M408D Final Exam Review – Day 2 - SOLUTIONS

1) If y_0 is the particular solution of the differential equation $\frac{dy}{dx} - \frac{\ln x}{xy} = 0$ satisfying the initial condition $y(1) = 2$, find the value of $y_0(3)$.

$$\frac{dy}{dx} - \frac{\ln x}{xy} = 0 \qquad \left[\frac{dy}{dx} = \frac{\ln x}{xy}\right]$$

$$\int y\,dy = \int \frac{\ln x}{x}\,dx \qquad u = \ln x \quad du = \frac{dx}{x}$$

$$y(1) = 2 \qquad x=1 \quad y=2$$

$$\frac{1}{2}y^2 = \int u\,du = \frac{1}{2}u^2 + C$$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$$

$$y(3) = \qquad x=3$$

$$2 = \frac{1}{2}\cdot 0 + C \Rightarrow C = 2$$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln(x))^2 + 2$$

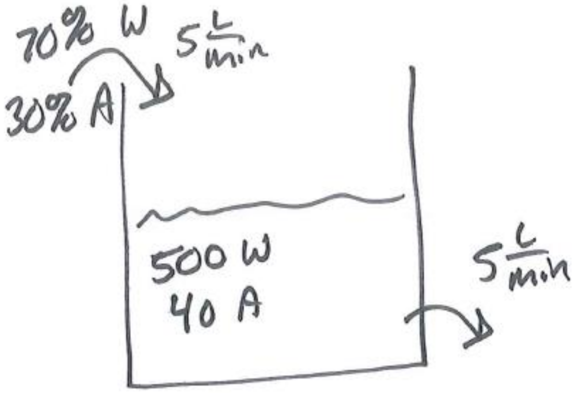
$$y(1) = -2$$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln 3)^2 + 2$$

$$y\cancel{^2} = \sqrt{(\ln 3)^2 + 4}$$

$$\ln(3^2) = 2\ln 3$$

2) A tank contains a mixture of 500 L of water and 40 L of alcohol. A solution of 70% water and 30% alcohol enters the tank at a rate of 5 L/min. The solution is continuously mixed with the fluid in the tank, which is drained from the tank at a rate of 5 L/min. What volume of alcohol remains in the tank after 10 minutes?



$y =$ alcohol in tank

$\frac{dy}{dt}$ $\qquad y(0) = 40 \text{ L}$

$$\frac{dy}{dt} = .3(5) - \left(\frac{y}{540}\right)(5)$$

$$\frac{dy}{dt} - \frac{1}{108} y = 1.5 \qquad \Big| \quad y' + p(t)y = \sim$$

$$e^{\int_0^t p(u)\,du}$$

$$\cdot\left(e^{\int_0^t -\frac{1}{108}\,du} = e^{-\frac{1}{108}t}\right)$$

$$e^{-\frac{t}{108}} \frac{dy}{dt} - \frac{1}{108} e^{-\frac{t}{108}} y = 1.5 e^{-\frac{1}{108}t}$$

$$\left[e^{-\frac{t}{108}} y\right]' = 1.5 e^{-\frac{1}{108}t}$$

integrate

$$\underline{e^{-\frac{t}{108}}} y = \int 1.5 e^{-\frac{1}{108}t}\,dt$$

$$= 1.5 \cdot (-108) e^{-\frac{1}{108}t} + C$$

mult. by $e^{\frac{t}{108}} \rightsquigarrow y = -162 + C e^{\frac{t}{108}}$

$$40 = y(0) = -162 + C \qquad 202 = C$$

$$y = -162 + 202 e^{\frac{t}{108}}$$

3) Find the general solution for the differential equation $(y+1)\frac{dy}{dx} = xy\sin x$

$$(y+1)\frac{dy}{dx} = xy\sin x$$

$$\int \frac{(y+1)}{y}\,dy = \int x\sin x\,dx \qquad u = x \quad v = -\cos x$$

$$du = dx \quad dv = \sin x\,dx$$

$$= -x\cos x + \underbrace{\int \cos x\,dx}_{\sin x}$$

$$\int 1 + \frac{1}{y}\,dy$$

$$\underline{\underline{y + \ln y}} = -x\cos x + \sin x + C$$

4) A function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = e^{y}(y^{2} - 3y - 10)$. Find the intervals on which $y(t)$ is increasing, and those on which it is decreasing.

5) A function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = y^{3} - y$. Find $\lim_{t\to\infty} y(t)$.

6) For what values of k does the function $y(t) = \cos kt + \sin kt$ satisfy the differential equation $5y'' + 16y = 0$?

7) A freshly-poured cup of coffee has a temperature of $90°$ C in a room with temperature $20°$ C. After 5 minutes, the temperature of the coffee was $85°$C. Newton's law of cooling states that the rate of cooling of an object is proportional to the difference in temperature between the object and its surroundings. Find the function $y(t)$ representing the temperature of the coffee at time t, where $t = 0$ is the time at which the coffee is poured.

$y =$ temperature of coffee at t

$y(0) = 90°C \quad y(5) = 85°C$

$$\frac{dy}{dt} = k(y-20) \qquad \int \frac{dy}{y-20} = \int k\,dt$$

$$\ln(y-20) = kt + C$$

$$\rightarrow y = 20 + e^{kt} \cdot \cancel{e^{C}}\, C_1 \leftarrow \qquad y-20 = e^{kt+C}$$

$$y(0) = 20 + C_1 = 90 \qquad y = 20 + e^{kt+C}$$

$$y(5) = 20 + e^{5k} \cdot C_1 = 85$$

$$C_1 = 70 \qquad 20 + e^{5k} \cdot 70 = 85$$

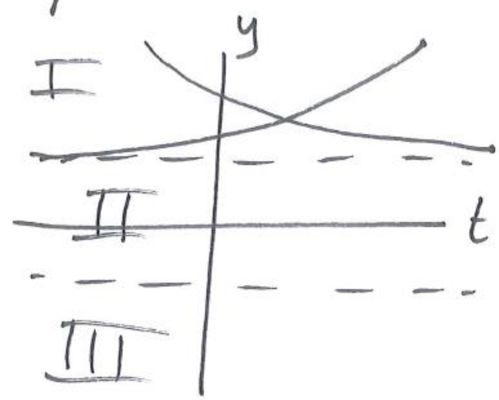
$$e^{5k} = \frac{65}{70}$$

$$\cancel{5}k = \frac{\ln(65/70)}{5}$$

8) Find the particular solution, $y_0(t)$ of the differential equation $(1+\cos x)y' = (1+e^{-y})\sin x$ satisfying the initial condition $y(0) = 0$.

$$\frac{dy}{dt} = 0$$

$y^3 - y = 0 \quad y = 0, 1, -1$



$$(1+\cos x)\frac{dy}{dx} = (1+e^{-y})\sin x$$

$$\frac{1}{1+e^{-y}}\,dy \qquad \frac{\sin x}{1+\cos x}$$

$$\int \frac{1}{1+e^{-y}}\,dy = \int \frac{1}{(1+u)}\frac{-du}{(u)}$$

$u = e^{-y}$

$du = -e^{-y}\,dy$

$du = -u\,dy$

$$\frac{du}{u} = -dy$$

9) Use Euler's method with step size 0.2 to estimate y(1), where $y(x)$ is the solution to the initial-value problem $y' = y + xy,\ y(0) = 1$.

Euler's $\quad h = 0.2 \quad y(1) \quad y(x)$

$y' = y + xy$

$x=0 \quad y=1$

$y(0) = 1$

$y(0.2) = 1 + 0.2(1) = 1.2$

$y(0.4) = 1.2 + 0.2(1.44)$

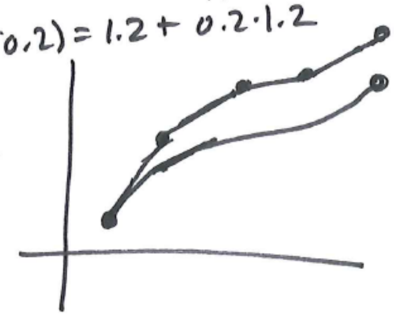
$= 1.2 + .288 = 1.488$

$y(0.6) = y(0.4) + h\,y'(0.4)$

$y(0) = 1$

$1.2 + .24 = 1.44$

$y'(0.2) = 1.2 + 0.2 \cdot 1.2$



$$Y_{n+1} = Y_n + \underbrace{\Delta Y}_{ss}$$

$$h \cdot \frac{dy}{dx}$$

10) Find the solution to the linear initial-value problem $x^2y' + 2xy = \ln x,\ y(1) = 2$.

11) Sketch the slope field corresponding to the differential equation $\frac{dy}{dx} = 2 - y$, and draw the solution curve passing through the point (0, 1).

12) Eliminate the parameter to find a Cartesian equation of the curve given by the parametric equations $x(t) = 3\sin 2t,\ y(t) = 5\cos 2t,\ 0 \le t \le \pi$. Then graph the curve, indicating orientation.

$\sin^2\theta + \cos^2\theta = 1 \leftarrow$ $\qquad \theta = 2t$

$x(t) = 3\sin(2t) \rightsquigarrow \frac{x(t)}{3} = \sin(2t)$ $\qquad 0 < \theta < 2\pi$

$y(t) = 5\cos(2t) \rightsquigarrow \frac{y(t)}{5} = \cos(2t)$

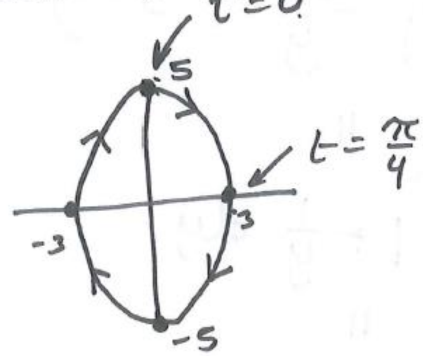
$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$

$0 < t < \pi$

$t = 0: \ (0, 5)$

$t = \frac{\pi}{4}: \ (3, 0)$

$t = \pi: (0, 5)$



sint cost

$0 < t < 2\pi$

13) Eliminate the parameter to find a Cartesian equation of the curve given by the parametric equations $x(t) = \tan^2 t,\ y(t) = \sec t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$. Then graph the curve, indicating orientation.

14) Write parametric equations of the line segment starting at the point P(-1, 3), and ending at the point Q(2, -6), with parameter $t \in [0,1]$.

$0 \le t \le 1$

$$x(t) = P_x(\overset{1-t}{\bullet}) + Q_x(t)$$

$$x(0) = P_x \qquad x(1) = Q_x$$

$$x(t) = -1(1-t) + 2(t) = -1 + t + 2t = \boxed{-1+3t}$$

$$y(t) = 3(1-t) + (-6)(t) = 3 - 3t - 6t = \boxed{3-9t}$$

$$(1-t)\big|_{t=0} = 1 \qquad (1-t)\big|_{t=1} = 0$$

$$[3,5] \qquad y = mx + b$$

$$x(t) = \cancel{P_x(\;)}\,4(t)$$
$$= \cancel{P_x(\;)}$$

$x(t)$ $\quad -1 \longrightarrow 2$

3 units

$-1 + 3t$

$y(t)$ $\quad 3 \longrightarrow -6$

-9 units

$3 - 9t$

15) Write parametric equations corresponding to the path of the particle traversing the circle of radius 2, centered at (0, 1), counterclockwise, completing the circle once in the interval $0 \le t < 4\pi$.

$r=2 \quad (0,1) \qquad (x-0)^2+(y-1)^2=4$

$0<t<4\pi \qquad x^2+(y-1)^2=4$

$4\sin^2\theta+4\cos^2\theta=4 \qquad 2\cos t \quad 2\sin t \qquad x=2\cos(t/2)$

$\qquad\qquad \overset{\shortparallel}{x} \qquad \overset{\shortparallel}{y-1} \qquad y=1+2\sin(t/2)$

$t=0 \quad (2,1)$

$t=\pi \quad (0,3)$

$-\sin\theta=\sin(-\theta)$

16) Find the length of the curve $x(t)=e^{t}+e^{-t}$, $y(t)=1-2t$, 1£t£3.

$x(t)=e^t+e^{-t} \quad y(t)=1-2t \qquad 1<t<3$

$\int\sqrt{(dx)^2+(dy)^2}\,\frac{dt}{dt}$

$\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$x'(t)=e^t-e^{-t}$

$y'(t)=-2$

$\int_a^b\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

$(x'(t))^2=(e^t-e^{-t})^2=e^{2t}-2+e^{-2t}$) add

$(y'(t))^2=4$

$(x')^2+(y')^2=e^{2t}+2+e^{-2t}=(e^t+e^{-t})^2$

$\text{Length}=\int_1^3 e^t+e^{-t}\,dt=e^t-e^{-t}\Big|_1^3=(e^3-e^{-3})-(e^1-e^{-1})$

17) Graph the curve and find its length $x(t)=e^{t}\sin t, y(t)=e^{t}\cos t$, 0£$t$£2.

18) Find the points on the curve $x(t)=t^3-4t$, $y(t)=t^2-t$ where the tangent line is vertical.

19) Find the points on the curve $x(t)=e^{-t}+e^{t}$, $y(t)=\sin t+\cos t$, 0£t£2p, corresponding to a horizontal tangent line.

20) Find the equation of the tangent line to the graph of $x(t) = 2t^2 - 1,\ y(t) = 3t - 2$ at the point (7, -8).

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

$x(t) = 2t^2 - 1 \quad x'(t) = 4t$

$y(t) = 3t - 2 \quad y'(t) = 3$

$$\frac{dy}{dx} = \frac{3}{4t}$$

$(7, -8)$

$x \quad y$

$y(t) = 3t - 2 = -8$

$3t = -6$

$t = -2$

$$m = \frac{3}{4(-2)} = -\frac{3}{8}$$

point slope

$$y + 8 = -\frac{3}{8}(x - 7)$$

21) Find $\dfrac{d^2y}{dx^2}$ for the parametric curve given by $x(t) = t^2 + 5, y(t) = \sin 2t$.

$x(t) = t^2 + 5 \quad y(t) = \sin 2t$

$$\frac{d^2y}{dx^2}$$

$$z(t) = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{2}\cos 2t}{\cancel{2}t} = \frac{\cos(2t)}{t}$$

$$z'(t) = \frac{-2\sin(2t)t - \cos 2t}{t^2}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx} = \frac{dz/dt}{dx/dt}$$

$x'(t) = 2t$

$$\cancel{\frac{d/dx(dy/dx)}{dx/dt}} \qquad \frac{d/dt\,(dy/dx)}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{-2\sin(2t)t - \cos 2t}{2t^3}$$

22) Find the area under the curve enclosed by the x-axis and the curve $x(t) = e^t - 1,\ y(t) = t^2 - 4$.

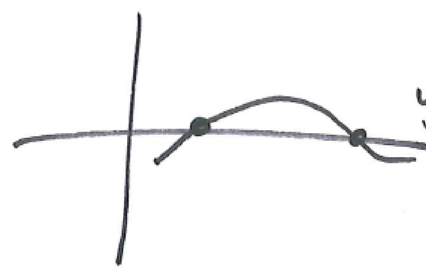
$x(t) = e^t - 1 \qquad y(t) = t^2 - 4 = 0$ when $t = \pm 2$

$\int y(x)\,dx \qquad x(t) = e^t - 1$

$dx(t) = e^t\,dt \qquad x'(t) = e^t$

$\int y(t)\,e^t\,dt \qquad \boxed{\int y(t)x'(t)\,dt}$

$\int_{-2}^{2} (t^2-4)\,e^t\,dt$



$y \overset{?}{=} 0$

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23) The equation in polar coordinates $r = \cos^2\theta$ defines a curve in the xy plane. Find the equation of the line tangent to the curve at the point corresponding to $\theta = \pi/4$.

Let $r=\cos^2\theta$ be a curve in the xy-plane
Find eqn of line tangent to curve @ $\theta=\frac{\pi}{4}$

$x=r\cos\theta$ @ $\frac{\pi}{4}$ $\rightarrow$ $x=\frac{1}{2}\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{4}$

$y=r\sin\theta$ @ $\frac{\pi}{4}$ $\rightarrow$ $y=\frac{1}{2}\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{4}$

$r=\cos^2\theta$ @ $\frac{\pi}{4}$ $\rightarrow$ $r=\cos^2\left(\frac{\pi}{4}\right)=\left(\frac{\sqrt{2}}{2}\right)^2=\frac{1}{2}$

pt in x, y coordinates is: $\left(\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4}\right)$

$m=\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, $y(\theta)=(\cos^2\theta)\cdot\sin\theta=\sin\theta-\sin^3\theta$

$x(\theta)=(\cos^2\theta)\cdot\cos\theta=\cos^3\theta$

So, $y'(\theta)=\cos\theta-3\sin^2\theta\cos\theta$

$x'(\theta)=-3\cos^2\theta\sin\theta$

And m @ $\frac{\pi}{4}=\frac{y'\left(\frac{\pi}{4}\right)}{x'\left(\frac{\pi}{4}\right)}=\frac{\cos\left(\frac{\pi}{4}\right)-3\sin^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)}{-3\cos^2\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}$

$$=\frac{\frac{\sqrt{2}}{2}-\frac{3}{2}\frac{\sqrt{2}}{2}}{-\frac{3}{2}\frac{\sqrt{2}}{2}}=\frac{-\frac{1}{2}\frac{\sqrt{2}}{2}}{-\frac{3}{2}\frac{\sqrt{2}}{2}}=\frac{1}{3}$$

So, $m=\frac{1}{3}$ & pt is $\left(\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4}\right)$, use $y-y_1=m(x-x_1)$

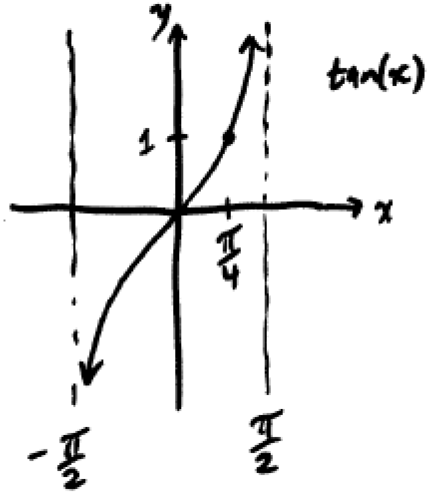
to get: $y-\frac{\sqrt{2}}{4}=\frac{1}{3}\left(x-\frac{\sqrt{2}}{4}\right)$

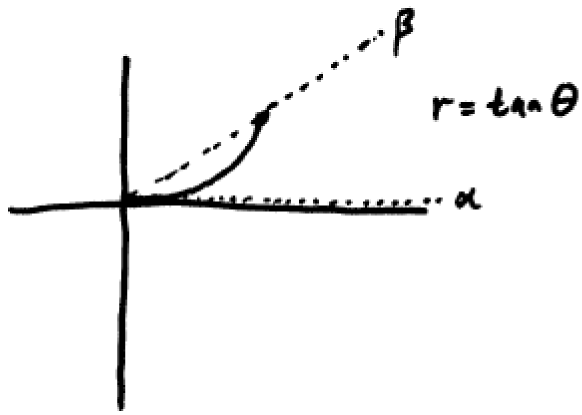
Sketch the bounded region and then calculate the area.

24) $r=\tan\theta$, $\alpha=0$ and $\beta=\pi/4$

Sketch the bounded region & calculate the area

$r = \tan\theta$, $\alpha = 0$, $\beta = \frac{\pi}{4}$





$$\int_\alpha^\beta \frac{1}{2} f^2(\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2} \tan^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \left(\sec^2\theta - 1\right) d\theta$$

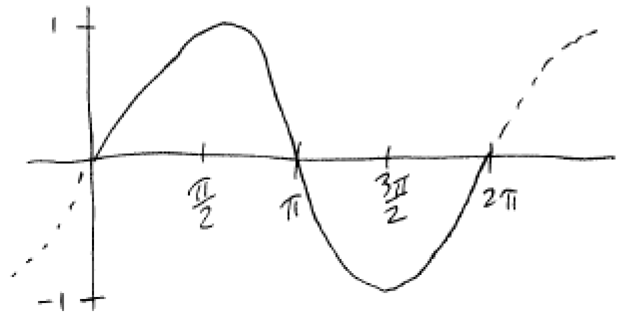
$$= \frac{1}{2} \left(\tan\theta - \theta\right) \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \left[\left(1 - \frac{\pi}{4}\right) - (0 - 0)\right]$$

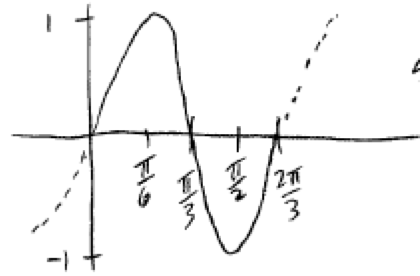
$$= \frac{1}{2} - \frac{\pi}{8}$$

25) One loop of $r = \sin 3\theta$

$r = \sin 3\theta$ (one loop)



$\sin(x)$



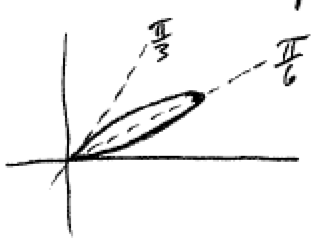
$\sin(3x)$

The "3" compresses the wavelength of the sine function.

Note: For $r = \sin 3\theta$, one loop is generated in polar coordinates when r goes from zero to positive values (or negative values) then back to zero.

Notice in the above graph of $\sin(3x)$ that $\sin(3x)$ goes from zero (at $x=0$) to one (at $x = \frac{\pi}{6}$) back to zero (at $x = \frac{\pi}{3}$)

So, one loop is generated in polar coordinates for θ in $[0, \frac{\pi}{3}]$



$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2}\left(\sin 3\theta\right)^2 d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2}\left(\frac{1-\cos 6\theta}{2}\right) d\theta \qquad \left[\sin^2\phi = \frac{1-\cos 2\phi}{2}\right]$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 6\theta\right) d\theta$$

$$= \left[\frac{1}{4}\theta - \frac{1}{4}\left(\frac{1}{6}\sin 6\theta\right)\right]_0^{\frac{\pi}{3}} = \frac{\pi}{12} - \frac{1}{24}(0) = \boxed{\frac{\pi}{12}}$$

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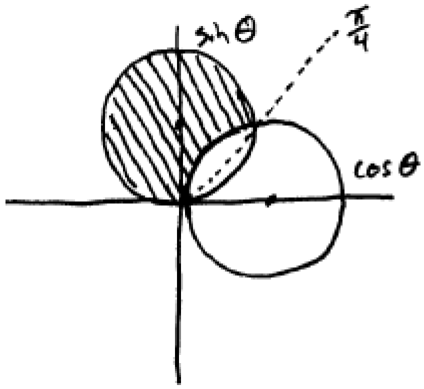
26) Inside $r = \sin\theta$ and outside $r = \cos\theta$

Inside $r = \sin\theta$ & outside $r = \cos\theta$



$$A_{circle} = \pi r^2 = \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$$

$$A_{wedge} = 2 \int_0^{\frac{\pi}{4}} \frac{1}{2} (\sin\theta)^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} \, d\theta$$

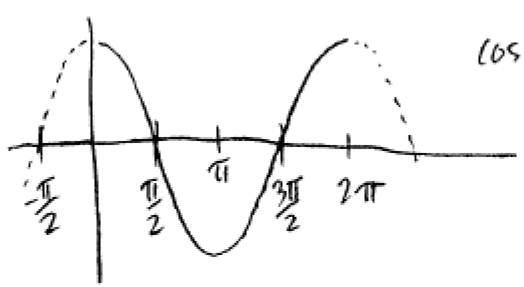
$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \cos 2\theta) d\theta$$

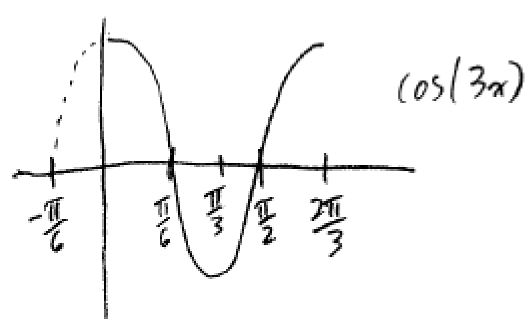
$$= \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$$

$$A_{circle} - A_{wedge} = \left(\frac{\pi}{4}\right) - \left(\frac{\pi}{8} - \frac{1}{4}\right) = \boxed{\frac{\pi}{8} + \frac{1}{4}}$$

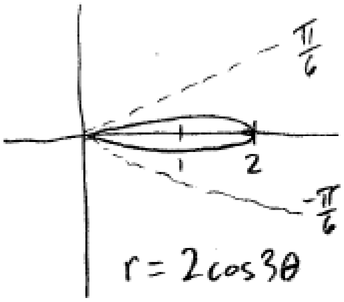
27) One loop of $r = 2\cos(3\theta)$

$r = 2\cos(3\theta)$ (one loop)





Again, note that one loop is generated in polar coordinates when r ranges from zero, to positive (or negative) values, then back to zero. Given the graph of $\cos(3x)$, we see that $r = 2\cos(3\theta)$ ranges from value zero to one and back to zero for θ in $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$



$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(2\cos 3\theta\right)^2 d\theta$$

$$= 2\int_{0}^{\frac{\pi}{6}} \frac{1}{2}\left(2\cos 3\theta\right)^2 d\theta$$

$$= 4\int_{0}^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta$$

$$= 4\int_{0}^{\frac{\pi}{6}} \frac{1+\cos 6\theta}{2}\, d\theta \qquad \left[\cos^2\varphi = \frac{1+\cos 2\varphi}{2}\right]$$

$$= 2\int_{0}^{\frac{\pi}{6}} \left(1+\cos 6\theta\right) d\theta$$

$$= 2\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big]_{0}^{\frac{\pi}{6}} = 2\left[\left(\frac{\pi}{6}+0\right)-\left(0+0\right)\right] = \boxed{\frac{\pi}{3}}$$

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