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Exam 4 427J

1. (10 points) Consider the system

$$\frac{d}{dt}\vec{x} = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 4 & 13 \\ -2 & -6 \end{bmatrix}$$

(a) Compute the general solution to the system.

(b) Compute the matrix exponential e^{At}.

2. (15 points) Consider the Heat Equation

$$u_t = \alpha^2 u_{xx}$$

$$u(x,0) = \begin{cases} -1, & 0 < x \le \pi \\ 1, & \pi < x < 2\pi \end{cases}$$

$$u(0,t) = u(2\pi,t) = 0, \quad t \ge 0$$

(a) Give the **Sine Series** for $u(x,0)$. Include the first four nonzero terms.
Each coefficient should be a single reduced fraction with no trig functions and no decimals.

(b) Give the solution to the Heat Equation, $u(x,t)$. Include the first four nonzero terms.
Each coefficient should be a single reduced fraction with no trig functions and no decimals.

2a) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin\left(\frac{nx}{2}\right)dx = \frac{1}{\pi}\left(\underbrace{\int_0^{\pi} -\sin\left(\frac{nx}{2}\right)dx}_{} + \underbrace{\int_{\pi}^{2\pi} \sin\left(\frac{nx}{2}\right)dx}_{}\right)$

$= \left[\frac{2}{n}\cos\left(\frac{nx}{2}\right)\right]_0^{\pi} \quad \left[-\frac{2}{n}\cos\left(\frac{nx}{2}\right)\right]_{\pi}^{2\pi}$

$= \left(\frac{2}{n}\cos\left(\frac{n\pi}{2}\right)\right) - \left(\frac{2}{n}\right) \quad \underbrace{\left(-\frac{2}{n}\cos(\pi n)\right) - \left(-\frac{2}{n}\cos\left(\frac{n\pi}{2}\right)\right)}$

$b_n = \frac{1}{\pi}\left(\frac{2}{n}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n} + -\frac{2}{n}\cos(\pi n) + \frac{2}{n}\cos\left(\frac{n\pi}{2}\right)\right)$

$b_n = \frac{2}{\pi n}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} - \frac{2}{n\pi}\cos(\pi n) + \frac{2}{n\pi}\cos\left(\frac{\pi n}{2}\right)$

$= \frac{2}{\pi n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n + \cos\left(\frac{\pi n}{2}\right)\right) = \frac{2}{\pi n}\left(2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right) = b_n$

(-1)

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{2}\right)$

$b_1 = 0 \quad b_6 = -\frac{4}{3\pi}$

$b_2 = -\frac{4}{\pi}$ By assumption of the pattern the remaining b_n's are:

$b_3 = 0$

$b_4 = 0 \quad b_{10} = -\frac{4}{5\pi}$

$b_{14} = -\frac{4}{7\pi}$

$$\boxed{\therefore u(x,0) = -\frac{4}{\pi}\sin(x) - \frac{4}{3\pi}\sin(3x) - \frac{4}{5\pi}\sin(5x) - \frac{4}{7\pi}\sin(7x) - \ldots}$$

2b) If $u(x,0) = -\frac{4}{\pi}\sin(x) - \frac{4}{3\pi}\sin(3x) - \frac{4}{5\pi}\sin(5x) - \frac{4}{7\pi}\sin(7x) - \ldots$

$e^{-\frac{\cancel{4\pi^2}\alpha^2}{\cancel{4\pi^2}}\cdot t} = e^{-\alpha^2 t}$

$e^{-\frac{36\pi^2\alpha^2}{4\pi^2}\cdot t} = e^{-9\alpha^2 t}$

$e^{-\frac{100\pi^2\alpha^2}{4\pi^2}\cdot t} = e^{-25\alpha^2 t}$

$e^{-\frac{196\pi^2\alpha^2}{4\pi^2}\cdot t} = e^{-49\alpha^2 t}$

$$\boxed{\therefore u(x,t) = -\frac{4}{\pi}\sin(x)e^{-\alpha^2 t} - \frac{4}{3\pi}\sin(3x)e^{-9\alpha^2 t} - \frac{4}{5\pi}\sin(5x)e^{-25\alpha^2 t} - \frac{4}{7\pi}\sin(7x)e^{-49\alpha^2 t} - \ldots}$$

Consider the system:

$$\frac{d}{dt}\vec{x} = A\vec{x}, \text{ where } A = \begin{bmatrix} 4 & 13 \\ -2 & -6 \end{bmatrix}$$

1a. Compute the general solution to the system:

$$p(\lambda) = \det\left(\begin{bmatrix} 4-\lambda & 13 \\ -2 & -6-\lambda \end{bmatrix}\right) = (4-\lambda)(-6-\lambda) - (13)(-2)$$

$$-(-26)$$

$$= (4-\lambda)(-6-\lambda) + 26$$

$$= \lambda^2 + 2\lambda - 24 + 26$$

$$= \lambda^2 + 2\lambda + 2$$

	-6	$-\lambda$
4	-24	-4λ
$-\lambda$	6λ	λ^2

$$\lambda = -1 \pm i$$

$$A - (-1+i)I = \begin{bmatrix} 5-i & 13 \\ -2 & -5-i \end{bmatrix} \qquad \text{rref}(A-(-1+i)I) = \begin{bmatrix} 1 & \frac{5}{2}+\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$$

$$x + \left(\frac{5}{2} + \frac{1}{2}i\right)y = 0 \longmapsto \begin{Bmatrix} x = -\left(\frac{5}{2}+\frac{1}{2}i\right)y \\ y = y \end{Bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -\frac{5}{2}-\frac{1}{2}i \\ 1 \end{bmatrix}$$

$$y = y$$

$$\vec{V}_{(-1+i)} = \begin{bmatrix} -\frac{5}{2}-\frac{1}{2}i \\ 1 \end{bmatrix}$$

$$\vec{z}(t) = e^{(-1+i)t}\begin{bmatrix} -\frac{5}{2}-\frac{1}{2}i \\ 1 \end{bmatrix} = e^{-t}\cdot(\cos t + i\sin(t))\left(\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix} + i\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\right)$$

$$= e^{-t}\left(\begin{bmatrix} -\frac{5}{2}\cos t \\ \cos t \end{bmatrix} + i\begin{bmatrix} -\frac{1}{2}\cos t \\ 0 \end{bmatrix} + i\begin{bmatrix} -\frac{5}{2}\sin(t) \\ \sin(t) \end{bmatrix} + i^2\begin{bmatrix} -\frac{1}{2}\sin t \\ 0 \end{bmatrix}\right)$$

$$= e^{-t}\left(\begin{bmatrix} -\frac{5}{2}\cos t + \frac{1}{2}\sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} -\frac{1}{2}\cos t - \frac{5}{2}\sin(t) \\ \sin(t) \end{bmatrix}\right)$$

$$\boxed{\therefore \vec{x} = c_1 e^{-t}\begin{bmatrix} -\frac{5}{2}\cos t + \frac{1}{2}\sin t \\ \cos t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -\frac{1}{2}\cos t - \frac{5}{2}\sin(t) \\ \sin(t) \end{bmatrix}}$$

1b. Compute the matrix exponential e^{At}

$$X(t) = \begin{bmatrix} -\frac{5}{2}\cos(t) + \frac{1}{2}\sin(t) & -\frac{1}{2}\cos(t) - \frac{5}{2}\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix}$$

$$X(0) = \begin{bmatrix} -\frac{5}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}, \quad \det(X(0)) = \frac{1}{2}$$

$$\Rightarrow (X(0))^{-1} = 2\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -5 \end{bmatrix},$$

Now, we do $X(t) \cdot (X(0))^{-1}$:

$$= \begin{bmatrix} -\frac{5}{2}\cos(t) + \frac{1}{2}\sin(t) & -\frac{1}{2}\cos(t) - \frac{5}{2}\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -5 \end{bmatrix}$$

$$\text{entry}_1 = \underbrace{\cancel{\left(-\frac{5}{2}\cos(t) + \frac{1}{2}\sin(t)\right)(0)}}_{\to 0} + (-2)\left(-\frac{1}{2}\cos t - \frac{5}{2}\sin(t)\right)$$
$$= \cos(t) + 5\sin(t)$$

$$\text{entry}_2 = \left(-\frac{5}{2}\cos(t) + \frac{1}{2}\sin(t)\right)(1) + \left(-\frac{1}{2}\cos(t) - \frac{5}{2}\sin(t)\right)(-5)$$
$$= \cancel{-\frac{5}{2}\cos t} + \frac{1}{2}\sin(t) + \cancel{\frac{5}{2}\cos(t)} + \frac{25}{2}\sin(t)$$
$$= 13\sin(t)$$

$$\text{entry}_3 = (-2)\sin(t) = -2\sin t$$

$$\text{entry}_4 = \cos(t) - 5\sin(t)$$

$$\boxed{\therefore e^{At} = e^{-t}\begin{bmatrix} \cos(t) + 5\sin(t) & 13\sin(t) \\ -2\sin(t) & \cos(t) - 5\sin(t) \end{bmatrix}}$$