Partial differential equations involving functions of three variables are also very important in science and engineering. The three-dimensional Laplace equation is

$$\boxed{5} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and one application is in geophysics. If $u(x, y, z)$ represents magnetic field strength at position (x, y, z), then it satisfies Equation 5. The strength of the magnetic field indicates the distribution of iron-rich minerals and reflects different rock types and the location of faults.

14.3 Exercises

1. At the beginning of this section we discussed the function $I = f(T, H)$, where I is the heat index, T is the actual temperature, and H is the relative humidity. Use Table 1 to estimate $f_T(92, 60)$ and $f_H(92, 60)$. What are the practical interpretations of these values?

2. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Wind speed (knots) v \ Duration (hours) t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

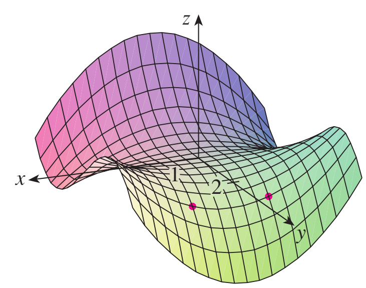
$$\lim_{t \to \infty} \frac{\partial h}{\partial t}$$

3. The temperature T (in °C) at a location in the Northern Hemisphere depends on the longitude x, latitude y, and time t, so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

(a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$?

(b) Honolulu has longitude 158°W and latitude 21°N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.

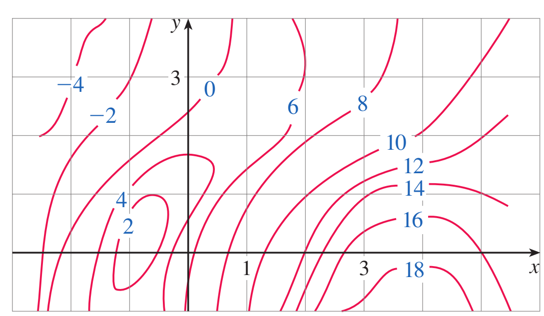
4–5 Determine the signs of the partial derivatives for the function f whose graph is shown.



4. (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

5. (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$

6. A contour map is given for a function f. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



7. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

8. If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

9–36 Find the first partial derivatives of the function.

9. $f(x, y) = x^4 + 5xy^3$

10. $f(x, y) = x^2y - 3y^4$

11. $g(x, y) = x^3 \sin y$

12. $g(x, t) = e^{xt}$

13. $z = \ln(x + t^2)$

14. $w = \dfrac{u}{v^2}$

15. $f(x, y) = ye^{xy}$

16. $g(x, y) = (x^2 + xy)^3$

17. $g(x, y) = y(x + x^2y)^5$

18. $f(x, y) = \dfrac{x}{(x + y)^2}$

19. $f(x, y) = \dfrac{ax + by}{cx + dy}$

20. $w = \dfrac{e^v}{u + v^2}$

21. $g(u, v) = (u^2v - v^3)^5$

22. $u(r, \theta) = \sin(r \cos \theta)$

23. $R(p, q) = \tan^{-1}(pq^2)$

24. $f(x, y) = x^y$

25. $F(x, y) = \int_y^x \cos(e^t)\, dt$

26. $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

27. $f(x, y, z) = x^3yz^2 + 2yz$

28. $f(x, y, z) = xy^2e^{-xz}$

29. $w = \ln(x + 2y + 3z)$

30. $w = y \tan(x + 2z)$

31. $p = \sqrt{t^4 + u^2 \cos v}$

32. $u = x^{y/z}$

33. $h(x, y, z, t) = x^2y \cos(z/t)$

34. $\phi(x, y, z, t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$

35. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

36. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

37–40 Find the indicated partial derivative.

37. $R(s, t) = te^{s/t}$; $R_t(0, 1)$

38. $f(x, y) = y \sin^{-1}(xy)$; $f_y\left(1, \frac{1}{2}\right)$

39. $f(x, y, z) = \ln \dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}$; $f_y(1, 2, 2)$

40. $f(x, y, z) = x^{yz}$; $f_z(e, 1, 0)$

41–44 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

41. $x^2 + 2y^2 + 3z^2 = 1$

42. $x^2 - y^2 + z^2 - 2z = 4$

43. $e^z = xyz$

44. $yz + x \ln y = z^2$

45–46 Find $\partial z/\partial x$ and $\partial z/\partial y$.

45. (a) $z = f(x) + g(y)$ (b) $z = f(x + y)$

46. (a) $z = f(x)g(y)$ (b) $z = f(xy)$
(c) $z = f(x/y)$

47–52 Find all the second partial derivatives.

47. $f(x, y) = x^4y - 2x^3y^2$

48. $f(x, y) = \ln(ax + by)$

49. $z = \dfrac{y}{2x + 3y}$

50. $T = e^{-2r} \cos \theta$

51. $v = \sin(s^2 - t^2)$

52. $z = \arctan \dfrac{x + y}{1 - xy}$

53–56 Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

53. $u = x^4y^3 - y^4$

54. $u = e^{xy} \sin y$

55. $u = \cos(x^2y)$

56. $u = \ln(x + 2y)$

57–64 Find the indicated partial derivative(s).

57. $f(x, y) = x^4y^2 - x^3y$; f_{xxx}, f_{xyx}

58. $f(x, y) = \sin(2x + 5y)$; f_{yxy}

59. $f(x, y, z) = e^{xyz^2}$; f_{xyz}

60. $g(r, s, t) = e^r \sin(st)$; g_{rst}

61. $W = \sqrt{u + v^2}$; $\dfrac{\partial^3 W}{\partial u^2\, \partial v}$

62. $V = \ln(r + s^2 + t^3)$; $\dfrac{\partial^3 V}{\partial r\, \partial s\, \partial t}$

63. $w = \dfrac{x}{y + 2z}$; $\dfrac{\partial^3 w}{\partial z\, \partial y\, \partial x}$, $\dfrac{\partial^3 w}{\partial x^2\, \partial y}$

64. $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3}$

65–66 Use Definition 4 to find $f_x(x, y)$ and $f_y(x, y)$.

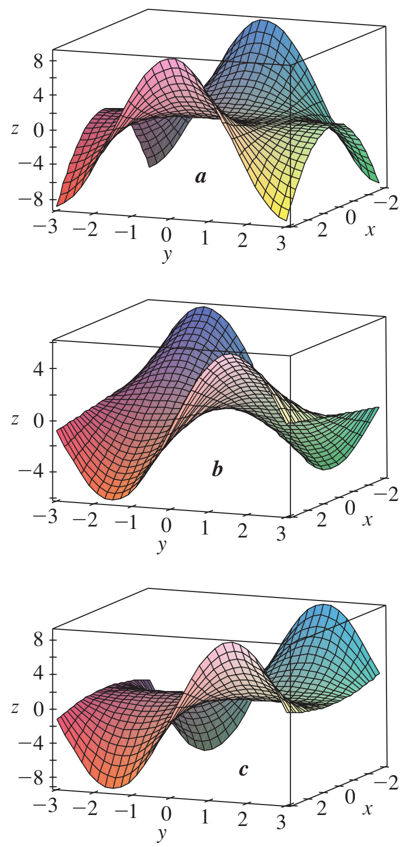
65. $f(x, y) = xy^2 - x^3y$

66. $f(x, y) = \dfrac{x}{x + y^2}$

67. If $f(x, y, z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right)$, find f_{xzy}. [*Hint:* Which order of differentiation is easiest?]

68. If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find g_{xyz}. [*Hint:* Use a different order of differentiation for each term.]

69. The following surfaces, labeled a, b, and c, are graphs of a function f and its partial derivatives f_x and f_y. Identify each surface and give reasons for your choices.



70–71 Find f_x and f_y and graph f, f_x, and f_y with domains and viewpoints that enable you to see the relationships between them.

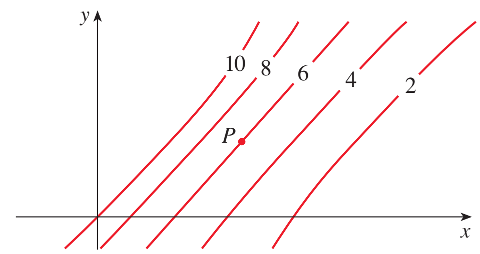
70. $f(x, y) = \dfrac{y}{1 + x^2y^2}$ **71.** $f(x, y) = x^2y^3$

72. Determine the signs of the partial derivatives for the function f whose graph is shown in Exercises 4–5.
(a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$
(c) $f_{xy}(1, 2)$ (d) $f_{xy}(-1, 2)$

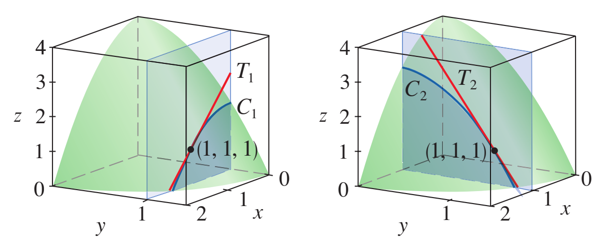
73. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$.

x \ y	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

74. Level curves are shown for a function f. Determine whether the following partial derivatives are positive or negative at the point P.
(a) f_x (b) f_y (c) f_{xx} (d) f_{xy} (e) f_{yy}



75. (a) In Example 3 we found that $f_x(1, 1) = -2$ for the function $f(x, y) = 4 - x^2 - 2y^2$. We interpreted this result geometrically as the slope of the tangent line to the curve C_1 at the point $P(1, 1, 1)$, where C_1 is the trace of the graph of f in the plane $y = 1$. (See the figure.) Verify this interpretation by finding a vector equation for C_1, computing the tangent vector to C_1 at P, and then finding the slope of the tangent line to C_1 at P in the plane $y = 1$.
(b) Use a similar method to verify that $f_y(1, 1) = -4$.



76. If $u = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$, where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u$$

77. Show that the function $u = u(x, t)$ is a solution of the wave equation $u_{tt} = a^2u_{xx}$.
(a) $u = \sin(kx)\,\sin(akt)$
(b) $u = t/(a^2t^2 - x^2)$
(c) $u = (x - at)^6 + (x + at)^6$
(d) $u = \sin(x - at) + \ln(x + at)$

78. Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.
(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$
(c) $u = x^3 + 3xy^2$ (d) $u = \ln\sqrt{x^2 + y^2}$
(e) $u = \sin x \cosh y + \cos x \sinh y$
(f) $u = e^{-x}\cos y - e^{-y}\cos x$

79. Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

80. The Heat Equation Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the *heat conduction equation* $u_t = \alpha^2 u_{xx}$.

81. The Diffusion Equation The *diffusion equation*

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

where D is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time t at a distance x from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)}$$

is a solution of the diffusion equation.

82. The temperature at a point (x, y) on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where T is measured in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in (a) the x-direction and (b) the y-direction.

83. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\partial R/\partial R_1$.

84. Ideal Gas Law The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant.

(a) Show that $\dfrac{\partial P}{\partial V} \dfrac{\partial V}{\partial T} \dfrac{\partial T}{\partial P} = -1$.

(b) Show that $T \dfrac{\partial P}{\partial T} \dfrac{\partial V}{\partial T} = mR$.

85. Van der Waals Equation The *Van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. Calculate $\partial T/\partial P$ and $\partial P/\partial V$.

86. The wind-chill index is modeled by the function

$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature (°C) and v is the wind speed (in km/h). When $T = -15$°C and $v = 30$ km/h, by how much would you expect the apparent temperature W to drop if the actual temperature decreases by 1°C? What if the wind speed increases by 1 km/h?

87. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. Calculate and interpret the partial derivatives.

(a) $\dfrac{\partial S}{\partial w}(160, 70)$ (b) $\dfrac{\partial S}{\partial h}(160, 70)$

88. One of Poiseuille's laws states that the resistance of blood flowing through an artery is

$$R = C\frac{L}{r^4}$$

where L and r are the length and radius of the artery and C is a positive constant determined by the viscosity of the blood. Calculate $\partial R/\partial L$ and $\partial R/\partial r$ and interpret them.

89. In the project following Section 4.7 we expressed the power needed by a bird during its flapping mode as

$$P(v, x, m) = Av^3 + \frac{B(mg/x)^2}{v}$$

where A and B are constants specific to a species of bird, v is the velocity of the bird, m is the mass of the bird, and x is the fraction of the flying time spent in flapping mode. Calculate $\partial P/\partial v$, $\partial P/\partial x$, and $\partial P/\partial m$ and interpret them.

90. In a study of frost penetration it was found that the temperature T at time t (measured in days) at a depth x (measured in feet) can be modeled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and λ is a positive constant.

(a) Find $\partial T/\partial x$. What is its physical significance?
(b) Find $\partial T/\partial t$. What is its physical significance?
(c) Show that T satisfies the heat equation $T_t = kT_{xx}$ for a certain constant k.
(d) Graph $T(x, t)$ for $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$.
(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

91. The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

92. The average energy E (in kcal) needed for a lizard to walk or run a distance of 1 km has been modeled by the equation

$$E(m, v) = 2.65m^{0.66} + \frac{3.5m^{0.75}}{v}$$

where m is the body mass of the lizard (in grams) and v is its speed (in km/h). Calculate $E_m(400, 8)$ and $E_v(400, 8)$ and interpret your answers.

Source: C. Robbins, *Wildlife Feeding and Nutrition*, 2d ed. (San Diego: Academic Press, 1993).

93. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

94. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

95. You are told that there is a function f whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

96. If a, b, c are the sides of a triangle and A, B, C are the opposite angles, find $\partial A/\partial a$, $\partial A/\partial b$, $\partial A/\partial c$ by implicit differentiation of the Law of Cosines.

97. Use Clairaut's Theorem to show that if the third-order partial derivatives of f are continuous, then

$$f_{xyy} = f_{yxy} = f_{yyx}$$

98. (a) How many nth-order partial derivatives does a function of two variables have?

(b) If these partial derivatives are all continuous, how many of them can be distinct?

(c) Answer the question in part (a) for a function of three variables.

99. If

$$f(x, y) = x(x^2 + y^2)^{-3/2} e^{\sin(x^2 y)}$$

find $f_x(1, 0)$. [*Hint:* Instead of finding $f_x(x, y)$ first, note that it's easier to use Equation 1 or Equation 2.]

100. If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

101. Let

$$f(x, y) = \begin{cases} \dfrac{x^3 y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Graph f.

(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.

(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

T (e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

DISCOVERY PROJECT | DERIVING THE COBB-DOUGLAS PRODUCTION FUNCTION

In Example 14.1.4 we described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of the amount of labor L and the capital investment K. If the production function is denoted by $P = P(L, K)$, then $\partial P/\partial L$, the rate at which production changes with respect to the amount of labor, is called the **marginal productivity of labor**. Similarly, $\partial P/\partial K$ is the **marginal productivity of capital**.

Here we use these partial derivatives to show how the particular form of the model used by Cobb and Douglas follows from the following assumptions they made about the economy.

(i) If either labor or capital vanishes, then so will production.

(ii) The marginal productivity of labor is proportional to the amount of production per unit of labor (P/L).

(iii) The marginal productivity of capital is proportional to the amount of production per unit of capital (P/K).

1. Assumption (ii) says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α. If K is held constant ($K = K_0$), then this partial differential equation becomes the ordinary differential equation

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

Solve this separable differential equation by the methods of Section 9.3 to get $P(L, K_0) = C_1(K_0)\, L^\alpha$, where the constant C_1 is written as $C_1(K_0)$ because it could depend on the value of K_0.

(*continued*)