15.1 Exercises

1. (a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \leqslant x \leqslant 6, 0 \leqslant y \leqslant 4\}$$

Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each square.

(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

2. If $R = [0, 4] \times [-1, 2]$, use a Riemann sum with $m = 2$, $n = 3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Take the sample points to be (a) the lower right corners and (b) the upper left corners of the rectangles.

3. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0, 2] \times [0, 1]$. Take the sample points to be upper right corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

4. (a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2] \times [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.

(b) Use the Midpoint Rule to estimate the volume in part (a).

5. Let V be the volume of the solid that lies under the graph of $f(x, y) = \sqrt{52 - x^2 - y^2}$ and above the rectangle given by $2 \leqslant x \leqslant 4$, $2 \leqslant y \leqslant 6$. Use the lines $x = 3$ and $y = 4$ to divide R into subrectangles. Let L and U be the Riemann sums computed using lower left corners and upper right corners, respectively. Without calculating the numbers V, L, and U, arrange them in increasing order and explain your reasoning.

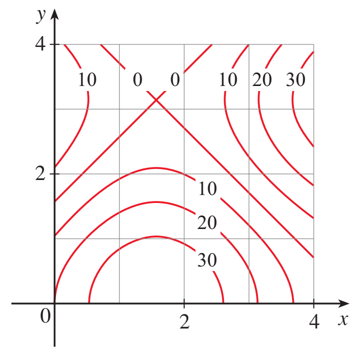
6. A 20-ft by 30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table. Estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

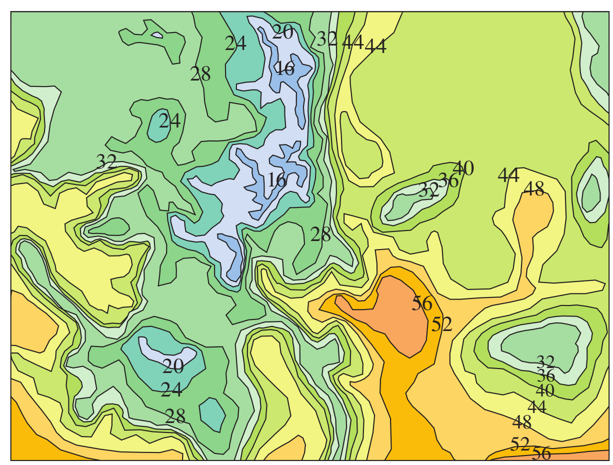
7. A contour map is shown for a function f on the square $R = [0, 4] \times [0, 4]$.

(a) Use the Midpoint Rule with $m = n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$.

(b) Estimate the average value of f.



8. The contour map shows the temperature, in degrees Fahrenheit, at 4:00 PM on a day in February in Colorado. (The state measures 388 mi west to east and 276 mi south to north.) Use the Midpoint Rule with $m = n = 4$ to estimate the average temperature in Colorado at that time.



9–11 Evaluate the double integral by first identifying it as the volume of a solid.

9. $\iint_R \sqrt{2}\, dA$, $\quad R = \{(x, y) \mid 2 \leqslant x \leqslant 6, -1 \leqslant y \leqslant 5\}$

10. $\iint_R (2x + 1)\, dA$, $\quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 4\}$

11. $\iint_R (4 - 2y)\, dA$, $\quad R = [0, 1] \times [0, 1]$

12. The integral $\iint_R \sqrt{9 - y^2}\, dA$, where $R = [0, 4] \times [0, 2]$, represents the volume of a solid. Sketch the solid.

13–14 Find $\int_0^2 f(x, y)\, dx$ and $\int_0^3 f(x, y)\, dy$

13. $f(x, y) = x + 3x^2y^2$

14. $f(x, y) = y\sqrt{x + 2}$

15–26 Calculate the iterated integral.

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$

16. $\int_0^1 \int_0^1 (x + y)^2\, dx\, dy$

17. $\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy$

18. $\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\, dy\, dx$

19. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\, dx\, dy$

20. $\int_1^3 \int_1^5 \dfrac{\ln y}{xy}\, dy\, dx$

21. $\int_1^4 \int_1^2 \left(\dfrac{x}{y} + \dfrac{y}{x}\right) dy\, dx$

22. $\int_0^1 \int_0^2 ye^{x-y}\, dx\, dy$

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\, d\phi\, dt$

24. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\, dy\, dx$

25. $\int_0^1 \int_0^1 v(u + v^2)^4\, du\, dv$

26. $\int_0^1 \int_0^1 \sqrt{s + t}\, ds\, dt$

27–34 Calculate the double integral.

27. $\iint_R x \sec^2 y\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \pi/4\}$

28. $\iint_R (y + xy^{-2})\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

29. $\iint_R \dfrac{xy^2}{x^2 + 1}\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1, -3 \leqslant y \leqslant 3\}$

30. $\iint_R \dfrac{\tan\theta}{\sqrt{1 - t^2}}\, dA, \quad R = \left\{(\theta, t) \mid 0 \leqslant \theta \leqslant \pi/3, 0 \leqslant t \leqslant \tfrac{1}{2}\right\}$

31. $\iint_R x \sin(x + y)\, dA, \quad R = [0, \pi/6] \times [0, \pi/3]$

32. $\iint_R \dfrac{x}{1 + xy}\, dA, \quad R = [0, 1] \times [0, 1]$

33. $\iint_R ye^{-xy}\, dA, \quad R = [0, 2] \times [0, 3]$

34. $\iint_R \dfrac{1}{1 + x + y}\, dA, \quad R = [1, 3] \times [1, 2]$

35–37 Sketch the solid whose volume is given by the iterated integral.

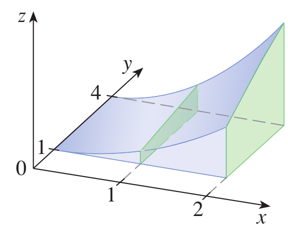
35. $\int_0^1 \int_0^1 (4 - x - 2y)\, dx\, dy$

36. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\, dy\, dx$

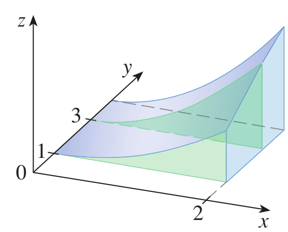
37. $\int_{-2}^2 \int_{-1}^3 (4 - x^2)\, dy\, dx$

38. Consider the solid region S that lies under the surface $z = x^2\sqrt{y}$ and above the rectangle $R = [0, 2] \times [1, 4]$.

(a) Find a formula for the area of a cross-section of S in the plane perpendicular to the x-axis at x for $0 \leqslant x \leqslant 2$. Then use the formula to compute the areas of the cross-sections illustrated.



(b) Find a formula for the area of a cross-section of S in the plane perpendicular to the y-axis at y for $1 \leqslant y \leqslant 4$. Then use the formula to compute the areas of the cross-sections illustrated.



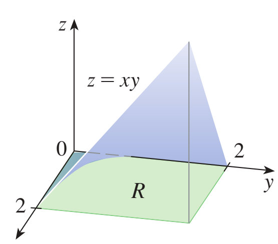
(c) Find the volume of S.

39–42 The figure shows a surface and a rectangle R in the xy-plane.

(a) Set up an iterated integral for the volume of the solid that lies under the surface and above R.

(b) Evaluate the iterated integral to find the volume of the solid.

39.



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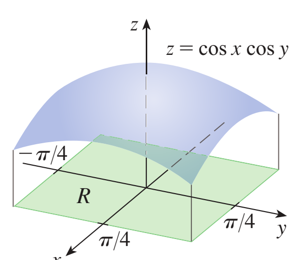

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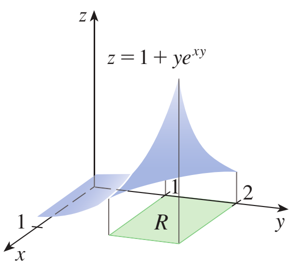

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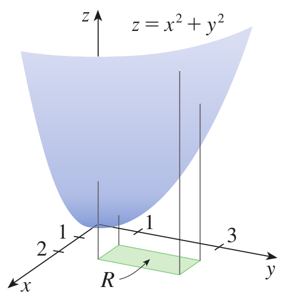

43. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x, y) \mid -1 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1\}$.

44. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

45. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

46. Find the volume of the solid enclosed by the surface $z = x^2 + xy^2$ and the planes $z = 0$, $x = 0$, $x = 5$, and $y = \pm 2$.

47. Find the volume of the solid enclosed by the surface $z = 1 + x^2ye^y$ and the planes $z = 0$, $x = \pm 1$, $y = 0$, and $y = 1$.

48. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

49. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

50. Graph the solid that lies between the surface $z = 2xy/(x^2 + 1)$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$. Then find its volume.

T **51.** Use a computer algebra system to find the exact value of the integral $\iint_R x^5y^3e^{xy}\,dA$, where $R = [0, 1] \times [0, 1]$. Then use the CAS to draw the solid whose volume is given by the integral.

T **52.** Graph the solid that lies between the surfaces $z = e^{-x^2}\cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \leqslant 1$, $|y| \leqslant 1$. Use a computer algebra system to approximate the volume of this solid correct to four decimal places.

53–54 Find the average value of f over the given rectangle.

53. $f(x, y) = x^2y$,
R has vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

54. $f(x, y) = e^y\sqrt{x + e^y}$, $\quad R = [0, 4] \times [0, 1]$

55–56 Use symmetry to evaluate the double integral.

55. $\displaystyle\iint_R \frac{xy}{1 + x^4}\,dA, \quad R = \{(x, y) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

56. $\displaystyle\iint_R (1 + x^2\sin y + y^2\sin x)\,dA, \quad R = [-\pi, \pi] \times [-\pi, \pi]$

T **57.** Use a computer algebra system to compute the iterated integrals

$$\int_0^1\int_0^1 \frac{x - y}{(x + y)^3}\,dy\,dx \quad \text{and} \quad \int_0^1\int_0^1 \frac{x - y}{(x + y)^3}\,dx\,dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

58. (a) In what way are the theorems of Fubini and Clairaut similar?
(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\,dt\,ds$$

for $a < x < b$, $c < y < d$, show that

$$g_{xy} = g_{yx} = f(x, y)$$

15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function not just over rectangles but also over regions of more general shape.