**FIGURE 19**

Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle $[-4, 4]$ by $[-4, 4]$

When $a < -1$, both branches are smooth; but when a reaches -1 , the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0 . When $a = 0$, both branches come together and form a circle (see Example 2). For a between 0 and 1 , the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y -axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell.

10.1 Exercises

1–2 For the given parametric equations, find the points (x, y) corresponding to the parameter values $t = -2, -1, 0, 1, 2$.

1. $x = t^2 + t, \quad y = 3^{t+1}$
2. $x = \ln(t^2 + 1), \quad y = t/(t + 4)$

3–6 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

3. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \leq t \leq 2$
4. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \leq t \leq 2$
5. $x = 2^t - t, \quad y = 2^{-t} + t, \quad -3 \leq t \leq 3$
6. $x = \cos^2 t, \quad y = 1 + \cos t, \quad 0 \leq t \leq \pi$

7–12

(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve.

7. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$
8. $x = 3t + 2, \quad y = 2t + 3$
9. $x = t^2 - 3, \quad y = t + 2, \quad -3 \leq t \leq 3$
10. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi$
11. $x = \sqrt{t}, \quad y = 1 - t$
12. $x = t^2, \quad y = t^3$

13–22

(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

13. $x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq \pi$
14. $x = \sin 4\theta, \quad y = \cos 4\theta, \quad 0 \leq \theta \leq \pi/2$
15. $x = \cos \theta, \quad y = \sec^2 \theta, \quad 0 \leq \theta < \pi/2$

16. $x = \csc t$, $y = \cot t$, $0 < t < \pi$

17. $x = e^{-t}$, $y = e^t$

18. $x = t + 2$, $y = 1/t$, $t > 0$

19. $x = \ln t$, $y = \sqrt{t}$, $t \geq 1$

20. $x = |t|$, $y = |1 - |t||$

21. $x = \sin^2 t$, $y = \cos^2 t$

22. $x = \sinh t$, $y = \cosh t$

23–24 The position of an object in circular motion is modeled by the given parametric equations, where t is measured in seconds. How long does it take to complete one revolution? Is the motion clockwise or counterclockwise?

23. $x = 5 \cos t$, $y = -5 \sin t$

24. $x = 3 \sin\left(\frac{\pi}{4}t\right)$, $y = 3 \cos\left(\frac{\pi}{4}t\right)$

25–28 Describe the motion of a particle with position (x, y) as t varies in the given interval.

25. $x = 5 + 2 \cos \pi t$, $y = 3 + 2 \sin \pi t$, $1 \leq t \leq 2$

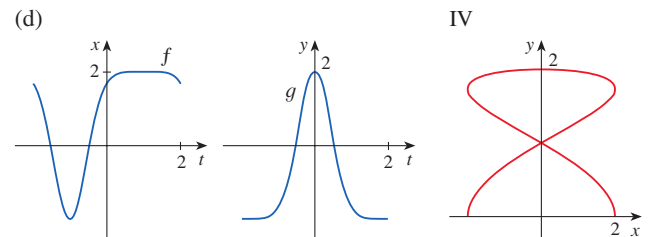
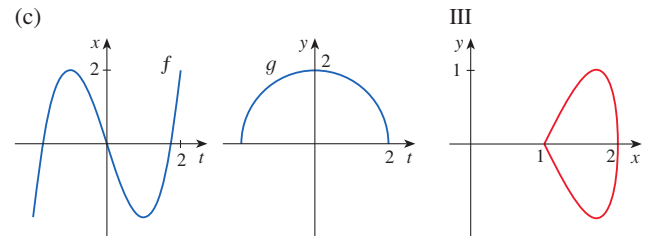
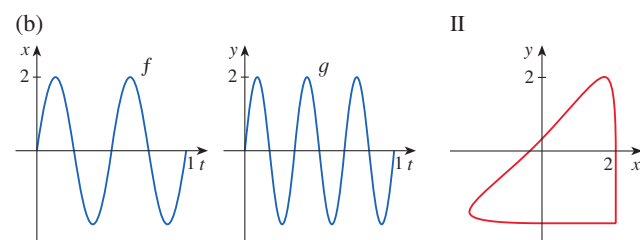
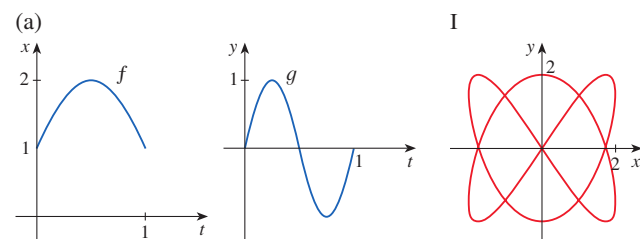
26. $x = 2 + \sin t$, $y = 1 + 3 \cos t$, $\pi/2 \leq t \leq 2\pi$

27. $x = 5 \sin t$, $y = 2 \cos t$, $-\pi \leq t \leq 5\pi$

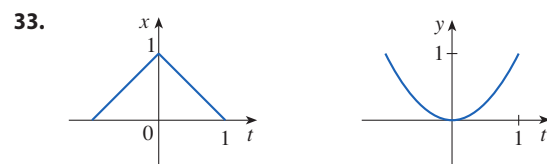
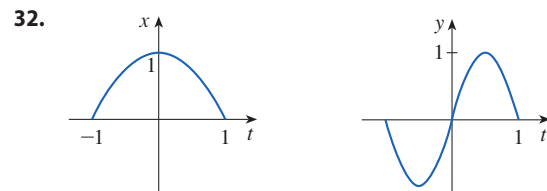
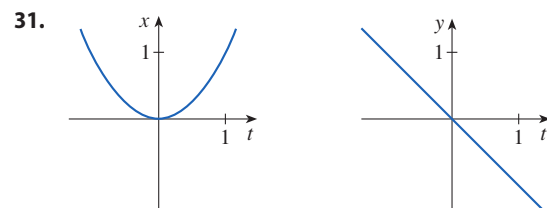
28. $x = \sin t$, $y = \cos^2 t$, $-2\pi \leq t \leq 2\pi$

29. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?

30. Match each pair of graphs of equations $x = f(t)$, $y = g(t)$ in (a)–(d) with one of the parametric curves $x = f(t)$, $y = g(t)$ labeled I–IV. Give reasons for your choices.



31–33 Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



34. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices.

(a) $x = t^4 - t + 1$, $y = t^2$

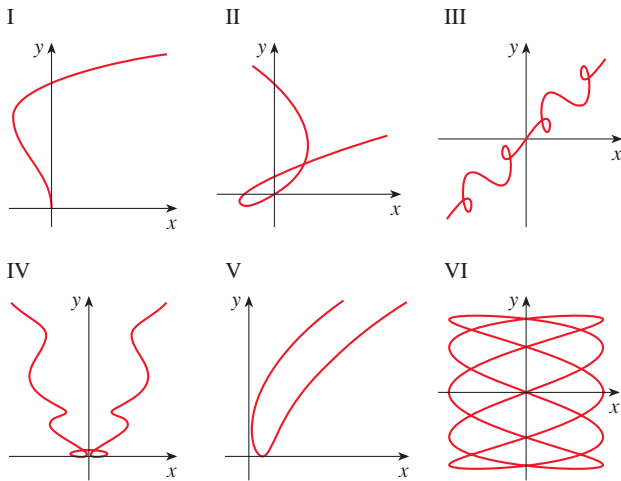
(b) $x = t^2 - 2t$, $y = \sqrt{t}$

(c) $x = t^3 - 2t$, $y = t^2 - t$

(d) $x = \cos 5t$, $y = \sin 2t$

(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$

(f) $x = t + \sin 2t, \quad y = t + \sin 3t$



35. Graph the curve $x = y - 2 \sin \pi y$.

36. Graph the curves $y = x^3 - 4x$ and $x = y^3 - 4y$ and find their points of intersection correct to one decimal place.

37. (a) Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t \quad y = y_1 + (y_2 - y_1)t$$

where $0 \leq t \leq 1$, describe the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

(b) Find parametric equations to represent the line segment from $(-2, 7)$ to $(3, -1)$.

38. Use a graphing calculator or computer and the result of Exercise 37(a) to draw the triangle with vertices $A(1, 1)$, $B(4, 2)$, and $C(1, 5)$.

39–40 Find parametric equations for the position of a particle moving along a circle as described.

39. The particle travels clockwise around a circle centered at the origin with radius 5 and completes a revolution in 4π seconds.

40. The particle travels counterclockwise around a circle with center $(1, 3)$ and radius 1 and completes a revolution in three seconds.

41. Find parametric equations for the path of a particle that moves along the circle $x^2 + (y - 1)^2 = 4$ in the manner described.

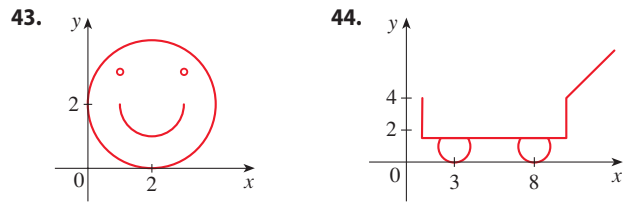
- Once around clockwise, starting at $(2, 1)$
- Three times around counterclockwise, starting at $(2, 1)$
- Halfway around counterclockwise, starting at $(0, 3)$

42. (a) Find parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$. [Hint: Modify the equations of the circle in Example 2.]

(b) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1, 2, 4$, and 8 .

(c) How does the shape of the ellipse change as b varies?

43–44 Use a graphing calculator or computer to reproduce the picture.



45. (a) Show that the points on all four of the given parametric curves satisfy the same Cartesian equation.

$$(i) \quad x = t^2, \quad y = t \quad (ii) \quad x = t, \quad y = \sqrt{t}$$

$$(iii) \quad x = \cos^2 t, \quad y = \cos t \quad (iv) \quad x = 3^{2t}, \quad y = 3^t$$

(b) Sketch the graph of each curve in part (a) and explain how the curves differ from one another.

46–47 Compare the curves represented by the parametric equations. How do they differ?

$$46. (a) \quad x = t, \quad y = t^{-2} \quad (b) \quad x = \cos t, \quad y = \sec^2 t$$

$$(c) \quad x = e^t, \quad y = e^{-2t}$$

$$47. (a) \quad x = t^3, \quad y = t^2 \quad (b) \quad x = t^6, \quad y = t^4$$

$$(c) \quad x = e^{-3t}, \quad y = e^{-2t}$$

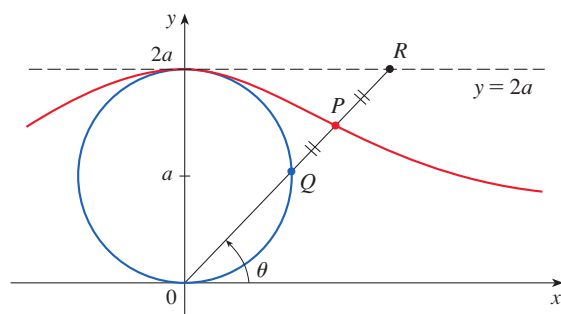
48. Derive Equations 1 for the case $\pi/2 < \theta < \pi$.

49. Let P be a point at a distance d from the center of a circle of radius r . The curve traced out by P as the circle rolls along a straight line is called a **trochoid**. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with $d = r$. Using the same parameter θ as for the cycloid, and assuming the line is the x -axis and $\theta = 0$ when P is at one of its lowest points, show that parametric equations of the trochoid are

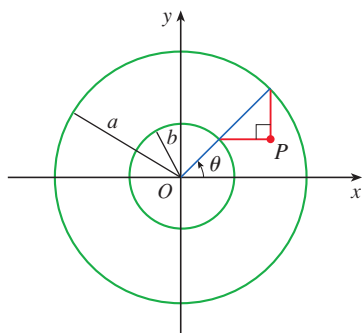
$$x = r\theta - d \sin \theta \quad y = r - d \cos \theta$$

Sketch the trochoid for the cases $d < r$ and $d > r$.

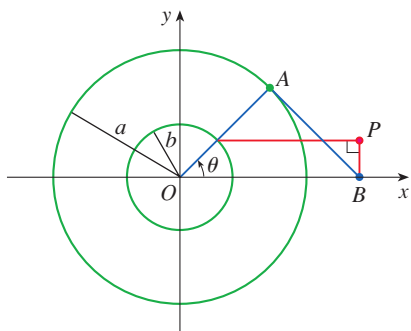
50. In the figure, the circle of radius a is stationary, and for every θ , the point P is the midpoint of the segment QR . The curve traced out by P for $0 < \theta < \pi$ is called the **longbow curve**. Find parametric equations for this curve.



51. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. Then eliminate the parameter and identify the curve.



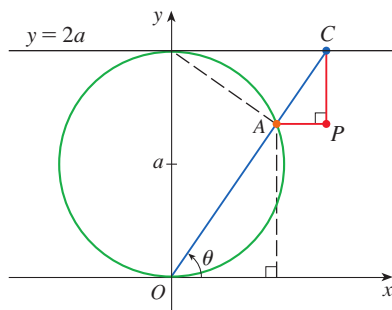
52. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. The line segment AB is tangent to the larger circle.



53. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P in the figure. Show that parametric equations for this curve can be written as

$$x = 2a \cot \theta \quad y = 2a \sin^2 \theta$$

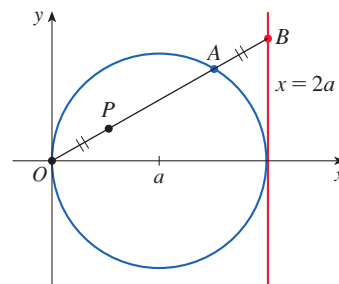
Sketch the curve.



54. (a) Find parametric equations for the set of all points P as shown in the figure such that $|OP| = |AB|$. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical

method for constructing the edge of a cube whose volume is twice that of a given cube.)

- (b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.



55–57 Intersection and Collision Suppose that the position of each of two particles is given by parametric equations. A *collision point* is a point where the particles are at the same place at the same time. If the particles pass through the same point but at different times, then the paths intersect but the particles don't collide.

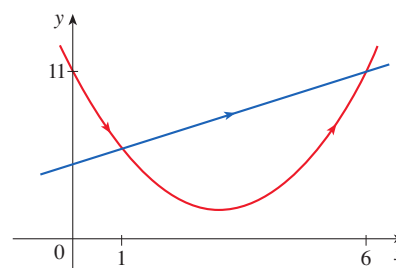
55. The position of a red particle at time t is given by

$$x = t + 5 \quad y = t^2 + 4t + 6$$

and the position of a blue particle is given by

$$x = 2t + 1 \quad y = 2t + 6$$


Their paths are shown in the graph.



- (a) Verify that the paths of the particles intersect at the points $(1, 6)$ and $(6, 11)$. Is either of these points a collision point? If so, at what time do the particles collide?
(b) Suppose that the position of a green particle is given by

$$x = 2t + 4 \quad y = 2t + 9$$

Show that this particle moves along the same path as the blue particle. Do the red and green particles collide? If so, at what point and at what time?

-  56. The position of one particle at time t is given by

$$x = 3 \sin t \quad y = 2 \cos t \quad 0 \leq t \leq 2\pi$$

and the position of a second particle is given by

$$x = -3 + \cos t \quad y = 1 + \sin t \quad 0 \leq t \leq 2\pi$$

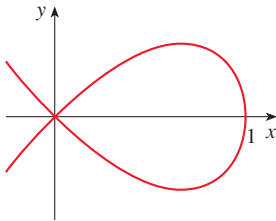
- (a) Graph the paths of both particles. At how many points do the graphs intersect?

- (b) Do the particles collide? If so, find the collision points.
 (c) Describe what happens if the path of the second particle is given by

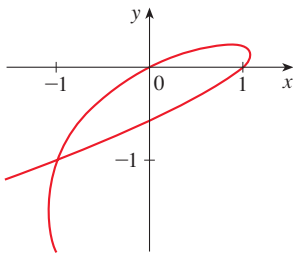
$$x = 3 + \cos t \quad y = 1 + \sin t \quad 0 \leq t \leq 2\pi$$

57. Find the point at which the parametric curve intersects itself and the corresponding values of t .

(a) $x = 1 - t^2, \quad y = t - t^3$



(b) $x = 2t - t^3, \quad y = t - t^2$



58. If a projectile is fired from the origin with an initial velocity of v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity (9.8 m/s^2).

- (a) If a gun is fired with $\alpha = 30^\circ$ and $v_0 = 500 \text{ m/s}$, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?
 (b) Use a graph to check your answers to part (a). Then graph the path of the projectile for several other values of the angle α to see where it hits the ground. Summarize your findings.
 (c) Show that the path is parabolic by eliminating the parameter.

59. Investigate the family of curves defined by the parametric equations $x = t^2, y = t^3 - ct$. How does the shape change as c increases? Illustrate by graphing several members of the family.

60. The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when c increases?

61. Graph several members of the family of curves with parametric equations $x = t + a \cos t, y = t + a \sin t$, where $a > 0$. How does the shape change as a increases? For what values of a does the curve have a loop?

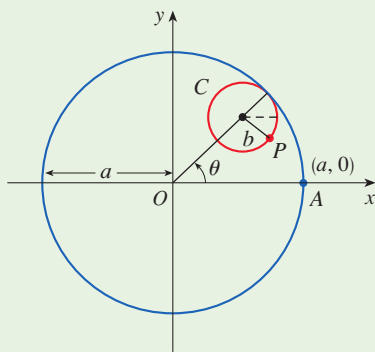
62. Graph several members of the family of curves $x = \sin t + \sin nt, y = \cos t + \cos nt$, where n is a positive integer. What features do the curves have in common? What happens as n increases?

63. The curves with equations $x = a \sin nt, y = b \cos t$ are called **Lissajous figures**. Investigate how these curves vary when a, b , and n vary. (Take n to be a positive integer.)

64. Investigate the family of curves defined by the parametric equations $x = \cos t, y = \sin t - \sin ct$, where $c > 0$. Start by letting c be a positive integer and see what happens to the shape as c increases. Then explore some of the possibilities that occur when c is a fraction.

DISCOVERY PROJECT

RUNNING CIRCLES AROUND CIRCLES



In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A **hypocycloid** is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a . Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are

$$x = (a - b) \cos \theta + b \cos \left(\frac{a - b}{b} \theta \right) \quad y = (a - b) \sin \theta - b \sin \left(\frac{a - b}{b} \theta \right)$$