INFINITE SERIES TESTS: CONVERGENT OR DIVERGENT?

First, it is important to recall some key definitions and notations before discussing the individual series tests.

Sequence: a function that takes on real number values a_n and is defined on the set of positive integers n = 1, 2, 3,…

Listed Notation: $\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$

For example: $\{a_n\} = \dfrac{n}{n+1} = \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5} \ldots$ for n = 1, 2, 3, 4, ...

Series: a summation of a sequence of numbers

Notation: $\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

Note that a sequence is a list of values, each value occurring for different value of n
However a series is the sum of these values.

Convergence - whenever a sequence or a series has a limit

Divergence - whenever a sequence or a series does not have a limit

As a matter of notation, in the following pages, let $\sum a_n = \displaystyle\sum_{n=c}^{\infty} a_n$ for some integer $c \geq 0$

The first type of series test is used only for a series with a specific form:

TEST FOR CONVERGENCE OF A <u>GEOMETRIC SERIES</u>:

A **geometric series** is a series which follows the pattern:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots$$

where a is the initial term and r is a constant ratio (i.e., $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{3}$, etc.).

If $|r| < 1$ (that is, $-1 < r < 1$), then the series converges.

If $|r| \geq 1$, the series diverges.

Sum of the series: Further, when convergent, this series converges to: $\dfrac{a}{1-r}$

Example:

Determine whether the series $2+1+\frac{1}{2}+\frac{1}{4}+\ldots$ converges.

Solution:

To find r divide any term by the term preceding it.

The series follows the pattern of a geometric series with $a=2$ and $r=\frac{1}{2}$.

So, $2+1+\frac{1}{2}+\frac{1}{4}+\ldots=\sum_{n=0}^{\infty}2\left(\frac{1}{2}\right)^{n}$

By the test for convergence, this series does converge, since $\left|\frac{1}{2}\right|=\frac{1}{2}<1$.

It follows that the series converges to 4 since:

$$\frac{a}{1-r}=\frac{2}{1-\frac{1}{2}}=\frac{2}{\frac{1}{2}}=4$$

POSITIVE SERIES

An infinite series with no negative terms is often referred to as a **positive series**.
There are several tests used to determine whether or not a positive series converges or diverges.

THE NTH-TERM TEST:

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

Otherwise, the test is inconclusive.

This test is useful in quickly determining whether a given series diverges.

Example:

Does the following series converge or diverge: $\sum_{n=1}^{\infty}\frac{2n+3}{5n-1}$

Solution:

Notice that the terms (the sequence values) get smaller as n gets larger. However, the terms do not approach zero. In fact:

$$\lim_{n\to\infty} a_n=\frac{2}{5}$$

Therefore, for very large values of n , the series is (more or less) adding on $\frac{2}{5}$ with each successive term. And so, the sum (series) will grow infinitely large.

Therefore, since: $\lim_{n\to\infty} a_n=\frac{2}{5}\neq 0$, we have that the series $\sum_{n=1}^{\infty}\frac{2n+3}{5n-1}$ diverges.

THE INTEGRAL TEST:

Let $f(x) > 0$ for $x \geq 1$, and $f(x)$ is a continuous decreasing function.

Given $a_n = f(n)$,

if $\int_1^{\infty} f(x)dx$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$;

if $\int_1^{\infty} f(x)dx$ is divergent, then so is $\sum_{n=1}^{\infty} a_n$.

This test is useful for the determining the convergence or divergence of a series that has a sequence $\{a_n\}$ that looks like it might be easy enough to integrate.

Example:

Determine whether or not the series converges
(This special series is called the **Harmonic Series**)

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

Solution:

The nth-term test doesn't help here since $\lim_{n \to \infty} a_n = 0$ (the terms approach a value of zero).

Since $\{a_n\} = \frac{1}{n}$, Let $f(x) = \frac{1}{x}$

Using the integral test

$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &= \lim_{b \to \infty} \int_1^{b} \frac{1}{x} dx \\ &= \lim_{b \to \infty} \ln x \Big|_1^b \\ &= \lim_{b \to \infty} (\ln b - \ln 1) \\ &= \infty - 0 \\ &= \infty \end{aligned}$$

Since the integral is divergent, then the series is divergent

P-SERIES TEST:

Given a series of the form: $\sum \frac{1}{n^p}$

If $0 < p \leq 1$, then the series diverges. If $p > 1$, then the series converges.

This test is derived from the integral test and is useful only for the series which contain this specific form. However, it will be quite helpful during the second portion of the comparison tests.

DIRECT COMPARISON TEST:

Given three series, b_n, c_n, d_n:
if $0 \le c_n \le d_n$ for each n and $\sum d_n$ converges, then $\sum c_n$ converges;
if $0 \le b_n \le c_n$ for each n and $\sum b_n$ diverges, then $\sum c_n$ diverges.

This test is used quite often and is one of the more important ones to know. It requires familiarity with the convergence or divergence of other series, in addition to being able to compare them with a given series. This test proves very useful when a series appears "untestable."

Example #1:

Does the series $\sum_{n=1}^{\infty} \frac{n+2}{n+3} \cdot \frac{1}{n^3}$ converge or diverge?

Solution:

This series closely resembles the p-series, $\frac{1}{n^3}$, which converges. In order to compare the two series using the direct comparison test, determine which series is larger. Since $\frac{n+2}{n+3} < 1$ for any n, $\frac{n+2}{n+3} \cdot \frac{1}{n^3} < \frac{1}{n^3}$. (Multiplying each side by $\frac{1}{n^3}$ preserves the inequality since $\frac{1}{n^3}$ is a positive term for all n greater than or equal to 1). So, by the comparison test, $\sum_{n=1}^{\infty} \frac{n+2}{n+3} \cdot \frac{1}{n^3}$ converges.

Example #2:

Does the series $\sum_{n=1}^{\infty} \frac{n+5}{n^2+3n}$ converge or diverge?

Solution:

Factor the denominator, so the series can be represented as $\sum_{n=1}^{\infty} \frac{n+5}{n(n+3)}$

Then, split up the fraction into $\sum_{n=1}^{\infty} \frac{n+5}{n+3} \cdot \frac{1}{n}$. Note that $\frac{n+5}{n+3} > 1$, so $\frac{n+5}{n+3} \cdot \frac{1}{n} > \frac{1}{n}$. Thus, by comparing this series with the divergent harmonic series, $\sum_{n=1}^{\infty} \frac{n+5}{n^2+3n}$ diverges.

LIMIT COMPARISON TEST:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = K$

and K is a non-zero, finite number $(0 < K < \infty)$
then either both series converge or both series diverge.

Similar to the Direct Comparison Test, this test is also helpful when confronted with a series that appears untestable. Most often, this test can be applied to rational functions or with a combination of polynomials and exponentials.

Example:

Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^2 + 3n + 7}{4n^4 + 3n^2 + 10}$

Solution:

As n increases, the highest powers of the numerator and the denominator become the significant terms, in this case n^2 and $4n^4$. So $\frac{n^2 + 3n + 7}{4n^4 + 3n^2 + 10}$ is similar to $\frac{n^2}{4n^4} = \frac{1}{4n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{4n^2}$ is a convergent p-series. Dividing the series by $\frac{1}{4n^2}$ (i.e., multiply by the reciprocal) results in $\frac{4n^4 + 12n^3 + 28n^2}{4n^4 + 3n^2 + 10}$. Now $\lim_{n \to \infty} \frac{4n^4 + 12n^3 + 28n^2}{4n^4 + 3n^2 + 10} = \frac{4}{4} = 1$. So, since the limit is a positive, finite value, the series $\sum_{n=1}^{\infty} \frac{n^2 + 3n + 7}{4n^4 + 3n^2 + 10}$ behaves the same way as $\sum_{n=1}^{\infty} \frac{1}{4n^2}$. That is to say, $\sum_{n=1}^{\infty} \frac{n^2 + 3n + 7}{4n^4 + 3n^2 + 10}$ converges.

RATIO TEST:

For a given series $\sum a_n$:

if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ < 1, then the series converges;

if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ > 1 or goes to ∞, then the series diverges; and

if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$ = 1, then the test is inconclusive.

This test proves particularly helpful for series in which exponentials and/or factorials appear. The absolute-ratio test below will prove to be a more encompassing test, but the ratio test can certainly be useful.

Example:

Determine the convergence for the series, $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

Solution:

This series has both a factorial and an exponential. First, it is important to determine the $n+1$ term, which is, $a_{n+1} = \frac{3^{n+1}}{(n+1)!}$. Now, place it in the ratio test format, such that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3^n \cdot 3}{(n+1)n!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}$$

.

So, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$ thus the series converges

ROOT TEST:

For a given series, $\sum a_n$:

if $\lim_{n\to\infty} \sqrt[n]{a_n}$ < 1, then the series converges; if $\lim_{n\to\infty} \sqrt[n]{a_n}$ > 1 or ∞, then the series diverges;

and if $\lim_{n\to\infty} \sqrt[n]{a_n}$ = 1, then the test is inconclusive.

This test will assist in determining the convergence or divergence of monomials of degree n, especially when the term n^n appears in the series. The only terms that appear in such a series should be various degrees of n alone and no other terms- this test's effectiveness is limited to a small array of series.

Example:

Determine the convergence of the series, $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n}$.

Solution:

Since the series has both exponentials in the top and bottom, the root test should be applied. So, $\sqrt[n]{\frac{2^{n^2}}{n^n}} = \frac{2^n}{n}$ and $\lim_{n\to\infty} \frac{2^n}{n} = \infty$. Thus, by the root test, this series diverges.

SERIES WITH BOTH POSITIVE AND NEGATIVE TERMS

The next set of series tests are those that apply to series with both negative and positive terms (known as alternating series). All the previous tests apply only to those tests whose terms are all positive. An **alternating series** is a series of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - a_5 + \cdots ,$$

in other words, a series whose terms alternate between positive and negative values. Note that a_n is a sequence of positive numbers and the $(-1)^n$ determines the sign.

ALTERNATING SERIES TEST:

For a given alternating series, consider a_n (the sequence of positive numbers without the $(-1)^n$)

If (i) $a_{n+1} < a_n$ for all n, and
(ii) $\lim_{n\to\infty} a_n = 0$

Then the series $\sum (-1)^n a_n$ converges.

This test will immediately give us the convergence of an alternating series, just by looking at its basic behavior. If each term gets progressively smaller, and the terms approach 0, then the series converges. If an alternating series fails one of the two criteria, the test is inconclusive, however a version of the N-th Term Test can often be used to determine the behavior of the series.

Example:

Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges or diverges.

Solution:

For this series, the terms continually decrease:
$\frac{1}{n+1} < \frac{1}{n}$ (for all n) and the terms decrease toward zero: $\lim_{n\to\infty} \frac{1}{n} = 0$

Therefore, because these 2 criteria are met, the series converges.

ABSOLUTE CONVERGENCE TEST:

If $\sum |a_n|$ converges, then so does $\sum a_n$

Like the alternating series test, this test checks only for the convergence of an alternating series and says nothing about the divergence.

Example:

Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n+2)^2}$ converge?

Solution:

Take the absolute value of the series, and get $\sum_{n=1}^{\infty} \frac{2^n}{(n+2)^2}$. By applying the nth term test, this series diverges (the series continues to grow without bound), so the absolute convergence test does not conclude whether the original series converges or diverges.

Before explaining the last two tests involved with alternating series, a few definitions will help clarify certain types of convergence.

A convergent series $\sum a_n$ is said to **converge absolutely** if $\sum |a_n|$ converges as well.

A convergent series $\sum a_n$ is said to **converge conditionally** if $\sum |a_n|$ diverges.

For example, the alternating harmonic series converges conditionally because:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, while $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

While any of the above tests may be used when determining whether a given alternating series is absolutely convergent or conditionally convergent, the following two tests are very helpful, especially when determining the Interval of Convergence and Radius of Convergence of a power series.

ABSOLUTE-RATIO TEST:

Given the series $\sum a_n$, (whose terms may be positive or mixed positive and negative)

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series $\sum a_n$ is absolutely convergent.

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or goes to ∞, then the series $\sum a_n$ is divergent.

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the test is inconclusive.

The absolute-ratio test can be used in similar situations to the ratio test. (i.e., factorials and exponentials)

Example:

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Solution:

Use the Absolute-Ratio Test with $a_n = (-1)^n \frac{n^3}{3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n (n)^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

So, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{1}{3}\left(\dfrac{n+1}{n}\right)^3 = \dfrac{1}{3} < 1$

Therefore, the series $\sum\limits_{n=1}^{\infty}(-1)^n \dfrac{n^3}{3^n}$ is absolutely convergent

ABSOLUTE-ROOT TEST:

Given the series $\sum a_n$, (whose terms may be positive or mixed positive and negative)

If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}$ < 1, then the series $\sum a_n$ is absolutely convergent.

If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}$ > 1 or ∞, then the series $\sum a_n$ is divergent.

If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}$ = 1, the test is inconclusive.

Example:

Test the convergence of the series: $\sum\limits_{n=1}^{\infty}(-1)^n\left(\dfrac{2n+3}{3n+2}\right)^n$

Solution:

Use the Absolute-Root Test with $a_n = (-1)^n\left(\dfrac{2n+3}{3n+2}\right)^n$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|(-1)^n\left(\frac{2n+3}{3n+2}\right)^n\right|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \left(\frac{2n+3}{3n+2}\right)$$

So, $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\left(\dfrac{2n+3}{3n+2}\right) = \dfrac{2}{3} < 1$

Therefore, the series $\sum\limits_{n=1}^{\infty}(-1)^n\left(\dfrac{2n+3}{3n+2}\right)^n$ is absolutely convergent.