

+

-

÷

M427J: Diffy Egn

June 6, 2025

$$\ln(y) - \frac{1}{\sqrt{t}} + y'' = \tan^{-1}(\frac{t}{1-y}) + y^{(3)} \quad (\text{Non-Linear})$$

What is $y'(t)$?
 $y' = y$ (1st order) $y(t) = e^t, 0, C \cdot e^t$
 $y^{(3)} = y$ (3rd Order)

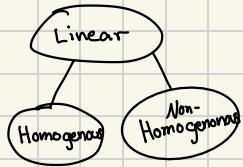
Non-Linear
 $y \cdot y' = 0$

Linear vs. Non-Linear DEs

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = b$$

$$\frac{1}{t} y^{(n)} + e^t y^{(n-1)} + \dots + t^3 y' + 7 y = \arcsin(\sqrt{t}) \quad (\text{Linear DE})$$

$$y' = y \rightarrow -y' - y = 0$$



5th Order, Linear, homogeneous

$$y^{(5)} - \frac{1}{t^2} y'' + e^{t^2} y = 0$$

5th Order, Linear, Non-homogeneous

$$y^{(5)} - t^3 y' = t^2$$

$$\begin{cases} \text{Brown 1} \\ \text{I 2} \end{cases} \begin{cases} a(t) y' + b(t) y = 0 & 1^{\text{st}}, \text{ homogeneous} \\ a(t) y' + b(t) y = g(t) & 1^{\text{st}}, \text{ non-homogeneous} \end{cases}$$

Systems of DEs

$$P'(t) = g(t) + f(t) \quad \text{More Functions}$$

$$g'(t) = 3f(t) \quad \text{More variables} \rightarrow u(x, t) \Rightarrow \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}$$

June 9, 2025 - Brown 1 2 (Linear 1st Order ODE)

$$\text{Find } y(t) \longrightarrow a(t) y' + b(t) y = g(t)$$

$$a_n(t) y^{(n)} + \dots$$

$$\text{First order, linear: } y' + a(t) y = b(t)$$

$$\text{homogen: } y' + a(t) y = 0$$

$$\text{Non-homogen: } y' + a(t) y = b(t) \neq 0$$

First order, homogen. ODEs:

$$y' + a(t) y = 0$$

$$(\ln(f(x)))' = \frac{1}{f(x)} \cdot f'(x)$$

$$\text{Proof #1: } y' + a(t) y = 0 \rightarrow \frac{y'}{y} = -a(t)$$

$$\rightarrow y' y = -a(t)$$

$$\rightarrow (\ln(y))' = -a(t)$$

$$\rightarrow \frac{d}{dt} (\ln(y(t))) = -a(t)$$

$$\rightarrow \int \frac{d}{dt} (\ln(y(t))) dt = \int -a(t) dt$$

$$\rightarrow \ln(y(t)) = \int -a(t) dt + C$$

$$e^{\ln(y(t))} = e^{\int -a(t) dt + C}$$

$$y(t) = e^{\int -a(t) dt + C} \equiv e^{\int -a(t) dt} \cdot e^C$$

$$= C e^{\int -a(t) dt}$$

$$\text{Example #1: } y' = y \rightarrow y' - y = 0$$

$$\equiv y' + \underline{a(t)y} = 0$$

$$y' y = 1 \rightarrow (\ln y)' = 1$$

$$\rightarrow \ln y = t + C$$

$$\rightarrow y = e^t \rightarrow y = e^t \cdot e^C$$

$$y = C e^t \quad (\text{gen. solution})$$

$$\text{Example #2: } y' + 2t y = 0$$

$$y' y = -2t$$

$$(\ln y)' = -2t$$

$$\ln y = -t^2 + C$$

$$y = e^{-t^2} \cdot e^C$$

$$y = A e^{-t^2}$$

I.V.P (Initial Value Problem):

$$\text{Exp. 1: } y' + 2t y = 0, \quad y(0) = 3$$

$$y = C e^{-t^2} \leftarrow \text{Solve ODE}$$

$$3 = C e^{-(0)^2} \quad y = 3e^{-t^2} \quad (\text{unique solution})$$

$$3 = C$$

$$\text{Exp 2: } y' + (\sin t) y = 0, \quad y(0) = \frac{3}{2} \quad (1^{\text{st}} \text{ Order, hom., IVP})$$

$$y' = -\sin t \cdot y$$

$$\int (\ln y)' = \int -\sin t \quad \frac{3}{2} = e^{\cos(t)} C$$

$$\ln y = \cos t + C$$

$$\frac{3}{2} = C$$

$$y = e^{\cos t \cdot C}$$

$$\frac{3}{2} e^{\cos t}$$

$$y = e^{\cos t \cdot A}$$

$$\frac{3}{2} e^{\cos t - 1}$$

Non-homogeneous, 1st Order ODE: $y' + a(t) y = b(t)$

$$(f(t) \cdot g(t))' = f(t)g'(t) + f'(t)g(t) \quad (y')' = b(t)$$

$$(f \cdot g)' = f' \cdot g + g \cdot f$$

What if I could multiply by $\mu(t)$:

$$y' + a(t) y = b(t)$$

$$\mu(t) y' + \mu(t) a(t) y = \mu(t) b(t)$$

$$(\mu y)' = \mu(t) b(t)$$

$$? y' + ? y = \mu(t) b(t)$$

$$\text{If } ?' = a(t) \mu(t) = \mu'(t) \quad | \quad \text{If } \mu'(t) = a(t) \mu(t)$$

$$? = \mu(t)$$

$$\frac{\mu'(t)}{\mu(t)} = a(t)$$

$$(\log(\mu))' = a(t)$$

$$\ln \mu = \int a(t)$$

$$\mu = e^{\int a(t)}$$

$$\mu' = e^{\int a(t)} \cdot (\int a(t))'$$

$$= \mu \cdot a(t) \Rightarrow$$

$$y' + a(t) y = b(t) \neq 0$$

$$1^{\text{st}} \quad \mu(t) = e^{\int a(t)}$$

$$\mu y' + a(t) \cdot \mu \cdot y = \mu \cdot b(t)$$

$$(\mu \cdot y)' = \mu \cdot b(t)$$

$$\text{Example #3: } y' - 2t y = t \quad \mu = e^{\int -2t}$$

$$\mu = e^{-t^2}$$

$$y' e^{-t^2} - 2t e^{-t^2} y = t e^{-t^2}$$

$$(e^{-t^2} y)' = t e^{-t^2}$$

$$e^{-t^2} y = \int t e^{-t^2} dt + C$$

$$\text{let } u = -t^2 \quad -\frac{1}{2} du = -2t \quad -\frac{1}{2} du = \frac{t}{-2}$$

$$e^{-t^2} y = -\frac{1}{2} e^u du$$

$$e^{-t^2} y = -\frac{1}{2} e^u + C$$

$$\frac{e^{-t^2} y}{e^{-t^2}} = \frac{-\frac{1}{2} e^u + C}{e^{-t^2}}$$

$$y = -\frac{1}{2} + \frac{C}{e^{t^2}}$$

$$= -\frac{1}{2} + C e^{t^2}$$

$$y(t) \equiv -\frac{1}{2}$$

$$y' + 2t y = t, \quad y(1) = 2$$

$$\mu = e^{\int 2t dt} = e^{t^2}$$

$$e^{t^2} y' + 2e^{t^2} y = t e^{t^2}$$

$$(e^{t^2} y)' = t e^{t^2}$$

$$e^{t^2} y = \int t e^{t^2} dt + C = \frac{1}{2} e^{t^2} + C$$

$$y = \frac{1}{2} + \frac{C}{e^{t^2}} \equiv \frac{1}{2} + C e^{-t^2}$$

$$2 = \frac{1}{2} + C e^{-(1)^2} \quad \boxed{y = \frac{1}{2} + \frac{3e^{-t^2} + 1}{2}}$$

$$2 = \frac{1}{2} + C e^{-1}$$

$$\frac{3}{2} = C e^{-1}$$

$$\frac{3e}{2} = C$$

$$y' + \frac{2t}{1+t^2} y = \frac{1}{1+t^2} \quad \text{let } u = 1+t^2$$

$$du = 2t$$

$$\mu = e^{\int \frac{2t}{1+t^2} dt}$$

$$u = e^{\ln(1+t^2)}$$

$$u = 1+t^2$$

$$(1+t^2)y' + \frac{2t}{1+t^2} y = \frac{1}{1+t^2}(1+t^2)$$

$$(1+t^2) \cdot y' + 2t y = 1$$

$$(1+t^2) y = t + C$$

$$y = \frac{t + C}{1+t^2}$$

Brown 1.4 Separable

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$$\int f(y) dy = \int g(t) dt$$

$$\text{Ex 1. } \frac{dy}{dt} = \frac{t^2}{y^2}$$

$$\int y^2 dy = \int t^2 dt$$

$$\frac{y^3}{3} = \left(\frac{t^3}{3} + C\right)$$

$$y^3 = t^3 + 3C$$

$$\sqrt[3]{y^3} = \sqrt[3]{t^3 + C}$$

$$y = \sqrt[3]{t^3 + C}$$

$$\text{Example #2: } e^y \frac{dy}{dt} = -t^3, \quad y(1) = 1$$

$$e^y \frac{dy}{dt} = -t^3$$

$$\int e^y dy = \int (-t^3) dt$$

$$e^y = \frac{1}{2}t^2 + \frac{1}{4}t^4 + C$$

$$y = \ln\left(\frac{1}{2}t^2 + \frac{1}{4}t^4 + C\right)$$

$$1 = \ln\left(\frac{1}{2} + \frac{1}{4} + C\right)$$

$$1 = \ln\left(\frac{1}{2} + \frac{1}{4} + C\right)$$

$$1 = \ln\left(\frac{3}{4} + C\right)$$

$$e^1 = \frac{3}{4} + C$$

$$e^1 - \frac{3}{4} = C$$

$$\text{Example #3: } y \frac{dy}{dt} + (1+y^2) \sin(t) = 0, \quad y(0) = 1$$

$$y \frac{dy}{dt} = -(1+y^2) \sin(t)$$

$$\int \frac{y}{1+y^2} dy = \int \sin(t) dt$$

$$\frac{1}{2} \ln(1+y^2) = \cos(t) + C$$

ODE is separable if

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$$\int f(y) dy = \int g(t) dt$$

$$\frac{dy}{dt} = t \cdot y \quad \text{Yes}$$

$$= \frac{t}{1/y}$$

$$\ln(a \cdot b) = \ln(a) + \ln(b)$$

$$\text{Question #2: } \frac{dy}{dt} + \sqrt{t} \sin(t) y = 0$$

$$\text{Question #3: } \frac{dy}{dt} + \frac{2t}{1+t^2} y = \frac{1}{1+t^2}$$

$$\text{Gen. sol.} \quad \frac{1}{2} \ln(1+y^2) = \cos(t) + \ln(\sqrt{2}) - 1$$

Implicit Eqn

Discussion Section

June 10, 2025

$$\frac{dy}{dt} + a(t)y = b(t) \quad \text{if } b(t) = 0 \text{ then:}$$

$$\frac{dy}{dt} + a(t)y = 0 \quad (\text{separable})$$

$$\rightarrow \frac{dy}{dt} = -a(t)y$$

$$\frac{d\mu(t)}{dt} = \mu(t)a(t)$$

$$\frac{d(\mu(t))}{\mu(t)} = a(t) dt$$

$$\ln(\mu(t)) = \int a(t) dt$$

$$\mu(t) = e^{\int a(t) dt}$$

Assume $\exists \mu(t)$ such that

$$\mu(t) \frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t) \rightarrow \mu(t) = e^{\int a(t) dt}$$

$$\frac{d}{dt}(\mu(t)y) = \mu(t)b(t)$$

$$\mu(t)y = \int \mu(t)b(t) dt + C$$

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t) dt + C$$

Initial Condition vs. Boundary Condition

$$y(t_0) = y_0, \quad (t_0, y_0), \quad t_0 = ? \quad y_0 = ?$$

$$y(t_0) = \frac{1}{\mu(t_0)} \left(\int_{t_0}^{t_0} \mu(s) b(s) ds + C \right) = y_0$$

$$\rightarrow C = \mu(t_0)y_0 = \mu(t_0)y(t_0)$$

I.V.P. \equiv (D.E. + I.C.)

$$\frac{dy}{dt} + a(t)y = b(t), \quad y(t_0) = y_0$$

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^t \mu(s) b(s) ds + \mu(t_0)y_0 \right)$$

Question #1: $\frac{dy}{dt} + t \cos(t) = 0$

$$\text{Question #1: } \frac{dy}{dt} + y = te^t$$

$$\text{Question #8: } \frac{dy}{dt} + \sqrt{1+t^2} y = 0, \quad y(0) = \sqrt{5}$$

$$\text{Question #5: } \frac{dy}{dt} + t^2 y = 1$$

$$\text{Question #6: } \frac{dy}{dt} + t^2 y = t^2$$

$$\text{Question #7: } \frac{dy}{dt} + \frac{1}{1+t^2} y = -\frac{t^3}{1+t^4} y$$

Linear Algebra Concepts for Differential Equations

Vectors:

$$\vec{u}, \vec{v} \in \mathbb{R}^n$$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i$$

Matrices:

$$M_{r \times c}(\mathbb{R})$$

$$= \left\{ \begin{bmatrix} a_{ij} & \dots & a_{ic} \\ \vdots & \ddots & \vdots \\ a_{rn} & \dots & a_{rc} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \right\}$$

$$A \cdot B =$$

$$i^m \begin{bmatrix} \cdots & \cdots & \cdots \\ a & \cdots & a_c \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = i^m \begin{bmatrix} \cdots & \cdots & \cdots \\ 0 & & 0 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$A \in M_{r \times c}(\mathbb{R}), B \in M_{m \times n}(\mathbb{R})$$

If $C = M$, then

$$A \cdot B \in M_{r \times n}(\mathbb{R})$$

Example #1:

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 3(2) + 0(2) + (-1)5 \\ = 6 + 0 - 5 \\ = 1$$

Example:

$$\begin{bmatrix} -2 & 0 & 5 \\ 8 & 8 & 1 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$$

Complex Arithmetic:

$$z = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$$

$$\operatorname{Re}(z) = \alpha, \operatorname{Im}(z) = \beta$$

Example (multiplication):

$$(1+i) \cdot (3-2i) = 3-2i+3i-2i^2 \\ = 3+i+2 \\ = 5+i$$

Complex conjugate and inverse:

$$\overline{3-2i} = 3+2i \text{ (conjugate)}$$

$$\frac{1}{3-2i} \cdot \frac{(3+2i)}{(3+2i)} = \frac{3+2i}{9-4i^2} \text{ (Inversion)} \\ = \frac{3+2i}{13} = \frac{3}{13} + i \frac{2}{13}$$

Division of complex #'s:

$$\frac{1+i}{3-2i} \cdot \frac{(3+2i)}{(3+2i)} = \frac{3+2i}{9-4i^2} \\ = \frac{3+5i-2}{9+4} = \frac{1+5i}{13} \\ \equiv \frac{1}{13} + i \frac{5}{13}$$

Example:

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \in M_{3 \times 2}(\mathbb{R})$$

$$B = \begin{bmatrix} 2 & 8 & 1 \\ -2 & 0 & 4 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$$

$$A \cdot B = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 & 1 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 24 & 23 \\ -2 & 0 & 4 \\ -6 & -8 & 7 \end{bmatrix} \begin{array}{l} | -1 \cdot 2 + 2(-2) = -6 \\ | -1(8) + 2(0) = -8 \\ | -1(1) + 2(4) = 7 \end{array}$$

Computing the inverse of a square matrix

$$A \in M_n(\mathbb{R}) \quad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in M_n(\mathbb{R})$$

The inverse of A is an $n \times n$ matrix B so that

$$A \cdot B = B \cdot A = I_n \Rightarrow B = A^{-1}$$

Let $A \in M_n(\mathbb{R})$

$$[A|I_n] \rightarrow \dots$$

$$\rightarrow [I_n|A^{-1}]$$

$$\text{Example: } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

$$[A|I_n] = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1} A = I_n \text{ for correctness}$$

$$A \in M_n(\mathbb{R})$$

The following are equivalent:

$$1) \det(A) \neq 0$$

2) A is invertible (there is an $A^{-1} \in M_n(\mathbb{R})$) \Rightarrow non-singular

$$3) \text{RREF}(A) = I_n$$

$$\text{When } n=2, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Example: } A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1-2} \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

Homogeneous system:

$$\begin{aligned} -2x_1 - 4x_2 + 3x_3 - 13x_4 - 5x_5 &= 0 \\ -x_1 - 2x_2 + x_3 - 5x_4 - x_5 &= 0 \end{aligned} \quad \left\{ \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ -1 & -2 & 1 & -5 & -1 \end{bmatrix} = A \right.$$

$$\text{rref}(A) = \left[\begin{array}{ccccc|c} -2 & -4 & 3 & -13 & -5 & 0 \\ 1 & 2 & -1 & 5 & 1 & 0 \\ -1 & -2 & 1 & -5 & -1 & 0 \\ 0 & 0 & 3 & -9 & -9 & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 5 & 1 & 0 \\ -2 & -4 & 3 & -13 & -5 & 0 \\ 1 & 2 & -1 & 5 & 1 & 0 \\ 0 & 0 & 3 & -9 & -9 & 0 \end{array} \right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 5 & 1 & 0 \\ -2 & -4 & 3 & -13 & -5 & 0 \\ 0 & 0 & 3 & -9 & -9 & 0 \end{array} \right] \xrightarrow{R_4 - 3R_3} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 5 & 1 & 0 \\ -2 & -4 & 3 & -13 & -5 & 0 \\ 0 & 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = x_2 \\ x_4 = x_1 \\ x_5 = x_1$$

$$x_2 = x_3 \\ x_4 = x_3 \\ x_5 = x_3$$

$$x_3 = x_4 \\ x_5 = x_4$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_3 = x_3$$

$$x_4 = x_3 \\ x_5 = x_3$$

$$x_1 = x_3 \\ x_2 = x_3 \\ x_$$

Linear Algebra Concepts continued

Vector Spaces:

V_L, V_A

$$V_{y''-y} = \{c_1 e^t + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R}\}$$

$$L[y] = y'' + y \quad (\text{Lin. ODE})$$

$$V_y'' = \{c_1 + c_2 \mid c_1, c_2 \in \mathbb{R}\}$$

$$\text{If } L[y] = g(t) \quad (\text{Non-homo.})$$

$$L[y] = 0 \quad (\text{homo.}) \quad A \in M_n(\mathbb{R}),$$

$$V_L = \{y(t) \in C(\mathbb{R}) \mid L[y] = 0\} \quad V_A = \{\vec{x}(t) \in C^\infty(\mathbb{R}, \mathbb{R}^n) \mid \frac{d}{dt} \vec{x} = A \vec{x}\}$$

$$= \{c_1 \cos t + c_2 \sin t \mid c_1, c_2 \in \mathbb{R}\}$$

$$V_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \left\{ c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

$$V_{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \left\{ c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

Subspaces:

$$A \in M_{r \times n}(\mathbb{R}) \quad \cap \left\{ \underbrace{\begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix}}_{C} \cdot \begin{bmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{bmatrix} \right\}$$

$$\text{Ker}(A) \equiv \text{Null}(A)$$

"kernel" "Nullspace"

$$= \{\vec{v} \in \mathbb{R}^r \mid A \cdot \vec{v} = \vec{0}\}$$

$$\text{Example: Let } A = \begin{bmatrix} 4 & -12 & -3 & 18 \\ -1 & 3 & 1 & -5 \\ 1 & -3 & 2 & -1 \\ 1 & -12 & -3 & 18 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 1 & 3 & 1 & -5 \\ -1 & 3 & 2 & -1 \\ 1 & -12 & -3 & 18 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 3 & -6 \\ -1 & 3 & 2 & -1 \\ 1 & -12 & -3 & 18 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & -2 \\ 1 & -12 & -3 & 18 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 3x_2 + 3x_4 &= 0 \\ x_2 &= x_2 \\ x_3 - 2x_4 &= 0 \\ x_4 &= x_4 \end{aligned} \quad \begin{aligned} x_1 &= 3x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned} \quad \left\{ \begin{aligned} &= x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \end{aligned} \right.$$

$$\text{Ker}(A) = \left\{ x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \mid x_2, x_4 \in \mathbb{R} \right\}$$

Application: Population Modeling

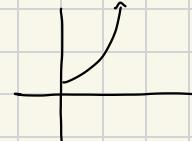
$y(t)$ = # of individuals at time t .
 y' = rate of growth of a population
 y' related to y ($y' \leftrightarrow y$)
 $y' = k \cdot y$ Model

To solve:
 $y' - ky = 0 \rightarrow \frac{dy}{y} = k dt$
 $\int (\ln y)' = \int k$
 $\ln y = kt + C$
 $y = Ce^{kt}$ $y(0) = \beta$
 $\Rightarrow P(t) = P_0 e^{kt}$

A mathematical model is a differential equation.

A, B

A is directly proportional to B, $A = kB$
A is inversely proportional to B, $A = \frac{k}{B}$



Exponential Pop. Growth

$$y' = ky \rightarrow y = Ce^{kt}$$

Carrying capacity = N (the max capacity of the environment).

$$\begin{cases} y' = ky, y(0) \\ y' = ky(1-y/N) \end{cases} \quad k = \text{growth rate}$$

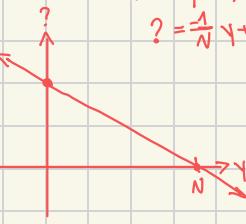
$$\begin{cases} y' = 0, y \sim N \\ y' = 0, y \sim 0 \end{cases}$$

$y' = k \cdot y$ (?) when $y \sim 0$, ? = 1

when $y \sim N$, ? = 0
 $? = \frac{1}{N} y + 1 \approx 1 - y/N$

$$\begin{aligned} y' &= y(1-y) = y - y^2 \\ \frac{dy}{dt} &= ky(1-y/N) \quad \left\{ \begin{array}{l} \frac{1}{y(1-y/N)} dy = \sqrt{k} dt \\ = kt + C \end{array} \right. \\ y &= \frac{N}{N + e^{-kt} + C} \end{aligned}$$

Logistic Growth

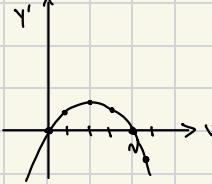


$$y' = \frac{dy}{dt} = ky(1-y/N)$$

$$y' = y(1-y)$$

$$= y - y^2$$

$$y' = ky(1-y/N)$$



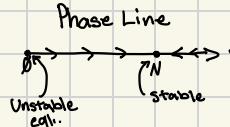
Graph y' as a function of y , value of y

where $y' = 0$:

$$y(t) \equiv 0, y(t) \equiv N$$

Solutions to the ODEs

Equilibrium points.



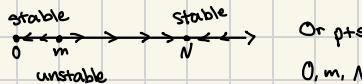
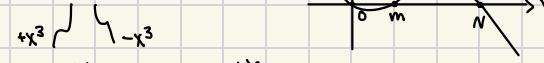
m is minimum population for reproduction, allee

$$y' = ky(1-y/N) \text{ if } y > m, \text{ then } y' > 0$$

$$y < m, \text{ then } y' < 0$$

$$y' = ky(1-y/N) \cdot (y-m)$$

$$= \frac{k}{N} y^3 + \dots$$



Or pts
 $0, m, N$

Braun 2.0 \rightarrow 2nd order ODEs and IVPs (Linear)

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

$$\text{or } y'' + p(t)y' + q(t)y = g(t)$$

$$ay'' + by' + cy = g(t)$$

$$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$$

$$\text{Example #1: } y'' - y = 0 \quad a=1, b=0, c=-1$$

$$(1)y'' + (0)y' + (-1)y = 0$$

$$y'' = y \mapsto y = \underbrace{e^{rt}}_{C_1} + \underbrace{e^{-rt}}_{C_2}$$

$C_1 y_1 + C_2 y_2$

Linear combinations of y_1, y_2

$$y = Ce^{rt}, de^{rt}, a \cdot \sinh(rt), b \cosh(rt)$$

$$y' = \cosh(rt), y'' = \sinh(rt)$$

$$y = \sinh(rt), y' = \cosh(rt)$$

$$(Ce^t + Ce^{-t}) + a \sinh(rt) + b \cosh(rt)$$

$$\sinh(rt) = \frac{e^{rt} - e^{-rt}}{2}, \cosh(rt) = \frac{e^{rt} + e^{-rt}}{2}$$

$$= \frac{1}{2}(e^{rt} - e^{-rt}) + \frac{1}{2}(e^{rt} + e^{-rt})$$

$$= \frac{1}{2}e^{rt} + \frac{1}{2}e^{-rt} = y$$

$$y'' - y = 0 \quad y_1 = e^{rt}, y_2 = -e^{rt}$$

$$y = C_1 y_1 + C_2 y_2 \mapsto C_1 e^{rt} + C_2 (-e^{rt})$$

Wronskian:

$$W[y_1, y_2] = y_1 \cdot y_2' - y_1' \cdot y_2 = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

If y_1, y_2 solve $ay'' + by' + cy = 0$ and $W[\ln t, \sinh t] = (\ln t) \cos t - \frac{1}{4} \sin t$

$$W[y_1, y_2] \neq 0$$

$$\text{GOOD: } y = C_1 y_1 + C_2 y_2$$

if $W[y_1, y_2] = 0$, BAD, keep working

$$\text{Example: } y'' - y = 0 \quad y_1 = e^{rt}, y_2 = -e^{rt}$$

$$W[e^{rt}, -e^{rt}] = (e^{rt})(-e^{rt}) - (e^{rt})(-e^{rt}) = -e^{2rt} + e^{2rt} = 0 \quad \text{BAD}$$

$$\text{Try } y_1 = e^{rt}, y_2 = e^{-rt}$$

$$W[e^{rt}, e^{-rt}] = (e^{rt})(e^{-rt})' - (e^{-rt})(e^{rt})' = -1 - 1 \equiv -2 \quad \text{GOOD}$$

$$y = C_1 e^{rt} + C_2 e^{-rt} \quad \text{General sol.}$$

$$\text{to } y'' - y = 0$$

High level view:

$$ay'' + by' + cy = 0 \quad y = \ln(t)$$

$$y' = \frac{1}{t}$$

$$\text{Exponential fronts. } y'' = -\frac{1}{t^2}$$

Sine/cosine fronts.

Polynomial

$$\text{Guess: } y = e^{rt} \quad y' = e^{rt} \cdot r, y'' = r^2 e^{rt}$$

$$ar^2 e^{rt} + br e^{rt} + ce^{rt} = 0$$

$$\text{Characteristic polynomial } ar^2 + br + c = 0$$

Quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad r_1, r_2$$

$$e^{r_1 t}, e^{r_2 t}$$

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$$

$$W[y_1, y_2] \neq 0$$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Example: $y'' + y' - 6y = 0 \rightarrow r^2 + r - 6 = 0$

$$(r+3)(r-2) = 0 \rightarrow r = \{-3, 2\}$$

$$y_1 = e^{2t}, y_2 = e^{-3t}$$

$$\begin{aligned} W[y_1, y_2] &= (e^{2t})(e^{-3t})' - (e^{2t})'(e^{-3t}) \\ &= e^{2t}(-3e^{-3t}) - (2e^{2t})e^{-3t} \\ &= -3e^{-t} - 2e^{-t} = -5e^{-t} \neq 0 \end{aligned}$$

$$y = C_1 e^{2t} + C_2 e^{-3t}$$

Example: $y'' - y = 0 \rightarrow r^2 - 1 = 0$

$$(r+1)(r-1) = 0 \rightarrow r = -1, 1$$

$$y = C_1 e^t + C_2 e^{-t}$$

$$y(0) = 2, y'(0) = 1$$

$$y(0) = C_1 e^{(0)} + C_2 e^{(0)} = C_1 + C_2 = 2$$

$$y' = C_1 e^t - C_2 e^{-t}$$

$$y'(0) = C_1 e^{(0)} - C_2 e^{(0)} = C_1 - C_2 = 1$$

$$C_1 = \frac{3}{2}, C_2 = \frac{1}{2}$$

$$ay'' + by' + cy = 0$$

Discriminant:

$$\Delta = b^2 - 4ac$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Brown $\left\{ \begin{array}{l} \Delta > 0 \text{ 2 real roots} \\ 2.2 \quad \left\{ \begin{array}{l} r_1, r_2 \\ y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \end{array} \right. \end{array} \right.$

Brown $\left\{ \begin{array}{l} \Delta < 0 \text{ complex roots} \\ 2.2.1 \end{array} \right.$

Brown $\left\{ \begin{array}{l} \Delta = 0 \text{ double roots} \\ 2.2.2 \end{array} \right.$

Example: $y'' + y = 0 \rightarrow r^2 + 1 = 0 \rightarrow r^2 = -1$

$$z(t) = e^{it} = \cos t + i \sin t \quad r = \pm \sqrt{1} = \pm i$$

$$y_1 = \cos t$$

$$y_2 = \sin t$$

$$y' = -\sin t$$

$$y'' = -\cos t$$

$$y'' + y = -\cos t + \cos t = 0$$

$$y'' + y = -\sin t + \sin t = 0$$

$$y = \sin t$$

$$W[\cos t, \sin t] = (\cos t)'(\sin t) - (\cos t)(\sin t)' = \cos t \sin t - \cos t \sin t = 0$$

$$y' = \cos t$$

$$(\cos t)'(\sin t)$$

$$y'' = -\sin t$$

$$= \cos t \cos t - \sin t \sin t$$

Example: $y'' + 2y' + 2y = 0 \rightarrow r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4-4 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i \rightarrow \alpha \pm i\beta$$

Pick $r = -1 + i$

$$z(t) = e^{(-1+i)t} = e^{-t} \cdot e^{it}$$

$$z(t) = \frac{\cos t + i \sin t}{e^t}$$

$$y = \frac{C_1 \cos t + C_2 \sin t}{e^t} = e^{-t} C_1 \cos t + e^{-t} C_2 \sin t$$

$$r = \alpha \pm i\beta$$

$$\rightarrow y = C_1 e^{-t} \cos bt + C_2 e^{-t} \sin bt \quad \left\{ \text{Short-cut!} \right.$$

Example: $y'' - 4y' + 13 = 0$

$$r^2 - 4r + 13 = 0$$

$$r = \frac{4 \pm \sqrt{16-4 \cdot 13}}{2}$$

$$r = \frac{4 \pm 2\sqrt{9-13}}{2}$$

$$= 2 \pm 3i$$

$$\alpha \pm \beta i$$

Example: $y'' + y' + y = 0 \quad r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

Example: $y'' + 4y' + 4y = 0 \rightarrow r^2 + 4r + 4 = 0$

$$y_1 = e^{-2t} \quad (r+2)(r+2) = 0$$

$$f(t)v'' + g(t)v' = 0 \quad v = -2 \quad (\text{double})$$

$$\rightarrow y_2 = v(t) y_1(t)$$

$$y_2 = t e^{-2t}$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t}$$

Review:

$$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$$

$$ar^2 + br + c = 0 \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (3) \Delta = 0, r \text{ double roots}$$

$$\text{Discriminant: } \Delta = b^2 - 4ac$$

$$(1) \Delta > 0, r_1, r_2 \text{ real}$$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$(2) \Delta < 0, \alpha \pm i\beta \text{ complex}$$

$$y = C_1 e^{\alpha t} + C_2 e^{\alpha t} \cos(\beta t) + C_3 e^{\alpha t} \sin(\beta t)$$

$$\frac{1+ti}{1-2i} \cdot \frac{(1+2i)}{(1+2i)} = \frac{1+ti+2i+(i)(2i)}{1-2i+2i-(2i)(2i)} = \frac{1+3i+2i^2}{1-4i^2} = \frac{1+3i-2}{1+4} = \frac{-1+3i}{5}$$

$$C = \left\{ \alpha + i\beta \mid \alpha, \beta \in \mathbb{R} \right\} = \left(-\frac{1}{5} \right) + i \left(\frac{3}{5} \right)$$

$$y'' - y = +$$

$$y = e^t \quad y = e^t \quad y'' = e^t$$

$$y'' - y = e^t - e^t = 0$$

Particular sol.

$$y = -t + C_1 e^t \rightarrow y \text{ or } y_p$$

$$y = -t + C_1 e^t + C_2 e^{-t}$$

solutions \rightarrow Comp./Hom.

$$y'' - y = 0$$

sol.

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right) + i \left(-\frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right)$$

$$= \cos t + i \sin t \quad \text{Euler's Formula.}$$

$$e^{it} = \cos t + i \sin t \quad \theta \text{ real}$$

$ay'' + by' + cy = q(t)$ 2.3 How to write solutions

2.5 Solving

$$y'' - y = +$$

$$y = \text{trig } x$$

$$1 \text{ solution}$$

$$X \quad y = t^2 \quad y' = 2t \quad y'' = 2 \quad y = -t, y' = -1, y'' = 0 \quad \checkmark$$

$$y'' - y = 0 = 2 - t^2 \neq + \quad y'' - y = 0 - (-t) = + = +$$

$$y = t, y' = 1, y'' = 0$$

$$X \quad y'' - y = 0 - t = -t \neq +$$

$$y = -t + e^t$$

$$y' = -1 + e^t$$

$$y'' = 0 + e^t$$

$$y = -t + e^t \checkmark$$

$$y'' - y =$$

To solve $a\psi'' + b\psi' + c\psi = g(t)$

1. Find ψ_c , solve $a\psi'' + b\psi' + c\psi = 0$

2. Find ψ_p , 1 particular solution

3. $\psi = \psi_p + \psi_c = \psi_p + C_1\psi_1 + C_2\psi_2$

Example: $\psi'' + \psi = t$

$$1. \psi'' + \psi = 0 \rightarrow r^2 + 1 = 0, r = \pm i \rightarrow \alpha = 0, \beta = 1$$

$$\psi_c = C_1 e^{it} \cos(t) + C_2 e^{it} \sin(t) \\ = C_1 \cos(t) + C_2 \sin(t)$$

$$2. \psi = t, \psi' = 1, \psi'' = 0$$

$$\psi'' + \psi = 0 + t = t \rightarrow \psi_p = \psi = t$$

$$3. \psi = \psi_p + \psi_c = t + C_1 \cos(t) + C_2 \sin(t)$$

Example: $\psi'' + \psi = t^2$

$$1. \psi'' + \psi = 0 \rightarrow \psi_c = C_1 \cos(t) + C_2 \sin(t)$$

$$2. \psi = t^2, \psi' = 2t, \psi'' = 2$$

$$\psi = t^2 - 2, \psi' = 2t, \psi'' = 2$$

$$\psi'' + \psi = 2t + t^2 - 2 = t^2 \checkmark$$

$$3. \psi = \psi_p + \psi_c$$

$$\text{Example: } \psi'' + \psi' - 2\psi = t^2 \quad r^2 + r - 2 = 0$$

$$1. \psi'' + \psi' - 2\psi = 0 \quad r = 1, -2$$

$$(r+2)(r-1) = 0$$

$$\psi_c = C_1 e^{-2t} + C_2 e^{t}$$

$$2. \psi = t^2, \psi' = 2t, \psi'' = 2$$

$$\psi'' + \psi' - 2\psi = 2t + 2t^2 - 2t^2 = t^2$$

$$\psi = t^2 - 2, \psi' = 2t, \psi'' = 2$$

$$\psi'' + \psi' - 2\psi = 2t + 2t^2 - 2(t^2 - 2)$$

$$= 2 + 2t - 2t^2 + 4$$

$$= -2t^2 + 2t + 6$$

$$2. \psi = A t^2 + B t + C$$

$$\psi' = 2At + B$$

$$\psi'' = 2A$$

$$2A + 2At + B$$

$$\psi'' + \psi' - 2\psi = -2(A t^2 + B t + C)$$

$$= -2At^2 + (2A - 2B)t + (2A + B - 2C) = t^2 + 0 \cdot t + 0$$

$$-2A = 1 \quad 2A - 2B = 0 \quad -2C = 0$$

$$A = -\frac{1}{2}$$

$$A = B$$

$$B = -\frac{1}{2}$$

$$-2C = \frac{3}{2}$$

$$C = -\frac{3}{4}$$

$$\text{Example: } \psi'' - 6\psi' + 9\psi = 3t^2 - 4t + 1$$

$$\psi'' - 6\psi' + 9\psi = 0 \rightarrow r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0 \quad r = 3, \text{ double}$$

$$\psi = \psi_c + \frac{1}{27}$$

$$\rightarrow \psi_c = C_1 e^{3t} + C_2 t e^{3t} \quad ③ \quad \psi = \psi_c + \frac{1}{27} = \frac{1}{3} t^2 + \frac{1}{27} + C_1 e^{3t} + C_2 t e^{3t}$$

$$2. \psi = A t^2 + B t + C \quad \psi' = 2At + B \quad \psi'' = 2A$$

$$\psi'' - 6\psi' + 9\psi = -6(2At + B) + 9(A t^2 + B t + C)$$

$$+ 9(A t^2 + B t + C) = 2(\psi_c) - 6(0) + (\psi_c) = 1$$

$$y'' - 2y' + 2y = (2t-1)e^{-t} \quad \text{R.H.S}$$

$$\textcircled{1} \quad y'' - 2y' + 2y = 0 \rightarrow r^2 - 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = 1 \pm i \rightarrow \alpha \pm i\beta$$

$$y_1 = e^{\alpha t} \cos \beta t$$

$$y_2 = e^{\alpha t} \sin \beta t$$

$$y_c = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\textcircled{2} \quad y = (At+B)e^{-t}$$

$$y'' - 2y' + 2y$$

$$y' = Ae^{-t} + (At+B)(-e^{-t})$$

$$= [(-A)t + (A-B)]e^{-t}$$

$$y'' = (-A)e^{-t} - [(-A)t + (A-B)]e^{-t}$$

$$= [(A)t + (-2A+B)]e^{-t}$$

$$+ [(2A) + (-2A+2B)]e^{-t} \cancel{-2y'}$$

$$+ [(2A)t + (2B)]e^{-t} \cancel{-2y}$$

$$[(5A)t + (-4A+3B)]e^{-t} \leftarrow \text{L.H.S}$$

$$= [2(t) + (-1)]e^{-t} \rightarrow \begin{bmatrix} 5 & 0 & 2 \\ -4 & 5 & -1 \end{bmatrix} \text{ RREF} \rightarrow \begin{matrix} A = 2/5 \\ B = 3/25 \end{matrix}$$

$$y = (\frac{2}{5}t + \frac{3}{25})e^{-t}$$

$$\textcircled{3} \quad y = \Psi + y_c$$

$$y(t) = (\frac{2}{5}t + \frac{3}{25})e^{-t} + C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\text{Example: } y'' - 2y' + 2y = \cos t$$

$$\textcircled{1} \quad y_c = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\textcircled{2} \quad \Psi = A \cos t + B \sin t$$

$$y' = -A \sin t + B \cos t$$

$$y'' = -A \cos t - B \sin t$$

$$\begin{aligned} y'' - 2y' + 2y &= -A \cos t - B \cos t + A \cos t + B \sin t \\ &+ 2A \sin t - 2B \cos t \cancel{-2y'} \\ &+ 2A \cos t + 2B \sin t \cancel{-2y} \end{aligned}$$

$$(A-2B) \cos t + (2A+B) \sin t \cancel{\downarrow \text{LHS}}$$

$$= (1) \cos t + (0) \sin t$$

$$\rightarrow \begin{bmatrix} 1 & -2 & | & 1 \\ 2 & 1 & | & 0 \end{bmatrix} \rightarrow \text{RREF} \rightarrow \begin{matrix} A = 1/5 \\ B = -2/5 \end{matrix}$$

$$\textcircled{3} \quad \Psi = \frac{1}{5} \cos t - \frac{2}{5} \sin t$$

$$y = \Psi + y_c$$

$$y = \frac{1}{5} \cos t - \frac{2}{5} \sin t + C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\text{Example: } y'' - y' - 2y = (\frac{1}{2}t^3 + \sqrt{2}t)e^{-7t} \sin(\pi t)$$

$$\textcircled{1} \quad y'' - y' - 2y = 0$$

$$r^2 - r - 2 = 0 \quad r = 2, -1$$

$$(r-2)(r+1) = 0 \quad y_c = C_1 e^{2t} + C_2 e^{-t}$$

$$\textcircled{2} \quad \Psi = (At^3 + Bt^2 + Ct + D)e^{-7t} \cos(\pi t)$$

$$+ (E + F + G + H)e^{-7t} \sin(\pi t)$$

Edge cases:

$$y'' - y = e^t$$

$$\textcircled{1} \quad y'' - y = 0 \rightarrow r^2 - 1 = 0$$

$$(r+1)(r-1) = 0 \quad r = 1, -1$$

$$y_c = C_1 e^t + C_2 e^{-t} \rightarrow \text{Double trouble}$$

$$\textcircled{2} \quad \Psi = A \cdot e^t, \quad \Psi' = A e^t, \quad \Psi'' = A e^t$$

$$y'' - \Psi \cancel{= A e^t} \quad A e^t = C_1 e^t \leftarrow \text{L.H.S}$$

$$= e^t \leftarrow \text{R.H.S}$$

$$y'' + 2y' + 2y = te^{-t} \cos t$$

$$\textcircled{1} \quad y'' + 2y' + 2y = 0 \rightarrow r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-4 \cdot 2}}{2} = -1 \pm i$$

$$y_c = C_1 e^{-t} \cos \theta + C_2 e^{-t} \sin \theta$$

$$\textcircled{2} \quad \Psi = (At + B)e^{-t} \cos t +$$

$$y'' - 2y' + 2y = te^{-t} \cos t$$

$$\textcircled{1} \quad y_c = C_1 e^{-t} \cos \theta + C_2 e^{-t} \sin \theta$$

Trouble!

$$\textcircled{2} \quad \Psi = \{(At + B)e^{-t} \cos t + (Ct + D)e^{-t} \sin t\} +$$

$$= (At^2 + Bt + Ct^2 + Dt)e^{-t} \cos t + (Ct^2 + Dt + Ct + D)e^{-t} \sin t$$

$$y'' - 2y' + y = e^t$$

$$\textcircled{1} \quad y'' - 2y' + y = 0 \rightarrow r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0 \quad r = 1$$

$$y_c = C_1 e^t + C_2 te^t$$

$$\textcircled{2} \quad \Psi = A e^t \times \text{Trouble}$$

$$\Psi = A t e^t \times \text{Double trouble}$$

$$\Psi = At^2 e^t$$

Review for Exam 1

June 21, 2025

$$\begin{aligned} 5A + 2B &= 1 \\ 2A - B &= 0 \end{aligned} \rightarrow \begin{bmatrix} 5 & 2 & | & 1 \\ 2 & -1 & | & 0 \end{bmatrix}$$

$$\text{GOAL} \rightarrow \begin{bmatrix} 1 & 0 & | & A \\ 0 & 1 & | & B \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & | & X \\ 0 & 1 & | & Y \end{bmatrix}$$

$$1 \cdot A + B \cdot 0 = X$$

$$0 \cdot A + B \cdot 1 = Y$$

$$\begin{aligned} x + z &= 1 \\ 3x + 6y - 3z &= 3 \\ -2x - 3y + 3z &= 2 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 1 & | & 1 \\ 3 & 6 & -3 & | & 3 \\ -2 & -3 & 3 & | & 2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ -2 & -3 & 3 & | & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & 2 & -1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 1 & 1 & | & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -3 & | & -7 \\ 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \text{ (RREF)}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 - 3z \\ 4 - 2z \\ z \end{bmatrix} \rightarrow \begin{bmatrix} -7 \\ 4 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & -3 & | & -7 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Parametric form

$$x_1 - 3x_2 = -3$$

$$x_2 = x_2$$

$$x_3 = 2$$

$$\begin{aligned} x_1 &= -3 + 3x_2 \\ x_2 &= x_2 \\ x_3 &= 2 \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 + 3x_2 \\ x_2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

2nd Order non-homogeneous:

$$ay'' + by' + cy = g(t) \leftarrow p(t)$$

$$p(t)e^{at}$$

$$p(t)e^{at}(\sin \beta t + \cos \beta t)$$

$$\textcircled{1} \quad ay'' + by' + cy = 0$$

$$ar^2 + br + c = 0 \rightarrow y_c = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\Delta > 0$$

Guess Ψ (pdf)

$$\Delta < 0$$

'Check for trouble'

Grind

$$\textcircled{3} \quad \Psi = \Psi + \Psi_c$$

$$y'' + 2y' + 2y = 3e^{-t} \cos t$$

$$r^2 + 2r + 2 = 0 \leftrightarrow y'' + 2y' + 2y = 0$$

$$r = -2 \pm \sqrt{-2+4} = -1 \pm i$$

$$y_c = C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t)$$

$$\begin{aligned} \textcircled{2} \quad \Psi &= (Ae^{-t} \cos(t) + Be^{-t} \sin(t)) + \\ &= Ate^{-t} \cos(t) + Bte^{-t} \sin(t) \end{aligned}$$

$$(r-3)^2 = r^2 - 6r + 9$$

e^{3t} not allowed

$$y'' - 6y' + 9y = (-t+2)e^{2t}$$

$$\textcircled{1} \quad y'' - 6y' + 9y = 0 \rightarrow r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0 \rightarrow r=3, \text{ double}$$

$$y_c = C_1 e^{3t} + C_2 te^{3t}$$

$$\textcircled{2} \quad \Psi = (At+B)e^{2t}$$

$$\Psi' = Ae^{2t} + 2(At+B)e^{2t}$$

$$= [(2A)t + (A+2B)]e^{2t}$$

$$\Psi'' = 2Ae^{2t} + 2[2At + (A+2B)]e^{2t}$$

$$= [(4A)t + (4A+4B)]e^{2t}$$

$$\Psi'' - 6\Psi' + 9\Psi = [(4A)t + (4A+4B)]e^{2t}$$

$$+ [(-12A)t + (-6A-12B)]e^{2t}$$

$$+ [(9A)t + (9B)]e^{2t}$$

$$[(1A)t + (-2A+B)]e^{2t} =$$

$$[(-1)t + (2)]e^{2t} \quad \Psi = (-t)e^{2t}$$

$$\begin{aligned} A &= -1 \\ -2A+B &= 2 \end{aligned}$$

$$\Psi = \Psi + \Psi_c = -te^{2t} + C_1 e^{3t} + C_2 te^{3t}$$

Lecture: June 27, 2025

System of ODEs has more functions

$$f(x), g(x) \quad f' = \frac{1}{g} + fF \quad f' = Af + B \cdot g$$

$$1^{\text{st}} \text{ Order, Linear, Homo., } g' = x^3 - g \cdot f \quad g' = Cf + D \cdot g$$

and constant coefficient

Variable change:

$$\begin{aligned} f, g, h \\ f' = Af + Bg + Ch \end{aligned}$$

$$\begin{aligned} x_1(t), x_2(t) \\ x_1(t) = Ax_1 + Bx_2 \\ x_2(t) = Cx_1 + Dx_2 \end{aligned}$$

A system looks like:

$$\begin{aligned} x_1' = -\frac{1}{2}x_1 + \frac{1}{2}x_2 \\ x_2' = \frac{1}{2}x_1 - \frac{1}{2}x_2 \end{aligned} \text{ or } \frac{d}{dt} \vec{x} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \vec{x} \Rightarrow \frac{d}{dt} \vec{x} = A \vec{x}, A \in M_2(\mathbb{R})$$

$$\vec{x}(0) = \vec{v} \in \mathbb{R}^2$$

For an I.V.P., we need $x_1(0) = ?$ and $x_2(0) = ?$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \vec{x}$$

$$\text{I.V.P.: } \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then the solution is $\vec{x}^1 = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-t} \\ \frac{1}{2} - \frac{1}{2}e^{-t} \end{bmatrix}$

$$\text{I.V.P.: } \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \vec{v} \in \mathbb{R}^2, \text{ solution is } \vec{x}^2 = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-t} \\ \frac{1}{2} + \frac{1}{2}e^{-t} \end{bmatrix}$$

General solution for the example:

$$\vec{x}(t) = C_1 \vec{x}^1(t) + C_2 \vec{x}^2(t) = C_1 \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-t} \\ \frac{1}{2} - \frac{1}{2}e^{-t} \end{bmatrix} + C_2 \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-t} \\ \frac{1}{2} + \frac{1}{2}e^{-t} \end{bmatrix}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} \text{ (Case 1)}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} \rightarrow \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2x_1 + 0x_2 \\ 0x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 3x_2 \end{bmatrix}$$

$$x_1' = -2x_1 \quad x_1' = -2x_1$$

$$x_2' = 3x_2 \quad x_1' = -2$$

$$(\ln x_2)' = 3 \quad (\ln x_1)' = -2$$

$$\ln x_2 = 3t + C \quad \ln x_1 = -2t + C$$

$$x_2 = e^{3t} \cdot D \quad x_1 = e^{-2t} \cdot C$$

$$\begin{aligned} \text{Gen. solution: } \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Ce^{-2t} \\ De^{3t} \end{bmatrix} \\ &= C \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} + D \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} \end{aligned}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \vec{x}$$

$$\vec{x} = C_1 \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{\frac{1}{2}t} \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ \sqrt{3} \end{bmatrix}$$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} \rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1' = x_2 \quad \xleftarrow{\qquad\qquad\qquad} \quad x_2' = 2x_1 - x_2$$

$$x_1'' = (x_1')' = 2x_1 - x_1 \rightarrow x_1'' = 2x_1 - x_1' \quad x_1'' = 2x_1 - x_1' = 0$$

$$x_2'' = 2x_1 - x_2 \quad x_1'' + x_1' - 2x_1 = 0$$

$$r^2 + r - 2 = 0 \quad (r+2)(r-1) = 0$$

$$r = 1, -2$$

$$x_1 = C_1 e^t + C_2 e^{-2t}$$

$$x_1 = e^t \text{ or } e^{-2t}$$

Lecture (Linear Algebra)

July 2nd, 2025

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$$

\mathbb{R}^n with vector addition and scalar multiplication

$$\mathbb{R}^2: \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \quad 2 \in \mathbb{R}, \quad 2 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

$$\mathbb{R}^3: \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$$

Abstract Vector Spaces

A set V together with a rule for adding two elements (or vectors) $x, y \in V$ and a rule for scaling any element (or vector) $x \in V$ by a real number $\alpha \in \mathbb{R}$

\mathbb{R}^n is a vector space

$$P_3 = \{ p(t) \mid \deg(p(t)) \leq 3 \}$$

$$= \{ a+bt+ct^2+dt^3 \mid a, b, c, d \in \mathbb{R} \}$$

$$\dim(P_3) = 4$$

$$P = \{ p(t) \mid p(t) \text{ polynomial} \}$$

$$\dim(P) = \infty \leftrightarrow P_2 \subseteq P, P_3 \subseteq P, \dots$$

$$\dim(P) = 1 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, \dots$$

$$\dim(P) = 2 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, \dots$$

$$\dim(P) = 3 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, \dots$$

$$\dim(P) = 4 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, \dots$$

$$\dim(P) = 5 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, \dots$$

$$\dim(P) = 6 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, \dots$$

$$\dim(P) = 7 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, \dots$$

$$\dim(P) = 8 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, \dots$$

$$\dim(P) = 9 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, \dots$$

$$\dim(P) = 10 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, \dots$$

$$\dim(P) = 11 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, \dots$$

$$\dim(P) = 12 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, \dots$$

$$\dim(P) = 13 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, \dots$$

$$\dim(P) = 14 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, \dots$$

$$\dim(P) = 15 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, \dots$$

$$\dim(P) = 16 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, \dots$$

$$\dim(P) = 17 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, \dots$$

$$\dim(P) = 18 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, \dots$$

$$\dim(P) = 19 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, \dots$$

$$\dim(P) = 20 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, \dots$$

$$\dim(P) = 21 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, \dots$$

$$\dim(P) = 22 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, \dots$$

$$\dim(P) = 23 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, P_{23} \subseteq P, \dots$$

$$\dim(P) = 24 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, P_{23} \subseteq P, P_{24} \subseteq P, \dots$$

$$\dim(P) = 25 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, P_{23} \subseteq P, P_{24} \subseteq P, P_{25} \subseteq P, \dots$$

$$\dim(P) = 26 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, P_{23} \subseteq P, P_{24} \subseteq P, P_{25} \subseteq P, P_{26} \subseteq P, \dots$$

$$\dim(P) = 27 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, P_{23} \subseteq P, P_{24} \subseteq P, P_{25} \subseteq P, P_{26} \subseteq P, P_{27} \subseteq P, \dots$$

$$\dim(P) = 28 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P_4 \subseteq P, P_5 \subseteq P, P_6 \subseteq P, P_7 \subseteq P, P_8 \subseteq P, P_9 \subseteq P, P_{10} \subseteq P, P_{11} \subseteq P, P_{12} \subseteq P, P_{13} \subseteq P, P_{14} \subseteq P, P_{15} \subseteq P, P_{16} \subseteq P, P_{17} \subseteq P, P_{18} \subseteq P, P_{19} \subseteq P, P_{20} \subseteq P, P_{21} \subseteq P, P_{22} \subseteq P, P_{23} \subseteq P, P_{24} \subseteq P, P_{25} \subseteq P, P_{26} \subseteq P, P_{27} \subseteq P, P_{28} \subseteq P, \dots$$

$$\dim(P) = 29 \leftrightarrow P_1 \subseteq P, P_2 \subseteq P, P_3 \subseteq P, P$$

Example:
 $C(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$
 Real-valued functions
 Vector Space
 $f(x) + g(x)$ is continuous
 $f(x), c \in \mathbb{R} \rightarrow cf(x)$ is continuous
 $\dim(C(\mathbb{R})) = \infty$ $P \subseteq C(\mathbb{R})$
 $C'(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}$
 $(f+g)' = f'+g'$
 $(cf)' = c \cdot f'$
 $C^2(\mathbb{R}), C^3(\mathbb{R}), \dots, C^\infty(\mathbb{R})$
 $e^x, \sin(x), \dots$
 Solutions to linear differential equations live in $C^\infty(\mathbb{R})$
 Solutions to homogenous differential equations live in $C^\infty(\mathbb{R})$
 AND are vector spaces.
 $V = \{y(t) \mid y'' - y = 0\}$ $r^2 - 1 = 0$
 $\mathbb{R}^2 = \{c_1 e^t + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R}\} \subseteq C^\infty(\mathbb{R})$
 Linear Algebra 5 → Vector Spaces
 Linear Algebra 6 → Subspaces
 Let $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$
 $\dim(V) = 2$
 Example: $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$ $A: \mathbb{R}^3 \rightarrow \mathbb{R}^2$
 Think about \vec{v} , and $A \cdot \vec{v}$; $\vec{v} \in \mathbb{R}^3$
 $A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2 \\ x+y+3z \end{bmatrix}$
 $\text{Kernel}(A) = \{\vec{v} \in \mathbb{R}^3 \mid A \cdot \vec{v} = \vec{0} \in \mathbb{R}^2\}$
 (Null(A))
 What $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ solve
 $\begin{cases} x-2=0 \\ -x+y+3z=0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$
 $R_2 \rightarrow R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$
 $x-2=0 \rightarrow x=2$
 $y+2z=0 \rightarrow y=-2z$
 $z=z \rightarrow z=2$
 Example: $\begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} = B$
 $\text{Ker}(B) = \{\vec{v} \in \mathbb{R}^5 \mid B \cdot \vec{v} = \vec{0}\}$
 $\begin{cases} x_1 - 2x_2 + 7x_5 = 0 \\ x_3 + 8x_5 = 0 \\ x_4 - 3x_5 = 0 \\ x_2 = x_3 \\ x_5 = x_6 \end{cases} \quad \begin{cases} x_1 = 2x_2 - 7x_5 \\ x_2 = x_3 \\ x_3 = -8x_5 \\ x_4 = 3x_5 \\ x_5 = x_6 \end{cases}$
 $\left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 8 \\ 3 \\ 0 \\ 0 \end{bmatrix} \mid x_2, x_5 \in \mathbb{R} \right\} = \text{Ker}(B) \subseteq \mathbb{R}^5$
 Lecture
 July 7th, 2025
 Online covers kernel and span while in class covers 2nd order non-homogeneous
 and 2x2's ($\begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$)
 V a vector space, $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$
 $\text{Span}(S) = \{a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_k \vec{v}_k \mid a_1, \dots, a_k \in \mathbb{R}\}$
 $S \subseteq \text{Span}(S) \subseteq V$
 subvector subspace
Example:
 $\text{Ker}(\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 7 \\ 1 & 1 & 1 & 26 \end{bmatrix})$
 $x_1 - 2x_3 + 3x_4 = 0$
 $x_2 - 4x_3 + 7x_4 = 0$
 $x_3 = x_3$
 $x_4 = x_4$
 $\left\{ x_3 \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 7 \\ 1 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R} \right\}$
 $= \text{Span}(\{ \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 1 \\ 1 \end{bmatrix} \})$
Example:
 $V = \{y(t) \mid y'' - y = 0\}$
 $\begin{cases} P_2 = \{a + bt + ct^2 \mid a, b, c \in \mathbb{R}\} \\ = \{C_1 e^t + C_2 e^{-t} \mid C_1, C_2 \in \mathbb{R}\} \\ = \text{Span}(\{e^t, e^{-t}\}) \end{cases}$
Example:
 $\vec{b} = \begin{bmatrix} -17 \\ 11 \\ 11 \end{bmatrix} \in \text{Span}(\{ \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix} \})$
 Is \vec{b} a linear combination of these 3 vectors?
 Are there $a, b, c \in \mathbb{R}$ so that:
 $\begin{bmatrix} -17 \\ 11 \\ 11 \end{bmatrix} = a \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix}$
 $= \begin{bmatrix} 6a \\ -3a \\ -3a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -2b \end{bmatrix} + \begin{bmatrix} 9c \\ -3c \\ -3c \end{bmatrix} = \begin{bmatrix} 6a+2b+9c \\ -3a+b \\ -3a-2b-3c \end{bmatrix}$
 $= \begin{bmatrix} 6 & 2 & 9 \\ 2 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
 If free x_3 variable, then
 $\begin{bmatrix} 6 & 2 & 9 & -17 \\ 2 & 1 & -2 & 11 \\ -3 & -2 & -3 & -7 \end{bmatrix} \xrightarrow{\text{RREF}(A)} \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
 no solution.
 $\begin{cases} a+3c = -3 \\ a = -3-3c \\ b-3c = -1 \rightarrow b = 3c-1 \\ c = c \end{cases} \quad \text{sol} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$
 scalars
 $\begin{bmatrix} -17 \\ 11 \\ 11 \end{bmatrix} = -3 \begin{bmatrix} 6 \\ -3 \\ -3 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix}$
 Get pts. on the exam.
Example: Use parametric form to show that
 $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \text{Span}(\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \})$
 $\begin{bmatrix} -1 & -1 & 2 & -5 & -1 \\ -1 & -1 & 1 & -2 & -1 \\ -2 & -2 & -1 & 5 & -6 \\ 1 & 1 & 2 & -7 & 7 \end{bmatrix} \xrightarrow{\text{RREF}(A)} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
 $\begin{cases} a+b-d=3 \\ b=b \\ c-3d=2 \\ d=d \end{cases} \quad \begin{cases} a=3-b+d \\ b=b \\ c=3d+2 \\ d=d \end{cases} \quad \text{sol} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$
 $3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix} = \vec{b} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
Example: $\text{Ker}(\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & -9 & -26 \\ 5 & 5 & 17 & -26 \end{bmatrix})$
 $\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}(A)} \begin{bmatrix} x_1 - x_3 - 3x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \\ x_3 = x_3 \end{bmatrix}$
 $\begin{cases} x_1 - x_3 - 3x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \\ x_3 = x_3 \end{cases} \quad \begin{cases} x_1 = x_3 + 3x_4 \\ x_2 = -2x_3 + x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$
Example: $y'' - 2y' + y = 2e^t$ ①
 $y'' - 2y' + y = 2t$ ②
 1. Solve the homo. eqn.
 $y'' - 2y' + y = 0 \xrightarrow{\text{check}} r^2 - 2r + 1 = 0$
 $(r-1)(r-1) = 0 \quad r=1 \text{ (double)}$
 $y_c = C_1 e^t + C_2 t e^t$
 $y_p = A e^{2t} + B t e^{2t}$
 $y'' =$
 $2. y'' - 2y' + y = 2t$
 $\xrightarrow{\text{check}} (r-1)(r-1) = 0 \quad r=1 \text{ double}$
 $y_c = C_1 e^t + C_2 t e^t$
 $y = A t + B \quad (A t + B) - 2(A t + B) = 0 =$
 $y' = A \quad -2A + A + B = 2t$
 $y'' = 0 \quad (A t + B) - (-2A + B) = 2t + 0$
 $y \rightarrow 2t + 4 \quad A=2$
 $2A + B = 0 \quad B=4$
 $y'' - 2y' = 2t \quad B=4$
 $\xrightarrow{\text{check}} r^2 - 2r = 0 \equiv r(r-2) \equiv r=0$
 $y_c = C_1 e^0 + C_2 t e^{2t}$
 $= C_1 + C_2 t e^{2t}$
 $y = (A t^2 + B t + C) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 $\xrightarrow{\text{check}} r^2 = r=2$
 $y_c = C_1 e^{2t} + C_2 t e^{2t}$
 $y = (A t^2 + B t + C_1 e^{2t} + C_2 t e^{2t}) e^{2t}$
 $y' = 2A t + B + 2C t e^{2t} + 2C_1 e^{2t} + 2C_2 t e^{2t}$
 $y'' = 2A + 2B + 4C t e^{2t} + 4C_1 e^{2t} + 4C_2 t e^{2t}$
 $y'' - 2y' = 2t$
 \x

$S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$, vector space

$$\text{Span}(S) = \{\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{R}\}$$

$$(3d) \quad S_2 = \{a+bz+cz^2 \mid a, b, c \in \mathbb{R}\} = \text{Span}(\{1, z, z^2\})$$

$$(2d) \quad V = \{y \mid y'' - y = 0\} = \{c_1 e^x + c_2 e^{-x} \mid c_1, c_2 \in \mathbb{R}\} = \text{Span}(\{e^x, e^{-x}\})$$

$$(3d) \quad \mathbb{R}^3 = \text{Span}(\{i, j, k\})$$

$$\vec{i} = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \{x\vec{i} + y\vec{j} + z\vec{k} \mid x, y, z \in \mathbb{R}\}$$

$$\vec{j} = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \{x\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \mid x, y, z \in \mathbb{R}\}$$

$$\vec{k} = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \{x\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3$$

$$C = \{\alpha + i\beta \mid \alpha, \beta \in \mathbb{R}\} = \text{Span}(\{1, i\})$$

V a vector space

Conjecture: If $V = \text{Span}(\{\vec{v}_1, \dots, \vec{v}_k\})$,

then V is dimension k , "V is like \mathbb{R}^k "

Conjecture for $k=1$: If $V = \text{span}(\{\vec{v}\})$,

then the dimension of V is 1. ($\vec{v}, \neq \vec{0}$)

$$\text{Span}(\{\vec{v}\}) = \{\alpha \vec{v} \mid \alpha \in \mathbb{R}\}$$

$$\text{Span}(\{\vec{0}\}) = \{\alpha \vec{0} \mid \alpha \in \mathbb{R}\} = \{\vec{0}\}$$

Conjecture for $k=2$: If $V = \text{Span}(\{\vec{v}_1, \vec{v}_2\})$,

then dim(V) is 2. ($\vec{v}_1, \vec{v}_2 \neq \vec{0}$)

$$V = \text{Span}(\{\vec{v}_1, \vec{v}_2\}) = \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$\text{Span}(\{\{1\}, \{z\}\}) = \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ z \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\beta \begin{bmatrix} 1 \\ z \end{bmatrix} \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{\alpha + 2\beta \mid \alpha, \beta \in \mathbb{R}\}$$

Conjecture for $k=3$: If $V = \text{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$, the dim(V) is 3.

$$V = \text{Span}(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\})$$

= x-y plane



If $V = \text{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$, it's bad if:

$$\vec{v}_1 = \alpha_1 \vec{v}_2 + \alpha_2 \vec{v}_3$$

$$\vec{v}_2 = \beta_1 \vec{v}_1 + \beta_3 \vec{v}_3$$

$$\vec{v}_3 = \gamma_1 \vec{v}_1 + \gamma_2 \vec{v}_2$$

it's BAD if:

$$-\vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 = \vec{0}$$

$$-\vec{v}_2 + \beta_1 \vec{v}_1 + \beta_3 \vec{v}_3 = \vec{0}$$

$$-\vec{v}_3 + \gamma_1 \vec{v}_1 + \gamma_2 \vec{v}_2 = \vec{0}$$

it's BAD if there are $\alpha, \beta, \gamma \in \mathbb{R}$, NOT ALL ZERO.

$$\alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 = \vec{0}$$

The set $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ is BAD, i.e linearly dependent,

if there are $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

NOT ALL ZERO

such that $\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k = \vec{0}$.

The set $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ is good, i.e Linearly independent, if

$$\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k = \vec{0} \implies \alpha_1 = \alpha_2 = \dots = \alpha_k = 0.$$

(The only linear combination of the vectors in S that equals $\vec{0}$ is the trivial linear combination).

$$\text{Example: } S = \left\{ \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$\alpha \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -3 & 1 \\ -3 & 1 & 0 \\ 8 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -3 & 1 \\ -3 & 1 & 0 \\ 8 & 0 & -1 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{array}{l} a - c = 0 \\ b - 3c = 0 \\ c = c \end{array}$$

$$\implies c \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\implies (1) \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix} + (3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$$

Example: Use parametric form to show that

$$S = \left\{ \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix} \right\}$$

is lin. dependent

$$\begin{bmatrix} -5 & -3 & 7 & -6 \\ 2 & 1 & -3 & 3 \\ 3 & 1 & -5 & 6 \\ 0 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{array}{l} a - 2c + 3d = 0 \\ b + c - 3d = 0 \\ c = c \\ d = d \end{array} \implies \begin{array}{l} a = 2c - 3d \\ b = -c + 3d \\ c = c \\ d = d \end{array}$$

$$\begin{array}{l} \text{Two answers, both are legal.} \\ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \implies (2) \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix} = \vec{0} \\ (-3) \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix} = \vec{0} \end{array}$$

Lecture

July 11th, 2025

Thrm: $A \in M_n(\mathbb{R})$, write $A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n] \rightarrow \vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$

① $A\vec{x} = \vec{0}$ has non-trivial (non-zero) solutions.

② $\text{Ker}(A) > \{\vec{0}\} \iff (\dim(\text{Ker}(A) > 0))$

③ $\{\vec{v}_1, \dots, \vec{v}_n\}$ linearly dependent

④ $\text{RREF}(A) \neq I_n$

⑤ $\text{rank}(A) < n$

⑥ A is not invertible \therefore singular

⑦ $\det(A) = 0$, then A is singular

① $A\vec{x} = \vec{0}$ has only the trivial solution ($\vec{x} = \vec{0}$).

② $\text{Ker}(A) = \{\vec{0}\}$

③ $\{\vec{v}_1, \dots, \vec{v}_n\}$ linearly independent

④ $\text{RREF}(A) = I_n$

⑤ $\text{rank}(A) = n$

⑥ A is invertible (A^{-1} is OK)

⑦ $\det(A) \neq 0$

Goal: Solve $\frac{d}{dt} \vec{x} = A\vec{x}$, $A \in M_n(\mathbb{R})$

$$\text{Example: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \vec{x} \quad \vec{x}^1 = \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}, \quad \vec{x}^2 = \begin{bmatrix} 0 \\ e^{4t} \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \vec{x} \quad \vec{x}^1 = \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}, \quad \vec{x}^2 = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$\vec{x}(+) = \begin{bmatrix} x_1(+) \\ \vdots \\ x_n(+) \end{bmatrix} = \begin{bmatrix} a_1 e^{\lambda_1 t} \\ \vdots \\ a_n e^{\lambda_n t} \end{bmatrix} \quad \text{My guess}$$

$$\frac{d}{dt} \vec{x} = A\vec{x} \quad A \in M_n(\mathbb{R}) \quad \text{Guess: } \vec{x} = e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\frac{d}{dt} (e^{\lambda t} \cdot \vec{v}) = A \cdot (e^{\lambda t} \cdot \vec{v}) \quad = e^{\lambda t} \vec{v}, \quad \lambda \in \mathbb{R} \wedge \vec{v} \in \mathbb{R}^n$$

$$\lambda \vec{v} = A \cdot \vec{v} \quad \frac{d}{dt} (e^{\lambda t} \begin{bmatrix} -2 \\ 5 \end{bmatrix}) = \frac{d}{dt} \begin{bmatrix} -2e^{\lambda t} \\ 5e^{\lambda t} \end{bmatrix} = \begin{bmatrix} -2 \cdot 3e^{\lambda t} \\ 5 \cdot 3e^{\lambda t} \end{bmatrix} = 3e^{\lambda t} \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Def: $A \in M_n(\mathbb{R})$, if $\lambda \in \mathbb{R}$, $\vec{v}_\lambda \in \mathbb{R}^n$ ($\vec{v}_\lambda \neq \vec{0}$) such that $A \cdot \vec{v}_\lambda = \lambda \cdot \vec{v}_\lambda$, then λ is an eigenvalue for A and \vec{v}_λ is an eigenvector for A with eigenvalue λ

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0} \rightarrow [A] \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} - \lambda \begin{bmatrix} \vec{v} \end{bmatrix} = \vec{0}$$

$$\vec{v}(A - \lambda I) = \vec{0} \rightarrow \boxed{[A - \lambda I]} \begin{bmatrix} \vec{v} \end{bmatrix} = \vec{0}$$

$$A \cdot \vec{v} - \lambda \cdot I_n \cdot \vec{v} = \vec{0} \rightarrow [A] \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} - \lambda \begin{bmatrix} I_n \end{bmatrix} \begin{bmatrix} \vec{v} \end{bmatrix} = \vec{0}$$

$$(A - \lambda I_n) \cdot \vec{v} = \vec{0}$$

Pretend I know an eigenvalue, λ

Inside $\text{ker}(A - \lambda I_n)$ is the eigenvectors.

$$\begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{bmatrix} \quad \begin{array}{l} 7-\lambda \\ -5-3\lambda-5\lambda \\ -x=\lambda \\ \lambda^2 \end{array}$$

$$\det \begin{pmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{pmatrix} = (7-\lambda)(-5-\lambda) - (8)(-4)$$

$$\lambda^2 - 2\lambda - 35 + 32 = \lambda^2 - 2\lambda - 3$$

$$\lambda = 3 \quad \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix} \quad \begin{array}{l} \lambda^2 - 2\lambda - 3 \\ +1 \quad \lambda^2 - 3\lambda \\ \hline \lambda = 3, -1 \end{array}$$

$$\text{rref} \begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - x_2 = 0 \rightarrow x_1 = x_2 \\ x_2 = x_2 \quad x_2 = x_2 \end{array}$$

$$\vec{x}' = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = 3$$

$$\lambda = -3 \quad \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 8 & -4 \end{bmatrix} \quad \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \lambda = -1 \quad \left\{ \begin{array}{l} e^{-t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \end{array} \right\}$$

$$\text{rref} \begin{pmatrix} 8 & -4 \\ 8 & -4 \end{pmatrix} = \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x - 1/2 x_2 = 0 \\ x_2 = x_2 \end{array} \rightarrow \begin{array}{l} x_1 = 1/2 x_2 \\ x_2 = x_2 \end{array}$$

$$\vec{y}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Lecture

July 26, 2025

$$\text{Solve } \frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} \vec{x}$$

1. Characteristic polynomial: $p(\lambda) = \det(A - \lambda I)$

$$A - \lambda I = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{pmatrix} = (\lambda - 7)(\lambda + 5) + 32 = \lambda^2 - 2\lambda - 3 = \begin{cases} \lambda = 3 \\ \lambda = -1 \end{cases}$$

$$2. \lambda = 3: \text{Ker}(A - 3I)$$

$$A - 3I = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 8 & -8 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x - y = 0 \rightarrow x = y \\ y = y \end{array} \rightarrow \begin{array}{l} y \\ y \end{array} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}' = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} | e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1: \text{Ker}(A - (-1)I) = \text{Ker}(A + I)$$

$$A + I = \begin{bmatrix} 8 & -4 \\ 8 & -4 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x - 1/2 y = 0 \rightarrow x = 1/2 y \\ y = y \end{array} \rightarrow \begin{array}{l} y \\ 1 \end{array} \rightarrow \vec{v}_{-1} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \vec{x}'' = e^{-t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$3. \vec{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\text{Solve: } \frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & 5 \\ 0 & 3 \end{bmatrix} \vec{x}$$

$$p(\lambda) = \det \begin{pmatrix} -2-\lambda & 5 \\ 0 & 3-\lambda \end{pmatrix} = (-2-\lambda)(3-\lambda) - (5) \cdot 0$$

$$= (-2-\lambda)(3-\lambda) \\ = \lambda = \{3, -2\}$$

$$\lambda = 3: \text{Ker}(A - 3I)$$

$$A - 3I = \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x - y = 0 \rightarrow x = y \\ y = y \end{array} \rightarrow \begin{array}{l} y \\ y \end{array}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}' = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -2: \text{Ker}(A - (-2)I)$$

$$A + 2I = \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - 5R_1 \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ y = 0 \end{array} \quad \begin{array}{l} x = x \\ y = 0 \end{array} \rightarrow \begin{array}{l} x \\ y \end{array} = \begin{bmatrix} x \\ 0 \end{bmatrix} \equiv x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \vec{v}_{-2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x} = C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{Solve: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 10 & -4 \\ 24 & -10 \end{bmatrix} \vec{x}$$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 10-\lambda & -4 \\ 24 & -10-\lambda \end{pmatrix}$$

$$= (10-\lambda)(-10-\lambda) - (-4)(24)$$

$$= (\lambda - 10)(\lambda + 10) + 96 = \lambda^2 - 100 + 96$$

$$= \lambda^2 - 4$$

$$= (\lambda + 2)(\lambda - 2) \rightarrow \lambda = \{-2\}$$

$$\lambda = 2: A - 2I = \begin{bmatrix} 8 & -4 \\ 24 & -12 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} x - 1/2 y = 0 \\ y = y \end{array} \rightarrow \begin{array}{l} x = 1/2 y \\ y = y \end{array} \rightarrow \begin{array}{l} y \\ 1 \end{array} \rightarrow \vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \vec{x}' = e^{2t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\lambda = -2: A + 2I = \begin{bmatrix} 12 & -4 \\ 28 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x - 1/3 y = 0 \\ y = y \end{array} \rightarrow \begin{array}{l} x = 1/3 y \\ y = y \end{array} \rightarrow \begin{array}{l} y \\ 1 \end{array} = y \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\vec{v}_{-2} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \vec{x}'' = e^{-2t} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

If $\text{Ker}(A - \lambda I_n) = \{ \vec{0} \}$, there are no eigenvectors;

then λ is NOT an eigenvalue

$$\iff \det(A - \lambda I) \neq 0$$

If $\text{Ker}(A - \lambda I_n)$ is "bigger" than $\{ \vec{0} \}$,

then λ is an eigenvalue $\iff \det(A - \lambda I) = 0$

$$\text{Solve: } \frac{d}{dt} \vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

$$e^{it} \vec{v}_i$$

$$\lambda = 2+i: A - (\lambda+i)\mathbf{I} = A + (-2-i)\mathbf{I}$$

$$= \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix} + \begin{bmatrix} -2-i & 0 \\ 0 & -2-i \end{bmatrix} = \begin{bmatrix} -5-i & 13 \\ -2 & 5-i \end{bmatrix}$$

ref(result):
 $\rightarrow \begin{bmatrix} -2 & 5-i \\ -5-i & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5/2 + 1/2i \\ 0 & 0 \end{bmatrix}$

$$\left. \begin{array}{l} x + \left(\frac{5}{2} + \frac{1}{2}i\right)y = 0 \\ y = y \end{array} \right\} \rightarrow \begin{bmatrix} 5/2 - 1/2i & \\ 1 & 1 \end{bmatrix} \Rightarrow \vec{v}_{2+i} = \begin{bmatrix} 5/2 - 1/2i \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{z} &= e^{(2+i)t} \cdot \begin{bmatrix} 5/2 - 1/2i & \\ 1 & 1 \end{bmatrix} \\ &= e^{2t} \left[\begin{bmatrix} \frac{5}{2} \cos t & \frac{1}{2} \sin t \\ \cos t & 0 \end{bmatrix} + i \begin{bmatrix} -\frac{1}{2} \cos t & \frac{1}{2} \sin t \\ \sin t & 0 \end{bmatrix} \right] + i \begin{bmatrix} \frac{5}{2} \sin t & \frac{1}{2} \cos t \\ \sin t & 0 \end{bmatrix} \\ \vec{x} &= C_1 e^{2t} \begin{bmatrix} \frac{5}{2} \cos t & \frac{1}{2} \sin t \\ \cos t & 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\frac{1}{2} \cos t & \frac{1}{2} \sin t \\ \sin t & 0 \end{bmatrix} \end{aligned}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}$$

$$\begin{aligned} p(\lambda) &= \det \begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix} = (-1-\lambda)(5-\lambda) - 9(-1) \\ &= (\lambda+1)(\lambda-5) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda-2)(\lambda-2) \end{aligned}$$

$\lambda = 2$, double root

$$\lambda = 2: A - 2\mathbf{I} = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\left. \begin{array}{l} x - 3y = 0 \\ y = y \end{array} \right\} x = 3y \rightarrow y \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Jordan Cycle:

- Find \vec{z} such that $(A - \lambda\mathbf{I})\vec{z} = \vec{v}_1$
- $\vec{x}^2 = e^{2t}(\vec{z} + t\vec{v}_1)$

Find \vec{z} such that $\begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \vec{z} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -3 & 9 & 3 \\ -1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left. \begin{array}{l} x - 3y = -1 \\ y = y \end{array} \right\} x = 3y - 1 \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \vec{z} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{x}^2 &= e^{2t} \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \\ &= e^{2t} \begin{bmatrix} -1 + 3t \\ t \end{bmatrix} \end{aligned}$$

$$\vec{x} = C_1 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -1 + 3t \\ t \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}$$

$$\begin{aligned} p(\lambda) &= \det(A - \lambda\mathbf{I}) = (-1-\lambda)(-1-\lambda) - 1 \cdot 0 \\ &= (\lambda+1)(\lambda+1) = \lambda = -1, \text{ double root} \end{aligned}$$

$$\lambda = -1: A - (-1)\mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} x = x \\ y = 0 \end{array} \right\} \Rightarrow x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\vec{x}^1 = e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Find \vec{z} :
 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} y = 1 \\ x = x \end{array} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{x}^2 = e^{-t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\vec{x} = C_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix} \vec{x}$$

$$\begin{aligned} ① p(\lambda) &= \det \begin{bmatrix} -3-\lambda & 13 \\ -2 & 7-\lambda \end{bmatrix} = (-3-\lambda)(7-\lambda) - (-13)(-2) \\ &= (\lambda+3)(\lambda-7) + 26 = \lambda^2 - 4\lambda + 5 \end{aligned}$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Continued on top right!

July 25, 2025

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix} \vec{x}, \rho(\lambda) = \det \begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix}$$

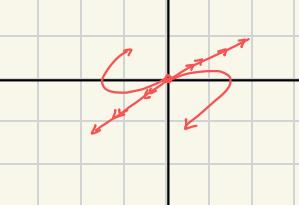
$$= (\lambda+1)(\lambda-5) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

 $\lambda=2$, double root

Degenerate Nodal source

$$\lambda=2: A-2I = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{T} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} \vec{x}, \rho(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix}$$

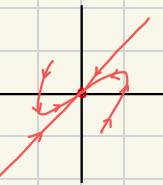
$$\text{Degenerate Nodal Sink} = (1-1)(1+3) + 4 = \lambda^2 + 2\lambda + 1$$

 $\lambda=-1$, double

$$= (1+1)^2$$

$$\lambda=-1: A-I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \vec{T} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$



$$\frac{d}{dt} \vec{x} = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} \vec{x} \quad \text{linear } 2 \times 2$$

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} = \begin{bmatrix} 7x_1(t) - 4x_2(t) \\ 8x_1(t) - 5x_2(t) \end{bmatrix} \Rightarrow \begin{cases} x'_1 = 7x_1 - 4x_2 \\ x'_2 = 8x_1 - 5x_2 \end{cases}$$

$$X' = X^2 - Y^2, Y' = Y - 1$$

Step 1: Equilibrium points

where $f(x,y) = x'(t)$ and $g(y) = y'(t)$

$$x' = 0 \text{ and } y' = 0$$

$$f(x,y) = 0 \text{ and } g(x,y) = 0$$

$$x^2 - y^2 = 0 \quad \leftarrow x^2 - y^2 = 0$$

$$(x+y)(x-y) = 0 \quad y-1=0 \rightarrow y=1$$

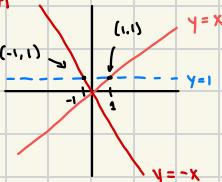
$$x+y=0 \quad \leftarrow y=-x$$

$$\text{or } y=x$$

$$x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

$$x=1, -1$$



Step 2:

$$\begin{aligned} J(-1,1) &= \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix} \\ J(1,1) &= \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Step 2: $x' = X^2 - Y^2 = f(x,y)$

$$Y' = Y - 1 = g(x,y)$$

Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$J = \begin{bmatrix} 2x & -2y \\ 0 & 1 \end{bmatrix}$$

Step 4:

$$\frac{d}{dt} \vec{x} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix} \vec{x} @ (-1,1)$$

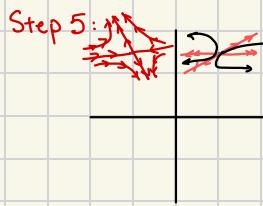
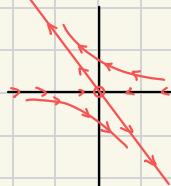
$$\rho(\lambda) = \det \begin{bmatrix} -2-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix} = (\lambda+2)(\lambda-1)$$

$$\lambda = 1, -2$$

$$\lambda = 1: A - I = \begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2/3 \\ 0 & 0 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$\lambda = -2: A + 2I = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} x=x \\ y=0 \end{array} \right\} \rightarrow \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_2$$

$$\vec{v}_{-2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$@ (1,1) \frac{d}{dt} \vec{x} = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$\rho(\lambda) = \det \begin{bmatrix} 2-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}$$

$$\text{Nodal source} = (1-2)(1-1)$$

$$\lambda = 2, 1$$

$$\lambda = 1: A - I = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = 2: A - 2I = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left\{ \begin{array}{l} x=x \\ y=0 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_2$$

