This print-out should have 12 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Locate and classify all the local extrema of

$$f(x, y) = x^3 - y^3 - 3xy - 3 .$$

1. local min at $(-1, 1)$,
local max at $(0, 0)$

2. local min at $(0, 0)$,
saddle point at $(-1, 1)$

3. local min at $(-1, 1)$,
saddle point at $(0, 0)$

4. local max at $(-1, 1)$,
saddle point at $(0, 0)$ **correct**

5. local max at $(0, 0)$,
saddle point at $(-1, 1)$

Explanation:

Since f has derivatives everywhere, the critical points occur at the solutions of

$$\nabla f(x, y) = f_x \mathbf{i} + f_y \mathbf{j} = 0 .$$

But $f_x = 0$ when

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0 , \quad i.e., \quad y = x^2 ,$$

while $f_y = 0$ when

$$\frac{\partial f}{\partial y} = -3y^2 - 3x = 0 , \quad i.e., \quad x = -y^2 .$$

Substituting the first into the second yields

$$x = -(x^2)^2 = -x^4 ,$$

which can be re-written as

$$x(1 + x^3) = 0 , \quad i.e., x = 0, \quad -1 .$$

Thus, the critical points are

$$(0, 0), \qquad (-1, 1) ,$$

and to classify these critical points we use the Second Derivative test. Now

$$f_{xx} = 6x , \quad f_{yy} = -6y , \text{ and } \quad f_{xy} = -3 .$$

But then, at $(0, 0)$,

$$A = f_{xx}(0, 0) = 0, \quad B = f_{xy}(0, 0) = -3 ,$$

while

$$C = f_{yy}(0, 0) = 0 .$$

Consequently,

$$D = AC - B^2 = -9 < 0 ,$$

and so there is a

$$\boxed{\text{saddle point at } (0, 0)} .$$

On the other hand, at $(-1, 1)$,

$$A = f_{xx}(-1, 1) = -6, \quad B = f_{xy}(-1, 1) = -3 ,$$

while

$$C = f_{yy}(-1, 1) = -6 .$$

Thus

$$D = AC - B^2 = 27 > 0 ,$$

and so, since $A, C < 0$, there is a

$$\boxed{\text{local maximum at } (-1, 1)} .$$

keywords:

002 10.0 points

Which one of the following properties does the function

$$f(x, y) = x^3 + 2xy^2 - 5x - 4y + 20$$

have?

1. local min value 14 at $(1, 1)$ **correct**

2. saddle point at $(1, 1)$

3. local max value 14 at $(-1, 1)$

4. local max value 14 at $(1, 1)$

5. local min value 14 at $(-1, 1)$

6. saddle point at $(-1, 1)$

Explanation:

First we have to locate and classify the critical points of f. Now after differentiation,

$$f_x = 3x^2 + 2y^2 - 5, \quad f_y = 4xy - 4.$$

Thus the critical points of f are the solutions of the equations

$$(\dagger) \qquad 3x^2 + 2y^2 = 5, \quad xy = 1.$$

One solution of $(\dagger)$ is $(1, 1)$, while $(1, -1)$ is not a solution, so we can concentrate on the critical point $(1, 1)$. But after differentiating once again, we see that

$$f_{xx} = 6x, \quad f_{yy} = 4x, \quad f_{xy} = 4y.$$

At $(1, 1)$, therefore,

$$A = f_{xx}\Big|_{(1,1)} = 6 > 0, \quad B = f_{xy}\Big|_{(1,1)} = 4,$$

and

$$C = f_{yy}\Big|_{(1,1)} = 4 > 0;$$

in particular,

$$AC - B^2 \;=\; 24 - 16 > 0.$$

Thus by the second derivative test, f has a

$$\boxed{\text{local minimum at } (1, 1),}$$

for which

$$\boxed{f(1, 1) \;=\; 14.}$$

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003 10.0 points

Locate and classify the local extremum of f when

$$f(x, y) \;=\; 3x + \frac{y}{3} + \frac{1}{xy} + 1, \quad (x, y > 0).$$

1. local min at $\left(\frac{1}{3}, 3\right)$ **correct**

2. local min at $(3, 3)$

3. local max at $\left(\frac{1}{3}, 3\right)$

4. saddle at $(3, 3)$

5. saddle at $\left(\frac{1}{3}, 3\right)$

6. local max at $(3, 3)$

Explanation:

Differentiating once we see that

$$f_x \;=\; 3 - \frac{1}{x^2 y}, \quad f_y \;=\; \frac{1}{3} - \frac{1}{xy^2}.$$

At a local extremum these first partial derivatives are zero. Thus f has a local extremum at $\left(\frac{1}{3}, 3\right)$.

To classify the local extremum we use the second derivative test. Now

$$f_{xx} = \frac{2}{x^3 y}, \quad f_{xy} = \frac{1}{x^2 y^2}, \quad f_{yy} = \frac{2}{xy^3}.$$

But then

$$A \;=\; f_{xx}\Big|_{\left(\frac{1}{3}, 3\right)} \;=\; 18 > 0,$$

$$C \;=\; f_{yy}\Big|_{\left(\frac{1}{3}, 3\right)} \;=\; \frac{2}{9} > 0,$$

and

$$B \;=\; f_{xy}\Big|_{\left(\frac{1}{3}, 3\right)} \;=\; 1.$$

Thus

$$AC - B^2 \;=\; 3 \;>\; 0.$$

Hence by the second derivative test, f has a

$$\boxed{\text{local min at } \left(\frac{1}{3},\, 3\right)}.$$

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004 10.0 points

Which of the following most correctly describes the behaviour of the graph of the function

$$f(x,\, y) \;=\; 2(x+y)(xy+9)+4.$$

1. saddle-points at $(3,\,-3),\;(-3,\,3)$ **correct**

2. local max at $(3,\,-3),\;(-3,\,3)$

3. saddle-points at $(3,\,3),\;(-3,\,-3)$

4. local max at $(3,\,3),\;(-3,\,-3)$

5. saddle $(3,\,-3),$ local max $(-3,\,3)$

Explanation:

After expansion

$$f(x,\, y) \;=\; 2x^2y + 2xy^2 + 18x + 18y + 4.$$

Thus

$$\frac{\partial f}{\partial x} \;=\; 4xy + 2y^2 + 18,$$

while

$$\frac{\partial f}{\partial y} \;=\; 2x^2 + 4xy + 18.$$

Consequently, the critical points of f occur at the solutions of the equations

$$2(2xy + y^2 + 9) \;=\; 0, \quad 2(x^2 + 2xy + 9) \;=\; 0,$$

i.e., when

$$x^2 + 2xy + 9 \;=\; 0,$$

$$y^2 + 2xy + 9 \;=\; 0\,.$$

After subtraction, we obtain $x^2 = y^2$, so $x = \pm y$. Now

$$x \;=\; y \;\implies\; 3x^2 + 9 \;=\; 0,$$

while

$$x \;=\; -y \;\implies\; -x^2 + 9 \;=\; 0.$$

Hence the critical points of f are

$$(3,\,-3), \qquad (-3,\,3).$$

To determine the behaviour of f we need to compute the values of the second derivatives at these critical points. But

$$\frac{\partial^2 f}{\partial x^2} \;=\; 4y, \;\; \frac{\partial^2 f}{\partial y^2} \;=\; 4x,$$

and

$$\frac{\partial^2 f}{\partial x \partial y} \;=\; 4(x+y).$$

Thus at $(3,\,-3)$

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \;=\; -144 \;<\; 0$$

and at $(-3,\,3)$

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \;=\; -144 \;<\; 0.$$

Hence the graph of f has

$$\boxed{\text{saddle-points at } (3,\,-3),\;(-3,\,3)}.$$

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005 10.0 points

Locate and classify the critical point of

$$f(x, y) \;=\; \ln(xy) + 4y^2 - 2y - 2xy + 4,$$

for $x, y > 0$.

1. saddle-point at $\left(\frac{1}{4},\, 2\right)$

2. saddle-point at $\left(2,\, \frac{1}{4}\right)$ **correct**

3. local minimum at $\left(2,\, \frac{1}{4}\right)$

4. local maximum at $\left(2,\, \frac{1}{4}\right)$

5. local maximum at $\left(\frac{1}{4},\, 2\right)$

6. local minimum at $\left(\frac{1}{4},\, 2\right)$

Explanation:

The critical point of f is the common solution of the equations

$$\frac{\partial f}{\partial x} = \frac{1}{x} - 2y = 0\,,$$

$$\frac{\partial f}{\partial y} = \frac{1}{y} + 8y - 2 - 2x = 0\,.$$

By the first equation, $2x = 1/y$. Using this in the second equation, we see that

$$8y - 2 = 0 \quad i.e.,\ y = \frac{1}{4}\,.$$

So f has a critical point at

$$\left(2,\, \frac{1}{4}\right).$$

Now after differentiation,

$$f_{xx} = -\frac{1}{x^2}, \quad f_{xy} = -2, \quad f_{yy} = 8 - \frac{1}{y^2}\,.$$

Thus at the critical point $\left(2,\, \frac{1}{4}\right)$,

$$A = f_{xx}\Big|_{\left(2,\, \frac{1}{4}\right)} = -\frac{1}{4} < 0, \qquad B = -2\,,$$

$$C = f_{yy}\Big|_{\left(2,\, \frac{1}{4}\right)} = -8 < 0\,,$$

in which case

$$AC - B^2 = -2 < 0\,,$$

Consequently, by the second derivative test f has a

$$\boxed{\text{saddle-point at } \left(2,\, \frac{1}{4}\right)}\,.$$

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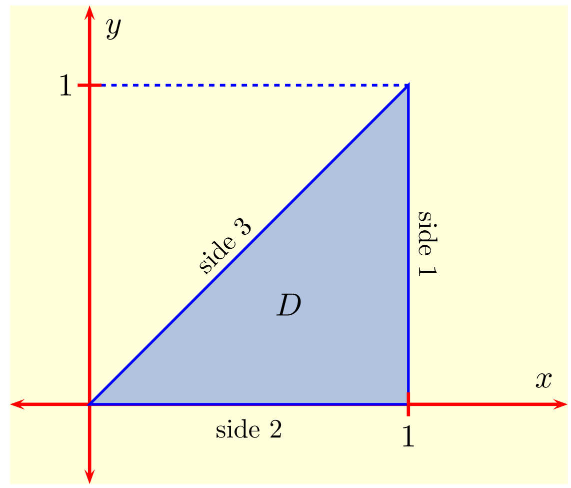
006 10.0 points

Locate the point at which the function

$$f(x,\, y) = x^2 - 2y^2 - x + y$$

has its absolute maximum on the shaded triangular region D shown in



1. on side 1 but not at an end-point **correct**

2. at a critical point inside D

3. on side 2 but not at an end-point

4. on side 3 but not at an end-point

5. at a vertex of D

Explanation:

Now the absolute maximum of

$$f(x,\, y) = x^2 - 2y^2 - x + y$$

on D occurs either at a critical point of f inside D or at a point on the sides of D.

But

$$\frac{\partial f}{\partial x} = 2x - 1\,, \qquad \frac{\partial f}{\partial y} = 1 - 4y\,,$$

so f has only one critical point and it occurs at $(1/2,\, 1/4)$, a point inside D since the graph shows that

$$D = \Big\{(x,\, y) : 0 \le y \le x\,,\ 0 \le x \le 1\Big\}.$$

At this critical point

$$f\Big(\frac{1}{2},\,\frac{1}{4}\Big) \;=\; -\frac{1}{8}\,.$$

We next check the maximum value of f on the sides of D. At the vertices of D,

$$f(0,\,0) \;=\; 0, \quad f(1,\,0) \;=\; 0, \quad f(1,\,1) \;=\; -1\,,$$

while on

(i) side 1: *here* $x = 1$ *and*

$$f(1,\,y) \;=\; y - 2y^2 \;=\; -2\Big(y^2 - \frac{1}{2}y\Big)$$

$$= -2\Big(\Big(y-\frac{1}{4}\Big)^2 - \frac{1}{16}\Big) \;=\; \frac{1}{8} - 2\Big(y - \frac{1}{4}\Big)^2,$$

in which case

$$\max f \;=\; \max_{0<y<1}\Big(\frac{1}{8} - 2\Big(y-\frac{1}{4}\Big)^2\Big)$$

$$= f\Big(1,\,\frac{1}{4}\Big) \;=\; \frac{1}{8}\,,$$

(ii) side 2: *here* $y = 0$ *and*

$$f(x,\,0) \;=\; x^2 - x \;=\; \Big(x - \frac{1}{2}\Big)^2 - \frac{1}{4}\,,$$

in which case

$$\max f \;=\; \max_{0\le x\le 1}\Big(\Big(x-\frac{1}{2}\Big)^2 - \frac{1}{4}\Big)$$

$$= f(0,\,0) \;=\; f(1,\,0) \;=\; 0\,.$$

(iii) side 3: *here* $y = x$ *and*

$$f(x,\,x) \;=\; -x^2\,,$$

in which case

$$\max f \;=\; \max_{0\le x\le 1}(-x^2) \;=\; f(0,\,0) \;=\; 0\,,$$

Consequently, on D

$$\max f \;=\; \max\Big\{-1\,,\,-\frac{1}{8}\,,\,0\,,\,\frac{1}{8}\Big\}$$

$$= \frac{1}{8} \;=\; f\Big(1\,,\,\frac{1}{4}\Big)\,,$$

and this occurs

$\boxed{\text{on side 1 but not at an end-point}}$.

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keywords:

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007 10.0 points

Determine the absolute maximum of

$$f(x,\,y) \;=\; x^2 + y^2 - x - y + 2$$

on the unit disk

$$D \;=\; \{(x,\,y) :\, x^2 + y^2 \le 1\,\}\,.$$

1. absolute max $=\; 4$

2. absolute max $=\; 3$

3. absolute max $=\; 3 + \sqrt{2}$ **correct**

4. absolute max $=\; \dfrac{3}{2}$

5. absolute max $=\; 3 - \sqrt{2}$

Explanation:

The Absolute Extrema occur either at a critical point inside the disk, or on the boundary of the disk.

1. *Inside* D: when

$$f(x,\,y) \;=\; x^2 + y^2 - x - y + 2\,,$$

the critical points occur at the solutions to

$$\frac{\partial f}{\partial x} \;=\; 2x - 1\,, \quad \frac{\partial f}{\partial y} \;=\; 2y - 1 \;=\; 0\,.$$

So $(1/2,\,1/2)$ is the only critical point, and this lies inside the disk. At this critical point

$$f\Big(\frac{1}{2},\,\frac{1}{2}\Big) \;=\; \frac{3}{2}\,.$$

2. *On the boundary of D*: the boundary of D is the unit circle which can be parametrized by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi .$$

Thus the restriction of f to the boundary is the function

$$f(\mathbf{r}(t)) = 3 - \cos t - \sin t .$$

Finding the absolute extrema of $f(\mathbf{r}(t))$ on $[0, 2\pi]$ is a single variable problem: they will occur either at a critical point in $(0, 2\pi)$, or at the endpoints $t = 0, 2\pi$. But

$$\frac{df}{dt}(\mathbf{r}(t)) = \sin t - \cos t ,$$

so the critical points of $f(\mathbf{r}(t))$ on $(0, 2\pi)$ occur when $\tan t = 1$, *i.e.* at $t = \pi/4, 5\pi/4$. But

$$f\Big(\mathbf{r}\Big(\frac{\pi}{4}\Big)\Big) = 3 - \sqrt{2}, \quad f\Big(\mathbf{r}\Big(\frac{5\pi}{4}\Big)\Big) = 3 + \sqrt{2},$$

while

$$f(\mathbf{r}(0)) = 2 = f(\mathbf{r}(2\pi)) .$$

Consequently, on D, $f(x, y)$ has

$$\boxed{\text{Absolute max} = 3 + \sqrt{2}}\,.$$

keywords:

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008 10.0 points

Determine the absolute maximum value of

$$f(x, y) = -2\cos x \cos y$$

on the square $0 \le x, y \le \pi$.

1. absolute max $= 2\pi$

2. absolute max $= 2$ **correct**

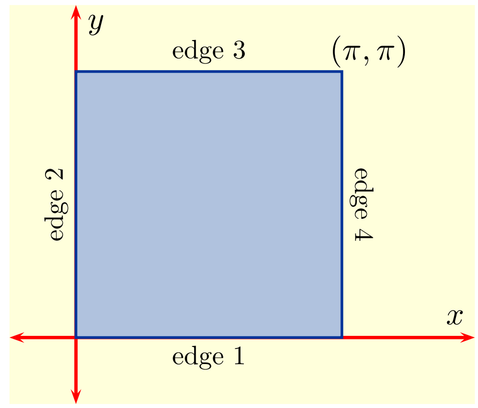
3. absolute max $= -2\pi$

4. absolute max $= 0$

5. absolute max $= -2$

Explanation:

The absolute extrema of $f(x, y)$ occur either at a critical point inside the square, or on the boundary of the square as shown in



Now

$$\frac{\partial f}{\partial x} = 2\sin x \cos y , \quad \frac{\partial f}{\partial y} = 2\cos x \sin y .$$

Thus the critical points are at the solutions of

$$2\sin x \cos y = 0 , \quad 2\cos x \sin y = 0 .$$

Since

$$0 < x, y < \pi \implies \sin x, \sin y > 0 ,$$

the only critical points inside the square occur when

$$\cos x = 0, \quad \cos y = 0 ,$$

i.e., at $(\pi/2, \pi/2)$, and at this critical point

$$f\Big(\frac{\pi}{2}, \frac{\pi}{2}\Big) = 0 .$$

We look at the edges separately:

(i) on Edge 1, $f(x, 0) = -2\cos x$ so

$$\max_{0 \le x \le \pi} f(x, 0) = 2 .$$

(i) on Edge 2, $f(0, y) = -2\cos y$ so

$$\max_{0 \le y \le \pi} f(0, y) = 2 .$$

(i) on Edge 3, $f(x, \pi) = 2\cos x$ so

$$\max_{0 \le x \le \pi} f(x, \pi) = 2 .$$

(i) on Edge 4, $f(\pi, y) = 2\cos x$ so

$$\max_{0 \le y \le \pi} f(\pi, y) = 2.$$

Consequently, on the square, f has

$$\boxed{\text{absolute maximum} = 2}.$$

keywords:

009 10.0 points

Use the method of Lagrange multipliers to minimize

$$f(x, y) = \sqrt{3x^2 + y^2}$$

subject to the constraint

$$x + y = 1.$$

1. min value $= \frac{1}{2}\sqrt{3}$ **correct**

2. min value $= \sqrt{3}$

3. min value $= 1$

4. no min value exists

5. min value $= \frac{1}{2}$

Explanation:

Set

$$g(x, y) = x + y - 1.$$

Then by the method of Lagrange multipliers the extreme values of f under the constraint $g = 0$ occur at the solutions of

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0.$$

But when

$$f(x, y) = \sqrt{3x^2 + y^2}$$

we see that

$$\nabla f = \frac{3x}{\sqrt{3x^2 + y^2}}\,\mathbf{i} + \frac{y}{\sqrt{3x^2 + y^2}}\,\mathbf{j}.$$

Since

$$\nabla g = \mathbf{i} + \mathbf{j},$$

the equation $\nabla f = \lambda \nabla g$ thus becomes

$$\frac{1}{\sqrt{3x^2 + y^2}}\Big(3x\,\mathbf{i} + y\,\mathbf{j}\Big) = \lambda(\mathbf{i} + \mathbf{j}).$$

After comparing coefficients this reduces to the pair of equations

$$\lambda = \frac{3x}{\sqrt{3x^2 + y^2}}, \quad \lambda = \frac{y}{\sqrt{3x^2 + y^2}},$$

i.e., $y = 3x$. But we still have the constraint equation

$$g(x, y) = x + y - 1 = 0.$$

Substituting $y = 3x$ gives

$$x + 3x - 1 = 4x - 1 = 0.$$

Consequently, the only solution of

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0,$$

occurs at

$$(x, y) = \Big(\frac{1}{4}, \frac{3}{4}\Big),$$

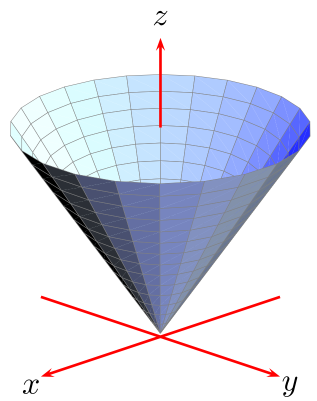
and at this point

$$f\Big(\frac{1}{4}, \frac{3}{4}\Big) = \frac{1}{2}\sqrt{3}.$$

But is this a maximum or a minimum value? We can decide algebraically or graphically, the best choice depending on f and g. Let's do it graphically because the graphs in 3-space of f and $g = 0$ are easy to describe. Indeed, the graph of

$$z = f(x, y) = \sqrt{3x^2 + y^2}$$

is a cone



while the graph of

$$g(x, y) = x + y - 1 = 0$$

in 3-space is a vertical plane. Minimizing f on $g = 0$ corresponds to finding the height of the lowest point on the *intersection of this vertical plane with the cone*. Since the intersection will be half of a hyperbola opening upwards, we see that f has a minimum on $g = 0$ and this

$$\boxed{\text{min value} = \frac{1}{2}\sqrt{3}}.$$

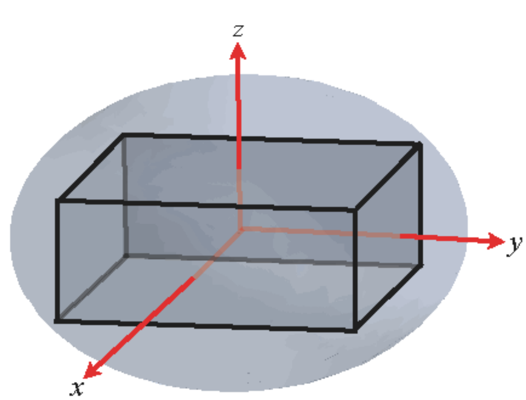
keywords: Lagrange multipliers, optimization, gradient, quadratic function, constraint, constrained optimization,

010 10.0 points

A rectangular box with edges parallel to the axes is inscribed in the ellipsoid

$$3x^2 + y^2 + z^2 = 9$$

similar to the one shown in



Use Lagrange multipliers to determine the maximum volume of this box.

Note: *all 8 vertices of the box will lie on the ellipsoid when the volume is maximized.*

1. volume = 24 cu. units **correct**
2. volume = 72 cu. units
3. volume = 36 cu. units
4. volume = 12 cu. units
5. volume = 18 cu. units

Explanation:

The rectangular box will be centered at the origin, so if the corner lying in the first octant is $P(x, y, z)$, then the box will have sidelengths $2x$, $2y$ and $2z$. Thus the box will have volume

$$V = 8xyz\,.$$

But P lies on the ellipsoid

$$3x^2 + y^2 + z^2 = 9\,,$$

so we have to maximize V subject to this last restriction on x, y, and z. To use Lagrange multipliers, set

$$F(x, y, z, \lambda) = 8xyz - \lambda(3x^2 + y^2 + z^2 - 9)\,.$$

Then F will have a critical point when

$$8yz = 6\lambda x\,, \quad 8zx = 2\lambda y\,, \quad 8xy = 2\lambda z\,,$$

and

$$3x^2 + y^2 + z^2 = 9\,.$$

From the first set of equations we see that

$$8xyz = 6\lambda x^2, \;\; 8yzx = 2\lambda y^2, \;\; 8zxy = 2\lambda z^2\,,$$

in which case

$$6x^2 = 2y^2 = 2z^2\,.$$

From the fact that P lies on the ellipsoid, therefore, it thus follows that F has a critical point at

$$x^2 = 1, \quad y^2 = 3, \quad z^2 = 3.$$

At this critical point V will be maximized. Consequently, the box has

$$\boxed{\text{maximum volume} = 24 \text{ cu. units}}.$$

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011 10.0 points

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 2x^2y$, subject to the constraint

$$2x^2 + y^2 = 300.$$

1. $f_{max} = 2000$, $f_{min} = 0$

2. $f_{max} = 1000$, $f_{min} = -1000$

3. $f_{max} = 500$, $f_{min} = -500$

4. $f_{max} = 2000$, $f_{min} = -2000$ **correct**

5. $f_{max} = 0$, $f_{min} = -1000$

Explanation:

The Lagrange condition is

$$\langle 4xy, 2x^2 \rangle = \lambda \langle 4x, 2y \rangle$$

This yields

$$4xy = \lambda \cdot 4x, \quad 2x^2 = \lambda \cdot 2y.$$

Solving the first equation implies that either $x = 0$ or $\lambda = y$.

If $x = 0$, then from the constraint equation:

$$2(0)^2 + y^2 = 300$$
$$y = \pm\sqrt{300} = \pm 10\sqrt{3}$$

If $\lambda = y$, then from the second Lagrange equation:

$$2x^2 = 2y^2$$

and from the constraint equation:

$$2y^2 + y^2 = 300$$
$$y = \pm 10$$

We also need to find x:

$$2x^2 = 2y^2$$
$$2x^2 = 2(10)^2$$
$$x = \pm\sqrt{100}$$

There are six candidates for the points where the extreme values of f occur: $\left(0, \pm 10\sqrt{3}\right)$ and $\left(\pm\sqrt{100}, \pm 10\right)$. We just need to plug all four into f to check values, and we get maximum and minimum values of 2000 and -2000.

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012 10.0 points

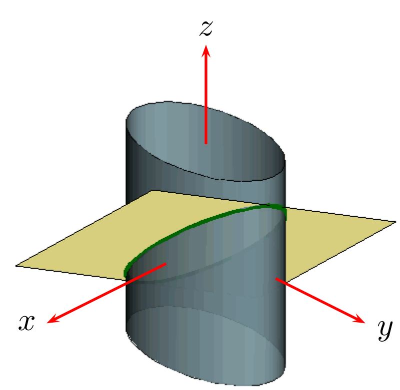
Finding the minimum value of

$$f(x, y) = x + 2y - 1$$

subject to the constraint

$$g(x, y) = 3x^2 + 4y^2 - 3 = 0$$

is equivalent to finding the height of the lowest point on the curve of intersection of the graphs of f and g shown in



Use Lagrange multipliers to determine this minimum value.

1. min value $= -4$

2. min value $= -2$

3. min value $= -1$

4. min value $= -3$ **correct**

5. min value $= -5$

Explanation:

The extreme values occur at solutions of

$$(\nabla f)(x, y) = \lambda(\nabla g)(x, y).$$

Now

$$(\nabla f)(x, y) = \langle 1, 2 \rangle,$$

while

$$(\nabla g)(x, y) = \langle 6x, 8y \rangle.$$

Thus

$$1 = 6\lambda x, \qquad 2 = 8\lambda y,$$

and so

$$\lambda = \frac{1}{6x} = \frac{1}{4y}, \quad i.e., \ y = \frac{3}{2}x.$$

But

$$g\left(x, \frac{3}{2}x\right) = 12x^2 - 3 = 0, \quad i.e., \ x = \pm\frac{1}{2}.$$

Consequently, the extreme points are

$$\left(\frac{1}{2}, \frac{3}{4}\right), \quad \left(-\frac{1}{2}, -\frac{3}{4}\right).$$

Since

$$f\left(\frac{1}{2}, \frac{3}{4}\right) = 1, \quad f\left(-\frac{1}{2}, -\frac{3}{4}\right) = -3,$$

we thus see that

$$\boxed{\text{min value} = -3}.$$

keywords: