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# M427J: Diffy Egn

June 6, 2025

$$\ln(y) - \frac{1}{\sqrt{t}} + y'' = \tan^{-1}(\frac{t}{1-y}) + y^{(3)} \quad (\text{Non-Linear})$$

What is  $y'(t)$ ?  
 $y' = y$  (1st order)  $y(t) = e^t, 0, C \cdot e^t$   
 $y^{(3)} = y$  (3rd Order)

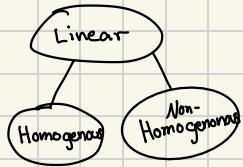
Non-Linear  
 $y \cdot y' = 0$

Linear vs. Non-Linear DEs

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = b$$

$$\frac{1}{t} y^{(n)} + e^t y^{(n-1)} + \dots + t^3 y' + 7 y = \arcsin(\sqrt{t}) \quad (\text{Linear DE})$$

$$y' = y \rightarrow -y' - y = 0$$



5th Order, Linear, homogeneous

$$y^{(5)} - \frac{1}{t^2} y'' + e^{t^2} y = 0$$

5th Order, Linear, Non-homogeneous

$$y^{(5)} - t^3 y' = t^2$$

$$\begin{cases} \text{Brown 1} \\ \text{I 2} \end{cases} \begin{cases} a(t) y' + b(t) y = 0 & 1^{\text{st}}, \text{homogeneous} \\ a(t) y' + b(t) y = g(t) & 1^{\text{st}}, \text{non-homogeneous} \end{cases}$$

## Systems of DEs

$$P'(t) = g(t) + f(t) \quad \text{More Functions}$$

$$g'(t) = 3f(t) \quad \text{More variables} \rightarrow u(x, t) \Rightarrow \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}$$

June 9, 2025 - Brown 1 2 (Linear 1st Order ODE)

$$\text{Find } y(t) \longrightarrow a(t) y' + b(t) y = g(t)$$

$$a_n(t) y^{(n)} + \dots$$

$$\text{First order, linear: } y' + a(t) y = b(t)$$

$$\text{homogen: } y' + a(t) y = 0$$

$$\text{Non-homogen: } y' + a(t) y = b(t) \neq 0$$

First order, homogen. ODEs:

$$y' + a(t) y = 0$$

$$(\ln(f(x)))' = \frac{1}{f(x)} \cdot f'(x)$$

$$\text{Proof #1: } y' + a(t) y = 0 \rightarrow \frac{y'}{y} = -a(t)$$

$$\rightarrow y' y = -a(t)$$

$$\rightarrow (\ln(y))' = -a(t)$$

$$\rightarrow \frac{d}{dt} (\ln(y(t))) = -a(t)$$

$$\rightarrow \int \frac{d}{dt} (\ln(y(t))) dt = \int -a(t) dt$$

$$\rightarrow \ln(y(t)) = \int -a(t) dt + C$$

$$e^{\ln(y(t))} = e^{\int -a(t) dt + C}$$

$$y(t) = e^{\int -a(t) dt + C} \equiv e^{\int -a(t) dt} \cdot e^C$$

$$= C e^{\int -a(t) dt}$$

$$\text{Example #1: } y' = y \rightarrow y' - y = 0$$

$$\equiv y' + \underline{a(t)y} = 0$$

$$a(t) = 1 \rightarrow (\ln y)' = 1$$

$$\rightarrow \ln y = t + C$$

$$\rightarrow y = e^t \rightarrow y = e^t \cdot e^C$$

$$y = C e^t \quad (\text{gen. solution})$$

$$\text{Example #2: } y' + 2t y = 0$$

$$y'/y = -2t$$

$$(\ln y)' = -2t$$

$$\ln y = -t^2 + C$$

$$y = e^{-t^2} \cdot e^C$$

$$y = A e^{-t^2}$$

I.V.P (Initial Value Problem):

$$\text{Exp. 1: } y' + 2t y = 0, \quad y(0) = 3$$

$$y = C e^{-t^2} \leftarrow \text{Solve ODE}$$

$$3 = C e^{-(0)^2} \quad y = 3e^{-t^2} \quad (\text{unique solution})$$

$$3 = C$$

$$\text{Exp 2: } y' + (\sin t) y = 0, \quad y(0) = \frac{3}{2} \quad (1^{\text{st}} \text{ Order, hom., IVP})$$

$$y' = -\sin t \cdot y$$

$$\int (\ln y)' = \int -\sin t \quad \frac{3}{2} = e^{\cos(t)} C$$

$$\ln y = \cos t + C$$

$$\frac{3}{2} = C$$

$$y = e^{\cos t \cdot C}$$

$$\frac{3}{2} e^{\cos t}$$

$$y = e^{\cos t \cdot A}$$

$$\frac{3}{2} e^{\cos t - 1}$$

Non-homogeneous, 1st Order ODE:  $y' + a(t) y = b(t)$

$$(f(t) \cdot g(t))' = f(t)g'(t) + f'(t)g(t) \quad (y')' = b(t)$$

$$(f \cdot g)' = f' \cdot g + g \cdot f$$

What if I could multiply by  $\mu(t)$ :

$$y' + a(t) y = b(t)$$

$$\mu(t) y' + \mu(t) a(t) y = \mu(t) b(t)$$

$$(\mu y)' = \mu(t) b(t)$$

$$? y' + ? y = \mu(t) b(t)$$

$$\text{If } ?' = a(t) \mu(t) = \mu'(t) \quad | \quad \text{If } \mu'(t) = a(t) \mu(t)$$

$$? = \mu(t)$$

$$\frac{\mu'(t)}{\mu(t)} = a(t)$$

$$(\log(\mu))' = a(t)$$

$$\ln \mu = \int a(t)$$

$$\mu = e^{\int a(t)}$$

$$\mu' = e^{\int a(t)} \cdot (\int a(t))'$$

$$= \mu \cdot a(t) \Rightarrow$$

$$y' + a(t) y = b(t) \neq 0$$

$$1^{\text{st}} \quad \mu(t) = e^{\int a(t)}$$

$$\mu y' + a(t) \cdot \mu \cdot y = \mu \cdot b(t)$$

$$(\mu \cdot y)' = \mu \cdot b(t)$$

$$\text{Example #3: } y' - 2t y = t \quad \mu = e^{\int -2t}$$

$$\mu = e^{-t^2}$$

$$y' e^{-t^2} - 2t e^{-t^2} y = t e^{-t^2}$$

$$(e^{-t^2} y)' = t e^{-t^2}$$

$$e^{-t^2} y = \int t e^{-t^2} dt + C$$

$$\text{let } u = -t^2 \quad -\frac{1}{2} du = -\frac{1}{2} t \quad \frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} u^{\frac{1}{2}} + C$$

$$e^{-t^2} y = -\frac{1}{2} t e^{-t^2} + C$$

$$e^{-t^2} y = -\frac{1}{2} t e^{-t^2} + C$$

$$\frac{e^{-t^2} y}{e^{-t^2}} = \frac{-\frac{1}{2} t e^{-t^2} + C}{e^{-t^2}}$$

$$y = -\frac{1}{2} t + \frac{C}{e^{t^2}}$$

$$= -\frac{1}{2} t + C e^{t^2}$$

$$y(t) \equiv -\frac{1}{2}$$

$$y' + 2t y = t, \quad y(1) = 2$$

$$\mu = e^{\int 2t dt} = e^{t^2}$$

$$e^{t^2} y' + 2e^{t^2} y = t e^{t^2}$$

$$(e^{t^2} y)' = t e^{t^2}$$

$$e^{t^2} y = \int t e^{t^2} dt + C = \frac{1}{2} e^{t^2} + C$$

$$y = \frac{1}{2} + \frac{C}{e^{t^2}} \equiv \frac{1}{2} + C e^{-t^2}$$

$$2 = \frac{1}{2} + C e^{-(1)^2} \quad \boxed{y = \frac{1}{2} + \frac{3e^{-t^2} + 1}{2}}$$

$$2 = \frac{1}{2} + C e^{-1}$$

$$\frac{3}{2} = C e^{-1}$$

$$\frac{3}{2} = C$$

$$y' + \frac{2t}{1+t^2} y = \frac{1}{1+t^2} \quad \text{let } u = 1+t^2$$

$$u = e^{\int \frac{2t}{1+t^2} dt} \quad du = 2t$$

$$u = e^{\ln(1+t^2)}$$

$$u = 1+t^2$$

$$(1+t^2)y' + \frac{2t}{1+t^2} y = \frac{1}{1+t^2}(1+t^2)$$

$$(1+t^2) \cdot y' + 1 = 1$$

$$(1+t^2) y = t + C$$

$$y = \frac{t + C}{1+t^2}$$

Brown 1.4 Separable

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$$\int f(y) dy = \int g(t) dt$$

$$\text{Ex 1. } \frac{dy}{dt} = \frac{t^2}{y^2}$$

$$\int y^2 dy = \int t^2 dt$$

$$\frac{y^3}{3} = \left(\frac{t^3}{3} + C\right)$$

$$y^3 = t^3 + 3C$$

$$\sqrt[3]{y^3} = \sqrt[3]{t^3 + C}$$

$$y = \sqrt[3]{t^3 + C}$$

$$\text{Example #2: } e^y \frac{dy}{dt} = -t^3, \quad y(1) = 1$$

$$e^y \frac{dy}{dt} = -t^3$$

$$\int e^y dy = \int -t^3 dt$$

$$e^y = \frac{1}{2}t^2 + \frac{1}{4} + C$$

$$y = \ln\left(\frac{1}{2}t^2 + \frac{1}{4} + C\right)$$

$$1 = \ln\left(\frac{1}{2} + \frac{1}{4} + C\right)$$

$$1 = \ln\left(\frac{1}{2} + \frac{1}{4} + C\right)$$

$$1 = \ln\left(\frac{3}{4} + C\right)$$

$$e^1 = \frac{3}{4} + C$$

$$e^1 - \frac{3}{4} = C$$

$$\text{Example #3: } y \frac{dy}{dt} + (1+y^2) \sin(t) = 0, \quad y(0) = 1$$

$$y \frac{dy}{dt} = -(1+y^2) \sin(t)$$

$$\int \frac{1}{1+y^2} dy = \int \sin(t) dt$$

$$\frac{1}{2} \ln(1+y^2) = \cos(t) + C$$

$$\text{Gen. sol.} \quad \frac{1}{2} \ln(1+y^2) = \cos(t) + \ln(\sqrt{2}) - 1$$

$$\text{Implicit Eqn}$$

## Discussion Section

June 10, 2025

$$\frac{dy}{dt} + a(t)y = b(t) \quad \text{if } b(t) = 0 \text{ then:}$$

$$\frac{dy}{dt} + a(t)y = 0 \quad (\text{separable})$$

$$\rightarrow \frac{dy}{dt} = -a(t)y$$

$$\frac{d\mu(t)}{dt} = \mu(t)a(t)$$

$$\frac{d(\mu(t))}{\mu(t)} = a(t) dt$$

$$\ln(\mu(t)) = \int a(t) dt$$

$$\mu(t) = e^{\int a(t) dt}$$

$$\begin{aligned} \text{Assume } \exists \mu(t) \text{ such that} \\ \mu(t) \frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t) \\ \frac{d}{dt}(\mu(t)y) = \mu(t)b(t) \end{aligned}$$

$$\mu(t)y = \int \mu(t)b(t) dt + C$$

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t) dt + C$$

Initial Condition vs. Boundary Condition

$$y(t_0) = y_0, \quad (t_0, y_0), \quad t_0? \quad y_0?$$

$$y(t_0) = \frac{1}{\mu(t_0)} \left( \int_{t_0}^{t_0} \mu(s) b(s) ds + C \right) = y_0$$

$$\rightarrow C = \mu(t_0)y_0 = \mu(t_0)y(t_0)$$

I.V.P.  $\equiv$  (D.E. + I.C.)

$$\frac{dy}{dt} + a(t)y = b(t), \quad y(t_0) = y_0$$

$$y(t) = \frac{1}{\mu(t)} \left( \int_{t_0}^t \mu(s) b(s) ds + \mu(t_0)y_0 \right)$$

Question #1:  $\frac{dy}{dt} + t \cos(t) = 0$

O.D.E. is separable if

$$\begin{aligned} \frac{dy}{dt} &= \frac{g(t)}{f(y)} \quad \text{NO} \\ \int f(y) dy &= \int g(t) dt \\ \frac{dy}{dt} &= t \cdot y = \text{Yes} \\ &= \frac{t}{y} \end{aligned}$$

$$\begin{aligned} \ln(a \cdot b) &= \ln(a) + \ln(b) \\ \text{Question #2: } \frac{dy}{dt} + \sqrt{t} \sin(t) y &= 0 \end{aligned}$$

$$\text{Question #3: } \frac{dy}{dt} + \frac{2t}{1+t^2} y = \frac{1}{1+t^2}$$

$$\text{Question #1: } \frac{dy}{dt} + y = te^t$$

$$\text{Question #8: } \frac{dy}{dt} + \sqrt{1+t^2} y = 0, \quad y(0) = \sqrt{5}$$

$$\text{Question #5: } \frac{dy}{dt} + t^2 y = 1$$

$$\text{Question #6: } \frac{dy}{dt} + t^2 y = t^2$$

$$\text{Question #7: } \frac{dy}{dt} + \frac{1}{1+t^2} y = -\frac{t^3}{1+t^4} y$$

# Linear Algebra Concepts for Differential Equations

Vectors:

$$\vec{u}, \vec{v} \in \mathbb{R}^n$$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i$$

Matrices:

$$M_{r \times c}(\mathbb{R})$$

$$= \left\{ \begin{bmatrix} a_{ij} & \dots & a_{ic} \\ \vdots & \ddots & \vdots \\ a_{rn} & \dots & a_{rc} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \right\}$$

$$A \cdot B =$$

$$i^m \begin{bmatrix} \cdots & \cdots & \cdots \\ a & \cdots & a_c \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = i^m \begin{bmatrix} \cdots & \cdots & \cdots \\ 0 & & 0 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$A \in M_{r \times c}(\mathbb{R}), B \in M_{m \times n}(\mathbb{R})$$

If  $C = M$ , then

$$A \cdot B \in M_{r \times n}(\mathbb{R})$$

Example #1:

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 3(2) + 0(2) + (-1)5 \\ = 6 + 0 - 5 \\ = 1$$

Example:

$$\begin{bmatrix} -2 & 0 & 5 \\ 8 & 8 & 1 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$$

Complex Arithmetic:

$$z = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$$

$$\operatorname{Re}(z) = \alpha, \operatorname{Im}(z) = \beta$$

Example (multiplication):

$$(1+i) \cdot (3-2i) = 3-2i+3i-2i^2 \\ = 3+i+2 \\ = 5+i$$

Complex conjugate and inverse:

$$\overline{3-2i} = 3+2i \text{ (conjugate)}$$

$$\frac{1}{3-2i} \cdot \frac{(3+2i)}{(3+2i)} = \frac{3+2i}{9-4i^2} \text{ (Inversion)} \\ = \frac{3+2i}{13} = \frac{3}{13} + i \frac{2}{13}$$

Division of complex #'s:

$$\frac{1+i}{3-2i} \cdot \frac{(3+2i)}{(3+2i)} = \frac{3+2i+3i+2i^2}{9-4i^2} \\ = \frac{3+5i-2}{9+4} = \frac{1+5i}{13} \\ \equiv \frac{1}{13} + i \frac{5}{13}$$

Example:

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \in M_{3 \times 2}(\mathbb{R})$$

$$B = \begin{bmatrix} 2 & 8 & 1 \\ -2 & 0 & 4 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$$

$$A \cdot B = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 & 1 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 24 & 23 \\ -2 & 0 & 4 \\ -6 & -8 & 7 \end{bmatrix} \begin{array}{l} | -1 \cdot 2 + 2(-2) = -6 \\ | -1(8) + 2(0) = -8 \\ | -1(1) + 2(4) = 7 \end{array}$$

Computing the inverse of a square matrix

$$A \in M_n(\mathbb{R}) \quad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in M_n(\mathbb{R})$$

The inverse of  $A$  is an  $n \times n$  matrix  $B$  so that

$$A \cdot B = B \cdot A = I_n \Rightarrow B = A^{-1}$$

Let  $A \in M_n(\mathbb{R})$

$$[A|I_n] \rightarrow \dots$$

$$\rightarrow [I_n|A^{-1}]$$

$$\text{Example: } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

$$[A|I_n] = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 + 4R_1} \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1} A = I_n \text{ for correctness}$$

$$A \in M_n(\mathbb{R})$$

The following are equivalent:

$$1) \det(A) \neq 0$$

2)  $A$  is invertible (there is an  $A^{-1} \in M_n(\mathbb{R})$ )  $\Rightarrow$  non-singular

$$3) \text{RREF}(A) = I_n$$

$$\text{When } n=2, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Example: } A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1-2} \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

Homogeneous system:

$$\begin{aligned} -2x_1 - 4x_2 + 3x_3 - 13x_4 - 5x_5 &= 0 \\ -x_1 - 2x_2 + x_3 - 5x_4 - x_5 &= 0 \end{aligned} \quad \left\{ \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ -1 & -2 & 1 & -5 & -1 \end{bmatrix} = A \right.$$

$$\text{rref}(A) = \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ 1 & 2 & -1 & -5 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 & -5 & -1 \\ -2 & -4 & 3 & -13 & -5 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & -3 & -13 & -5 \\ -2 & 0 & -1 & -13 & -5 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & -3 & -13 & -5 \\ 0 & 0 & -2 & -13 & -5 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & -3 & -13 & -5 \\ 0 & 1 & -1 & -6.5 & -2.5 \end{bmatrix}$$

$$x_1 = x_2 \\ x_4 = x_1 \\ x_5 = x_1$$

$$x_3 = x_2 \\ x_4 = x_3 \\ x_5 = x_3$$

Free variables

$$x_2, x_3, x_4, x_5$$

$$\downarrow x_2, x_3, x_4, x_5$$

$$\text{rank}(A)=2$$

$$0 \ 0 \ 0 \ 0 \ 0$$

$$0 \ 0 \ 0 \ 0 \ 0$$

$$0 \ 0 \ 0 \ 0 \ 0$$

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Complex Arithmetic:

$$z = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$$

$$\operatorname{Re}(z) = \alpha, \operatorname{Im}(z) = \beta$$

Example (multiplication):

$$(1+i) \cdot (3-2i) = 3-2i+3i-2i^2 \\ = 3+i+2 \\ = 5+i$$

Matrices:

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$$A \cdot B =$$

$$i^m \begin{bmatrix} \cdots & \cdots & \cdots \\ a & \cdots & a_c \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = i^m \begin{bmatrix} \cdots & \cdots & \cdots \\ 0 & & 0 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$A \in M_{r \times c}(\mathbb{R}), B \in M_{m \times n}(\mathbb{R})$$

If  $C = M$ , then

$$A \cdot B \in M_{r \times n}(\mathbb{R})$$

Example:

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \in M_{3 \times 2}(\mathbb{R})$$

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The inverse of  $A$  is an  $n \times n$  matrix  $B$  so that

$$A \cdot B = B \cdot A = I_n \Rightarrow B = A^{-1}$$

Let  $A \in M_n(\mathbb{R})$

$$[A|I_n] \rightarrow \dots$$

$$\rightarrow [I_n|A^{-1}]$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1} A = I_n \text{ for correctness}$$

$$\text{When } n=2, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Homogeneous system:

$$\begin{aligned} -2x_1 - 4x_2 + 3x_3 - 13x_4 - 5x_5 &= 0 \\ -x_1 - 2x_2 + x_3 - 5x_4 - x_5 &= 0 \end{aligned} \quad \left\{ \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ -1 & -2 & 1 & -5 & -1 \end{bmatrix} = A \right.$$

$$\text{rref}(A) = \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ 1 & 2 & -1 & -5 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 & -5 & -1 \\ -2 & -4 & 3 & -13 & -5 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & -3 & -13 & -5 \\ 0 & 0 & -1 & -13 & -5 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & -3 & -13 & -5 \\ 0 & 1 & -1 & -6.5 & -2.5 \end{bmatrix}$$

$$x_1 = x_2 \\ x_4 = x_1 \\ x_5 = x_1$$

$$x_3 = x_2 \\ x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_2 \\ x_3 = x_2 \\ x_4 = x_3 \\ x_5 = x_3$$

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# Linear Algebra Concepts continued

## Vector Spaces:

$V_L, V_A$

$$V_{y''-y} = \{c_1 e^t + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R}\}$$

$$L[y] = y'' + y \quad (\text{Lin. ODE})$$

$$V_y'' = \{c_1 + c_2 \mid c_1, c_2 \in \mathbb{R}\}$$

$$\text{If } L[y] = g(t) \quad (\text{Non-homo.})$$

$$L[y] = 0 \quad (\text{homo.}) \quad A \in M_n(\mathbb{R}),$$

$$V_L = \{y(t) \in C(\mathbb{R}) \mid L[y] = 0\} \quad V_A = \{\vec{x}(t) \in C^\infty(\mathbb{R}, \mathbb{R}^n) \mid \frac{d}{dt} \vec{x} = A \vec{x}\}$$

$$= \{c_1 \cos t + c_2 \sin t \mid c_1, c_2 \in \mathbb{R}\}$$

$$V_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \left\{ c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

$$V_{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \left\{ c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

## Subspaces:

$$A \in M_{r \times n}(\mathbb{R}) \quad \cap \left\{ \underbrace{\begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix}}_{C} \cdot \begin{bmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{bmatrix} \right\}$$

$$\text{Ker}(A) \equiv \text{Null}(A)$$

"kernel" "Nullspace"

$$= \{\vec{v} \in \mathbb{R}^r \mid A \cdot \vec{v} = \vec{0}\}$$

$$\text{Example: Let } A = \begin{bmatrix} 4 & -12 & -3 & 18 \\ -1 & 3 & 1 & -5 \\ 1 & -3 & 2 & -1 \\ 1 & -12 & -3 & 18 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 1 & 3 & 1 & -5 \\ -1 & 3 & 2 & -1 \\ 1 & -12 & -3 & 18 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 3 & -6 \\ -1 & 3 & 2 & -1 \\ 1 & -12 & -3 & 18 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & -2 \\ 1 & -12 & -3 & 18 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 3x_2 + 3x_4 &= 0 \\ x_2 &= x_2 \\ x_3 - 2x_4 &= 0 \\ x_4 &= x_4 \end{aligned} \quad \begin{aligned} x_1 &= 3x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned} \quad \left\{ \begin{aligned} &= x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \end{aligned} \right.$$

$$\text{Ker}(A) = \left\{ x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \mid x_2, x_4 \in \mathbb{R} \right\}$$

# Application: Population Modeling

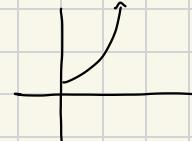
$y(t)$  = # of individuals at time  $t$ .  
 $y'$  = rate of growth of a population  
 $y'$  related to  $y$  ( $y' \leftrightarrow y$ )  
 $y' = k \cdot y$  Model

To solve:  
 $y' - ky = 0 \rightarrow y'/y = k$   
 $\int (\ln y)' = \int k$   
 $\ln y = kt + C$   
 $y = Ce^{kt} \rightarrow y(0) = C = P_0$   
 $\sim P(t) = P_0 e^{kt}$

A mathematical model is a differential equation.

A, B

A is directly proportional to B,  $A = kB$   
A is inversely proportional to B,  $A = \frac{k}{B}$



Exponential Pop. Growth

$$y' = ky \rightarrow y = Ce^{kt}$$

Carrying capacity =  $N$  (the max capacity of the environment).

$$\begin{cases} y' = ky, y(0) \\ y' = ky(1-y/N) \end{cases} \quad k = \text{growth rate}$$

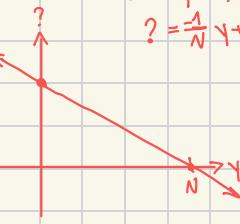
$$\begin{cases} y' = 0, y \sim N \\ y' = 0, y \sim 0 \end{cases}$$

$y' = k \cdot y$  (?) when  $y \sim 0, ? = 1$

when  $y \sim N, ? = 0$   
 $? = \frac{1}{N} y + 1 \approx 1 - y/N$

$$\begin{aligned} y' &= y(1-y) = y - y^2 \\ \frac{dy}{dt} &= ky(1-y/N) \quad \left\{ \begin{array}{l} \frac{1}{y(1-y/N)} dy = \sqrt{k} dt \\ = kt + C \end{array} \right. \\ y &= \frac{N}{N + e^{-kt} + C} \end{aligned}$$

Logistic Growth

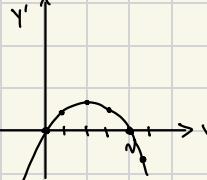


$$y' = \frac{dy}{dt} = ky(1-y/N)$$

$$y' = y(1-y)$$

$$= y - y^2$$

$$y' = ky(1-y/N)$$



Graph  $y'$  as a function of  $y$ , value of  $y$

where  $y' = 0$ :

$$y(t) \equiv 0, y(t) \equiv N$$

Solutions to the ODEs

Equilibrium points.

Phase Line

Unstable eqn.

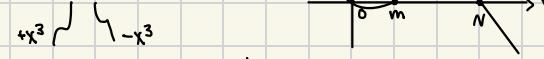
$m$  is minimum population for reproduction, allee

$$y' = ky(1-y/N) \text{ if } y > m, \text{ then } y' > 0$$

$$y < m, \text{ then } y' < 0$$

$$y' = ky(1-y/N) \cdot (y-m)$$

$$= \frac{k}{N} y^3 + \dots$$



Or pts  
 $0, m, N$

stable  
unstable

Braun 2.0  $\rightarrow$  2<sup>nd</sup> order ODEs and IVPs (Linear)

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

$$\text{or } y'' + p(t)y' + q(t)y = g(t)$$

$$ay'' + by' + cy = g(t)$$

$$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$$

$$\text{Example #1: } y'' - y = 0 \quad a=1, b=0, c=-1$$

$$(1)y'' + (0)y' + (-1)y = 0$$

$$y'' = y \mapsto y = \underbrace{e^{rt}}_{C_1} + \underbrace{e^{-rt}}_{C_2}$$

$C_1 y_1 + C_2 y_2$

Linear combinations of  $y_1, y_2$

$$y = Ce^{rt}, de^{rt}, a \cdot \sinh(rt), b \cosh(rt)$$

$$y' = \cosh(rt), y'' = \sinh(rt)$$

$$y = \sinh(rt), y' = \cosh(rt)$$

$$(Ce^t + Ce^{-t}) + a \sinh(rt) + b \cosh(rt)$$

$$\sinh(rt) = \frac{e^{rt} - e^{-rt}}{2}, \cosh(rt) = \frac{e^{rt} + e^{-rt}}{2}$$

$$e^t + e^{-t} \mapsto y = -7e^t + 2e^{-t}$$

$$y' = -7e^t - 2e^{-t}$$

$$y'' = -7e^t + 2e^{-t} = y$$

$$y'' = y$$

$$y'' - y = 0 \quad y_1 = e^t, y_2 = -e^{-t}$$

$$y = C_1 y_1 + C_2 y_2 \mapsto Ce^t + C_2 e^{-t}$$

Wronskian:

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

$$= \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

If  $y_1, y_2$  solve  $ay'' + by' + cy = 0$  and  $W[\ln t, \sinh t] = (\ln t) \cos t - \frac{1}{4} \sin t$

$$W[y_1, y_2] \neq 0$$

$$\text{GOOD: } y = C_1 y_1 + C_2 y_2$$

if  $W[y_1, y_2] = 0$ , BAD, keep working

$$\text{Example: } y'' - y = 0 \quad y_1 = e^t, y_2 = -e^{-t}$$

$$W[e^t, -e^{-t}] = (e^t)(-e^{-t}) - (e^t)(-e^{-t})$$

$$= -e^{2t} + e^{2t} = 0 \quad \text{BAD}$$

$$\text{Try } y_1 = e^t, y_2 = e^{-t}$$

$$W[e^t, e^{-t}] = (e^t)(e^{-t})' - (e^{-t})(e^t)'$$

$$= -1 - 1 \equiv -2 \quad \text{GOOD}$$

$$y = C_1 e^t + C_2 e^{-t} \quad \text{General sol.}$$

$$\text{to } y'' - y = 0$$

High level view:

$$ay'' + by' + cy = 0 \quad y = \ln(t)$$

$$y' = \frac{1}{t}$$

$$\text{Exponential fronts. } y'' = -\frac{1}{t^2}$$

Sine/cosine fronts.

Polynomial

$$\text{Guess: } y = e^{rt} \quad y' = e^{rt} \cdot r, y'' = r^2 e^{rt}$$

$$ar^2 e^{rt} + br e^{rt} + ce^{rt} = 0$$

$$\text{Characteristic polynomial } ar^2 + br + c = 0$$

Quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad r_1, r_2$$

$$e^{r_1 t}, e^{r_2 t}$$

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$$

$$W[y_1, y_2] \neq 0$$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Example:  $y'' + y' - 6y = 0 \rightarrow r^2 + r - 6 = 0$

$$(r+3)(r-2) = 0 \rightarrow r = \{-3, 2\}$$

$$y_1 = e^{2t}, y_2 = e^{-3t}$$

$$\begin{aligned} W[y_1, y_2] &= (e^{2t})(e^{-3t})' - (e^{2t})'(e^{-3t}) \\ &= e^{2t}(-3e^{-3t}) - (2e^{2t})e^{-3t} \\ &= -3e^{-t} - 2e^{-t} = -5e^{-t} \neq 0 \end{aligned}$$

$$y = C_1 e^{2t} + C_2 e^{-3t}$$

Example:  $y'' - y = 0 \rightarrow r^2 - 1 = 0$

$$(r+1)(r-1) = 0 \rightarrow r = -1, 1$$

$$y = C_1 e^t + C_2 e^{-t}$$

$$y(0) = 2, y'(0) = 1$$

$$y(0) = C_1 e^{(0)} + C_2 e^{(0)} = C_1 + C_2 = 2$$

$$y' = C_1 e^t - C_2 e^{-t}$$

$$y'(0) = C_1 e^{(0)} - C_2 e^{(0)} = C_1 - C_2 = 1$$

$$C_1 = \frac{3}{2}, C_2 = \frac{1}{2}$$

$$ay'' + by' + cy = 0$$

Discriminant:

$$\Delta = b^2 - 4ac$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Braun  $\left\{ \begin{array}{l} \Delta > 0 \text{ 2 real roots} \\ 2.2 \quad \left\{ \begin{array}{l} r_1, r_2 \\ y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \end{array} \right. \end{array} \right.$

Braun  $\left\{ \begin{array}{l} \Delta < 0 \text{ complex roots} \\ 2.2.1 \end{array} \right.$

Braun  $\left\{ \begin{array}{l} \Delta = 0 \text{ double roots} \\ 2.2.2 \end{array} \right.$

Example:  $y'' + y = 0 \rightarrow r^2 + 1 = 0 \rightarrow r^2 = -1$

$$z(t) = e^{it} = \cos t + i \sin t \quad r = \pm \sqrt{1} = \pm i$$

$$y_1 = \cos t$$

$$y_2 = \sin t$$

$$y' = -\sin t \quad y'' + y = -\cos t + \cos t = 0$$

$$y'' = -\cos t \quad y'' + y = -\sin t + \sin t = 0$$

$$y = \sin t$$

$$W[\cos t, \sin t] = (\cos t)'(\sin t) -$$

$$y' = \cos t$$

$$(\cos t)'(\sin t)$$

$$y'' = -\sin t \quad = \cos t \cos t - \sin t \sin t$$

Example:  $y'' + 2y' + 2y = 0 \rightarrow r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4-4 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i \rightarrow \alpha \pm i\beta$$

Pick  $r = -1 + i$

$$z(t) = e^{(-1+i)t} = e^{-t} \cdot e^{it}$$

$$z(t) = \frac{\cos t + i \sin t}{e^t}$$

$$y = \frac{C_1 \cos t + C_2 \sin t}{e^t} \equiv e^{-t} C_1 \cos t + e^{-t} C_2 \sin t$$

$$r = \alpha \pm i\beta$$

$$\rightarrow y = C_1 e^{-t} \cos bt + C_2 e^{-t} \sin bt \quad \left\{ \text{Short-cut!} \right.$$

Example:  $y'' - 4y' + 13 = 0$

$$r^2 - 4r + 13 = 0 \quad r = \frac{4 \pm \sqrt{16-4 \cdot 13}}{2}$$

$$r = \frac{4 \pm 2\sqrt{13}}{2}$$

$$= 2 \pm 3i$$

$$\alpha \pm \beta i$$

Example:  $y'' + y' + y = 0 \quad r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

Example:  $y'' + 4y' + 4y = 0 \rightarrow r^2 + 4r + 4 = 0$

$$y_1 = e^{-2t} \quad (r+2)(r+2) = 0$$

$$f(t)v'' + g(t)v' = 0 \quad v = -2 \quad (\text{double})$$

$$\rightarrow y_2 = V(t) y_1(t)$$

$$y_2 = t e^{-2t}$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t}$$

Review:

$$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$$

$$ar^2 + br + c = 0 \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (3) \Delta = 0, r \text{ double roots}$$

$$\text{Discriminant: } \Delta = b^2 - 4ac$$

$$(1) \Delta > 0, r_1, r_2 \text{ real}$$

$$(2) \Delta < 0, \alpha \pm i\beta \text{ complex}$$

$$y = C_1 e^{\alpha t} + C_2 e^{\alpha t} \cos(\beta t)$$

$$y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$$

$$\frac{1+ti}{1-2i} \cdot \frac{(1+2i)}{(1+2i)} = \frac{1+ti+2i+(i)(2i)}{1-2i+2i-(2i)(2i)} = \frac{1+3i+2i^2}{1-4i^2} = \frac{1+3i-2}{1+4} = \frac{-1+3i}{5}$$

$$C = \left\{ \alpha + i\beta \mid \alpha, \beta \in \mathbb{R} \right\} = \left( -\frac{1}{5} \right) + i \left( \frac{3}{5} \right)$$

$$y'' - y = +$$

$$y = e^t \quad y = e^t \quad y'' = e^t$$

$$y'' - y = e^t - e^t = 0$$

Particular sol.

$$y = -t + C_1 e^t \rightarrow y^* \text{ or } y_p$$

$$y = -t + C_1 e^t + C_2 e^{-t}$$

solutions  $\rightarrow$  Comp./Hom.

$$y'' - y = 0$$

sol.

$$= \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right) + i \left( -\frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right)$$

$$= \cos t + i \sin t \quad \text{Euler's Formula.}$$

$$e^{it} = \cos t + i \sin t \quad \theta \text{ real}$$

$ay'' + by' + cy = q(t)$     2.3 How to write solutions

### 2.5 Solving

$$y'' - y = + \quad y = \text{trig } x \quad 1 \text{ solution}$$

$$X \quad y = t^2 \quad y' = 2t \quad y'' = 2 \quad y = -t, y' = -1, y'' = 0 \quad \checkmark$$

$$y'' - y = 0 = 2 - t^2 \neq + \quad y'' - y = 0 - (-t) = + = +$$

$$y = t, y' = 1, y'' = 0$$

$$X \quad y'' - y = 0 - t = -t \neq +$$

$$y = -t + e^t$$

$$y' = -1 + e^t$$

$$y'' = 0 + e^t$$

$$y = -t + e^t \checkmark$$

$$y'' - y =$$

To solve  $a\psi'' + b\psi' + c\psi = g(t)$

1. Find  $\psi_c$ , solve  $a\psi'' + b\psi' + c\psi = 0$

2. Find  $\psi_p$ , 1 particular solution

3.  $\psi = \psi_p + \psi_c = \psi_p + C_1\psi_1 + C_2\psi_2$

Example:  $\psi'' + \psi = t$

$$1. \psi'' + \psi = 0 \rightarrow r^2 + 1 = 0, r = \pm i \rightarrow \alpha = 0, \beta = 1$$

$$\psi_c = C_1 e^{it} \cos(t) + C_2 e^{it} \sin(t) \\ = C_1 \cos(t) + C_2 \sin(t)$$

$$2. \psi = t, \psi' = 1, \psi'' = 0$$

$$\psi'' + \psi = 0 + t = t \rightarrow \psi_p = \psi = t$$

$$3. \psi = \psi_p + \psi_c = t + C_1 \cos(t) + C_2 \sin(t)$$

Example:  $\psi'' + \psi = t^2$

$$1. \psi'' + \psi = 0 \rightarrow \psi_c = C_1 \cos(t) + C_2 \sin(t)$$

$$2. \psi = t^2, \psi' = 2t, \psi'' = 2$$

$$\psi = t^2 - 2, \psi' = 2t, \psi'' = 2$$

$$\psi'' + \psi = 2t + t^2 - 2 = t^2 \checkmark$$

$$3. \psi = \psi_p + \psi_c$$

$$\text{Example: } \psi'' + \psi' - 2\psi = t^2 \quad r^2 + r - 2 = 0$$

$$1. \psi'' + \psi' - 2\psi = 0 \quad r = 1, -2$$

$$(r+2)(r-1) = 0$$

$$\psi_c = C_1 e^{-2t} + C_2 e^{t}$$

$$2. \psi = t^2, \psi' = 2t, \psi'' = 2$$

$$\psi'' + \psi' - 2\psi = 2t + 2t^2 - 2t^2 = t^2$$

$$\psi = t^2 - 2, \psi' = 2t, \psi'' = 2$$

$$\psi'' + \psi' - 2\psi = 2t + 2t^2 - 2(t^2 - 2)$$

$$= 2 + 2t - 2t^2 + 4$$

$$= -2t^2 + 2t + 6$$

$$2. \psi = A t^2 + B t + C$$

$$\psi' = 2At + B$$

$$\psi'' = 2A$$

$$2A + 2At + B$$

$$\psi'' + \psi' - 2\psi = -2(A t^2 + B t + C)$$

$$= -2At^2 + (2A - 2B)t + (2A + B - 2C) = t^2 + 0 \cdot t + 0$$

$$-2A = 1 \quad 2A - 2B = 0 \quad -2C = 0$$

$$A = -\frac{1}{2}$$

$$A = B$$

$$B = -\frac{1}{2}$$

$$-2C = \frac{3}{2}$$

$$C = -\frac{3}{4}$$

$$\text{Example: } \psi'' - 6\psi' + 9\psi = 3t^2 - 4t + 1$$

$$\psi'' - 6\psi' + 9\psi = 0 \rightarrow r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0 \quad r = 3, \text{ double}$$

$$\psi = \psi_c + \frac{1}{27}$$

$$\rightarrow \psi_c = C_1 e^{3t} + C_2 t e^{3t} \quad ③ \quad \psi = \psi_c + \frac{1}{27} = \frac{1}{3} t^2 + \frac{1}{27} + C_1 e^{3t} + C_2 t e^{3t}$$

$$2. \psi = A t^2 + B t + C \quad \psi' = 2At + B \quad \psi'' = 2A$$

$$\psi'' - 6\psi' + 9\psi = -6(2At + B) + 9(A t^2 + B t + C)$$

$$+ 9(A t^2 + B t + C) = 2(\psi_c) - 6(0) + (\psi_c) = 1$$

$$y'' - 2y' + 2y = (2t-1)e^{-t} \quad \text{R.H.S}$$

$$\textcircled{1} \quad y'' - 2y' + 2y = 0 \rightarrow r^2 - 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = 1 \pm i \rightarrow \alpha \pm i\beta$$

$$y_1 = e^{\alpha t} \cos \beta t$$

$$y_2 = e^{\alpha t} \sin \beta t$$

$$y_c = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\textcircled{2} \quad y = (At+B)e^{-t}$$

$$y'' - 2y' + 2y$$

$$y' = Ae^{-t} + (At+B)(-e^{-t})$$

$$= [(-A)t + (A-B)]e^{-t}$$

$$y'' = (-A)e^{-t} - [(-A)t + (A-B)]e^{-t}$$

$$= [(A)t + (-2A+B)]e^{-t}$$

$$+ [(2A) + (-2A+2B)]e^{-t} \cancel{-2y'}$$

$$+ [(2A)t + (2B)]e^{-t} \cancel{-2y}$$

$$[(5A)t + (-4A+3B)]e^{-t} \leftarrow \text{L.H.S}$$

$$= [2(t) + (-1)]e^{-t} \rightarrow \begin{bmatrix} 5 & 0 & 2 \\ -4 & 5 & -1 \end{bmatrix} \text{ RREF} \rightarrow \begin{matrix} A = 2/5 \\ B = 3/25 \end{matrix}$$

$$y = (\frac{2}{5}t + \frac{3}{25})e^{-t}$$

$$\textcircled{3} \quad y = \Psi + y_c$$

$$y(t) = (\frac{2}{5}t + \frac{3}{25})e^{-t} + C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\text{Example: } y'' - 2y' + 2y = \cos t$$

$$\textcircled{1} \quad y_c = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\textcircled{2} \quad \Psi = A \cos t + B \sin t$$

$$y' = -A \sin t + B \cos t$$

$$y'' = -A \cos t - B \sin t$$

$$\begin{aligned} y'' - 2y' + 2y &= -A \cos t - B \cos t + A \cos t + B \sin t \\ &+ 2A \sin t - 2B \cos t \cancel{-2y'} \\ &+ 2A \cos t + 2B \sin t \cancel{-2y} \end{aligned}$$

$$(A-2B) \cos t + (2A+B) \sin t \cancel{\downarrow \text{LHS}}$$

$$= (1) \cos t + (0) \sin t$$

$$\rightarrow \begin{bmatrix} 1 & -2 & | & 1 \\ 2 & 1 & | & 0 \end{bmatrix} \rightarrow \text{RREF} \rightarrow \begin{matrix} A = 1/5 \\ B = -2/5 \end{matrix}$$

$$\textcircled{3} \quad \Psi = \frac{1}{5} \cos t - \frac{2}{5} \sin t$$

$$y = \Psi + y_c$$

$$y = \frac{1}{5} \cos t - \frac{2}{5} \sin t + C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\text{Example: } y'' - y' - 2y = (\frac{1}{2}t^3 + \sqrt{2}t)e^{-7t} \sin(\pi t)$$

$$\textcircled{1} \quad y'' - y' - 2y = 0$$

$$r^2 - r - 2 = 0 \quad r = 2, -1$$

$$(r-2)(r+1) = 0 \quad y_c = C_1 e^{2t} + C_2 e^{-t}$$

$$\textcircled{2} \quad \Psi = (A t^3 + B t^2 + C t + D) e^{-7t} \cos(\pi t)$$

$$+ (E t^3 + F t^2 + G t + H) e^{-7t} \sin(\pi t)$$

Edge cases:

$$y'' - y = e^t$$

$$\textcircled{1} \quad y'' - y = 0 \rightarrow r^2 - 1 = 0$$

$$(r+1)(r-1) = 0 \quad r = 1, -1$$

$$y_c = C_1 e^t + C_2 e^{-t} \rightarrow \text{Double trouble}$$

$$\textcircled{2} \quad \Psi = A \cdot e^t, \quad \Psi' = A e^t, \quad \Psi'' = A e^t$$

$$y'' - \Psi \cancel{= A e^t} \quad A e^t = C_1 e^t \leftarrow \text{L.H.S}$$

$$= e^t \leftarrow \text{R.H.S}$$

$$y'' + 2y' + 2y = t e^{-t} \cos t$$

$$\textcircled{1} \quad y'' + 2y' + 2y = 0 \rightarrow r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-4 \cdot 2}}{2} = -1 \pm i$$

$$y_c = C_1 e^{-t} \cos \theta + C_2 e^{-t} \sin \theta$$

$$\textcircled{2} \quad \Psi = (At+B)e^{-t} \cos t +$$

$$y'' - 2y' + 2y = t e^{-t} \cos t$$

$$\textcircled{1} \quad y_c = C_1 e^{-t} \cos \theta + C_2 e^{-t} \sin \theta$$

Trouble!

$$\textcircled{2} \quad \Psi = \{(At+B)e^{-t} \cos t + (Ct+D)e^{-t} \sin t\} +$$

$$= (At^2 + Bt + Ct^2 + Dt)e^{-t} \cos t + (Ct^2 + Dt + Ct + D)e^{-t} \sin t$$

$$y'' - 2y' + y = e^t$$

$$\textcircled{1} \quad y'' - 2y' + y = 0 \rightarrow r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0 \quad r = 1$$

$$y_c = C_1 e^t + C_2 t e^t$$

$$\textcircled{2} \quad \Psi = A e^t \times \text{Trouble}$$

$$\Psi = A t e^t \times \text{Double trouble}$$

$$\Psi = At^2 e^t$$

Review for Exam 1

June 21, 2025

$$\begin{aligned} 5A + 2B &= 1 \\ 2A - B &= 0 \end{aligned} \rightarrow \begin{bmatrix} 5 & 2 & | & 1 \\ 2 & -1 & | & 0 \end{bmatrix}$$

$$\text{GOAL} \rightarrow \begin{bmatrix} 1 & 0 & | & A \\ 0 & 1 & | & B \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & | & X \\ 0 & 1 & | & Y \end{bmatrix}$$

$$1 \cdot A + B \cdot 0 = X$$

$$0 \cdot A + B \cdot 1 = Y$$

$$\begin{aligned} x + z &= 1 \\ 3x + 6y - 3z &= 3 \\ -2x - 3y + 3z &= 2 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 1 & | & 1 \\ 3 & 6 & -3 & | & 3 \\ -2 & -3 & 3 & | & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ -2 & -3 & 3 & | & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 1 & 1 & | & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -3 & | & -7 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \text{ (RREF)}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 - 3z \\ 4 - 2z \\ z \end{bmatrix} \rightarrow \begin{bmatrix} -7 \\ 4 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & -3 & | & -7 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Parametric form

$$x_1 - 3x_2 = -3$$

$$x_2 = x_2$$

$$x_3 = 2$$

$$\begin{aligned} x_1 &= -3 + 3x_2 \\ x_2 &= x_2 \\ x_3 &= 2 \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 + 3x_2 \\ x_2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

2<sup>nd</sup> Order non-homogeneous:

$$ay'' + by' + cy = g(t) \leftarrow p(t)$$

$$p(t)e^{at}$$

$$p(t)e^{at}(\sin \beta t + \cos \beta t)$$

$$\textcircled{1} \quad ay'' + by' + cy = 0$$

$$ar^2 + br + c = 0 \rightarrow y_c = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\Delta > 0$$

Guess  $\Psi$  (pdf)

$$\Delta < 0$$

'Check for trouble'

Grind

$$\textcircled{3} \quad \Psi = \Psi + \Psi_c$$

$$y'' + 2y' + 2y = 3e^{-t} \cos t$$

$$r^2 + 2r + 2 = 0 \leftrightarrow y'' + 2y' + 2y = 0$$

$$r = -2 \pm \sqrt{-2+4} = -1 \pm i$$

$$y_c = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$$

$$\begin{aligned} \textcircled{2} \quad \Psi &= (Ae^{-t} \cos(t) + Be^{-t} \sin(t)) + \\ &= Ate^{-t} \cos(t) + Bt e^{-t} \sin(t) \end{aligned}$$

$$(r-3)^2 = r^2 - 6r + 9$$

$e^{3t}$  not allowed

$$y'' - 6y' + 9y = (-t+2)e^{2t}$$

$$\textcircled{1} \quad y'' - 6y' + 9y = 0 \rightarrow r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0 \rightarrow r=3, \text{ double}$$

$$y_c = C_1 e^{3t} + C_2 te^{3t}$$

$$\textcircled{2} \quad \Psi = (At + B)e^{2t}$$

$$\Psi' = Ae^{2t} + 2(At+B)e^{2t}$$

$$= [(2A)t + (A+2B)]e^{2t}$$

$$\Psi'' = 2Ae^{2t} + 2[2At + (A+2B)]e^{2t}$$

$$= [(4A)t + (4A+4B)]e^{2t}$$

$$\Psi'' - 6\Psi' + 9\Psi = [(4A)t + (4A+4B)]e^{2t}$$

$$+ [(-12A)t + (-6A-12B)]e^{2t}$$

$$+ [(9A)t + (9B)]e^{2t}$$

$$[(1A)t + (-2A+B)]e^{2t} =$$

$$[(-1)t + (2)]e^{2t} \quad \Psi = (-t)e^{2t}$$

$$\begin{aligned} A &= -1 \\ -2A + B &= 2 \end{aligned}$$

$$\Psi = \Psi + \Psi_c = -te^{2t} + C_1 e^{3t} + C_2 te^{3t}$$

Lecture: June 27, 2025

System of ODEs has more functions

$$f(x), g(x) \quad f' = \frac{1}{g} + fF \quad f' = Af + B \cdot g$$

$$1^{\text{st}} \text{ Order, Linear, Homo., } g' = x^3 - g \cdot f \quad g' = Cf + D \cdot g$$

and constant coefficient

Variable change:

$$\begin{aligned} f, g, h \\ f' = Af + Bg + Ch \end{aligned}$$

$$\begin{aligned} x_1(t), x_2(t) \quad \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Rightarrow \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} \\ x'_1(t) = Ax_1 + Bx_2 \quad \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix} \quad \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \\ x'_2(t) = Cx_1 + Dx_2 \end{aligned}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \vec{x}$$

A system looks like:

$$\begin{aligned} x'_1 &= -\frac{1}{2}x_1 + \frac{1}{2}x_2 \\ x'_2 &= \frac{1}{2}x_1 - \frac{1}{2}x_2 \end{aligned} \quad \text{or} \quad \frac{d}{dt} \vec{x} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \vec{x} \Rightarrow \frac{d}{dt} \vec{x} = A \vec{x}, A \in M_2(\mathbb{R})$$

$$\vec{x}(0) = \vec{v} \in \mathbb{R}^2$$

For an I.V.P., we need  $x_1(0) = ?$  and  $x_2(0) = ?$

$$\text{System} \quad \frac{d}{dt} \vec{x} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \vec{x}$$

$$\text{I.V.P.: } \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then the solution is  $\vec{x}^1 = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{t} \\ \frac{1}{2} - \frac{1}{2}e^{t} \end{bmatrix}$

$$\text{I.V.P.: } \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \vec{v} \in \mathbb{R}^2 \quad \text{solution is } \vec{x}^2 = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-t} \\ \frac{1}{2} + \frac{1}{2}e^{-t} \end{bmatrix}$$

General solution for the example:

$$\vec{x}(t) = C_1 \vec{x}^1(t) + C_2 \vec{x}^2(t) = C_1 \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{t} \\ \frac{1}{2} - \frac{1}{2}e^{t} \end{bmatrix} + C_2 \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-t} \\ \frac{1}{2} + \frac{1}{2}e^{-t} \end{bmatrix}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} \quad (\text{Case 1})$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} \rightarrow \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + 0x_2 \\ 0x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 3x_2 \end{bmatrix}$$

$$x'_1 = -2x_1 \quad \frac{x'_1}{x_1} = -2$$

$$(ln x_2)' = 3 \quad (ln x_1)' = -2$$

$$\ln x_2 = 3t + C \quad \ln x_1 = -2t + C$$

$$x_2 = e^{3t+C} \quad x_1 = e^{-2t+C}$$

$$\begin{aligned} \text{Gen. solution: } \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-2t+C} \\ e^{3t+C} \end{bmatrix} \\ &= C \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} + D \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} \end{aligned}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \vec{x}$$

$$\vec{x} = C_1 \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{\frac{1}{2}t} \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ \sqrt{3}e^{\frac{1}{2}t} \end{bmatrix}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} \rightarrow \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x'_1 = x_2 \quad \leftarrow \quad x'_2 = 2x_1 - x_2$$

$$x'_1 = (x_1)' = 2x_1 - x_1 \rightarrow x''_1 = 2x_1 - x'_1$$

$$x''_1 + x'_1 - 2x_1 = 0 \quad x''_1 + x'_1 - 2x_1 = 0$$

$$r^2 + r - 2 = 0 \quad (r+2)(r-1) = 0$$

$$r = 1, -2$$

$$x_1 = C_1 e^t + C_2 e^{-2t}$$

$$x_1 = e^t \text{ or } e^{-2t}$$

## Lecture (Linear Algebra)

July 2<sup>nd</sup>, 2025

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$$

$\mathbb{R}^n$  with vector addition and scalar multiplication

$$\mathbb{R}^2: \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \quad 2 \in \mathbb{R}, \quad 2 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

$$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\}$$

$$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

$\mathbb{R}^n$  is a vector space

$$P_3 = \{p(t) \mid \deg(p(t)) \leq 3\}$$

$$= \{a+bt+ct^2+dt^3 \mid a, b, c, d \in \mathbb{R}\}$$

$$\dim(P_3) = 4$$

$$P = \{p(t) \mid p(t) \text{ polynomial}\}$$

$$\dim(P) = \infty \leftrightarrow P_2 \subseteq P, P_3 \subseteq P, \dots$$

$$P_2 = \{a+bt+c t^2 \mid a, b, c \in \mathbb{R}\}$$

$$= \{p(t) \mid p(t) \text{ polynomial} \}$$

$$\deg(p(t)) \leq 2 \quad \dim(p(t)) = 3, \mathbb{R}^3$$

$$(-t^2 + 3t + 1) + (t^2 + t)$$

$$= 4t + 1$$

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \dim(M_2(\mathbb{R})) = ? = 4$$

$$M_{2 \times 3}(\mathbb{R}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \mid a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R} \right\}$$

$$\dim(A) = 6$$

**Example:**  
 $C(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$   
 Real-valued functions  
 Vector Space  
 $f(x) + g(x)$  is continuous  
 $f(x), c \in \mathbb{R} \rightarrow cf(x)$  is continuous  
 $\dim(C(\mathbb{R})) = \infty$   $P \subseteq C(\mathbb{R})$   
 $C'(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}$   
 $(f+g)' = f'+g'$   
 $(cf)' = c \cdot f'$   
 $C^2(\mathbb{R}), C^3(\mathbb{R}), \dots, C^\infty(\mathbb{R})$   
 $e^x, \sin(x), \dots$   
 Solutions to linear differential equations live in  $C^\infty(\mathbb{R})$   
 Solutions to homogenous differential equations live in  $C^\infty(\mathbb{R})$   
 AND are vector spaces.  
 $V = \{y(t) \mid y'' - y = 0\}$   $r^2 - 1 = 0$   
 $\mathbb{R}^2 = \{c_1 e^t + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R}\} \subseteq C^\infty(\mathbb{R})$   
 Linear Algebra 5 → Vector Spaces  
 Linear Algebra 6 → Subspaces  
 Let  $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$   
 $\dim(V) = 2$   
 Example:  $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$   $A: \mathbb{R}^3 \rightarrow \mathbb{R}^2$   
 Think about  $\vec{v}$ , and  $A \cdot \vec{v}$ ;  $\vec{v} \in \mathbb{R}^3$   
 $A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2 \\ x+y+3z \end{bmatrix}$   
 $\text{Kernel}(A) = \{\vec{v} \in \mathbb{R}^3 \mid A \cdot \vec{v} = \vec{0} \in \mathbb{R}^2\}$   
 (Null(A))  
 What  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$  solve  
 $\begin{cases} x-2=0 \\ -x+y+3z=0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$   
 $R_2 \rightarrow R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$   
 $x-2=0 \rightarrow x=2$   
 $y+2z=0 \rightarrow y=-2z$   
 $z=z \rightarrow z=2$   
 Example:  $\begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} = B$   
 $\text{Ker}(B) = \{\vec{v} \in \mathbb{R}^6 \mid B \cdot \vec{v} = \vec{0}\}$   
 $\begin{cases} x_1 - 2x_2 + 7x_5 = 0 \\ x_3 + 8x_5 = 0 \\ x_4 - 3x_5 = 0 \\ x_2 = x_3 \\ x_5 = x_6 \\ x_6 = x_5 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2 - 7x_5 \\ x_2 = x_3 \\ x_3 = -8x_5 \\ x_4 = 3x_5 \\ x_5 = x_6 \\ x_6 = x_5 \end{cases}$   
 $\left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 8 \\ 3 \\ 1 \\ 0 \end{bmatrix} \mid x_2, x_5 \in \mathbb{R} \right\} = \text{Ker}(B) \subseteq \mathbb{R}^6$   
 Lecture  
 July 7<sup>th</sup>, 2025  
 Online covers kernel and span while in class covers 2<sup>nd</sup> order non-homogeneous  
 and 2x2's ( $\begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$ )  
 V a vector space,  $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$   
 $\text{Span}(S) = \{a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_k \vec{v}_k \mid a_1, \dots, a_k \in \mathbb{R}\}$   
 $S \subseteq \text{Span}(S) \subseteq V$   
 subvector subspace  
**Example:**  
 $\text{Ker}(\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 7 \\ 1 & 1 & 1 & 26 \end{bmatrix})$   
 $x_1 - 2x_3 + 3x_4 = 0$   
 $x_2 - 4x_3 + 7x_4 = 0$   
 $x_3 = x_3$   
 $x_4 = x_4$   
 $\left\{ x_3 \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 7 \\ 1 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R} \right\}$   
 $= \text{Span}(\{ \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 1 \\ 1 \end{bmatrix} \})$   
**Example:**  
 $V = \{y(t) \mid y'' - y = 0\}$   
 $\begin{cases} P_2 = \{a + bt + ct^2 \mid a, b, c \in \mathbb{R}\} \\ = \{C_1 e^t + C_2 e^{-t} \mid C_1, C_2 \in \mathbb{R}\} \\ = \text{Span}(\{e^t, e^{-t}\}) \end{cases}$   
**Example:**  
 $\vec{b} = \begin{bmatrix} -17 \\ 11 \\ 11 \end{bmatrix} \in \text{Span}(\{ \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix} \})$   
 Is  $\vec{b}$  a linear combination of these 3 vectors?  
 Are there  $a, b, c \in \mathbb{R}$  so that:  
 $\begin{bmatrix} -17 \\ 11 \\ 11 \end{bmatrix} = a \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix}$   
 $= \begin{bmatrix} 6a \\ -3a \\ -3a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -2b \end{bmatrix} + \begin{bmatrix} 9c \\ -3c \\ -3c \end{bmatrix} = \begin{bmatrix} 6a+2b+9c \\ -3a+b \\ -3a-2b-3c \end{bmatrix}$   
 $= \begin{bmatrix} 6 & 2 & 9 \\ 2 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$   
 If free  $x_3$  variable, then  
 $\begin{bmatrix} 5 & 2 & 9 & -17 \\ 2 & 1 & 3 & 11 \end{bmatrix} \xrightarrow{\text{RREF}(A)} \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -3 & 1 \end{bmatrix}$   
 $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$  no solution.  
 $\begin{cases} a+3c = -3 \\ b-3c = 11 \end{cases} \rightarrow \begin{cases} a = -3-3c \\ b = 3c+11 \end{cases}$   
 $\begin{cases} c = c \\ c = c \end{cases}$   
 $\begin{bmatrix} -17 \\ 11 \\ 11 \end{bmatrix} = -3 \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix}$   
 scalars  
 Get pts. on the exam.  
**Example:** Use parametric form to show that  
 $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix} \in \text{Span}(\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \})$   
 $\begin{bmatrix} -1 & -1 & 2 & -5 & -1 \\ -1 & -1 & 1 & -2 & -1 \\ -2 & -2 & -1 & 5 & -6 \end{bmatrix} \xrightarrow{\text{RREF}(A)} \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$   
 $\begin{cases} a+b-d=3 \\ b=b \\ c-3d=2 \\ d=d \end{cases} \rightarrow \begin{cases} a=3-b+d \\ b=b \\ c=3d+2 \\ d=d \end{cases}$   
 $\text{Sol} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$   
 $3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix} = \vec{b} = \begin{bmatrix} -1 \\ -1 \\ 8 \end{bmatrix}$   
**Example:**  $\text{Ker}(\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & -9 & -26 \\ 5 & 5 & 17 & -26 \end{bmatrix})$   
 $\text{RREF}(A) \rightarrow \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_3 - 3x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$   
 $\begin{cases} x_3 = x_3 \\ x_4 = x_4 \end{cases}$   
 $\begin{bmatrix} x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R} \end{bmatrix}$   
**Example:**  $y'' - 2y' + y = 2e^t$  ①  
 $y'' - 2y' + y = 2t$  ②  
 1. Solve the homo. eqn.  
 $y'' - 2y' + y = 0 \xrightarrow{\text{check}} r^2 - 2r + 1 = 0$   
 $(r-1)(r-1) = 0 \quad r=1 \text{ (double)}$   
 $y_c = C_1 e^t + C_2 t e^t$   
 $y_p = A e^{2t} + B t e^t$   
 $y = A e^{2t} + A t e^t + C_1 e^t + C_2 t e^t$   
 $y'' = 2A e^{2t} + A t^2 e^t + C_1 e^t + 2C_2 t e^t$   
 $2. y'' - 2y' + y = 2t$   
 $\rightarrow (r-1)(r-1) = 0 \quad r=1 \text{ double}$   
 $y_c = C_1 e^t + C_2 t e^t$   
 $y = A t e^t + B (A + B - 2(A) + 0) = -2A + A t + B = 2t$   
 $y'' = 0 \quad (A) + (-2A+B) = 2t + 0$   
 $y \rightarrow 2t + 4$   
 $A=2$   
 $2A+B=0$   
 $B=4$   
 $y'' - 2y' = 2t$   
 $\rightarrow r^2 - 2r = 0 \equiv r(r-2) \equiv r=0$   
 $y_c = C_1 e^0 + C_2 t e^{2t}$   
 $y = A t^2 + B$   
 $y'' = 2A$

$S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$ , vector space

$$\text{Span}(S) = \{\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$$

$$(3d) \quad S_2 = \{a+bz+cz^2 \mid a, b, c \in \mathbb{R}\} = \text{Span}(\{1, z, z^2\})$$

$$(2d) \quad V = \{y \mid y'' - y = 0\} = \{c_1 e^x + c_2 e^{-x} \mid c_1, c_2 \in \mathbb{R}\} = \text{Span}(\{e^x, e^{-x}\})$$

$$(3d) \quad \mathbb{R}^3 = \text{Span}(\{i, j, k\})$$

$$\vec{i} = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \{x\vec{i} + y\vec{j} + z\vec{k} \mid x, y, z \in \mathbb{R}\}$$

$$\vec{j} = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \{x\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mid x, y, z \in \mathbb{R}\}$$

$$\vec{k} = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \{x\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3$$

$$C = \{\alpha + i\beta \mid \alpha, \beta \in \mathbb{R}\} = \text{Span}(\{1, i\})$$

$V$  a vector space

Conjecture: If  $V = \text{Span}(\{\vec{v}_1, \dots, \vec{v}_k\})$ ,

then  $V$  is dimension  $k$ , "V is like  $\mathbb{R}^k$ "

Conjecture for  $k=1$ : If  $V = \text{span}(\{\vec{v}\})$ ,

then the dimension of  $V$  is 1. ( $\vec{v} \neq \vec{0}$ )

$$\text{Span}(\{\vec{v}\}) = \{\alpha \vec{v} \mid \alpha \in \mathbb{R}\}$$

$$\text{Span}(\{\vec{0}\}) = \{\alpha \vec{0} \mid \alpha \in \mathbb{R}\} = \{\vec{0}\}$$

Conjecture for  $k=2$ : If  $V = \text{Span}(\{\vec{v}_1, \vec{v}_2\})$ ,

then dim(V) is 2. ( $\vec{v}_1, \vec{v}_2 \neq \vec{0}$ )

$$V = \text{Span}(\{\vec{v}_1, \vec{v}_2\}) = \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$\text{Span}(\{\{1\}, \{z\}\}) = \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ z \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\beta \begin{bmatrix} 1 \\ z \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{\alpha + 2\beta \mid \alpha, \beta \in \mathbb{R}\}$$

Conjecture for  $k=3$ : If  $V = \text{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ , the dim(V) is 3.

$$V = \text{Span}(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\})$$

= x-y plane



If  $V = \text{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ , it's bad if:

$$\vec{v}_1 = \alpha_1 \vec{v}_2 + \alpha_2 \vec{v}_3$$

$$\vec{v}_2 = \beta_1 \vec{v}_1 + \beta_3 \vec{v}_3$$

$$\vec{v}_3 = \gamma_1 \vec{v}_1 + \gamma_2 \vec{v}_2$$

it's BAD if:

$$-\vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 = \vec{0}$$

$$-\vec{v}_2 + \beta_1 \vec{v}_1 + \beta_3 \vec{v}_3 = \vec{0}$$

$$-\vec{v}_3 + \gamma_1 \vec{v}_1 + \gamma_2 \vec{v}_2 = \vec{0}$$

it's BAD if there are  $\alpha, \beta, \gamma \in \mathbb{R}$ , NOT ALL ZERO.

$$\alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 = \vec{0}$$

The set  $S = \{\vec{v}_1, \dots, \vec{v}_k\}$  is BAD, i.e linearly dependent,

if there are  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

NOT ALL ZERO

such that  $\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k = \vec{0}$ .

The set  $S = \{\vec{v}_1, \dots, \vec{v}_k\}$  is good, i.e Linearly independent, if

$$\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k = \vec{0} \implies \alpha_1 = \alpha_2 = \dots = \alpha_k = 0.$$

(The only linear combination of the vectors in  $S$  that equals  $\vec{0}$  is the trivial linear combination).

$$\text{Example: } S = \left\{ \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$\alpha \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -3 & 1 \\ -3 & 1 & 0 \\ 8 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -3 & 1 \\ -3 & 1 & 0 \\ 8 & 0 & -1 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \alpha - c = 0 \\ b - 3c = 0 \\ c = c \end{array}$$

$$\implies c \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\implies (1) \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix} + (3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$$

Example: Use parametric form to show that

$$S = \left\{ \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix} \right\}$$

is lin. dependent

$$\begin{bmatrix} -5 & -3 & 7 & -6 \\ 2 & 1 & -3 & 3 \\ 3 & 1 & -5 & 6 \\ 0 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} a - 2c + 3d = 0 \\ b + c - 3d = 0 \\ c = c \\ d = d \end{array} \begin{array}{l} a = 2c - 3d \\ b = -c + 3d \\ c = c \\ d = d \end{array}$$

$$\begin{array}{l} \text{Two answers, both are legal.} \\ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \implies (2) \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 7 \\ -5 \\ 1 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} -6 \\ 3 \\ -3 \\ 0 \end{bmatrix} = \vec{0} \\ (-3) \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 7 \\ -5 \\ 1 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} -6 \\ 3 \\ -3 \\ 0 \end{bmatrix} = \vec{0} \end{array}$$

## Lecture

July 11<sup>th</sup>, 2025

Thrm:  $A \in M_n(\mathbb{R})$ , write  $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n] \rightarrow \vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$

①  $A\vec{x} = \vec{0}$  has non-trivial (non-zero) solutions.

②  $\text{Ker}(A) > \{\vec{0}\} \iff (\dim(\text{Ker}(A) > 0))$

③  $\{\vec{v}_1, \dots, \vec{v}_n\}$  linearly dependent

④  $\text{RREF}(A) \neq I_n$

⑤  $\text{rank}(A) < n$

⑥  $A$  is not invertible  $\therefore$  singular

⑦  $\det(A) = 0$ , then  $A$  is singular

①  $A\vec{x} = \vec{0}$  has only the trivial solution ( $\vec{x} = \vec{0}$ ).

②  $\text{Ker}(A) = \{\vec{0}\}$

③  $\{\vec{v}_1, \dots, \vec{v}_n\}$  linearly independent

④  $\text{RREF}(A) = I_n$

⑤  $\text{rank}(A) = n$

⑥  $A$  is invertible ( $A^{-1}$  is OK)

⑦  $\det(A) \neq 0$

Goal: Solve  $\frac{d}{dt} \vec{x} = A\vec{x}$ ,  $A \in M_n(\mathbb{R})$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \vec{x} \quad \vec{x}^1 = \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}, \quad \vec{x}^2 = \begin{bmatrix} 0 \\ e^{4t} \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \vec{x} \quad \vec{x}^1 = \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}, \quad \vec{x}^2 = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$\vec{x}(+) = \begin{bmatrix} x_1(+) \\ \vdots \\ x_n(+) \end{bmatrix} = \begin{bmatrix} a_1 e^{\lambda_1 t} \\ \vdots \\ a_n e^{\lambda_n t} \end{bmatrix} \quad \text{My guess}$$

$$\frac{d}{dt} \vec{x} = A\vec{x} \quad A \in M_n(\mathbb{R}) \quad \text{Guess: } \vec{x} = e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\frac{d}{dt} (e^{\lambda t} \cdot \vec{v}) = A \cdot (e^{\lambda t} \cdot \vec{v}) \quad = e^{\lambda t} \vec{v}, \quad \lambda \in \mathbb{R} \wedge \vec{v} \in \mathbb{R}^n$$

$$\lambda \vec{v} = A \cdot \vec{v} \quad \frac{d}{dt} (e^{\lambda t} \begin{bmatrix} -2 \\ 5 \end{bmatrix}) = \frac{d}{dt} \begin{bmatrix} -2e^{\lambda t} \\ 5e^{\lambda t} \end{bmatrix} = \begin{bmatrix} -2 \cdot 3e^{\lambda t} \\ 5 \cdot 3e^{\lambda t} \end{bmatrix} = 3e^{\lambda t} \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Def:  $A \in M_n(\mathbb{R})$ , if  $\lambda \in \mathbb{R}$ ,  $\vec{v}_\lambda \in \mathbb{R}^n$  ( $\vec{v}_\lambda \neq \vec{0}$ ) such that  $A \cdot \vec{v}_\lambda = \lambda \cdot \vec{v}_\lambda$ , then  $\lambda$  is an eigenvalue for  $A$  and  $\vec{v}_\lambda$  is an eigenvector for  $A$  with eigenvalue  $\lambda$

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0} \rightarrow [A] \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} - \lambda \begin{bmatrix} \vec{v} \end{bmatrix} = \vec{0}$$

$$\vec{v}(A - \lambda I) = \vec{0} \rightarrow \boxed{[A - \lambda I]} \begin{bmatrix} \vec{v} \end{bmatrix} = \vec{0}$$

$$A \cdot \vec{v} - \lambda \cdot I_n \cdot \vec{v} = \vec{0} \rightarrow [A] \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} - \lambda \begin{bmatrix} I_n \end{bmatrix} \begin{bmatrix} \vec{v} \end{bmatrix} = \vec{0}$$

$$(A - \lambda I_n) \cdot \vec{v} = \vec{0}$$

Pretend I know an eigenvalue,  $\lambda$

Inside  $\text{Ker}(A - \lambda I_n)$  is the eigenvectors.

$$\begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{bmatrix} \quad \begin{array}{l} 7-\lambda \\ -5-3\lambda-5\lambda \\ -x=\lambda \\ \lambda^2 \end{array}$$

$$\det \begin{pmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{pmatrix} = (7-\lambda)(-5-\lambda) - (8)(-4)$$

$$\lambda^2 - 2\lambda - 35 + 32 = \lambda^2 - 2\lambda - 3$$

$$\lambda = 3 \quad \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix} \quad \begin{array}{l} \lambda^2 - 2\lambda - 3 \\ +1 \quad \lambda^2 - 3\lambda \\ \hline \lambda = 3, -1 \end{array}$$

$$\text{rref} \begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - x_2 = 0 \rightarrow x_1 = x_2 \\ x_2 = x_2 \quad x_2 = x_2 \end{array}$$

$$\vec{x}' = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{array}{l} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = 3 \end{array}$$

$$\lambda = -3 \quad \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 8 & -4 \end{bmatrix} \quad \begin{array}{l} \vec{v}_{(-1)} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \lambda = -1 \\ \end{array} \quad e^{-3t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\text{rref} \begin{pmatrix} 8 & -4 \\ 8 & -4 \end{pmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x - 1/2 x_2 = 0 \\ x_2 = x_2 \end{array} \rightarrow \begin{array}{l} x_1 = 1/2 x_2 \\ x_2 = x_2 \end{array}$$

$$\vec{y}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

## Lecture

July 26, 2025

$$\text{Solve } \frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} \vec{x}$$

1. Characteristic polynomial:  $p(\lambda) = \det(A - \lambda I)$

$$A - \lambda I = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{pmatrix} = (\lambda - 7)(\lambda + 5) + 32 = \lambda^2 - 2\lambda - 3 = \begin{cases} \lambda = 3 \\ \lambda = -1 \end{cases}$$

$$2. \lambda = 3: \text{Ker}(A - 3I)$$

$$A - 3I = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x - y = 0 \rightarrow x = y \\ y = y \end{array} \quad \begin{array}{l} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x} = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$

$$\lambda = -1: \text{Ker}(A - (-1)I) = \text{Ker}(A + I)$$

$$A + I = \begin{bmatrix} 8 & -4 \\ 8 & -4 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x - 1/2 y = 0 \rightarrow x = 1/2 y \\ y = y \end{array} \rightarrow \begin{array}{l} \vec{v}_{-1} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \vec{x} = e^{-t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \\ y = y \end{array}$$

$$3. \vec{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\text{Solve: } \frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix} \vec{x}$$

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -2-\lambda & 5 \\ 0 & 3-\lambda \end{pmatrix} = (-2-\lambda)(3-\lambda) - (5) \cdot 0 \\ &= (-2-\lambda)(3-\lambda) \\ &= \lambda = \{3, -2\} \end{aligned}$$

$$\lambda = 3: \text{Ker}(A - 3I)$$

$$A - 3I = \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x - y = 0 \rightarrow x = y \\ y = y \end{array} \quad \begin{array}{l} \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x} = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$

$$\lambda = -2: \text{Ker}(A - (-2)I)$$

$$A + 2I = \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - 5R_1 \quad x = x \\ y = 0 \end{array} \quad \begin{array}{l} x = x \\ y = 0 \end{array} \rightarrow \begin{array}{l} \vec{v}_1 = \begin{bmatrix} x \\ 0 \end{bmatrix} = \vec{x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}$$

$$\vec{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{Solve: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 10 & -4 \\ 24 & -10 \end{bmatrix} \vec{x}$$

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} 10-\lambda & -4 \\ 24 & -10-\lambda \end{pmatrix} \\ &= (10-\lambda)(-10-\lambda) - (-4)(24) \\ &= (\lambda-10)(\lambda+10) + 96 = \lambda^2 - 100 + 96 \end{aligned}$$

$$= \lambda^2 - 4$$

$$= (\lambda+2)(\lambda-2) \rightarrow \lambda = \{-2, 2\}$$

$$\lambda = 2: A - 2I = \begin{bmatrix} 8 & -4 \\ 24 & -12 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x - 1/2 y = 0 \\ y = y \end{array} \rightarrow \begin{array}{l} x = 1/2 y \\ y = y \end{array} \rightarrow \begin{array}{l} \vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \vec{x} = e^{2t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \end{array}$$

$$\lambda = -2: A + 2I = \begin{bmatrix} 12 & -4 \\ 28 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x - 1/3 y = 0 \\ y = y \end{array} \rightarrow \begin{array}{l} x = 1/3 y \\ y = y \end{array} \rightarrow \begin{array}{l} \vec{v}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \vec{x} = e^{-2t} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \\ \vec{v}_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \end{array}$$

If  $\text{Ker}(A - \lambda I_n) = \{\vec{0}\}$ , there are no eigenvectors;

then  $\lambda$  is NOT an eigenvalue

$$\iff \det(A - \lambda I) \neq 0$$

If  $\text{Ker}(A - \lambda I_n)$  is "bigger" than  $\{\vec{0}\}$ ,

then  $\lambda$  is an eigenvalue  $\iff \det(A - \lambda I) = 0$

$$\text{Solve: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

$$e^{it} \vec{v}_i$$

$$\lambda = 2+i: A - (\lambda+i)\mathbf{I} = A + (-2-i)\mathbf{I}$$

$$= \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix} + \begin{bmatrix} -2-i & 0 \\ 0 & -2-i \end{bmatrix} = \begin{bmatrix} -5-i & 13 \\ -2 & 5-i \end{bmatrix}$$

ref(result):  
 $\rightarrow \begin{bmatrix} -2 & 5-i \\ -5-i & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5/2 + 1/2i \\ 0 & 0 \end{bmatrix}$

$$x + \left(\frac{5}{2} + \frac{1}{2}i\right)y = 0 \quad \left. \begin{array}{l} \\ y=y \end{array} \right\} \rightarrow y \begin{bmatrix} 5/2 - 1/2i \\ 1 \end{bmatrix} \Rightarrow \vec{v}_{2+i} = \begin{bmatrix} 5/2 - 1/2i \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{z} &= e^{(2+i)t} \cdot \begin{bmatrix} 5/2 - 1/2i \\ 1 \end{bmatrix} \\ &= e^{2t} \left[ \begin{bmatrix} \frac{5}{2} \cos t & \frac{1}{2} \sin t \\ \cos t & 0 \end{bmatrix} + i \begin{bmatrix} \frac{1}{2} \cos t & \frac{1}{2} \sin t \\ \sin t & 0 \end{bmatrix} \right] \\ \vec{z} &= e^{2t} \begin{bmatrix} \frac{5}{2} \cos t + \frac{1}{2}i \sin t & \frac{1}{2} \cos t + \frac{1}{2}i \sin t \\ \cos t & \sin t \end{bmatrix} \\ \vec{x} &= C_1 e^{2t} \begin{bmatrix} \frac{5}{2} \cos t & \frac{1}{2} \sin t \\ \cos t & 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2}i \sin t & \frac{1}{2} \cos t + \frac{1}{2}i \sin t \\ \sin t & 0 \end{bmatrix} \end{aligned}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}$$

$$\begin{aligned} p(\lambda) &= \det \begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix} = (-1-\lambda)(5-\lambda) - 9(-1) \\ &= (\lambda+1)(\lambda-5) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda-2)(\lambda-2) \end{aligned}$$

$\lambda = 2$ , double root

$$\lambda = 2: A - 2\mathbf{I} = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$x - 3y = 0 \quad \left. \begin{array}{l} x=3y \\ y=y \end{array} \right\} x = 3y \rightarrow y \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Jordan Cycle:

- Find  $\vec{z}$  such that  $(A - \lambda \mathbf{I}) \vec{z} = \vec{v}_1$
- $\vec{x}^2 = e^{2t} (\vec{z} + t \vec{v}_1)$

Find  $\vec{z}$  such that  $\begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \vec{z} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -3 & 9 & 3 \\ -1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x - 3y = -1 \\ y = y \end{cases} \rightarrow \begin{cases} x = 3y - 1 \\ y = y \end{cases} \Rightarrow \vec{z} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{x}^2 &= e^{2t} \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \\ &= e^{2t} \begin{bmatrix} -1 + 3t \\ t \end{bmatrix} \end{aligned}$$

$$\vec{x} = C_1 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -1 + 3t \\ t \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}$$

$$\begin{aligned} p(\lambda) &= \det(A - \lambda \mathbf{I}) = (-1-\lambda)(-1-\lambda) - 1 \cdot 0 \\ &= (\lambda+1)(\lambda+1) = \lambda = -1, \text{ double root} \end{aligned}$$

$$\lambda = -1: A - (-1)\mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \left. \begin{array}{l} x=x \\ y=0 \end{array} \right\} \Rightarrow x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\vec{x}^1 = e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Find  $\vec{z}$ :  
 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} y=1 \\ x=x \end{cases} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{x}^2 = e^{-t} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = C_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix} \vec{x}$$

$$\begin{aligned} ① \quad p(\lambda) &= \det \begin{bmatrix} -3-\lambda & 13 \\ -2 & 7-\lambda \end{bmatrix} = (-3-\lambda)(7-\lambda) - (-13)(-2) \\ &= (\lambda+3)(\lambda-7) + 26 = \lambda^2 - 4\lambda + 5 \end{aligned}$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Continued on top right!

### Section 3.11 - Matrix Exponential

$$e^{\begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix}t} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix}t + \frac{(-3)(13)t^2}{2!} + \frac{+3}{3!} \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix}^3 + \frac{t^4}{4!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$= \left[ \frac{\sum a_n t^n}{2!}, \frac{\sum b_n t^n}{2!} \right] = \frac{[f(t) g(t)]}{[h(t) j(t)]}$$

$e^{At}$ :

$$\text{Solve } \frac{d}{dt} \vec{x} = A\vec{x}, \vec{x}', \vec{x}'' \quad \textcircled{2}$$

$$\lambda(t) = [\vec{x}' \vec{x}''] \quad \textcircled{3}$$

$$e^{At} = \lambda(t) \cdot (\lambda(0))^{-1}$$

$$e^{\begin{bmatrix} -3 & 17 \\ -2 & 7 \end{bmatrix}t}$$

$$\textcircled{1} \text{ Solve } \frac{d}{dt} \vec{x} = \begin{bmatrix} -3 & 17 \\ -2 & 7 \end{bmatrix} \vec{x}$$

$$\vec{x} = C_1 e^{2t} \begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2} \sin t \\ \cos t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\frac{1}{2} \cos t + \frac{1}{2} \sin t \\ \sin t \end{bmatrix}$$

$$\textcircled{2} \quad \lambda(t) = e^{2t} \begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2} \sin t & -\frac{1}{2} \cos t + \frac{1}{2} \sin t \\ \cos t & \sin t \end{bmatrix}$$

$$\textcircled{3} \quad \lambda(t) \cdot (\lambda(0))^{-1} = \begin{bmatrix} \frac{1}{2} \cos t + \frac{1}{2} \sin t & -\frac{1}{2} \cos t + \frac{1}{2} \sin t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$

$$\lambda(0) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \rightarrow \det(\lambda(0)) = \frac{1}{2}$$

$$(\begin{bmatrix} a & b \\ c & d \end{bmatrix})^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(\lambda(0))^{-1} = 2 \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$

$$\therefore e^{2t} \begin{bmatrix} \cos t - 5 \sin t & 13 \sin t \\ -2 \sin t & \cos t + 5 \sin t \end{bmatrix}$$

Double root:

$$e^{\begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix}t}$$

$$\textcircled{1} \text{ Solve } \frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}$$

$$p(\lambda) = \det \begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)(\lambda-2)$$

$\lambda = 2$ , double

$$\lambda = 2: A - 2I = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$x - 3y = 0 \rightarrow x = 3y \quad \left\{ \begin{array}{l} \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{x}' = e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ y = y \end{array} \right.$$

$$y = y \quad \left\{ \begin{array}{l} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}'' = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array} \right.$$

Jordan Cycle:

$$\text{Solve } (A - 2I)^{-1} \vec{x} = \vec{v}_1 \quad \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 9 & 3 \\ -1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x - 3y = -1 \\ y = y \end{cases} \rightarrow \begin{cases} x = 3y - 1 \\ y = y \end{cases}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x} = e^{2t} \left( \vec{v}_1 + t \vec{v}_2 \right)$$

$$= e^{2t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} 1 & 1 \\ 0 & 1+t \end{bmatrix}$$

$$\lambda(t) \cdot (\lambda(0))^{-1} = \dots$$

$$\lambda(0) = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, \det(\lambda(0)) = 1$$

$$\lambda(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

$$e^{2t} \begin{bmatrix} 3 & -1+3t \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = e^{2t} \begin{bmatrix} 1-3t & 9t \\ -t & 1+3t \end{bmatrix}$$

$$1-3+3(-1+3t) = 3-3+9t = 9t$$

### Chapter 4

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

$$\vec{x}' = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}'' = e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{Graph } \vec{x}'(+) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

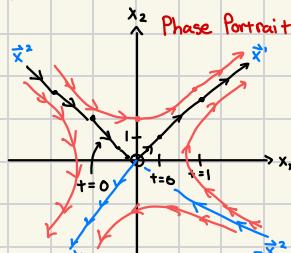
$$t \quad \vec{x}'(+)$$

$$-1 \quad \vec{x}'(-) = \begin{bmatrix} 1/e \\ 1/e \end{bmatrix}$$

$$0 \quad \vec{x}'(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$1 \quad \vec{x}'(1) = \begin{bmatrix} e \\ e \end{bmatrix}$$

$$2 \quad \vec{x}'(2) = \begin{bmatrix} e^2 \\ e^2 \end{bmatrix}$$



Phase Plane (Saddle)

$$\text{Graph } \vec{x}^2(+) = \begin{bmatrix} -e^t \\ e^{-t} \end{bmatrix}$$

$$\text{Example: } \vec{x}^1 + \vec{x}^2 = \begin{bmatrix} e^t - e^{-t} \\ e^t + e^{-t} \end{bmatrix}$$

$$t=0, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$t=1, \begin{bmatrix} e - 1/e \\ e + 1/e \end{bmatrix}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -9 \\ 8 & -5 \end{bmatrix} \vec{x}$$

$$\lambda = 3, -1 \text{ (saddle)}$$

$$\text{Sol: } e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Lecture

July 23<sup>rd</sup>, 2025

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -9 \\ 8 & -5 \end{bmatrix} \vec{x}, \lambda: 3, -1 \text{ Saddle}$$

$$\vec{x} = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}^2 = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \vec{x}$$

$$p(\lambda) = \det \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = (\lambda-2)(\lambda-1) \text{ Nodal source}$$

$$\lambda = 1: A - I = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = 2: A - 2I = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Experimental case:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{cases} x_1 = 1, \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ x_2 = 2, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases} \Rightarrow \begin{cases} \vec{x}^1 = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{x}^2 = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$$\vec{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x} + \vec{x}^2 = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}, t=0: \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

$$t=0: \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$t=\pi/2: \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$t=\pi: \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$t=\pi/2: \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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$$t=\pi: \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

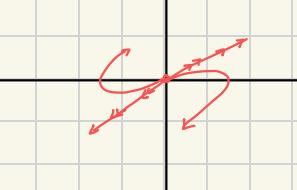
July 25, 2025

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}, p(\lambda) = \det \begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix} = (\lambda+1)(\lambda-5) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$\lambda=2$ , double root

Degenerate Nodal source

$$\lambda=2: A-2I = \begin{bmatrix} -3 & 9 \\ 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix} \vec{x}, p(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix}$$

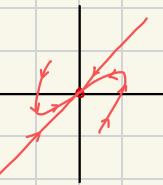
$$\text{Degenerate Nodal Sink} = (1-1)(1+3) + 4 = \lambda^2 + 2\lambda + 1$$

 $\lambda=-1$ , double

$$= (1+1)^2$$

$$\lambda=-1: A-I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$



$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} \vec{x} \quad \text{linear } 2 \times 2$$

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} = \begin{bmatrix} 7x_1(t) - 4x_2(t) \\ 8x_1(t) - 5x_2(t) \end{bmatrix} \Rightarrow \begin{cases} x'_1 = 7x_1 - 4x_2 \\ x'_2 = 8x_1 - 5x_2 \end{cases}$$

$$x' = x^2 - y^2, y' = y - 1$$

Step 1: Equilibrium points

where  $f(x,y) = x'(t)$  and  $g(x,y) = y'(t)$ 

$$x' = 0 \text{ and } y' = 0$$

$$f(x,y) = 0 \text{ and } g(x,y) = 0$$

$$x^2 - y^2 = 0 \quad x^2 - y^2 = 0$$

$$(xy)(x-y) = 0 \quad y-1 = 0 \rightarrow y=1$$

$$x+y=0 \quad y=-x$$

$$\text{or } x-y=0 \quad y=x$$

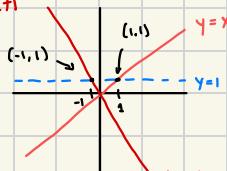
$$(x+1)(x-1)=0$$

$$x=1, -1$$

Step 3:

$$J(-1, 1) = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

$$J(1, 1) = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$$



$$\text{Step 2: } x' = x^2 - y^2 = f(x,y), \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y' = y - 1 = g(x,y)$$

Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$J = \begin{bmatrix} 2x & -2y \\ 0 & 1 \end{bmatrix}$$

Step 4:

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix} \vec{x} @ (-1, 1)$$

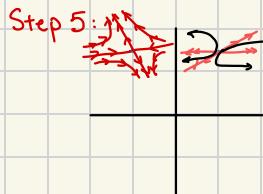
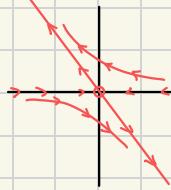
$$p(\lambda) = \det \begin{bmatrix} -2-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix} = (\lambda+2)(\lambda-1)$$

$$\lambda = 1, -2$$

$$\lambda = 1: A - I = \begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2/3 \\ 0 & 0 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$\lambda = -2: A + 2I = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} x=x \\ y=0 \end{array} \right\} \Rightarrow \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_2$$

$$\vec{v}_{-2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$@ (1, 1) \frac{d}{dt} \vec{x} = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$p(\lambda) = \det \begin{bmatrix} 2-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}$$

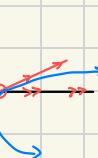
$$\text{Nodal source} = (1-2)(1-1)$$

$$\lambda = 2, 1$$

$$\lambda = 1: A - I = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = 2: A - 2I = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} x=x \\ y=0 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_2$$

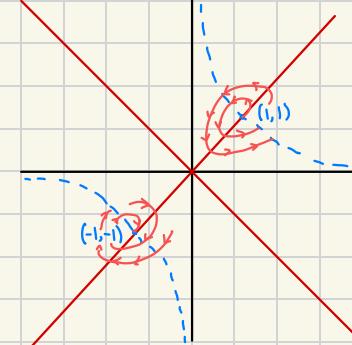


## Lecture

July 28<sup>th</sup>, 2025

(1) Equilibrium Points

$$\begin{aligned} x' &= x^2 - y^2 \\ y' &= xy - 1 \\ \begin{cases} (x+y)(x-y)=0 \\ x+y=0 \vee x-y=0 \end{cases} \\ y &= -x, y=x \\ x+y &= 0 \\ y &= 1 \\ y &= \frac{1}{x} \end{aligned}$$



$$\textcircled{2} \quad J = \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}$$

$$\textcircled{3} \quad J(-1, -1) = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$J(1, 1) = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\textcircled{4} \quad @ (-1, -1), \frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix} \vec{x}$$

$$p(\lambda) = \det \begin{bmatrix} -2-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}$$

$$= (1+2)(\lambda+1) + 2$$

$$= \lambda^2 + 3\lambda + 9$$

$$\lambda = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm i\sqrt{7}}{2}$$

$$= \frac{-3}{2} \pm i \frac{\sqrt{7}}{2}$$

Spiral sink

Clockwise

$$\begin{aligned} @ (1, 1) \frac{d}{dt} \vec{x} &= \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \vec{x} \\ p(\lambda) &= \det \begin{bmatrix} 2-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = \lambda = (1-2)(1-1) + 2 \\ &= \lambda^2 - 3\lambda + 9 \\ \lambda &= \frac{3 \pm \sqrt{9-16}}{2} = \frac{3}{2} \pm i \frac{\sqrt{7}}{2} \quad \text{Spiral source} \end{aligned}$$

Online Exam 3:  $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 1. Linear Dependence  $\rightarrow$  a) solve2.  $2 \times 2$  systems  $\rightarrow$  b)  $e^{At}$ 

In class Exam 3: c) Phase

3.  $2 \times 2$  system  $\rightarrow$ 4.  $2 \times 2$  system  $\rightarrow$

Example #1:  $S = \left\{ \begin{bmatrix} -2 \\ 3 \\ -12 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 33 \end{bmatrix} \right\}$  Use parametric form to show S is linearly dependent.

$$\begin{bmatrix} -2 & -1 & 5 \\ 3 & 1 & -8 \\ -12 & -2 & 33 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} r_1 - 3r_3 = 0 \\ r_2 + r_3 = 0 \rightarrow r_3 = -r_2 \\ r_3 = r_3 \end{cases}$$

$$r_1 + r_2 = 0 \rightarrow r_1 = -r_2$$

$$r_1 + r_2 + r_3 = 0 \rightarrow r_1 = -r_2 - r_3$$

$$\therefore r_1 + r_2 + r_3 = 0 \rightarrow \text{S is linearly dependent}$$

Example #2:  $S = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 3 & 6 & 1 & 7 \\ 2 & 1 & 1 & 1 \\ -1 & -2 & -1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} a+2b+3d=0 \\ c-2d=0 \\ d=d \rightarrow b=b \\ d=d \rightarrow c=2d \\ d=d \end{cases}$$

$$b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$(-2) \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + (1) \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} = \vec{0}$$

Example #3:  $\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}$

$$p(\lambda) = \det \begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix} = (-1+\lambda)(5-\lambda) + 9 = \lambda^2 - 4\lambda + 9$$

$$= (1-2)(\lambda-2) \rightarrow \lambda = 2, \text{ double}$$

$$\lambda = 2: A - 2I = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{x} = e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Jordan Cycle:  $(A - \lambda I) \vec{z} = \vec{v}_\lambda \rightarrow \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \vec{z} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -3 & 9 & 3 \\ -1 & 3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{cases} x - 3y = -1 \\ y = y \end{cases} \quad \begin{cases} x = -1 + 3y \\ y = y \end{cases}$$

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} y \Rightarrow \vec{z} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\therefore \vec{x}^2 = e^{4t} (\vec{z} + t \vec{v}_\lambda) = e^{2t} \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} -1+3t \\ t \end{bmatrix}$$

a)  $c_1 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1+3t \\ t \end{bmatrix}$

b)  $e^{At}, \vec{x} = e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{x}^2 = e^{2t} \begin{bmatrix} -1+3t \\ t \end{bmatrix}$

$$x(t) = e^{2t} \begin{bmatrix} 3 & -1+3t \\ 1 & t \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, \det(x(0)) = 1$$

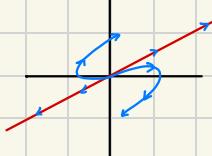
$$(x(0))^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

$$e^{2t} \begin{bmatrix} 3 & -1+3t \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = e^{2t} \begin{bmatrix} 3 & -1+3t \\ 1 & t \end{bmatrix} = e^{2t}$$

c)  $\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}, \lambda = 2, \text{ double, degenerate nodal source}$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{t} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



Example #4:  $\frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -1 \\ 8 & -5 \end{bmatrix}, e^{At}, e^{3t}$

$$e^{At}:$$

$$X(t) = \begin{bmatrix} e^{2t} & 1/2 e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix}$$

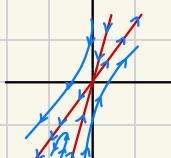
$$X(0) = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}, \det(X(0)) = 1/2$$

$$(X(0))^{-1} = \frac{1}{1/2} \begin{bmatrix} 1 & -1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$e^{At} = X(t) \cdot (X(0))^{-1}$$

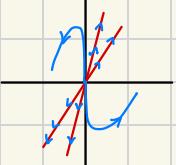
$$= \begin{bmatrix} e^{2t} & 1/2 e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} - e^{2t} \\ 2e^{2t} \end{bmatrix}$$



$$\frac{d}{dt} \vec{x} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \vec{x}, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{1/2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Nodal source



Lecture

August 1st, 2025

Predator-Prey Modeling

$x(t) = \# \text{ of prey} @ \text{time } t, y(t) = \# \text{ of predator} @ \text{time } t$

$X' = \alpha \cdot x - \beta \cdot xy, Y' = \gamma \cdot xy - \delta \cdot y$  (Lotka-Volterra System)

Step 2: Jacobian Matrix

$$J = \begin{bmatrix} -\alpha & -\beta \\ \gamma & -\delta \end{bmatrix}$$

Step 3:

$$@ (0,0) = J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, @ (1,1) = J(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Step 4:

$$@ (0,0), \frac{d}{dt} \vec{x} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\delta \end{bmatrix} = (\lambda-1)(\lambda+1)$$

$$\lambda = \{1, -1\}$$

$$\lambda = 1: A - I = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \rightarrow x = 0$$

$$\therefore \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1: A + I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \rightarrow y = 0$$

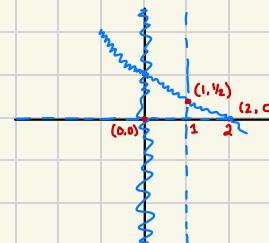
$$\therefore \vec{v}_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Adding carrying capacity tweak:

$$x' = \alpha x - \beta xy \quad x' = \alpha x \cdot (1 - \frac{x}{N}) - \beta xy$$

$$y' = \gamma xy - \delta y \quad y' = \gamma xy - \delta y$$

$$\text{Let } X' = x(1 - \frac{x}{N}) - xy, Y' = xy - \delta y$$



Step 1 (MOD):

$$\begin{aligned} x(1 - \frac{x}{N}) - xy &= 0 \\ x(1 - \frac{1}{2} - y) &= 0 \rightarrow y = -\frac{1}{2} + 1 \vee x = 0 \\ xy - y &= 0 \\ (x-1)y &= 0 \rightarrow y = 0 \vee x = 1 \end{aligned}$$

Step 2 (MOD):

$$J = \begin{bmatrix} 1-x-y & -x \\ y & x-1 \end{bmatrix}$$

Step 3 (MOD):

$$@ (0,0) = J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$@ (1,1/2) = J(1,1/2) = \begin{bmatrix} -1/2 & -1 \\ 1/2 & 0 \end{bmatrix}$$

$$@ (2,0) = J(2,0) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Example #4:  $\frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -1 \\ 8 & -5 \end{bmatrix}, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{1/2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$e^{At}:$$

$$X(t) = \begin{bmatrix} e^{2t} & 1/2 e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}, \det(X(0)) = 1/2$$

$$(X(0))^{-1} = \frac{1}{1/2} \begin{bmatrix} 1 & -1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$e^{At} = X(t) \cdot (X(0))^{-1}$$

$$= \begin{bmatrix} e^{2t} & 1/2 e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} - e^{2t} \\ 2e^{2t} \end{bmatrix}$$



# Lecture

August 8<sup>th</sup>, 2025

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = f(x), 0 < x < l$$

$$u(0, t) = u(l, t) = 0, t \geq 0$$

$$\text{If } \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

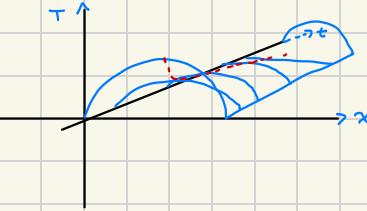
$$u(x, 0) = \sin \frac{n\pi x}{l}, 0 < x < l$$

$$u(0, t) = u(l, t) = 0, t \geq 0$$

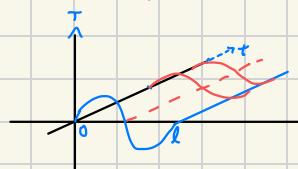
then:

$$u(x, t) = \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2\alpha^2}{l^2} t}$$

Example #1:  $n=1$  HE:  $u(x, 0) = \sin \frac{\pi x}{l}, 0 \leq x \leq l$



$$n=2, u(x, 0) = \sin \frac{2\pi x}{l}, 0 \leq x \leq l$$



$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{If: } u(x, 0) = \sum_{n=1}^{\infty} (b_n \sin \frac{n\pi x}{l}), 0 < x < l$$

$$u(0, t) = u(l, t) = 0, t \geq 0$$

then:

$$u(x, t) = \sum_{n=1}^{\infty} b_n \left( \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2\alpha^2}{l^2} t} \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, 0 < x < l$$

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} dx$$

Example #1: If  $f(x) = 1, 0 < x < 2$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}, 0 < x < 2$$

$$b_n = \frac{2}{2} \int_0^2 1 \cdot \sin \frac{n\pi x}{2} dx = \int_0^2 \sin \frac{(n\pi x)}{2} dx$$

$$= \int_0^2 \sin \left( \frac{n\pi x}{2} \right) dx$$

$$\left[ \frac{2}{n\pi} \cos \left( \frac{n\pi x}{2} \right) \right]_0^2 = \left( \frac{2}{n\pi} \cos(n\pi) \right) - \left( \frac{2}{n\pi} \right)$$

$$= \frac{2}{n\pi} (1 - \cos(n\pi))$$

$$= \frac{2}{n\pi} (1 - (-1)^n)$$

$$1 = b_1 \sin \frac{\pi x}{2} + b_2 \sin \frac{2\pi x}{2} + \dots$$

$$1 = \frac{1}{\pi} \sin \frac{\pi x}{2} + \frac{1}{3\pi} \sin \frac{3\pi x}{2} + \frac{1}{5\pi} \sin \frac{5\pi x}{2} + \frac{1}{7\pi} \sin \frac{7\pi x}{2} + \dots$$

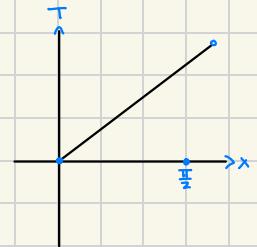
# Lecture

August 13, 2025

Sine Series for

$$u(x, 0) = f(x) = x$$

$$0 < x < \frac{\pi}{2}$$



$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = x, 0 < x < \frac{\pi}{2}$$

$$u(0, t) = u(\frac{\pi}{2}, t) = 0, t \geq 0$$

$$b_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(x) \cdot \sin \frac{n\pi x}{\pi} dx$$

$$b_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} x \cdot \sin(2nx) dx, u = x, dv = \sin(2nx) dx$$

$$du = dx, v = -\frac{1}{2n} \cos(2nx)$$

$$= \frac{4}{\pi} \left[ -\frac{1}{2n} x \cdot \cos(2nx) - \int -\frac{1}{2n} \cos(2nx) dx \right]$$

$$= \frac{4}{\pi} \left[ -\frac{1}{2n} x \cdot \cos(2nx) - 0 \cdot \cos(0) - \frac{1}{2n} \sin(2nx) \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi n} \left[ \frac{\pi}{2} (-1)^n - \frac{1}{2n} (\sin(2n \cdot \frac{\pi}{2}) - \sin(2n \cdot 0)) \right]$$

$$= -\frac{1}{n} (-1)^n = -\frac{(-1)^n}{n} = b_n$$

$$b_1 = 1$$

$$b_2 = -1/2$$

$$b_3 = 1/3$$

$$b_4 = -1/4$$

$$b_5 = 1/5$$

$$b_6 = -1/6$$

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\pi} = \sum_{n=1}^{\infty} b_n \sin(2nx)$$

$$= b_1 \sin(2x) + b_2 \sin(4x) + b_3 \sin(6x) + \dots$$

$$= \sin(2x) - \frac{\sin(4x)}{2} + \frac{\sin(6x)}{3} - \frac{\sin(8x)}{4} + \frac{\sin(10x)}{5} - \frac{\sin(12x)}{6} + \dots$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\pi} \cdot e^{-\frac{n^2\pi^2\alpha^2}{4} t}$$

$$= \sin(2x) \cdot e^{-4\alpha^2 t} - \frac{1}{2} \sin(4x) \cdot e^{-16\alpha^2 t} + \frac{1}{3} \sin(6x) \cdot e^{-36\alpha^2 t} - \frac{1}{4} \sin(8x) \cdot e^{-64\alpha^2 t} + \dots$$

## Complex Fourier Series

$$\int f(x) e^{inx} dx$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = \begin{cases} z & 0 < x \leq \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

$$u(0, t) = u(2\pi, t) = 0, t \geq 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin \frac{n\pi x}{\pi} dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin \left( \frac{n\pi x}{2} \right) dx$$

$$= \frac{1}{\pi} \left[ \int_0^{\pi} 2 \sin \left( \frac{n\pi x}{2} \right) dx + \int_{\pi}^{2\pi} 0 \cdot \sin \left( \frac{n\pi x}{2} \right) dx \right]$$

$$u(x, t) = \frac{4}{\pi} \sin \left( \frac{x}{2} \right) \cdot e^{-1/4\alpha^2 t} + \frac{4}{\pi} \sin(x) \cdot e^{-\alpha^2 t} + \dots$$