Using the formula in entry 98 of the Table of Integrals (or a computer), we have

$$e^{3t}I = \int 15e^{3t}\sin 30t\,dt = 15\,\frac{e^{3t}}{909}(3\sin 30t - 30\cos 30t) + C$$

$$I = \tfrac{5}{101}(\sin 30t - 10\cos 30t) + Ce^{-3t}$$

Since $I(0) = 0$, we get

$$-\tfrac{50}{101} + C = 0$$

so $$I(t) = \tfrac{5}{101}(\sin 30t - 10\cos 30t) + \tfrac{50}{101}e^{-3t}$$ ■

9.5 Exercises

1–4 Determine whether the differential equation is linear. If it is linear, then write it in the form of Equation 1.

1. $y' + x\sqrt{y} = x^2$ **2.** $y' - x = y\tan x$

3. $ue^t = t + \sqrt{t}\,\dfrac{du}{dt}$ **4.** $\dfrac{dR}{dt} + t\cos R = e^{-t}$

5–16 Solve the differential equation.

5. $y' + y = 1$ **6.** $y' - y = e^x$

7. $y' = x - y$ **8.** $4x^3y + x^4y' = \sin^3 x$

9. $xy' + y = \sqrt{x}$ **10.** $2xy' + y = 2\sqrt{x}$

11. $xy' - 2y = x^2, \quad x > 0$ **12.** $y' - 3x^2y = x^2$

13. $t^2\dfrac{dy}{dt} + 3ty = \sqrt{1 + t^2}, \quad t > 0$

14. $t\ln t\,\dfrac{dr}{dt} + r = te^t$

15. $y' + y\cos x = x$ **16.** $y' + 2xy = x^3e^{x^2}$

17–24 Solve the initial-value problem.

17. $xy' + y = 3x^2, \quad y(1) = 4$

18. $xy' - 2y = 2x, \quad y(2) = 0$

19. $x^2y' + 2xy = \ln x, \quad y(1) = 2$

20. $t^3\dfrac{dy}{dt} + 3t^2y = \cos t, \quad y(\pi) = 0$

21. $t\dfrac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

22. $xy' + y = x\ln x, \quad y(1) = 0$

23. $xy' = y + x^2\sin x, \quad y(\pi) = 0$

24. $(x^2 + 1)\dfrac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$

25–26 Solve the differential equation and graph several members of the family of solutions. How does the solution curve change as C varies?

25. $xy' + 2y = e^x$ **26.** $xy' = x^2 + 2y$

27–29 Bernoulli Differential Equations A *Bernoulli differential equation* (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

27. Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

28. Solve the differential equation $xy' + y = -xy^2$.

29. Solve the differential equation $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$.

30. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

31. In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10 Ω, and $I(0) = 0$.
(a) Find $I(t)$.
(b) Find the current after 0.1 seconds.

32. In the circuit shown in Figure 4, a generator supplies a voltage of $E(t) = 40\sin 60t$ volts, the inductance is 1 H, the resistance is 20 Ω, and $I(0) = 1$ A.
(a) Find $I(t)$.
(b) Find the current after 0.1 seconds.
(c) Graph the current function.

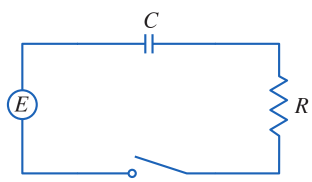
33. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = dQ/dt$ (see Example 3.7.3), so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, a battery gives a constant voltage of 60 V, and the initial charge is $Q(0) = 0$ C. Find the charge and the current at time t.



34. In the circuit of Exercise 33, $R = 2\ \Omega$, $C = 0.01$ F, $Q(0) = 0$, and $E(t) = 10 \sin 60t$. Find the charge and the current at time t.

35. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a *learning curve*. In Exercise 9.1.27 we proposed the differential equation

$$\frac{dP}{dt} = k[M - P(t)]$$

as a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

36. Two new workers were hired for an assembly line. Jim processed 25 units during the first hour and 45 units during the second hour. Mark processed 35 units during the first hour and 50 units the second hour. Using the model of Exercise 35 and assuming that $P(0) = 0$, estimate the maximum number of units per hour that each worker is capable of processing.

37. In Section 9.3 we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable differentiable equations. (See Example 9.3.6.) If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable.

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$$

Solve this equation and find the concentration after 20 minutes.

38. A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

39. An object with mass m is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after t seconds, then the speed is $v = s'(t)$ and the acceleration is $a = v'(t)$. If g is the acceleration due to gravity, then the downward force on the object is $mg - cv$, where c is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt} = mg - cv$$

(a) Solve this as a linear equation to show that

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

(b) What is the limiting velocity?
(c) Find the distance the object has fallen after t seconds.

40. If we ignore air resistance, we can conclude that heavier objects fall no faster than lighter objects. But if we take air resistance into account, our conclusion changes. Use the expression for the velocity of a falling object in Exercise 39(a) to find dv/dm and show that heavier objects *do* fall faster than lighter ones.

41. (a) Show that the substitution $z = 1/P$ transforms the logistic differential equation $P' = kP(1 - P/M)$ into the linear differential equation

$$z' + kz = \frac{k}{M}$$

(b) Solve the linear differential equation in part (a) and thus obtain an expression for $P(t)$. Compare with Equation 9.4.7.

42. To account for seasonal variation in the logistic differential equation, we could allow k and M to be functions of t:

$$\frac{dP}{dt} = k(t)P\left(1 - \frac{P}{M(t)}\right)$$

(a) Verify that the substitution $z = 1/P$ transforms this equation into the linear equation

$$\frac{dz}{dt} + k(t)z = \frac{k(t)}{M(t)}$$

(b) Write an expression for the solution of the linear equation in part (a) and use it to show that if the carrying