$+ \quad \times$

$- \quad \div$

M427J: Diffy Eqn

June 6, 2025

$\ln(y) \cdot \frac{1}{\sqrt{t}} + y'' = \tan^{-1}\left(\frac{t}{1-y}\right) + y^{(3)}$ (Non-Linear)

What is $y'(t)$?

$y' = y$ (1^{st} order) $y(t) = e^t, 0, c\cdot e^t$

$y^{(3)} = y$ (3^{rd} Order)

Non-Linear

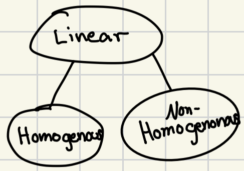
$y \cdot y' = 0$

Linear vs. Non-Linear DEs

$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = b$

$\frac{1}{t} y^{(n)} + e^t y^{(n-1)} + \ldots + t t^3 y' + 7y = \arcsin(\sqrt{t})$ (Linear DE)

$y' = y \to -y' - y = 0$



5^{th} Order, Linear, homogenous

$y^{(5)} - \frac{1}{t^3} y'' + e^{t^2} y = 0$

5^{th} Order, Linear, Non-homogeneous

$y^{(5)} - t^3 y' = t^2$

Brown 1.2
$\begin{cases} a(t)\, y' + b(t)\, y = 0 & 1^{st}, \text{ homogeneous} \\ a(t)\, y' + b(t)\, y = g(t) & 1^{st}, \text{ non-homogeneous} \end{cases}$

Systems of DEs

$f'(t) = g(t) + f(t)$

$g'(t) = 3f(t)$

More functions

More variables $\to u(x,t) \Rightarrow \frac{\partial u}{\partial x} \; \frac{\partial u}{\partial t}$

PDE's

June 9, 2025 - Brown 1.2 (Linear 1^{st} Order ODE)

Find $y(t) \longrightarrow a(t)y' + b(t)y = g(t)$

$a_n(t) y^{(n)} + \ldots$

First order, linear: $y' + a(t)y = b(t)$

homogen: $y' + a(t)y = 0$

Non-homogen: $y' + a(t)y = b(t) \not\equiv 0$

First order, homogen. ODEs:

$y' + a(t)y = 0$

$(\ln(f(x)))' = \frac{1}{f(x)} \cdot f'(x)$

Proof #1: $y' + a(t)y = 0 \to \frac{y'}{y} = \frac{-a(t)y}{y}$

$\to y'/y = -a(t)$

$\to (\ln(y))' = -a(t)$

$\to \frac{d}{dt}(\ln(y(t))) = -a(t)$

$\to \int \frac{d}{dt}(\ln(y(t))) = \int -a(t)\,dt$

$\to \ln(y(t)) = \int -a(t)\,dt + C$

$\cancel{e^{\ln(y(t))}} = e^{\int -a(t) + C}$

$y(t) = e^{\int -a(t)\,dt + C} \equiv e^{\int -a(t)\,dt} \cdot e^C$

$= Ce^{\int -a(t)\,dt}$

Example #1: $y' = y \to y' - y = 0$

$\equiv y' + \underbrace{a(t)}_{a(t)=-1} y = 0$

$y'/y = 1 \to (\ln y)' = 1$

$\to \ln y = t + C$

$\to y = e^t \to y = e^t \cdot e^C$

$y = Ce^t$ (gen. solution)

Example #2: $y' + 2t\,y = 0$

$y'/y = -2t$

$(\ln y)' = -2t$

$\ln y = -t^2 + C$

$y = e^{-t^2} \cdot e^C$

$y = Ae^{-t^2}$

I.V.P (Initial Value Problem):

Exp. 1: $y' + 2t\,y = 0, \quad y(0) = 3$

$y = Ce^{-t^2} \leftarrow$ Solve ODE

$3 = Ce^{-(0)^2}$ $\quad y = 3e^{-t^2}$ (unique solution)

$3 = C$

Exp 2: $y' + (\sin t)y = 0, \quad y(0) = 3/2$ (1^{st} Order, hom., IVP)

$y' = (-\sin t)y$

$\int (\ln y)' = \int -\sin t$

$\ln y = \cos t + C$

$y = e^{\cos t} \cdot e^C$

$y = e^{\cos t} \cdot A$

$3/2 = e^{\cos(0)} C$

$\frac{3}{2e} = C$

$y = \frac{3e^{\cos t}}{2e}$

$y = \frac{3e^{\cos t - 1}}{2}$

Non-homogenous, 1^{st} Order ODE: $y' + a(t)y = b(t)$

$(f(t)\cdot g(t))' = f(t)g'(t) + f'(t)g(t)$ $\qquad (\quad y)' = b(t)$

$(f \cdot g)' = f' \cdot g + g' \cdot f$

What if I could multiply by $\mu(t)$:

$y' + a(t)y = b(t)$

$\mu(t)\,y' + \mu(t)\,a(t)\,y = \mu(t)\,b(t)$

$(\quad y)' = \mu(t)\,b(t)$

$?y' + ?y = \mu(t)\,b(t)$

If $?' = a(t)\mu(t) = \mu'(t)$

$? = \mu(t)$

If $\mu'(t) = a(t)\mu(t)$

$\frac{\mu'(t)}{\mu(t)} = a(t)$

$(\log(\mu))' = a(t)$

$\ln \mu = \int a(t)$

$\mu = e^{\int a(t)}$

$\mu' = e^{\int a(t)} \cdot \left(\int a(t)\right)'$

$= \mu \cdot a(t)$ ∎

$y' + a(t)y = b(t) \not\equiv 0$

1^{st} $\mu(t) = e^{\int a(t)}$ (Integrating Factor)

$\mu y' + a(t) \cdot \mu \cdot y = \mu \cdot b(t)$

$(\mu \cdot y)' = \mu \cdot b(t)$

Example #3: $y' - 2t\,y = t$ $\qquad \mu = e^{\int -2t}$

$\mu = e^{-t^2}$

$y'e^{-t^2} - 2te^{-t^2}y = te^{-t^2}$

$(e^{-t^2}y)' = te^{-t^2}$

$e^{-t^2}y = \int te^{-t^2}\,dt + C$

let $u = -t^2$

$-\frac{1}{2}du = \cancel{-2t}\,\frac{}{-2}$

$e^{-t^2}y = -\frac{1}{2}\int e^u\,du$

$e^{-t^2}y = -\frac{1}{2}e^u + C$

$\frac{e^{-t^2}y}{e^{-t^2}} = \frac{-\frac{1}{2}e^{-t^2} + C}{e^{-t^2}}$

$y = -\frac{1}{2} + \frac{C}{e^{-t^2}}$

$= -\frac{1}{2} + Ce^{t^2}$

$y(t) \equiv -\frac{1}{2}$

Discussion Section

June 10, 2025

$$\frac{dy}{dt} + a(t)y = b(t) \quad \text{if } b(t) = 0 \text{ then:}$$

$$\frac{dy}{dt} + a(t)y = 0 \quad \text{(Seperable)}$$

$$\to \frac{dy}{dt} = -a(t)y$$

Assume $\exists\, \mu(t)$ such that

$$\left.\begin{aligned} \mu(t)\frac{dy}{dt} + \mu(t)a(t)y &= \mu(t)b(t) \\ \frac{d}{dt}\left(\mu(t)\,y\right) &= \mu(t)\,b(t) \end{aligned}\right\} \to \mu(t) = e^{\int a(t)\,dt}$$

$$\frac{d\mu(t)}{dt} = \mu(t)a(t)$$

$$\frac{d(\mu(t))}{\mu(t)} = a(t)\,dt$$

$$\ln \mu(t) = \int a(t)\,dt$$

$$\mu(t) = e^{\int a(t)\,dt} \;\blacksquare$$

$$\mu(t)y = \int \mu(t)\,b(t) + C$$

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)\,b(t) + C \;\blacksquare$$

Initial Condition vs. Boundary Condition

$$y(t_0) = y_0, \quad (t_0, y_0), \quad t_0 = ? \quad y_0 = ?$$

$$y(t_0) = \frac{1}{\mu(t_0)}\left(\int_{t_0}^{t_0} \mu(s)\,b(s)\,ds + C\right) = y_0$$

$$\to C = \mu(t_0)\,y_0 = \mu(t_0)\,y(t_0)$$

$$\text{I.V.P} \equiv (\text{DE} + \text{I.C})$$

$$\frac{dy}{dt} + a(t)y = b(t), \quad y(t_0) = y_0$$

$$y(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(t)\,b(t)\,dt + \mu(t_0)\,y_0\right)$$

Question #1: $\frac{dy}{dt} + y\cos(t) = 0$

$$y' + 2t\,y = t, \quad y(1) = 2$$

$$\mu = e^{\int 2t} = e^{t^2}$$

$$e^{t^2}y' + 2e^{t^2}y = te^{t^2}$$

$$\left(e^{t^2}y\right)' = te^{t^2}$$

$$e^{t^2}y = \int te^{t^2}\,dt + C = \frac{1}{2}e^{t^2} + C$$

$$y = \frac{1}{2} + \frac{C}{e^{t^2}} \equiv \frac{1}{2} + Ce^{-t^2}$$

$$2 = \frac{1}{2} + Ce^{-(1)^2}$$

$$2 = \frac{1}{2} + Ce^{-1}$$

$$3/2 = Ce^{-1}$$

$$\frac{3e}{2} = C$$

$$\boxed{y = \frac{1}{2} + \frac{3e^{-t^2+1}}{2}}$$

$$y' + \frac{2t}{1+t^2}\,y = \frac{1}{1+t^2}$$

$$\mu = e^{\int \frac{2t}{1+t^2}\,dt} \to \text{let } u = 1+t^2, \quad du = 2t$$

$$\mu = e^{\ln(1+t^2)}$$

$$\mu = 1+t^2$$

$$(1+t^2)y' + \cancel{(1+t^2)}\frac{2t}{\cancel{1+t^2}}\,y = \frac{1}{\cancel{1+t^2}}\cancel{(1+t^2)}$$

$$\left((1+t^2)\cdot y\right)' = 1$$

$$(1+t^2)y = t + C$$

$$y = \frac{t+C}{1+t^2}$$

Braun 1.4 Separable

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$$\int f(y)\,dy = \int g(t)\,dt$$

$$\frac{dy}{dt} = t + y \quad \text{NO}$$

$$\frac{dy}{dt} = t \cdot y = \text{Yes}$$

$$= \frac{t}{1/y}$$

ODE is seperable if

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$$\to \int f(y)\,dy = \int g(t)\,dt$$

Exp 1. $\frac{dy}{dt} = \frac{t^2}{y^2}$

$$\int y^2\,dy = \int t^2\,dt$$

$$3 \cdot \frac{y^3}{3} = \left(\frac{t^3}{3} + c\right)3$$

$$y^3 = t^3 + \cancel{3c}\,C$$

$$\sqrt[3]{y^3} = \sqrt[3]{t^3 + C}$$

$$y = \sqrt[3]{t^3 + C}$$

Example #2: $e^y\frac{dy}{dt} - t - t^3 = 0, \quad y(1) = 1$

$$e^y\frac{dy}{dt} = t + t^3$$

$$\int e^y\,dy = \int (t + t^3)\,dt$$

$$e^y = \frac{1}{2}t^2 + \frac{t^4}{4} + C$$

$$y = \ln\left(\frac{t^2}{2} + \frac{t^4}{4} + C\right)$$

$$1 = \ln\left(\frac{1^2}{2} + \frac{1^4}{4} + C\right)$$

$$1 = \ln\left(\frac{1}{2} + \frac{1}{4} + C\right)$$

$$1 = \ln\left(\frac{3}{4} + C\right)$$

$$e^1 = \frac{3}{4} + C$$

$$e^1 - \frac{3}{4} = C$$

$$\ln(a \cdot b) = \ln(a) + \ln(b)$$

$$y = \ln\left(\frac{t^2}{2} + \frac{t^4}{4} + \left(e - \frac{3}{4}\right)\right)$$

Question #2: $\frac{dy}{dt} + \sqrt{t}\,\sin(t)\,y = 0$

Question #3: $\frac{dy}{dt} + \frac{2t}{1+t^2}\,y = \frac{1}{1+t^2}$

Example #3: $y\frac{dy}{dt} + (1+y^2)\sin(t) = 0, \quad y(0) = 1$

$$y\frac{dy}{dt} = -(1+y^2)\sin(t)$$

$$\int \frac{y}{1+y^2}\,dy = \int -\sin(t)\,dt$$

$$\boxed{\frac{1}{2}\ln(1+y^2) = \cos(t) + C}$$

Gen. Sol.

Implicit Eqn

$$\frac{1}{2}\ln(2) = \cos(0) + C$$

$$\frac{1}{2}\ln(2) - 1 = C$$

$$\frac{1}{2}\ln(1+y^2) = \cos(t) + \ln(\sqrt{2}) - 1$$

Question #4: $\frac{dy}{dt} + y = te^t$

Question #8: $\frac{dy}{dt} + \sqrt{1+t^2}\, y = 0$, $y(0) = \sqrt{5}$

Question #5: $\frac{dy}{dt} + t^2 y = 1$

Question #6: $\frac{dy}{dt} + t^2 y = t^2$

Question #7: $\frac{dy}{dt} + \frac{t}{1+t^2} y = -\frac{t^3}{1+t^4} y$

Linear Algebra Concepts for Differential Equations

Vectors:

$\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_n v_n$$

Example #1:

$$\vec{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 3(2) + 0(2) + (-1)5 = 6 + 0 - 5 = 1$$

Complex Arithmetic:

$z = \alpha + i\beta, \quad \alpha, \beta \in \mathbb{R}$

$\mathrm{Re}(z) = \alpha, \quad \mathrm{Im}(z) = \beta$

Example (multiplication):

$$(1+i)\cdot(3-2i) = 3 - 2i + 3i - 2i^2 = 3 + i + 2 = 5 + i$$

Complex conjugate and inverse:

$\overline{3-2i} = 3 + 2i$ (conjugate)

$$\frac{1}{3-2i} \cdot \frac{(3+2i)}{(3+2i)} = \frac{3+2i}{9-4i^2} \text{ (Inversion)} = \frac{3+2i}{13} \equiv \frac{3}{13} + i\frac{2}{13}$$

Division of complex #'s:

$$\frac{1+i}{3-2i} \cdot \frac{(3+2i)}{(3+2i)} = \frac{3+2i+3i+2i^2}{9-4i^2} = \frac{3+5i-2}{9+4} = \frac{1+5i}{13} \equiv \frac{1}{13} + i\frac{5}{13}$$

Matrices:

$$M_{r \times c}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{1,1} & \cdots & a_{1,c} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,c} \end{bmatrix} \middle| \, a_{i,j} \in \mathbb{R} \right\}$$

Example:

$$\begin{bmatrix} -2 & 0 & 5 \\ 8 & 8 & 1 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$$

$$A \cdot B = \begin{array}{c} \\ i^{th} \end{array}\begin{bmatrix} \cdots & \cdots \\ a_1 \cdots & a_c \\ \cdots & \cdots \end{bmatrix} \begin{array}{c} j^{th} \\ \begin{bmatrix} \vdots & b_1 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & b_r & \vdots \end{bmatrix} \end{array} = \begin{array}{c} \\ i^{th} \end{array}\begin{array}{c} j^{th} \\ \begin{bmatrix} & 0 & \end{bmatrix} \end{array}$$

$A \in M_{r\times c}(\mathbb{R})$, $B \in M_{m \times n}(\mathbb{R})$

if $c = m$, then

$A \cdot B \in M_{r \times n}(\mathbb{R})$

Example:

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \in M_{3\times 2}(\mathbb{R})$$

$$B = \begin{bmatrix} 2 & 8 & 1 \\ -2 & 0 & 4 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$$

$3\cdot 2 + 5\cdot(-2) = -4$ | $0\cdot 2 + 1\cdot(-2) = -2$

$3\cdot 8 + 0\cdot 5 = 24$ | $0\cdot 8 + 1\cdot 0 = 0$

$3\cdot 1 + 5\cdot 4 = 23$ | $0\cdot 1 + 1\cdot 4 = 4$

$$A \cdot B = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 & 1 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 24 & 23 \\ -2 & 0 & 4 \\ -6 & -8 & 7 \end{bmatrix}$$

$-1\cdot 2 + 2(-2) = -6$

$-1(8) + 2(0) = -8$

$-1(1) + 2(4) = 7$

Computing the inverse of a square matrix

$A \in M_n(\mathbb{R})$

$$I_n = \overbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}}^{n} \Big\} n$$

The inverse of A is an $n \times n$ matrix B so that

$A \cdot B = B \cdot A = I_n \Rightarrow B = A^{-1}$

Let $A \in M_n(\mathbb{R})$

$[A | I_n] \to \ldots$

$\to [I_n | A^{-1}]$

Example: $A = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$

$$[A|I_n] = \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \xrightarrow{R_2 - 2R_1} \\ \to R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 1 & 4 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 3R_2]{-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & -2 & -3 & 0 \end{array}\right] \to R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & -2 & -3 & 0 \end{array}\right] \equiv [I_n | A^{-1}] \Rightarrow A^{-1}A = I_n \text{ for correctness}$$

$$A^{-1}A = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ -2 & -3 & 0 \end{bmatrix}\begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

When $n = 2$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \equiv \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example: $A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

Determinants:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when $n = 2$

$\det(A) = ad - bc$

$\det\left(\begin{bmatrix} -1 & 5 \\ 3 & 1 \end{bmatrix}\right) = -1 - 15 = -16$

Theory:

$$A \in M_3(R) = \begin{bmatrix} i & j & k \\ A & B & C \\ D & E & F \end{bmatrix}$$

Then $\det(A) = i(BF - EC) - j(AF - DC) + k(AE - BD)$

If $n = 2$, then $\det(A) = ad - bc$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A = \begin{array}{c} + \quad - \quad + \\ \begin{bmatrix} -1 & -2 & 3 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\det(A) = (-1)\det\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} - (-2)\det\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} + (3)\det\begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix}$$

$$= (-1)(2\cdot 1 - (-1)(-1)) - (-2)(0\cdot -1 - (-2)1) + 3((-2)(-1) - 2(0))$$

$$= -1 - 2 + 6 = 3$$

$A \in M_n(\mathbb{R})$

The following are equivalent:

1) $\det(A) \neq 0$

2) A is invertible (there is an $A^{-1} \in M_n(\mathbb{R})$) $\Rightarrow$ non-singular

3) $RREF(A) = I_n$

Solving Linear Systems: Non-homogenous system

Example #1:

$$\begin{aligned} -3x_1 - 2x_2 + 7x_3 &= 5 \\ 2x_1 + x_2 - 5x_3 &= -3 \\ -7x_1 - 2x_2 + 19x_3 &= 9 \end{aligned}$$

$$A = \left[\begin{array}{ccc|c} -3 & -2 & 7 & 5 \\ 2 & 1 & -5 & -3 \\ -7 & -2 & 19 & 9 \end{array}\right]$$

$$\mathrm{rref}(A) = \left[\begin{array}{ccc|c} -3 & -2 & 7 & 5 \\ 2 & 1 & -5 & -3 \\ -7 & -2 & 19 & 9 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 2 & 1 & -5 & -3 \\ -7 & -2 & 19 & 9 \end{array}\right] \xrightarrow[R_3 \to R_3 + 7R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right] \xrightarrow[R_3 + 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(columns: $x_1 \; x_2 \; x_3$)

x_1 and x_2 are dependent, based on the pivots. While x_3 is free

$x_1 - 3x_3 = -1 \longrightarrow x_1 = -1 + 3x_3$

$x_2 + x_3 = -1 \longrightarrow x_2 = -1 - x_3$

$x_3 = x_3 \longrightarrow x_3 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 3x_3 \\ -1 - x_3 \\ 0 + x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Homogenouns system:

$$\left.\begin{aligned} -2x_1 - 4x_2 + 3x_3 - 13x_4 - 5x_5 &= 0 \\ -x_1 - 2x_2 + x_3 - 5x_4 - x_5 &= 0 \\ 3x_3 - 9x_4 - 9x_5 &= 0 \end{aligned}\right\} = \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ -1 & -2 & 1 & -5 & -1 \\ 0 & 0 & 3 & -9 & -9 \end{bmatrix} = A$$

$$\mathrm{rref}(A) = \begin{bmatrix} -2 & -4 & 3 & -13 & -5 \\ -1 & -2 & 1 & -5 & -1 \\ 0 & 0 & 3 & -9 & -9 \end{bmatrix} \xrightarrow[-R_1,\ \frac{1}{3}R_3]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 & 5 & 1 \\ -2 & -4 & 3 & -13 & -5 \\ 0 & 0 & 1 & -3 & -3 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & -1 & 5 & 1 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & -3 & -3 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & 5 & 1 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \boxed{\begin{array}{c} \text{Free variables} \\ x_2 \quad x_4 \quad x_5 \\ \begin{bmatrix} 1 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}}$$

$\mathrm{rank}(A) = 2$

$x_2 = x_2$

$x_4 = x_4$

$x_5 = x_5$

$x_1 + 2x_2 + 2x_4 - 2x_5 = 0$

$x_3 - 3x_4 - 3x_5 = 0$

Homo. linear sys.

$$\left.\begin{aligned} x_1 &= -2x_2 - 2x_4 + 2x_5 \\ x_2 &= x_2 \\ x_3 &= 3x_4 + 3x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}\right\} \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}$$

Linear Algebra Concepts continued

Vector Spaces:

V_L, V_A

$L[y] = y'' + y$ (Lin. ODE)

If $L[y] = g(t)$ (Non-homo.)

$L[y] = 0$ (homo.)

$V_L = \{y(t) \in C(\mathbb{R}) \mid L[y] = \emptyset\}$

$= \{c_1 \cos t + c_2 \sin t \mid c_1, c_2 \in \mathbb{R}\}$

$V_{y''-y} = \{c_1 e^t + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R}\}$

$V_{y''} = \{c_1 + c_2 t \mid c_1, c_2 \in \mathbb{R}\}$

$A \in M_n(\mathbb{R})$,

$V_A = \{\vec{x}(t) \in C^\infty(\mathbb{R}, \mathbb{R}^n) \mid \frac{d}{dt}\vec{x} = A\vec{x}\}$

$$V_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \left\{ c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

$$V_{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \left\{ c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

Subspaces:

$A \in M_{r \times c}(\mathbb{R})$ $\quad r\{\overbrace{[\quad\;]}^{c} \cdot [\;]$

$Ker(A) \equiv Null(A)$

"kernel" "Nullspace"

$= \{\vec{v} \in \mathbb{R}^c \mid A \cdot \vec{v} = \vec{0}\}$

Example: Let $A = \begin{bmatrix} 4 & -12 & -3 & 18 \\ -1 & 3 & 1 & -5 \\ 1 & -3 & 2 & -1 \end{bmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -3 & 2 & -1 \\ -1 & 3 & 1 & -5 \\ 4 & -12 & -3 & 18 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 + R_1} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & -11 & 22 \end{bmatrix} \xrightarrow[-1/11 R_3]{1/3 R_2} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{array}{c} \begin{array}{cccc} & x_2 & & x_4 \text{ free} \end{array} \\ \begin{bmatrix} \underline{1} & -3 & 0 & 3 \\ 0 & 0 & \underline{1} & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{aligned} x_1 - 3x_2 + 3x_4 &= 0 \\ x_2 &= x_2 \\ x_3 - 2x_4 &= 0 \\ x_4 &= x_4 \end{aligned} \longmapsto \left.\begin{aligned} x_1 &= 3x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned}\right\} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$Ker(A) = \left\{ x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \,\middle|\, x_2, x_4 \in \mathbb{R} \right\}$$

Application: Population Modeling

$y(t) = $ # of individuals at time t.

A mathematical model is a differential equation.

$y' = $ rate of growth of a population

y' related to y ($y' \leftrightarrow y$)

A, B

$\boxed{y' = k \cdot y}$ Model

A is directly proportional to B, $A = k \cdot B$

A is inversely proportional to B, $A = \frac{k}{B}$

To solve:

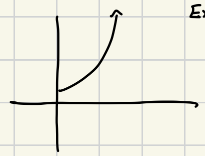
$y' - ky = 0 \longrightarrow y'/y = k$

$\int (\ln y)' = \int k$

$\ln y = kt + C$

$y = Ce^{kt} \Rightarrow y(0) = \beta$

$\rightsquigarrow P(t) = P_0 e^{kt}$



Exponential Pop. Growth

$y' = k \cdot y \rightarrow y = Ce^{kt}$

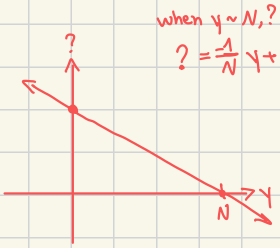
Carrying capacity $= N$ (the max capacity of the environment).

$\begin{cases} y' = k \cdot y, & y \sim 0 \\ y' = 0, & y \sim N \end{cases}$ $\quad y' = ky(1 - y/N)$

$y' = k \cdot y(?)$ when $y \sim 0$, $? = 1$

when $y \sim N$, $? = 0$

$? = \frac{-1}{N} y + 1 = 1 - y/N$



$k = $ growth rate

$N = $ Carrying capacity

Solve:

$y' = y(1-y) = y - y^2$

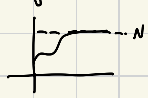
$\frac{dy}{dt} = ky(1 - y/N)$

$\int \frac{1}{y(1-y/N)} dy = \int k \, dt$

$= kt + C$

$y = \frac{N}{\# e^{-kt} + \#}$

Logistic Growth

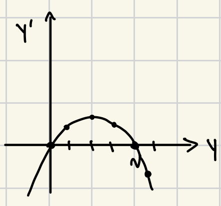


$y' = \frac{dy}{dt} = ky(1 - y/N)$

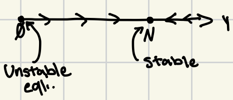
$y' = y(1-y)$

$= y - y^2$

$y' = \underbrace{ky(1 - y/N)}$



Phase Line



Graph y' as a function of y, value of y where $y' = 0$.

$y(t) \equiv 0$, $y(t) \equiv N$

- Solutions to the ODEs
- Equilibrium points.

m is minimum population for reproduction, allee

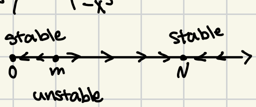
$y' = ky(1 - y/N)$ if $y > m$, then $y' > 0$

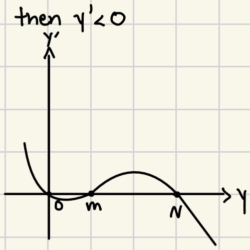
$y < m$, then $y' < 0$

$y' = ky(1 - y/N) \cdot (y - m)$

$= \frac{k}{N} y^3 + \ldots$

$+x^3$ $\quad$ $-x^3$





Cr pts

$0, m, N$

Braun 2.0 $\rightarrow$ 2nd order ODEs and IVPs (Linear)

$a(t) y'' + b(t) y' + c(t) y = g(t)$

or $y'' + p(t) y' + q(t) y = g(t)$

$ay'' + by' + cy = g(t)$

$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$

Example #1: $y'' - y = 0 \quad a = 1, b = 0, c = -1$

$(1) y'' + (0) y' + (-1) y = 0$

$y'' = y \longmapsto y = \{e^t, 0, e^{-t}\}$

$ce^t \quad de^t$

$\boxed{C_1 y_1 + C_2 y_2}$ Linear combinations of y_1, y_2

$y = ce^t, de^t, a \cdot \sinh(t), b\cosh(t)$

$y' = \cosh(t) \quad y' = \sinh(t)$

$y = \sinh(t) \quad y = \cosh(t)$

$ce^t + ce^{-t} + a\sinh(t) + b\cosh(t)$

$e^t + e^{-t} \longmapsto y = -7e^t + 2e^{-t}$

$y' = -7e^t - 2e^{-t}$

$y'' = -7e^t + 2e^t = y$

$\sinh(t) = \frac{e^t - e^{-t}}{2} = \frac{1}{2}(e^t - e^{-t})$

$\cosh(t) = \frac{e^t + e^{-t}}{2} = \frac{1}{2}(e^t + e^{-t})$

$y'' - y = 0 \qquad y_1 = e^t, \; y_2 = -e^t$

$y = C_1 y_1 + C_2 y_2 \longmapsto Ce^t + C_2(-e^t)$

Wronskian:

$W[y_1, y_2] = y_1 \cdot y_2' - y_1' \cdot y_2$

$= \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

If y_1, y_2 solve $ay'' + by' + cy = 0$ and $W[y_1, y_2] \not\equiv 0$

GOOD: $y = C_1 y_1 + C_2 y_2$

if $W[y_1, y_2] \equiv 0$, BAD, keep working

$W[\ln t, \sin t] = (\ln t)\cos t - \frac{1}{t}\sin t$

Example: $y'' - y = 0 \quad y_1 = e^t, \; y_2 = -e^t$

$W[e^t, -e^t] = (e^t)(-e^t) - (e^t)(-e^t)$

$= -e^{2t} + e^{2t} \equiv 0$ BAD

Try $y_1 = e^t, \; y_2 = e^{-t}$

$W[e^t, e^{-t}] = (e^t)(e^{-t})' - (e^{-t})(e^t)'$

$= -1 - 1 \equiv -2$ GOOD

$y = C_1 e^t + C_2 e^{-t}$ General sol. to $y'' - y = 0$

High level view:

$ay'' + by' + cy = 0$

$y = \ln(t)$

$y' = \frac{1}{t}$

$y'' = -\frac{1}{t^2}$

Exponential fcnts.

Sine/cosine fcnts.

Polynomial

Guess: $y = e^{rt} \quad y' = e^{rt} \cdot r, \quad y'' = r^2 e^{rt}$

$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

Characteristic polynomial $\rightarrow ar^2 + br + c = 0$

Quadratic formula

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

r_1, r_2

$\underset{y_1}{e^{r_1 t}}, \underset{y_2}{e^{r_2 t}}$

$y_1 = e^{r_1 t} \quad y_2 = e^{r_2 t}$

$W[y_1, y_2] \not\equiv 0$

$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Example: $y'' + y' - 6y = 0 \longrightarrow r^2 + r - 6 = 0$

$(r+3)(r-2) = 0 \longmapsto r = \{2, -3\}$

$y_1 = e^{2t}, \; y_2 = e^{-3t}$

$W[y_1, y_2] = (e^{2t})(e^{-3t})' - (e^{2t})'(e^{-3t})$

$= e^{2t}(-3e^{-3t}) - (2e^{2t})e^{-3t}$

$= -3e^{-t} - 2e^{-t} = -5e^{-t} \neq 0$

$y = C_1 e^{2t} + C_2 e^{-3t}$

Example: $y'' - 4y' + 13 = 0$

$r^2 - 4r + 13 = 0$

$r = \dfrac{4 \pm \sqrt{4^2 - 4 \cdot 13}}{2}$

$r = \dfrac{4 \pm 2\sqrt{4 - 13}}{2}$

$r = \dfrac{4 \pm 2\sqrt{-9}}{2}$

$= 2 \pm 3i$

$\alpha \pm \beta i$

Example: $y'' - y = 0 \longmapsto r^2 - 1 = 0$

$(r+1)(r-1) = 0 \quad r = -1, 1$

$y = C_1 e^{t} + C_2 e^{-t}$

$y(0) = 2, \; y'(0) = 1.$

$y(0) = C_1 e^{(0)} + C_2 e^{(0)} = C_1 + C_2 = 2$

$y' = C_1 e^{t} - C_2 e^{-t}$

$y'(0) = C_1 e^{(0)} - C_2 e^{0} = C_1 - C_2 = 1$

$C_1 = 3/2, \; C_2 = 1/2$

Example: $y'' + y' + y = 0 \quad r^2 + r + 1 = 0$

$r = \dfrac{-1 \pm \sqrt{1-4}}{2} \quad = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$y = C_1 e^{-1/2 t}\cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{-1/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$

Example: $y'' + 4y' + 4y = 0 \longrightarrow r^2 + 4r + 4 = 0$

$y_1 = e^{-2t}$

$(r+2)(r+2) = 0$

$f(t)v'' + g(t)v' = 0 \quad r = -2$ (double)

$\rightarrow y_2 = v(t)\, y_1(t)$

$y_2 = t e^{-2t}$

$y = C_1 e^{-2t} + C_2 \cdot t e^{-2t}$

$ay'' + by' + cy = 0$

$ar^2 + br + c = 0$

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Discriminant:

$\Delta = b^2 - 4ac$

Braun 2.2 $\begin{cases} \Delta > 0 & \text{2 real roots} \\ r_1, r_2 & y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \end{cases}$

Braun 2.2.1 $\Big\}\; \Delta < 0$ complex roots

Braun 2.2.2 $\Big\}\; \Delta = 0$ double roots

Review:

$ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$

$ar^2 + br + c = 0 \quad r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Discriminant: $\Delta = b^2 - 4ac$

(1) $\Delta > 0$, r_1, r_2 real

$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

(2) $\Delta < 0$, $\alpha \pm i\beta$ complex

$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$

(3) $\Delta = 0$, r double roots

$y = C_1 e^{rt} + C_2 t e^{rt}$

Example: $y'' + y = 0 \longmapsto r^2 + 1 = 0 \longmapsto r^2 = -1$

$z(t) = e^{it} = \underbrace{\cos t}_{y_1} + i\underbrace{\sin(t)}_{y_2} \qquad r = \pm\sqrt{1} = \pm i$

$y = \cos t$

$y' = -\sin t$

$y'' = -\cos t$

$y'' + y = -\cos t + \cos t = 0$

$y'' + y = -\sin t + \sin t = 0$

$y = \sin t$

$y' = \cos t$

$y'' = -\sin t$

$W[\cos t, \sin t] = (\cos t)(\sin t)' - (\cos t)'(\sin t)$

$= \cos t \cos t - \sin t \sin t$

$\dfrac{1+i}{1-2i} \cdot \dfrac{(1+2i)}{(1+2i)} = \dfrac{1+i+2i+(i)(2i)}{1-2i+2i+(-2i)(2i)} = \dfrac{1+3i+2i^2}{1-4i^2} = \dfrac{1+3i-2}{1+4} = \dfrac{-1+3i}{5}$

$\mathbb{C} = \{\alpha + i\beta \mid \alpha, \beta \in \mathbb{R}\} \qquad = \left(-\frac{1}{5}\right) + i\left(\frac{3}{5}\right)$

$z(t) = e^{it}$

$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{x!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

$e^{it} = 1 + it + \dfrac{(it)^2}{2!} + \dfrac{(it)^3}{3!} + \dfrac{(it)^4}{4!} + \ldots$

$= 1 + it + i^2 \cdot \dfrac{t^2}{2!} + i^2 \dfrac{t^3}{3!} + i^4 \dfrac{t^4}{4!}$

$= 1 + it - \dfrac{t^2}{2!} - \dfrac{it^3}{3!} + \dfrac{t^4}{4!} + i\dfrac{t^5}{5!} + \ldots$

$= \left(1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \dfrac{t^6}{6!} + \ldots\right) + i\left(t - \dfrac{t^3}{3!} + \dfrac{t^5}{5!} - \dfrac{t^7}{7!} + \ldots\right)$

$= \cos t + i\sin(t)$ Euler's Formula.

$e^{i\theta} = \cos\theta + i\sin\theta \quad \theta$ real

$y'' - y = t$

$y = e^t \quad y' = e^t \quad y'' = e^t$

$y'' - y = e^t - e^t = 0$

$y = -t + C_1 e^t$

Particular sol. ψ or y_p

$y = -t + C_1 e^t + C_2 e^{-t}$

solutions $y'' - y = 0$

Comp./Hom. sol. y_c

Example: $y'' + 2y' + 2y = 0 \longrightarrow r^2 + 2r + 2 = 0$

$r = \dfrac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \dfrac{-2 \pm \sqrt{-4}}{2}$

$= \dfrac{-2 \pm 2i}{2} = -1 \pm i \longmapsto \alpha \pm i\beta$

Pick $r = -1 + i$

$z(t) = e^{(-1+i)t} = \underbrace{e^{-t}}_{y_2} \cdot \underbrace{e^{it}}_{y_1}$

$z(t) = \dfrac{\cos t + i\sin(t)}{e^t}$

$y = \dfrac{C_1 \cos t + C_2 \sin(t)}{e^t} = e^{-t} C_1 \cos t + e^{-t} C_2 \sin(t)$

$r = \alpha \pm i\beta$

$\rightarrow y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$ } Short-cut!

$ay'' + by' + cy = g(t)$

2.3 How to write solutions

2.5 Solving

$y'' - y = t \qquad y =$ trig x

1 solution

$y = -t + e^t$

X $y = t^2 \quad y' = 2t \quad y'' = 2$

$y'' - y = 0 = 2 - t^2 \neq t$

$y = -t, \; y' = -1, \; y'' = 0$ ✓

$y'' - y = 0 - (-t) = t = t$

$y' = -1 + e^t$

$y'' = 0 + e^t$

X $y = t, \; y' = 1, \; y'' = 0$

$y'' - y = 0 - t = -t \neq t$

$y = -t + e^t$ ✓

$y'' - y =$

To solve $ay''+by'+cy=g(t)$

1. Find y_c, solve $ay''+by'+cy=0$
2. Find y_p, 1 particular solution
3. $y=y_p+y_c=y_p+C_1y_1+C_2y_2$

$ay''+by'+cy=g(t) \leftarrow p(t)$ polynomial

ψ = polynomial of some degree

$\{\, p(t)e^{\alpha t}$

$y''-6y'+9y=e^{-t}$

$\psi=Ae^{-t}$

$\psi'=-Ae^{-t}$

$\psi''=Ae^{-t}$

$\psi''-6\psi+9\psi=Ae^{-t}-6(-Ae^{-t})+9(Ae^{-t})$

$=Ae^{-t}+6Ae^{-t}+9Ae^{-t}=16Ae^{-t}=e^{-t}$

$16A=1 \to A=1/16$: $\psi=1/16\, e^{-t}$

$y=\psi+y_c=\frac{1}{16}e^{-t}+c_1e^{3t}+c_2te^{3t}$

Example: $y''+y=t$

1. $y''+y=0 \to r^2+1=0,\ r=\pm i \nearrow \alpha\pm i\beta$ $\alpha=0, \beta=1$

$y_c=C_1e^{0t}\cos(1\cdot t)+C_2e^{0t}\sin(1\cdot t)$

$=C_1\cos t+C_2\sin t$

2. $\psi=t,\ \psi'=1,\ \psi''=0$

$\psi''+\psi=0+t=t \longrightarrow y_p=\psi=t$

3. $y=\psi+y_c=t+c_1\cos t+c_2\sin(t)$

Example: $y''+y=t^2$

1. $y''+y=0 \to y_c=c_1\cos t+c_2\sin t$
2. $\psi=t^2 \quad \psi'=2t \quad \psi''=2$

$\psi=t^2-2 \quad \psi'=2t,\ \psi''=2$

$\psi''+\psi=2t+t^2-2=t^2$ ✓

3. $y=\psi+y$

Example: $y''+y'-2y=t^2 \qquad r^2+r-2=0$

1. $y''+y'-2y=0 \qquad r=1,-2$

$(r+2)(r-1)=0$

$y_c=C_1e^{t}+c_2e^{-2t}$

2. $\psi=t^2,\ \psi'=2t,\ \psi''=2$

$\psi''+\psi'-2\psi=2+2t-2t^2\neq t^2$

$\psi=t^2-2 \quad \psi'=2t \quad \psi''=2$

$\psi''+\psi'+2\psi=2+2t-2(t^2-2)$

$=2+2t-2t^2+4$

$=-2t^2+2t+6$

2. $\psi=At^2+Bt+C$

$\psi'=2A+B$

$\psi''=2A \qquad 2A+2A+B$

$\psi''+\psi'-2\psi=-2(At^2+Bt+C)$

$=-2At^2+(2A-2B)t+(2A+B-2C)=t^2+0\cdot t+0$

$-2A=1 \qquad 2A-2B=0 \qquad -3/2-2C=0$

$A=-1/2 \qquad A=B \qquad -2C=3/2$

$B=-1/2$

$C=-3/4$

Example: $y''+2y'+2y=(-t+2)e^{3t} \rightsquigarrow p(t)e^{\alpha t}$

1. $r^2+2r=0 \to r=-1\pm i$

$y_c=c_1e^{-t}\cos t+c_2e^{-t}\sin(t)$

2. $\psi=(At+B)e^{3t},\ \psi'=(A)e^{3t}+3(At+B)e^{3t},\ \psi''=[9(A)t+(6A+9B)]e^{3t}$

$\equiv(A+3At+B)e^{3t}$

$\equiv((3A)t+(A+3B))e^{3t}$

$\to \psi''+2\psi'+2\psi=[(9A)t+(6A+9B)]e^{3t}$

$+[(6A)t+(2A+6B)]e^{3t}$

$+[(2A)t+(2B)]e^{3t}$

$=[(17A)t+(8A+17B)]e^{3t}=(-t+2)e^{3t}$

... (Lin Alg)

$A=-1/17,\ B=\frac{42}{289}$

3. $y=\psi+y_c$

$y=(-1/17\,t+\frac{42}{289})e^{3t}+c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$

Example: $y''-6y'+9y=3t^2-4t+1$

$y''-6y'+9y=0 \longrightarrow r^2-6r+9=0$

$(r-3)(r-3)=0 \quad r=3$, double

$\to y_c=c_1e^{3t}+c_2te^{3t}$

2. $\psi=At^2+Bt+C \quad \psi'=2At+B \quad \psi''=2A$

$2A$

$\psi''-6\psi'+9\psi=-6(2At+B) \qquad =(3)t^2+(-4)t+1$

$+9(At^2+Bt+C) \qquad =2(1/3)-6(0)+(1/27)=1$

$\psi=1/3\,t^2+\frac{1}{27}$

③ $y=\psi+y_c=\frac{1}{3}t^2+\frac{1}{27}+c_1e^{3t}+c_2te^{3t}$

$y'' - 2y' + 2y = (2t-1)e^{-t}$ ← R.H.S

1. $y''-2y'+2y=0 \to r^2-2r+2=0$

$r = \frac{2\pm\sqrt{4-2\cdot 4}}{2} = 1\pm i \to \alpha \pm i\beta$

$y_1 = e^{\alpha t}\cos\beta t$

$y_2 = e^{\alpha t}\sin\beta t$

$y_c = C_1e^t\cos t + C_2 e^t \sin t$

2. $\varphi = (At+B)e^{-t}$

$\varphi'' - 2\varphi' + 2\varphi$

$\varphi' = Ae^{-t} + (At+B)(-e^{-t})$

$= [(-A)t + (A-B)]e^{-t}$

$\varphi'' = (-A)e^{-t} - [(-A)t + (A-B)]e^{t}$

$= [(A)t + (-2A+B)]e^{-t}$

$+ [(2A)t + (-2A+2B)]e^{-t}$ ← $-2\varphi'$

$+ [(2A)t + (2B)]e^{-t}$ ← 2φ

$[(5A)t + (-4A+5B)]e^{-t}$ ← L.H.S

$= [2(t)+(-1)]e^t \to \left[\begin{array}{cc|c}5&0&2\\-4&5&-1\end{array}\right]$ RREF $\to$ $A = 2/5$, $B = 3/25$

$\varphi = (2/5\, t + 3/25)e^{-t}$

3. $y = \varphi + y_c$

$y(t) = (2/5\, t + 3/25)e^{-t} + C_1e^t\cos t + C_2e^t\sin t$

Edge cases:

$y''-y=e^t$

① $y''-y=0 \to r^2-1=0$

$(r+1)(r-1)=0$

$r=1,-1$

$y_c = C_1e^t + C_2e^{-t}$ → Detect trouble

2. $\varphi = A\cdot e^t$, ~~$\varphi' = Ae^t$, $\varphi''=Ae^t$~~

~~$\varphi'' - \varphi = Ae^t - Ae^t = 0$~~ ← L.H.S

$= e^t$ ← R.H.S

$\varphi = Ate^t$ (Just works!)

$\varphi' = Ae^t + Ate^t$

$=(At+A)e^t$

$\varphi'' = Ae^t + (At+A)e^t$

$= [At + (2A)]e^t$

$\varphi''-\varphi = [\cancel{At} + (2A)]e^t$

$- [\cancel{At} + (0)]e^t$

Will always cancel

$= 2Ae^t = e^t$

$2A = 1 \to A = 1/2$

$\varphi = \frac{1}{2}te^t$

$y''+2y'+2y = te^t\cos t$

① $y''+2y'+2y=0 \to r^2+2r+2=0$

$r = \frac{-2\pm\sqrt{4-4\cdot 2}}{2} = -1\pm i$

$y_c = C_1e^{-t}\cos t + C_2e^{-t}\sin t$

② $\varphi = (At+B)e^t\cos t +$

$y''+2y'+2y = te^{-t}\cos t$

① $y_c = C_1e^{-t}\cos t + C_2e^{-t}\sin t$

② $\varphi = \{(At+B)e^{-t}\cos t + (Ct+D)e^{-t}\sin t + \}\cdot t$ Trouble!

$= (At^2+Bt)e^{-t}\cos t + (Ct^2+Dt)e^{-t}\sin t$

$y''-2y'+y=e^t$

① $y''-2y'+y=0 \to r^2-2r+1=0$

$(r-1)(r-1)=0$

$r=1$

$y_c = C_1e^t + C_2te^t$

② $\varphi = Ae^t$ ✗ Trouble

$\varphi = Ate^t$ ✗ Double trouble

$\varphi = At^2e^t$

Example: $y''-2y'+2y=\cos t$

① $y_c = C_1e^t\cos t + C_2e^t\sin t$

② $\varphi = A\cos t + B\sin t$

$\varphi' = -A\sin t + B\cos t$

$\varphi'' = -A\cos t - B\sin t$

$\varphi''-2\varphi'+2\varphi = -A\cos t - B\cos t$ ← φ''

$+ 2A\sin t - 2B\cos t$ ← $2\varphi'$

$+ 2A\cos t + 2B\sin t$ ← 2φ

$(A-2B)\cos t + (2A+B)\sin t$ ← LHS

$= (1)\cos t + (0)\sin t$

$\to \left[\begin{array}{cc|c}1&-2&1\\2&1&0\end{array}\right] \to$ RREF $\to$ $A = 1/5$, $B = -2/5$

3. $\varphi = 1/5\cos t - 2/5\sin t$

$y = \varphi + y_c$

$y = 1/5\cos t - 2/5\sin t + C_1e^t\cos t + C_2e^t\sin t.$

$y''+y=\cos t$

$y_c = C_1\cos t + C_2\sin t$

$\varphi = (A\cos t + B\sin t)\cdot t$

$= At\cos t + Bt\sin t$

Example: $y''-y'-2y = (\frac{1}{2}t^3 + \sqrt{2}t)e^{-7t}\sin(\pi t)$

① $y''-y'-2y=0$

$r^2-r-2=0$ $r=2,-1$

$(r-2)(r+1)=0$ $y_c = C_1e^{2t}+C_2e^{-t}$

② $\varphi = (At^3+Bt^2+Ct+D)e^{-7t}\cos(\pi t)$

$+ (Et^3+Ft^2+Gt+H)e^{-7t}\sin(\pi t)$

Review for Exam 1

June 24, 2025

$5A+2B=1$

$2A-B=0$ $\to \left[\begin{array}{cc|c}5&2&1\\2&-1&0\end{array}\right]$

GOAL $\to \left[\begin{array}{cc|c}1&0&A\\0&1&B\end{array}\right]$ or $\left[\begin{array}{cc|c}1&0&x\\0&1&y\end{array}\right]$

$1\cdot A + B\cdot 0 = x$

$0\cdot A + B\cdot 1 = y$

$y+z=4$

$3x+6y-3z=3$

$-2x-3y+3z=2$

$\left[\begin{array}{ccc|c}0&1&1&4\\3&6&-3&3\\-2&-3&3&2\end{array}\right]$

$\xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c}3&6&-3&3\\0&1&1&4\\-2&-3&3&2\end{array}\right]$

$\xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c}1&2&-1&1\\0&1&1&4\\-2&-3&3&2\end{array}\right]$

$\xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|c}1&2&-1&1\\0&1&1&4\\0&1&1&4\end{array}\right]$

$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c}1&0&-3&-7\\0&1&1&4\\0&1&1&4\end{array}\right]$

$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c}1&0&-3&-7\\0&1&1&4\\0&0&0&0\end{array}\right]$ (RREF)

z_f

$x-3z=-7$ | $x=-7-3z$

$y+z=4$ | $y=4-z$

$z=z$ | $z=z$

$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-7-3z\\4-z\\z\end{bmatrix} \to \underbrace{\begin{bmatrix}-7\\4\\0\end{bmatrix} + z\begin{bmatrix}-3\\-1\\1\end{bmatrix}}_{\text{Parametric form}}$

$x_1-3x_2=-3$

$x_2=x_2$

$x_3=2$

$x_1 = -3+3x_2$

$x_2 = x_2$

$x_3 = 2$

$\to \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-3+3x_2\\x_2\\2\end{bmatrix} \to \begin{bmatrix}-3\\0\\2\end{bmatrix} + x_2\begin{bmatrix}3\\1\\0\end{bmatrix}$

2nd Order non-homogenous:

$ay'' + by' + cy = g(t) \leftarrow p(t)$

$p(t)e^{t}$

$p(t)e^{\alpha t}(\sin\beta t \wedge \cos\beta t)$

① $ay'' + by' + cy = 0$

$ar^2 + br + c = 0 \rightarrow y_c = C_1 Y_1 + C_2 Y_2$

$\Delta > 0$

$\Delta < 0$

$\Delta = 0$

② Guess φ (pdf)

"Check for trouble"

Grind

③ $Y = \varphi + Y_c$

$y'' + 2y' + 2y = 3e^{-t}\cos t$

$r^2 + 2r + 2 = 0 \leftrightarrow y'' + 2y' + 2y = 0$

$r = \frac{-2 \pm \sqrt{4 - 2\cdot 4}}{2} = -1 \pm i$

$Y_c = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t)$

② $\varphi = (Ae^{-t}\cos(t) + Be^{-t}\sin(t))t$

$= Ate^{-t}\cos(t) + Bte^{-t}\sin(t)$

$(r-3)^2 = r^2 - 6r + 9$ $\qquad e^{3t}$ not allowed

$y'' - 6y' + 9y = (-t+2)e^{2t}$

① $y'' - 6y' + 9y = 0 \rightarrow r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0 \rightarrow r = 3$, double

$Y_c = C_1 e^{3t} + C_2 te^{3t}$

② $\varphi = (At + B)e^{2t}$

$\varphi' = Ae^{2t} + 2(At+B)e^{2t}$

$= [(2A)t + (A + 2B)]e^{2t}$

$\varphi'' = 2Ae^{2t} + 2[2At + (A + 2B)]e^{2t}$

$= [(4A)t + (4A + 4B)]e^{2t}$

$\varphi'' - 6\varphi' + 9\varphi = [(4A)t + (4A + 4B)]e^{2t}$

$+ [(-12A)t + (-6A - 12B)]e^{2t}$

$+ [(9A)t + (9B)]e^{2t}$

$[(A)t + (-2A + B)]e^{2t} =$

$[(-1)t + (2)]e^{2t}$ $\qquad \varphi = (-t)e^{2t}$

$A = -1$

$-2A + B = 2$

$y = \varphi + y_c = -te^{2t} + C_1 e^{3t} + C_2 te^{3t}$

Lecture: June 27, 2025

System of ODEs has more functions

$f(x), g(x)$ $\qquad f' = \frac{1}{g} + \sqrt{f}$ $\qquad f' = Af + B\cdot g$

1st Order, Linear, Homo., $\quad g' = x^3 - g\cdot f$ $\qquad g' = Cf + D\cdot g$

and constant coefficient

f, g, h

$f' = A\cdot f + Bg + C\cdot h$

$\vdots$

Variable change:

$x_1(t),\ x_2(t)$

$x_1'(t) = Ax_1 + Bx_2$

$x_2'(t) = Cx_1 + Dx_2$

$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Longrightarrow \frac{d}{dt}\vec{x}(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$ $\qquad \frac{d}{dt}\vec{x}(t) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$

$$\boxed{\frac{d}{dt}\vec{X} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\vec{X}}$$

A system looks like:

$x_1' = -\frac{1}{2}x_1 + \frac{1}{2}x_2$

$x_2' = \frac{1}{2}x_1 - \frac{1}{2}x_2$

or $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\vec{x} \Longrightarrow \frac{d}{dt}\vec{x} = A\vec{x}$ $\quad A \in M_2(\mathbb{R})$

$\vec{x}(0) = \vec{v} \in \mathbb{R}^2$

For an I.V.P, we need $x_1(0) = ?$ and $x_2(0) = ?$

a $\qquad b$

$y = -t + Ce^{t} + C_2 e^{-t}$

$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \overset{\text{I.V.P}}{\Longrightarrow} \frac{d}{dt}\vec{x} = A\cdot\vec{x}$ where $A \in M_n(\mathbb{R})$

$\vec{x}(0) = \vec{v} \in \mathbb{R}^n$

System

$\frac{d}{dt}\vec{x} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\vec{x}$

I.V.P: $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then the solution is $\vec{x}^1 = \begin{bmatrix} 1/2 + 1/2 e^{-t} \\ 1/2 - 1/2 e^{-t} \end{bmatrix}$

IVP: $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution is $\vec{x}^2 \begin{bmatrix} 1/2 - 1/2 e^{-t} \\ 1/2 + 1/2 e^{-t} \end{bmatrix}$

General solution for the example:

$\vec{x}(t) = c_1\vec{x}^1(t) + c_2\vec{x}^2 = c_1\begin{bmatrix} 1/2 + 1/2 e^{-t} \\ 1/2 - 1/2 e^{-t} \end{bmatrix} + c_2\begin{bmatrix} 1/2 - 1/2 e^{-t} \\ 1/2 + 1/2 e^{-t} \end{bmatrix}$

Example: $\frac{d}{dt}\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\vec{x}$ (Case 1)

$\frac{d}{dt}\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\vec{x} \rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2x_1 + 0x_2 \\ 0x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 3x_2 \end{bmatrix}$

$x_1' = -2x_1$ $\qquad \frac{x_1'}{x_1} = \frac{-2x_1}{x_1}$

$x_2' = 3x_2$ $\qquad (\ln x_1)' = -2$

$(\ln x_2)' = 3$ $\qquad \ln x_1 = -2t + C$

$\ln x_2 = 3t + C$ $\qquad x_1 = e^{-2t}\cdot C$

$x_2 = e^{3t}\cdot D$

Gen. solution: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c\,e^{-2t} \\ d\,e^{3t} \end{bmatrix}$

$= c\begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$

Example: $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \pi/2 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}\vec{x}$

$\vec{x} = c_1\begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{\pi/2 t} \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ e^{\sqrt{3}t} \end{bmatrix}$

Example: $\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ A & B \end{bmatrix}\vec{x}$

$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\vec{x} \longrightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_2 \\ 2x_1 - x_2 \end{bmatrix}$

$x_1' = x_2$

$x_2' = 2x_1 - x_2$

$x_1'' = (x_1')' = 2x_1 - x_1' \longrightarrow x_1'' = 2x_1 - x_1'$

$x_1'' + x_1' - 2x_1 = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r = 1, -2$

$x_1 = C_1 e^{t} + C_2 e^{-2t}$

$x_1 = e^{t}$ or e^{-2t}

$\vec{x} = C_1\begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} + C_2\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$

Example: $\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x} \rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

$x_1' = x_1$ $\qquad x_1'' + x_1 = 0$

$x_2' = -x_2$ $\qquad r^2 + 1 = 0 \rightarrow r = \pm i$

$x_1'' = -x_1$ $\qquad x_1 = \cos t,\ \sin t$

$\vec{x} = C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

Lecture (Linear Algebra)

July 2nd, 2025

$\mathbb{R}^3 = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R}\right\}$

$\mathbb{R}^n = \left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, \ldots, x_n \in \mathbb{R}\right\}$

$\mathbb{R}^n$ with vector addition and scalar multiplication

$\mathbb{R}^2$: $\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $\ 2 \in \mathbb{R}$, $\begin{bmatrix} 2 \\ -3 \end{bmatrix} \in \mathbb{R}^2$

$2\cdot\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

$M_2(\mathbb{R}) = \left\{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \middle| \alpha, \beta, \gamma, \delta \in \mathbb{R}\right\}$

$\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

$-3 \in \mathbb{R}, \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \in M_2(\mathbb{R}) \Longrightarrow -3\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix}$

$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad \dim(M_2(\mathbb{R}))? = 4$

$M_{2\times 3}(\mathbb{R}) = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \middle| a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}\right\}$

$\dim(A) = 6$

Abstract Vector Spaces

A set V together with a rule for adding two elements (or vectors) $x, y \in \mathbb{R}$ and a rule for scaling any element (or vector) $x \in V$ by a real number $\alpha \in \mathbb{R}$

$\mathbb{R}^n$ is a vector space

$P_3 = \{p(t) \mid \deg(p(t)) \leq 3\}$

$= \{a + bt + ct^2 + dt^3 \mid a, b, c, d \in \mathbb{R}\}$

$\dim(P_3) = 4$

$\dim(P_n) = n + 1$

$P = \{p(t) \mid p(t)$ polynomial$\}$

$\dim(P) = \infty \leftrightarrow P_2 \subseteq P,\ P_3 \subseteq P, \ldots$

$P_2 = \{a + bt + ct^2 \mid a, b, c \in \mathbb{R}$

$= \{p(t) \mid p(t)$ polynomial $\deg(p(t)) \leq 2\}$ $\qquad \dim(p(t)) = 3,\ \mathbb{R}^3$

$(-t^2 + 3t + 1) + (t^2 + t)$

$= 4t + 1$

$C(\mathbb{R}) = \{\underbrace{f:\mathbb{R}\to\mathbb{R}}_{\text{Real-valued functions}} \mid f \text{ is continous}\}$

Vector Space

$f(x)+g(x)$ is continous

$f(x),\ c\in\mathbb{R} \to c\cdot f(x)$ is continuous

$\dim(C(\mathbb{R}))=\infty \quad P \subseteq C(\mathbb{R})$

$C'(\mathbb{R}) = \{f:\mathbb{R}\to\mathbb{R} \mid f \text{ is differentiable}\}$

$(f+g)' = f'+g'$

$(c\cdot f)' = c\cdot f'$

$C^2(\mathbb{R}),\ C^3(\mathbb{R}), \dots, C^\infty(\mathbb{R})$

$e^x, \sin(x), \dots$

Solutions to linear differential equations live in $C^\infty(\mathbb{R})$

Solutions to homogenouous differential equations live in $C^\infty(\mathbb{R})$

AND are vector spaces.

$V = \{y(t) \mid y''-y=0\} \quad r^2-1=0$

$\mathbb{R}^2 \quad = \{C_1e^t + C_2e^{-t} \mid C_1, C_2 \in\mathbb{R}\} \subseteq C^\infty(\mathbb{R})$

Linear Algebra 5 → Vector Spaces

Linear Algebra 6 → Subspaces

Let $V = \left\{\begin{bmatrix}x\\y\\0\end{bmatrix} \middle| x,y\in\mathbb{R}\right\} \subseteq \mathbb{R}^3$

$\dim(V)=2$

Example: $A = \begin{bmatrix}1&0&-1\\-1&1&3\end{bmatrix} \in M_{2\times3}(\mathbb{R}) \quad A:\mathbb{R}^3\mapsto\mathbb{R}^2$

Think about $\vec{v}$, and $A\cdot\vec{v}$; $\vec{v}\in\mathbb{R}^3$

$A\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x-z\\-x+y+3z\end{bmatrix}$

$\text{Kernel}(A) = \{\vec{v}\in\mathbb{R}^3 \mid A\cdot\vec{v} = \vec{0}\in\mathbb{R}^2\}$

(Null(A))

What $\begin{bmatrix}x\\y\\z\end{bmatrix}\in\mathbb{R}^2$ solve

$$\left.\begin{aligned}x-z&=0\\-x+y+3z&=0\end{aligned}\right\} \to \left[\begin{array}{ccc|c}1&0&-1&0\\-1&1&3&0\end{array}\right] \xrightarrow{R_2\to R_2+R_1} \begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$$

z is free

$\text{Kernel}(A) = \left\{z\begin{bmatrix}1\\-2\\1\end{bmatrix} \middle| z\in\mathbb{R}\right\} \subseteq \mathbb{R}^3$

$$\left.\begin{aligned}x-z=0 &\to x=z\\y+2z=0 &\to y=-2z\\z=z &\to z=z\end{aligned}\right\} \to z\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

Example: $\begin{array}{c}\begin{bmatrix}1&-2&0&0&7\\0&0&1&0&8\\0&0&0&1&-3\end{bmatrix}\\ x_1 \quad\quad x_3\ x_4\end{array} = B$

$\text{Ker}(B) = \{\vec{v}\in\mathbb{R}^5 \mid B\cdot\vec{v} = \vec{0}\}$

$$\begin{aligned}x_1-2x_2+7x_5&=0\\x_3+8x_5&=0\\x_4-3x_5&=0\\x_2&=x_2\\x_5&=x_5\end{aligned} \qquad \begin{aligned}x_1&=2x_2-7x_5\\x_2&=x_2\\x_3&=-8x_5\\x_4&=3x_5\\x_5&=x_5\end{aligned}$$

$\left\{x_2\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + x_5\begin{bmatrix}-7\\0\\-8\\3\\1\end{bmatrix} \middle| x_2, x_5\in\mathbb{R}\right\} = \text{Kern}(B) \subseteq \mathbb{R}^5$

Lecture

July 7th, 2025

Online covers kernel and span while in class covers 2nd order non-homogenounous

and 2x2's $\left(\begin{bmatrix}0&1\\ *&*\end{bmatrix}\right)$

V a vector space, $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$

$\text{Span}(S) = \{\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k \mid \alpha_1, \dots, \alpha_k \in\mathbb{R}\}$

$\underset{\text{subset}}{S \subseteq} \text{Span}(S) \underset{\text{subspace}}{\subseteq} V$

Example:

$\text{Ker}\left(\begin{bmatrix}1&0&-2&3\\0&1&4&7\end{bmatrix}\right)$

$$\left.\begin{aligned}x_1-2x_3+3x_4&=0\\x_2-4x_3+7x_4&=0\\x_3&=x_3\\x_4&=x_4\end{aligned}\right\} \mapsto \left\{x_3\begin{bmatrix}2\\-4\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-3\\7\\0\\1\end{bmatrix} \middle| x_3, x_4\in\mathbb{R}\right\}$$

$= \text{Span}\left(\left\{\begin{bmatrix}2\\-4\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\7\\0\\1\end{bmatrix}\right\}\right)$

Example:

$V = \{y(t) \mid y''-y=0\}$

$= \{C_1e^t + C_2e^{-t} \mid C_1, C_2\in\mathbb{R}\}$

$= \text{Span}(\{e^t, e^{-t}\})$

$P_2 = \{a+bt+ct^2 \mid a,b,c\in\mathbb{R}\}$

$= \text{Span}(\{1, t, t^2\})$

Example: Is $\vec{b} = \begin{bmatrix}-17\\-7\\11\end{bmatrix} \in \text{Span}\left(\left\{\begin{bmatrix}5\\2\\-3\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}, \begin{bmatrix}9\\3\\-3\end{bmatrix}\right\}\right)$

Is $\vec{b}$ a linear combination of these 3 vectors?

Are there $a,b,c\in\mathbb{R}$ so that:

$\begin{bmatrix}-17\\-7\\11\end{bmatrix} = a\begin{bmatrix}5\\2\\-3\end{bmatrix} + b\begin{bmatrix}2\\1\\-2\end{bmatrix} + c\begin{bmatrix}9\\3\\-3\end{bmatrix}$?

$= \begin{bmatrix}5a\\2a\\-3a\end{bmatrix} + \begin{bmatrix}2b\\b\\-2b\end{bmatrix} + \begin{bmatrix}9c\\3c\\-3c\end{bmatrix} \equiv \begin{bmatrix}5a+2b+9c\\2a+b+3c\\-3a-2b-3c\end{bmatrix}$

$= \begin{bmatrix}5&2&9\\2&1&3\\-3&-2&-3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$

$\left[\begin{array}{ccc|c}5&2&9&-17\\2&1&3&-7\\-3&-2&-3&11\end{array}\right] \xrightarrow{\text{RREF}(A)} \left[\begin{array}{ccc|c}1&0&3&-3\\0&1&-3&-1\\0&0&0&0\end{array}\right]$ x_3 free

If free x_n variable, then

no solution.

$$\left.\begin{aligned}a+3c&=-3\\b-3c&=-1\\c&=c\end{aligned} \to \begin{aligned}a&=-3-3c\\b&=3c-1\\c&=c\end{aligned}\right\} \text{sol} = \begin{bmatrix}-3\\-1\\0\end{bmatrix} + c\begin{bmatrix}-3\\3\\1\end{bmatrix}$$

scalars

$\begin{bmatrix}-17\\-7\\11\end{bmatrix} \overset{\checkmark}{=} -3\begin{bmatrix}5\\2\\-3\end{bmatrix} - 1\begin{bmatrix}2\\1\\-2\end{bmatrix} + 0\begin{bmatrix}9\\3\\-3\end{bmatrix}$ → Get pts. on the exam.

Example: Use parametric form to show that

$\vec{b} = \begin{bmatrix}-1\\-1\\-8\\7\end{bmatrix} \in \text{Span}\left(\left\{\begin{bmatrix}-1\\-1\\-2\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-5\\-2\\5\\-7\end{bmatrix}\right\}\right)$

$\left[\begin{array}{cccc|c}-1&-1&2&-5&-1\\-1&-1&1&-2&-1\\-2&-2&-1&5&-8\\1&1&2&-7&7\end{array}\right] \longrightarrow \begin{array}{c}\quad x_2 \quad\ x_4\ \text{free}\\ \left[\begin{array}{cccc|c}1&1&0&-1&3\\0&0&1&-3&2\\0&0&0&0&0\\0&0&0&0&0\end{array}\right]\end{array}$

$$\left.\begin{aligned}a+b-d&=3\\b&=b\\c-3d&=2\\d&=d\end{aligned} \longrightarrow \begin{aligned}a&=3-b+d\\b&=b\\c&=3d+2\\d&=d\end{aligned}\right\} \text{sol} = \begin{bmatrix}3\\0\\2\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + d\begin{bmatrix}1\\0\\3\\1\end{bmatrix}$$

$3\begin{bmatrix}-1\\-1\\-2\\1\end{bmatrix} + 0\begin{bmatrix}-1\\-1\\-2\\1\end{bmatrix} + 2\begin{bmatrix}-2\\1\\-1\\2\end{bmatrix} + 0\begin{bmatrix}-5\\-2\\5\\7\end{bmatrix} = \vec{b} = \begin{bmatrix}-1\\-1\\-8\\7\end{bmatrix}$

Example: $\text{Kern}\left(\begin{bmatrix}1&1&1&-4\\2&3&4&-9\\5&11&17&-26\end{bmatrix}\right)$

$\xrightarrow{\text{RREF}(A)} \begin{bmatrix}1&0&-1&-3\\0&1&2&-1\\0&0&0&0\end{bmatrix} \mapsto \begin{aligned}x_1-x_3-3x_4&=0\\x_2+2x_3-x_4&=0\\x_3&=x_3\\x_4&=x_4\end{aligned} \to \begin{aligned}x_1&=x_3+3x_4\\x_2&=-2x_3+x_4\\x_3&=x_3\\x_4&=x_4\end{aligned}$

$\left\{x_3\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + x_4\begin{bmatrix}3\\1\\0\\1\end{bmatrix} \middle| x_3, x_4\in\mathbb{R}\right\}$

Example: $y''-2y'+y = 2e^t$ ①

$y''-2y'+y = 2t$ ②

1. Solve the homo. eqn.

$y''-2y'+y=0 \xmapsto{r} r^2-2r+1=0$

$(r-1)(r-1)=0 \quad r=1$ (double)

Check

$Y_c = C_1e^t + C_2te^t$

$\psi = Ae^t\cdot t\cdot t = At^2e^t$

$\psi' = 2Ate^t + At^2e^t$

$\psi'' =$

2. $y''-2y'+y = 2t$

$\xmapsto{r} (r-1)(r-1)=0$; $r=1$ double

$Y_c = C_1e^t + C_2te^t$

$\psi = At + B \qquad (At+B) - 2(A) + 0 =$

$\psi' = A \qquad -2A + At + B = 2t$

$\psi'' = 0 \qquad (A)t + (-2A+B) = 2t + 0$

$\psi \to 2t + 4 \qquad A=2$

$\boxed{y = 2t + 4 + C_1e^t + C_2te^t}$ $\quad -2A+B=0$

$B=4$

$y''-2y' = 2t$

$\xmapsto{r} r^2-2r=0 \equiv r(r-2) \equiv \begin{matrix}r=0\\r=2\end{matrix}$

$Y_c = C_1e^0 + C_2e^{2t}$

$= C_1 + C_2e^{2t}$

$\psi = (At+B)t = At^2+Bt$

$\psi' = 2tA + B$

$\psi'' = 2A$

Lecture

July 10th, 2025

$S=\{\vec v_1,\dots,\vec v_k\}\subseteq V$, vector space

$\text{Span}(S)=\{\alpha_1\vec v_1+\dots+\alpha_k\vec v_k \mid \alpha_1,\dots,\alpha_k\in\mathbb{R}\}$

(3d) $P_2=\{a+bt+ct^2 \mid a,b,c\in\mathbb{R}\}=\text{Span}(\{1,t,t^2\})$

(2d) $V=\{y \mid y''-y=0\}=\{c_1e^t+c_2e^{-t} \mid c_1,c_2\in\mathbb{R}\}$
$=\text{Span}(\{e^t,e^{-t}\})$

(3d) $\mathbb{R}^3=\text{Span}(\{\vec i,\vec j,\vec k\})$

$\vec i=e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}=\{x\vec i+y\vec j+z\vec k \mid x,y,z\in\mathbb{R}\}$

$\vec j=e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}=\left\{x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix} \,\middle|\, x,y,z\in\mathbb{R}\right\}$

$\vec k=e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \,\middle|\, x,y,z\in\mathbb{R}\right\}=\mathbb{R}^3$

$\mathbb{C}=\{\alpha+i\beta \mid \alpha,\beta\in\mathbb{R}\}=\text{Span}(\{1,i\})$

V a vector space

Conjecture: If $V=\text{Span}(\{\vec v_1,\dots,\vec v_k\})$, then V is dimension k, "V is like $\mathbb{R}^k$"

Conjecture for $k=1$: If $V=\text{span}(\{\vec v\})$, then the dimension of V is 1. $(\vec v_1\neq\vec 0)$

$\text{Span}(\{\vec v_1\})=\{\alpha_1\vec v_1 \mid \alpha_1\in\mathbb{R}\}$

$\text{Span}(\{\vec 0\})=\{\alpha_1\cdot\vec 0 \mid \alpha_1\in\mathbb{R}\}=\{\vec 0\}$

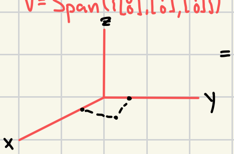
Conjecture for $k=2$: If $V=\text{Span}(\{\vec v_1,\vec v_2\})$, then $\dim(V)$ is 2. $(\vec v_1,\vec v_2\neq\vec 0)$

$V=\text{Span}(\{\vec v_1,\vec v_2\})=\{\alpha_1\vec v_1+\alpha_2\vec v_2 \mid \alpha_1,\alpha_2\in\mathbb{R}\}$

$\text{Span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}\right)=\left\{\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}2\\2\end{bmatrix} \,\middle|\, \alpha,\beta\in\mathbb{R}\right\}$
$=\left\{\alpha\begin{bmatrix}1\\1\end{bmatrix}+2\beta\begin{bmatrix}1\\1\end{bmatrix} \,\middle|\, \alpha,\beta\in\mathbb{R}\right\}$
$=\left\{(\alpha+2\beta)\begin{bmatrix}1\\1\end{bmatrix} \,\middle|\, \alpha,\beta\in\mathbb{R}\right\}$

Conjecture for $k=3$: If $V=\text{Span}(\{\vec v_1,\vec v_2,\vec v_3\})$, the $\dim(V)$ is 3.

$V=\text{Span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}\right)$



= x-y plane

If $V=\text{Span}(\{\vec v_1,\vec v_2,\vec v_3\})$, it's bad if:

$\vec v_1=a_2\vec v_2+a_3\vec v_3$

$\vec v_2=b_1\vec v_1+b_3\vec v_3$

$\vec v_3=c_1\vec v_1+c_2\vec v_2$

it's BAD if:

$-\vec v_1+a_2\vec v_2+a_3\vec v_3=\vec 0$

$-\vec v_2+b_1\vec v_1+b_3\vec v_3=\vec 0$

$-\vec v_3+c_1\vec v_1+c_2\vec v_2=\vec 0$

it's BAD if there are $\alpha,\beta,\gamma\in\mathbb{R}$, NOT ALL ZERO.

$\alpha\vec v_1+\beta\vec v_2+\gamma\vec v_3=\vec 0$

The set $S=\{\vec v_1,\dots,\vec v_k\}$ is BAD, i.e linearly dependent, if there are $\alpha_1,\dots,\alpha_k\in\mathbb{R}$

NOT ALL ZERO

such that $\alpha_1\vec v_1+\dots+\alpha_k\vec v_k=\vec 0$.

The set $S=\{\vec v_1,\dots,\vec v_k\}$ is good, ie Linearly independent, if $\alpha_1\vec v_1+\dots\alpha_k\vec v_k=\vec 0\implies\alpha_1=\alpha_2=\dots=\alpha_k=0$.

(The only linear combination of the vectors in S that equals $\vec 0$ is the trivial linear combination).

Example: $S=\left\{\begin{bmatrix}10\\-3\\8\end{bmatrix},\begin{bmatrix}-3\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-1\end{bmatrix}\right\}$

$a\begin{bmatrix}10\\-3\\8\end{bmatrix}+b\begin{bmatrix}-3\\1\\0\end{bmatrix}+c\begin{bmatrix}-1\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\begin{bmatrix}10&-3&-1\\-3&1&0\\8&0&-1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\begin{bmatrix}10&-3&-1\\-3&1&0\\8&0&-1\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&0&-1\\0&1&-3\\0&0&0\end{bmatrix}\equiv\begin{matrix}a-c=0\\b-3c=0\\c=c\end{matrix}$

$\implies c\begin{bmatrix}1\\3\\1\end{bmatrix}$

$\longrightarrow(1)\begin{bmatrix}10\\-3\\8\end{bmatrix}+(3)\begin{bmatrix}-3\\1\\0\end{bmatrix}+(1)\begin{bmatrix}-1\\0\\-1\end{bmatrix}=\vec 0$

Example: Use parametric form to show that

$S=\left\{\begin{bmatrix}-5\\2\\3\\0\end{bmatrix},\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},\begin{bmatrix}7\\-3\\-5\\1\end{bmatrix},\begin{bmatrix}-6\\3\\6\\-3\end{bmatrix}\right\}$ is lin. dependent

$\begin{bmatrix}-5&-3&7&-6\\2&1&-3&3\\3&1&-5&6\\0&1&1&-3\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&-2&3\\0&1&1&-3\\0&0&0&0\\0&0&0&0\end{bmatrix}\equiv\begin{matrix}a-2c+3d=0\\b+c-3d=0\\c=c\\d=d\end{matrix}\longrightarrow\begin{matrix}a=2c-3d\\b=-c+3d\\c=c\\d=d\end{matrix}$

Two answers, both are legal.

$c\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}+d\begin{bmatrix}-3\\3\\0\\1\end{bmatrix}\implies(2)\begin{bmatrix}-5\\2\\3\\0\end{bmatrix}+(-1)\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}+(1)\begin{bmatrix}7\\-3\\-5\\1\end{bmatrix}+(0)\begin{bmatrix}-6\\3\\6\\-3\end{bmatrix}=\vec 0$

$(-3)\begin{bmatrix}-5\\2\\3\\0\end{bmatrix}+(-1)\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}+(1)\begin{bmatrix}7\\-3\\-5\\1\end{bmatrix}+(0)\begin{bmatrix}-6\\3\\6\\3\end{bmatrix}=\vec 0$

Lecture

July 14th, 2025

Thrm: $A\in M_n(\mathbb{R})$, write $A=[\vec v_1\ \vec v_2\ \dots\ \vec v_n]\to\vec v_1,\dots,\vec v_n\in\mathbb{R}^n$

① $A\vec x=\vec 0$ has non-trivial (non-zero) solutions.	① $A\vec x=\vec 0$ has only the trivial solution $(\vec x=0)$.
② $\text{Ker}(A)>\{\vec 0\}\longleftrightarrow(\dim(\text{Ker}(A)>0))$	② $\text{Ker}(A)=\{\vec 0\}$
③ $\{\vec v_1,\dots,\vec v_n\}$ linearly dependent	③ $\{\vec v_1,\dots,\vec v_n\}$ linearly independent
④ $\text{RREF}(A)\neq I_n$	④ $\text{RREF}(A)=I_n$
⑤ $\text{rank}(A)<n$	⑤ $\text{rank}(A)=n$
⑥ A is not invertible $\therefore$ singular	⑥ A is invertible (A^{-1} is OK)
⑦ $\det(A)=0$, then A is singular	⑦ $\det(A)\neq 0$

Goal: Solve $\frac{d}{dx}\vec x=A\vec x$, $A\in M_n(\mathbb{R})$

$\vec x(t)=\begin{bmatrix}x_1(t)\\\dots\\x_n(t)\end{bmatrix}\begin{bmatrix}a_1e^{\lambda_1t}\\\dots\\a_ne^{\lambda_nt}\end{bmatrix}=e^{\lambda t}\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}\leftarrow$ My guess

Example: $\frac{d}{dt}\vec x=\begin{bmatrix}-3&0\\0&4\end{bmatrix}\vec x$ $\quad\vec x^1=\begin{bmatrix}e^{-3t}\\0\end{bmatrix}$, $\vec x^2=\begin{bmatrix}0\\e^{4t}\end{bmatrix}$

$\frac{d}{dt}\vec x=\begin{bmatrix}0&1\\-2&3\end{bmatrix}\vec x$ $\quad\vec x^1=\begin{bmatrix}e^t\\e^t\end{bmatrix}$ $\quad\vec x^2=\begin{bmatrix}e^{2t}\\2e^{2t}\end{bmatrix}$

$\frac{d}{dt}\vec{x} = A\vec{x}$ $A \in M_n(\mathbb{R})$ Guess: $\vec{x} = e^{\lambda t}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$\frac{d}{dt}(e^{\lambda t}\cdot\vec{v}) = A\cdot(e^{\lambda t}\cdot\vec{v})$ $= e^{\lambda t}\vec{v},\ \lambda \in \mathbb{R} \wedge \vec{v} \in \mathbb{R}^n$

$\lambda \cancel{e^{\lambda t}}\cdot\vec{v} = \cancel{e^{\lambda t}}\cdot A\cdot\vec{v}$

$\lambda\cdot\vec{v} = A\cdot\vec{v}$ $\frac{d}{dt}\left(e^{3t}\begin{bmatrix}-2\\5\end{bmatrix}\right) = \frac{d}{dt}\left(\begin{bmatrix}-2e^{3t}\\5e^{3t}\end{bmatrix}\right) = \begin{bmatrix}-2\cdot 3e^{3t}\\5\cdot 3e^{3t}\end{bmatrix} = 3\cdot e^{3t}\begin{bmatrix}-2\\5\end{bmatrix}$

$\lambda \in \mathbb{R}^n,\ \vec{v} \in \mathbb{R}^n$

$\vec{x} = e^{\lambda t}\vec{v}$

Defn: $A \in M_n(\mathbb{R})$, if $\lambda \in \mathbb{R}$, $\vec{v}_\lambda \in \mathbb{R}^n$ ($\vec{v}_\lambda \neq \vec{0}$) such that $A\cdot\vec{v}_\lambda = \lambda\cdot\vec{v}_\lambda$, then λ is an eigenvalue for A and $\vec{v}_\lambda$ is an eigenvector for A with eigenvalue λ

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = 0 \longrightarrow [A][\vec{v}_1] - \lambda[\vec{v}] = 0$

$\vec{v}(A-\lambda) = \vec{0} \rightarrow \cancel{[A-\lambda]}[\vec{v}] = \vec{0}$

$A\cdot\vec{v} - \lambda I_n\cdot\vec{v} = \vec{0} \longrightarrow [A][\vec{v}_1] - \lambda[I_n][\vec{v}] = \vec{0}$

$(A - \lambda I_n)\cdot\vec{v} = \vec{0}$

Pretend I know an eigenvalue, λ

Inside $\ker((A-\lambda_n I_n))$ is the eigenvectors.

$\begin{bmatrix}7 & -4\\8 & -5\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} = \begin{bmatrix}7-\lambda & -4\\8 & -5-\lambda\end{bmatrix}$

$\begin{array}{ccc} & 7 & -\lambda \\ -5 & -35 & -5\lambda \\ -\lambda & -7\lambda & \lambda^2\end{array}$

$\det\begin{pmatrix}7-\lambda & -4\\8 & -5-\lambda\end{pmatrix} = (7-\lambda)(-5-\lambda) - (8)(-4)$

$\lambda^2 - 2\lambda - 35 + 32$ $\quad \lambda - 3$

$\lambda = 3$ $\quad \lambda^2 - 2\lambda - 3$ $\quad \begin{array}{ccc} & \lambda & \lambda^2 & -3\lambda \\ +1 & \lambda & -3\end{array}$

$\begin{bmatrix}7 & -4\\8 & -5\end{bmatrix} - \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix} = \begin{bmatrix}4 & -4\\8 & -8\end{bmatrix}$ $\quad (\lambda-3)(\lambda+1)$

$\boxed{\lambda = 3, -1}$

$\text{rref}\begin{pmatrix}4 & -4\\8 & -8\end{pmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$ $\quad \begin{array}{l}x_1 - x_2 = 0\\x_2 = x_2\end{array} \rightarrow \begin{array}{l}x_1 = x_2\\x_2 = x_2\end{array}$

$\vec{x}' = e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad x_2\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ $\vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix}$, $\lambda = 3$

$\lambda = -3$ $\quad \begin{bmatrix}7 & -4\\8 & -5\end{bmatrix} - \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} =$

$= \begin{bmatrix}8 & -4\\8 & -4\end{bmatrix}$ $\quad \vec{v}_{(-1)} = \begin{bmatrix}1/2\\1\end{bmatrix}$, $\lambda = -1 \Big\}$ $e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$

$\text{rref}\begin{pmatrix}8 & -4\\8 & -4\end{pmatrix} = \begin{bmatrix}1 & -1/2\\0 & 0\end{bmatrix}$ $\quad \begin{array}{l}x - \frac{1}{2}x_2 = 0\\x_2 = x_2\end{array} \rightarrow \begin{array}{l}x_1 = \frac{1}{2}x_2\\x_2 = x_2\end{array}$

$y(t) = C_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$

Lecture

July 26, 2025

Solve $\frac{d}{dt}\vec{x} = \begin{bmatrix}7 & -4\\8 & -5\end{bmatrix}\vec{x}$

1. Characteristic polynomial: $p(\lambda) = \det(A - \lambda I$

$A - \lambda I = \begin{bmatrix}7 & -4\\8 & -5\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}7-\lambda & -4\\8 & -5-\lambda\end{bmatrix}$

$\det(A - \lambda I) = \det\left(\begin{bmatrix}7-\lambda & -4\\8 & -5-\lambda\end{bmatrix}\right) = (\lambda-7)(\lambda+5) + 32 = \lambda^2 - 2\lambda - 3 = \begin{cases}\lambda = 3\\\lambda = -1\end{cases}$

2. $\lambda = 3$: $\ker(A - 3I)$

$A - 3I = \begin{bmatrix}7 & -4\\8 & -5\end{bmatrix} - 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}4 & -4\\8 & -8\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$

$\left.\begin{array}{l}x - y = 0 \rightarrow x = y\\y = y \rightarrow y = y\end{array}\right\}$ $y\begin{bmatrix}1\\1\end{bmatrix} \rightarrow \vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix}$ $\vec{x}' = e^{3t}\begin{bmatrix}1\\1\end{bmatrix} \Big| e^{\lambda t}[\vec{v}_\lambda]$

$\lambda = -1$: $\ker(A - (-1)I) = \ker(A + I)$

$A + 1 = \begin{bmatrix}8 & -4\\8 & -4\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & -1/2\\0 & 0\end{bmatrix}$

$\left.\begin{array}{l}x - \frac{1}{2}y = 0 \rightarrow x = \frac{1}{2}y\\y = y \longrightarrow y = y\end{array}\right\} \rightarrow y\begin{bmatrix}1/2\\1\end{bmatrix} \longrightarrow \vec{v}_{-1} = \begin{bmatrix}1/2\\1\end{bmatrix}$, $\vec{x}^2 = e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$

3. $\vec{x} = C_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$

Solve: $\frac{d}{dt}\vec{x} = \begin{bmatrix}-2 & 5\\0 & 3\end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix}-2-\lambda & 5\\0 & 3-\lambda\end{bmatrix}\right) = (-2-\lambda)(3-\lambda) - (5)\cdot 0$

$= (-2-\lambda)(3-\lambda)$

$= \lambda = \{3, -2\}$

$\lambda = 3$: $\ker(A - 3I)$

$A - 3I = \begin{bmatrix}-5 & 5\\0 & 0\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \Big\}$ $\begin{array}{l}x - y = 0 \rightarrow x = y\\y = y \rightarrow y = y\end{array}$

$y\begin{bmatrix}1\\1\end{bmatrix} \rightarrow \vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{x}' = e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$

$\lambda = -2$: $\ker(A - (-2)I)$

$A + 2I = \begin{bmatrix}0 & 5\\0 & 5\end{bmatrix} \xrightarrow{\frac{1}{5}R_1} \begin{bmatrix}0 & 1\\0 & 5\end{bmatrix} \xrightarrow{R_2 - 5R_1} \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \Big\}$ $\begin{array}{l}x = x\\y = 0\end{array} \rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \vec{v}_{-2} = \begin{bmatrix}1\\0\end{bmatrix}$,

$e^{-2t}\begin{bmatrix}1\\0\end{bmatrix}$

$\vec{x} = C_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-2t}\begin{bmatrix}1\\0\end{bmatrix}$

Solve: $\frac{d}{dt}\vec{x} = \begin{bmatrix}10 & -4\\24 & -10\end{bmatrix}\vec{x}$

$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix}10-\lambda & -4\\24 & -10-\lambda\end{bmatrix}\right)$

$= (10-\lambda)(-10-\lambda) - (-4)(24)$

$= (\lambda-10)(\lambda+10) + 96 = \lambda^2 - 100 + 96$

$= \lambda^2 - 4$

$= (\lambda+2)(\lambda-2) \rightarrow \lambda = \begin{Bmatrix}-2\\2\end{Bmatrix}$

$\lambda = 2$: $A - 2I = \begin{bmatrix}8 & -4\\24 & -12\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & -1/2\\0 & 0\end{bmatrix}$

$\begin{array}{l}x - \frac{1}{2}y = 0\\y = y\end{array} \longrightarrow \left.\begin{array}{l}x = \frac{1}{2}y\\y = y\end{array}\right\} \rightarrow y\begin{bmatrix}1/2\\1\end{bmatrix} \longrightarrow \vec{v}_2 = \begin{bmatrix}1/2\\1\end{bmatrix}$, $\vec{x}' = e^{2t}\begin{bmatrix}1/2\\1\end{bmatrix}$

$\lambda = -2$: $A + 2I = \begin{bmatrix}12 & -4\\28 & -8\end{bmatrix} \rightarrow \begin{bmatrix}1 & -1/3\\0 & 0\end{bmatrix} \rightarrow \left\{\begin{array}{l}x - \frac{1}{3}y = 0\\y = y\end{array}\right. \rightarrow \left.\begin{array}{l}x = \frac{1}{3}y\\y = y\end{array}\right\} = y\begin{bmatrix}1/3\\1\end{bmatrix}$

$\vec{v}_{-2} = \begin{bmatrix}1/3\\1\end{bmatrix}$, $\vec{x}^2 = e^{-2t}\begin{bmatrix}1/3\\1\end{bmatrix}$

If $\ker(A - \lambda I_n) = \{\vec{0}\}$, there are no eigenvectors;
then λ is NOT an eigenvalue
$\Longleftrightarrow \det(A - \lambda I) \neq 0$

If $\ker(A - \lambda I_n)$ is "bigger" than $\{\vec{0}\}$,
then λ is an eigenvalue $\Longleftrightarrow \det(A - \lambda I) = 0$

Solve: $\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x}$

$e^{it} \cdot \vec{v}_i$

① $p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right)$

$= (-\lambda)(-\lambda) - (1)(-1)$

$= \lambda^2 + 1 = 0 \to \lambda^2 = -1 \implies \lambda = \pm\sqrt{-1} = \pm i$

② $\lambda = i$: $\ker(A - iI)$

$A - iI = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix} \xrightarrow{\frac{1}{-1}R_1} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \xrightarrow{R_2 + iR_1} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

$\left.\begin{matrix} x + iy = 0 \\ y = y \end{matrix}\right. \to \left.\begin{matrix} x = -i \cdot y \\ y = y \end{matrix}\right\} y\begin{bmatrix} -i \\ 1 \end{bmatrix} \to \vec{v}_z = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\implies \vec{z} = e^{\lambda t}\vec{v}_\lambda = e^{it}\vec{v}_i = e^{it} \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$= (\cos t + i \cdot \sin(t)) \cdot \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

$= \begin{bmatrix} 0 \\ \cos t \end{bmatrix} + i\begin{bmatrix} -\cos t \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ \sin(t) \end{bmatrix} + \overset{-1}{\cancel{i^2}}\begin{bmatrix} -\sin t \\ 0 \end{bmatrix}$

$= \underset{\vec{x}^1}{\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}} + i\underset{\vec{x}^2}{\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}}$

③ $\vec{x} = C_1\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + C_2\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$

Solve: $\frac{d}{dt}\vec{x} = \begin{bmatrix} 5 & -4 \\ 10 & -7 \end{bmatrix}\vec{x}$

$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 5-\lambda & -4 \\ 10 & -7-\lambda \end{bmatrix}\right)$

$= \lambda^2 + 2\lambda - 35 + 40 = \lambda^2 + 2\lambda + 5$

$\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$\lambda = -1 + 2i$: $\ker(A - (-1+2i)iI) = \ker(A + (1-2i)I)$

$\begin{bmatrix} 5 & -4 \\ 10 & -7 \end{bmatrix} + \begin{bmatrix} 1-2i & 0 \\ 0 & 1-2i \end{bmatrix} = \begin{bmatrix} 6-2i & -4 \\ 10 & -6-2i \end{bmatrix}$

$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 10 & -6-2i \\ 6-2i & -4 \end{bmatrix} \xrightarrow[R_2 + (\text{img})R_1]{\frac{1}{10}R_1} \begin{bmatrix} 1 & \frac{-6-2i}{10} \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -\frac{3}{5} - \frac{1}{5}i \\ 0 & 0 \end{bmatrix}$

assump $R_2 = (0,0)^T$

$\left.\begin{matrix} x + \left(-\frac{3}{5} - \frac{1}{5}i\right)y = 0 \\ y = y \end{matrix}\right\} \begin{matrix} x = \left(\frac{3}{5} + \frac{1}{5}i\right)y \\ y = y \end{matrix} \to y\begin{bmatrix} \frac{3}{5} + \frac{1}{5}i \\ 1 \end{bmatrix} \implies \vec{v}_{-1+2i} = \begin{bmatrix} \frac{3}{5} + \frac{1}{5}i \\ 1 \end{bmatrix}$

$\vec{z} = e^{\lambda t}\vec{v}_\lambda = e^{(-1+2i)t} \cdot \begin{bmatrix} \frac{3}{5} + \frac{1}{5}i \\ 1 \end{bmatrix}$

$\equiv e^{-t} \cdot e^{(2i)t} \cdot \begin{bmatrix} \frac{3}{5} + \frac{1}{5}i \\ 1 \end{bmatrix}$

$= e^{-t}(\cos(2t) + i\sin(2t))\left(\begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} + i\begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}\right)$

$= e^{-t}\left(\begin{bmatrix} \frac{3}{5}\cos(2t) \\ \cos(2t) \end{bmatrix} + i\begin{bmatrix} \frac{1}{5}\cos 2t \\ 0 \end{bmatrix} + i\begin{bmatrix} \frac{3}{5}\sin 2t \\ \sin 2t \end{bmatrix} + \begin{bmatrix} -\frac{1}{5}\sin 2t \\ 0 \end{bmatrix}\right)$

$= e^{-t}\begin{bmatrix} \frac{3}{5}\cos 2t - \frac{1}{5}\sin(2t) \\ \cos 2t \end{bmatrix} + i\,e^{-t}\begin{bmatrix} \frac{1}{5}\cos 2t + \frac{3}{5}\sin 2t \\ \sin 2t \end{bmatrix}$

$= C_1e^{-t}\begin{bmatrix} \frac{3}{5}\cos 2t - \frac{1}{5}\sin 2t \\ \cos 2t \end{bmatrix} + C_2e^{-t}\begin{bmatrix} \frac{1}{5}\cos 2t + \frac{3}{5}\sin 2t \\ \sin 2t \end{bmatrix}$

Lecture

July 18th, 2025

$\frac{d}{dt}\vec{x} = \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix}$

① $p(\lambda) = \det\begin{bmatrix} -3-\lambda & 13 \\ -2 & 7-\lambda \end{bmatrix} = (-3-\lambda)(7-\lambda) - (13)(-2)$

$= (\lambda+3)(\lambda-7) + 26 = \lambda^2 - 4\lambda + 5$

$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

Continued on top right!

$\lambda = 2+i$: $A - (2+i)I = A + (-2-i)I$

$= \begin{bmatrix} -3 & 13 \\ -2 & 7 \end{bmatrix} + \begin{bmatrix} -2-i & 0 \\ 0 & -2-i \end{bmatrix} = \begin{bmatrix} -5-i & 13 \\ -2 & 5-i \end{bmatrix}$

rref(result):

$\longrightarrow \begin{bmatrix} -2 & 5-i \\ -5-i & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{5}{2} + \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$

$\left.\begin{matrix} x + \left(-\frac{5}{2} + \frac{1}{2}i\right)y = 0 \\ y = y \end{matrix}\right\} \longrightarrow y\begin{bmatrix} \frac{5}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} \implies \vec{v}_{2+i} = \begin{bmatrix} \frac{5}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}$

$\vec{z} = e^{(2+i)t} \cdot \begin{bmatrix} \frac{5}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}$

$= e^{2t}\left(\begin{bmatrix} \frac{5}{2}\cos t \\ \cos t \end{bmatrix} + i\begin{bmatrix} -\frac{1}{2}\cos t \\ 0 \end{bmatrix} + i\begin{bmatrix} \frac{5}{2}\sin t \\ \sin t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\sin t \\ 0 \end{bmatrix}\right)$

$\vec{z} = e^{2t}\begin{bmatrix} \frac{5}{2}\cos t \\ \cos t \end{bmatrix} + i \cdot e^{2t}\begin{bmatrix} -\frac{1}{2}\cos t + \frac{5}{2}\sin t \\ \sin t \end{bmatrix}$

$\vec{x} = C_1e^{2t}\begin{bmatrix} \frac{5}{2}\cos t + \frac{1}{2}\sin t \\ \cos t \end{bmatrix} + C_2e^{2t}\begin{bmatrix} -\frac{1}{2}\cos t + \frac{5}{2}\sin t \\ \sin t \end{bmatrix}$

$\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix}$

$p(\lambda) = \det\begin{bmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{bmatrix} = (-1-\lambda)(5-\lambda) - 9(-1)$

$= (\lambda+1)(\lambda-5) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda-2)(\lambda-2)$

$\lambda = 2$, double root

$\lambda = 2$: $A - 2I = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$\left.\begin{matrix} x - 3y = 0 \\ y = y \end{matrix}\right\} \begin{matrix} x = 3y \\ y = y \end{matrix} \implies y\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \implies \vec{x}^1 = e^{2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\boxed{\begin{array}{l} \text{Jordan Cycle:} \\ 1.\ \text{Find } \vec{J} \text{ such that } (A - \lambda I)\vec{J} = \vec{v}_\lambda \\ 2.\ \vec{x}^2 = e^{\lambda t}(\vec{J} + t\vec{v}_\lambda) \end{array}}$$

Find $\vec{J}$ such that $\begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix}\vec{J} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} -3 & 9 & 3 \\ -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 3 & 1 \\ 3 & 9 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$\left\{\begin{matrix} x - 3y = -1 \\ y = y \end{matrix}\right. \longrightarrow \left.\begin{matrix} x = 3y - 1 \\ y = y \end{matrix}\right\} = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\vec{J}} + y\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{x}^2 = e^{2t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$

$= e^{2t}\begin{bmatrix} -1 + 3t \\ t \end{bmatrix}$

$\vec{x} = C_1e^{2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2e^{2t}\begin{bmatrix} -1 + 3t \\ t \end{bmatrix}$

$\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\vec{x}$

$p(\lambda) = \det(A - \lambda I) = (-1-\lambda)(-1-\lambda) - 1 \cdot 0$

$= (\lambda+1)(\lambda+1) = \lambda = -1$, double root

$\lambda = -1$: $A - (-1)I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\left\}\begin{matrix} x = x \\ y = 0 \end{matrix}\right. \implies x\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$

$\vec{x}^1 = e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Find $\vec{J}$:

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{J} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \left.\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]\right\} \to \begin{matrix} y = 1 \\ x = x \end{matrix} \implies \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{J}} + x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}^2 = e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} t \\ 1 \end{bmatrix}$

$\vec{x} = C_1e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2e^{-t}\begin{bmatrix} t \\ 1 \end{bmatrix}$

Section 3.11 - Matrix Exponential

$$e^{\begin{bmatrix}-3&13\\-2&7\end{bmatrix}t} \Rightarrow I + \begin{bmatrix}-3&13\\-2&7\end{bmatrix}t + \frac{\left(\begin{bmatrix}-3&13\\-2&7\end{bmatrix}t\right)^2}{2!} + \frac{t^3}{3!}\begin{bmatrix}-3&13\\-2&7\end{bmatrix}^3 + \frac{t^4}{4!} + \dots \qquad I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$= \begin{bmatrix}\sum_0^\infty a_n t^n & \sum_0^\infty b_n t^n\\ \sum_0^\infty c_n t^n & \sum_0^\infty d_n t^n\end{bmatrix} = \begin{bmatrix}f(t) & g(t)\\ h(t) & j(t)\end{bmatrix}$$

e^{At}:

Solve $\frac{d}{dt}\vec{x} = A\vec{x}$, $\vec{x}^1$, $\vec{x}^2$ ①

$\mathcal{X}(t) = [\vec{x}^1 \ \vec{x}^2]$ ②

$e^{At} = \mathcal{X}(t)\cdot(\mathcal{X}(0))^{-1}$

$e^{\begin{bmatrix}-3&17\\-2&7\end{bmatrix}t}$

① Solve $\frac{d}{dt}\vec{x} = \begin{bmatrix}-3&17\\-2&7\end{bmatrix}\vec{x}$

$$\vec{x} = c_1\underbrace{e^{2t}\begin{bmatrix}\frac{5}{2}\cos t + \frac12 \sin t\\ \cos t\end{bmatrix}}_{\vec{x}^1} + c_2\underbrace{e^{2t}\begin{bmatrix}-\frac12\cos t + \frac52 \sin t\\ \sin t\end{bmatrix}}_{\vec{x}^2}$$

② $\mathcal{X}(t) = e^{2t}\begin{bmatrix}5/2\cos t + 1/2 \sin t & -1/2 \cos t + 5/2 \sin t\\ \cos t & \sin t\end{bmatrix}$

③ $\mathcal{X}(t)\cdot(\mathcal{X}(0))^{-1} = e^{2t}\cdot\begin{bmatrix}5/2\cos t + 1/2\sin t & -1/2\cos t + 5/2 \sin t\\ \cos t & \sin t\end{bmatrix}\begin{bmatrix}0&1\\-2&5\end{bmatrix}$

$\mathcal{X}(0) = \begin{bmatrix}5/2 & -1/2\\ 1 & 0\end{bmatrix} \longrightarrow \det(\mathcal{X}(0)) = 1/2$

$\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)^{-1} = \frac{1}{\det A}\cdot\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$(\mathcal{X}(0))^{-1} = 2\begin{bmatrix}0 & 1/2\\ -1 & 5/2\end{bmatrix}$

$= \begin{bmatrix}0&1\\-2&5\end{bmatrix}$

$\therefore e^{2t}\begin{bmatrix}\cos t - 5\sin t & 13 \sin t\\ -2\sin t & \cos t + 5 \sin t\end{bmatrix}$

Double root:

$e^{\begin{bmatrix}-1&9\\-1&5\end{bmatrix}t}$

I. Solve $\frac{d}{dt}\vec{x} = \begin{bmatrix}-1&9\\-1&5\end{bmatrix}$

$p(\lambda) = \det\begin{bmatrix}-1-\lambda & 9\\ -1 & 5-\lambda\end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)(\lambda-2)$

$\lambda = 2$, double

$\lambda=2: A - 2I = \begin{bmatrix}-3&9\\-1&3\end{bmatrix} \longrightarrow \begin{bmatrix}1&-3\\0&0\end{bmatrix}$

$\left.\begin{matrix}x - 3y = 0\\ y = y\end{matrix}\right. \mapsto \left.\begin{matrix}x = 3y\\ y=y\end{matrix}\right\} \mapsto \vec{v}_2 = \begin{bmatrix}3\\1\end{bmatrix}$, $\vec{x}^1 = e^{2t}\begin{bmatrix}3\\1\end{bmatrix}$

Jordan Cycle:

Solve $(A - \lambda I)\vec{\jmath} = \vec{v}_\lambda$ $\quad \begin{bmatrix}-3&9\\-1&3\end{bmatrix}\vec{\jmath} = \begin{bmatrix}3\\1\end{bmatrix}$

$\left[\begin{matrix}-3&9\\-1&3\end{matrix}\middle|\begin{matrix}3\\1\end{matrix}\right] \longrightarrow \left[\begin{matrix}1&-3\\0&0\end{matrix}\middle|\begin{matrix}-1\\0\end{matrix}\right] \Longrightarrow \left\{\begin{matrix}x - 3y = -1\\ y=y\end{matrix}\right. \rightarrow \left.\begin{matrix}x = 3y-1\\ y=y\end{matrix}\right\}$

$\begin{bmatrix}-1\\0\end{bmatrix} + y\begin{bmatrix}3\\1\end{bmatrix} \Rightarrow \vec{\jmath} = \begin{bmatrix}-1\\0\end{bmatrix}$

$\vec{x}^2 = e^{\lambda t}(\vec{\jmath} + t\vec{v}_\lambda)$

$= e^{2t}\left(\begin{bmatrix}-1\\0\end{bmatrix} + t\begin{bmatrix}3\\1\end{bmatrix}\right) = e^{2t}\begin{bmatrix}-1+3t\\ t\end{bmatrix}$

$\mathcal{X}(t)\cdot(\mathcal{X}(0))^{-1} = \dots$

$\mathcal{X}(0) = \begin{bmatrix}3&-1\\1&0\end{bmatrix}$, $\det(\mathcal{X}(0)) = 1$

$\mathcal{X}(0)^{-1} = \begin{bmatrix}0&1\\-1&3\end{bmatrix}$

$e^{2t}\begin{bmatrix}3 & -1+3t\\ 1 & t\end{bmatrix}\begin{bmatrix}0&1\\-1&3\end{bmatrix} = e^{2t}\begin{bmatrix}1-3t & 9t\\ -t & 1+3t\end{bmatrix}$

$1\cdot 3 + 3\cdot(-1+3t) = 3 - 3 + 9t = 9t$

Chapter 4

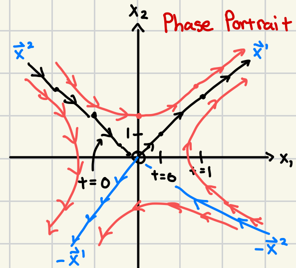
$\frac{d}{dt}\vec{x} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\vec{x}$ $\qquad \vec{x}^1 = e^t\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{x}^2 = e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$

$\vec{x} = c_1 e^t\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$

Graph $\vec{x}^1(t) = e^t\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}e^t\\e^t\end{bmatrix}$

t	$\vec{x}^1(t)$
-1	$\vec{x}^1(-1) = \begin{bmatrix}1/e\\1/e\end{bmatrix}$
0	$\vec{x}^1(0) = \begin{bmatrix}1\\1\end{bmatrix}$
1	$\vec{x}^1(1) = \begin{bmatrix}e\\e\end{bmatrix}$
2	$\vec{x}^1(2) = \begin{bmatrix}e^2\\e^2\end{bmatrix}$

Phase Portrait



Phase Plane (Saddle)

Graph $\vec{x}^2(t) = \begin{bmatrix}-e^{-t}\\ e^{-t}\end{bmatrix}$

t	$\vec{x}^2(t)$
0	$\vec{x}^2(0) = \begin{bmatrix}-1\\1\end{bmatrix}$
1	$\vec{x}^2(1) = \begin{bmatrix}-1/e\\1/e\end{bmatrix}$
2	$\vec{x}^2(2) = \begin{bmatrix}-1/e^2\\1/e^2\end{bmatrix}$
-1	$\vec{x}^2(-1) = \begin{bmatrix}-e\\e\end{bmatrix}$

Example: $\vec{x}^1 + \vec{x}^2 = \begin{bmatrix}e^t - e^{-t}\\ e^t + e^{-t}\end{bmatrix}$

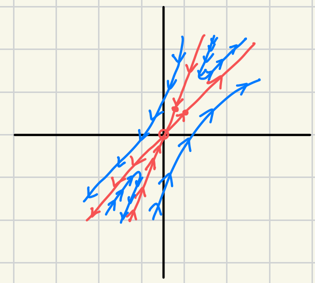
$t = 0, \begin{bmatrix}0\\2\end{bmatrix}$

$t = 1, \begin{bmatrix}e - 1/e\\ e + 1/e\end{bmatrix}$

Example: $\frac{d}{dt}\vec{x} = \begin{bmatrix}7&-4\\8&-5\end{bmatrix}\vec{x}$

$\lambda = 3, -1$ (saddle)

Sol: $e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$, $e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$

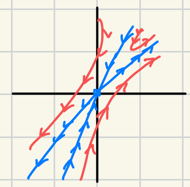


Lecture

July 23th, 2025

$\frac{d}{dt}\vec{x} = \begin{bmatrix}7&-4\\8&-5\end{bmatrix}\vec{x}$, $\lambda: \{3, -1\}$ Saddle

$\vec{x}^1 = e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad \vec{x}^2 = e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$



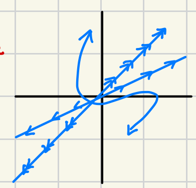
$\frac{d}{dt}\vec{x} = \begin{bmatrix}0&2\\-1&3\end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix}0&2\\-1&3\end{bmatrix}\right) = (\lambda-2)(\lambda-1)$ Nodal source

$\lambda=1: A - I = \begin{bmatrix}-1&2\\-1&2\end{bmatrix} \rightarrow \begin{bmatrix}1&-2\\0&0\end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$

$\lambda=2: A - 2I = \begin{bmatrix}-2&2\\-1&1\end{bmatrix} \rightarrow \begin{bmatrix}1&-1\\0&0\end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

$\vec{x} = c_1 e^t\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$

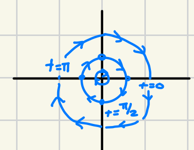


Complex eigenvalues

$\frac{d}{dt}\vec{x} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\vec{x}$

$p(\lambda) = \lambda^2 + 1 = \lambda = \pm i$, $\vec{x}^1 = \begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}$, $\vec{x}^2 = \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$

t	$\vec{x}^1(t)$
0	$\vec{x}^1(0) = \begin{bmatrix}1\\0\end{bmatrix}$
$\pi/2$	$\vec{x}^1(\pi/2) = \begin{bmatrix}0\\-1\end{bmatrix}$
$3\pi/2$	$\vec{x}^1(3\pi/2) = \begin{bmatrix}0\\1\end{bmatrix}$
π	$\vec{x}^1(\pi) = \begin{bmatrix}-1\\0\end{bmatrix}$



Center, Clockwise

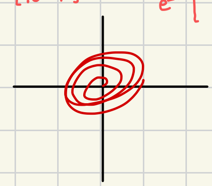
Experimental case:

$\begin{bmatrix}1&0\\0&2\end{bmatrix} \rightarrow \left\{\begin{matrix}\lambda=1, \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}\\ \lambda=2, \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}\end{matrix}\right\} \rightarrow \left\{\vec{x}^1 = e^t\begin{bmatrix}1\\0\end{bmatrix}, \vec{x}^2 = e^{2t}\begin{bmatrix}0\\1\end{bmatrix}\right\}$

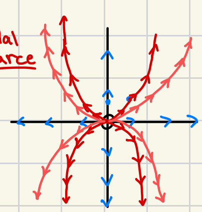
$\vec{x} = c_1 e^t\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}0\\1\end{bmatrix}$

$\vec{x}^1 + \vec{x}^2 = \begin{bmatrix}e^t\\ e^{2t}\end{bmatrix}$, $t = 0: \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}e\\e^2\end{bmatrix}\right\}$

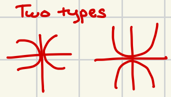
$\frac{d}{dt}\vec{x} = \begin{bmatrix}5&-4\\10&-7\end{bmatrix}\vec{x} \Longrightarrow e^{-t}\begin{bmatrix}2/5\cos 2t - 1/5 \sin 2t\\ \cos 2t\end{bmatrix}$, $e^{2t}\begin{bmatrix}1/5\cos 2t + 3/5 \sin 2t\\ \sin 2t\end{bmatrix}$



Nodal source



Two types



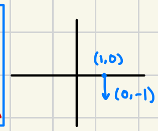
If α of $\alpha + i\beta i = 0$, then center/no spiral

If $\alpha > 0$, then spiral sink

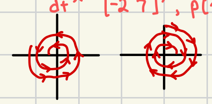
If $\alpha < 0$, then spiral source

Trick:

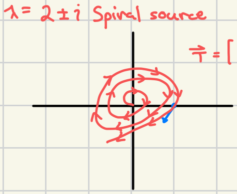
- Start at $\begin{bmatrix}1\\0\end{bmatrix}$
- Graph $\vec{T} = 1^{st}$ column of A



$\frac{d}{dt}\vec{x} = \begin{bmatrix}-3&13\\-2&7\end{bmatrix}\vec{x}$, $p(\lambda) = \lambda = 2 \pm i$ Spiral source



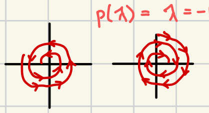
$\vec{T} = \begin{bmatrix}-3\\-2\end{bmatrix}$, $P = \begin{bmatrix}1\\0\end{bmatrix}$



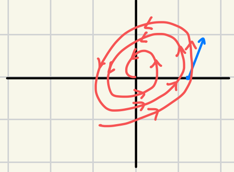
Example: $\frac{d}{dt}\vec{x} = \begin{bmatrix}5&-4\\10&-7\end{bmatrix}\vec{x}$

$p(\lambda) = \lambda = -1 \pm 2i$ (Spiral sink)

Counterclockwise



$\vec{T} = \begin{bmatrix}5\\10\end{bmatrix}$



Lecture

July 25, 2025

$\frac{d}{dt}\vec{x} = \begin{bmatrix}-1 & 9\\ -1 & 5\end{bmatrix}\vec{x}$, $p(\lambda) = \det\left(\begin{bmatrix}-1-\lambda & 9\\ -1 & 5-\lambda\end{bmatrix}\right)$

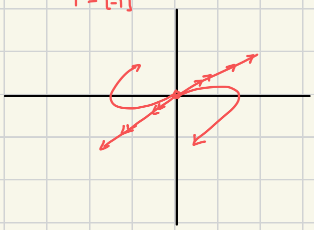
$= (\lambda+1)(\lambda-5)+9 = \lambda^2-4\lambda+4 = (\lambda-2)^2$

$\lambda=2$, double root

Degenerate Nodal source

$\lambda=2$: $A-2I = \begin{bmatrix}-3 & 9\\ -1 & 3\end{bmatrix} \to \begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}3\\1\end{bmatrix}$

$\vec{T} = \begin{bmatrix}-1\\-1\end{bmatrix}$



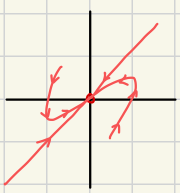
$\frac{d}{dt}\vec{x} = \begin{bmatrix}1 & -1\\ 4 & -3\end{bmatrix}\vec{x}$, $p(\lambda) = \det\left(\begin{bmatrix}1-\lambda & -1\\ 4 & -3-\lambda\end{bmatrix}\right)$

Degenerate Nodal Sink $= (\lambda-1)(\lambda+3)+4 = \lambda^2+2\lambda+1$

$\lambda=-1$, double $= (\lambda+1)^2$

$\lambda=-1$: $A+I = \begin{bmatrix}2 & -1\\ 4 & -2\end{bmatrix} \to \begin{bmatrix}1 & -1/2\\ 0 & 0\end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix}1/2\\1\end{bmatrix}$, $\vec{T} = \begin{bmatrix}1\\4\end{bmatrix}$



$\frac{d}{dt}\vec{x} = \begin{bmatrix}7 & -4\\ 8 & -5\end{bmatrix}\vec{x}$ linear 2×2

$\frac{d}{dt}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}7 & -4\\ 8 & -5\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$

$\begin{bmatrix}x_1'(t)\\ x_2'(t)\end{bmatrix} = \begin{bmatrix}7x_1(t)-4x_2(t)\\ 8x_1(t)-5x_2(t)\end{bmatrix} \Big\} \Rightarrow x_1' = 7x_1-4x_2$, $x_2' = 8x_1-5x_2$

$X' = x^2-y^2$, $y' = y-1$

Step 1: Equilibrium points

where $f(x,y) = x'(t)$ and $g'(x,y) = y'(t)$

$x'=0$ and $y'=0$

$f(x,y)=0$ and $g(x,y)=0$

$x^2-y^2=0 \leftarrow x^2-y^2=0$

$(x+y)(x-y)=0$ $y-1=0 \to y=1$

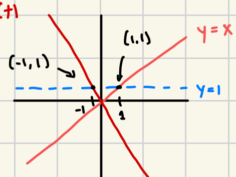
$\left.\begin{matrix}x+y=0\\ \text{or}\\ x-y=0\end{matrix}\right\} \Rightarrow \begin{matrix}y=-x\\ \text{or}\\ y=x\end{matrix}$

$x^2-y^2=0$

$x^2-1=0$

$(x+1)(x-1)=0$

$x=1,-1$



Step 2: $x' = x^2-y^2 = f(x,y)$

$y' = y-1 = g(x,y)$

Jacobian Matrix:

$J = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}$

$J = \begin{bmatrix}2x & -2y\\ 0 & 1\end{bmatrix}$

Step 3:

$J(-1,1) = \begin{bmatrix}-2 & -2\\ 0 & 1\end{bmatrix}$

$J(1,1) = \begin{bmatrix}2 & -2\\ 0 & 1\end{bmatrix}$

Step 4:

$\frac{d}{dt}\vec{x} = \begin{bmatrix}-2 & -2\\ 0 & 1\end{bmatrix}\vec{x}$ @ $(-1,1)$

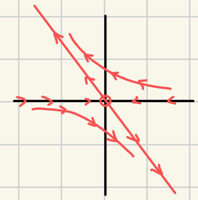
$p(\lambda) = \det\begin{bmatrix}-2-\lambda & -2\\ 0 & 1-\lambda\end{bmatrix} = (\lambda+2)(\lambda-1)$

$\lambda = 1,-2$

$\lambda=1$: $A-I = \begin{bmatrix}-3 & -2\\ 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 2/3\\ 0 & 0\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}-2/3\\ 1\end{bmatrix}$

$\lambda=-2$: $A+2I = \begin{bmatrix}0 & -2\\ 0 & 3\end{bmatrix} \to \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \Big\}\begin{matrix}x=x\\ y=0\end{matrix} \to \begin{bmatrix}x\\0\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$



Step 5:

@(1,1) $\frac{d}{dt}\vec{x} = \begin{bmatrix}2 & -2\\ 0 & 1\end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix}2-\lambda & -2\\ 0 & 1-\lambda\end{bmatrix}\right)$

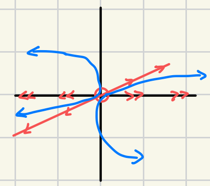
Nodal source $= (\lambda-2)(\lambda-1)$

$\lambda = 2,1$

$\lambda=1$: $A-I = \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$

$\lambda=2$: $A-2I = \begin{bmatrix}0 & -2\\ 0 & -1\end{bmatrix} \to \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \Big\}\begin{matrix}x=x\\ y=0\end{matrix} \Rightarrow \begin{bmatrix}1\\0\end{bmatrix} = \vec{v}_2$



Lecture

July 28th, 2025

$x' = x^2-y^2$

$y' = xy-1$

$(x+y)(x-y)=0$

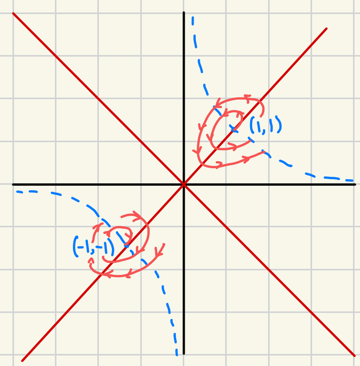
$x+y=0 \vee x-y=0$

$y=-x$, $y=x$

$xy=1$

$y=\frac{1}{x}$

① Equilibrium Points



② $J = \begin{bmatrix}2x & -2y\\ y & x\end{bmatrix}$

③ $J(-1,-1) = \begin{bmatrix}-2 & 2\\ -1 & -1\end{bmatrix}$

$J(1,1) = \begin{bmatrix}2 & -2\\ 1 & 1\end{bmatrix}$

④ @$(-1,-1)$, $\frac{d}{dt}\vec{x} = \begin{bmatrix}-2 & 2\\ -1 & -1\end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix}-2-\lambda & 2\\ -1 & -1-\lambda\end{bmatrix}\right)$

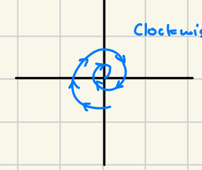
$= (\lambda+2)(\lambda+1)+2$

$= \lambda^2+3\lambda+4$

$\lambda = \frac{-3\pm\sqrt{9-16}}{2} = \frac{-3\pm i\sqrt{7}}{2}$

$= \frac{-3}{2} \pm i\frac{\sqrt{7}}{2}$ Spiral sink

Clockwise



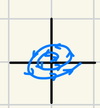
@(1,1) $\frac{d}{dt}\vec{x} = \begin{bmatrix}2 & -2\\ 1 & 1\end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix}2-\lambda & -2\\ 1 & 1-\lambda\end{bmatrix}\right) = \lambda = (\lambda-2)(\lambda-1)+2$

$= \lambda^2-3\lambda+4$

$\lambda = \frac{3\pm\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$ Spiral source

$\vec{T} = \begin{bmatrix}3\\1\end{bmatrix}$



Online Exam 3: $[\ ]+\_[\ ]+\_[\ ] = \begin{bmatrix}0\\0\\0\end{bmatrix}$

1. Linear Dependence
2. 2 x 2 systems → a) Solve, b) e^{At}, c) Phase

In class Exam 3:

3. 2 x 2 system
4. 2 x 2 system

Example #1: $S=\left\{\begin{bmatrix}-2\\3\\-12\end{bmatrix},\begin{bmatrix}-1\\1\\-3\end{bmatrix},\begin{bmatrix}5\\-8\\33\end{bmatrix}\right\}$ Use parametric form to show S is linearly dependent.

$$\left\{\begin{bmatrix}-2&-1&5\\3&1&-8\\-12&-3&33\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&0&-3\\0&1&1\\0&0&0\end{bmatrix}\right\}\longrightarrow\left\{\begin{matrix}r-3t=0\\s+t=0\\t=t\end{matrix}\right.\longrightarrow\left.\begin{matrix}r=3t\\s=-t\\t=t\end{matrix}\right\}$$

$$\therefore t\begin{bmatrix}3\\-1\\1\end{bmatrix}\Longrightarrow(3)\begin{bmatrix}-2\\3\\-12\end{bmatrix}+(-1)\begin{bmatrix}-1\\1\\-3\end{bmatrix}+\begin{bmatrix}5\\-8\\33\end{bmatrix}=\vec{0}$$

Example #2: $S=\left\{\begin{bmatrix}3\\2\\-1\end{bmatrix},\begin{bmatrix}6\\4\\-2\end{bmatrix},\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}7\\4\\1\end{bmatrix}\right\}$

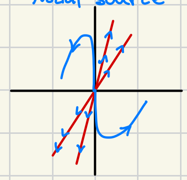
$$\left\{A=\begin{bmatrix}3&6&1&7\\2&4&1&4\\-1&-2&-2&1\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&2&0&3\\0&0&1&-2\\0&0&0&0\end{bmatrix}\right\}\longrightarrow\left\{\begin{matrix}a+2b+3d=0\\c-2d=0\\d=d\\b=b\end{matrix}\right.\longrightarrow\left.\begin{matrix}a=-2b-3d\\b=b\\c=2d\\d=d\end{matrix}\right\}$$

$$b\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}-3\\0\\2\\1\end{bmatrix}$$

$$(-2)\begin{bmatrix}3\\2\\-1\end{bmatrix}+(1)\begin{bmatrix}6\\4\\-2\end{bmatrix}+(0)\begin{bmatrix}1\\1\\2\end{bmatrix}+(0)\begin{bmatrix}7\\4\\1\end{bmatrix}=\vec{\varnothing}$$

$\frac{d}{dt}\vec{x}=\begin{bmatrix}?\end{bmatrix}\vec{x}$, $e^{3t}\begin{bmatrix}1\\1\end{bmatrix}, e^{t}\begin{bmatrix}1/2\\1\end{bmatrix}$

Nodal source



Lecture

August 1st, 2025

Predator–Prey Modeling

$x(t)$ = # of prey @ time t, $y(t)$ = # of predator @ time t.

$x'=\alpha\cdot x-\beta xy,\ y'=\gamma\cdot x\cdot y-\delta y$ (Lotka-Volterra System)

Example #3: $\frac{d}{dt}\vec{x}=\begin{bmatrix}-1&9\\-1&5\end{bmatrix}$

$$p(\lambda)=\det\left(\begin{bmatrix}-1-\lambda&9\\-1&5-\lambda\end{bmatrix}\right)=(\lambda+1)(\lambda-5)+9=\lambda^2-4\lambda+4$$

$$=(\lambda-2)(\lambda-2)\to\lambda=2,\text{ double}$$

$$\lambda=2:\ A-2I=\begin{bmatrix}-3&9\\-1&3\end{bmatrix}\xrightarrow{RREF}\begin{bmatrix}1&-3\\0&0\end{bmatrix}$$

$$\vec{v}_2=\begin{bmatrix}3\\1\end{bmatrix},\ \vec{x}^1=e^{2t}\begin{bmatrix}3\\1\end{bmatrix}$$

Jordan Cycle: $(A-\lambda I)\vec{J}=\vec{v}_\lambda\Longrightarrow\begin{bmatrix}-3&9\\-1&3\end{bmatrix}\vec{J}=\begin{bmatrix}3\\1\end{bmatrix}$

$$\left[\begin{array}{cc|c}-3&9&3\\-1&3&1\end{array}\right]\xrightarrow{RREF}\left[\begin{array}{cc|c}1&-3&-1\\0&0&0\end{array}\right]\to\left\{\begin{matrix}x-3y=-1\\y=y\end{matrix}\right.\longrightarrow\left.\begin{matrix}x=-1+3y\\y=y\end{matrix}\right\}$$

$$\begin{bmatrix}-1\\0\end{bmatrix}+\begin{bmatrix}3\\1\end{bmatrix}y\Longrightarrow\therefore\vec{J}=\begin{bmatrix}-1\\0\end{bmatrix}$$

$$\therefore\ \vec{x}^2=e^{\lambda t}(\vec{J}+t\vec{v}_\lambda)=e^{2t}\left(\begin{bmatrix}-1\\0\end{bmatrix}+t\begin{bmatrix}3\\1\end{bmatrix}\right)$$

$$=e^{2t}\begin{bmatrix}-1+3t\\t\end{bmatrix}$$

a) $\therefore\ c_1e^{2t}\begin{bmatrix}3\\1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}-1+3t\\t\end{bmatrix}$

b) e^{At}, $\vec{x}^1=e^{2t}\begin{bmatrix}3\\1\end{bmatrix}$, $\vec{x}^2=e^{2t}\begin{bmatrix}-1+3t\\t\end{bmatrix}$

$$\mathcal{X}(t)=e^{2t}\begin{bmatrix}3&-1+3t\\1&t\end{bmatrix}$$

$$\mathcal{X}(0)=\begin{bmatrix}3&-1\\1&0\end{bmatrix},\ \det(\mathcal{X}(0))=1$$

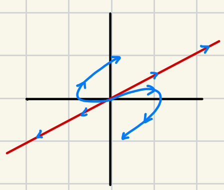
$$(\mathcal{X}(0))^{-1}=\begin{bmatrix}0&1\\-1&3\end{bmatrix}$$

$$e^{2t}\begin{bmatrix}3&-1+3t\\1&t\end{bmatrix}\begin{bmatrix}0&1\\-1&3\end{bmatrix}=e^{2t}\begin{bmatrix}3&-1+3t\\1&t\end{bmatrix}=e^{At}$$

c) $\frac{d}{dt}\vec{x}=\overset{\vec{T}}{\begin{bmatrix}-1&9\\-1&5\end{bmatrix}}\vec{x}$, $\lambda=2$, double, degenerate nodal source

$$\vec{v}_2=\begin{bmatrix}3\\1\end{bmatrix}$$

$$\vec{T}=\begin{bmatrix}-1\\-1\end{bmatrix}$$



Example #1: $x'=x-xy$

$y'=xy-y$

Step 1: Equilibrium points:

$$x-xy=0\longrightarrow x(1-y)=0$$

$$\Rightarrow\boxed{\begin{matrix}x=0\vee 1-y=0\\x=0\vee y=1\end{matrix}}$$

$$\boxed{\begin{matrix}y=0\vee x-1=0\\y=0\vee x=1\end{matrix}}$$

$x(t)\equiv0$

$y(t)\equiv0$

Step 2: Jacobian Matrix

$$J=\begin{bmatrix}1-y&-x\\y&x-1\end{bmatrix}$$

Step 3:

$$@(0,0)=J(0,0)=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ @(1,1)=J(1,1)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

Step 4:

$$@(0,0),\ \frac{d}{dt}\vec{x}=\begin{bmatrix}1-\lambda&0\\0&-1-\lambda\end{bmatrix}=(\lambda-1)(\lambda+1)$$

$$\lambda=\{1,-1\}$$

$$\lambda=1:\ A-I=\begin{bmatrix}0&0\\0&-2\end{bmatrix}\to\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}\to\left\{\begin{matrix}x=x\\y=0\end{matrix}\right\}\to x\begin{bmatrix}1\\0\end{bmatrix}$$

$$\therefore\vec{v}_1=\begin{bmatrix}1\\0\end{bmatrix}$$

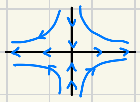
$$\lambda=-1:\ A+I=\begin{bmatrix}2&0\\0&0\end{bmatrix}\to\begin{bmatrix}1&0\\0&0\end{bmatrix}\to\left\{\begin{matrix}x=0\\y=y\end{matrix}\right\}\to y\begin{bmatrix}0\\1\end{bmatrix}$$

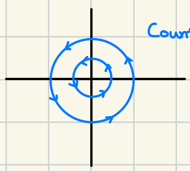
$$\therefore\vec{v}_{-1}=\begin{bmatrix}0\\1\end{bmatrix}$$

Step 4 continued:

$$@(1,1)\ \frac{d}{dt}\vec{x}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\vec{x},\ p(\lambda)=\det\left(\begin{bmatrix}-\lambda&-1\\1&-\lambda\end{bmatrix}\right)=\lambda^2+1$$

$$\lambda=\pm i\ \text{(Center)}\qquad\vec{T}=\begin{bmatrix}0\\1\end{bmatrix}$$



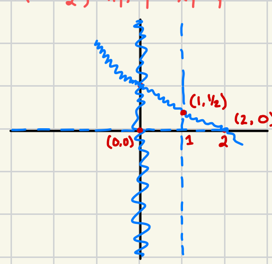


Counterclockwise

Adding carrying capacity tweak:

$$\begin{matrix}x'=\alpha x-\beta xy\\y'=\gamma xy-\delta y\end{matrix}\longmapsto\begin{matrix}x'=\alpha x\cdot(1-\frac{x}{N})-\beta xy\\y'=\gamma xy-\delta y\end{matrix}$$

Let $x'=x(1-\frac{x}{2})-xy,\ y'=xy-y$



Step 1 (MOD):

$$x(1-\tfrac{x}{2})-xy=0$$

$$x(1-\tfrac{1}{2}-y)=0\longrightarrow y=-\tfrac{x}{2}+1\vee x=0$$

$$xy-y=0$$

$$(x-1)y=0\longrightarrow y=0\vee x=1$$

Step 2 (MOD):

$$J=\begin{bmatrix}1-x-y&-x\\y&x-1\end{bmatrix}$$

Step 3 (MOD):

$$@(0,0)=J(0,0)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$@(1,\tfrac{1}{2})=J(1,\tfrac{1}{2})=\begin{bmatrix}-1/2&-1\\1/2&0\end{bmatrix}$$

$$@(2,0)=J(2,0)=\begin{bmatrix}-1&-2\\0&1\end{bmatrix}$$

Example #4: $\frac{d}{dt}\vec{x}=\begin{bmatrix}7&-4\\8&-5\end{bmatrix}$, $e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$, $e^{-t}\begin{bmatrix}1/2\\1\end{bmatrix}$

e^{At}:

$$\mathcal{X}(t)=\begin{bmatrix}e^{3t}&\frac{1}{2}e^{-t}\\e^{3t}&e^{-t}\end{bmatrix}$$

$$\mathcal{X}(0)=\begin{bmatrix}1&1/2\\1&1\end{bmatrix},\ \det(\mathcal{X}(0))=1/2$$

$$(\mathcal{X}(0))^{-1}=\frac{1}{1/2}\begin{bmatrix}1&-1/2\\-1&1\end{bmatrix}=\begin{bmatrix}2&-1\\-2&2\end{bmatrix}$$

$$e^{At}=\mathcal{X}(t)\cdot(\mathcal{X}(0))^{-1}$$

$$=\begin{bmatrix}e^{3t}&\frac{1}{2}e^{-t}\\e^{3t}&e^{-t}\end{bmatrix}\cdot\begin{bmatrix}2&-1\\-2&2\end{bmatrix}$$

$$=\begin{bmatrix}2e^{3t}-e^{-t}&\\2e^{3t}-&\end{bmatrix}$$

Lecture

August 4^{th}, 2025

$x(t) = \#$ of pray, $y(t) = \#$ of predators

$x' = \alpha x\left(1 - \frac{x}{N}\right) - \beta \cdot x \cdot y$

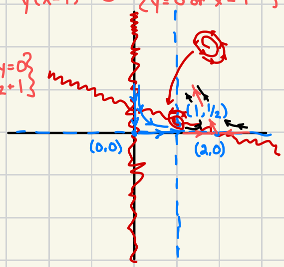
$y' = \gamma xy - \delta y$

Example #1: $x' = x \cdot \left(1 - \frac{x}{2}\right) - xy$, $y' = xy - y$

$xy - y = 0 \longrightarrow y(x-1) = 0 : \left\{\begin{matrix} y = 0 \text{ or } x-1 = 0 \\ y = 0 \text{ or } x = 1 \end{matrix}\right\}$

$x(1 - x/2 - y) = 0$

$\left\{\begin{matrix} x = 0 \text{ or } 1 - x/2 - y = 0 \\ x = 0 \text{ or } y = -x/2 + 1 \end{matrix}\right\}$



$x' = x - \frac{x^2}{2} - xy \qquad J = \begin{bmatrix} 1-x-y & -x \\ y & x-1 \end{bmatrix}$

$y' = xy - y$

$@(0,0) = J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$@(1, 1/2) = J(1, 1/2) = \begin{bmatrix} -1/2 & -1 \\ 1/2 & 0 \end{bmatrix}$

$@(2,0) = J(2,0) = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$

#1: $@(0,0)$, $\frac{d}{dt}\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right)$

$= (\lambda - 1)(\lambda + 1)$

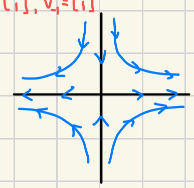
$\lambda = -1, 1$ (Saddle)

$\lambda = 1$: $A - I = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{RREF} \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\} \rightarrow \left\{\begin{matrix} x = x \\ y = 0 \end{matrix}\right\}$

$\therefore x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = -1$: $A + I = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{RREF} \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} \rightarrow \left\{\begin{matrix} x = 0 \\ y = y \end{matrix}\right\}$

$\therefore y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



#2: $@(2,0)$, $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix} -1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}\right)$

$= (\lambda + 1)(\lambda - 1)$

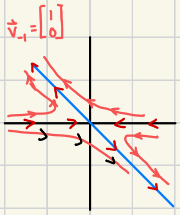
$\lambda = (1, -1)$ [Saddle]

$\lambda = 1$: $A - I = \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\} \rightarrow \left\{\begin{matrix} x + y = 0 \\ y = y \end{matrix} \rightarrow \begin{matrix} x = -y \\ y = y \end{matrix}\right\}$

$\therefore y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = -1$: $A + I = \begin{bmatrix} 0 & -2 \\ 0 & 2 \end{bmatrix} \longrightarrow \left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\} \rightarrow \left\{\begin{matrix} x = x \\ y = 0 \end{matrix}\right\}$

$\therefore x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



#3: $@(1, 1/2)$, $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1/2 & -1 \\ 1/2 & 0 \end{bmatrix}\vec{x}$

$p(\lambda) = \det\left(\begin{bmatrix} -1/2-\lambda & -1 \\ 1/2 & -\lambda \end{bmatrix}\right)$

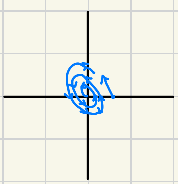
$= \lambda^2 + 1/2\,\lambda + 1/2 = 0$

$= 2\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-1 \pm \sqrt{-7}}{4}$

$= -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$ (Spiral sink)

$\vec{T} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$



func $u(x,t)$

$\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial t} \quad \frac{\partial^2 u}{\partial x \partial t}$ (PDE's)

Heat equation:

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial u}{\partial t}$$