10 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y)\, dA \leqslant M \cdot A(D)$$

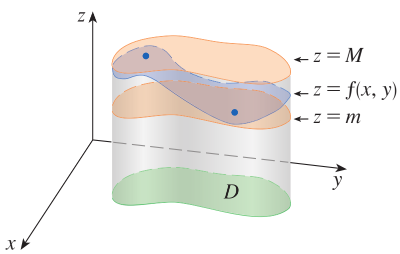


FIGURE 20

Figure 20 illustrates Property 10 for the case $m > 0$. The volume of the solid below the graph of $z = f(x, y)$ and above D is between the volumes of the cylinders with base D and heights m and M. (Compare to Figure 5.2.17, which illustrates the analogous property for single integrals.)

EXAMPLE 6 Use Property 10 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and, because the natural exponential function is increasing, we have

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 10, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$ ■

15.2 Exercises

1–6 Evaluate the iterated integral.

1. $\displaystyle\int_1^5 \int_0^x (8x - 2y)\, dy\, dx$

2. $\displaystyle\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy$

3. $\displaystyle\int_0^1 \int_0^y x e^{y^3}\, dx\, dy$

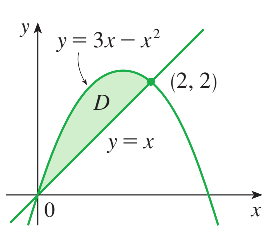
4. $\displaystyle\int_0^{\pi/2} \int_0^x x \sin y\, dy\, dx$

5. $\displaystyle\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$

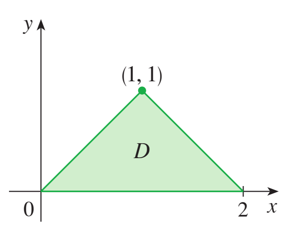
6. $\displaystyle\int_0^1 \int_0^{e^v} \sqrt{1 + e^v}\, dw\, dv$

7–10
(a) Express the double integral $\iint_D f(x, y)\, dA$ as an iterated integral for the given function f and region D.
(b) Evaluate the iterated integral.

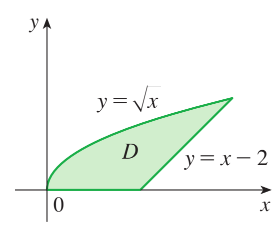
7. $f(x, y) = 2y$



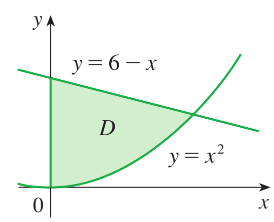
8. $f(x, y) = x + y$



9. $f(x, y) = xy$



10. $f(x, y) = x$



11–14 Evaluate the double integral.

11. $\displaystyle\iint_D \frac{y}{x^2 + 1}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant \sqrt{x}\}$

12. $\displaystyle\iint_D (2x + y)\, dA, \quad D = \{(x, y) \mid 1 \leqslant y \leqslant 2,\ y - 1 \leqslant x \leqslant 1\}$

13. $\displaystyle\iint_D e^{-y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 3,\ 0 \leqslant x \leqslant y\}$

14. $\displaystyle\iint_D y\sqrt{x^2 - y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x\}$

15. Draw an example of a region that is
(a) type I but not type II
(b) type II but not type I

16. Draw an example of a region that is
(a) both type I and type II
(b) neither type I nor type II

17–18 Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

17. $\displaystyle\iint_D x\, dA$, D is enclosed by the lines $y = x$, $y = 0$, $x = 1$

18. $\displaystyle\iint_D xy\, dA$, D is enclosed by the curves $y = x^2$, $y = 3x$

19–22 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

19. $\displaystyle\iint_D y\, dA$, D is bounded by $y = x - 2$, $x = y^2$

20. $\displaystyle\iint_D y^2 e^{xy}\, dA$, D is bounded by $y = x$, $y = 4$, $x = 0$

21. $\displaystyle\iint_D \sin^2 x\, dA$,
D is bounded by $y = \cos x$, $0 \leqslant x \leqslant \pi/2$, $y = 0$, $x = 0$

22. $\displaystyle\iint_D 6x^2\, dA$, D is bounded by $y = x^3$, $y = 2x + 4$, $x = 0$

23–28 Evaluate the double integral.

23. $\displaystyle\iint_D x\cos y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

24. $\displaystyle\iint_D (x^2 + 2y)\, dA$, D is bounded by $y = x$, $y = x^3$, $x \geqslant 0$

25. $\displaystyle\iint_D y^2\, dA$,
D is the triangular region with vertices (0, 1), (1, 2), (4, 1)

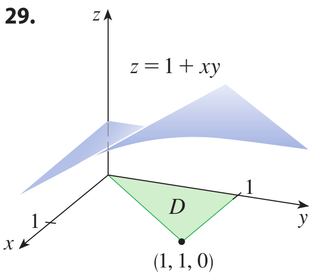
26. $\displaystyle\iint_D xy\, dA$, D is enclosed by the quarter-circle
$y = \sqrt{1 - x^2}$, $x \geqslant 0$, and the axes

27. $\displaystyle\iint_D (2x - y)\, dA$,
D is bounded by the circle with center the origin and radius 2

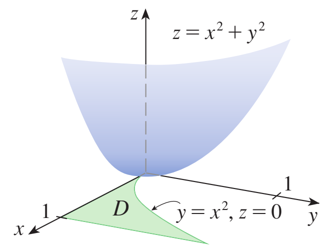
28. $\displaystyle\iint_D y\, dA$, D is the triangular region with vertices (0, 0), (1, 1), and (4, 0)

29–30 The figure shows a surface and a region D in the xy-plane.
(a) Set up an iterated double integral for the volume of the solid that lies under the surface and above D.
(b) Evaluate the iterated integral to find the volume of the solid.

29.



30.



31–40 Find the volume of the given solid.

31. Under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

32. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

33. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

34. Enclosed by the paraboloid $z = x^2 + y^2 + 1$ and the planes $x = 0$, $y = 0$, $z = 0$, and $x + y = 2$

35. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

36. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

37. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

38. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

39. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

40. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

41. Use a graph to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

42. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graph to estimate the points of intersection.)

43–46 Find the volume of the solid by subtracting two volumes.

43. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

44. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

45. The solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$

46. The solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$

47–50 Sketch the solid whose volume is given by the iterated integral.

47. $\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$

48. $\int_0^1 \int_0^{1-x^2} (1 - x)\, dy\, dx$

49. $\int_0^3 \int_0^y \sqrt{9 - x^2}\, dx\, dy$

50. $\int_{-2}^2 \int_{-1}^{3-x^2} e^{-y}\, dy\, dx$

T **51–54** Use a computer algebra system to find the exact volume of the solid.

51. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

52. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

53. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

54. Enclosed by $z = x^2 + y^2$ and $z = 2y$

55–60 Sketch the region of integration and change the order of integration.

55. $\int_0^1 \int_0^y f(x, y)\, dx\, dy$

56. $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx$

57. $\int_0^{\pi/2} \int_{\sin x}^1 f(x, y)\, dy\, dx$

58. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy$

59. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$

60. $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

61–66 Evaluate the integral by reversing the order of integration.

61. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

62. $\int_0^1 \int_{x^2}^1 \sqrt{y}\, \sin y\, dy\, dx$

63. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\, dy\, dx$

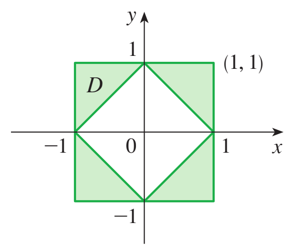
64. $\int_0^2 \int_{y/2}^1 y \cos(x^3 - 1)\, dx\, dy$

65. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

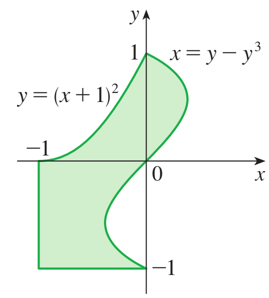
66. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

67–68 Express D as a union of regions of type I or type II and evaluate the integral.

67. $\iint\limits_D x^2\, dA$



68. $\iint\limits_D y\, dA$



69–70 Use Property 10 to estimate the value of the integral.

69. $\iint\limits_S \sqrt{4 - x^2y^2}\, dA$,

$S = \{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$

70. $\iint\limits_T \sin^4(x + y)\, dA$, $\quad T$ is the triangle enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$

71–72 Find the average value of f over the region D.

71. $f(x, y) = xy$,
D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

72. $f(x, y) = x \sin y$,
D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

73. Prove Property 10.

74. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint\limits_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

75–79 Use geometry or symmetry, or both, to evaluate the double integral.

75. $\iint\limits_D (x + 2)\, dA$,

$D = \left\{(x, y) \;\middle|\; 0 \leqslant y \leqslant \sqrt{9 - x^2}\right\}$

76. $\iint\limits_D \sqrt{R^2 - x^2 - y^2}\, dA$,

D is the disk with center the origin and radius R