(9) $\frac{dy}{dt} + \sqrt{1+t^2}\, e^{-t} y = 0$ $\quad y(0) = 1$ $\quad t_0=0$, $y_0=1$

$\mu(t) = e^{\int \sqrt{1+t^2} e^{-t} dt} \Rightarrow \underline{\mu(t) = e^{\int_0^t \sqrt{1+s^2}\, e^{-s} ds}}$

$y(t) = e^{-\int_0^t \sqrt{1+s^2}\, e^{-s} ds} \left(\underbrace{\cancel{\int_0^t \mu(t)\,(0)}}_{=0} + \mu(0)\, y_0 \right)$ $\quad t_0=0$

$\mu(0) = e^{\int_0^0 \sqrt{1+s^2} e^{-s} ds} = e^0 = 1$

$y(t) = e^{-\int_0^t \sqrt{1+s^2}\, e^{-s}} (1)$

(10) $\frac{dy}{dt} + \sqrt{1+t^2}\, e^{-t} y = \underline{0}, \quad y(0) = \underline{0}$

(11) $\frac{dy}{dt} - 2ty = t, \quad y(0) = 1$ $\quad t_0=0$, $y_0=1$

$\mu(0) = e^{-(0)^2} = e^0 = 1$

$\mu(t) = e^{-\int 2t\,dt} = e^{-t^2}$

$y(t) = \frac{1}{e^{-t^2}} \left(\frac{1}{-2} \int_0^t -2s\, e^{-s^2} ds + \mu(0)\, y_0 \right)$ $\quad y_0 = 1$

$-\frac{1}{2} \left[e^{-s^2} \right]_0^t = -\frac{1}{2}\left(e^{-t^2} - e^0\right) = \underline{\frac{1}{2}\left(1 - e^{-t^2}\right)}$

$y(t) = e^{t^2} \left(\frac{1}{2} \left(1 - e^{-t^2}\right) + 1 \right)$

$y(t) = e^{t^2} + \frac{1}{2} e^{t^2} - \frac{1}{2}$

$y(t) = \frac{3}{2} e^{t^2} - \frac{1}{2}$

(12) $\frac{dy}{dt} + ty = 1 + t \qquad y\left(\frac{3}{2}\right) = 0$ $\quad t_0$, y_0

$\mu\left(\frac{3}{2}\right) = e^{\frac{1}{2}\left(\frac{3}{2}\right)^2} = e^{\frac{9}{8}}$

$\mu(t) = e^{\int t\,dt} = e^{\frac{1}{2}t^2}$

$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^t \mu(s)\, b(s)\, ds + \mu(t_0)\, y_0 \right)$ Formula

$t_0 \quad y_0 = 0$

$$y(t) = e^{-\frac{1}{2}t^2}\left(\int_{\frac{3}{2}}^{t} e^{\frac{1}{2}s^2}(1+s)\,ds + \mu\left(\tfrac{3}{2}\right)(0)\right)$$

$$y(t) = e^{-\frac{1}{2}t^2}\left(\int_{\frac{3}{2}}^{t} e^{\frac{1}{2}s^2} s\,ds + \int_{\frac{3}{2}}^{t} e^{\frac{1}{2}s^2}\,ds\right) \quad (E)$$

$$\left[e^{\frac{1}{2}s^2}\right]_{\frac{3}{2}}^{t} \quad +$$

(13) $\frac{dy}{dt} + y = \frac{1}{1+t^2} \qquad y(1) = 2$

$$\mu(t) = e^{\int dt} = e^{t} \qquad y_0 \qquad \mu(1) = e^1 = e$$

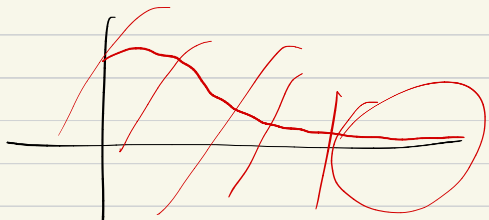
$$y(t) = e^{-t}\left(\int_{1}^{t} \frac{e^{s}}{1+s^2}\,ds + 2e\right)$$

(14) $\frac{dy}{dt} - 2ty = 1, \qquad y(0) = 1$

$$\mu(t) = e^{-\int 2t\,dt} = e^{-t^2} \qquad \mu(0) = e^{-(0)^2} = e^0 = 1$$

$$y(t) = e^{t^2}\left(\int_{0}^{t} \underbrace{e^{-s^2}}_{\mu(t)}\,ds + 1\right)$$

$$\int_{0}^{\infty} e^{-s^2}\,ds \overset{?}{=} \frac{\sqrt{\pi}}{2}$$



(15) $(1+t^2)\frac{dy}{dt} + ty = (1+t^2)^{\frac{5}{2}} \qquad \div (1+t^2)$

$$\frac{dy}{dt} + \frac{t}{1+t^2}y = \frac{(1+t^2)^{\frac{5}{2}}}{1+t^2} = (1+t^2)^{\frac{3}{2}}$$

$$\mu(t) = e^{\frac{1}{2}\int \frac{2t}{1+t^2}dt} = e^{\frac{1}{2}\ln(1+t^2)} = e^{\ln(1+t^2)^{\frac{1}{2}}} = (1+t^2)^{\frac{1}{2}}$$

$$y(t) = \frac{1}{(1+t^2)^{\frac{1}{2}}}\left(\int (1+t^2)^{\frac{1}{2}}\,(1+t^2)^{\frac{3}{2}} + c\right)$$

$$y(t) = \frac{1}{(1+t^2)^{\frac{1}{2}}}\left(\int (1+t^2)^2 + C\right)$$

$$\int 1 + 2t^2 + t^4$$

$$y(t) = \frac{1}{(1+t^2)^{\frac{1}{2}}}\left(t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + C\right)$$

(16) $(1+t^2)\frac{dy}{dt} + 4ty = t, \quad y(1) = \frac{1}{4}$

$\mu(t_0)$ $\mu(1) = (1+1^2)^2 = 4$

$$\frac{dy}{dt} + \frac{4t}{1+t^2}\,y = \frac{t}{1+t^2} \qquad y(1) = \frac{1}{4}$$

$$\mu(t) = e^{2\int \frac{2t}{1+t^2}dt} = e^{2\ln(1+t^2)} = e^{\ln(1+t^2)^2} = (1+t^2)^2$$

$$y(t) = \frac{1}{(1+t^2)^2}\left(\int_1^t (1+s^2)^{\cancel{2}\,1}\,\frac{s}{\cancel{1+s^2}}\,ds + 4\left(\frac{1}{4}\right)\right)$$

$$\int_1^t s + s^3\,ds$$

$$\left[\frac{s^2}{2} + \frac{s^4}{4}\right]_1^t$$

$$y(t) = \frac{1}{(1+t^2)^2}\left(\frac{t^2}{2} + \frac{t^4}{2} - \left(\frac{1}{2} + \frac{1}{4}\right) + 1\right)$$

$$y' + ty = g(t) \qquad y(0) = 0, \qquad g(t) = \begin{cases} 2 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

$$g : [0,\infty) \longrightarrow \{0,2\}$$

$$\mu(t) = e^t$$

$$y(t) = e^{-t}\left(\int_0^1 g(s)e^s ds + \int_1^t \underbrace{g(s)}_{=0} e^s ds + 0\right)$$

$$0$$

$$y(t) = e^{-t}\left(2\int_0^1 e^s ds\right) \qquad e-1$$

$$y(t) = e^{-t}(2(e-1))$$

$$y(t) = 2(e^{1-t} - e^{-t})$$

(18) $\frac{dy}{dt} + ay = b\, e^{-ct}$, $\quad \frac{a, c > 0}{b \in \mathbb{R}}$ (we want to show that the solution approaches zero at $t \to \infty$

$\lim_{t\to\infty} y(t) = 0$

$$\mu(t) = e^{\int a\,dt} = e^{at}$$

$$y(t) = e^{-at}\left(\int e^{at}\, b\, e^{-ct}\, dt + C\right)$$

$$y(t) = \frac{1}{e^{at}}\left(b \frac{1}{(a-c)}\int (a-c)\, e^{(a-c)t}\, dt + C\right)$$

$$y(t) = \frac{1}{e^{at}}\left(\frac{b}{a-c}\, e^{(a-c)t} + C\right)$$

$$\infty$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{1}{e^{at}}\right) \lim_{t\to\infty}\left(\frac{b}{a-c}\, e^{(a-c)t} + C\right)$$

$$= 0$$

(19) $\frac{dy}{dt} + a(t)y = f(t)$, $\quad f(t), a(t)$ continuous functions.

$a(t) \geq c > 0$ and $\lim_{t\to\infty} f(t) = 0$, show that $y(t) \to 0$ as $t \to \infty$.

$$\mu(t) = e^{\int a(t)\,dt}$$

$$\lim_{t\to t_0} \int g(t)\,dt = \int \lim_{t\to t_0} g(t)\,dt \qquad \text{cont.}$$

$$\lim_{t\to t_0} g(f(t)) = g\left(\lim_{t\to t_0} f(t)\right)$$

DCT

$$y(t) = e^{-\int a(t)\,dt}\left(\int \mu(t)\, f(t)\, dt + c\right)$$

$$\lim_{t\to\infty} y(t) = \left(\lim_{t\to\infty} e^{-\int a(t)dt}\right)\left(\lim_{t\to\infty} \int \mu(t) f(t) dt\right) + \left(\lim_{t\to\infty} e^{-\int a(t)dt}\right) c$$

$$= \left(e^{-\int \lim_{t\to\infty} a(t)\, dt}\right)\left(\int \left(\lim_{t\to\infty}\mu(t)\right)\underbrace{\left(\lim_{t\to\infty} f(t)\right)}_{=0} dt\right)$$

$$+ e^{-\int \lim_{t\to\infty} a(t)\, dt} (c)$$

$$c \le a(t)$$

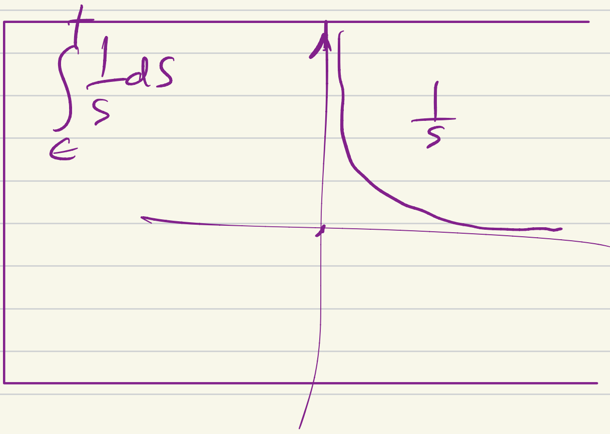
$$-c \ge -a(t)$$

$$-\int a(t)\, dt \le -c \int dt = -ct$$

$$e^{-\int a(t)dt} \le e^{-ct}$$

$$\lim_{t\to\infty} y(t) = 0 + c\, e^{\lim_{t\to\infty} -\int a(t) dt}$$

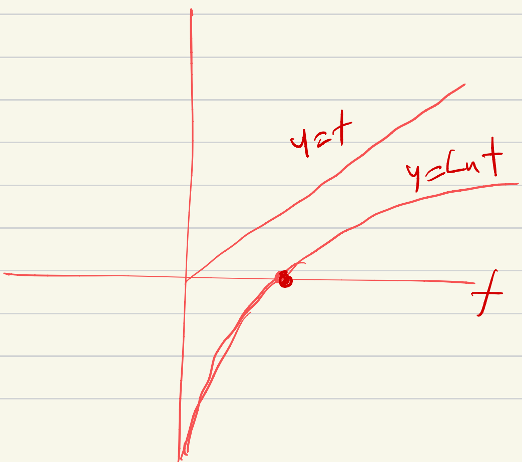
$$\le e^{\lim_{t\to\infty} (-ct)} = 0$$



$$\Rightarrow \lim_{t\to\infty} y(t) = 0.$$

(20) $\frac{dy}{dt} + \frac{1}{t} y = \frac{1}{t^2}$

$$\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$$



$$y(t) = \frac{1}{t}\left(\int t \left(\frac{1}{t^2}\right) dt + c\right)$$

$$y(t) = \frac{1}{t}(\ln t + c)$$

$$\lim_{t\to 0} y(t) = \frac{\ln t + C}{t} = -\infty.$$

(21) $\dfrac{dy}{dt} + \dfrac{1}{\sqrt{t}}\, y = e^{\sqrt{t}/2}$

$$\mu(t) = e^{\int t^{-\frac{1}{2}}dt} = e^{2t^{\frac{1}{2}}}$$

$\frac{1}{2}t^{2} + 2\sqrt{t} +$

$$y(t) = e^{-2t^{\frac{1}{2}}}\left(\int e^{2t^{\frac{1}{2}}}\, e^{\frac{t^{\frac{1}{2}}}{2}}\,dt + C\right)$$

$$\int e^{\frac{3}{2}\sqrt{t}}\,dt$$