cos^2\theta\,d\theta$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{16 - x^2}}$

9. $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt$

10. $\displaystyle\int_0^1 \frac{\sqrt{\arctan x}}{1 + x^2}\,dx$

11. $\displaystyle\int x(\ln x)^2\,dx$

12. $\displaystyle\int \sin x\cos x\ln(\cos x)\,dx$

13. $\displaystyle\int_1^2 \frac{\sqrt{x^2 - 1}}{x}\,dx$

14. $\displaystyle\int \frac{e^{2x}}{1 + e^{4x}}\,dx$

15. $\displaystyle\int e^{\sqrt[3]{x}}\,dx$

16. $\displaystyle\int \frac{x^2 + 2}{x + 2}\,dx$

17. $\displaystyle\int x^2\tan^{-1}x\,dx$

18. $\displaystyle\int (x + 2)^2(x + 1)^{20}\,dx$

19. $\displaystyle\int \frac{x - 1}{x^2 + 2x}\,dx$

20. $\displaystyle\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$

21. $\displaystyle\int x\cosh x\,dx$

22. $\displaystyle\int \frac{x^2 + 8x - 3}{x^3 + 3x^2}\,dx$

23. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4x}}$

24. $\displaystyle\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

25. $\displaystyle\int \frac{x + 1}{9x^2 + 6x + 5}\,dx$

26. $\displaystyle\int \tan^5\theta\,\sec^3\theta\,d\theta$

27. $\displaystyle\int_0^2 \sqrt{x^2 - 2x + 2}\,dx$

28. $\displaystyle\int \cos\sqrt{t}\,dt$

29. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + 1}}$

30. $\displaystyle\int e^x\cos x\,dx$

31. $\displaystyle\int \frac{x\sin\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx$

32. $\displaystyle\int \frac{dx}{x^{1/2}+x^{1/4}}$

33. $\displaystyle\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx$

34. $\displaystyle\int x\sin x\cos x\,dx$

35. $\displaystyle\int_0^{\pi/2} \cos^3 x\sin 2x\,dx$

36. $\displaystyle\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx$

37. $\displaystyle\int_{-3}^{3} \frac{x}{1+|x|}\,dx$

38. $\displaystyle\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

39. $\displaystyle\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx$

40. $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx$

41. $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

42. $\displaystyle\int (\arcsin x)^2\,dx$

43. $\displaystyle\int \frac{1}{\sqrt{x+x^{3/2}}}\,dx$

44. $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta$

45. $\displaystyle\int (\cos x+\sin x)^2\cos 2x\,dx$

46. $\displaystyle\int x\cos^3(x^2)\sqrt{\sin(x^2)}\,dx$

47. $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx$

48. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta$

49. $\displaystyle\int \frac{1}{\sqrt{e^x-4}}\,dx$

50. $\displaystyle\int x\sin\left(\sqrt{1+x^2}\right)dx$

51–60 Evaluate the integral or show that it is divergent.

51. $\displaystyle\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx$

52. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^4}\,dx$

53. $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$

54. $\displaystyle\int_2^{6} \frac{y}{\sqrt{y-2}}\,dy$

55. $\displaystyle\int_0^{4} \frac{\ln x}{\sqrt{x}}\,dx$

56. $\displaystyle\int_0^{1} \frac{1}{2-3x}\,dx$

57. $\displaystyle\int_0^{1} \frac{x-1}{\sqrt{x}}\,dx$

58. $\displaystyle\int_{-1}^{1} \frac{dx}{x^2-2x}$

59. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5}$

60. $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx$

61–62 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C=0$).

61. $\displaystyle\int \ln(x^2+2x+2)\,dx$

62. $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

63. Graph the function $f(x)=\cos^2 x\sin^3 x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

64. (a) How would you evaluate $\int x^5e^{-2x}\,dx$ by hand? (Don't actually carry out the integration.)
(b) How would you evaluate $\int x^5e^{-2x}\,dx$ using a table of integrals? (Don't actually do it.)
(c) Use a computer to evaluate $\int x^5e^{-2x}\,dx$.
(d) Graph the integrand and the indefinite integral on the same screen.

65–68 Use the Table of Integrals on Reference Pages 6–10 to evaluate the integral.

65. $\displaystyle\int \sqrt{4x^2-4x-3}\,dx$

66. $\displaystyle\int \csc^5 t\,dt$

67. $\displaystyle\int \cos x\sqrt{4+\sin^2 x}\,dx$

68. $\displaystyle\int \frac{\cot x}{\sqrt{1+2\sin x}}\,dx$

69. Verify Formula 33 in the Table of Integrals (a) by differentiation and (b) by using a trigonometric substitution.

70. Verify Formula 62 in the Table of Integrals.

71. Is it possible to find a number n such that $\int_0^{\infty} x^n\,dx$ is convergent?

72. For what values of a is $\int_0^{\infty} e^{ax}\cos x\,dx$ convergent? Evaluate the integral for those values of a.

73–74 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with $n=10$ to approximate the given integral. Round your answers to six decimal places.

73. $\displaystyle\int_2^{4} \frac{1}{\ln x}\,dx$

74. $\displaystyle\int_1^{4} \sqrt{x}\cos x\,dx$

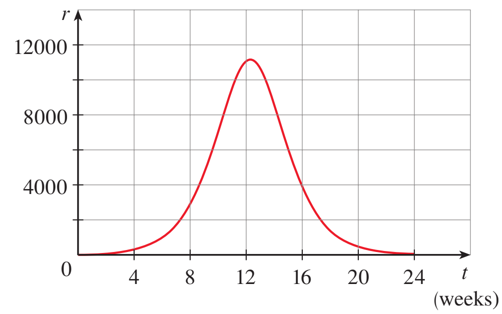
75. Estimate the errors involved in Exercise 73, parts (a) and (b). How large should n be in each case to guarantee an error of less than 0.00001?

76. Use Simpson's Rule with $n=6$ to estimate the area under the curve $y=e^x/x$ from $x=1$ to $x=4$.

77. The speedometer reading (v) on a car was observed at 1-minute intervals and recorded in the chart. Use Simpson's Rule to estimate the distance traveled by the car.

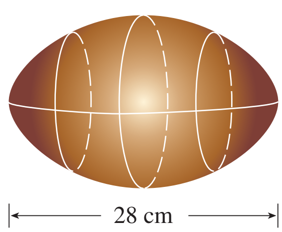
t (min)	v (mi/h)	t (min)	v (mi/h)
0	40	6	56
1	42	7	57
2	45	8	57
3	49	9	55
4	52	10	56
5	54		

78. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



79. (a) If $f(x) = \sin(\sin x)$, use a computer algebra system to compute $f^{(4)}(x)$ and then use a graph to find an upper bound for $|f^{(4)}(x)|$.
(b) Use Simpson's Rule with $n = 10$ to approximate $\int_0^{\pi} f(x)\,dx$ and use part (a) to estimate the error.
(c) How large should n be to guarantee that the size of the error in using S_n is less than 0.00001?

80. Suppose you are asked to estimate the volume of a football. You measure and find that a football is 28 cm long. You use a piece of string and measure the circumference at its widest point to be 53 cm. The circumference 7 cm from each end is 45 cm. Use Simpson's Rule to make your estimate.



81. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

(a) $\displaystyle\int_1^{\infty} \frac{2 + \sin x}{\sqrt{x}}\,dx$ (b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1 + x^4}}\,dx$

82. Find the area of the region bounded by the hyperbola $y^2 - x^2 = 1$ and the line $y = 3$.

83. Find the area bounded by the curves $y = \cos x$ and $y = \cos^2 x$ between $x = 0$ and $x = \pi$.

84. Find the area of the region bounded by the curves $y = 1/(2 + \sqrt{x})$, $y = 1/(2 - \sqrt{x})$, and $x = 1$.

85. The region under the curve $y = \cos^2 x$, $0 \leqslant x \leqslant \pi/2$, is rotated about the x-axis. Find the volume of the resulting solid.

86. The region in Exercise 85 is rotated about the y-axis. Find the volume of the resulting solid.

87. If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, show that

$$\int_0^{\infty} f'(x)\,dx = -f(0)$$

88. We can extend our definition of average value of a continuous function to an infinite interval by defining the average value of f on the interval $[a, \infty)$ to be

$$f_{\text{avg}} = \lim_{t\to\infty} \frac{1}{t - a}\int_a^t f(x)\,dx$$

(a) Find the average value of $y = \tan^{-1}x$ on the interval $[0, \infty)$.
(b) If $f(x) \geqslant 0$ and $\int_a^{\infty} f(x)\,dx$ is divergent, show that the average value of f on the interval $[a, \infty)$ is $\lim_{x\to\infty} f(x)$, if this limit exists.
(c) If $\int_a^{\infty} f(x)\,dx$ is convergent, what is the average value of f on the interval $[a, \infty)$?
(d) Find the average value of $y = \sin x$ on the interval $[0, \infty)$.

89. Use the substitution $u = 1/x$ to show that

$$\int_0^{\infty} \frac{\ln x}{1 + x^2}\,dx = 0$$

90. The magnitude of the repulsive force between two point charges with the same sign, one of size 1 and the other of size q, is

$$F = \frac{q}{4\pi\varepsilon_0 r^2}$$

where r is the distance between the charges and ε_0 is a constant. The *potential* V at a point P due to the charge q is defined to be the work expended in bringing a unit charge to P from infinity along the straight line that joins q and P. Find a formula for V.