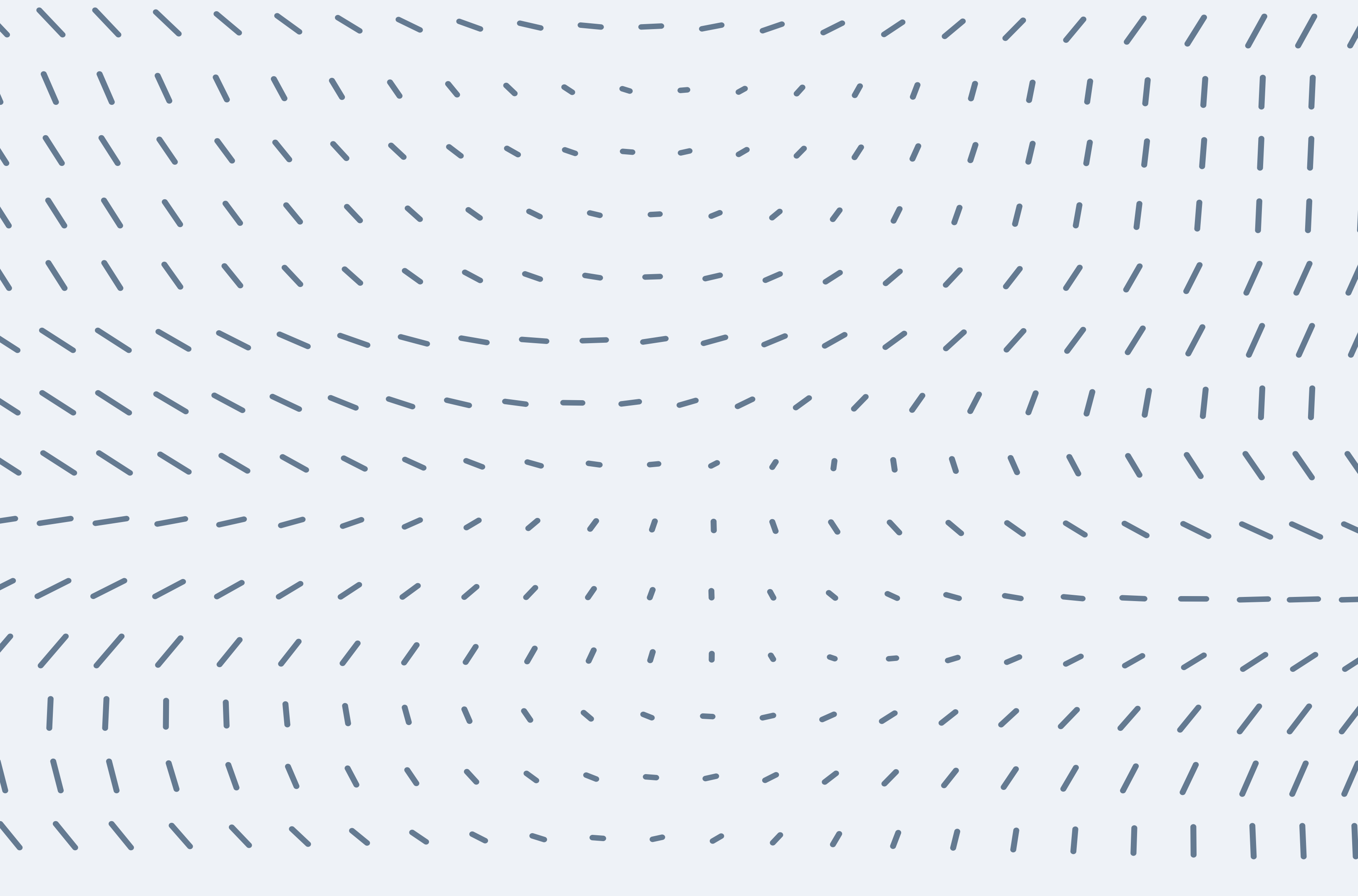
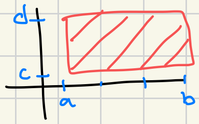


M 427L Vector Calculus

June 6^{th}, 2025

$$\int_c^d\int_a^b f(x,y)\,dx\,dy$$



Not the real world, only part!

Reviewing Vectors:

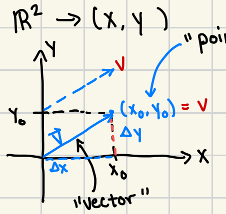
$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\mathbb{R}^n = \{(a_1, \dots, a_n) : a_i \text{ are real numbers}\}$

$f(x_1, \dots, x_n) = (f_1(x_1,\dots,x_n), \dots, f_n(x_1,\dots,x_n))$

$\mathbb{R}^n \to (x_1, \dots, x_n)$ "Vectors" "points"

$\mathbb{R}^2 \to (x, y)$



Points Arithmetic

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ Scalar Addition

$\lambda(x,y) = (\lambda x, \lambda y)$ Scalar Multiplication

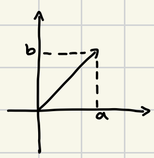
Example:

$$\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-5\\-7\end{bmatrix} \to (-5, 7)$$

Point = TIP − TAIL (Cords)

- $V = (a,b)$

Norm of $v = \|v\| = \sqrt{a^2+b^2}$

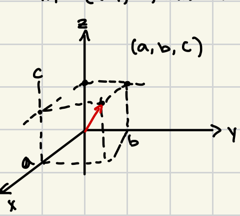


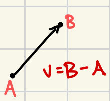
- $V = (a_1, b_1)$ $\quad W = (a_2, b_2)$

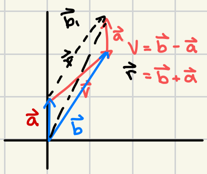
Dot Product: $V \cdot W = a_1 \cdot a_2 + b_1 \cdot b_2$

June 9^{th}, 2025

$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \text{ is a real number}\}$







Norm:

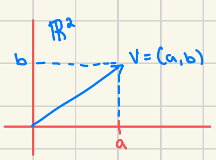
$V = (x_1, \dots, x_n)$

$$\|v\| = \sqrt{\sum_{i=0}^{n} x_n^2}$$

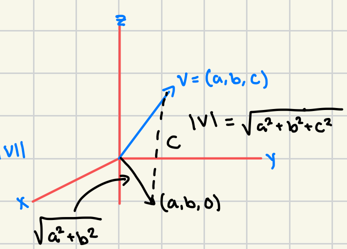
Example #1: $|(-1,2,3)| = \sqrt{(-1)^2 + 2^2 + 3^2}$

$= \sqrt{1+4+9}$

$= \sqrt{14}$



hyp $= \sqrt{a^2+b^2} = \|v\|$ length of the vector



Unit vector

If $\vec{v} \neq 0$, then $|\vec{v}| \neq 0$ and

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$|\hat{v}| = \frac{|v|}{|v|} = 1$$

$\lambda(a,b) = (\lambda a, \lambda b)$

$\sqrt{a^2+b^2}$

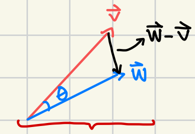
$\sqrt{\lambda^2 a^2 + \lambda^2 b^2} = |\lambda|\sqrt{a^2+b^2}$

We say two vectors $\vec{v}, \vec{w}$ have the same direction if $\hat{v} = \hat{w}$.

Example #2: $\vec{v} = (1,2,3)$ and $\vec{w} = (-1,2,1)$

$$\hat{v} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \frac{(1,2,3)}{\sqrt{14}} = \left(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}\right)$$

$$\hat{w} = \frac{(-1,2,1)}{\sqrt{(-1)^2+2^2+1^2}} = \frac{(-1,2,1)}{\sqrt{6}} = \frac{1}{\sqrt{6}}(-1,2,1)$$



$|w - v|^2 = |v|^2 + |w|^2 - 2|v||w|\cos\theta$

"Vector = $\underbrace{\text{direction}}_{\hat{v}}$ + $\underbrace{\text{magnitude}}_{|\vec{v}|}$" $\quad \hat{v} = \frac{v}{|v|}$

$\boxed{\to \vec{v} = \hat{v} \cdot |v|}$

Way 2: $|w-v|^2 = (w-v)\cdot(w-v)$

$= |w|^2 - 2 \cdot w \cdot v + |v|^2$

$\to v \cdot w = |v||w|\cos\theta$

$\to \cos\theta = \frac{v \cdot w}{|v||w|}$

Dot Product:

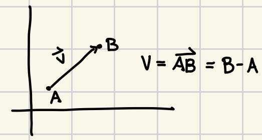
$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

NOTE! $\vec{v} = (a_1, \dots, a_n)$: $\vec{v} \cdot \vec{v} = a_1^2 + \dots + a_n^2$

$\sqrt{v^2} = |v|$

$v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2 \to (\lambda v) \cdot w = \lambda(v \cdot w)$

$v \cdot w = w \cdot v$



$V = \vec{AB} = B - A$

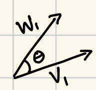
Discussion Section

June 10^{th}, 2025.

$W_1 = (2, -1, 0, 2)$ $\qquad W_1 \cdot V_1 = \sum_{i=1}^{n} a_n \cdot b_n$ where $n = 4$

$V_1 = (1, 2, -1, 2)$

$= 4$



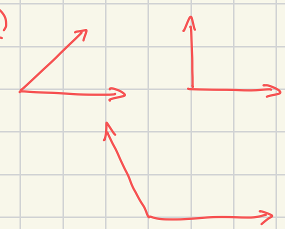
$$\to \cos\theta = \frac{V \cdot W}{|V||W|}$$

$$= \frac{4}{3\sqrt{10}}$$

proj $w_1 = V_1 \cdot \hat{V_2} = |V_1|\cos\theta$

$$= \cancel{3}\left(\frac{4}{\cancel{3}\sqrt{10}}\right) = \boxed{\frac{4}{\sqrt{10}}}$$

$$\hat{V_2} = \frac{(-1,2,-1,2)}{\sqrt{10}}$$

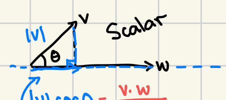


Dot Products:

$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^{n} a_n b_n$

$v \cdot v = v^2 = \sum_{i=1}^{n} a_n^2 = |v|^2$

if $v = (a_1, \dots, a_n)$



$v \cdot w = |v||w|\cos\theta$

$|v|\cos\theta = \frac{v \cdot w}{|w|}$

V·W	Angle
0	$\pi/2$
+	$<\frac{\pi}{2}$
−	$>\frac{\pi}{2}$

$|v|\cos\theta \underbrace{\hat{w}}_{\frac{w}{|w|}} = \frac{v \cdot w}{|w|^2} \cdot w$ (Vector Projection)

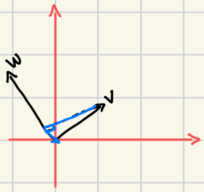
Lecture

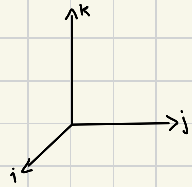
June 13^{th}

Example #1: Find the projection of $v: (2,1)$ onto $w=(-1,3)$

Scalar: $\frac{v\cdot w}{|w|} = \frac{1}{\sqrt{10}}$ $\quad |w| = \sqrt{(-1)^2+(3)^2} = \sqrt{10}$

Vector: $\underbrace{\underbrace{\frac{v\cdot w}{|w|}}_{\text{Length}} \cdot \underbrace{\frac{w}{|w|}}_{\text{Direction}}}_{|w|^2} = \frac{1}{10}(-1,3) \equiv \left(\frac{-1}{10}, \frac{3}{10}\right)$





$i \times j = k \qquad j \times i = -k$

$\vec{i} \times \vec{i} =$

$v \times v = -v \times v = \vec{0}$

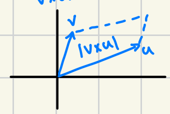
$v \times (w+u) = v\times w + v \times u$

$v \times (\lambda u) = (\lambda v) \times u \equiv \lambda (v \times u)$

$(1,2,3)\times(2,3,4) = (i+2j+3k)\times(2i+3j+4k)$

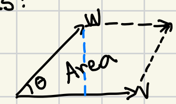
$(v_1,v_2,v_3)\times(u_1,u_2,u_3) = \det\left(\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3\end{bmatrix}\right)$

Application: $|v\times u|$ = area of the parallelogram spanned by $\vec{v}$ and $\vec{u}$



Areas:

SOH
CAH
TOA



$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Area $= |v||w| \sin\theta$

$A^2 = |v|^2|w|^2 \sin^2\theta$

$= |v^2||w^2| (1-\cos^2\theta)$

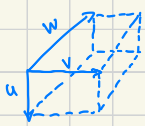
$\equiv |v^2||w^2| - |v^2||w^2|\cos^2\theta$

$A^2 = v^2w^2 - (v\cdot w)^2 = \begin{vmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w\end{vmatrix}$

$A_{obj} = \sqrt{\begin{vmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w\end{vmatrix}}$

"Gram determinant"

3D:



$\text{Vol} = \sqrt{\begin{vmatrix} u\cdot v & u\cdot u & u\cdot w \\ v\cdot u & v\cdot v & v\cdot w \\ w\cdot u & w\cdot v & w\cdot w\end{vmatrix}}$

Example: $V_1 = (1,1,1)$ $\quad V_1\cdot V_2 = -1+1+2 = 2$

$V_2 = (-1,1,2)$ $\quad V_1\cdot V_3 = 0+1+0 = 1$

$V_3 = (0,1,0)$ $\quad V_2 \cdot V_3 = 0+1+0 = 1$

$6(1)-(1)(1) = 5$

$2(1)-(1)(1) = 1$

$2(1)-6(1) = -4$

$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1\end{vmatrix} = a\left(\begin{vmatrix} 6 & 1 \\ 1 & 1\end{vmatrix}\right) - b\left(\begin{vmatrix} 2 & 1 \\ 1 & 1\end{vmatrix}\right) + c\left(\begin{vmatrix} 2 & 6 \\ 1 & 1\end{vmatrix}\right)$

$= 3\det\begin{pmatrix} 6 & 1 \\ 1 & 1\end{pmatrix} - 2\det\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix} + 1\det\begin{pmatrix} 2 & 6 \\ 1 & 1\end{pmatrix}$

$3\cdot 5 - 2\cdot 1 + (-4)$

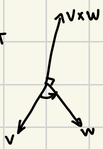
$= 15 - 2 - 4$

$= 15 - 6$

$= \sqrt{9}$

$\text{Vol} = \boxed{3}$

$\mathbb{R}^3$: Cross Product



$i = (1,0,0)$

$j = (0,1,0)$

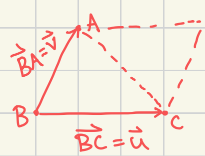
$k = (0,0,1)$

$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{vmatrix}$

$= i\det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3\end{pmatrix} - j\det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3\end{pmatrix} + k\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2\end{pmatrix}$

$\equiv \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3\end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3\end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2\end{vmatrix}\right)$

Example: $A=(1,0,1)$, $B=(1,0,0)$, $C=(-1,1,1)$



$V = A - B = (0,0,1)$

$u = C - B = (-2,1,1)$

$v\times u = \det\left(\begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ -2 & 1 & 1\end{bmatrix}\right) = i\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1\end{bmatrix}\right) - j\det\left(\begin{bmatrix} 0 & 1 \\ -2 & 1\end{bmatrix}\right) + k\det\left(\begin{bmatrix} 0 & 0 \\ -2 & 1\end{bmatrix}\right)$

$= \langle -1, -2, 0\rangle$

$A_{triangle} = \frac{1}{2}\text{norm}(-1,-2,0)$

$= \frac{\sqrt{5}}{2}$

Discussion Section

June 12^{th}, 2025

$2\cdot(-1) + 1(-1) + 0(-1) + 0(1) + 1(-1)$

$= -2-1-1 = -4$

$1+1+1+1$

$4+1+1+1$

$v = (-1,1,0,1,-1)$

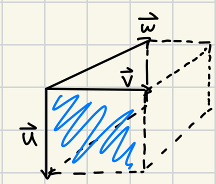
$w = (2,-1,-1,0,1)$

$\begin{bmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w\end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 7\end{bmatrix}$

$\det\left(\begin{bmatrix} 4 & -4 \\ -4 & 7\end{bmatrix}\right) = 4(7) - (-4(-4))$

$28 - 16 = \boxed{12} = A^2$

$\sqrt{12} = A$



$V = |v\times w| |w| \cos\alpha$

$= (u\times v)\cdot w$

$A = (1,1,1)$

$B = (-1,2,1)$ $\quad \vec{AB} = (-2,1,0) = \vec{B} - \vec{A}$

$C = (3,1,-1)$ $\quad \vec{AC} = (2,0,-2) = \vec{C} - \vec{A}$

$\vec{AB}\times\vec{AC} =$

$\left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ 2 & 0 & -2\end{bmatrix}\right| = \hat{i}\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -2\end{bmatrix}\right) - \hat{j}\det\left(\begin{bmatrix} -2 & 0 \\ 2 & -2\end{bmatrix}\right) + \hat{k}\det\left(\begin{bmatrix} -2 & 1 \\ 2 & 0\end{bmatrix}\right)$

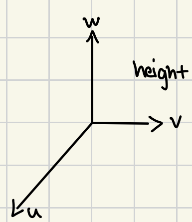
$-2+0=-2 \qquad 4-0=4 \qquad -2$

$-2\hat{i} - 4\hat{j} - 2\hat{k}$

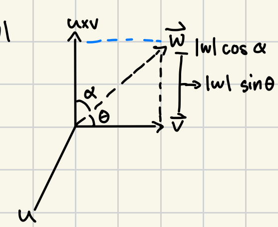
$\equiv (-2,-4,-2)$

$= \sqrt{24} = |(-2,-4,-2)|$

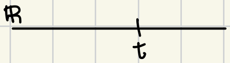
$= \frac{1}{2}\cdot\sqrt{24} = \sqrt{6}$

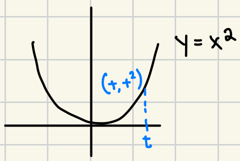


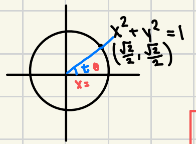
height $= |w|$



Curve:







$C: [a,b] \longrightarrow \mathbb{R}^3$

$\boxed{\begin{array}{l}\text{Example: } y = x^2: \; C(x) = (x, x^2) \\ \text{Example: } x^2+y^2=1: \; C(\theta) = (\cos\theta, \sin\theta)\end{array}}$

Example: $x^2+y^2 = y^3 \longmapsto r^2 = r^3 \sin^3\theta$

$1 = r\sin^3\theta$

$\frac{1}{\sin^3\theta} = r$

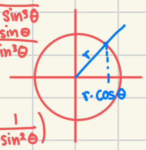
$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$x = r\cos\theta = \frac{\cos\theta}{\sin^3\theta}$

$y = r\sin\theta = \frac{\sin\theta}{\sin^3\theta} = \frac{1}{\sin^2\theta}$

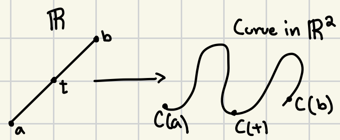


$C(\theta) = \left(\frac{\cos\theta}{\sin^3\theta}, \frac{1}{\sin^2\theta}\right)$

Lecture

June 16, 2025

$C: [a,b] \longmapsto \mathbb{R}^n$



Example: $c(t) = t$ $\quad c: [0,1] \to \mathbb{R}$

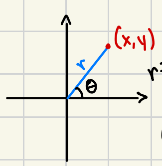
0 1

$c(t) = -t$ $\quad c: [-1,0] \to \mathbb{R}$

$c(-1) = 1$

END Start $c(0) = 0$

Polar:



$r^2 = x^2+y^2$

$r \in [0, \infty)$

$\theta \in [0, 2\pi]$

$x = r\cos\theta$

$y = r\sin\theta$

Example: $y^2 = x^4 + y^4$ $\qquad 0 \le \theta \le 2\pi$

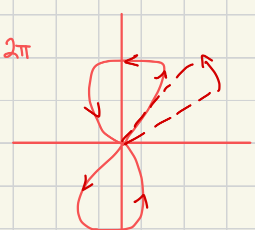
$r^2\sin^2\theta = r^4\cos^4\theta + r^4\sin^4\theta$

$r^2\sin^2\theta = r^4(\cos^4\theta + \sin^4\theta)$

$\sin^2\theta = r^2(\cos^4\theta + \sin^4\theta)$

$\frac{\sin^2\theta}{\cos^4\theta + \sin^4\theta} = r^2 \longrightarrow r = \frac{|\sin\theta|}{\sqrt{\cos^4\theta + \sin^4\theta}}$

$C(\theta) = \left(\frac{|\sin\theta|\cos\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}, \frac{|\sin\theta|\sin\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}\right) \quad \theta \in [0, 2\pi]$

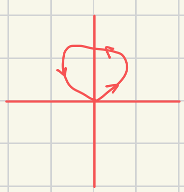


Example: $y^3 = x^4 + y^4$

$r^3\sin^3\theta = r^4\cos^4\theta + r^4\sin^4\theta$

$r = \frac{\sin^3\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$

$c(\theta) = \left(\frac{\sin^3\theta\cos\theta}{r^4\cos^4\theta + r^4\sin^4\theta}, \frac{\sin^4\theta}{r^4\cos^4\theta + r^4\sin^4\theta}\right)$



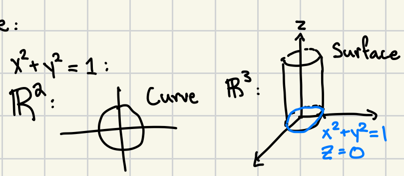
No problem restriction

No algebra restriction

Polar restriction $\Rightarrow r \ge 0$ $\quad \sin^3\theta \ge 0 \to \theta \in [0, \pi]$ $\quad 0 \le \theta \le \pi$

Surface:

$x^2+y^2 = 1$:



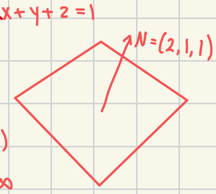
Example: $2x+y+z=1$

$N = (2,1,1)$

$z = 1-2x-y$

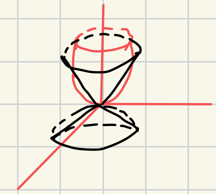
$\Phi(x,y) = (x, y, 1-2x-y)$

$, -\infty < x, y < \infty$



Example: $z = x^2+y^2$

$z^2 = x^2+y^2$

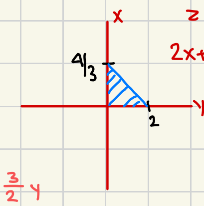


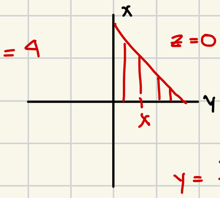
Friday: ZOOM!!!

Example: $2x+3y+2z = 4$ $\qquad N = (2,3,2)$ normal

$2z = 4-2x-3y \longrightarrow z = 2 - x - \frac{3}{2}y$

$\Phi(x,y) = (x, y, 2-x-\frac{3}{2}y) \longrightarrow 2-x-\frac{3}{2}y \ge 0 \cup x \ge 0 \cup y \ge 0$

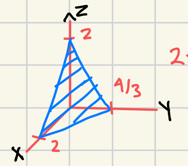




$y = \frac{4-2x}{3}$

First octant: $(x, y, z \ge 0)$

$\Phi(x,y) = (x, y, 2-x-\frac{3}{2}y) = \underbrace{(0,0,2)}_{\text{Point}} + \underbrace{x(1,0,-1) + y(0,1,-\frac{3}{2})}_{\text{tangent direction}}$



$2x+3y+2z = 4$

Vertical lines:

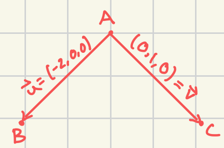
$\{(x,y): 0 \le x \le 2, \; 0 \le y \le \frac{4-2x}{3}\}$ $\qquad$ Similar to a PDF???

Example: $A = (1,0,1)$, $B = (-1,0,1)$, $C = (1,1,1)$

$\Phi(r,s) = (1,0,1) + (-2,0,0)r$

$+ (0,1,0)s$

$= (1-2r, s, 1)$



$u \times v = \begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i \det\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - j\det\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} + k\det\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

$= (0i, -0j, -2k) = (0, 0, -2)$

Standard form: $0(x-1) + 0(y-0) + -2(z-1) = 0$ $\quad (P_A)$

$\longrightarrow z = 1$

Example: Line: $\ell(t) =$ point $+ t$(direction) $\nearrow N = (1,2,3)$

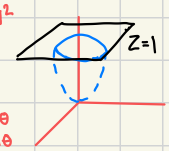
Line perpend. to $x+2y+3z=1$ thru $(-1,1,1)$

$\ell(t) = (-1,1,1) + t(1,2,3) \equiv (-1+t, 1+2t, 1+3t)$

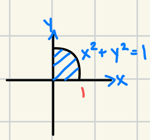
Cylindrical

Example: $z = x^2+y^2$

$\Phi(x, y, x^2+y^2)$



Restriction of 1st octant below $z=1$



$0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$ $\quad \mapsto z = z, \; x = r\cos\theta, \; y = r\sin\theta$

$z = x^2+y^2 \to z = r^2$

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$: $\{z \le 1, \; x, y, z \ge 0\}$

$\to r^2 \le 1$, $r\cos\theta \ge 0$, $r\sin\theta \ge 0$, $\cancel{r^2 \ge 0}$

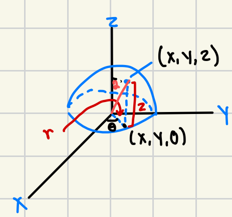
$\hookrightarrow 0 \le r \le 1 \quad 0 \le \theta \le \frac{\pi}{2}$

Example: $x^2+y^2+z^2 = 4$

$x = r\cos\theta$ $\qquad r^2+z^2 = 4$

$y = r\sin\theta$ $\qquad z = 2\cos\varphi$

$r = 2\sin\varphi$



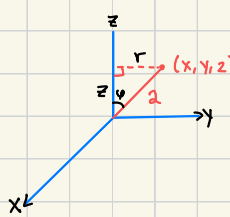
$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$x = 2\sin\varphi\cos\theta$

$y = 2\sin\varphi\sin\theta$

$z = 2\sin\varphi$



Lecture
June 20^{th}

Review: Cylindrical and Spherical Coordinates

$C(t) = (x(t), y(t), z(t))$

$\Phi(r,s) = (x(r,s), y(r,s), z(r,s))$

Definition The *spherical coordinates* of points (x, y, z) in space are the triples (ρ, θ, ϕ), defined as follows:

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi, \qquad (3)$$

where

$$\rho \geq 0, \qquad 0 \leq \theta < 2\pi, \qquad 0 \leq \phi \leq \pi.$$

Definition The *cylindrical coordinates* (r, θ, z) of a point (x, y, z) are defined by (see Figure 1.4.2)

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \qquad (1)$$

Example: $x + y + 2z = 4$

$x = 4 - y - 2z$

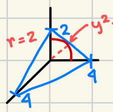
$\Phi(y,z) = (4 - y - 2z, y, z) \quad -\infty < y, z < \infty$

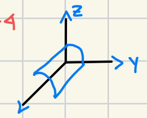
Example: $x + y + 2z = 4$ inside $y^2 + z^2 = 4$

$y = r\cos\theta$

$z = r\sin\theta$

$y^2 + z^2 = r^2$





$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$x + y + 2z = 4, \quad y^2 + z^2 \leq 4 \to r^2 \leq 4$

$r \leq 2$

$x + r\cos\theta + 2r\sin\theta = 4$

$x = 4 - r\cos\theta - 2r\sin\theta$

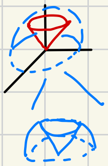
$\boxed{\Phi(r,\theta) = (4 - r\cos\theta - 2r\sin\theta, r\cos\theta, r\sin\theta): 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2}$

Example: Solid bounded by:

(inside) $x^2 + y^2 + z^2$

below $z = \sqrt{x^2+y^2}$

above xy-plane



$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

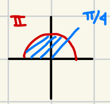
$z = \rho\cos\varphi$

$x^2 + y^2 + z^2 = \rho^2 \to x^2 + y^2 = \rho^2\sin^2\varphi : 0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$

$x^2 + y^2 + z^2 \leq 4 \to \rho^2 = 4 \Rightarrow 0 \leq \rho \leq 2$

$z \leq \sqrt{x^2+y^2} \to \rho\cos\varphi \leq \rho\sin\varphi$

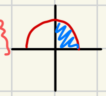
$\cos\varphi \leq \sin\varphi$



$\frac{\pi}{4} \leq \varphi \leq \pi$

$z \geq 0 \to \rho\cos\varphi \geq 0 : 0 \leq \varphi \leq \frac{\pi}{2}$

$W = \{(\rho, \theta, \varphi): 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\}$



Applications of Surfaces and curves:

$T(x,y,z) = x^2 + y^2 + z^2 \overset{s}{=} \rho^2$

Limit:

$\lim_{(x,y,z)\to p} f(x,y,z) =$ expected value for $f(p)$

If $\lim_{(x,y,z)\to p} f(x,y,z) = f(p)$, we say f is continuous

Example:

$$\lim_{(x,y)\to(3,0)} y e^{x - \sqrt{x^2 + \ln(y^2+1)}} = 0$$

Example: $\lim_{(x,y)\to 0} \frac{xy^2}{x^2+y^2}$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

$\frac{r^3\cos\theta\sin^2\theta}{r^2} = r\cos\theta\sin^2\theta$

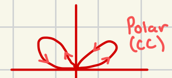
$\lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r\cos\theta\sin^2\theta = 0$

Exam Review:

1. Vectors
2. Parametrizations
3. Limits

Parametrization:

a) $x^2 y = x^4 + y^4$



$x = r\cos\theta$
$y = r\sin\theta$

$r^3\cos^2\theta\sin\theta = r^4(\cos^4\theta + \sin^4\theta)$

$\frac{\cos^2\theta\sin\theta}{\cos^4\theta + \sin^4\theta} = r$

$C(\theta) = \left(\frac{\cos^3\theta\sin\theta}{\cos^4\theta + \sin^4\theta}, \frac{\cos^3\theta\sin^2\theta}{\cos^4\theta + \sin^4\theta}\right)$

Algebraic restriction: None

Polar Restriction: $r \geq 0$: $\sin\theta \geq 0$

$0 \leq \theta \leq \pi$

Problem restriction: none

$0 \leq \theta \leq \pi$

b). Surfaces:

$y - z = x^2 + z^2$ inside $x^2 + z^2 = 1$ $\quad x^2 + z^2 \leq 1$

$y = x^2 + z^2 - z$

No restrictions:
$y = z + x^2 + z^2$
$(x, z + x^2 + z^2, z)$
$-\infty < x, z < \infty$

$x = r\cos\theta$

$z = r\sin\theta$

$x^2 + z^2 = r^2$

$y - r\sin\theta = r^2 \to y = r^2 + r\sin\theta$

$r^2 \leq 1$

$0 \leq r \leq 1$

$$\boxed{\begin{gathered}\Phi(r,\theta) = (r\cos\theta, r^2 + r\sin\theta, r\sin\theta)\\ 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\end{gathered}}$$

1^{st} Octant

$x, y, z \geq 0$

$0 \leq \theta \leq \pi/2$

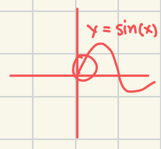
$r^2 + r\sin\theta \geq 0$

Lecture: Limits

June 23, 2025

Example: $\lim_{(x,y)\to 0} \frac{y^3\sin(x^2+y^2)}{(x^2+y^2)^k}$, find $\forall k$ for which $\lim = 0$

$\lim_{t\to 0} \frac{\sin(t)}{t} \overset{L.H}{=} \lim_{t\to 0} \frac{\cos t}{1} = 1$



When $t \sim 0$: $\sin(t) = t$

$\lim_{r\to 0} \frac{r^3\sin^3\theta\cancel{\sin(r^2)}}{r^{2k}}$ $\;\sin(r^2) \to r^2$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

$\lim_{r\to 0} \frac{r^3\sin^3\theta\, r^2}{r^{2k}} = \lim_{r\to 0} \underbrace{r^{5-2k}}_{✓}\underbrace{\sin^3\theta}_{\times} = 0$

$5 - 2k > 0 \to k < 5/2$

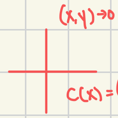
Example: $\lim_{(x,y)\to 0} \frac{xy}{x^2+y^2}$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

$\lim_{(x,y)\to 0} \frac{r^2\cos\theta\sin\theta}{r^2}$

$\lim_{(x,y)\to 0} \cos\theta\sin\theta =$ $\theta = 0 \to 0$; $\theta = \frac{\pi}{4} \to 1/2$

$\theta = 0$



$C(x) = (x, 0)$

On $(x, 0)$:

$\frac{xy}{x^2+y^2} = 0 \to 0$

$\theta = \frac{\pi}{4}$: $C(x) = (x, x)$

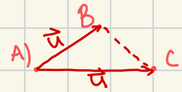
On (x, x):

$\frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = 1/2 \to 1/2$

Vectors:

$A = (1, -1, 2), \ B = (1, 2, 3), \ C = (2, 1, -1)$

a) Why are they not collinear?



Scalar multiple

$\vec{u} = \vec{B} - \vec{A} = (0, 3, 1)$

$\vec{v} = \vec{C} - \vec{A} = (1, 2, -3)$

Not a multiple of each other.

B) Find the area of ΔABC using the appropriate Gram's Determinant.

Let $u = (0, 3, 1)$
$v = (1, 2, -3)$

$\frac{1}{2}\sqrt{\begin{vmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v\end{vmatrix}} = $ Area ΔABC

$0^2 + 3^2 + 1^2 = 10$

$1^2 + 2^2 + (-3)^2 = 1 + 4 + 9 = 14$

$0 + 6 - 3 = 3$

$\frac{1}{2}\sqrt{\begin{vmatrix} 10 & 3 \\ 3 & 14\end{vmatrix}} = \boxed{\frac{\sqrt{131}}{2} = \text{Area } \Delta ABC}$

$ad - bc = 140 - 9 = 131$

C) Find the cartesian equation of the plane containing A, B, C. $\quad -9-2$

$u \times v = \begin{vmatrix} i & j & k \\ 0 & 3 & 1 \\ 1 & 2 & -3\end{vmatrix} = i\det\begin{pmatrix} 3 & 1 \\ 2 & -3\end{pmatrix} - j\det\begin{pmatrix} 0 & 1 \\ 1 & -3\end{pmatrix} + k\det\begin{pmatrix} 0 & 3 \\ 1 & 2\end{pmatrix}$

$-11i + j - 3k$

$N = (-11, 1, -3)$ or $(11, -1, 3)$

Standard form: $11(x-1) - 1(y-2) + 3(z-3)$

$11x - 11 - y + 2 + 3z - 9 = 0$

$11x - y + 3z - 18 = 0$

$\to \boxed{11x - y + 3z = 18}$

Limits:

$$f(x,y,z) = \frac{(x^4+y^4)^k\cos(x+z)}{\sqrt{x^2+y^2+z^2}}, \quad k > 0$$

a) Rewrite f using spherical coordinates

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$(x^4 + y^4)^k = \rho^{4k}\sin^{4k}\phi\,(\cos^4\theta + \sin^4\theta)^k$

$\cos(x+z) = \cos(\rho(\sin\phi\cos\theta + \cos\phi))$

$x^2 + y^2 + z^2 = \rho^2$

$x^2 + y^2 = \rho^2\sin^2\phi$ $\quad \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

$f = \rho^{4k-1}\sin^{4k}\phi(\cos^4\theta + \sin^4\theta)^k\cos(\rho(\sin\phi\cos\theta + \cos\phi))$

b) Find all $k > 0$ for which $\lim_{(x,y,z)\to 0} f(x,y,z) = 0$

$f = \underbrace{\rho^{4k-1}}_{0}\underbrace{\sin^{4k}\phi}_{\text{bounded}} \times \underbrace{(\cos^4\theta + \sin^4\theta)^k}_{\text{bounded}} \times \underbrace{\cos(\overbrace{\rho(\sin\phi\cos\theta + \cos\phi)}^{\overset{\varnothing}{\overbrace{0 \quad \text{bounded}}}})}_{1} \times$

We want $\rho \to 0$

We want $4k - 1 > 0$

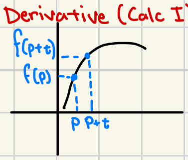
$\boxed{k > 1/4}$

$\rho^2 \to 0$
$\rho^3 \to 0$
$\rho^0 \to 1 \neq 0$

$\rho^{1/2} \to 0 \equiv \sqrt{\rho} \to 0$
$\rho^{-1} \to 0$
$\frac{1}{\rho} \to 0$

Applications of Limits:

Derivative (Calc I)



$$\lim_{t\to 0}\frac{f(p+t)-f(p)}{t}=f'(p)$$

Partial Derivatives (Calc II)

$$f_x=\lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t}$$
$$f_y=\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}$$

Discussion Section

June 24th, 2025

$$\lim_{(x,y)\to 0}\frac{y^k\ln(x^2+y^2)}{x^4+y^4}$$ $y=r\sin\theta$, $x=r\cos\theta$, $x^2+y^2=r^2$

$$\lim_{r\to 0}\frac{r^k\sin^k\theta\ln(r^2)}{r^4(\cos^4\theta+\sin^4\theta)}$$

$$\to\frac{r^{k-4}\sin^k\theta\ln(r^2)}{\cos^4\theta+\sin^4\theta}:\begin{cases}\lim_{r\to\infty}\ln(r)=\text{DNE}\ -\infty\\ \frac{\sin^k\theta}{\sin^4\theta+\cos^4\theta}\to\text{constant}\end{cases}$$

Fix θ, $r\to 0$

$\to r^{k-4}\begin{cases}0 & k>4\\ 1 & k=4\\ \text{DNE} & k<4\end{cases}$

$\lim_{(x,y)\to 0}$ "=" $-\infty\cdot\text{const}\cdot\begin{cases}0\\ \infty\end{cases}$

$-\infty\cdot\text{const}\cdot 0$

$\frac{r^k}{r^4}\cdot\log(r^2)$

$=\frac{r^k\log(r^2)}{r^4}\approx\frac{\infty}{\infty}$ or $\frac{0}{0}$

$=\lim_{r\to 0}\frac{\log(r^2)}{r^{4-k}}$

$=\lim_{r\to 0}$

Lecture

June 27, 2025

x-direction: $f:\mathbb{R}^2\to\mathbb{R}$

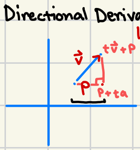
$$\lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t}=f_x=\frac{\partial f}{\partial x}$$

y-direction:

$$\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}=f_y=\frac{\partial f}{\partial y}$$

Example: $f=\frac{xy}{x^2+y^2}$ $\quad f_x=\frac{y(x^2+y^2)-xy(2x)}{(x^2+y^2)^2}$

Directional Derivative



$$\lim_{t\to 0}\frac{f(P+t\vec{v})-f(P)}{t}=f_{\vec{v}}(P)$$

Remark: $f_{-\vec{v}}=-f_{\vec{v}}$

Computation of the directional derivative:

$\vec{v}=(a,b)$

Then $f_{\vec{v}}(P)=af_x(P)+bf_y(P)$

Example: $f=\frac{x}{\sqrt{x^2+y^2+z^2+3}}$

Find all directions in which f increases when moving away from $(2,-1,1)$? $\to f_{\vec{v}}(2,-1,1)>0$

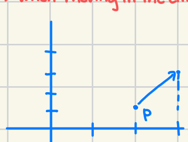
$$f_x=\frac{\sqrt{x^2+y^2+z^2+3}-\frac{x^2}{\sqrt{x^2+y^2+z^2+3}}}{(x^2+y^2+z^2+3)}$$

$$\left.\begin{aligned}&f_x(2,-1,1)=\frac{3-\frac{4}{3}}{9}=\frac{5}{27}\\&f_y=\frac{-2x}{(x^2+y^2+z^2+3)^{3/2}}=f_y(2,-1,1)=\frac{2}{27}\\&f_z=\frac{-2x}{(x^2+y^2+z^2+3)^{3/2}}=f_z(2,-1,1)=-\frac{2}{27}?\end{aligned}\right\}\ \frac{5}{27}a+\frac{2}{27}b-\frac{2}{27}c>0$$

$\boxed{\text{All directions }(a,b,c)\text{ with }5a+2b-2c>0}$

Example:

$f: 2x^5-76y^3$, is f increasing or decreasing at $P=(2,1)$ when moving in the direction $\vec{v}=(1,2)$? $\}\ f_{\vec{v}}(P)=+$ or $-$?



Decreasing!

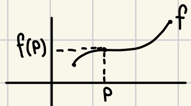
$f_x=10x^4$ $\quad\hat{v}=\frac{(1,2)}{\|(1,2)\|}=\frac{1}{\sqrt{5}}(1,2)$

$f_y=-228y^2$ $\quad f_{\vec{v}}=\frac{10x^4}{\sqrt{5}}-\frac{2(228y^2)}{\sqrt{5}}$

$f_{\vec{v}}(P)=f_{\vec{v}}(2,1)=\frac{160-456}{\sqrt{5}}=\frac{-296}{\sqrt{5}}$

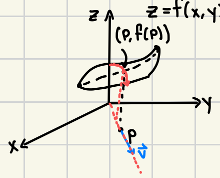
Lecture

June 30th, 2025



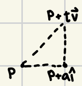
Calculus I

Multivariate:



$\vec{v}$ = unit vector (direction) $\quad$ Where $\vec{v}=(a,b)$

$$f_{\vec{v}}(P)=\lim_{t\to 0}\frac{f(P+t\vec{v})-f(P)}{t}\equiv af_x(P)+bf_y(P)$$



When $t\approx 0$: $f(P+t\vec{v})\approx f(P)+f_{\vec{v}}(P)t$

$f(P+h)\approx f(P)+f_{\vec{v}}(P)\cdot|h|$

$f(P+ta\hat{\imath})\approx f(P)+f_x(P)ta$

$f(P+t\vec{v})\approx f(P+ta\hat{\imath})+f_y(P+ta\hat{\imath})tb$

$\approx f(P)+(f_x(P)a+f_y(P+ta\hat{\imath})b)t$

$\approx f(P)+(\underbrace{f_x(P)a+f_y(P)b}_{f_{\vec{v}}(P)})t$

$$f_x(P)a+f_y(P)b\equiv\underbrace{\overbrace{(f_x(P),f_y(P))}^{\nabla f(P)}}_{\text{the gradient of }f}\cdot\overbrace{(a,b)}^{\vec{v}}$$

$f(P+\overbrace{t\vec{v}}^{\vec{h}})\approx f(P)+\nabla f(P)\cdot\vec{v}t$

$f(P+\vec{h})=f(P)+\nabla f(P)\cdot\vec{h}$ (Linear Approximation)

Example: Q near P

$f(Q)\approx f(P)+\nabla f(P)\cdot\vec{h}$

$\vec{h}=Q-P$

Example: $f(x,y)=(e^x-y,\ \sin(x-y))$

a) Write down the linear approximation near $P=(0,0)$

$f(x,y)=\overbrace{(1,0)}^{f(0,0)}+(\Delta x-\Delta y,\ \Delta x-\Delta y)$

$\boxed{f(x,y)=(1+x-y,x-y)}$

$f_1=e^x-y\approx 1+\overbrace{(1,-1)}^{\nabla f(P)}\cdot\overbrace{(\Delta x,\Delta y)}^{\vec{h}}$

$f_x=e^x$ $\quad\nabla f(P)=(e^0,-1)$

$f_y=-1$

$e^x-y=(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots)-y$

$\sin(x-y)=x-y$

$\sin(t)\approx t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots$

Lecture - Jacobian Matrix

July 2nd, 2025

$f:\mathbb{R}^n\to\mathbb{R}$

$f(x_1,\ldots,x_n)\approx f(P)+f_{x_1}(P)\Delta x_1+f_{x_2}(P)\Delta x_2+\ldots+f_{x_n}(P)\Delta x_n$ (near P)

$\equiv f(P)+\nabla f(P)\cdot h$ where $\nabla f(P)=(f_{x_1}(P),\ldots,f_{x_n}(P))$ and $h=(\Delta x_1,\ldots,\Delta x_n)$

Example: $f(x,y)=(\cos(x+y),\ e^{2-x+y})$ near $P=(1,-1)$

$f_1=\cos(x+y)$ $\quad\nabla f_1(1,-1)=(-\sin(x+y),-\sin(x+y))|_{(1,-1)}=(0,0)$

$f_1\approx f(1,-1)+0\Delta x+0\Delta y=1$

$f_2=e^{2-x+y}$ $\quad\Delta f_2(1,-1)=(-e^{2-x+y},e^{2-x+y})|_{(1,-1)}=(-1,1)$

$f_2\approx f_2(1,-1)+(-1)\Delta x+(1)\Delta y=1-\Delta x+\Delta y$

$\Rightarrow f\approx(1,1-\Delta x+\Delta y)$ $\quad\Delta x=x-1$, $\Delta y=y+1$

Examples: $2x+3y-2z=1$ $\quad$ normal $\vec{v}=(2,-3,-2)$

$P=(1/2,0,0)$

$x^2+y^2+z^2=1\longrightarrow x^3+y^2+z^2=1$, $P=(-1,1,1)$

$f(P)=0$ $\quad x^3+y^2+z^2-1=0=f$ $\quad\Delta x=x+1$, $\Delta y=y-1$, $\Delta z=z-1$

$f\approx 0+3\Delta x+2\Delta y+2\Delta z$ $\quad\nabla f=(3x^2,2y,2z)$

$\nabla f(P)=(3,2,2)$ $\quad$ Actual surface $f=0$

Linear Approx.

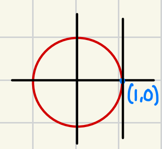
Tangent Plane $\to 3(x+1)+2(y-1)+2(z-1)=0$

$f=0$ curve/surface, P

$\nabla f(P)$ = normal vector

$c(\theta)=(\cos\theta,\sin\theta)$, $0\le\theta\le 2\pi$

$\theta=0$



$L(\theta)=(1,\theta)$

$f(x,y,z)=(f_1,f_2)$

$\approx(f_1(P)+\nabla f_1(P)\cdot h,\ f_2(P)+\nabla f_2(P)\cdot h)$

$=(f_1(P),f_2(P))+(f_{1x}\Delta x+f_{1y}\Delta y+f_{1z}\Delta z,\ f_{2x}\Delta x+f_{2y}\Delta y+f_{2z}\Delta z)$

$$=\begin{bmatrix}f_1(P)\\f_2(P)\end{bmatrix}+\underbrace{\begin{bmatrix}f_{1x}(P)&f_{1y}(P)&f_{1z}(P)\\f_{2x}(P)&f_{2y}(P)&f_{2z}(P)\end{bmatrix}}_{D_Pf}\begin{bmatrix}\Delta x\\\Delta y\\\Delta z\end{bmatrix}$$

Lecture

July 7^{th}, 2025

Linear Approximation

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f = f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$

Near P:

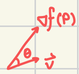
$f(x_1, \dots, x_n) \approx f(P) + f_{x_1}(P)\Delta x_1 + \dots + f_{x_n}(P)\Delta x_n$

When $m=1$: $f(x_1, \dots, x_n) \approx f(P) + \nabla f(P)\cdot h$, where $h = (\Delta x_1, \dots, \Delta x_n)$

$m>1$: $f(x_1, \dots, x_n) \approx f(P) + (D_P f)(h)$, where $D_P f = \begin{bmatrix} \nabla f_1(P) \\ \vdots \\ \nabla f_m(P) \end{bmatrix}$

Ways to interpret $D_P f$

$f: \mathbb{R}^n \longmapsto \mathbb{R}$ (temperature, etc.



$f_{\vec{v}}(P) = \nabla f(P)\cdot \vec{v} = |\nabla f(P)| \cos\theta$

$\nabla f(P)$ = the direction of largest increase

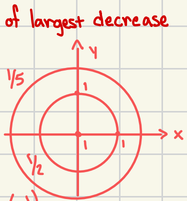
$-\nabla f(P)$ = the direction of largest decrease

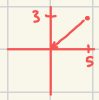
Example: $f(x,y) = \frac{1}{1+x^2+y^2}$

$\nabla f(5,3) =$ $(-5,-3)$

$\nabla f = \left(-\frac{2x}{(1+x^2+y^2)^2}, -\frac{2y}{(1+x^2+y^2)^2}\right)$

$\nabla f(5,3) = \left(-\frac{2}{35^2}(5), \frac{-2}{35^2}(3)\right)$





Second Interpretation of $D_P f$

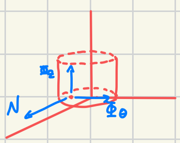
$f(x,y,z) = 0$

$0 = f_x(P)(x-P_x) + f_y(P)(y-P_y) + f_z(P)(z-P_z)$

$\nabla f(P)$ = normal vector to the surface.

Final Intepretation of $D_P f$

Example: $x^2+y^2=1$ in $\mathbb{R}^3$



$r^2 = 1 \longleftrightarrow r=1$

$\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$

$P = (0,1) \to \theta = 0, z=1 \longmapsto (x=1, y=0, z=1)$

$\Phi(\theta, z) \approx \Phi(P) + \Phi_\theta(P)\Delta\theta + \Phi_z(P)\Delta z$

$N = \Phi_\theta \times \Phi_z = (0,1,0)\times(0,0,1) = (1,0,0)$ outward normal.

Tangent Vectors:

$\Phi_\theta = (-\sin\theta, \cos\theta, 0)\big|_{\substack{\theta=0\\z=1}} = (0,1,0)$

$\Phi_z(P) = (0,0,1)\big|_{\substack{\theta=0\\z=1}}$

$f(x,y,z) = (x - e^{x^2-y}, z\cos(x), y\sin(z))$

Quadratic Approximation:

at $(0,0,0)$

$f = (x - e^{x^2-y}, z\cos(x), y\sin(z))$ near $P = (0,0,0)$

$e^t = 1 + t + \frac{t^2}{2} + \dots$

$\cos(t) = 1 - \frac{t^2}{2} + \dots$

$\sin(t) = t - \frac{t^3}{3!} + \dots$

$x - e^{x^2-y} = x\left(1 + (x^2-y) + \frac{(x^2-y)^2}{2} + \dots\right)$ ($\underbrace{\cancel{x^4} - \cancel{2x^2y} + y^2}$)

$\approx x - 1 - x^2 + y - \frac{1}{2}y^2$

$z\cos(x) = z\left(1 - \frac{x^2}{2} + \dots\right) \approx z - \cancel{\ }\approx z$

$y\sin(z) = y(z - \dots) \approx yz$

$F = (x - 1 - x^2 + y - \frac{1}{2}y^2, z, yz)$

$f: \mathbb{R}^n \to \mathbb{R}$

To simplify: $n=2$

near $P = (x_0, y_0)$:

$f(x,y) \approx f(P) + f_x(P)\Delta x + f_y(P)\Delta y$

$f(P) + \frac{1}{2}\left(f_{xx}(P)\Delta x^2 + 2f_{xy}(P)\Delta x\Delta y + f_{yy}(P)\Delta y^2\right)$

$D^2 f = D\nabla f = H(f(P))$ $\quad h$

$f(x,y) \approx f(P) + \nabla f(P)\cdot h + \frac{1}{2}[\Delta x \; \Delta y]\begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$

$\equiv f(P) + \nabla f(P)\cdot h + \frac{1}{2} h^T\cdot h \cdot \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{bmatrix}$

$\frac{1}{2} h^T H(f(P)) h$

Determinants:

$$Hf = \begin{array}{cc} D_1 & D_2 \\ \end{array} \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$x^2 + 2xy + 2y^2 \equiv (x^2+y^2) + y^2$

Max: When
D_1 = negative # , D_2 = positive # $\quad \begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix}$

Min: When
D_1 = positive # , D_2 = positive # $\quad \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$

$z = f(x,y)$ $\quad \nabla f(P) = 0$

$H(f(P)) > 0$

Min

$\therefore$ P is a critical point $\quad$ Must be positive

$f(x,y) \approx f(P) + \frac{1}{2} h^T H(f(P)) h$

Lets look at possible quadractic portions:

- $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = H(f(P)) \longleftrightarrow a\Delta x^2 + b\Delta y^2$

Positive: When (min) $a, b > 0$ $\quad$ Negative: When (max) $a, b < 0$

Or a saddle point when

$a > \emptyset, b < \emptyset \vee a < \emptyset, b > \emptyset$

$2\Delta x^2 - \Delta y^2$

$\boxed{\text{If } H > 0, \text{ and } f_x(x_0, y_0) > 0 \qquad H = 0 \text{ then test is inconclusive}}$

Lecture

July 11^{th}, 2025

$z = f(x,y)$

$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$ $\quad a\Delta x^2 + b\Delta y^2$

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ $\quad a\Delta x^2 + b\Delta y^2 + c\Delta z^2$

min: $\begin{bmatrix} + & & \\ & + & \\ 0 & & + \end{bmatrix}$ (D_1, D_2, D_3)

$\{D_1, D_2, D_3\} = +, +, +$

max: $\begin{bmatrix} (-) & & 0 \\ & (-) & \\ 0 & & (-) \end{bmatrix}$ (D_1, D_2, D_3)

$\{D_1, D_2, D_3\} = -, +, -$

$\nabla f(P) = 0$

$H(f(P)) = \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{bmatrix}$

$f(P) + h^T H(f(P)) h$

$> 0 \Rightarrow$ min

$< 0 \Rightarrow$ max

otherwise $\Rightarrow$ saddle

saddle: else if

det $H(f) \neq 0$.

If det $H(f) = 0$: inconclusive.

Quiz

b) $f = x^3 + 3y^2 + z^2 + 3xz + 3xy$

$f_x = 3x^2 + 3z + 3y = 0 \to 4y^2 + 3y + y = 0$

$f_y = 6y + 3x = 0 \longrightarrow x = -2y$

$f_z = 2z + 3x = 0 \to 2z - 6y = 0 \to z = 3y$

$\nabla f = (3x^2 + 3z + 3y, \; 6y + 3x, \; 2z + 3x)$

$\to 4y^2 + 4y = 0 \longrightarrow y^2 + y = 0$

$y(y+1) = 0 \begin{cases} y = 0 \\ y = -1 \end{cases}$

$y = 0$ $\quad$ $y = -1$

$x = -2y = 0$ $\quad$ $x = 2$

$z = 3y = 0$ $\quad$ $z = -3$

$(0,0,0)$ $\quad$ $(2,-1,3)$

$H(f(P)) = \begin{bmatrix} 6x & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

$H(f(0)) = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

$\det(H(f(0))) = 0\cancel{\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix}} - 3\begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix} = (-)$ $\quad$ (0, 6, -18)

$D_1 = 0$?

$D_2 = -9$

$\{?, -, -\}$: Saddle (D_1, D_2, D_3)

$H(f(2,-1,-3)) = \begin{bmatrix} 12 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

$D_1 = +$

$D_2 = +$

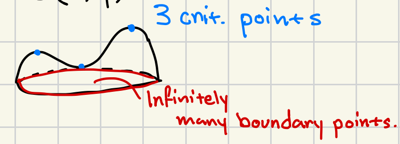
$D_3 = 12\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix}$ $\quad$ (12, 6, -18)

$= 144 - 18 - 3(18) = +$

Positive-definite

$\therefore$ minimum

$z = f(x,y)$



LaGrange Multiples:

Q_1: Find max. of $f(x,y)$ when $g(x,y)=0$

Example: $f(x,y)=y-x^2$ subject to $x^2+y^2=1$
$\equiv g=x^2+y^2-1$

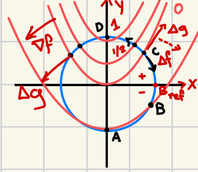
$f=y-x^2=0$
$y=x^2$

$f=1=y-x^2$
$y=x^2+1$

$f=-1=y-x^2$
$y=x^2-1$

Candidate: Tangential intersection

$\boxed{\nabla f=\lambda\nabla g}$
$g=0$



C → B (>0)
C → D (<0) } Transversal intersect.

C → Saddle
B_{ref} → Saddle } Transversal

{A, D} = tangential intersect.

A → Local max
D → Local max

$f=xy$ on $x^2+4y^2=4$

$x^2+(2y)^2=4$

$x=2\cos\theta$
$y=2\sin\theta$ } $C(\theta)=(2\cos\theta,\ \sin\theta),\ 0\le\theta\le 2\pi$

Max/min:

$f=xy=2\cos\theta\cdot\sin\theta$ when $0\le\theta\le 2\pi$

C.P: $f'=2\cos^2\theta-2\sin^2\theta \Longrightarrow \cos^2\theta=\sin^2\theta$
$\left\{\frac{5\pi}{4},\ \frac{\pi}{4},\ \frac{7\pi}{4},\ \frac{3\pi}{4}\right\}$
$\left\{\left(-\sqrt2,-\frac{\sqrt2}{2}\right),\left(\sqrt2,\frac{\sqrt2}{2}\right),\left(\sqrt2,\frac{-\sqrt2}{2}\right),\left(-\sqrt2,\frac{\sqrt2}{2}\right)\right\}$

B.P.: $\theta=0,2\pi$
$(1,0)$

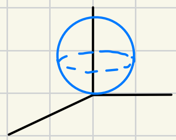
Lecture
July 14, 2025

Review of Lagrange Multipliers: W

Example: $f=x^2-y^2+z^2$ on $x^2+y^2+z^2\le 1$

$\text{int } W=\{(x,y,z): x^2+y^2+z^2<1\}$

$\partial W=\{(x,y,z): x^2+y^2+z^2=1\}$



1 In int W:

$\nabla f=(2x,-2y,2z)$

$\nabla f(0)=(0,0,0)$

2 In ∂W ($x^2+y^2+z^2=1 \longrightarrow g=x^2+y^2+z^2-1$)

$\nabla f=\lambda\nabla g$: $2x=\lambda\cdot 2x \longrightarrow x=\lambda x$
$\cancel{2y=\lambda\cdot 2y}$ $\quad -y=\lambda y \longrightarrow -yx=\lambda xy=xy$
$2z=\lambda\cdot 2z \quad z=\lambda z$ $\quad 2xy=0 \Longrightarrow \{0,0\}$
$\hookrightarrow xz=\lambda xz=xz$
$yz=\lambda yz=-yz$
$yz=0\} \mapsto y=0,\ z=0$

$x^2+y^2+z^2=1$

$x^2+z^2=1$
$y=0$

$(0,0,1)$ $\quad (0,1,0)$
$(0,0,-1)$ $\quad (0,-1,0)$
$(1,0,0)$
$(-1,0,0)$

3 List

	(x,y,z)	$f=x^2-y^2+z^2$
int W	$(0,0,0)$	0
∂W	$(0,\pm1,0)$	-1
	$(x,0,z)$, $x^2+z^2=1$	1

MAX = 1
MIN = −1

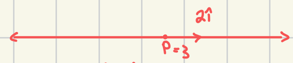
Differential Forms:

1-forms:

In $\mathbb{R}$: $dx:\mathbb{R}\longrightarrow\mathbb{R}$

$dx(P)(\vec v)$ = how much you walk in the x-direction

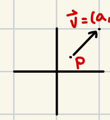
$dx(2\hat\imath)=2$



$dx(P)(2\hat\imath)=2$

$W=\int f(x)\,dx$

In $\mathbb{R}^2$:



$dx(a\hat\imath+b\hat\jmath)=a \equiv dx(P)(a\hat\imath+b\hat\jmath)$

$dy(a\hat\imath+b\hat\jmath)=b \equiv dy(P)(a\hat\imath+b\hat\jmath)$

$w=A\,dx+B\,dy \equiv A(x,y)+B(x,y)\,dy,\quad A,B:\mathbb{R}^2\longrightarrow\mathbb{R}$

"Fake 1-form"

$ds=\sqrt{dx^2+dy^2}$

$ds(\vec v)=|\vec v|$

Example:

$(x^2,-3xy)\cdot(dx,dy)=\omega=x^2\,dx-3xy\,dy$

$\omega(1,2)(\vec v)=3+6=9$

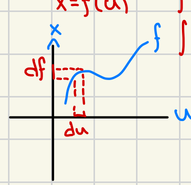
$\vec v=(3,-1)$

$(x^2,-3xy)$

Pullback:

$\mathbb{R}$: $dx \longleftrightarrow df=f'(u)\,du$

$x=f(u)$ $\quad \int f'(u)\,du=\int df$



$\int g(\underbrace{f(u)}_{x})\underbrace{f'(u)\,du}_{dx}=\int g(x)\,d(x)$

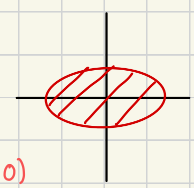
$\mathbb{R}^2$: $\omega=A\,dx+B\,dy$

$x=f(u,v) \longrightarrow dx\sim df=f_u\,du+f_v\,dv$

$y=g(u,v)$ $\quad dy\sim dg=g_u\,du+g_v\,dv$

Lecture
July 16th, 2025

$f(x,y)=xy$ when $x^2+4y^2\le 4$



int $x^2+4y^2<4$:

$\nabla f=(y,x)=0 \Longrightarrow (x,y)=(0,0)$

Boundary: $x^2+4y^2=4=g(x,y)$

$\nabla f=\lambda\nabla g$

$y=2\lambda x$

$x=8\lambda y \longrightarrow x^2=8\lambda yx=4y\overbrace{(2\lambda x)}^{y}=4y^2$

$x^2+4y^2=4$ $\qquad x^2=4y^2$

$4y^2+4y^2=4$ $\qquad x^2=2$

$y=\pm\sqrt{1/2}$ $\qquad x=\pm\sqrt2$

y { $-\sqrt{1/2}$ $\quad \sqrt{1/2}$

x { $\sqrt2$ $\quad -\sqrt2$ $\quad \sqrt2$ $\quad -\sqrt2$

(x,y)	$x\cdot y$
$(0,0)$	0
$\left(\sqrt2,\frac{\sqrt2}{2}\right)$	1
$\left(\sqrt2,-\frac{\sqrt2}{2}\right)$	−1
$\left(-\sqrt2,\frac{\sqrt2}{2}\right)$	−1
$\left(-\sqrt2,\frac{-\sqrt2}{2}\right)$	1

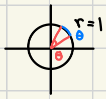
MAX = 1 at $\left\{\pm\sqrt2,\pm\frac{\sqrt2}{2}\right\}$

MIN = −1 at $\left\{\pm\sqrt2,\mp\frac{\sqrt2}{2}\right\}$

Example:

$x^2+y^2=1$

$r=1$



$dx = \cos\theta\, dr - r\sin\theta\, d\theta$ $\quad x = r\cos\theta$

$dy = \sin\theta\, dr + r\cos d\theta$ $\quad y = r\sin\theta$

On $x^2+y^2=1$ $(r=1)$ $\quad ds = \sqrt{\sin^2\theta\, d\theta^2 + \cos^2\theta\, d\theta}$

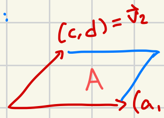
$= \sqrt{d\theta^2} = |d\theta|$

$dr = 0$ $\quad dx = -\sin\theta\, d\theta$

$r=1$ $\quad dy = \cos d\theta$

2-form:

FACT:



$\Longrightarrow A = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$(dx\wedge dy)(P)(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$dy\wedge dx = -dy\wedge dx \longrightarrow dx\wedge dx = 0$

M 427 Midterm II Review

July 19, 2025

1. Taylor Approximations
2. Tangent Lines and Planes
3. Minimum and Maximum

Min/Max Problems:

Problem #1: $f(x,y,z) = x^2 - xz + y^2 - z^2$

(a) Find and classify the critical points

$\nabla F = (2x-z,\ 2y,\ -x-2z)$

$\nabla F = 0 \to \begin{cases} 2x - z = 0 \to z = 2x \\ 2y = 0 \to y = 0 \\ -x - 2z \longrightarrow -x-(2x) = -x-4x = 0 \end{cases}$ $\quad x=0$, $z=0$

Critical point: $(0,0,0)$

$Hf = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -2 \end{bmatrix}$

$\boxed{\begin{array}{l} \{D_1, D_2, D_3\}: +,+,+ \text{ (min)} \\ \{D_1, D_2, D_3\}: -,+,- \text{ (max)} \\ \text{else: saddle point} \end{array}}$

Inconclusive if $D_3 = 0$

Test for inconclusion (D_3):

$\underbrace{2\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}}_{-4} - \cancel{0\begin{vmatrix} 0 & 0 \\ -1 & -2 \end{vmatrix}}^{0} + \underbrace{(-1)\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}}_{2}$

$= -10 = \det(D_3)$

$D_1 = 2$

Where $\{D_1, D_2, D_3\} = \{+,+,-\}$ Saddle point

b) Find max/min on $D = \{(x,y,z): x^2+y^2+z^2 \le 1\}$

$\text{int}(D)$: $x^2+y^2+z^2 < 1$ $(0,0,0)$

$\partial D: x^2+y^2+z^2 = 1 \xmapsto{G} G(x,y,z) = x^2+y^2+z^2-1$

$\nabla F = (2x-z,\ 2y,\ -x-2z)$, $\nabla G = (2x,\ 2y,\ 2z)$

$2x - z = 2\lambda x \longrightarrow y(2x-z) = 2(\lambda y)x \longrightarrow y(2x-z) = 2xy$

$2y = 2\lambda y \longrightarrow y = \lambda y$ $\quad \longrightarrow yz = 0$

$-x-2z = 2\lambda z$

$y(-x-2z) = 2(\lambda y)z$

$y(-x-2z) = 2yz$

$-yx - 2yz = 2yz$

$y(4z+x) = 0$

$y=0$: $4z+x=0 \to x = -4z$; $16z^2 + z^2 = 1$, $17z^2 = 1$, $z = \pm\frac{1}{\sqrt{17}}$ $\Rightarrow \left(\mp\frac{4}{\sqrt{17}}, 0, \pm\frac{1}{\sqrt{17}}\right)$

$z=0$: $4z+x=0 \to x=0$, $y^2=1$, $(0,\pm 1, 0)$; $y=0$

Branch: $4z+x=0$ $(y=0)$

$-\frac{9}{\sqrt{17}} = -\frac{8}{\sqrt{17}}\lambda$ where $\lambda = \frac{9}{8}$

$0 = \lambda\cdot 0 = 0$ ✓

$-\frac{6}{\sqrt{17}} = \frac{2}{\sqrt{17}}\cdot\lambda$, where $\lambda = -3$

Branch: $(y=0)\ (y=0)$

$2x - z = 2\lambda x$

$-x-2z = 2\lambda z$

$x^2 + z^2 = 1$

$-x^2 - 2xz = (\lambda x)z$ $\quad (-2x-z)$

$-x^2 - 2xz = -2xz - z^2$

$x^2 = z^2 \to |z| = |x|$

$x^2 + z^2 = 1 \longrightarrow 2x^2 = 1$

$x^2 = 1/2$

$\left(\frac{1}{\sqrt2}, 0, \pm\frac{1}{\sqrt2}\right)$ $\quad \left(\frac{-1}{\sqrt2}, 0, \pm\frac{1}{\sqrt2}\right)$

(x,y,z)	$f(x,y,z)$
$(0,0,0)$	0
$(0,\pm1,0)$	1
$\left(\pm\frac{1}{\sqrt2}, 0, \pm\frac{1}{\sqrt2}\right)$	$-1/2$
$\left(\pm\frac{1}{\sqrt2}, 0, \mp\frac{1}{\sqrt2}\right)$	$1/2$

Max: 1

Min: $-1/2$

Problem #2: Let $f = x^2 + x + y^2 - z^2$ on $\partial D = \{(x,y,z): x^2+y^2+z^2 = 1\}$

$\nabla F = \lambda\cdot\nabla G$ $\quad$ let $G = x^2+y^2+z^2-1$

$\nabla F = (2x+1,\ 2y,\ -2z)$, $\nabla G = (2x,\ 2y,\ 2z)$

$2x+1 = 2\lambda x$

$2y = 2\lambda y \longrightarrow y = \lambda y$

$-2z = 2z \longrightarrow -z = \lambda z$

$yz = \overset{-z}{\widetilde{\lambda z}}\, y$

$yz = -yz$

$yz = 0$

$z(2x+1) = (2\lambda x)z$ $\quad$ $(2x+1)y = 2\lambda y x$

$z(2x+1) = 2(\lambda z)x$ $\quad$ $(2x+1)y = 2yx$

$2xz + z = -2xz$ $\quad$ $y = 0$

$(4x+1)z = 0$

$y=0$: $z=0$ → $x^2=1$, $x=\pm1$, $(\pm1, 0, 0)$; $4x+1=0$, $x = -1/4$, $z^2 = \frac{15}{16}$, $z = \pm\frac{\sqrt{15}}{4}$, $\left(-\frac14, 0, \pm\frac{\sqrt{15}}{4}\right)$

$z=0$: $y=0$, $x^2=1$, $(\pm1, 0, 0)$

(x,y,z)	$f(x,y,z)$
$(1,0,0)$	$1+1+0-0=2$
$(-1,0,0)$	$1-1+0-0=0$
$\left(-1/4, 0, \pm\frac{\sqrt{15}}{4}\right)$	$1/16 - 1/4 + 0 - \frac{15}{16} = -9/8$

MAX: 2, MIN: $-9/8$

Tangent Lines and Planes:

Problem #1: Find the cartesian equation for the tangent plane to

$e^{y^2-z^2} = x^2 - \ln(1+xy-z)$

at point $P = (-1,-1,1)$

$f = e^{y^2-z^2} - x^2 + \ln(1+xy-z) = 0$

$\nabla f = \left(-2x + \frac{y}{1+xy-z},\ 2ye^{y^2-z^2} + \frac{x}{1+xy-z},\ -2ze^{y^2-z^2} - \frac{1}{1+xy-z}\right)$

$\nabla f(-1,-1,1) = (1,-3,-3)$

$\boxed{(x+1) - 3(y+1) - 3(z-1) = 0}$

b) Parametrization of the tangent line to the curve

$C: \begin{cases} z = x^2 + y^2 \\ z = y^2 + y + x \end{cases}$

at $P = (-1,2,5)$

$x^2 + \cancel{y^2} = \cancel{y^2} + y + x$

$y = x^2 - x$

$C(x) = (x,\ x^2 - x,\ x^4 - 2x^3 + 2x^2)$

$C'(x) = (1,\ 2x-1,\ 4x^3 - 6x^2 + 4x)$

$x=-1$: $\boxed{C'(-1) = (1,-3,-14)}$

$L(x) = (-1,2,5) + (1,-3,-14)(x+1)$

Taylor Approximations:

Problem #1: $f = (z e^{x-y^2}, x - \cos(z-x))$

Find the quadratic approximation at $P = (0,0,0)$

$$e^t = 1 + t + \frac{t^2}{2} + \dots$$
$$\cos(t) = 1 - \frac{t^2}{2} + \dots$$
$$\sin(t) = t - \cancel{\frac{t^3}{3!}} + \dots$$

$$z e^{x-y^2} = z\left(1 + (x-y^2) + \frac{(x-y^2)^2}{2}\right)$$
$$= z + zx - \cancel{zy^2}$$
$$= z + zx$$
$$x - \cos(z-x) = x - \Big(1 - \frac{\overbrace{(z-x)^2}^{z^2-2xz+x^2}}{2} + \dots\Big)$$
$$= x - 1 + \frac{z^2}{2} - xz + \frac{x^2}{2} + \dots$$
$$Q(x,y,z) = \left(z + xz, -1 + x + \frac{x^2}{2} - xz + \frac{z^2}{2}\right)$$

Lecture

July 23, 2025

1-forms:

$dx \rightsquigarrow \hat{i},\ dy \rightsquigarrow \hat{j},\ dz \rightsquigarrow \hat{k}$

$\mathbb{R}$: $\omega = f(x)\,dx$

$\mathbb{R}^2$: $\omega = A(x,y)\,dx + B(x,y)\,dy \rightsquigarrow F = (A,B)$; $F: \mathbb{R}^n \to \mathbb{R}^n$

$\mathbb{R}^3$: $\omega = A\,dx + B\,dy + C\,dz$ $\quad F = (A,B,C)$

2-forms:

$\mathbb{R}$: $\omega = 0$ $\qquad (dx \wedge dy)(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} \vec{v}_{1x} & \vec{v}_{2x} \\ \vec{v}_{1y} & \vec{v}_{2y} \end{vmatrix}$

$\mathbb{R}^2$: $(dx \wedge dy) f(x,y) = \omega$

$\mathbb{R}^3$: $\omega = A(dy\wedge dz) + B(dz \wedge dx) + C(dx \wedge dy)$

$dy \wedge dx = -dx \wedge dy$

$dx \wedge dx = 0$

$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$

$\det(r_1, r_{2,1} + r_{2,2}) = \det(r_1, r_{2,1}) + \det(r_1, r_{2,2})$

$\det(r_1, \lambda r_2) = \lambda \det(r_1, r_2)$

3-forms: (top/volume forms)

$\mathbb{R}$: $\omega = 0$

$\mathbb{R}^2$: $\omega = 0$

$\mathbb{R}^3$: $f(x,y,z)\,dx \wedge dy \wedge dz$

$dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$

$dz \wedge dx \wedge dy = +dx \wedge dy \wedge dz$

Defn: ω: top form

$\int_D \omega$ = integral from calc 2 you are used to

Example: $\omega = f(x)\,dx$
$$\int_{[a,b]} \omega = \int_a^b f(x)\,dx$$

Example: $\omega = f(x,y)(dx \wedge dy)$
$$\int_D \omega = \iint_D f(x,y)\,dx\,dy$$

Change of Variables:

1-forms:

$\omega = f(x)\,dx \longleftrightarrow \tilde{\omega} = f(g(u)) \cdot g'(u)\,du$

$x = g(u)$

$dg = g'(u)\,du$
$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u)) \cdot g'(u)\,du$$
$$\int_{g(b)}^{g(a)} f(x)\,dx = \int_a^b f(g(u)) \cdot |g'(u)|\,du$$

Theorem:

Let $\omega = f(x,y)\,dx \wedge dy$

$(x,y) = g(u,v) = (g_1(u,v), g_2(u,v))$

$dx = g_{1,u}\,du + g_{1,v}\,dv$

$dy = g_{2,u}\,du + g_{2,v}\,dv$

$g^*(dx \wedge dy) = (g_{1,u} g_{2,v} - g_{1,v} g_{2,u})\,du \wedge dv$

$$dx \wedge dy = \begin{vmatrix} g_{1,u} & g_{1,v} \\ g_{2,u} & g_{2,v} \end{vmatrix} du \wedge dv$$
$$= \det(Dg)\,du \wedge dv$$

Jacobian
$$\iint_{g(D)} f(x,y)\,dx\,dy = \iint_D f(g(u,v)) \cdot |\det(Dg)|\,du\,dv$$

Important Jacobians:

$\mathbb{R}^2/\mathbb{R}^3$: polar $\quad x = r\cos\theta$, $y = r\sin\theta$ $\quad \det Dg = r$

$\mathbb{R}^3$: spherical $\quad x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho\cos\varphi$ $\quad \det Dg = \rho^2 \sin\varphi$

Lecture

July 25, 2025

$\mathbb{R}$: $dx = |f'(u)|\,du$ $\quad x = f(u)$

$\mathbb{R}^2$: $dx\,dy = |\det Dg|\,du\,dv$ $\quad (x,y) = g(u,v)$

Important Jacobians:

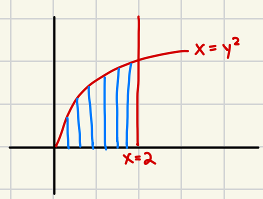
Polar/Spherical: r

Spherical: $\rho^2 \sin(\varphi)$

$$\int_a^b \int_{g(x)}^{h(x)} dy\,dx$$

$a \le x \le b$, $g(x) \le y \le h(x)$

Example: $\iint_A 14 y e^{x^2}\,dx\,dy$, $A: \begin{Bmatrix} x = y^2 \\ x = 2 \\ y = 0 \end{Bmatrix}$



$0 \le x \le 2$

$0 \le y \le \sqrt{x}$

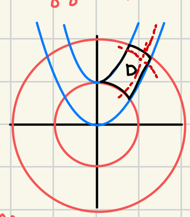
$$\int_0^2 \int_0^{\sqrt{x}} 14 y e^{x^2}\,dy\,dx = \int_0^2 7y^2 e^{x^2} \Big|_{y=0}^{y=\sqrt{x}} = \int_0^2 7x e^{x^2}\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$\frac{7}{2}\int e^u\,du = \frac{7}{2} e^u + C$$
$$= \left[\frac{7}{2} e^{x^2}\right]_0^2 = \boxed{\frac{7}{2} e^4}$$



Example: $\iint_D (2xy + x) e^{y^2+y}\,dA$

$D: \begin{Bmatrix} y = x^2 \to y - x^2 = 0 \\ y = x^2 + 1 \to y - x^2 = 1 \\ x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{Bmatrix}$

$u = y - x^2 \to u = \{1, 0\}$

$v = x^2 + y^2 \to v = \{1, 4\}$

$\tilde{D}: \{(u,v): 0 \le u \le 1, 1 \le v \le 4\}$

$$du\,dv = \left|\begin{vmatrix} -2x & 1 \\ 2x & 2y \end{vmatrix}\right| dx\,dy$$
$$= |-4xy - 2x|\,dx\,dy$$
$$= (4xy + 2x)\,dx\,dy$$

$$(2xy + x)e^{y^2+y}\,dx\,dy = (2xy + x)e^{y^2+y} \frac{du\,dv}{4xy + 2x} = \frac{1}{2} e$$

Lecture

July 28th, 2025

Midterm 2, Problem #1:

$F(x,y) = (e^{x-y^2} + \cos(x),\ x + y\sin(x+y),\ 1 + xe^y)$

$e^t \approx 1 + t + \frac{t^2}{2}$, $\cos(t) \approx 1 - \frac{t^2}{2}$, $\sin(t) \approx t$

$e^{x-y^2} + \cos(x) \approx \left(1 + x - y^2 + \frac{(x-y^2)^2}{2}\right) + \left(1 - \frac{x^2}{2}\right) \approx 2 + x - y^2$

$x + y\sin(x+y) \approx x + xy + y^2$

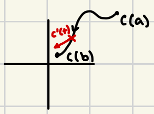
$1 + xe^y \approx 1 + x\left(1 + y + \frac{y^2}{2}\right) = 1 + x + xy$

$Q(x,y) = (2 + x - y^2,\ x + xy + y^2,\ 1 + x + xy)$

1-dimensional measurement $(\mathbb{R}^2)$

$\omega = \underbrace{A\,dx + B\,dy}_{\hookrightarrow F = (A,B)}$ $\qquad ds = \sqrt{(dx)^2 + (dy)^2}$

$C : [a,b] \longrightarrow \mathbb{R}^2$ (curve)



$C(t) = (x(t), y(t))$

$\omega \rightarrow A\,d(x(t)) + B\,d(y(t))$

$dx = x'(t)\,dt$, $dy = y'(t)\,dt$

$\omega = A(c(t))\,x'(t)\,dt + B(c(t))\,y'(t)\,dt$

$\equiv (A(c(t))\,x'(t) + B(c(t))\,y'(t))\,dt$

$\equiv (A(c(t)), B(c(t))) \cdot (x'(t), y'(t))\,dt$

Line Integral:

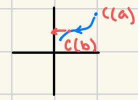
$\int_C \omega = \int_a^b F(c(t)) \cdot c'(t)\,dt$ = work done by $F = (A,B)$ to move along C.

Line Integrals

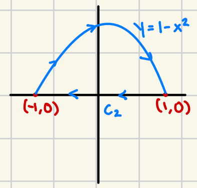
$C = C(t)$

$\omega = A\,dx + B\,dy \rightsquigarrow F = (A,B)$ $\qquad C : [a,b] \longrightarrow \mathbb{R}^2$

$\int_C \omega = \int_C F \cdot dr = \int_a^b F(c(t)) \cdot c'(t)\,dt$



Example #1: $F = (y^2, x)$



C_1: $C_1(x) = (x, 1-x^2)$ from $x = -1$ to $x = 1$

$C_1'(x) = (1, 2x)$

$F = (1 - 2x^2 + x^4,\ x)$

$\int_{-1}^{1} (1 - 2x^2 + x^4 - 2x^2)\,dx = \int_{-1}^{1} (1 - 4x^2 + x^4)\,dx$

$= 2\int_0^1 (1 - 4x^2 + x^4)\,dx$

$= 2\left[x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_0^1 = -\frac{4}{15}$

$\therefore \omega = -\frac{4}{15} + 0 = -\frac{4}{15}$

C_2: $C_2(x) = (x, 0)$ from $x = 1$ to $x = -1$

$C_2'(x) = (1, 0)$, $F = (0, x)$

$\int_1^{-1} (0,x)\cdot(1,0)\,dx = 0$

$ds = \sqrt{(dx)^2 + (dy)^2} \rightsquigarrow ds = \sqrt{(x'(t)\,dt)^2 + (y'(t)\,dt)^2}$

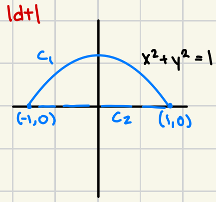
Where $c(t) = (x(t), y(t))$ $\quad = \sqrt{x'(t)^2 + y'(t)^2}\,|dt|$

$= |(x'(t), y'(t))|\,|dt|$

$= |c'(t)|\,|dt|$

$\int_C f(x,y)\,ds = \int_a^b f(c(t))\,|c'(t)|\,dt$ $\quad$ Smaller a to bigger b

Example #2: $\mu = \sqrt{x^2 + y^2}$



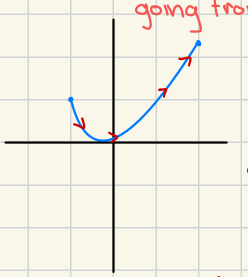
C_1: $C_1(\theta) = (\cos\theta, \sin\theta)$

$\theta \in [0, \pi]$

$C_1'(\theta) = (-\sin\theta, \cos\theta)$

$|C_1'(\theta)| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

Example #1: $\int_C y\,dx - x^2\,dy$ where C is the parabola $y = x^2$ going from $(-1,1)$ to $(2,4)$



$F = (y, -x^2)$

Step 1: $C(x) = (x, x^2)$

$x : [\underset{f}{-1}, \underset{t}{2}]$

Step 2, Option 1: $\int_C y\,dx - x^2\,dy = \int_{-1}^{2} x^2\,dx - x^2(2x)\,dx$

$= \int_{-1}^{2} x^2 - 2x^3\,dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{2}\right]_{-1}^{2} = \overbrace{\left(\frac{8}{3} - 8\right)}^{-16/3} - \left(-\frac{1}{3} - \frac{1}{2}\right)$

$= \boxed{-9/2}$

Step 2, Option 2: $\int_C F \cdot dr = \int (x^2, -x^2)\cdot(1, 2x)\,dx$

$c(x) = (x, x^2)$ $\quad = \int_{-1}^{2} (x^2 - 2x^3)\,dx$

$c'(x) = (1, 2x)$

Example #2: $F = (x - z,\ y^2,\ x + z)$ moving along the intersection between $z = x^2 + y^2$ and $z = y^2 + y$ going from $(1,1,2)$ to $(0,0,0)$. Find the work done by F.

Step 1: $x^2 + y^2 = y^2 + y \rightarrow x^2 = y$

$C(x) = (x, x^2, x^2 + x^4)$ from $x : [\underset{f}{1}, \underset{t}{0}]$

Step 2: work $= \int_1^0 (x - x^2 - x^4,\ x^4,\ x + x^2 + x^4) \cdot (1,\ 2x,\ 2x + 4x^3)\,dx$

$= \int_1^0 (x - x^2 - x^4 + 2x^5 + 2x^2 + 2x^3 + 2x^5 + 4x^4 + 4x^5 + 4x^7)\,dx$

$= \int_1^0 (x + x^2 + 2x^3 + 3x^4 + 8x^5 + 4x^7)\,dx$

$= -\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{8}{6} + \frac{1}{2}\right)$