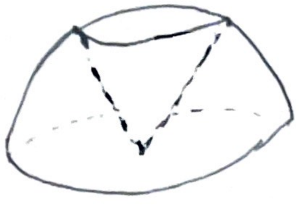
(1)



$$F = (3xz^2 + y,\ 3x^2y - \sin(z),\ 3y^2z - e^y)$$

$$\operatorname{div} F = (3z^2) + (3x^2) + (3y^2) = 3(x^2+y^2+z^2)$$

$$\text{Flux}_{out} = \iiint_W 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

We use spherical coordinates: $\tilde{W} = \{(\rho,\varphi,\theta) : \rho^2 \le 4,\ 0 \le \rho\cos\varphi \le \rho\sin\varphi\}$

$$\tilde{W} = \{(\rho,\varphi,\theta) : 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \tfrac{\pi}{4} \le \varphi \le \tfrac{\pi}{2}\}$$

$$\text{Flux}_{out} = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 (3\rho^2)(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 3\rho^4\sin\varphi\,d\rho\,d\varphi\,d\theta$$

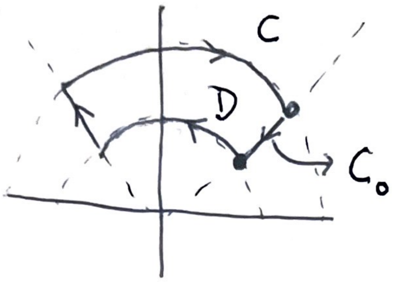
$$= 6\pi\int_{\pi/4}^{\pi/2}\int_0^2 \rho^4\sin\varphi\,d\rho\,d\varphi = 6\pi\int_0^2 \rho^4\left[-\cos\varphi\right]_{\pi/4}^{\pi/2} d\rho$$

$$= 3\pi\sqrt{2}\int_0^2 \rho^5\,d\rho = 3\pi\sqrt{2}\left[\rho^6/6\right]_0^2 = 32\pi\sqrt{2}$$

↳ positive means compatible with outword direction

$$\boxed{\text{Answer: } 32\pi\sqrt{2} \text{ in outward direction}}$$

② 




$C \cup C_0 = \ominus \partial D$

$C \cup C_0$ is CW but $\partial D$ is CCW

$C_0(x) = (x, x)$, $x$ from $\sqrt{2}$ to $\sqrt{2}/2$

SETUP: $\displaystyle\int_C F \cdot dr + \int_{C_0} F \cdot dr = -\iint_D (F_{2x} - F_{1y})\, dx\, dy$

$F = (y^3 - \cos(x-y), \cos(x-y))$

$F_{2x} - F_{1y} = (-\sin(x-y)) - (3y^2 - \sin(x-y)) = -3y^2$

$$\iint_D -3y^2\, dx\, dy = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^2 (-3r^2 \sin^2\theta)(r)\, dr\, d\theta = \int_{\pi/4}^{3\pi/4} \int_1^2 -3r^3 \sin^2\theta\, dr\, d\theta$$

$$= \left[-\frac{3r^4}{4}\right]_1^2 \int_{\pi/4}^{3\pi/4} \sin^2\theta\, d\theta = -\frac{45}{4} \int_{\pi/4}^{3\pi/4} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\frac{45}{8} \left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{3\pi/4} = -\frac{45}{8}\left(\frac{3\pi}{4} + \frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}\right)$$

$$= -\frac{45}{16}(\pi + 2)$$

$$\int_{C_0} F \cdot dr = \int_{\sqrt{2}}^{\sqrt{2}/2} (x^3 - \cos(x-x), \cos(x-x)) \cdot (1, 1)\, dx$$

$$= \int_{\sqrt{2}}^{\sqrt{2}/2} x^3 \, dx = \left[\frac{x^4}{4}\right]_{\sqrt{2}}^{\sqrt{2}/2} = \left(\frac{1}{16}\right) - \left(1\right) = -\frac{15}{16}$$

So:

$$\int_C F \cdot dr + \left(-\frac{15}{16}\right) = -\left(-\frac{45}{16}(\pi+2)\right)$$

$$\int_C F \cdot dr = \frac{45}{16}(\pi+2) + \frac{15}{16}$$

$$\boxed{\text{WORK} = \frac{45\pi + 105}{16}}$$

③ $F = \left(z - (1+xy)e^{xy-z},\ 1 - x^2 e^{xy-z},\ x + xe^{xy-z}\right)$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z-(1+xy)e^{xy-z} & 1-x^2e^{xy-z} & x+xe^{xy-z} \end{vmatrix} = (0,0,0)$$

Nice domain, so conservative.

$f_x = z - (1+xy)e^{xy-z}$

$f_y = 1 - x^2 e^{xy-z} \longrightarrow f = y - xe^{xy-z} + A(x,z)$

$f_z = x + xe^{xy-z}$

$f_x = -e^{xy-z} - xye^{xy-z} + A_x$

Comparing to $f_x = z - (1+xy)e^{xy-z}$:

$A_x = z \rightarrow A = xz + B(z)$

Rewrite: $f = y - xe^{xy-z} + xz + B(z)$

$f_z = xe^{xy-z} + x + B'(z)$

Comparing to $f_z = x + xe^{xy-z}$:

$B'(z) = 0 \rightarrow B(z) = \text{constant}$

So: $f = y - xe^{xy-z} + xz + \text{constant}.$

$$\text{WORK} = \Delta f = f(c(0)) - f(c(-1))$$

$$= f(1,-1,-1) - f(2,-2,0)$$

$$= (\cancel{-1}\cancel{-1}-1) - (\cancel{-2} - 2e^{-4} + 0)$$

$$= \boxed{2e^{-4} - 1}$$

(4) $W = \{(x,y,z): x^2+y^2 \le 4, \quad 0 \le z \le 1+x^2+y^2\}$

$F = (x^2-y,\ 3x^2y+\sin(z),\ e^y - 2xz)$

$\operatorname{div} F = (2x) + (3x^2) + (-2x) = 3x^2$

$$\text{Flux}_{out} = \iiint_W 3x^2\, dx\,dy\,dz$$

We use polar: $\tilde{W} = \{(r,\theta,z): 0 \le r \le 2,\ 0 \le z \le 1+r^2\}$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\int_0^{2\pi} \cos 2\theta\, d\theta = 0$

$\int_0^{2\pi} 1\, d\theta = 2\pi$

$$\text{Flux}_{out} = \int_0^{2\pi}\int_0^2\int_0^{1+r^2} (3r^2\cos^2\theta)(r)\, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^{1+r^2} \frac{3r^3}{2}(1+\cos(2\theta))\, dz\,dr\,d\theta$$

$$= \pi\int_0^2\int_0^{1+r^2} 3r^3\, dz\,dr = \pi\int_0^2 3r^3(1+r^2)\, dr$$

$$= \pi\int_0^2 (3r^3 + 3r^5)\, dr = \pi\left[\frac{3r^4}{4} + \frac{r^6}{2}\right]_0^2 = 44\pi$$

+ means compatible with outward orientation

Answer: $44\pi$ in outward orientation

(5)


$C \cup C_0 = -\partial D$

$C_0$ : portion from $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ to $(0,0)$

$C_0(x) = (x, x)$ with $x$ from $\frac{\sqrt{2}}{2}$ to $0$.

SETUP : $\displaystyle\int_C F\cdot dr + \int_{C_0} F\cdot dr = -\iint_D (F_{2x} - F_{1y})\,dx\,dy$

$F = (\cos(y-x), x^3 - \cos(y-x))$

$F_{2x} - F_{1y} = (3x^2 - \sin(y-x)) - (-\sin(y-x)) = 3x^2$

$$\iint_D 3x^2\,dx\,dy = \int_{\pi/4}^{\pi}\int_0^2 3r^3\cos^2\theta\,dr\,d\theta = \int_{\pi/4}^{\pi}\int_0^2 \frac{3r^3(1+\cos 2\theta)}{2}\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi} \left[\frac{3r^4}{8}\right]_0^2 (1+\cos 2\theta)\,d\theta = \int_{\pi/4}^{\pi} 6(1+\cos 2\theta)\,d\theta$$

$$= 6\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi} = 6\left(\pi - 0 - \frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{9\pi}{2} - 3$$

$$\int_{C_0} F\cdot dr = \int_{\frac{\sqrt{2}}{2}}^{0} \left(\cos(x-x), x^3 - \cos(x-x)\right)\cdot(1,1)\,dx = \int_{\frac{\sqrt{2}}{2}}^{0} x^3\,dx$$

$$= \left[ \frac{x^4}{4} \right]_{\sqrt{2}/2}^{0} = -\frac{1}{16}$$

So: $\int_C F \cdot dr + \left(\frac{-1}{16}\right) = -\left(\frac{9\pi}{2} - 3\right)$

$$\boxed{\int_C F \cdot dr = -\frac{9\pi}{2} + \frac{49}{16}}$$

⑥ $F = \left(\cos(xy-z) - xy\sin(xy-z),\ z - x^2\sin(xy-z),\ y + x\sin(xy-z)\right)$

$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \cos(xy-z)-xy\sin(xy-z) & z-x^2\sin(xy-z) & y+x\sin(xy-z) \end{vmatrix}$$

$$= (0,0,0)$$

$f_x = \cos(xy-z) - xy\sin(xy-z)$

$f_y = z - x^2\sin(xy-z) \longrightarrow f = yz + x\cos(xy-z) + A(x,z)$

$f_z = y + x\sin(xy-z)$

$f_x = \cos(xy-z) - xy\sin(xy-z) + A_x$

Compare to $f_x = \cos(xy-z) - xy\sin(xy-z)$:

$A_x = 0 \to A = B(z)$

Rewrite: $f = yz + x\cos(xy-z) + B(z)$

$f_z = y + x\sin(xy-z) + B'(z)$

Compare to $f_z = y + x\sin(xy-z)$:

$B'(z) = 0 \to B(z) = \text{constant}.$

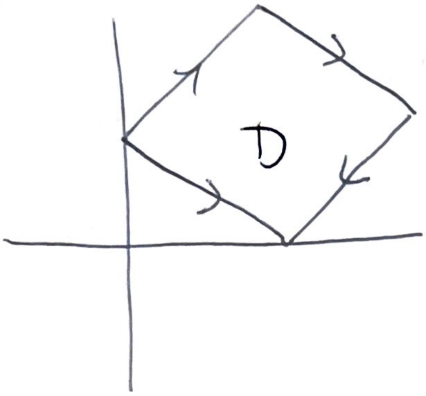
Conclusion: $f = yz + x\cos(xy - z)$ + constant.

$$\text{WORK} = \Delta f = f(c(0)) - f(c(-1))$$

$$= f(1,-1,-1) - f(2,-2,0)$$

$$= (1+1) - (0 + 2\cos(-4)) = \boxed{2 - 2\cos(4)}$$

(7)




$$C = -\partial D$$

$$\int_C F \cdot dr = -\iint_D \text{curl}(F)\,dx\,dy$$

$$F = \left(ye^{x-2y+2},\ xe^{x-2y+2}\right)$$

$$F_{2x} - F_{1y} = \left(e^{x-2y+2} + xe^{x-2y+2}\right) - \left(e^{x-2y+2} - 2ye^{x-2y+2}\right)$$

$$= (x+2y)e^{x-2y+2}$$

the lines bounding D are: $y = 1 + \frac{x}{2} \to x - 2y = -2$

$y = -1 + \frac{x}{2} \to x - 2y = 2$

$y = 3 - \frac{x}{2} \to x + 2y = 6$

$y = 1 - \frac{x}{2} \to x + 2y = 2$

Let $u = x - 2y$, $v = x + 2y$, so $\tilde{D} = \{(u,v) : -2 \leq u \leq 2,\ 2 \leq v \leq 6\}$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}\right| = 4 \to \text{Jacobian} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{4}$$

So: $\int_C F \cdot dr = -\iint_D (x+2y)e^{x-2y+2}\,dx\,dy$

$$= -\int_2^6 \int_{-2}^2 v e^{u+2} \frac{1}{4}\,du\,dv$$

$$= -\int_2^6 \frac{v}{4}\left[e^{u+2}\right]_{-2}^{2} dv = -\int_2^6 \frac{v}{4}(e^4-1)\,dv$$

$$= -(e^4-1)\left[\frac{v^2}{8}\right]_2^6 = (1-e^4)\left(\frac{9}{2}-\frac{1}{2}\right)$$

$$= \boxed{4-4e^4}$$