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Exam 3 Question 2 427J

2. Consider the system

$$\frac{d}{dt}\vec{x} = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 2 & -8 \\ 2 & -6 \end{bmatrix}$$

(a) Compute the general solution to the system.

(b) Compute the matrix exponential e^{At}.

(c) Draw the phase plane of the system $\frac{d}{dt}\vec{x} = A\,\vec{x}$ and say what type of phase plane it is.

a) $\frac{d}{dt}\vec{x} = \begin{bmatrix} 2 & -8 \\ 2 & -6 \end{bmatrix}\vec{x}$, $p(\lambda) = \det\left(\begin{bmatrix} 2-\lambda & 8 \\ 2 & -6-\lambda \end{bmatrix}\right) = (2-\lambda)(-6-\lambda) + 16$

$= \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = (\lambda+2)(\lambda+2)$

$\lambda = -2$, double root

$\lambda = -2$: $A + 2I = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{matrix} x - 2y = 0 \\ y = y \end{matrix} \rightarrow \begin{matrix} x = 2y \\ y = y \end{matrix} \right\} \Rightarrow y\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{v}_{-2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{x}^1 = e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Jordan Cycle: $(A - \lambda I)\vec{J} = \vec{v}_\lambda$

$\Rightarrow \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix}\vec{J} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 4 & -8 & 2 \\ 2 & -4 & 1 \end{array}\right]$

$\xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & -2 & 1/2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{ \begin{matrix} x - 2y = 1/2 \\ y = y \end{matrix} \rightarrow \begin{matrix} x = 2y + 1/2 \\ y = y \end{matrix} \right\}$

$\Rightarrow y\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \therefore \vec{J} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$\therefore \vec{x}^2 = e^{\lambda t}(\vec{J} + t\vec{v}_\lambda)$

$= e^{-2t}\left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$= e^{-2t}\begin{bmatrix} 1/2 + 2t \\ t \end{bmatrix}$

$$\boxed{\therefore C_1 e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} 1/2 + 2t \\ t \end{bmatrix}}$$

b) e^{At}, $\vec{x}^1 = e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{x}^2 = e^{-2t}\begin{bmatrix} 1/2 + 2t \\ t \end{bmatrix}$

$\mathcal{X}(t) = [\vec{x}^1 \; \vec{x}^2] = e^{-2t}\begin{bmatrix} 2 & 1/2 + 2t \\ 1 & t \end{bmatrix}$

$\mathcal{X}(0) = \begin{bmatrix} 2 & 1/2 \\ 1 & 0 \end{bmatrix}$, $\det(\mathcal{X}(0)) = -1/2$

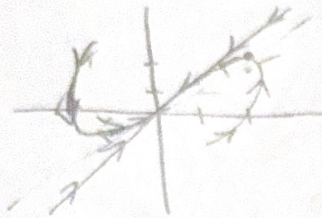
$(\mathcal{X}(0))^{-1} = -2 \cdot \begin{bmatrix} 2 & 1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -2 & 0 \end{bmatrix}$

$e^{At} = (\mathcal{X}(t) \cdot (\mathcal{X}(0))^{-1})$

$= e^{-2t}\begin{bmatrix} 2 & 1/2 + 2t \\ 1 & t \end{bmatrix} \cdot \begin{bmatrix} -4 & -1 \\ -2 & 0 \end{bmatrix}$

$$\boxed{= e^{-2t}\begin{bmatrix} -9 & -9t - 2 \\ -4 & -4t - 1 \end{bmatrix} = e^{At}}$$

c.) Let $\vec{T} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\lambda = -2$, double



Degenerate
Nodal
Sink