This print-out should have 10 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Calculate the flux of the vector field

$$\mathbf{F}(x,\, y,\, z) \;=\; 3\langle x+y,\, x-y,\, x^2+y^2-2z\rangle$$

through the surface S parametrized by

$$\boldsymbol{\Phi}(u,\, v) \;=\; \Big\langle\, u+2v,\, u-2v,\, u^2+2v^2 \,\Big\rangle$$

with $0 \le u,\, v \le 1$, and oriented by $\boldsymbol{\Phi}_u \times \boldsymbol{\Phi}_v$.

1. $I \;=\; -6$ **correct**
2. $I \;=\; -5$
3. $I \;=\; -9$
4. $I \;=\; -7$
5. $I \;=\; -8$

Explanation:

The flux of $\mathbf{F}$ through S is given by

$$I \;=\; \int_0^1 \int_0^1 \mathbf{F}(\boldsymbol{\Phi}(u,\, v)) \cdot \boldsymbol{\Phi}_u \times \boldsymbol{\Phi}_v \, dvdu\,.$$

Now

$$\boldsymbol{\Phi}_u \;=\; \langle\, 1,\, 1,\, 2u \,\rangle\,, \quad \boldsymbol{\Phi}_v \;=\; \langle\, 2,\, -2,\, 4v \,\rangle\,,$$

$$\boldsymbol{\Phi}_u \times \boldsymbol{\Phi}_v \;=\; 4\,\langle\, u+v,\, u-v,\, -1 \,\rangle\,.$$

On the other hand,

$$\mathbf{F}(\boldsymbol{\Phi}(u,\, v)) \;=\; 6\langle\, u,\, 2v,\, 2v^2 \,\rangle\,.$$

Thus

$$\begin{aligned}\mathbf{F}(\boldsymbol{\Phi}(u,\, v)) &\cdot \boldsymbol{\Phi}_u \times \boldsymbol{\Phi}_v \\ &\;=\; 24(u^2+3uv-4v^2)\,,\end{aligned}$$

as a simple calculation shows. Consequently,

$$I \;=\; 24\int_0^1 \left(\int_0^1 (u^2+3uv-4v^2)\, dv\right) du$$

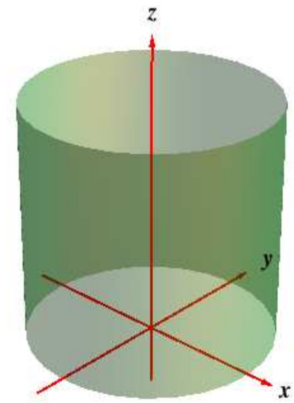
$$=\; \int_0^1 (24u^2+36u-32)\, du \;=\; -6\,.$$

002 10.0 points

Calculate the flux of the vector field

$$\mathbf{F}(x, y, z) \;=\; \Big\langle\, z(x^2+y^2),\, zy,\, zx \,\Big\rangle$$

through the outwardly-oriented open cylinder



having radius 1 and lying between the planes $z = 0$ and $z = 2$.

1. $I \;=\; \dfrac{3}{2}\pi$
2. $I \;=\; 0\pi$
3. $I \;=\; \dfrac{1}{2}\pi$
4. $I \;=\; 2\pi$ **correct**
5. $I \;=\; \pi$

Explanation:

In cylindrical polar coordinates the cylinder is parameterized by

$$\boldsymbol{\Phi}(\theta,\, z) \;=\; \langle\cos\theta,\, \sin\theta,\, z\rangle,$$

with $0 \le \theta \le 2\pi,\; 0 \le z \le 2$. But then

$$\boldsymbol{\Phi}_\theta = \langle -\sin\theta,\, \cos\theta,\, 0\rangle, \quad \boldsymbol{\Phi}_z = \langle 0,\, 0,\, 1\rangle,$$

$$\boldsymbol{\Phi}_\theta \times \boldsymbol{\Phi}_z \;=\; \langle\cos\theta,\, \sin\theta,\, 0\rangle\,.$$

Since θ is rotating counterclockwise, while z is increasing in the upwards vertical direction, the right hand rule shows that $\boldsymbol{\Phi}_\theta \times \boldsymbol{\Phi}_z$

points outward, so $\mathbf{\Phi}(\theta, z)$ is an orientation-preserving parametrization of the cylinder (as can also be seen from the direction of $\langle \cos\theta, \sin\theta, 0 \rangle$.

Thus the flux through S is given by the integral

$$I = \int_0^2 \int_0^{2\pi} \mathbf{F}(\mathbf{\Phi}(\theta, z)) \cdot \mathbf{\Phi}_\theta \times \mathbf{\Phi}_z \, dz d\theta \, .$$

Now

$$\mathbf{F}(\mathbf{\Phi}(\theta, z)) = \langle z, z\sin\theta, z\cos\theta \rangle ,$$
$$\mathbf{F}(\mathbf{\Phi}(\theta, z)) \cdot \mathbf{\Phi}_\theta \times \mathbf{\Phi}_z = z(\cos\theta + \sin^2\theta) \, .$$

Consequently,

$$I = \int_0^2 \int_0^{2\pi} z \left(\cos\theta + \frac{1}{2}(1 - \cos 2\theta) \right) d\theta dz$$
$$= \int_0^2 \pi z \, dz = 2\pi \, .$$

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003 10.0 points

Evaluate the integral

$$I = \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

for the vector field

$$\mathbf{F} = 3x\,\mathbf{i} + 2y\,\mathbf{j} - 2z\mathbf{k}$$

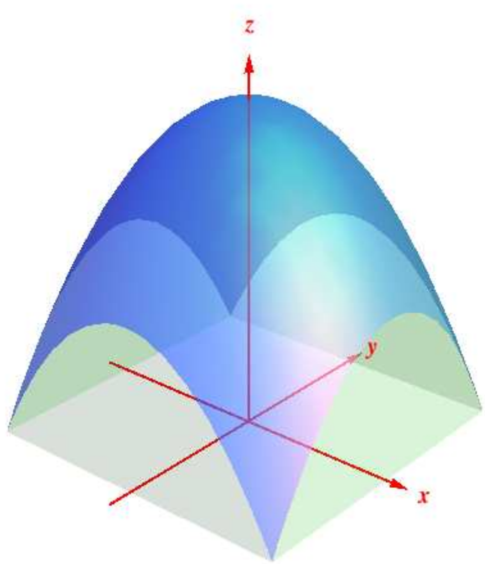
when S is the part of the paraboloid

$$z = 2 - x^2 - y^2 \, ,$$

oriented upwards, lying above the square

$$-1 \le x \le 1 \, , \quad -1 \le y \le 1 \, ,$$

as shown in



1. $I = 1$

2. $I = \dfrac{2}{3}$

3. $I = 2$

4. $I = \dfrac{4}{3}$

5. $I = \dfrac{8}{3}$ **correct**

Explanation:

If S is the graph of $z = f(x, y)$, then

$$d\mathbf{S} = (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}) dx dy \, .$$

So when $z = 2 - x^2 - y^2$, and

$$\mathbf{F} = 3x\,\mathbf{i} + 2y\,\mathbf{j} - 2z\mathbf{k}$$

we see that

$$d\mathbf{S} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k} \, ,$$

while

$$\mathbf{F} \cdot d\mathbf{S} = (6x^2 + 4y^2 - 2(2 - x^2 - y^2)) dx dy$$
$$= (8x^2 + 6y^2 - 4) dx dy \, .$$

Consequently,

$$\boxed{I = \int_{-1}^1 \int_{-1}^1 (8x^2 + 6y^2 - 4) \, dx dy = \frac{8}{3}} \, .$$

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keywords:

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004 10.0 points

Evaluate the surface integral

$$I = \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

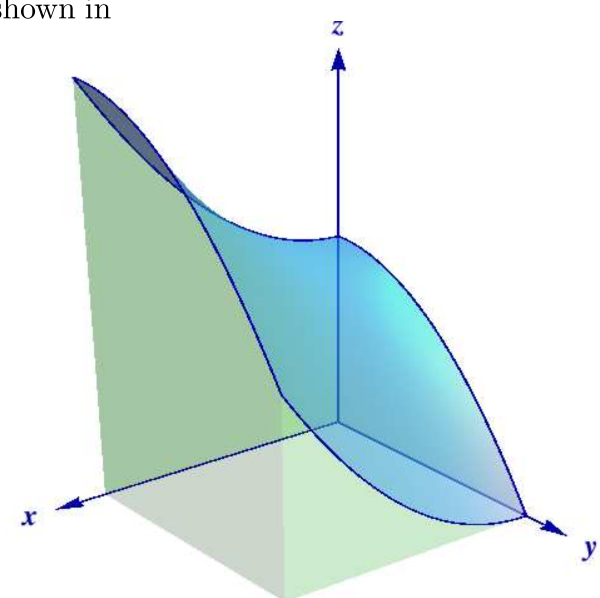
when

$$\mathbf{F}(x, y, z) = 2y^2\,\mathbf{i} + 3x^2\,\mathbf{j}$$

and S is the graph of

$$z = \frac{1}{2}(1 + x^2 - y^2), \quad 0 \le x, y \le 1,$$

shown in



1. $I = \dfrac{1}{3}$

2. $I = \dfrac{5}{6}$

3. $I = \dfrac{1}{6}$ **correct**

4. $I = \dfrac{2}{3}$

5. $I = \dfrac{1}{2}$

Explanation:

The vector surface area element of the graph of

$$z = f(x, y) = \frac{1}{2}(1 + x^2 - y^2)$$

is given by

$$\begin{aligned} d\mathbf{S} &= (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k})\,dxdy \\ &= (-x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k})\,dxdy\,. \end{aligned}$$

Thus

$$\begin{aligned} &\mathbf{F} \cdot d\mathbf{S} \\ &= (2y^2\,\mathbf{i} + 3x^2\,\mathbf{j}) \cdot (-x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k})dxdy \\ &= (3x^2y - 2xy^2)dxdy\,, \end{aligned}$$

and so as a repeated integral,

$$I = \int_0^1 \Big(\int_0^1 (3x^2y - 2xy^2)\,dy \Big) dx\,.$$

But

$$\int_0^1 (3x^2y - 2xy^2)\,dy = \Big[\frac{3}{2}x^2y^2 - \frac{2}{3}xy^3\Big]_0^1,$$

in which case

$$I = \int_0^1 \Big(\frac{3}{2}x^2 - \frac{2}{3}x\Big)\,dx = \Big[\frac{1}{2}x^3 - \frac{1}{3}x^2\Big]_0^1.$$

Consequently,

$$\boxed{I = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}}.$$

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005 10.0 points

Evaluate the integral

$$I = \int_S f\,dS$$

when

$$f(x, y, z) = 3(1 + y^2 + z^2)^{1/2}$$

and S is the surface given parametrically by

$$\mathbf{\Phi}(u, v) = (2uv, u + v, u - v)$$

for $u^2 + v^2 \le 1$.

1. $I = 12\pi$ **correct**

2. $I = 3\pi$

3. $I = 18\pi$

4. $I = 4\pi$

5. $I = 6\pi$

Explanation:

When S is parametrized by

$$\Phi(u, v) = (2uv, u+v, u-v)$$

for $u^2 + v^2 \leq 1$, then

$$I = \int\int_D f(\Phi(u, v))\|\mathbf{T}_u \times \mathbf{T}_v\| \, dudv,$$

where

$$f(\Phi(u, v)) = 3(1 + (u+v)^2 + (u-v)^2)^{1/2} = 3(1 + 2(u^2 + v^2))^{1/2},$$

and

$$D = \{(u, v) : u^2 + v^2 \leq 1\}.$$

On the other hand,

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = (2v, 1, 1),$$

while

$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = (2u, 1, -1).$$

In this case,

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2v & 1 & 1 \\ 2u & 1 & -1 \end{vmatrix} = -2\,\mathbf{i} + 2(u+v)\,\mathbf{j} + 2(v-u)\,\mathbf{k}.$$

Thus

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = 2(1 + (u+v)^2 + (v-u)^2)^{1/2} = 2(1 + 2(u^2 + v^2))^{1/2}.$$

So, finally, we arrive at

$$I = 6 \int\int_D (1 + 2(u^2 + v^2))\, dudv.$$

Because of the rotational symmetry, we'll use polar coordinates with

$$u = r\cos\theta, \qquad v = r\sin\theta,$$

to evaluate I. For then

$$I = 6\int_0^1 \int_0^{2\pi} (1 + 2r^2)\, r\, d\theta dr = 12\pi \int_0^1 (r + 2r^3)\, dr = 12\pi \Big[\frac{1}{2}r^2 + \frac{1}{2}r^4\Big]_0^1.$$

Consequently,

$$\boxed{I = 12\pi}.$$

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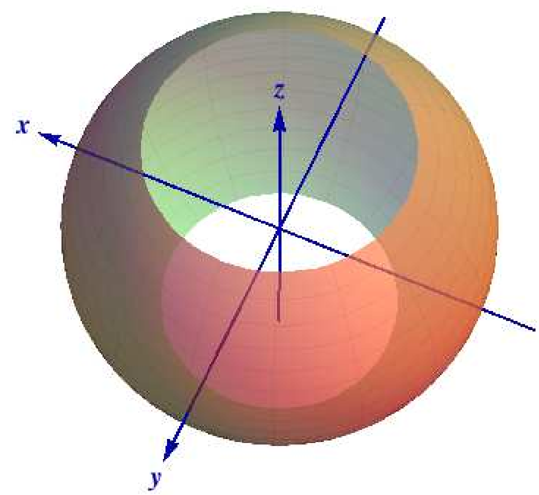
006 10.0 points

The surface S shown in



is the portion of the sphere

$$x^2 + y^2 + z^2 = 16$$

where

$$x^2 + y^2 \geq 12.$$

Determine the surface area of S.

1. Surface Area = 16 sq. units

2. Surface Area = 24 sq. units

3. Surface Area = 32 sq. units

4. Surface Area $= 32\pi$ sq. units **correct**

5. Surface Area $= 24\pi$ sq. units

6. Surface Area $= 16\pi$ sq. units

Explanation:

The sphere

$$x^2 + y^2 + z^2 = 16$$

is parametrized in spherical polar coordinates by

$$\mathbf{\Phi}(\theta,\, \phi) = 4(\cos\theta\sin\phi,\, \cos\theta\sin\phi,\, \cos\phi)$$

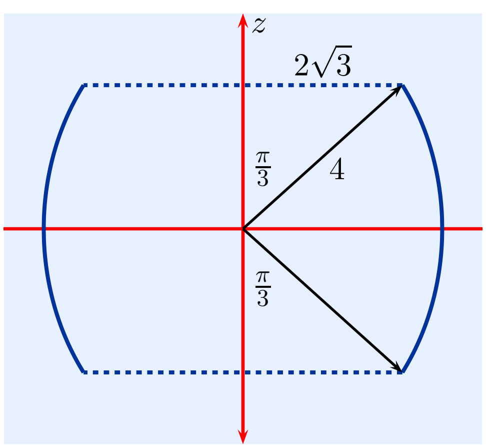
with

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi\,.$$

We have to determine what conditions need to be imposed on θ, ϕ so that only the points where

$$x^2 + y^2 \ge 12$$

are included. Since S doesn't change with rotation by θ from 0 to 2π around the z-axis, there will be no restriction on θ. On the other hand, spherical caps at the north and south poles are missing, so restrictions on ϕ are needed. Now a vertical cross-section of S through the origin looks like



and S is generated by rotating the arc of the circle between the two solid arrows around the z-axis. Thus S is parametrized by

$$\mathbf{\Phi}(\theta,\, \phi) = 4(\cos\theta\sin\phi,\, \cos\theta\sin\phi,\, \cos\phi)\,,$$

with

$$0 \le \theta \le 2\pi, \quad \frac{\pi}{3} \le \phi \le \frac{2\pi}{3}\,,$$

and the surface area element is

$$dS = 16\,\sin\phi\, d\theta d\phi\,.$$

Consequently, S has surface

$$\boxed{\text{area} = 16\int_{\pi/3}^{2\pi/3}\left(\int_0^{2\pi}\sin\phi\, d\theta\right)d\phi = 32\pi}\,.$$

007 10.0 points

Use the fact that

$$\mathbf{F}(x,\, y) = 2e^y\,\mathbf{i} + (2xe^y - 3)\,\mathbf{j}$$

is a gradient vector field to evaluate the line integral

$$I = \int_C \mathbf{F}\cdot d\mathbf{s}$$

along the curve C given parametrically by

$$\mathbf{c}(t) = te^t\,\mathbf{i} + (1+t)\,\mathbf{j}\,, \quad 0 \le t \le 1\,.$$

1. $I = -3e^2 - 2$

2. $I = 2e^2 - 3$

3. $I = -3e^3 + 2$

4. $I = 2e^3 - 3$ **correct**

5. $I = 2e^3 + 6$

6. $I = -3e^2 + 6$

Explanation:

Since $\mathbf{F}$ is a gradient vector field, the Fundamental Theorem for Line Integrals says that

$$I = \int_0^1 \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)\,dt = f(\mathbf{c}(1)) - f(\mathbf{c}(0))$$

for any function f such that $\mathbf{F} = \nabla f$.

To write

$$\mathbf{F}(x,\, y) \;=\; 2e^y\,\mathbf{i} + (2xe^y - 3)\,\mathbf{j}$$

as a gradient vector field, we have to find a function $f(x,\, y)$ such that

$$\frac{\partial f}{\partial x} \;=\; 2e^y, \quad \frac{\partial f}{\partial y} \;=\; 2xe^y - 3.$$

Now the first equation says that

$$f(x,\, y) \;=\; 2xe^y + D(y)\,,$$

for some arbitrary function $D(y)$, where by the second equation

$$\frac{\partial f}{\partial y} \;=\; 2xe^y + D'(y) \;=\; 2xe^y - 3\,,$$

i.e., $D'(y) = -3$. Thus $D(y) = -3y + K$, so

$$f(x,\, y) \;=\; 2xe^y - 3y + K\,,$$

for some arbitrary constant K.

On the other hand, when

$$\mathbf{c}(t) \;=\; te^t\,\mathbf{i} + (1+t)\,\mathbf{j}\,, \quad 0 \le t \le 1\,,$$

then

$$\mathbf{c}(0) \;=\; \mathbf{j}\,, \qquad \mathbf{c}(1) \;=\; e\,\mathbf{i} + 2\,\mathbf{j}\,.$$

Thus

$$f(\mathbf{c}(0)) \;=\; f(0,\, 1) \;=\; -3 + K\,,$$

while

$$f(\mathbf{c}(1)) \;=\; f(e,\, 2) \;=\; 2e.e^2 - 6 + K$$
$$= \; 2e^3 - 6 + K\,.$$

Consequently,

$$\boxed{I \;=\; f(\mathbf{c}(1)) - f(\mathbf{c}(0)) \;=\; 2e^3 - 3}\,.$$

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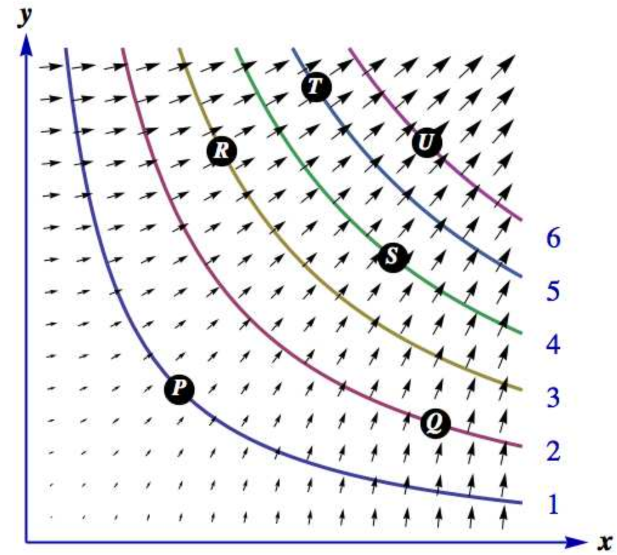
008 10.0 points

A gradient vector field $\mathbf{F} = \nabla f$ and points $P,\, Q,\, \ldots,\, U$ on contour lines of $z = f(x,\, y)$ are shown in



Determine the value of the line integral

$$I \;=\; \int_C \mathbf{F} \cdot d\mathbf{s}$$

when C is the line segment from S to R and the values of $f(x,\, y)$ on the contour lines are listed to the right.

1. $I \;=\; 3$
2. $I \;=\; 4$
3. $I \;=\; -4$
4. $I \;=\; 1$
5. $I \;=\; -1$ **correct**
6. $I \;=\; -3$

Explanation:

Since $\mathbf{F} = \nabla f$, the Fundamental Theorem for Line Integrals says that

$$I \;=\; \int_C \mathbf{F} \cdot d\mathbf{s} \;=\; f(R) - f(S)$$

for *any* smooth curve from S to R.

Reading off the values of $f(x,\, y)$ from the given contour values, we thus see that

$$\boxed{I \;=\; -1}\,.$$

keywords:

009 10.0 points

Use the fact that

$$\mathbf{F} = (6xy + 4\cos y)\,\mathbf{i} + (3x^2 - 4x\sin y)\,\mathbf{j}\,,$$

is a gradient vector field to evaluate the line integral

$$I = \int_C \mathbf{F}\cdot d\mathbf{s}$$

along a smooth curve C in the plane from

$$P = (1,\, \pi) \quad \text{to} \quad Q = \left(2,\, \frac{\pi}{2}\right).$$

1. $I = 3 + 4\pi$
2. $I = 6 + 4\pi$
3. $I = 6 - 4\pi$
4. $I = 6\pi - 4$
5. $I = 3\pi - 4$
6. $I = 3\pi + 4$ **correct**

Explanation:

Since $\mathbf{F}$ is a gradient vector field, the Fundamental Theorem for Line Integrals says that

$$I = \int_C \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,dt = f(Q) - f(P)$$

for any function f such that $\mathbf{F} = \nabla f$.

To write

$$\mathbf{F} = (6xy + 4\cos y)\,\mathbf{i} + (3x^2 - 4x\sin y)\,\mathbf{j}\,,$$

as a gradient vector field, we have to find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 6xy + 4\cos y\,, \qquad \frac{\partial f}{\partial y} = 3x^2 - 4x\sin y\,.$$

Now by the first equation,

$$f(x, y) = 3x^2y + 4x\cos y + D(y)$$

for an arbitrary function $D(y)$, which by the second equation satisfies

$$3x^2 - 4x\sin y + D'(y) = 3x^2 - 4x\sin y\,,$$

i.e., $D(y) = K$ for an arbitrary constant K. Thus

$$f(x, y) = 3x^2y + 4x\cos y + K\,.$$

But then

$$f(P) = 3\pi - 4 + K\,, \qquad f(Q) = 6\pi + K\,.$$

Consequently,

$$\boxed{I = 3\pi + 4}\,.$$

010 10.0 points

Find the work done by the force field

$$\mathbf{F}(x, y) = (xy^2 + 3)\,\mathbf{i} + (x^2y + 5)\,\mathbf{j}$$

in moving a particle along a smooth path in the plane from $A(1, 0)$ to $B(2, 1)$.

1. work done $=$ 10 units **correct**
2. work done $=$ 11 units
3. work done $=$ 12 units
4. work done $=$ 9 units
5. work done $=$ 8 units

Explanation:

The work done by a force field

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$$

in moving a particle along a smooth path C in the plane from $A(1, 0)$ to $B(2, 1)$ is given by the line integral

$$I = \int_C \mathbf{F}\cdot d\mathbf{s}\,.$$

This integral can be evaluated when $\mathbf{F}$ is a gradient vector field, *i.e.*, when there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = P(x, y), \qquad \frac{\partial f}{\partial y} = Q(x, y).$$

For then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A) = f(2, 1) - f(1, 0),$$

independently of the path C.

To write

$$\mathbf{F} = (xy^2 + 3)\,\mathbf{i} + (x^2y + 5)\,\mathbf{j},$$

as a gradient vector field we need to find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = xy^2 + 3, \quad \frac{\partial f}{\partial y} = x^2y + 5.$$

The first equation says that

$$f(x, y) = \frac{1}{2}x^2y^2 + 3x + D(y),$$

for some arbitrary function $D(y)$, where by the second equation

$$\frac{\partial f}{\partial y} = x^2y + D'(y) = x^2y + 5,$$

i.e., $D'(y) = 5$. Thus $D(y) = 5y + K$, so

$$f(x, y) = \frac{1}{2}x^2y^2 + 3x + 5y + K,$$

for some arbitrary constant K. But then

$$f(1, 0) = 3 + K,$$

while

$$f(2, 1) = \frac{1}{2}(4) + 6 + 5 + K = 13 + K.$$

Consequently,

$$\boxed{\text{work done} = 10 \text{ units}}.$$