$$\frac{dy}{dt} + \overset{p(t)}{a(t)}\, y = \overset{q(t)}{b(t)}$$

$\left(\text{if } b(t) = 0 \Rightarrow \frac{dy}{dt} + a(t)\,y = 0 \right.$

$\left. \frac{dy}{dt} = -a(t)\,y \quad (\text{separable}). \right)$

Assume $\exists\, \mu(t)$ such that

$$\mu(t)\frac{dy(t)}{dt} + \mu(t)\,a(t)\,y(t) = \mu(t)\,b(t)$$

we require

$$\mu(t) = e^{\int a(t)\,dt}$$

$$\frac{d\mu(t)}{dt} = \mu(t)\,a(t)$$

$$\frac{d\mu(t)}{\mu(t)} = a(t)\,dt$$

$$\ln \mu(t) = \int a(t)\,dt$$

$$\mu(t) = e^{\int a(t)\,dt}$$

$\#$

$$\frac{d}{dt}\left(\mu(t)\,y(t)\right) = \mu(t)\,b(t)$$

$$\mu(t)\,y(t) = \int \mu(t)\,b(t) + C$$

$$\boxed{y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)\,b(t)\,dt + C\right)}$$

Initial condition

Boundary conditions

$y(t_0) = y_0, \quad (t_0, y_0), \quad t_0 = ?, \; y_0 = ?$

$$y(t_0) = \frac{1}{\mu(t_0)}\left(\int_{t_0}^{t_0} \mu(s)\,b(s)\,ds + C\right) = y_0$$

$$\Rightarrow C = \mu(t_0)\,y_0 = \mu(t_0)\,y(t_0)$$

IVP $\equiv$ (DE + I.C.)

$$\frac{dy}{dt} + a(t)\,y = b(t), \qquad y(t_0) = y_0$$

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(t)\,b(t)\,dt + \mu(t_0)\,y_0\right)$$

① $\frac{dy}{dt} + y \underbrace{\cos t}_{a(t)} = \underbrace{0}_{b(t)}$

$$\mu(t) = e^{\int \cos(t)\,dt} = e^{\sin(t)}$$

$$y(t) = \frac{1}{e^{\sin(t)}}\left(\int \underbrace{(0)}_{b(t)} e^{\sin(t)}\,dt + \underline{c}\right)$$

$$\boxed{y(t) = c\,e^{-\sin(t)}}$$

② $\frac{dy}{dt} + y \underbrace{\sqrt{t}\sin(t)}_{a(t)} = 0$

$$\frac{1}{\mu(t)} = \frac{1}{e^{\int \sqrt{t}\sin(t)\,dt}} = e^{-\int \sqrt{t}\sin(t)\,dt}$$

$$\mu(t) = e^{\int \sqrt{t}\sin(t)\,dt}$$

$$y(t) = e^{-\int \sqrt{t}\sin(t)\,dt}\left(0 + c\right)$$

$$\int \frac{f'(t)}{f(t)}\,dt = \ln|f(t)| + C$$

$$y(t) = c\,e^{-\int \sqrt{t}\sin(t)\,dt}$$

③ $\frac{dy}{dt} + \underbrace{\frac{2t}{1+t^2}}_{a(t)}\,y = \underbrace{\frac{1}{1+t^2}}_{b(t)}$

$$\mu(t) = e^{\int \frac{2t}{1+t^2}\,dt} = \cancel{e}^{\cancel{\ln}(1+t^2)} = \underline{1+t^2}$$

$$y(t) = \frac{1}{1+t^2}\left(\int \underbrace{(1+t^2)}_{\mu(t)}\underbrace{\left(\frac{1}{1+t^2}\right)}_{b(t)}dt + c\right)$$

$$= \frac{1}{1+t^2}\left(\int dt + c\right)$$

$$y(t) = \frac{t+c}{1+t^2}$$

④ $\frac{dy}{dt} + y = t e^{t}$

$\mu(t) = e^{\int dt} = e^{t}$

$\int \underbrace{t}_{u} \underbrace{e^{2t} dt}_{dv}$ $u = t$, $du = dt$

$dv = e^{2t} dt = \frac{1}{2}(2e^{2t}) dt$

$v = \frac{1}{2} e^{2t}$

$\int u\,dv = uv - \int v\,du$

$\int t e^{2t} dt = \frac{t}{2} e^{2t} - \frac{1}{2} \int e^{2t} dt$

$= \frac{t}{2} e^{2t} - \frac{1}{4} e^{2t}$

$y(t) = \frac{1}{e^t} \left(\int \underbrace{e^{t}}_{\mu(t)} \underbrace{t e^{t}}_{b(t)} dt + C \right)$

$y(t) = \frac{1}{e^t} \left(\int t e^{2t} dt + C \right)$

$y(t) = \frac{1}{e^t} \left(\frac{t}{2} e^{2t} - \frac{1}{4} e^{2t} + C \right)$

$y(t) = \frac{1}{2} t e^{t} - \frac{1}{4} e^{t} + e^{-t} C$

$y(t) = \frac{1}{2}\left(t - \frac{1}{2}\right) e^{t} + e^{-t} C$

⑤ $\frac{dy}{dt} + t^2 y = 1$

$\mu(t) = e^{\int t^2 dt} = e^{\frac{t^3}{3}}$

$\frac{1}{\mu(t)}$

$y(t) = e^{-\frac{t^3}{3}} \left(\boxed{\int e^{\frac{t^3}{3}} (1)\, dt} + C \right)$

⑥ $\frac{dy}{dt} + t^2 y = t^2$

$\mu(t) = e^{\int t^2 dt} = e^{\frac{t^3}{3}}$

$\frac{d}{dt}\left(e^{\frac{t^3}{3}}\right) = t^2 e^{\frac{t^3}{3}}$

$y(t) = e^{-\frac{t^3}{3}} \left(\int e^{\frac{t^3}{3}} (t^2)\, dt + C \right)$

$y(t) = e^{-\frac{t^3}{3}} \left(e^{\frac{t^3}{3}} + C \right) \Rightarrow \boxed{y(t) = 1 + C e^{-\frac{t^3}{3}}}$

$$\left(\frac{dy}{dt} + a(t)\,y = b(t)\right)$$

⑦ $$\frac{dy}{dt} + \frac{t}{1+t^2}\,y = 1 - \frac{t^3}{1+t^4}\,y$$

$$\frac{dy}{dt} + \frac{t}{1+t^2}\,y + \frac{t^3}{1+t^4}\,y = 1$$

$$\frac{dy}{dt} + \underbrace{\left(\frac{t}{1+t^2} + \frac{t^3}{1+t^4}\right)}_{a(t)} y = \underbrace{1}_{b(t)}$$

$$\mu(t) = e^{\int\left(\frac{t}{1+t^2} + \frac{t^3}{1+t^4}\right)dt} = e^{\frac{1}{2}\int\frac{2t}{1+t^2}dt + \frac{1}{4}\int\frac{4t^3}{1+t^4}dt}$$

$$= e^{\frac{1}{2}\ln(1+t^2) + \frac{1}{4}\ln(1+t^4)}$$

$$= e^{\ln(1+t^2)^{\frac{1}{2}} + \ln(1+t^4)^{\frac{1}{4}}}$$

$$= \cancel{e}^{\cancel{\ln}(1+t^2)^{\frac{1}{2}}} \cdot \cancel{e}^{\cancel{\ln}(1+t^4)^{\frac{1}{4}}}$$

$$= (1+t^2)^{\frac{1}{2}}\,(1+t^4)^{\frac{1}{4}}$$

$$\boxed{y(t) = (1+t^2)^{-\frac{1}{2}}(1+t^4)^{-\frac{1}{4}}\left(\int (1+t^2)^{\frac{1}{2}}(1+t^4)^{\frac{1}{4}}\,dt + C\right)}$$

⑧ $$\frac{dy}{dt} + \underbrace{\sqrt{1+t^2}}_{a(t)}\,y = \underbrace{0}_{b(t)} \qquad y(\underset{t_0=0}{0}) = \underset{y_0=\sqrt{5}}{\sqrt{5}}$$

$$\mu(t) = e^{\int\sqrt{1+t^2}\,dt}$$

$$\int\sqrt{1+t^2}\,dt \qquad s^2 = 1+t^2$$

Let $t = \sinh(u)$

$$\sqrt{1+t^2} = \sqrt{1+\sinh^2(u)} = \sqrt{\cosh^2(u)} = \cosh(u)$$

$$\int \sec^m(u)\,du = \frac{\sin(u)\sec^{m-1}(u)}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2}(u)\,du, \quad m \geq 3$$

Reduction formula

you need it if you choose $t = \tan(u)$

$$dt = \cosh(u)\,du$$

$$\int \sqrt{1+t^2}\,dt = \int \cosh(u)\cosh(u)\,du = \int \cosh^2(u)\,du$$

$$= \int \frac{1+\cosh(2u)}{2}\,du$$

$\sinh^{-1}(t) = u$

$2\sinh(u)\cosh(u)$

$$\int \sqrt{1+t^2}\,dt = \frac{1}{2}u + \frac{1}{4}\sinh(2u)\,du,$$

$t = \sinh(u)$

$$= \frac{1}{2}\sinh^{-1}(t) + \frac{1}{2}t\sqrt{1+t^2}$$

$$\mu(t) = e^{\int \sqrt{1+t^2}\,dt} = e^{\frac{1}{2}\sin^{-1}(t) + \frac{1}{2}t\sqrt{1+t^2}}$$

$$\mu(t_0=0) = e^{\frac{1}{2}(0) + \frac{1}{2}(0)\sqrt{1+0}} = 1$$

if f is integrable

$$F(t) = \int_a^t f(s)\,ds$$

$$y(t) = \frac{1}{\mu(t)}\left(\underbrace{\int_{t_0}^{t} \mu(s)\underbrace{b(s)}_{=0}\,ds}_{=0} + \underbrace{\mu(t_0)}_{=1}\,y_0\right)$$

$$y(t) = e^{-\frac{1}{2}(\sinh^{-1}(t) + t\sqrt{1+t^2})}\left(0 + (1)\sqrt{5}\right)$$

$$\boxed{y(t) = \sqrt{5}\,e^{-\frac{1}{2}(\sin^{-1}(t) + t\sqrt{1+t^2})}}$$

if we find a value for the constant c then the solution is called (particular solution) otherwise, it is a general solution.