M 408D (Calculus II)
August 27, 2024

$\int \sqrt{\cot(x)}\, \csc^2(x)\, dx$

Let $u = \cot(x) = \frac{\cos x}{\sin x}$

$du = \frac{-\sin^2 x - \cos^2 x}{\sin^2(x)} = -\frac{1}{\sin^2(x)} dx$

$du = -\csc^2(x)\, dx$

$-du = \csc^2(x)\, dx$

$-1\int \sqrt{u}\, du \rightarrow u^{1/2} \ldots \frac{2}{3}u^{3/2}$

$I = \boxed{-\frac{2}{3}(\cot x)^{3/2} + C}$

$\int \sin(t)\sqrt{1+\cos(t)}\, dt$

Let $u = 1 + \cos(t)$

$\frac{du}{-1} = \frac{-\sin(t)\, dt}{-1}$

$-du = \sin(t)\, dt$

$-1\int \sqrt{u} \longrightarrow u^{1/2} \ldots \frac{2}{3}u^{3/2}$

$I = \boxed{-\frac{2}{3}(1+\cos(t))^{3/2} + C}$

$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

Let $u = \sqrt{x}$

$du = \frac{1}{2}x^{-1/2}\, dx$

$I = 2\int e^{-u}\, du \rightarrow \boxed{2(-e^{-\sqrt{x}}) + C = I}$

$\int \frac{t-1}{3t^2 - 6t - 5} dt$

Let $u = 3t^2 - 6t - 5$

$du = \frac{6t - 6\ dt}{6}$

$\frac{1}{6}\int \frac{1}{u}\, du$

$\frac{1}{6}\ln|(3t^2 - 6t - 5)| + C$

$\int \sin(x)\sin(\cos(x))$

Let $u = \cos(x)$

$\frac{du}{-1} = \frac{-\sin(x)\, dx}{-1}$

$-1\, du = \sin(x)\, dx$

$-1\int \sin(u)\, du$

$I = \boxed{\cos(\cos(x)) + C}$

$\int_0^1 x\sqrt{2x^2+1}\, dx$

Let $u = 2x^2 + 1$

$du = \frac{4x\, dx}{4}$

$\frac{1}{4} du = x\, dx$

$\frac{1}{4}\int_1^3 \sqrt{u} \qquad \frac{2}{3}u^{3/2}$

$\frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^3$

$\frac{1}{4}\left(\frac{2}{3}\sqrt{3^3}\right) - \frac{1}{4}\left(\frac{2}{3}\sqrt{1^3}\right)$

$I = \boxed{\frac{1}{4}\left(\frac{2\sqrt{27}}{3}\right) - \frac{1}{6}}$

August 29, 2024: Discussion section

Log
Inverse trig
Algebraic
Trig
Exponential

M 408D: Calculus II

August 30, 2024

Ex: $\int \ln(x)\, dx$

$u = \ln(x) \quad du = \frac{1}{x} dx$

$dv = x\, dx \quad v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$

$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

Ex: $\int \sin^2 x\, dx$

$\int \frac{1-\cos 2x}{2}\, dx$

$\frac{1}{2}\int 1 - \cos 2x\, dx$

$\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

Example #5: $\int \underbrace{\sin^3 x}\, \underbrace{\cos^2 x}\, dx$

$\int \sin^2 x \cdot \sin x\ \cos^2 x$

$\int (1-\cos^2 x) \sin x \cos^2 x\, dx$

Let $u = \cos x$

$-du = \sin x$

$-\int (1-u^2)\, u^2\, du$

$-\int u^2 - u^4\, du$

$-\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)$

$\boxed{-\left(\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right) + C}$

Example #6: $\int \underbrace{\tan^3}\, x \sec x\, dx$

$\int \tan^2 \cdot \tan x\ \sec x\, dx$

$(\sec^2 x - 1) \tan x \sec x\, dx$

Let $u = \sec x$

$du = \sec x \tan x\, dx$

$\int u^2 - 1\, du$

$\left(\frac{1}{3}u^3 - u\right)$

$\boxed{\frac{1}{3}\sec^3 x - \sec x + C}$

Ex: $\int \ln x\, dx$

Let $u = \ln x$, $du = \frac{1}{x} dx$

$v = x \qquad du = dx$

$= x \ln x - \int x \frac{1}{x}\, dx$

$x \ln x - x + C$

Ex: $\int \tan^{-1} x\, dx$

$\int \arctan(x)\, dx$

IBP $\quad u = \arctan(x) \quad du = \frac{1}{1+x^2} dx$

$v = x \qquad dv = dx$

$x \arctan(x) - \int \frac{x}{1+x^2} dx$

Let $u = 1 + x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$x \arctan(x) - \int \frac{1}{u}\, du$

$x \arctan(x) - \frac{1}{2}\ln|1+x^2| + C$

Ex: $\int e^x \sin x\, dx$

IBP $\quad$ let $u = e^x \quad dv = \sin x\, dx$

$du = e^x\, dx \quad v = -\cos x$

$-e^x \cos x + \int e^x \cos x\, dx$

IBP $\quad$ Let $u = e^x$, $dv = \cos x\, dx$

$du = e^x\, dx \quad v = \sin x$

$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$

$+\int e^x \sin x\, dx$

$\frac{2\int e^x \sin x\, dx}{2} = -e^x \cos x + e^x \sin x$

$\frac{1}{2}(-e^x \cos x + e^x \sin x) + C$

Final de 7.1

Empiezo de sección 7.2 Trig Integrals

Toolbox
1. MEM List
2. U-sub
3. IBP

Note: $\int \sin^3 x \cos x\, dx$

$(\sin x)^3 \cos x\, dx$

U-sub

Trig things to memorize

$\sec x = \frac{1}{\cos x}$

$\csc x = \frac{1}{\sin x}$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$\tan^2 + 1 = \sec^2 x$

Redux Identities

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

Double Angle Identities

$\sin 2x = 2\sin x \cos x$

$\int \sin x\, dx = -\cos x + c$

$\int \cos x\, dx = \sin x + C$

$\int \sec x \tan x\, dx = \sec x + c$

$\int \csc x \cot x\, dx = -\csc x + c$

$\int \sec^2 x\, dx = \tan x + C$

$\int \csc^2 x\, dx = -\cot x + c$

Sept 4, 2024 – M 408D Calc II – Trig Integral Cont'd

TOOLBOX
1. Mem List
2. U-sub
3. IBP
4. Trig Identities

Example #1

$\int \sin^3 x\, dx$

$\int \underbrace{\sin^2 x}\, \sin x\, dx$

$\int \frac{1-\cos 2x}{2} \sin x\, dx$

$\frac{1}{2}\int \sin x - \sin x \cos 2x\, dx$

$\int \underbrace{\sin^2 x}\, \sin x\, dx$

$\sin^2 x = 1 - \cos^2 x$

$\int (1-\cos^2 x) \sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$-\int 1 - u^2\, du$

$-(u - \frac{1}{3}u^3) + C$

$\boxed{-(\cos x - \frac{1}{3}\cos^2 x) + C}$

Example #7: $\int \tan^3 x \sec^2 x\, dx$

$\int \tan^2 x \tan x \sec^2 x\, dx$

$(\sec^2 x - 1) \tan x \sec^2 x\, dx$

Let $u = \sec x$

$du = \sec x \tan x$

$\int (u^2 - 1) u\, du$

$\int u^3 - u\, du$

$\frac{1}{4}u^4 - \frac{1}{2}u^2$

$\boxed{\frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C}$

$u = \tan x$

$du = \sec^2 x\, dx$

$\int u^3\, du$

$\boxed{\frac{1}{4}\tan^4 x + C}$

END of 7.2

Example #2: $\int \sin^4 x\, dx$

$\int \sin^2 x \cdot \sin^2 x\, dx$

$\int \frac{1-\cos 2x}{2} \cdot \frac{1-\cos 2x}{2}\, dx$

$\frac{1}{4}\int (1-\cos 2x)\cdot(1-\cos 2x)\, dx$

$\frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx$

$\frac{1 + \cos 2(2x)}{2} = \frac{1+\cos 4x}{2}$

$\frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx \longrightarrow \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2}$

$\frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\, dx$

$\boxed{\frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C}$

Example #3: $\int \sin^5 x\, dx$

if k is odd in $\sin^k x$, then use Pythagorean identities

$\int \sin^2 x \cdot \sin^2 x \cdot \sin x\, dx$

$(1-\cos^2 x)(1-\cos^2 x) \sin x$

Let $u = \cos x$

$du = -\sin x\, dx$

$-\int (1-u^2)(1-u^2)\, du$

$-\int 1 - 2u^2 + u^4\, du$

$-\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)$

$-\left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right) + C$

M 408D - Calculus II

Section 7.3:

A useful integral

$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx$

$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

Let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$\int \frac{1}{u} du$

$\ln|u|$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Add to the mem list:

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

$\int \tan x \, dx = \ln|\sec x| + C$

or $-\ln|\cos x| + C$

Example #1: $\int \frac{1}{\sqrt{x^2-4}} dx$

Let $x = 2\sec\theta$
$dx = 2\sec\theta\tan\theta \, d\theta$] Direct Replacement

$\sqrt{(2\sec\theta^2) - 2^2}$

$\sqrt{4\sec^2 - 4}$

$2\sqrt{\sec^2\theta - 1}$

$2\tan\theta$

$\int \frac{1}{\sqrt{(2\sec\theta)^2 - 2^2}} 2\sec\theta\tan\theta \, d\theta$

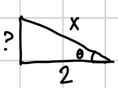
$\cancel{\frac{1}{2}}\int \frac{1}{\cancel{\tan\theta}} \cdot \cancel{2}\sec\theta \cancel{\tan\theta} \, d\theta$

$\int \sec\theta \, d\theta$

$\boxed{\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C}$

$\frac{x}{2} = \frac{2\sec\theta}{2}$

$\frac{x}{2} = \sec\theta$

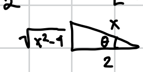


$\cos\theta = \frac{A}{H}$

$\tan\theta = \frac{O}{A} = \frac{\sqrt{x^2-4}}{2}$

$L^2 + 2^2 = x^2 - 2^2$
-2^2

$L^2 = \sqrt{x^2 - 2^2}$



Cases:

1. If $\sqrt{x^2 - a^2}$, you will always use $x = a\sec\theta$ $[\sec^2\theta - 1 = \tan^2\theta]$
2. If $\sqrt{a^2 - x^2}$, you will always use $x = a\sin\theta$ $[\sin^2\theta + \cos^2\theta = 1]$
3. If $\sqrt{x^2 + a^2}$, you will always use $x = a\cos\theta$ $[\tan^2\theta + 1 = \sec^2\theta]$

Example #2: $\int \frac{1}{\sqrt{x^2+9}} dx$

Let $x = 3\tan\theta$
$dx = 3\sec^2\theta \, d\theta$

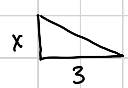
$\sqrt{9\tan^2\theta + 9}$

$3\sqrt{\tan^2\theta + 1}$

$3\sec\theta$

$\sqrt{x^2 + 3^2} = \sqrt{c^2}$

$\int \frac{1}{\sqrt{(3\tan\theta)^2 + 3^2}} 3\sec^2\theta \, d\theta$



$\int \frac{1}{\cancel{3\sec\theta}} \cancel{3}\sec^{\cancel{2}}\theta \, d\theta$

$\int \sec\theta \, d\theta$

$\ln|\sec\theta + \tan\theta| + C$

$\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$

Example #3: $\int \frac{1}{\sqrt{25 - x^2}}$

$x = 5\sin\theta$
$dx = 5\cos\theta \, d\theta$

$\frac{x}{5} = \sin\theta$

$\arcsin\left(\frac{x}{5}\right) = \theta$

$\int \frac{1}{\sqrt{25 - (5\sin\theta)^2}}$

$\sqrt{25 - 25\sin^2\theta}$

$5\sqrt{1 - \sin^2\theta}$

$5\sqrt{\cos^2\theta}$

$5\cos\theta$

$\int \frac{1}{\cancel{5\cos\theta}} \cdot \cancel{5\cos\theta} \, d\theta$

$\int d\theta$

$\theta + C \equiv \arcsin\left(\frac{x}{5}\right) + C$

Note: Algebra

$\int \frac{1}{\sqrt{2x^2 + 5}} dx$

$\sqrt{2\left(x^2 + \frac{5}{2}\right)}$

$\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2 + \frac{5}{2}}}$

$x = \sqrt{\frac{5}{2}}\tan\theta$

NOTE: The integral does not even need a square root.

$\int \frac{x^2}{x^2+7} dx$ $\quad x = \sqrt{7}\tan\theta$

M 408D - Calculus II - September 9, 2024

Toolbox
1. Mem List
2. U-sub
3. IBP
4. Trig Identites
5. Trig Sub

Trig Substitution cases
- $\sqrt{x^2 + a^2}$, $x = a\tan\theta$
- $\sqrt{x^2 - a^2}$, $x = a\sec\theta$
- $\sqrt{a^2 - x^2}$, $x = a\sin\theta$

Ex. $\int \frac{1}{x^2\sqrt{x^2-16}} dx$

Let $x = 4\sec\theta$

$dx = \cancel{4}\sec\theta \cancel{\tan\theta} \, d\theta$

$\int \frac{1}{(4\sec\theta)^2\sqrt{(4\sec\theta)^2 - 16}}$

$\sqrt{16\sec^2\theta - 16}$

$\sqrt{16(\sec^2\theta - 1)}$

$16\sec^2\theta \quad \sqrt{16\tan^2\theta}$

$16\sec^2\theta \quad \cancel{4\tan\theta}$

$\int 16\sec\theta$

$\frac{1}{16}\int \frac{1}{\sec\theta} d\theta$

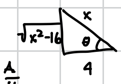
$\frac{1}{16}\int \cos\theta \, d\theta$

$\frac{1}{16}\sin\theta + C$

$\boxed{I = \frac{1}{16}\left(\frac{\sqrt{x^2-16}}{x}\right) + C}$

$\frac{x}{4} = \sec\theta$

$\frac{4}{x} = \cos\theta \quad \frac{A}{H}$



$c^2 + 4^2 = x^2$

$c = \sqrt{x^2 - 16}$

Ex. $\int \sqrt{4 - x^2} \, dx$

Let $x = 2\sin\theta$
$dx = 2\cos\theta \, d\theta$

$\int \sqrt{4 - 4\sin^2\theta} \; 2\cos\theta \, d\theta$

$2\int \sqrt{1 - \sin^2\theta}$

$\sqrt{\cos^2\theta}$

$2\int \cos\theta \; 2\cos\theta \, d\theta$

$4\int \cos^2\theta \, d\theta$

$4\int \frac{1 + \cos(2\theta)}{2} d\theta$

$2\int 1 + \cos 2\theta \, d\theta$

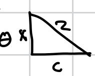
$2\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$

$2(\theta + \sin\theta\cos\theta) + C$

$\sin 2\theta = 2\sin\theta\cos\theta$

Go back to x

$\frac{x}{2} = \sin\theta$



$x^2 + c^2 = 2^2$

$c = \sqrt{4 - x^2}$

$2\left(\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{\sqrt{4-x^2}}{2}\right)\right) + C$

END OF SECTION 7.3

START OF 7.4 (Partial Fractions)

Ex. $\int \frac{3}{5x-2} dx$

Let $u = 5x - 2$
$du = 5 \, dx$

$\frac{3}{5}\int \frac{1}{u} du$

$\frac{3}{5}\ln|u| + C$

$\frac{3}{5}\ln|5x - 2| + C$

Ex. $\int \frac{3x+1}{x^2+1} dx$

$\int \left(\frac{3x}{x^2+1} + \frac{1}{x^2+1}\right)$

$\int \frac{3x}{x^2+1} + \int \frac{1}{x^2+1}$

Let $u = x^2 + 1$
$du = 2x \, dx$

$\frac{3}{2}\int \frac{1}{u} du$

$\frac{3}{2}\ln|x^2 + 1| + \tan x + C$

NOTE: $\int \frac{\text{CONST}}{\text{LINEAR}}$ (EASY!) NOTE: $\int \frac{\text{linear}}{\text{QUAD}}$ (Not that bad)

Partial Fractions (Alg. Streets)

Case I. Given a rational function $\frac{P(x)}{Q(x)}$ such that our denominator factors into distinct linear factors

Ex. 1 $\frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x-2}$

Ex. 2 $\frac{\sim\sim\sim}{(x-4)(x-2)(x+1)} = \frac{A}{(x-4)} + \frac{B}{(x-2)} + \frac{C}{x+1}$

CASE II. Given $\frac{P(x)}{Q(x)}$, $Q(x)$ has repeating linear factors.

Ex 1. $\frac{\sim\sim\sim}{(x+1)(x-3)^2} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

Ex 1. $\frac{}{(x+7)^4} = \frac{A}{(x+7)} + \frac{B}{(x+7)^2} + \frac{C}{(x+7)^3} + \frac{D}{(x+7)^4}$

Ex: $\int \frac{1}{x^3 + 4x^2 + 4x} \rightarrow \frac{1}{x(x^2 + 4x + 4)}$

$\frac{1}{x(x+2)(x+2)}$

$x(x+2)^2\left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\right)$

$\int \frac{1}{4}\left(\frac{1}{x}\right) - \frac{1}{4}\left(\frac{1}{x+2}\right) - \frac{1}{2}\left(\frac{1}{(x+2)^2}\right)$

$\frac{1}{4}(x+2)^2 + Bx(x+2) - \frac{1}{2}x$

$\frac{4}{4} + 3B \quad -\frac{1}{2}$

$B = -\frac{1}{4}$

$x = -2 \quad -\frac{1}{2}(x+2)^{-1}$
$x = 0$

$1 = A(x+2)^2 + Bx(x+2) + Cx$

$1 = 0 - 2C \quad 4A = 1$

$1 = -2C \quad A = \frac{1}{4}$

$\frac{1}{4}\ln|x| \quad 1 - \frac{1}{2} = C$

$-\frac{1}{4}\ln|x+2|$

$x - 3 = 0$
$x = +3$

$x + 3 = 0$
$x = -3$

Ex. $\int \frac{x+5}{x^2-9} dx$

$\frac{x+5}{(x+3)(x-3)} = \left(\frac{A}{x+3} + \frac{B}{x-3}\right)(x+3)(x-3)$

$x + 5 = A(x-3) + B(x+3)$

$x + 5 = Ax - 3A + Bx + 3B$

$1x + 5 = (A+B)x + (-3A + 3B)$

$A + B = 1$
$-3A + 3B = 5$

$A = \frac{1}{3} \quad -\frac{1}{3}\ln|x+3| + \frac{4}{3}\ln|x-3|$

$B = \frac{4}{3}$

$x + 5 = A(x-3) + B(x+3)$ when $x = 3$

$8 = A(3-3) + B(3+3)$

$6B = 8 \rightarrow \frac{4}{3} = B$

Case III: $\frac{P(x)}{Q(x)}$, if $Q(x)$ has an irreducible quadractic factor.

Ex. $\frac{\ldots}{(x-3)(x^2+x+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+x+1}$

Case IV: $Q(x)$ can have a repeated irreducible quadractic

Ex. $\frac{\ldots}{(x-3)(x^2+x+1)^2} = \frac{A}{x-3} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$

Ex. $\frac{\ldots}{(x^2+4)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3}$

Ex. $\frac{\ldots}{(x-1)(x+3)^3(x^2+9)(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)^3} + \frac{Ex+F}{x^2+9} + \frac{Gx+H}{x^2+1} + \frac{Ix+J}{(x^2+1)^2}$

$$\int\left(3x-10+\frac{23}{x+1}+\frac{-15}{(x+1)^2}\right)dx$$

Let $u = x+1$
$du = dx$

$$23\int\frac{1}{u} - 15\int\frac{1}{u^2}$$

$$\frac{3}{2}x^2 - 10x + 23\ln|x+1| - 15(x+1)^{-1} + C$$

Ex: $\frac{x^3-4x-10}{x^2-x-6}$

$x^2-x-6 \overline{)\, x^3+0x^2-4x-10}$

END of 7.4 (Partial Fractions)

Section 7.5, 7.6, 7.7

Review for exam | Table of Integrals | CA Integration (Numerical Integration)

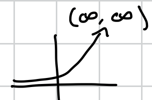
Begin Section 7.8 (Improper Integrals)

Recall Limits

① $\lim_{x\to\pm\infty} \frac{C}{x^n} = 0$, n is positive

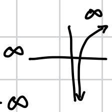
② $\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$



③ $\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$



④ $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

Ex. $\int \frac{7x}{x^3-2x^2+x-2}dx$

$x^2(x-2)+1(x-2)$

$(x-2)(x^2+1)\left(\frac{7x}{(x-2)(x^2+1)}\right) = \left(\frac{A}{x-2}+\frac{Bx+C}{x^2+1}\right)(x-2)(x^2+1)$

$7x = A(x^2+1) + Bx + C(x-2)$

$7x = (Ax+B)(x-2) + \frac{14}{5}(x^2+1)$ | $x=2$: $14 = 0 + \frac{C(5)}{5}$ → $\frac{14}{5} = C$

$x=0$

$0 = B(-2) + \frac{14}{5}$

$2B = 14/5$

$B = 7/5$

$7x = (Ax + 7/5)(x-2) + \frac{14}{5}(x^2+1)$

$x=1$

$7 = (A+7/5)(-1) + \frac{14}{5}(2)$

$= -A - 7/5 + \frac{28}{5}$

$A = -14/5$

$$= \int \frac{-\frac{14}{5}x + 7/5}{x^2+1} + \frac{14/5}{x-2}\,dx.$$

$$= \int\left(\frac{-14/5\,x}{x^2+1} + \frac{7/5}{x^2+1} + \frac{14/5}{x-2}\right)dx$$

$$= -\frac{14}{5}\int\frac{x}{x^2+1}dx + \frac{7}{5}\int\frac{1}{x^2+1}dx + \frac{14}{5}\int\frac{1}{x-2}dx$$

Let $u = x^2+1$, $du = 2x\,dx$ | Let $u = x-2$, $du = dx$

$\frac{-14}{10}\int\frac{1}{u}du$ | $+\frac{14}{5}\int\frac{1}{u}du$

$$= -\frac{7}{5}\ln|x^2+1| + 7/5\tan^{-1}x + 14/5\ln|x-2| + C$$

L'Hopital Rule

If the $\lim_{x\to a}\frac{f(x)}{g(x)}$ results in an indeterminate form then

I.F $= \{\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \frac{1}{\infty}, 0^\infty, \infty-\infty\}$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Type I improper integral

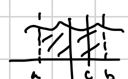
If f is continous on the interval of integration

① $\int_a^\infty f(x)\,dx \longrightarrow \lim_{b\to\infty}\int_a^b f(x)\,dx$

Ex. $\int_7^\infty f(x)\,dx = \int_7^b f(x)\,dx$

② $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$

③ $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$
② ①



Example #1: $\int_1^\infty \frac{1}{x}dx$

$= \lim_{b\to\infty}\int_1^b \frac{1}{x}dx$

$= \lim_{b\to\infty} \ln|x|\Big|_1^b$

$= \lim_{b\to\infty} \ln|b| - \ln|1|$

$= \infty$

Note! An improper integral converges if it's finite.

If the limit is infinite or is DNE, then it diverges.

Example #2: $\int_1^\infty \frac{1}{x^2}dx$

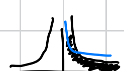
$= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$

$= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$

$= \lim_{b\to\infty} -\frac{1}{b} + 1$

$= \boxed{1}$

Converges



Note: Given $\frac{P(x)}{P(Q)}$, if $\deg(P) \geq \deg(Q)$, then we can use polynomial division to rewrite

$\frac{P(x)}{Q(x)} = f(x) + \boxed{\frac{P(x)}{Q(x)}} \to$ Partial fraction

$3\overline{)\,4381}$ = 1460

-3
13
-12
018
-18
01

$1460 + \frac{1}{3}$ — Quotient, remainder, Divisor

Example: $\int \frac{3x^3-4x^2+6x-2}{x^2+2x+1}dx$

$$\begin{array}{r} 3x-10 \\ x^2+2x+1 \overline{)\, 3x^3-4x^2+6x-2} \\ -3x^3 \mp 6x^2 \mp 3x + 0 \\ \hline 0 - 10x^2 + 3x - 2 \\ -(-10x^2-20x-10) \\ \hline 0 + 23x + 8 \end{array}$$

$\int\left(3x-10+\underbrace{\frac{23x+8}{x^2+2x+1}}_{\text{Partial Frac.}}\right)$

$\int\left(3x-10+\underbrace{\frac{23x+8}{x^2+2x+1}}_{\text{Partial Frac}}\right)$

$\int\left(3x-10+\frac{23x+8}{(x+1)(x+1)}\right) \Longrightarrow \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$23x + 8 = A(x+1) + B$, $x = -1$

$B = -15$

$23x + 8 = A(x+1) - 15$, $x = 0$

$8 = A - 15 \to A = 23$

Example #3: $\int_{-\infty}^\infty \frac{4}{1+x^2}dx \to \int_{-\infty}^0 \frac{4}{1+x^2}dx + \int_0^\infty$

$= \lim_{a\to-\infty}\int_a^0 \frac{4}{1+x^2} + \lim_{b\to\infty}\int_0^\infty \frac{4}{1+x^2}$

$\lim_{a\to-\infty} 4\arctan\Big|_a^0 + \lim_{b\to\infty} 4\arctan\Big|_0^b$

$\lim_{a\to-\infty} 4\arctan(0) - 4\arctan(a) + \lim_{b\to\infty} 4\tan^{-1}(b) + 4\tan^{-1}(0)$

$\cancel{4(0)}^0 - 4(-\frac{\pi}{2}) + 4(\frac{\pi}{2}) - \cancel{4(0)}^0$

$\frac{4\pi}{2} + \frac{4\pi}{2}$

$\boxed{4\pi}$

Type II improper integral

① If $f(x)$ is continuous on $(a,b]$; discontinuous at a.

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

$$\int_3^5 \frac{1}{x-3}\,dx$$

$$\lim_{c\to 3}\int_c^5 \frac{1}{x-3}\,dx$$

② If $f(x)$ is continous $[a,b)$; discontinous at b.

$$\int_a^b f(x)\,dx \to \lim_{c\to b^-}\int_a^c f(x)\,dx$$

③ Continuous on $[a,b]$ except there's a discontinuity at $x=C$, $a<c<b$

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{①} + \underbrace{\int_c^b f(x)\,dx}_{②}$$

Example #1: $\int_a^3 \frac{1}{x-3}\,dx$

$$\lim_{b\to 3^-}\int_a^b \frac{1}{x-3}\,dx$$

Let $u = x-3$, $du = dx$

$$\int \frac{1}{u}\,du$$

$$\ln|u|$$

$$\lim_{b\to 3}\ \ln|x-3|\Big]_0^b \quad [0,3)$$

$$\lim_{b\to 3} \ln|b-3| - \ln|-3|$$

$$\lim_{b\to 3} \ln|b-3| - \ln|3|$$

$$\boxed{-\infty}$$

Example #2: $\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}$

$$\lim_{a\to -2^+}\int_a^{14} \frac{1}{\sqrt[4]{x+2}}$$

let $u = x+2$, $du = dx$

$$\int \frac{1}{\sqrt[4]{u}}\,du \to \int u^{-1/4}\,du$$

$$\frac{4}{3}u^{3/4}$$

$$\lim_{a\to -2^+} \frac{4}{3}(x+2)^{3/4}\Big|_a^{14}$$

$$\lim_{a\to -2^+}\left(\frac{4}{3}(8) - \frac{4}{3}(a+2)^{3/4}\right)$$

$$\frac{32}{3} - 0 = \boxed{\frac{32}{3},\ \text{converges}}$$

END 7.8. - END Chapter 7

Chapter 9 - Intro to Diffy Q!

Recall: To solve any equation, is to find all possible values or functions that makes the equation true.

Example #1: Solve for x: $3x-2=7$

$$+2 \quad +2$$

$$\frac{3x}{3} = \frac{9}{3}$$

$$x=3$$

$$3(3)-2$$

$$9-2=7$$

$$7=7$$

Example #2: Solve for y: $\int \frac{dy}{dx} = \int 3x$

$$\int \frac{dy}{dx}\,dx = \int 3x\,dx$$

$$y = \frac{3}{2}x^2 + C$$

Def: Any equation with a derivative is a differential equation.

Q_1: $\frac{dy}{dx} = 3y$

$$\int \frac{dy}{dx} = \int 3y\,dx$$

$y =$

Defn: A differential equation that describes a physical process is a mathematical model

Defn: A differential equation with an initial condition is known as an initial value problem (IVP)

$$\frac{dy}{dx} = 3x \to \frac{3}{2}x^2 + C$$

Let's say $y(0) = 4$

$$4 = \frac{3}{2}\cancel{0^2}^{0} + C$$

$$\boxed{4 = C}$$

$$\boxed{y = \frac{3}{2}x^2 + 4}$$

Ex: Find all integer values of k for which $y = \sin kt$ is a solution to $y'' + 4y = 0$

$$y = \sin kt$$

$$y' = k\cos kt$$

$$y'' = -k^2 \sin kt$$

$$-k^2y + 4y = 0$$

$$-4y \quad -4y$$

$$\frac{-k^2\cancel{y}}{\cancel{y}} = \frac{-4y}{y}$$

$$\frac{-k^2}{-1} = \frac{-4}{-1}$$

$$\sqrt{k^2} = \sqrt{4}$$

$$\boxed{k = \pm 2}$$

or 0 [trivial]

Ex: Find all values of a for which $y = e^{at}$ is a solution to $y'' - y' - 12y = 0$

$$y = e^{at}$$

$$y' = ae^{at}$$

$$y'' = a^2e^{at}$$

$$a^2y - ay - 12y = 0$$

$$y(a^2 - a - 12) = 0$$

$$a = 4$$

$$a = -3$$

Ex: The solution to $y' = -4xy$ is $y = Ce^{-2x^2}$; find the solution that satisfies $y(0) = 2$.

$$y = Ce^{-2(0)^2}$$

$$2 = Ce^{-2(0)^2}$$

$$2 = C$$

$$\boxed{y = 2e^{-2x^2}}$$

Ex: Find all costant solutions to $y' = y^3 - 4y$

Answers $= y = C$

$$y' = 0$$

$$0 = y^3 - 4y$$

$$y(y^2-4)$$

$$y = 0$$

$$y^2 - 4 = 0$$

$$+4 \quad 4$$

$$\sqrt{y^2} = \sqrt{4}$$

$$\boxed{y = \pm 2}$$

$$y = \{-2, 0, 2\}$$

END 9.1 – Start 9.2

9.2 Euler's Method

END 9.2 –

9.3 Separable Differential Equations

Sept 23, 2024

Defn: A differential equation is separable if it can be written in the form of

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Seperable

$$\frac{dy}{dx} = 3x^2y^3 \rightarrow \frac{3x^2}{y^{-3}}$$

Not Seperable

$$\frac{dy}{dx} = 3x^2 + 7y^3 + \sin(xy)$$

Note: A separable differential equation can be solved using integration.

$$g(y)\cdot\frac{dy}{dx} = \frac{f(x)}{\cancel{g(y)}}\cdot \cancel{g(y)} \rightarrow g(y)\frac{d(y)}{dx} = f(x)$$

$$\int g(y)\frac{d(y)}{dx}\cdot dx = \int f(x)\,dx$$

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \longrightarrow \int g(y)\,dy = \int f(x)\,dx$$

Example #1: Solve $y' = 2xy^2$

① $\frac{dy}{dx} = \frac{2x}{y^{-2}}$ Want to see these steps

$$\int y^{-2}\,dy = \int 2x\,dx \quad ②$$

$$\frac{-y^{-1}}{-1} = \frac{x^2 + C}{-1} \rightarrow \text{This is ok in 408D}$$

$$\frac{1}{y} = -x^2 + C$$

$$y = \frac{1}{-x^2+C}$$

Example #2: $\frac{dy}{dx} = x^3e^{2y}$

$$\frac{dy}{dx} = \frac{x^3}{e^{-2y}}$$

$$\int e^{-2y}\,dy = \int x^3\,dx$$

$$\boxed{-\frac{1}{2}e^{-2y} = \frac{1}{4}x^4 + C}$$

Example #3: Solve IVP; $\frac{dy}{dx} = 3y,\ y(0) = 2$

$$\frac{dy}{dx} = \frac{3}{y^{-1}}$$

$$\int y^{-1}\,dy = \int 3\,dx$$

$$\boxed{\ln|y| = 3x + C}$$

$$e^{\ln|y|} = e^{3x+C}$$

$$y = e^{3x+C}$$

$$y = e^{3x}\cdot e^{C}$$

$$y = Ce^{3x} \qquad \boxed{2 = C}$$

$$2 = C\cancel{e^{3(0)}}^{1}$$

$$\boxed{y = 2e^{3x}}$$

Checking answer:

$$6e^{3x} = 3(2e^{3x})$$

What if:

$$\left.\begin{aligned} \ln|y| &= 3x + C \\ \ln|2| &= 3(0) + C \\ \ln|2| &= C \end{aligned}\right] \rightarrow \ln|y| = 3x + \ln|2|$$

Example #4: $2yy' = xy^2 + x$

$$\frac{dy}{dx} = \frac{xy^2+x}{2y} \rightarrow \frac{x(y^2+1)}{2y} \rightarrow \frac{x}{2y(y^2+1)^{-1}}$$

$$\int 2y(y^2+1)^{-1}\,dy = \int x\,dx$$

Let $u = y^2+1$

$du = 2y\,dy$

$$\int u^{-1}\,du = \frac{1}{2}x^2 + C$$

$$\ln|u| \rightarrow \boxed{\ln|y^2+1| = \frac{1}{2}x^2 + C}$$

$$e^{y^2+1} = e^{\frac{1}{2}x^2+C}$$

$$y^2+1 = e^{\frac{1}{2}x^2+C}$$

$$y^2+1 = Ce^{\frac{1}{2}x^2}$$

$$\sqrt{y^2} = \sqrt{Ce^{\frac{1}{2}x^2} - 1}$$

$$y = \pm\sqrt{Ce^{\frac{1}{2}x^2} - 1}$$

END - 9.3

Start 9.5 - Integrating Factor

Example #1: Solve $\frac{dy}{dx} = -\frac{1}{4}y + \frac{1}{4}e^x$

Step 1, move anything with a y to the left side.

$$\frac{dy}{dx} + \frac{1}{4}y = \frac{1}{4}e^x$$

Step 2, multiply left side eqn. by e^x

$$e^{\frac{1}{4}x}\left(\frac{dy}{dx} + \frac{1}{4}y = \frac{1}{4}e^x\right)$$

$$e^{\frac{1}{4}x}\frac{dy}{dx} + \frac{1}{4}e^{\frac{1}{4}x}y = \frac{1}{4}e^{\frac{1}{4}x}e^x$$

③ Undo the product rule.

$$\frac{d}{dx}(f\cdot g) \rightarrow \frac{d}{dx}\left(e^{\frac{1}{4}x}\cdot y\right) = \frac{1}{4}e^{\frac{5}{4}x}$$

Step 4, integrate both sides

$$\int \frac{d}{dx}\left(e^{\frac{1}{4}x}y\right)dx = \int \frac{1}{4}e^{\frac{5}{4}x}\,dx$$

$$\boxed{e^{\frac{1}{4}x}y = \frac{1}{5}e^{\frac{5}{4}x} + C} \quad C \text{ cannot absorb operations!}$$

$$y = \frac{\frac{1}{5}e^{\frac{5}{4}x} + C}{e^{\frac{1}{4}x}}$$

Note: To solve a differential equation of the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

You multiply by the integrating factor [is always $\mu(x) = e^{\int p(x)\,dx}$]

Example #2: Solve $\frac{dy}{dx} + 3y = 2xe^{-3x}$

$p(x) = 3$

$\mu(x) = e^{\int 3\,dx} = e^{3x}$

$$\underbrace{e^{3x}}_{f}\underbrace{\frac{dy}{dx}}_{g'} + \underbrace{3e^{3x}}_{f'}\underbrace{y}_{g} = 2x\cancel{e^{-3x}e^{3x}}$$

$$\frac{d}{dx}(\underbrace{e^{3x}\cdot y}_{f\cdot g}) = 2x \longrightarrow \int \frac{d}{dx}(e^{3x}\cdot y)\,dx = \int 2x\,dx$$

$$\boxed{e^{3x}\cdot y = x^2 + C}$$

$$y = \frac{x^2+C}{e^{3x}}$$

Example #3: Solve $y' = xe^{2x} + y$

$$y' - y = xe^{2x}$$

$p(x) = -1 \quad \mu = e^{-x}$

$$\underbrace{e^{-x}}_{f}\underbrace{y'}_{g'} - \underbrace{e^{-x}}_{f'}\underbrace{y}_{g} = xe^{2x}e^{-x}$$

$$\frac{d}{dx}(e^{-x}y) = xe^{x}$$

$$e^{-x}y = \int xe^x\,dx$$

$$\int xe^x\,dx \overset{\text{IBP}}{\rightarrow} \quad u = x \quad dv = e^x, \quad du = dx \quad v = e^x$$

$$= xe^x - \int e^x\,dx$$

$$e^{-x}\cdot y = xe^x - e^x + C$$

$$y = \frac{xe^x - e^x + C}{e^{-x}}$$

IF: $\frac{dy}{dx} + p(x)y = q(x)$

Solve: Using integrating factor

$$\mu = e^{\int p(x)\,dx}$$

$$\text{Solve}: x^4\left(y' + \frac{4y}{x}\right) = \left(\frac{\sin x}{x^4}\right)x^4 \quad \mu = e^{\int \frac{4}{x}dx} = e^{\ln|x^4|} = x^4$$

$$\underbrace{x^4}_{f}\underbrace{\frac{dy}{dx}}_{g'} + \underbrace{4x^3}_{f'}\underbrace{y}_{g} = \sin x$$

$$\int \frac{d}{dx}(x^4y) = \int \sin x$$

$$x^4y = -\cos x + C$$

$$x^4y + C - C = -\cos x + C - C$$

$$\boxed{y = \frac{-\cos x + C}{x^4}}$$

End of 9.5 & Chapter 9 (Exam 1)

Start Chapter 10 / 10.1 Parametric Equations.

Note: All of the calculus we know is defined for functions only. $f(x) = y$

Defn: If your x and y values are given as functions

$$x = f(t) \qquad y = g(t)$$

Over an interval of t values $[a,b]$; then the set of points $(x,y) = (f(t), g(t))$ is a parametric curve

- f, g are parametric equation(s)
- t parameter(s)
- $(f(a), g(a))$ - Initial point

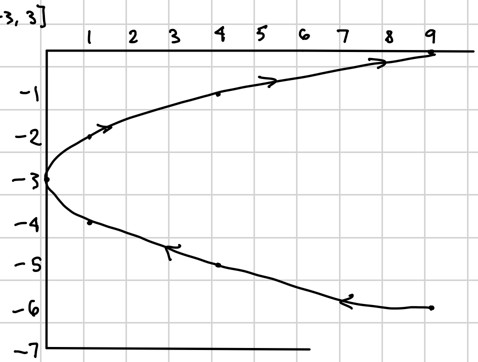
- $(f(b), g(b))$ Terminal point
- $f(t), g(t), [a, b]$ – Parameterization of a curve.

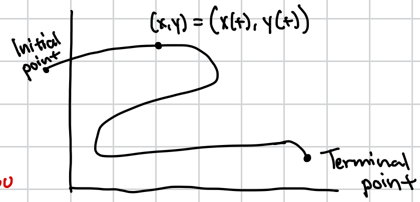
Example #1: Sketch the graph of

$x(t) = t^2$, $y(t) = t - 3$

$[-3, 3]$

t	x	y	(x, y)
-3	9	-6	9, -6
-2	4	-5	4, -5
-1	1	-4	1, -4
0	0	-3	0, -3
1	1	-2	1, -2
2	4	-1	4, -1
3	9	0	9, 0





Note!

Never graph by hand!

Find a computer to do it for you

Parametric equations cannot be expressed as a Cartesian equation (x, y)

Example #1.1

$x = t^2$, $y = t - 3$

$y + 3 = t$

$x = (y+3)^2$

$x = y^2 + 6y + 9$

Find a Cartesian equation given the parametric equation:

a) $x = t^2$, $y = t^3$ Note: $y = \pm x^{\frac{3}{2}}$

$y^{\frac{1}{3}} = t$

$x = (y^{1/3})^2 \to y^{2/3}$

$\boxed{x = y^{2/3}}$

b) $x = e^t - 1$, $y = e^{2t}$

$x = e^{\ln y^{1/2}} - 1$ $\ln|y| = 2t$

$\boxed{x = y^{1/2} - 1}$ $\frac{1}{2}\ln|y| = t$

$x = e^t - 1$, $y = (e^t)^2$

$x + 1 = e^t$

$\boxed{y = (x+1)^2}$

c) $x = 2\cos t$ $y = 2\sin t$

$\sin^2 t + \cos^2 t = 1$

$x^2 = 4\cos^2 t$ & $y^2 = 4\sin^2 t$

$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t$

$= 4(\cos^2 t + \sin^2 t)$

$= 4(1) = 4$

Made up this equation

Whipped it out of thin air.

$x^2 + y^2 = 4$

d) What if $x = 6\sin(3t)$ and $y = 2\cos(3t)$

$\frac{x}{6} = \sin(3t)$ $\left(\frac{x}{6}\right)^2 = \sin^2(3t)$

$\frac{y}{2} = \cos(3t)$ $+ \left(\frac{y}{2}\right)^2 = \cos^2(3t)$

$\boxed{\frac{x^2}{36} + \frac{y^2}{4}} = \sin^2(3t) + \cos^2(3t) = 1$

Identity: $\tan^2(t) + 1 = \sec^2(t)$

e) $x = \sec t$ $y = \tan t$

$1 = \sec^2(t) - \tan^2(t)$

$x^2 = \sec^2(t)$

$y^2 = \tan^2(t)$

$x^2 - y^2 = \sec^2(t) - \tan^2(t)$

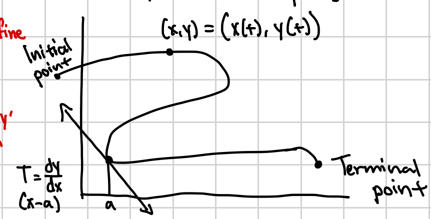
$\boxed{x^2 - y^2 = 1}$

END of 10.1 (Parametric Equations)

Section 10.2 (Calculus with Parametric Eqnts)

Note: When we define a curve as $(x, y) = (x(t), y(t))$

We can evaluate x' & y' but they do not mean the same thing as $\frac{dy}{dx}$



Note: To calculate $\frac{dy}{dx}$, which gives us (slope of the tangent line, rate of change, velocity)

We need a relationship between $\frac{dy}{dx}$ and $x(t), y(t)$

Also, $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ when $x'(t) \neq 0$

In addition, (2nd deriv) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)}$ and when $x'(t) \neq 0$

Example #1: Given $x = 2\cos t$, $y = 2\sin t$

1a.) Find $\frac{dy}{dx} = \frac{y'}{x'} = \frac{2\cos t}{-2\sin t} = \boxed{-\cot t}$

1b. Find the equation of the tangent line to the curve at $t = \frac{\pi}{4}$

$M = \frac{dy}{dx}\Big|_{t=\frac{\pi}{4}} = -\cot\left(\frac{\pi}{4}\right) = \frac{-\cos\left(\frac{\pi}{4}\right)}{\sin(\pi/4)} = -1$

$y = Mx + b$

$1 = -x + b$

$x = 2\cos\left(\frac{\pi}{4}\right)$

$= \cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} = \sqrt{2}$

$y = 2\sin\left(\frac{\pi}{4}\right)$

$= 2$

Recall: Given a curve $(x, y) = (x(t), y(t))$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$, $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)}$

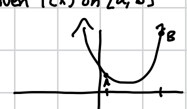
Example: $x = 2\cos\theta$, $y = 2\sin\theta$

$\frac{dy}{dx} = -\cot t$

Find $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-\cot t)}{-2\sin t} = \frac{\csc^2 t}{-2\sin t} = -\frac{1}{2}\csc^3 t$

Recall from Calc I:

Given $f(x)$ on $[a, b]$



$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Note: If a curve is defined by $(x, y) = (x(t), y(t))$

for $a \leq t \leq b$ where $x'(t)$ and $y'(t)$ are continuous and not simultaneously zero on $[a, b]$ and the curve is traversed exactly once on $[a, b]$

then:

$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

Example #1: Find the length of the curve $x = t^3$, $y = \frac{3t^2}{2}$

from $0 \leq t \leq \sqrt{3}$

$(x'(t))^2 = 9t^4$

$(y'(t))^2 = 9t^3$

$L = \int_0^{\sqrt{3}} \sqrt{9t^4 + 9t^2}\, dt$

$\int_0^{\sqrt{3}} \sqrt{9t^2(t^2+1)}\, dt$

$\int_0^{\sqrt{3}} 3t\sqrt{t^2+1}$

Let $u = t^2 + 1$

$du = 2t\, dt$

$\frac{1}{2}du = t\, dt$

$\frac{3}{2}\int_0^{\sqrt{3}} \sqrt{u}\, du$

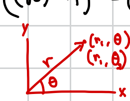
$\frac{3}{2}\left(\frac{2}{3}u^{3/2}\right)$

$\cancel{\frac{6}{6}}^{1} u^{3/2}$

$(t^2+1)^{3/2}\Big|_0^{\sqrt{3}}$

$\left((\sqrt{3})^2 + 1\right)^{3/2} - \left((0)^2 + 1\right)^{3/2}$

Polar vs. Cartesian Coordinate System



$x = r\cos\theta$ | $r^2 = x^2 + y^2$

$y = r\sin\theta$ | $\tan\theta = \frac{y}{x}$

Section 10.3 Polar Coordinate

Recall: A polar coordinate is a point (r, θ) where r is the distance from the origin and θ is the angle of direction

Example #2: Find a Polar Equation for ★ Test #2

ⓐ $x^2+y^2=2 \longrightarrow (r\cos\theta)^2+(r\sin\theta)^2=2$

$r^2\cos^2\theta + r^2\sin^2\theta$

$r^2(\cancel{\cos^2\theta+\sin^2\theta})^1 = 2$

$\boxed{r^2=2}$

$\sqrt{r^2}=\sqrt{2}$

$\boxed{r=\sqrt{2}}$

ⓑ $3x^2y=7$

$3(r\cos\theta)^2(r\sin\theta)=7$

$\boxed{3r^2\cos^2\theta\, r\sin\theta = 7}$

ⓒ $x=4$

$\boxed{r\cos\theta=4}$

Example #3: Find a Cartesian equation

ⓐ $r\sin\theta=2$

$\boxed{y=2}$

ⓑ $r^2=7$

$x^2+y^2=7$

ⓒ $r=4\sec\theta$

$\cos\theta\cdot r = 4\frac{1}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}$

$\boxed{x=4}$

ⓓ $r^2\sin(2\theta)=1$

$r^2(2\sin(\theta)\cos\theta)=1$

$r^2(2\sin(\theta))(r\cos(\theta))$

$2xy=1$

ⓔ $7=\csc\theta$

$7=\frac{1}{\sin\theta}\cdot\sin\theta$

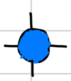
$7\sin\theta=1 \xrightarrow{\cdot r} 7r\sin\theta=r$

$7y=\pm\sqrt{x^2+y^2}$

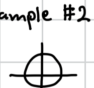
Sketching Regions

Example #1: $r^2\le4$

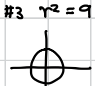
$x^2+y^2=4$



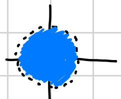
Example #2: $r=3$



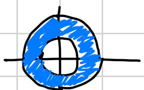
Ex. #3 $r^2=9$



Example #4: $r^2<16$

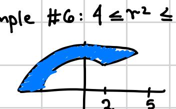


Example #5: $1\le r^2\le4$



Example #6: $r\le5,\ 0\le\theta\le\pi$

Example #6: $4\le r^2\le25,\ \frac{\pi}{4}\le\theta\le\pi$



END 10.3 (Polar Coordinates)

10.4 Derivatives in Polar Coordinates.

Note! Like parametric equations we want to find the slope of the tangent line $\left(\frac{dy}{dx}\right)$, 1st rate of change, velocity, and need a relationship between r, θ and $\frac{dy}{dx}$

Note! Given $r=r(\theta)$, we can calculate

$$\frac{dy}{dx}=\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \equiv \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$$

Example #1: Find the equation of the tangent line to the curve $r=-1+\sin\theta$ at $\theta=\pi$.

$y=x+b$

$y=x-1$

$y=mx+b$

$\left.\frac{dy}{dx}\right|_{\theta=\pi}$

$r=-1+\sin\theta$

$\frac{dr}{d\theta}=\cos\theta$

$$\frac{dy}{dx}=\frac{(\cos\theta)\sin\theta+(-1+\cos\theta)\cos\theta}{(\cos\theta)\cos\theta-(-1+\cos\theta)\sin\theta}$$

$$=\frac{(-1)(0)+(-1-0)(-1)}{(-1)(-1)-(-1+0)(0)}=\frac{1}{1}=1$$

$y=r\sin\theta$

$=(-1)(0)$

$=0$

$r=-1+\sin\theta$

$r=-1$

$x=r\cos\theta$

$-1(-1)$

$x=1$

$(r,\theta)\to(-1,\pi)$

$(x,y)\to(1,0)$

Length of a curve

$$r=r(\theta),\ L=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example #1: Find the length of the curve $r=\frac{1}{\sqrt{2}}e^\theta,\ 0\le\theta\le\frac{\pi}{2}$

$r=\left(\frac{1}{\sqrt{2}}e^\theta\right)^2$

$r^2=\frac{1}{2}e^{2\theta}$

$\left(\frac{dr}{d\theta}\right)^2=\left(\frac{1}{\sqrt{2}}e^\theta\right)=\frac{1}{2}e^{2\theta}$

$$L=\int_0^{\frac{\pi}{2}}\sqrt{\tfrac{1}{2}e^{2\theta}+\tfrac{1}{2}e^{2\theta}} \longrightarrow \sqrt{e^{2\theta}}=e^\theta$$

$$=\int_0^{\pi/2}e^\theta\,d\theta \longrightarrow e^\theta\Big|_0^{\pi/2}=e^{\pi/2}-1$$

End 10.4 (Derivatives in Polar)

End Chapter 10

Start 11.1 Sequences

Defn: A sequence is a list of #s in a given order;

$\{a_n\}_{n=1}^\infty$

Note!

Factorial:

$3!=3\cdot2\cdot1$

$6!=6\cdot5\cdot4\cdot3\cdot2\cdot1$

$N!=N\cdot(N-1)\cdot(N-2)\cdot\ldots\cdot3\cdot2\cdot1$

$(N+4)!=(N+4)(N+3)(N+2)(N+$

Example #1: $\{2n\}_1^\infty=2,4,\ldots$

Example #2: Write out the first 4 terms

a) $\{2n-1\}_{n=1}^\infty=1,3,7,9,\ldots$

b) $\{2n-1\}_{n=5}^\infty=9,11,13,15,\ldots$

c) $\{-1^n\cdot n^2\}_{n=1}^\infty=-1,4,-9,16,\ldots$ [Alternating Sequence]

d) $\left\{\frac{2^n}{(n+1)!}\right\}_{n=1}^\infty=1,2/3,1/3,2/15,\ldots$

Example #3: Find a formula a_n for:

a) $3,5,7,9,\ldots$

$\{2n-1\}_{n=2}^\infty$

$\{2n+1\}_{n=1}^\infty$

$\{2n+7\}_{n=-2}^\infty$

b) $1,-1/2,1/3,1/4$

$\left\{-1^{N+1}\cdot\frac{1}{N}\right\}_{N=1}^\infty$

c) $1/2,2/3,\frac{3}{4},\frac{4}{5},\ldots$

$\left\{\frac{N}{N+1}\right\}_{n=1}^\infty$

d) $1,1/3,1/9,\frac{1}{27},\frac{1}{81},\ldots$

$\left\{\frac{1}{3^N}\right\}_{n=0}^\infty$

e) $-3,1,-1/3,1/9,-1/27,\ldots$

$\left\{-1^N\cdot\frac{1}{3^N}\right\}_{N=-1}^\infty$

Defn: A sequence $\{a_n\}$ converges to a number L

① If $\lim_{n\to\infty}a_N=L$

If $\lim_{n\to\infty}a_n=\pm\infty$; The sequence $\{a_n\}$ diverges to $\pm\infty$.

If $\lim_{n\to\infty}a_n=DNE$; the sequence $\{a_n\}$ diverges

Example #4: Determine if the sequence converges or diverges

a) $\{2n+1\}_{n=1}^\infty$

$\lim_{2\to n}2n+1=\infty$

Diverges to ∞

b) $\{(-1)^N\}_{N=1}^\infty$

$\lim_{N\to\infty}(-1)^N=DNE$

Diverges

c) $\left\{\frac{2n-1}{N}\right\}_{N=1}^\infty$

$\lim_{N\to\infty}\frac{2N-1}{N}=\frac{\infty}{\infty}$

LH $\lim_{N\to\infty}=\frac{2}{1}=2$

Converges to 2

d) $\{2ne^{-n}\}_{n=7}^\infty$

$\lim_{n\to\infty}2ne^{-n}$

$\lim_{n\to\infty}\frac{2n}{e^n}=\frac{\infty}{\infty}$

LH $\lim_{n\to\infty}\frac{2}{e^n}=\frac{2}{\infty}=0$

Converges to 0.

Theorem: Let a_n be a sequence such that $\lim_{n\to\infty}a_n=L$.

Given a function $f(x)$ is continous at $X=L$ and defined for all $x=a_n$ then

$$\lim_{n\to\infty}f(a_n)=f\left(\lim_{n\to\infty}a_n\right)=f(L)$$

Example #5: $\left\{\sqrt{\frac{3n+4}{n}}\right\}_{n=1}^\infty$

$\lim_{n\to\infty}\sqrt{\frac{3n+4}{n}}$

Example #4: Determine if the sequence converges or diverges, to what?

$\left\{e^{\frac{2n+1}{n}}\right\}_{n=1}^\infty$: $\lim_{n\to\infty}e^{\frac{2n+1}{n}}=e^{\lim_{n\to\infty}\frac{2n+1}{n}}\xrightarrow{LH}e^{\lim_{N\to\infty}\frac{2}{1}}$

$=e^2$

$\boxed{\text{Converges to } e^2}$

Note! Common sequence [Memorize and important]

$\{\sqrt[n]{n}\}_{n=1}^\infty$

$\lim_{n\to\infty}\sqrt[n]{n}$

or

$\lim_{n\to\infty}n^{\frac{1}{n}}=y$ [Find y!]

$0=\ln(y)$

$e^0=y$

$1=y$

$\lim_{n\to\infty}\sqrt[n]{n}=1$

$\ln\left(\lim_{n\to\infty}n^{1/n}\right)=\ln(y)$

$\lim_{n\to\infty}\ln(n^{\frac{1}{n}})=\ln(y)$

$\lim_{n\to\infty}\frac{1}{n}\ln(n)=\ln(y)$

$\lim_{n\to\infty}\frac{\ln(n)}{n}=\ln(y)$

LH $\lim_{n\to\infty}\frac{1/n}{n}=\ln(y)$

Defn: A sequence is bounded from above if there exists a number M, such that for every N:

$a_n\le M$

We call M the upper bound.

Example #1: $\left\{\frac{1}{n}\right\}_{n=1}^\infty$, $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots0\}$

UB = 1

UB = 87

Defn: If M is the smallest upper bound, we say it is the least upper bound (LUB)

Defn: A sequence $\{a_n\}$ is bounded from below if there exists a number M such that for every n:

$a_n\ge M$

We get M a lower bound

Example #1: $\left\{\frac{1}{n}\right\}_{n=1}^\infty$ $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,0\}$

LB = 0

LB = -87

Defn: If M is the greatest lower bound, we call it the greatest lower bound (GLB)

Example #1: Find the LUB and GLB

$\left\{\frac{3}{n+1}\right\}_{n=1}^\infty$

GLB: 0

LUB: 3/2

$\lim_{n\to\infty}\frac{3}{n+1}=0$

$\frac{3}{4},\frac{3}{5},\frac{3}{6},\ldots,0$

Example #2: $\left\{\frac{n}{n+1}\right\}_{n=2}^\infty$ $\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6}$

$\lim_{n\to\infty}\frac{n}{n+1}$ LH $\lim_{n\to\infty}\frac{1}{1}=1$

GLB = $\frac{2}{3}$ LUB = 1

Example #3: $\{n^2+1\}_{n=1}^\infty$

$\lim_{n\to\infty}n^2+1=\infty$ $\{2,5,10,\ldots,\infty\}$

GLB = 2; LUB = none.

END 11.1

11.2 Series

Defn: A series is an infinite sum

$$\sum_{n=1}^\infty a_n=a_1+a_2+a_3+a_4\ldots a_n+\ldots$$

Note! Sequences and series are not the same!

Example #1: $\left\{\frac{1}{n}\right\}_{n=1}^\infty$ converges to 0

$\{1,\frac{1}{2},\frac{1}{3},\ldots,0\}$

$\sum_{n=1}^\infty\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots=\infty$ [Diverges to ∞]

Note! A series will either converge to a number or diverge to $\pm\infty$.

Defn: Given a series $\sum a_n$; the sequence of partial sums $\{S_n\}$ associated with:

$$\sum_{n=1}^\infty a_n=a_1+a_2+a_2+\ldots$$

$S_1=a_1$

$S_2=a_1+a_2$

$S_3=a_1+a_2+a_3$

$S_4=a_1+a_2+a_3+a_4$

$S_n=\{S_n\}_{n=1}^\infty$

Note! If this sequence $\{S_n\}$ converges to L then $\sum_{n=1}^\infty a_n$ converges to L:

$$\sum_{n=1}^\infty a_n=L, \text{ where } L \text{ is the sum.}$$

Example #1: Use a sequence of partial sums to find the sum:

$$\sum_{n=1}^\infty\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

$a_n=\frac{1}{2^n}$

$S_1=\frac{1}{2}$

$S_2=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

$S_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$

$S_4=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$

$\{\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\ldots\}$

$S_n=\left\{\frac{2^n-1}{2^n}\right\}_{n=1}^\infty$

Example #2:

$\lim_{n\to\infty}\frac{2^n-1}{2^n}=\frac{\infty}{\infty}$

$\lim_{n\to\infty}\underbrace{\frac{2^n}{2^n}-\frac{1}{2^n}}_{1-0=1}$

Since $S_n=\left\{\frac{2^n-1}{2^n}\right\}$ converges to 1; this means $\sum_{n=1}^\infty\frac{1}{2^n}$ converges to 1 $\therefore\sum_{n=1}^\infty\frac{1}{2^n}=1$

Example #3: Use a sequence of partial sums to find the sum.

$$\sum_{n=1}^\infty\frac{1}{n^2+n}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots$$

$S_1=\frac{1}{2}$

$S_2=\frac{1}{2}+\frac{1}{6}=\frac{4}{6}$

$S_3=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}$

$S_4=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}=\frac{4}{5}$

$S_n=\left\{\frac{n}{n+1}\right\}_{n=1}^\infty$

$\lim_{n\to\infty}\frac{n}{n+1}=\frac{\infty}{\infty}$

$\lim_{n\to\infty}\frac{1}{1}=1$

$\boxed{\sum_{n=1}^\infty\frac{1}{n^2+n}=1}$

Toolbox (Series)

1. Sequence of Partial Sums
2. Telescoping Series + Partial Sums

11.2 Series continued

Example #4: Same problem; different way.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} \longrightarrow \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$n=-1 \quad n=0$

$1 = A(n+1) + Bn$

$-1 = B$

$1 = A$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad S_n = \left\{1 - \frac{1}{n+1}\right\}_{n=1}^{\infty}$$

$S_1 = \left(1 - \frac{1}{2}\right)$

$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$

$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$

$S_4 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$

[Telescoping Series]

$\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)$

$= 1$

Defn: A geometric series is one that can be written as:

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \ldots$$

- First term is a
- Consecutive $\frac{a_{n+1}}{a_n} = r$
- If $|r| < 1$; the series converges at $\frac{a}{1-r}$
- If $|r| \geq 1$; the series diverges.

Example #1: $\overbrace{2+6}^{3}+\overbrace{18}^{3}+\overbrace{54}^{3}+\ldots \quad a = 2$

Check to see Geo

$\sum_{n=1}^{\infty} ar^{n-1} \rightarrow \sum_{n=1}^{\infty} 2\cdot 3^{n-1}$

① $\sum_{n=2}^{\infty} \frac{2^n}{5^{n+1}} = \frac{2^2}{5^3} + \frac{2^3}{5^4} + \frac{2^4}{5^5}$ ($n=2 \quad n=3 \quad n=4$)

$\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^{n+1}} = \frac{2^2}{5^3} + \frac{2^3}{5^4} + \frac{2^4}{5^5}$

$\sum_{n=1}^{\infty}\left(\frac{2^2}{5^3}\right)\left(\frac{2}{5}\right)^{n-1}$

$a = \frac{4}{125},\ r = \frac{2}{5}\} \rightarrow \frac{\frac{4}{125}}{1-\frac{2}{5}} = \boxed{\frac{4}{75}}$ converges

Example #2: Determine if the series converges or diverges?

a) $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$ $a = 2$ and $r = \frac{3}{4}$ (converges)

$\frac{a}{1-r} = \frac{2}{1-\frac{3}{4}} = \boxed{8}$

b) $\sum_{n=1}^{\infty} \frac{1}{3^n} \rightarrow \sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n \equiv \left(\frac{1}{3}\right)^1 \cdot \left(\frac{1}{3}\right)^{n-1}\right]$

$a = \frac{1}{3},\ r = \frac{1}{3} \qquad \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \boxed{\frac{1}{2}}$ (converges)

c) $\sum_{n=1}^{\infty}\left(\frac{2^n}{3^{n+1}}\right) = \left(\frac{2^1}{3^2}\right)\left(\frac{2}{3}\right)^{n-1}$ $\qquad \frac{\frac{2}{9}}{1-\frac{2}{3}} = \frac{\frac{2}{9}}{\frac{1}{3}} = \boxed{\frac{2}{3}}$ (converges)

$a = 2/9 \quad r = 2/3$

d) $\sum_{n=1}^{\infty} \frac{7\cdot 3^{n+1}}{5^{n+2}} \longrightarrow \sum_{n=1}^{\infty}\left(\frac{7\cdot 3^2}{5^3}\right)\left(\frac{3}{5}\right)^{n-1}$

$a = \frac{63}{125}\ r = \frac{3}{5}\} \rightarrow \frac{\frac{63}{125}}{1-\frac{3}{5}} = \frac{\frac{63}{125}}{\frac{4}{5}} = \boxed{\frac{63}{50}}$ converges

e) $\sum_{n=1}^{\infty}\left(\frac{-2}{3^2}\right)\left(\frac{-2}{3}\right)^{n-1}$

$a = \frac{-2}{9}\ r = -\frac{2}{3}\} \rightarrow \boxed{\frac{-2}{15}}$ converges

f) $\sum_{n=1}^{\infty} \frac{e^{n+1}}{4^{n-1}} \longrightarrow \sum_{n=1}^{\infty}(e^2)\left(\frac{e}{4}\right)^{n-1}$

$a = e^2,\ r = \frac{e}{4} \longrightarrow \frac{e^2}{1-\frac{e}{4}} = \frac{e^2}{\frac{4-e}{4}} = \boxed{\frac{4e^2}{4-e}}$ converges

g) $\sum_{n=1}^{\infty} \frac{1+4^n}{5^{n+1}} \longrightarrow \sum_{n=1}^{\infty}\left(\frac{1}{5^{n+1}} + \frac{4^n}{5^{n+1}}\right) \longrightarrow \sum_{n=1}^{\infty}\frac{1}{5^{n+1}} + \sum_{n=1}^{\infty}\frac{4^N}{5^{n+1}}$

$\sum_{n=1}^{\infty}\frac{1}{5^{n+1}} = \sum_{n=1}^{\infty}\left(\frac{1}{5^2}\right)\left(\frac{1}{5}\right)^{n-1} + \sum_{n=1}^{\infty}\left(\frac{4}{5^2}\right)\left(\frac{4}{5}\right)^{n-1}$

$a = \frac{1}{25}\ r = \frac{1}{5} \qquad a = \frac{4}{25}\ r = \frac{4}{5}$

$\frac{\frac{1}{25}}{1-\frac{1}{5}} = \frac{1}{20} \qquad \frac{\frac{4}{25}}{1-\frac{4}{5}} = \frac{4}{5}\}\ \frac{1}{20} + \frac{4}{5} = \frac{17}{20}$

h) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n-3}} \longrightarrow \sum_{n=1}^{\infty}\left(\frac{2}{3^{-2}}\right)\left(\frac{2}{3}\right)^{n-1}$

$a = \frac{2}{3^{-2}} = 18,\ r = \frac{2}{3}\} \longrightarrow \frac{18}{1-\frac{2}{3}} = \boxed{54}$ converges.

END 11.2 section

START 11.3 section (Integral Test)

Consider $\sum_{n=k}^{\infty} a_n$, if a_n can be considered as a continuous, positive, and decreasing function of $[K, \infty)$.

- If the $\int_k^{\infty} a_n\, dn$ converges, then series converges.
- If the $\int_k^{\infty} a_n\, dn$ diverges, then the series diverges.

Example(s): Determine if the series converges or diverges:

a) $\sum_{n=1}^{\infty}\frac{1}{n} \therefore \sum_{n=1}^{\infty}\frac{1}{n}$ diverges [IR]

$\int_1^{\infty}\frac{1}{n}\,dn$

$\lim_{b\to\infty} \ln|n|\Big|_1^b$

$\left(\lim_{b\to\infty}\ln|b|\right) - (\ln|1|)$

$\infty - 0 = \boxed{\infty}$ Infinity diverges

Note! Integral test does not determine what it converges to.

b) $\sum_{n=1}^{\infty}\frac{1}{n^2}\,dn \therefore \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

$\int_1^{\infty}\frac{1}{n^2}\,dn$

$\lim_{b\to\infty} -\frac{1}{n}\Big|_1^b$

$\left(\lim_{b\to\infty} -\frac{1}{b}\right) - (-1)$

$0 + 1 = 1$ [Converges]

c) $\sum_{n=1}^{\infty}\frac{1}{n^3}\,dn$

$\int_1^{\infty}\frac{1}{n^3}\,dn$

$\lim_{b\to\infty}\frac{-2}{n^2}\Big|_1^b \longrightarrow \left(\lim_{b\to\infty}\frac{-2}{b^2}\right) - \left(\frac{-2}{1}\right)$

$0 + 2 = \boxed{2}$ converges

d) $\sum_{n=1}^{\infty}\frac{8}{N^{27}}$

Converges via P-test

e) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

Diverges via P-test

f) $\sum_{n=1}^{\infty}\frac{1}{N^2+1}$

Cannot use p-test

INT $\int_1^{\infty}\frac{1}{n^2+1}\,dn$

$\lim_{b\to\infty}\arctan(n)\Big|_1^b \longrightarrow \left(\lim_{b\to\infty}\arctan(b)\right) - (\arctan(1))$

$\frac{\pi}{2} - \frac{\pi}{4}$ [Converges]

P-test

$\sum_{n=1}^{\infty}\frac{c}{N^p}$

- If $p > 1$, the series converges
- If $p < 1$, the series diverges.

Continued examples

g) $\sum_{N=1}^{\infty}\frac{1}{3n+5}$

$\int_1^{\infty}\frac{1}{3n+5}\,dn \longrightarrow$ Let $u = 3n+5$

$du = 3\,dn$

$\frac{1}{3}\int\frac{1}{u}\,du \qquad \frac{1}{3}du = dn$

$\frac{1}{3}\ln|u| \longrightarrow \lim_{b\to\infty}\frac{1}{3}\ln|3n+5|\Big|_1^b$

$\left(\lim_{b\to\infty}\frac{1}{3}\ln|3b+5|\right) - \left(\frac{1}{3}\ln|3(1)+5|\right)$

$\infty - \frac{1}{3}\ln|9|$

$\boxed{\infty}$ Diverges

h) $\sum_{n=1}^{\infty} 2^N e^{-N}$ [Geom.]

$\sum_{n=1}^{\infty}\frac{2^n}{e^n} \rightarrow \left(\frac{2}{e}\right)\left(\frac{2}{e}\right)^{n-1}$

$a = \frac{2}{e},\ r = \frac{2}{e}$

Converges!

END 11.3

i) $\sum_{n=1}^{\infty} 2ne^{-N}$

$\int_1^{\infty} 2ne^{-n}\,dn$

IBP: $u = 2n \quad dv = e^{-n}\,dn$

$du = 2\,dn \quad v = -e^{-n}$

$-2ne^{-n} + \int 2e^{-n}\,dn$

$-2ne^{-n} - 2e^{-n}$

$\lim_{b\to\infty} -2ne^{-n} - 2e^{-n}\Big|_1^b$

$\left(\lim_{b\to\infty} -2be^{-b} - 2e^{-b}\right) - \left(-2e^{-1} - 2e^{-1}\right)$

$\left(\lim_{b\to\infty}\frac{-2b}{e^b} - \cancel{\frac{2}{e^b}}\right) + 4e^{-1}$

$\Downarrow$ L.H.

$\lim_{b\to\infty}\cancel{\frac{-2}{e^b}}^{\,0} + 4e^{-1} = 4e^{-1}$ [Converges]

Section 11.4 (Comparison Tests)

$$\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

Limit Comparison Test

Given $\sum a_n$ and $\sum b_n$ such that $a_n, b_n > 0$

1. If you evaluate $\lim_{n\to\infty}\frac{a_n}{b_n} = C$
   - If the $\sum b_n$ converges, then $\sum a_n$ converges.
   - If the $\sum b_n$ diverges, then $\sum a_n$ diverges.
2. If you evaluate $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$
   - If $\sum b_n$ converges then $\sum a_n$ converges
   - If $\sum b_n$ diverges then the test failed
3. If you evaluate $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$
   - If $\sum b_n$ converges, then the test failed
   - If $\sum b_n$ diverges, then $\sum a_n$ diverges

$\lim = C \rightarrow$ Conv $\rightarrow$ Conv; $\rightarrow$ Div $\rightarrow$ Div

Example #1: $\sum_{n=1}^{\infty} n^2$

$\lim_{n\to\infty} n^2 = \infty \neq 0$

$\therefore$ Diverges by the n^{th} term test

Example #2: $\sum_{n=1}^{\infty}\frac{n}{2n+1}$

$\lim_{n\to\infty}\frac{n}{2n+1} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{1}{2} \neq 0$

$\therefore$ Diverges by the n^{th} term test.

Example #1:

a) $\sum_{n=1}^{\infty}\frac{N^3}{N^4+N}$ where $\sum b_n = \frac{1}{N} \rightarrow$ diverges via p-test

$\lim_{n\to\infty}\frac{\frac{N^3}{N^4+N}}{\frac{1}{N}} = \lim_{n\to\infty}\frac{N^4}{N^4+N} \stackrel{L.H}{=} \frac{4n^3}{4n^3+1} \stackrel{L.H}{=} \frac{4n^3}{4n^3+1} \stackrel{L.H}{=} \frac{12n^2}{12n^2} = \lim_{n\to\infty} 1 = \boxed{1}$ Converges

Diverges.

b) $\sum_{n=1}^{\infty}\frac{2n^2}{3n^4+5}$ $\quad \sum b_n = \sum\frac{1}{n^2}$ Converges via p-test

$\lim_{n\to\infty}\frac{\frac{2n^2}{3n^4+5}}{\frac{1}{n^2}} = \frac{2n^4}{3n^4+5} \stackrel{L.H}{=} \frac{8\cancel{n^3}}{12\cancel{n^3}} = \lim_{n\to\infty}\frac{2}{3} = \boxed{\frac{2}{3}}$ Converges

c) $\sum_{n=1}^{\infty}\frac{1}{3n^3+7n}$ where $\sum b_n = \sum\frac{1}{n^3}$ converges via ptest

$\lim_{n\to\infty}\frac{\frac{1}{3n^3+7n}}{\frac{1}{n^3}} = \frac{n^3}{3n^3+7n} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{3n^2}{9n^2+7} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{6\cancel{n}}{18\cancel{n}} = \boxed{\frac{1}{3}}$ Converges

Converges.

d) $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+4}}$ compare to $\sum b_n = \sum\frac{1}{n}$ diverges via p-test.

$\lim_{n\to\infty}\frac{\frac{n}{\sqrt{n^2+4}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{\sqrt{n^2+4}} = \lim_{n\to\infty}\frac{\sqrt{n^4}}{\sqrt{n^2+4}} = \lim_{n\to\infty}\sqrt{\frac{n^4}{n^2+4}} = \sqrt{\lim_{n\to\infty}\frac{n^4}{n^2+4}}$

$\stackrel{L.H}{=} \sqrt{\lim_{n\to\infty}\frac{4n^3}{3n^2}} = \sqrt{\lim_{n\to\infty}\frac{4}{3}n} = \infty$ diverges

e) $\sum_{n=1}^{\infty} 2ne^{-n}$ where $\sum b_n = \sum\frac{1}{n}$ diverges via p-test

$\lim_{n\to\infty}\frac{2ne^{-n}}{\frac{1}{n}} = 2n^2e^{-n} = \lim_{n\to\infty}\frac{2n^2}{e^n} \stackrel{L.H}{=} \frac{4n}{e^n} \stackrel{L.H}{=} \frac{4}{e^n} = \boxed{0}$] test failed

Let $\sum b_n = \frac{1}{n^2}$ converges via p-test

$\lim_{n\to\infty}\frac{2ne^{-n}}{\frac{1}{n^2}} = 2n^3e^{-n} = \lim_{n\to\infty}\frac{2n^3}{e^n} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{6n^2}{e^n} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{12n}{e^n} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{12}{e^n}$] test passed

f) $\sum_{n=1}^{\infty}\frac{e^n}{n^2+1}$ where $\sum b_n = \sum\frac{1}{n}$ diverges via p-test.

$\lim_{n\to\infty}\frac{\frac{e^n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{ne^n}{n^2+1} \stackrel{L.H}{=} \lim_{n\to\infty}\frac{e^n+ne^n}{2n} \stackrel{L.H}{=} \lim_{N\to\infty}\frac{e^n+ne^n+e^n}{2} = \boxed{\infty}$] Diverges

The Direct Comparison Test

Given $\sum a_n + \sum b_n$ such that $\{a_n, b_n > 0\}$ and for every n $a_n \leq b_n$.

1. If $\sum b_n$ converges then $\sum a_n$ converges
2. If $\sum a_n$ diverges then $\sum b_n$ diverges.
3. If $\sum b_n$ diverges then the test tells us nothing
4. If $\sum a_n$ converges then the test tells us nothing.

Examples: Determine if the series converges or diverges

a) $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$

1. Justify inequalities

   We know $\sin(n) \leq 1$

   then $\sin^2(n) \leq 1$

   $\therefore \frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$

2. Conclusion

   Since $\frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$ and $\sum\frac{1}{n^2}$ converges via p-test then $\sum\frac{\sin^2(n)}{n^2}$ converges

b) $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

$n^2+1 \geq n^2$

$\frac{1}{n^2+1} \leq \frac{1}{n^2}$

Since $\frac{1}{n^2+1} \leq \frac{1}{n^2}$ and $\sum\frac{1}{n^2}$ converges by p-test then $\sum\frac{1}{n^2+1}$ converges.

c) $\sum_{n=1}^{\infty}\frac{1}{n^n}$

	$n=1$	$n=2$	$n=3$	$n=4$
$\frac{1}{n^2}$	1	1/4	1/9	1/16
$\frac{1}{n^n}$	1	1/4	1/27	1/256

$\Longrightarrow \frac{1}{n^n} \leq \frac{1}{n^2}$

Since $\frac{1}{n^n} \leq \frac{1}{n^2}$ and $\sum\frac{1}{n^2}$ converges by p-test then $\sum\frac{1}{n^n}$ converges.

d) $\sum_{n=2}^{\infty}\frac{1}{n-1}$

	$n=2$	$n=3$	$n=4$
$\frac{1}{n}$	1/2	1/3	1/4
$\frac{1}{n-1}$	1	1/2	1/3

$\frac{1}{n} \leq \frac{1}{n-1}$

Since $\frac{1}{n} \leq \frac{1}{n-1}$ and $\sum\frac{1}{n}$ diverges by p-test then $\sum\frac{1}{n-1}$ diverges

END 11.4

N^{th} term Test

Given $\sum a_n$ if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

* This test says nothing about convergence!

Start 11.5 (Alternating Series)

Recall: The alternating series test:

$\sum (-1)^n = U_n$

- Converges if
  1. U_n eventually decreases
  2. $\lim_{n\to\infty} U_n = 0$
- Diverges if (2) fails
- ☆ Test is inconclusive if (1) fails.

Examples: Determine Series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ | $\sum \frac{1}{n^3}$ converges by p-test

Alternating Series $\rightarrow U_n = \frac{1}{n^3}$

1. $1, \frac{1}{8}, \frac{1}{27}, \ldots$
2. $\lim_{n\to\infty} \frac{1}{n^3} = 0$

∴ Absolute convergence

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ | $\sum \frac{1}{n}$ diverges by p-test

Alternating series $\longrightarrow U_n = \frac{1}{n}$

1. $1, \frac{1}{2}, \frac{1}{3}, \ldots$
2. $\lim_{n\to\infty} \frac{1}{n} = 0$

∴ Cond. convergence

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n+1}$

Alternating series: $U_n = \frac{n}{2n+1}$

2. $\lim_{n\to\infty} \frac{n}{2n+1} \overset{L.H}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$

∴ Diverges.

Defn: An alternating series $\sum a_n = \sum (-1)^n U_n$ is absolutely convergent if $\sum a_n$ converges and $\sum |a_n|$ converges.

Defn: An alternating series $\sum a_n = \sum (-1)^n U_n$ is conditionally convergent if $\sum a_n$ converges and $\sum |a_n|$ diverges

Examples: Determine if the series converges absolutely or conditionally, and/or diverges.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

Alternating Series: $U_n = \frac{1}{n^2+1}$

1. $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots$
2. $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$

∴ This series converges

Determining absolute or conditionally:

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Limit Comparison

$\sum b_n = \sum \frac{1}{n^2}$ converges by p-test

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \frac{n^2}{n^2+1} \overset{L.H}{=} \frac{2n}{2n} = 1$

∴ Converges

Then this series absolutely converges.

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

Alternating Series: $U_n = \frac{1}{\sqrt{n+1}}$

1. $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \ldots$
2. $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ∴ Converges

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \xrightarrow{\text{Lim Comp.}} \sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges by p-test

$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} \overset{L.H}{=} \sqrt{\lim_{n\to\infty} \frac{1}{1}} = 1$

∴ Diverges

This series is a conditionally convergent.

END

Section 11.5

Section 11.6 (Root and Ratio Test)

Ratio Test: Given any series $\sum a_n$ with

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$$

1. If $\alpha < 1$, then the series converges.
2. If $\alpha > 1$, then the series diverges.
3. If $\alpha = 1$, then the test is incomplete.

Example #1: Determine if the series converges or diverges.

$\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Ratio. $\lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty} \frac{2^{n+1} n!}{(n+1)! 2^n}$

$= \lim_{n\to\infty} \frac{2n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{2 \cdot \cancel{n}\cancel{(n-1)}(n-2)\ldots 3\cdot\cancel{2\cdot 1}}{(n+1)\cancel{(n)}\cancel{(n-1)}\ldots\cancel{3\cdot 2\cdot 1}}$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0 = \alpha$

Since $\alpha < 1$, it converges by the ratio test.

Converges!

Root Test: Given any series $\sum a_n$ with

$$\sqrt[n]{|a_n|} = \alpha$$

1. If $\alpha < 1$, series converges
2. If $\alpha > 1$, series diverges
3. If $\alpha = 1$, then the test is inconclusive.

Examples: Determine if the series diverges or converges.

a) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Root Test:

$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n}{n+1}\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \lim_{n\to\infty} \frac{2n}{n+1}$

$\overset{L.H}{=} \lim_{n\to\infty} 2 = 2 = \alpha$

∴ Diverges

Note! Because both root and ratio have absolutely if an alternating series converges with these tests we get absolute convergence for free

b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

Ratio Test:

$\lim_{n\to\infty} \left|\frac{\frac{(-2)^{n+1}}{(n+1)^2}}{\frac{(-2)^n}{n^2}}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(-2)^{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\cancel{(-2)^n}}\right| = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2}$

Breakup to cancel!

$= \lim_{n\to\infty} \frac{2n^2}{n^2+2n+1} \overset{L.H}{=} \lim_{n\to\infty} \frac{4n}{2n+2} = \lim_{n\to\infty} \frac{4}{2} = 2 = \alpha$

∴ Diverges via ratio test!

c) $\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{(n+1)!}$

Ratio Test:

$\lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1} e^{n+1}}{(n+2)!}}{\frac{(-1)^n e^n}{(n+1)!}}\right| = \lim_{n\to\infty} \left|\frac{\overset{-1}{\cancel{(-1)^{n+1}}}\overset{e'}{\cancel{e^{n+1}}}}{(n+2)!} \cdot \frac{(n+1)!}{\cancel{(-1)^n}\cancel{(e^n)}}\right| = \lim_{n\to\infty} \frac{e \cdot \cancel{(n+1)}\cdot\cancel{(n)}\cdot\cancel{(n-1)}\ldots\cancel{3\cdot 2\cdot 1}}{(n+2)\cdot\cancel{(n+1)}\cdot\cancel{(n)}\cdot\cancel{(n-1)}\ldots\cancel{3\cdot 2\cdot 1}}$

$\lim_{n\to\infty} \frac{e}{n+2} = 0 = \alpha$

∴ Absolutely converges!

d) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

Ratio Test

$\lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1} n+1}{n+2}}{\frac{(-1)^n n}{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(-1)^{n+1}} n+1}{n+2} \cdot \frac{n+1}{\cancel{(-1)^n} n}\right| = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2} \overset{LH}{=} \ldots = 1 = \alpha$

∴ Test fails

Test for alternating series

Condition #1: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ ✗

Condition #2: $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1$ ✗

∴ By condition #2 failure, the series diverges.

e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

Ratio Test:

$\lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}}{2(n+1)+1}}{\frac{(-1)^n}{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(-1)^{n+1}}}{2(n+1)+1} \cdot \frac{2n+1}{\cancel{(-1)^n}}\right| = \lim_{n\to\infty} \frac{2n+1}{2n+3} \overset{L.H}{=} \ldots = 1 = \alpha$

∴ Ratio Test fails

Test for alternating series:

Cond. 1: $\left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right\}$ ✓

Cond. 2: $\lim_{n\to\infty} \frac{1}{2n+1} = 0$ ✓

∴ Converges by Alternating Series Test.

$\sum_{n=1}^{\infty} \frac{1}{2n+1} \longrightarrow$ Limit Comparison Test: $\sum b_n = \sum \frac{1}{n}$ diverges by p-test

$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$ (Diverges)

∴ Conditionally converges.

f) $\sum_{n=1}^{\infty} \frac{n^4}{(-5)^n}$

Root test:

$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^4}{(-5)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^4}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^4}}{\sqrt[n]{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^4}}{5}$

$= \lim_{n\to\infty} \frac{(\sqrt[n]{n})^4}{5} = \frac{(1)^4}{5} = \frac{1}{5}$

∴ Absolutely converges via root test.

$\boxed{\text{Note! } \lim_{n\to\infty} \sqrt[n]{n} = 1}$

11.8 Power Series

We are adding a variable!

$$\sum_{n=1}^{\infty} (2n+1)x^n = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \dots$$

[infinite polynomial]

Defn: A power series about $x=0$, is a series that can be written as:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x^1 + c_2 x^2 + \dots$$

A power series about $x=a$. Is a series that can be written as

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a)^1 + c_2 (x-a)^2 + \dots$$

Example #1: $\underbrace{\sum_{n=1}^{\infty} x^{n-1}}_{\text{Geometric}} = 1 + x^1 + x^2 + x^3 + \dots$

$a=1 \quad r=x$

$|x|<1$ converge $[-1<x<1]$

$|x|>1$ diverges. $[x \le 1 \wedge x \ge 1]$

$\frac{a}{1-r} = \frac{1}{1-x}$

$$\boxed{\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x} \text{ on } -1<x<1}$$

Example #3: Interval of convergence

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3 3^n}$$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n^3 3^n}\right|}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{|x^n|}}{\sqrt[n]{n^3}\sqrt[n]{3^n}}$$

$$\lim_{n\to\infty} \frac{|x|}{(\sqrt[n]{n})^3 \cdot 3} \longrightarrow \frac{|x|}{3} = \alpha < 1 \text{ (Converges)}$$

$-3<|x|<3$

$x=3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3 3^n} \rightarrow \sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n^3 \cancel{3^n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ (converges by p-test)}$$

$x=-3$

$$\sum \frac{(-3)^n}{n^3 3^n} \rightarrow \sum \frac{1}{n^3}\left(\frac{-3}{3}\right)^n$$

$$\sum \frac{1}{n^3}(-1)^n$$

AST:
1. $1, \frac{1}{8}, \frac{1}{27}, \dots$
2. $\lim_{n\to\infty} \frac{1}{n^3} = 0$

Converges at $x=-3$.

$-3 \le x \le 3$

Example #4

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

Root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-1)^n}{n}\right|}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{|x-1|^n}}{\sqrt[n]{n}} = |x-1| < 1 \text{ converges}$$

$-1 < x-1 < 1$
$+1 \qquad +1$

$\boxed{0<x<2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

AST: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Converges at 0.

$x=2$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ Diverges by p-test}$$

$\therefore 0 \le x < 2.$

Example #5

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{(n+1)!}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x-4)^{n+1}}{(n+2)!}}{\frac{(x-4)^n}{(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x-4)^{n+1}}\ \cancel{(n+1)!}}{\underbrace{\cancel{(n+2)!}}_{n+2}\ \cancel{(x-4)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x-4|}{n+2} = |x-4| \lim_{n\to\infty} \frac{1}{n+2}$$

$$= |x-4| \cdot 0 = 0 = \alpha$$

$(-\infty, \infty)$

Converges everywhere.

Note! Since it converges on $-1<x<1$, we say that this is the interval of convergence

Example #2: Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{3^{n+1}}$$

Geom. series

$$\sum_{n=1}^{\infty} \left(\frac{x}{9}\right)\left(\frac{x}{3}\right)^{n-1}$$

$a = \frac{x}{9} \quad r = \frac{x}{3}$

$\left|\frac{x}{3}\right| < 1$

$\left|\frac{x}{3}\right| > 1$

$-1 < \frac{x}{3} < 1$

$-3 < x < 3$

Example #3: Int. of convergence

$$\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^{n+1}}$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)(3^{n+2})}}{\frac{x^n}{n \cdot 3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}\, n \cdot \cancel{3^n}\, 3}{(n+1)\cancel{(3)^{n+2}}\ \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x \cdot n}{(n+1)\cdot 3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{xn}{3n+3} \right|$$

$$\text{L.H} \lim_{n\to\infty} \left| \frac{x}{3} \right|$$

$\frac{|x|}{3} < 1$ Convergence

$|x| < 3$

$-3 < x < 3$

$\frac{|x|}{3} > 1$ Diverg.

$x<3$

$x>3$

$$\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n\cancel{3^{n+1}}}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n} \text{ Diverges by p-test @ } x=3$$

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n 3^{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n}\left(\frac{-3}{3}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{1}{3n}(-1)^n$$

AST: $a_n = \frac{1}{3n}$

Decreasing: $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}$

$\lim_{n\to\infty} \frac{1}{3n} = 0$

$\therefore -3 \le x < 3$

11.9 Fractions of a Power Series

- Now that we know series what can we do with them?
  1. Have an "ugly" function you want to integrate
  2. Turn into a power series (infinite polynomials)
  3. Test for convergence
  4. Integrate the power series instead.

Recall from 11.8

$$\sum_{n=0}^{\infty} x^{n-1} = \frac{1}{1-x} \text{ when } |x|<1$$

Note! This is the same as

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Example #2: Use the power series to evaluate: $\int \frac{1}{1+x^3}$

$$\int \frac{1}{1+x^3} \rightarrow \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n$$

$$\int \frac{1}{1+x^3}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{3n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{3n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3n+1} x^{3n+1} + C$$

Example #3: $\int \frac{1}{x-5}$

$$\int \frac{1}{x-5} \rightarrow \frac{1}{-(5-x)} = \frac{1}{-5(1-\frac{1}{5}x)} = -\frac{1}{5} \cdot \frac{1}{1-(\frac{1}{5}x)}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty} \left(\frac{1}{5}x\right)^n = -\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n+1} x^n$$

$$= \int \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n+1} x^n\,dx = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n+1} \int x^n\,dx$$

$$= -\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n+1} \frac{1}{n+1} x^{n+1} + C$$

Example #1: Given $f(x) = \frac{1}{1-x^3}$

ⓐ Write $f(x)$ as a power series

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}$$

ⓑ Test for convergence: Find the interval of convergence

$\sqrt[3]{|x^3|} \ \sqrt[3]{1} \longrightarrow -1 < x < 1$

ⓒ Now use the power series to evaluate

$$\int \frac{1}{1-x^3}\,dx = \int \sum_{n=0}^{\infty} x^{3n}\,dx$$

$$= \sum_{n=0}^{\infty} \int x^{3n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{3n+1} x^{3n+1} + C$$

Example #4: Find a power series

$$\frac{x^2}{1-x} = x^2 \cdot \frac{1}{1-x} = x^2 \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} x^{n+2}$$

11.10 Taylor Series

Let f be a function with derivatives of all orders on an interval containing $x=a$. Then the Taylor Series generated by $f(x)$ at $x=a$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Note! When $a=0$, we call it the Mclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots +$$

Example #1: Find the Taylor Series about $a=0$ for $f(x)=e^x$

$f = e^x \quad f(0)=1$

$f' = e^x \quad f'(0)=1$

$f'' = e^x \quad f''(0)=1$

$f''' = e^x \quad f'''(0)=1$

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n}$$

Example #2: Find the Mclaurin Series for $f(x) = \sin x$

$f = \sin(x) \quad f(0)=0$

$f' = \cos(x) \quad f'(0)=1$

$f'' = -\sin(x) \quad f''(0)=0$

$f''' = -\cos(x) \quad f'''(0)=1$

$f^{(4)} = \sin(x) \quad f^{(4)}=0$

$f^{(5)} = \cos(x) \quad f^{(5)}=1$

$$\sin x = 0 + \frac{x^1}{1!} + \frac{0x^2}{2!} + \frac{-x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!}$$

$$= \frac{x^1}{1!} + \frac{-x^3}{3!} + \frac{x^5}{5!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Example #3: Taylor Series about $a=\pi$ for $f(x) = \cos x$

$f = \cos x \quad f(\pi) = -1$

$f' = -\sin x \quad f'(\pi) = 0$

$f'' = -\cos x \quad f''(\pi) = 1$

$f''' = \sin x \quad f'''(\pi) = 0$

$f^{(4)} = \cos x \quad f^{(4)}(\pi) = -1$

$$\cos x = -1 + \frac{0}{1!}(x-\pi)^1 + \frac{(x-\pi)^2}{2!} + \frac{0}{3!}(x-\pi)^3 + \frac{-1}{4!}(x-\pi)^4$$

$$= -1 + \frac{1}{2!}(x-\pi)^2 + \frac{-1}{4}(x-\pi)^4$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}$$

Example #4: Find the McLaurin Series for $f(x) = \ln(1+x)$

$f = \ln(1+x) \quad f(0)=0$

$f' = \frac{1}{1+x} \quad f'(0)=1$

$f'' = \frac{-1}{(1+x)^2} \quad f''(0)=-1$

$f''' = \frac{2}{(1+x)^3} \quad f'''(0)=2$

$f^{(4)} = \frac{-6}{(1+x)^4} \quad f^{(4)}(0)=-6$

$f^{(5)} = \frac{24}{(1+x)^2} \quad f^{(5)} = 24$

$$\frac{d}{dx}(\ln(1+x)) = \frac{f'(x)}{f(x)}$$

$$f' = (1+x)^{-1}$$

$$f'' = -(1+x)^{-2}$$

$$\ln(1+x) = \frac{1}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + \frac{24}{5!}x^5$$

$$= x - \frac{1}{2!}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{n+1} \text{ or } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$$

END 11.10

11.11 Applications of Taylor Series

Recall Taylor Series: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4$

Example #1: Find the Taylor Series about $a=0$ for

$$g(x) = 3x\cos(x^3)$$

Instead of looking at: $g(x) = 3x\cos(x^3)$ $\longrightarrow$ Consider: $f(x) = \cos(x)$

$f = \cos x \quad f(0) = 1$

$f' = -\sin x \quad f'(0) = 0$

$f'' = -\cos x \quad f''(0) = -1$

$f''' = \sin x \quad f'''(0) = 0$

$f^{(4)} = \cos x \quad f^{(4)}(0) = 1$

$$\therefore \cos x = 1 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\therefore 3x\cos(x^3) = 3x\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^3)^{2n}$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{6n+1}$$

Use Part A to evaluate

$$\int 3x\cos x^3 = \int 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{6n+1}$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\int x^{6n+1}$$

$$= 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{6n+2}x^{6n+2} + C$$

Defn: The Taylor Polynomial of Degree k is

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note! Degree 4 $\rightarrow$ 4 derivatives.

Example #2: Find the Taylor Polynomial of degree 6 of $f(x) = \cos(2x)$ at $a = \frac{\pi}{2}$

$f = \cos 2x \quad f(\frac{\pi}{2}) = -1$

$f' = -2\sin(2x) \quad f'(\frac{\pi}{2}) = 0$

$f'' = -4\cos(2x) \quad f''(\frac{\pi}{2}) = 4$

$f''' = 8\sin(2x) \quad f'''(\frac{\pi}{2}) = 0$

$f^{(4)} = 16\cos(2x) \quad f^{(4)}(\frac{\pi}{2}) = -16$

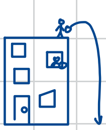
$f^{(5)} = -32\sin(2x) \quad f^{(5)}(\frac{\pi}{2}) = 0$

$f^{(6)} = -64\cos(2x) \quad f^{(6)}(\frac{\pi}{2}) = 64$

$$T_6(x) = -1 + \frac{4}{2!}(x-\tfrac{\pi}{4})^2 - \frac{16}{4!}(x-\tfrac{\pi}{2})^4 + \frac{64}{6!}(x-\tfrac{\pi}{2})^6$$

END 11.11

14.1 Multivariate Functions



$f(+)=y$

Defn: A real valued function f is a rule that assigns a unique number W to each element in the domain

$f(x) = y$, $f: \mathbb{R} \rightarrow \mathbb{R}$ 2-dim

$f(x,y) = z$, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ 3-dim

$f(x,y,z) = w$, $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ 4-dim

$f(x_1, x_2, x_3 \dots x_n) = w$, $f: \mathbb{R}^n \rightarrow \mathbb{R}$ $(n+1)$-dim

14.1 Multivariate Functions (continued)

Domain and Range

Recall: Given $f: \mathbb{R} \to \mathbb{R}$

By $f(x) = x^2$

Domain $(-\infty, \infty)$

Range $[0, \infty)$

Example #1: Find domain and range

a) $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 + y^2$

Range: $[0, \infty)$

Domain: $\mathbb{R}^2 \longleftrightarrow \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

b) $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \sqrt{x-y}$

Domain: $x - y \geq 0$

$\{(x,y) \mid x, y \in \mathbb{R} \wedge x - y \geq 0\}$

Range: $[0, \infty)$

c) $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \cos(x,y)$

Domain: $\mathbb{R}^2$

Range: $[-1, 1]$

d) $w = x^2 + y^2 + z^2 + 4$

Domain: $\mathbb{R}^3$

Range: $[4, \infty]$

Example #1: Find the domain

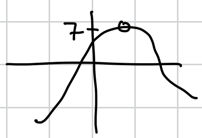
$$g(x,y) = \frac{2}{x+y-3}$$

Domain: $x + y - 3 \neq 0$

14.2 Limits and Continuity

Recall $f: \mathbb{R} \to \mathbb{R}$, $y = f(x)$

$\lim_{x \to x_0} f(x)$

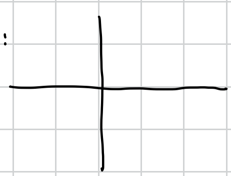


Domain:

Now: $f: \mathbb{R}^2 \to \mathbb{R}$, $z = f(x,y)$

$\lim_{(x,y) \to (x_0, y_0)} f(x,y)$

Domain:



Before:

1. Try to plug it in
2. Factor (algebra) and cancel; plug-in again.
3. Trig identities and cancel; plug-in again
4. Table
5. Graph

Example #1: Evaluate the following limit

ⓐ $\lim_{(x,y) \to (0,2)} \frac{x+y+4}{xy + x^2y^2 + 3}$

$= \frac{6}{3} = \boxed{2}$

ⓑ $\lim_{(x,y) \to (2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x} = \frac{0}{0} \to I.F.$

$x(xy - y + 4x - 4)$

$x(y(x-1) + 4(x-1))$

$x(x-1)(y+4) \longrightarrow \frac{\cancel{y+4}}{x(x-1)\cancel{(y+4)}}$

$\frac{1}{x(x-1)} = \frac{1}{x^2 - x}$

$\lim_{(x,y) \to (2,-4)} \frac{1}{x^2 - x} = \boxed{\frac{1}{2}}$

ⓒ $\lim_{(x,y) \to (0,0)} \frac{2xy - 2y^2}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

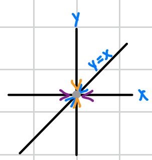
$\lim_{(x,y) \to (0,0)} \frac{(2xy - 2y^2)(\sqrt{x} + \sqrt{y})}{x - y}$

$\frac{2y\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{x-y}}$

$\lim_{(x,y) \to (0,0)} 2y(\sqrt{x} + \sqrt{y}) = \boxed{0}$

Example #2: Show the limit does not exist:

Domain:



$\lim_{(x,y) \to (0,0)} \frac{x^2 y^2}{x^4 + 3y^4}$

* You can evaluate the limit in different directions.

① x-axis → $y = 0$

$\lim_{(x,y) \to (0,0)} \frac{x^2 (0)^2}{x^4 + 3(0)^4} = \frac{0}{x^4}$

$\lim_{(x,y) \to (0,0)} \left[\frac{0}{x^4} \to 0\right] = \boxed{0}$

② y-axis → $x = 0$

$\lim_{(x,y) \to (0,0)} \frac{(0)^2 y^2}{(0)^4 + 3y^4}$

$\lim_{(x,y) \to (0,0)} \left[\frac{0}{3y^4} \to 0\right] = \boxed{0}$

③ $y = x$

$\lim_{(x,y) \to (0,0)} \frac{x^2 (x)^2}{x^4 + 3(x)^4} = \lim_{x \to 0} \frac{x^4}{4x^4} = \lim_{x \to 0} \frac{1}{4} = \boxed{\frac{1}{4}}$

Since $\emptyset \neq \frac{1}{4}$, then this limit does not exist [DNE].

Defn: A function $f(x,y)$ is continous at (x_0, y_0) if

① $f(x_0, y_0)$ exist

② $\lim_{(x,y) \to (0,0)} f(x,y)$ exist ☆

③ $f(x_0, y_0) = \lim_{(x,y) \to (0,0)} f(x,y)$

14.3 Partial Derivatives

Recall $y = f(x)$, $\frac{dy}{dx} = f'(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Defn: The partial derivative of $f(x,y)$ with respect to x is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

– X' is changing

– Y" stays the same

Defn: The partial derivative of $f(x,y)$ with respect to y is

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

– Y is changing

– X is the same

Note! $\frac{\partial f}{\partial x}$ you treat all other variables as constants!

Example #1: Given $f(x,y) = 2x^3y^2 + 4x^2 + 3y + 2xy + 4$

ⓐ $\frac{\partial f}{\partial x} = 6x^2y^2 + 8x + 2y$

ⓑ $\frac{\partial f}{\partial y} = 4x^3y + 3 + 2x$

Example #2: Given $g(x,y,z) = xyz + 3x^2y^3z^4 + 4xz$

Find $\frac{\partial g}{\partial y}$

$\frac{\partial g}{\partial y} = xz + 9x^2y^2z^4$

Notation

$\frac{\partial f}{\partial x} = f_x$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0)$$

Note! All of our derivative rules still hold!

Example #3: Given $f(x,y) = \sqrt{x^2+y^2}$

ⓐ Find $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = \left[(x^2+y^2)^{1/2}\right]'(y)$$

$$= \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \boxed{\frac{y}{\sqrt{x^2+y^2}}}$$

Example #4: Find $\left.\frac{\partial f}{\partial x}\right|_{(0,1)}$ for $f(x,y) = 2y\sin(x)$

$f_x = 2y\cos(x)$

$f_x(0,1) = \boxed{2}$

Integrated + Mixed Derivatives

$\frac{\partial^3 f}{\partial x^3} = f_{xxx}$ $\qquad$ $\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$

Example #1: $f(x,y) = 3x^2y^3 + 5x^6y^4$

ⓐ Find $\frac{\partial^2 f}{\partial x^2}$

$f_x = 6xy^3 + 30x^5y^4$

$\boxed{f_{xx} = 6y^3 + 150x^4y^4}$

ⓑ Find $\frac{\partial^2 f}{\partial x \partial y}$

$f_x = 6xy^3 + 30x^5y^4$

$\boxed{f_{xy} = 18xy^2 + 120x^5y^3}$

ⓒ Find $\frac{\partial^2 f}{\partial y \partial x}$

$f_y = 9x^2y^2 + 20x^6y^3$

$\boxed{f_{yx} = 18xy^2 + 120x^5y^3}$

Mixed Derivative Theorem

If $f(x,y)$ and all of its derivatives are defined throughout an open region in the domain then,

$$f_{xy} = f_{yx}$$

14.5 Multivariate Chain Rule

Recall $y=(3t(+)2)^3$

$\frac{dy}{dt}=3(3t(+)2)^2\cdot 3$

Regular Chain Rule

$\frac{d}{dt}(f(g(t)))=f'(g(t))\cdot g't$

OR

$y=f(x)$, $x=g(t)$ $\rightarrow$ $\boxed{\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}}$

$y=x^3$, $x=3t(+)2$ $\quad \frac{dy}{dt}=\boxed{3x^2\cdot 3}$ $=3(3t(+)2)^2\cdot 3$

Recall $z=(3xy+2)^2$

$\frac{\partial d}{\partial x}=2(3xy+2)(3y)$

If $w=f(x,y)$ with $x=x(t)$, $y=y(t)$ are differentiable then

$w=f(x(t),y(t))$ [Chain Rule]

is also differentiable then:

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

Example #1: Use the multivariate Chain Rule to find $\frac{dw}{dt}$ for $w=x^2+y^2+xy$, $x=\sin t$, $y=e^t$.

☆ = Wants to see this! Important!

$$\frac{dw}{dt}=\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial x}\frac{dx}{dt} \ ☆$$

$$=(2y+x)(e^t)+(2x+y)(\cos(t)) \ ☆$$

$$=(2e^t+\sin t)(e^t)+(2\sin t(+)e^t)(\cos t) \ ☆$$

Note! $w=f(x,y,z)$; $x=x(t)$, $y=y(t)$, $z(t)=z$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

Note! $w=f(x,y,z)$; $x=x(r,s)$, $y=y(r,s)$, $z=z(r,s)$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example #2: Multivariate Chain Rule to find $\frac{dw}{dt}$ for $\ln(x^2+y^2+z^2)$; $x=\sin t$, $y=\cos t$, $z=\tan t$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$=\frac{2x}{x^2+y^2+z^2}(\cos t)+\frac{2y}{x^2+y^2+z^2}(-\sin t)+\frac{2z}{x^2+y^2+z}(\sec^2 t)$$

$$=\frac{2x\cos t-2y\sin t+2z\sec^2 t}{x^2+y^2+z^2}=\frac{2\sin(t)\cos(t)-2\cos(t)\sin(t)+2\tan(t)\sec^2(t)}{\sin^2 t+\cos^2 t+\tan^2 t}=\frac{2\tan(t)\sec^2(t)}{1+\tan^2(t)}$$

$$=\frac{2\tan(t)\cancel{\sec^2(t)}}{\cancel{\sec^2(t)}}$$

$$=\boxed{2\tan(t)}$$

Example #3: Use multivariate chain rule to find $\frac{\partial h}{\partial r}$ for $h=a^2+ab^3+c^3$; $a=rs^2$, $b=r^2+s^2$, $c=1+rs$

$$\frac{\partial h}{\partial r}=\frac{\partial h}{\partial a}\frac{\partial a}{\partial r}+\frac{\partial h}{\partial b}\frac{\partial b}{\partial r}+\frac{\partial h}{\partial c}\frac{\partial c}{\partial r}$$

$$=(2a+b^3)(s^2)+(3ab^2)(2r)+(3c^2)(s)$$

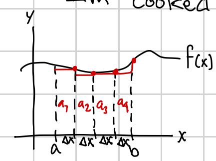
$$=\boxed{(2rs^2+(r^2+s^2)^3)s^2+(3rs^2(r^2+s^2)^2)2r+3(1+rs)^2(s)}$$

15.1 Multivariate Integrals

I'm cooked

Recall: $\int_a^b f(x)\,dx$

Notice! The first step is to partition the domain



$A_1=\Delta x\cdot f(a+\Delta x)$

$A_2=\Delta x\cdot f(a+2\Delta x)$

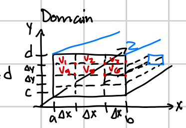
$A_n=\Delta x\cdot f(a+n\Delta x)$

The approximate area $=A_1+\ldots+A_n$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Delta x\cdot f(a+k\Delta x)=\text{exact area}$$

Double Integrals over triangles

$f(x,y)=z$



$c\le y\le d$

$a\le x\le b$

Approximate volume $=V_1+V_2+V_3+V_4+\ldots V_n$

$=V_k=l\cdot w\cdot h$

$=\Delta x\cdot\Delta y\cdot f(x^*,y^*)$

$=\Delta A\cdot f(x^*,y^*)$

Exact Volume $=\iint_R f(x,y)\,dA=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A\cdot f(x_{ij},y_{ij})$

Fubini's Theorem

If $f(x,y)$ is continuous throughout the rectangular region $R=[a,b]\times[c,d]$

$$\iint_R f(x,y)\,dA=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

$$\equiv\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

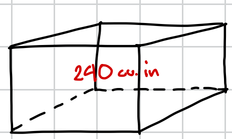
Example #1: $\iint_R(20+xy)\,dA$, let R: $0\le x\le 3$, $-2\le y\le 2$

$$\int_{-2}^{2}\int_0^3 20+xy\,dx\,dy$$

$$\left[20x+\frac{x^2y}{2}\right]_0^3=60+\frac{9}{2}y$$

$$\int_{-2}^{2}60+\frac{9}{2}y\,dy\rightarrow 60y+\frac{9}{4}y^2\Big|_{-2}^{2}=\left(60(2)+\cancel{\tfrac{9}{4}(2)^2}\right)-\left(60(-2)+\cancel{\tfrac{9}{4}(-2)^2}\right)$$

$$=\boxed{240}$$



Example #2: Evaluate $\int_1^2\int_0^1 xye^x\,dx\,dy$

$$=\int_0^1\int_1^2 xye^x\,dy\,dx$$

$$\frac{1}{2}xy^2e^x\Big|_1^2=\frac{1}{2}(2)^2xe^x-\frac{1}{2}(1)^2xe^x=\frac{3}{2}xe^x$$

$$=\int_0^1\frac{3}{2}xe^x\,dx$$

Let $u=\frac{3}{2}x$ $\quad dv=e^x\,dx$

$du=\frac{3}{2}\,dx$ $\quad v=e^x$

$$=\frac{3}{2}xe^x-\int\frac{3}{2}e^x\,dx=\frac{3}{2}\left(xe^x-e^x\right)\Big]_0^1$$

$$=\boxed{\frac{3}{2}}$$

END 15.1

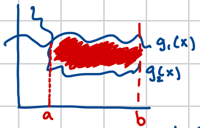
15.2 Double Integrals over General Regions

Strong Fubini's Theorem

If $f(x,y)$ is continuous on a domain D

1. If D is defined by

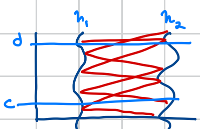
$a \le x \le b$
and
$g(x)_1 \le y \le g(x)_2$



$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1}^{g_2} f(x,y)\, dy\, dx$$

2. If D is defined by

$c \le y \le d$
and
$h_1(y) \le x \le h_2(y)$

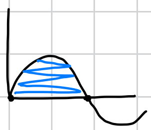


$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1}^{h_2} f(x,y)\, dx\, dy$$

Example #1: Find $\iint_D y\, dA$ where D: $0 \le x \le \pi$, $0 \le y \le \sin(x)$

$$\int_0^{\pi} \int_0^{\sin(x)} y\, dy\, dx$$

$$\frac{1}{2}y^2\Big|_0^{\sin x}$$



$$\int_0^{\pi} \frac{1}{2}\sin^2 x\, dx$$

$$= \frac{1}{4}\int_0^{\pi} 1 - \cos 2x\, dx$$

$$\frac{1}{4}(x - 2\sin(2x))\Big|_0^{\pi}$$

$$\frac{1}{4}x - \frac{1}{2}\sin(2x)\Big|_0^{\pi} = \boxed{\frac{\pi}{4}}$$

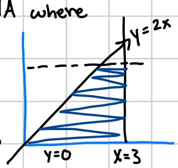
Example #2: Set up the integral for $\iint_D 3x \sin y\, dA$ when

D: $\{0 \le y \le 1,\ 0 \le x \le 4y^2 + 2$

$$\int_0^1 \int_0^{4y^2+2} 3x \sin y\, dx\, dy$$

Example #3: Set up the integral $\iint_D 3x^2y^2\, dA$ where

D: $\{ y = 2x,\ x = 3\}$ by x-axis



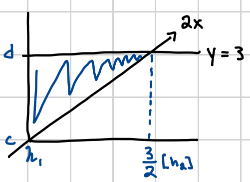
Part A By visual proof,

let x be bounded by $0 \le x \le 3$

and let y be bounded by $0 \le y \le 2x$

$$\therefore \int_0^3 \int_0^{2x} 3x^2y^2\, dy\, dx$$

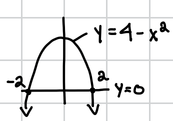
Part B: Let D be bounded by $y = 2x$, $y = 3$, and the y-axis



$$\int_0^{3/2} \int_{2x}^{3} 3x^2y^2\, dy\, dx$$

$2x \le y \le 3$
$0 \le x \le \frac{3}{2}$

Part C: Let D be bounded by $y = 4 - x^2$ and the x-axis

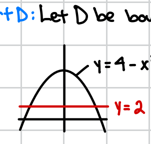


Let x be bounded by $-2 \le x \le 2$

and y be bounded by $0 \le y \le 4 - x^2$

$$\therefore \int_{-2}^{2} \int_0^{4-x^2} 3x^2y^3\, dy\, dx$$

Part D: Let D be bounded by $y = 4 - x^2$, $y = 2$, and the 1st quadrant



$2 = 4 - x^2$
$-4 \quad -4$
$\frac{-2}{-1} = \frac{-x^2}{-1}$
$\sqrt{2} = \sqrt{x^2}$
$\sqrt{2} = x$

Let y be bounded by $2 \le y \le 4 - x^2$

x be bounded by $0 \le x \le \sqrt{2}$

$$\therefore \int_0^{\sqrt{2}} \int_2^{4-x^2} 3x^2y^2\, dy\, dx$$

END 15.2

15.6 Triple Integration

Strong Fubini's Theorem on Triple Integration:

If $\iiint_D f(x,y,z)\, dV \equiv \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dz\, dy\, dx$

when:

$a \le x \le b$,
$g_1(x) \le y \le g_2(x)$
$h_1(x,y) \le z \le h_2(x,y)$

Evaluate $\iiint_D x\, dV$ where $D = \{[0 \le x \le 1], [0 \le y \le x^2], [2x + 2y \le z \le 5]\}$

$$\int_0^1 \int_0^{x^2} \underbrace{\int_{2x+2y}^{5} x\, dz}\, dy\, dx$$

$$xz\Big|_{2x+2y}^{5} = 5x - (x(2x+2y))$$
$$= 5x - (2x^2 + 2xy)$$
$$= 5x - 2x^2 - 2xy$$

$$\int_0^{x^2} 5x - 2x^2 - 2xy\, dy = 5xy - 2x^2y - xy^2\Big|_0^{x^2} = 5x^3 - 2x^4 - x^5$$

$$\int_0^1 5x^3 - 2x^4 - x^5 = \frac{5}{4}x^4 - \frac{2}{5}x^5 - \frac{1}{6}x^6\Big|_0^1 = \boxed{\frac{41}{60}}$$

END 15.6

15.3 Polar Integrals

Recall: $x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$
$\tan\theta = \frac{y}{x}$

Note! We want to change an integral from (x,y) to (r,θ).

Fubini's Polar Integration Theorem

If:

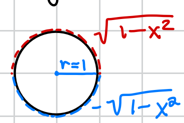
$$\iint_D f(x,y)\, dA$$

Then by the polar integration theorem

$$\iint_D f(r\cos, r\sin\theta)\cdot r\, \overbrace{d\theta\, dr}^{dA}$$

Example #1: Convert to Polar and integrate

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$



By Fubini's Polar Integration Theorem

$D = \{(0 \le r \le 1), (0 \le \theta \le 2\pi)\}$

$$\iint_D r^2\, r\, d\theta\, dr = \iint_D r^3\, d\theta\, dr$$

$$\theta r^3\Big|_0^{2\pi} = 2\pi r^3$$

$$= \int_0^1 2\pi r^3 = \frac{1}{2}\pi r^4\Big|_0^1 = \boxed{\frac{\pi}{2}}$$

Example #2: Convert to Polar

a) $\int_0^2 \int_0^{\sqrt{4-x^2}} x\, dy\, dx$

By FPT $\int_0^2 \int_0^{\frac{\pi}{2}} r\cos\theta \cdot r\, d\theta\, dr$

b) $\int_{-4}^{4} \int_0^{\sqrt{16-x^2}} y\, dy\, dx$

By FPT $\int_0^4 \int_0^{\pi} r^2 \sin\theta\, d\theta\, dr$

END 15.3