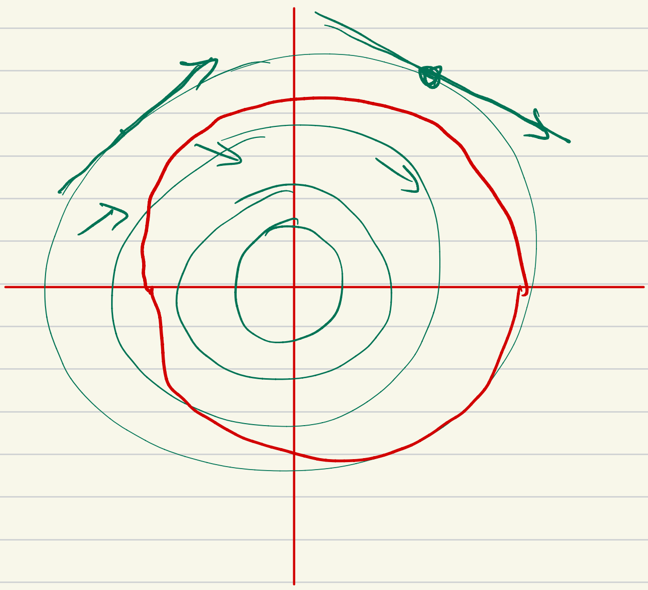
$$\frac{dy}{dt} = f(y)g(t) \Rightarrow \frac{dy}{f(y)} = g(t)dt$$

$$\frac{dy}{dt} = \frac{-t}{y} \Rightarrow y\,dy = -t\,dt$$



$$\int y\,dy = \int t\,dt + c$$

$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + c$$

$$y^2 + t^2 = 2c$$

$$y(t) = \pm\sqrt{2c - t^2}$$

① $(1+t^2)\frac{dy}{dt} = 1+y^2, \quad (\times dt)$

Hint: $\underline{\tan(x+y)} = \underline{\frac{\tan x + \tan y}{1-\tan x \tan y}}$

$$\underline{(1+t^2)}\,dy = \underline{(1+y^2)}\,dt \qquad \div\ (1+t^2)(1+y^2)$$

$$\frac{\cancel{(1+t^2)}\,dy}{\cancel{(1+t^2)}(1+y^2)} = \frac{\cancel{(1+y^2)}\,dt}{(1+t^2)\cancel{(1+y^2)}}$$

$$\int \frac{dy}{1+y^2} = \int \frac{dt}{1+t^2} + \cancel{c}\ \tan^{-1}(c)$$

* constant can be any function of constant

i.e. e^c, $\ln c$, $\sin c$

$$\tan^{-1}(y) = \tan^{-1}(t) + \tan^{-1} c$$

$$\underline{\tan}\,(\underline{\tan^{-1} y}) = \tan\Big(\underbrace{\tan^{-1}(t)}_{x} + \underbrace{\tan^{-1} c}_{y}\Big)$$

$$y(t) = \frac{\tan(\tan^{-1} t) + \tan(\tan^{-1} c)}{1 - \tan(\tan^{-1} t)\tan(\tan^{-1} c)}$$

$$y(t) = \frac{t+c}{1-ct}$$

(c) $\frac{dy}{dt} = (1+t)(1+y)$

$= \int t + \frac{t^2}{2}$

$n \neq -1$

$dy = (1+t)(1+y)\,dt \qquad \div (1+y)$

Recall $\int [g(t)]^n g'(t)\,dt$

$= \frac{[g(t)]^{n+1}}{n+1} + C$

$\frac{dy}{1+y} = (1+t)\,dt$

$= \int (1+t)'\,dt$

$= \frac{(1+t)^2}{2} + C$

$\int \frac{dy}{1+y} = \int (1+t)\,dt + \cancel{C}$ $\ln C$

$\ln(1+y) = \frac{(1+t)^2}{2} + \ln C$

$e^{\ln(1+y)} = e^{\frac{(1+t)^2}{2} + \ln C}$

$1+y = e^{\frac{1}{2}(1+t)^2} C$

$$\boxed{y(t) = C e^{\frac{1}{2}(1+t)^2} - 1}$$

(3) $\frac{dy}{dt} = 1 - t + y^2 - ty^2$

$\swarrow$ y^2 as Com. fac.

$\frac{dy}{dt} = (1-t) + y^2(1-t)$

$\swarrow$ $(1-t)$ as a comm. factor.

$\frac{dy}{dt} = (1+y^2)(1-t)$

$\int \frac{dy}{1+y^2} = -\int (1-t)\, dt + C$

$y = f(t)$ (explicit form)

$g(y, x) = c$ (implicit form)

$\tan^{-1} y = -\frac{1}{2}(1-t)^2 + C$ ✓ Implicit

$y(t) = \tan\left(-\frac{1}{2}(1-t)^2 + C\right)$ ✓ EX.

④ $\frac{dy}{dt} = e^{t+y+3}$

$$\frac{dy}{dt} = e^{t} e^{y} e^{3}$$

$$\frac{dy}{e^{y}} = e^{t+3}\,dt$$

$$-\int -e^{-y}\,dy = \int e^{t+3}\,dt + \overset{C}{\cancel{e^{C}}}$$

$$-e^{-y} = e^{t+3} + \overset{C}{\cancel{e^{C}}}$$

⑤ $\cos y \sin t \frac{dy}{dt} = \sin y \cos t$

$$\int \frac{g'(t)}{g(t)} dt = \ln|g(t)| + C$$

$$\cos y \sin t \, dy = \sin y \cos t \, dt$$

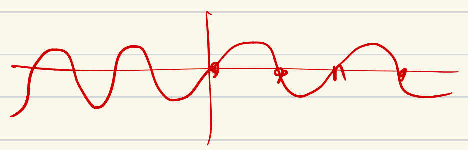
$$\int \frac{\cos y}{\sin y} dy = \int \frac{\cos t}{\sin t} dt + \ln C$$

$$\ln \sin y = \underline{\ln \sin t + \ln C} = \underline{\ln C \sin t}$$

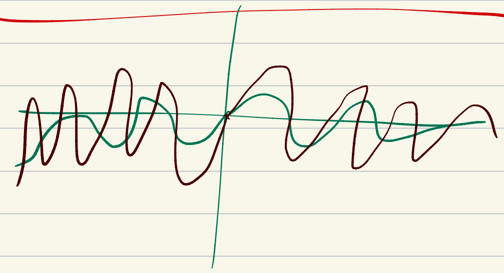
$$e^{\ln \sin y} = e^{\ln (C \sin t)}$$

$$\boxed{\sin y = C \sin t}$$

$\sin(t) = 0, \ t = n\pi, n \in \mathbb{Z}$



$$y(t) = \underline{\sin^{-1}(C \sin t)}$$



(6) $t^2(1+y^2)+2y\frac{dy}{dt}=0, \quad y(0)=1$

$$2y\frac{dy}{dt}=-t^2(1+y^2)$$

$$2y\,dy=-t^2(1+y^2)\,dt$$

$$\int\frac{2y}{1+y^2}\,dy=\int -t^2\,dt+c$$

$$\boxed{\ln|1+y^2|=\frac{-1}{3}t^3+c} \quad \text{G.S.}$$

$y(0)=1 \qquad t_0=0,\ y_0=1$

$$\ln(1+1^2)=\frac{-1}{3}(0)^3+c$$

$$\ln 2=c$$

$$e^{\ln(1+y^2)}=e^{\left(\frac{-1}{3}t^3+\ln 2\right)}$$

$e^{-\frac{1}{3}t^2+\ln 2}=e^{-\frac{1}{3}t^2}\cancel{e^{\ln 2}}$

$$1+y^2=2e^{-\frac{1}{3}t^3}$$

$$y^2=2e^{-\frac{1}{3}t^3}-1 \Rightarrow y(t)=\pm\sqrt{2e^{-\frac{1}{3}t^3}-1}$$

the Interval of existence

$$2e^{-\frac{1}{3}t^3}-1>0 \Rightarrow \ln e^{-\frac{1}{3}t^3}>\ln\frac{1}{2}$$

$$-\frac{1}{3}t^3>\ln\frac{1}{2} \Rightarrow t^3<-3\ln\frac{1}{2}=\ln\left(\frac{1}{2}\right)^{-3}$$

$$t^3\le\ln 8 \Rightarrow t\le\sqrt[3]{\ln 8}$$

$$t\in[0,\sqrt[3]{\ln 8}]$$

(7) $\frac{dy}{dt} = \frac{2t}{y+yt^2}$, $y(2) = 3$

$$dy = \frac{2t}{y(1+t^2)}\, dt$$

$$\int y\, dy = \int \frac{2t}{1+t^2}\, dt + C$$

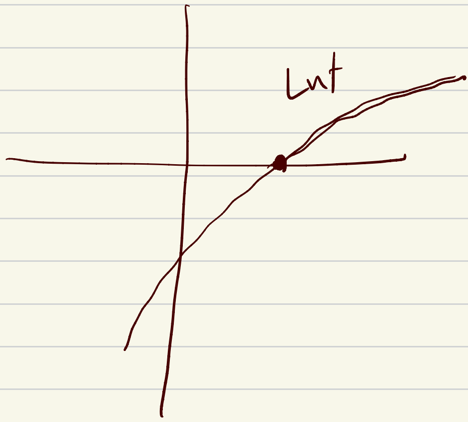
$y(2)=3$

$$\frac{1}{2} y^2 = \ln 1+t^2 + C \Rightarrow$$

$$\frac{1}{2}(3)^2 = \ln 1+2^2 + C$$

$$\frac{9}{2} = \ln 5 + C \Rightarrow C = \frac{9}{2} - \ln 5$$

$$\frac{1}{2} y^2(t) = \ln 1+t^2 + \frac{9}{2} - \ln 5$$



$$y(t) = \pm 2\sqrt{\ln 1+t^2 + \frac{9}{2} - \ln 5}$$

$$\ln 1+t^2 - \ln 5 + \frac{9}{2}$$

$$\ln \frac{1+t^2}{5} + \frac{9}{2} \geq 0$$

$$e^{\ln \frac{1+t^2}{5}} \geq e^{-\frac{9}{2}}$$

$$\frac{1+t^2}{5} \geq e^{-\frac{9}{2}} \Rightarrow 1+t^2 \geq 5e^{-\frac{9}{2}}$$

$$t^2 \geq 5e^{-\frac{9}{2}} - 1 \Rightarrow t \geq \sqrt{5e^{-\frac{9}{2}} - 1} \quad ?$$

(8) $(1+t^2)^{\frac{1}{2}} \frac{dy}{dt} = ty^3 (1+t^2)^{-\frac{1}{2}}$ $y(0)=1$

$$(1+t^2)^{\frac{1}{2}} dy = t\, y^3 (1+t^2)^{-\frac{1}{2}} dt$$

$$\int y^{-3} dy = \frac{1}{2}\int \frac{2t}{1+t^2} dt + C$$

$$-\frac{1}{2} y^{-2} = \frac{1}{2} \ln 1+t^2 + C$$

$y(0)=1$

$$-\frac{1}{2} \underbrace{(1)^{-2}}_{=1} = \frac{1}{2} \underbrace{\ln 1+0}_{=0} + C$$

$$C = -\frac{1}{2}$$

$$-\frac{1}{2} y^{-2}(t) = \ln\sqrt{1+t^2} - \frac{1}{2}$$

$$y^{-2}(t) = 1 - 2\ln\sqrt{1+t^2} = 1 - \ln 1+t^2$$

$$y^2(t) = \frac{1}{1-\ln 1+t^2}$$

$$y(t) = \pm \frac{1}{\sqrt{1-\ln 1+t^2}}$$

$$1-\ln 1+t^2 > 0$$

$$1+\ln\frac{1}{1+t^2} > 0 \Rightarrow e^1 \underline{e^{\ln\frac{1}{1+t^2}}} > 0$$

$\frac{1}{1+t^2} > 0 \Rightarrow$ the interval is $\mathbb{R}$

9. $\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)}$ $\quad y(0) = -1$

$$\int 2(y-1)\,dy = \int (3t^2+4t+2)\,dt + C$$

$$\cancel{2}\frac{(y-1)^2}{\cancel{2}} = t^3+2t^2+2t+C$$

$y(0)=1$

$$(-1-1)^2 = 0^3+2\cdot 0^2+2\cdot 0+C \Rightarrow C=4$$

$$(y-1)^2 = t^3+2t^2+2t+4$$

$$y-1 = \sqrt{t^3+2t^2+2t+4}$$

$$t^3+2t^2+2t+4 \geq 0$$

$t^3+2t^2+2t+4=0 \Rightarrow$ $t=-1 \Rightarrow -1+2-2+4$

$t=-2 \Rightarrow -8+8-4+4=0$

then $\overset{t+2}{t-(-2)}$ is a factor

$$\begin{array}{r|l} & t^2+2 \\ t+2 & t^3+2t^2+2t+4 \\ & \mp t^3 \mp 2t^2 \\ \hline & 2t+4 = 2(t+2) \end{array}$$

$$t^3+2t^2+2t+4 = (t+2)(t^2+2)$$

$$y-1 = \sqrt{(t+2)(t^2+2)} \geq 0$$

$$(t+2)(t^2+2) \geq 0 \Rightarrow t+2 \geq 0 \Rightarrow t \geq -2$$

$$t \in [-2, \infty)$$