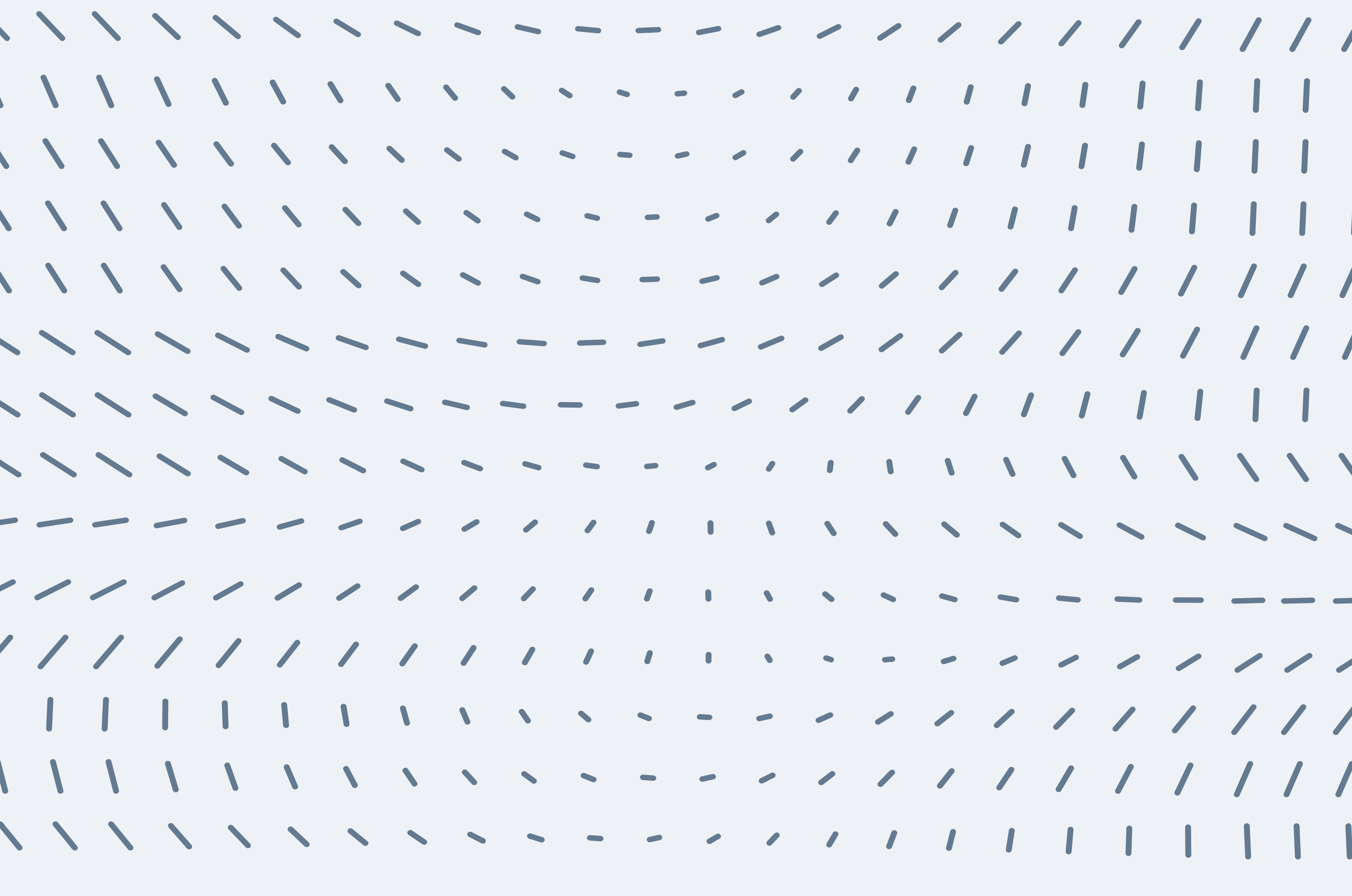
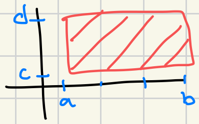


M 427L Vector Calculus

June 6th, 2025

$$\int_c^d\int_a^b f(x,y)\,dx\,dy$$



Not the real world, only part!

Reviewing Vectors:

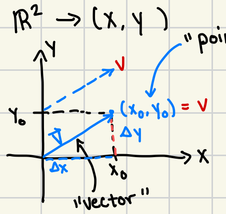
$f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\mathbb{R}^n = \{(a_1,\dots,a_n): a_i \text{ are real numbers}\}$

$f(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n))$

$\mathbb{R}^n \to (x_1,\dots,x_n)$ "Vectors" "points"

$\mathbb{R}^2 \to (x,y)$



Points Arithmetic

$(x_1,y_1)+(x_2,y_2) = (x_1+x_2,\ y_1+y_2)$ Scalar Addition

$\lambda(x,y) = (\lambda x, \lambda y)$ Scalar Multiplication

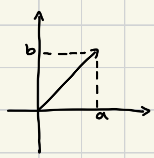
Example:

$\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-5\\-7\end{bmatrix} \to (-5,7)$

Point = TIP − TAIL (Coords)

- $V = (a,b)$

Norm of $v = \|v\| = \sqrt{a^2+b^2}$

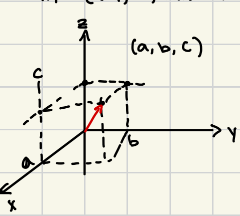


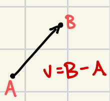
- $V = (a_1,b_1)$ $\quad W = (a_2,b_2)$

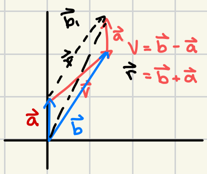
Dot Product: $V\cdot W = a_1\cdot a_2 + b_1\cdot b_2$

June 9th, 2025

$\mathbb{R}^n = \{(x_1,\dots,x_n): x_i \text{ is a real number}\}$





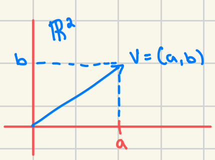


Norm:

$v = (x_1,\dots,x_n)$

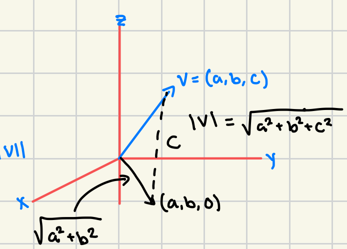
$\|v\| = \sqrt{\sum_{i=0}^{n} x_n^2}$

Example #1: $|(-1,2,3)| = \sqrt{(-1)^2+2^2+3^2}$
$= \sqrt{1+4+9}$
$= \sqrt{14}$



hyp $= \sqrt{a^2+b^2} = \|v\|$

length of the vector



Unit vector

If $\vec{v} \neq 0$, then $|\vec{v}| \neq 0$ and

$\hat{v} = \dfrac{\vec{v}}{|\vec{v}|}$

$|\hat{v}| = \dfrac{|v|}{|v|} = 1$

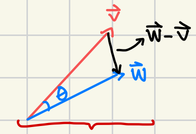
$\boxed{\begin{array}{l}\lambda(a,b) = (\lambda a, \lambda b)\\ \sqrt{a^2+b^2}\\ \sqrt{\lambda^2 a^2 + \lambda^2 b^2} = |\lambda|\sqrt{a^2+b^2}\end{array}}$

We say two vectors $\vec{v}, \vec{w}$ have the same direction if $\hat{v} = \hat{w}$.

Example #2: $\vec{v} = (1,2,3)$ and $\vec{w} = (-1,2,1)$

$\hat{v} = \dfrac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \dfrac{(1,2,3)}{\sqrt{14}} = \left(1/\sqrt{14},\ 2/\sqrt{14},\ 3/\sqrt{14}\right)$

$\hat{w} = \dfrac{(-1,2,1)}{\sqrt{(-1)^2+2^2+1^2}} = \dfrac{(-1,2,1)}{\sqrt{6}} = \dfrac{1}{\sqrt{6}}(-1,2,1)$



$|w-v|^2 = |v|^2 + |w|^2 - 2|v||w|\cos\theta$

"Vector = $\underbrace{\text{direction}}_{\hat{v}}$ + $\underbrace{\text{magnitude}}_{|\vec{v}|}$" $\quad \hat{v} = \dfrac{v}{|v|}$

$\boxed{\to \vec{v} = \hat{v}\cdot|v|}$

Way 2: $|w-v|^2 = (w-v)\cdot(w-v)$
$= |w|^2 - 2\,w\cdot v + |v|^2$

$\to v\cdot w = |v||w|\cos\theta$

$\to \cos\theta = \dfrac{v\cdot w}{|v||w|}$

Dot Product:

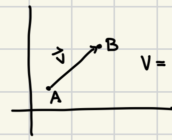
$(a_1,\dots,a_n)\cdot(b_1,\dots,b_n) = a_1b_1 + a_2b_2 + \dots + a_nb_n$

NOTE! $\vec{v} = (a_1,\dots,a_n)$: $\vec{v}\cdot\vec{v} = a_1^2+\dots+a_n^2$

$\sqrt{v^2} = |v|$

$v\cdot(w_1+w_2) = v\cdot w_1 + v\cdot w_2 \to (\lambda v)\cdot w = \lambda(v\cdot w)$

$v\cdot w = w\cdot v$



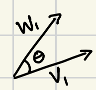
$V = \vec{AB} = B - A$

Discussion Section

June 10th, 2025.

$W_1 = (2,-1,0,2)$ $\qquad W_1\cdot V_1 = \sum_{i=1}^{n} a_n\cdot b_n$ where $n=4$

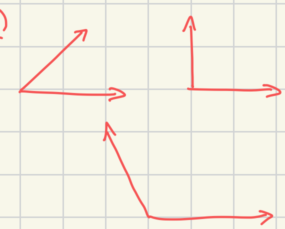
$V_1 = (1,2,-1,2)$ $\qquad = 4$



$\to \cos\theta = \dfrac{V\cdot W}{|V||W|}$
$= \dfrac{4}{3\sqrt{10}}$

$\text{proj}\ W_1 = V_1\cdot\hat{V}_2 = |V_1|\cos\theta$
$= \cancel{3}\left(\dfrac{4}{\cancel{3}\sqrt{10}}\right) = \boxed{\dfrac{4}{\sqrt{10}}}$

$\hat{V}_2 = \dfrac{(-1,2,-1,2)}{\sqrt{10}}$



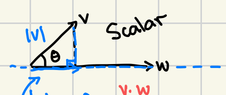
Dot Products:

$(a_1,\dots,a_n)\cdot(b_1,\dots,b_n) = \sum_{i=1}^{n} a_n b_n$

$v\cdot v = v^2 = \sum_{i=1}^{n} a_n^2 = |v|^2$

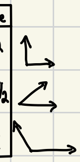
if $v = (a_1,\dots,a_n)$

$v\cdot w = |v||w|\cos\theta$



$|v|\cos\theta = \dfrac{v\cdot w}{|w|}$

V·W	Angle
0	$\pi/2$
+	$<\pi/2$
−	$>\frac{\pi}{2}$

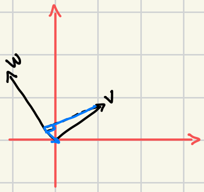


$|v|\cos\theta\,\underbrace{\hat{w}}_{\frac{w}{|w|}} = \dfrac{v\cdot w}{|w|^2}\cdot w$ (Vector Projection)

Example #1: Find the projection of $v: (2,1)$ onto $w = (-1,3)$

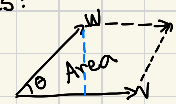
Scalar: $\frac{v \cdot w}{|w|} = \frac{1}{\sqrt{10}}$ $|w| = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}$

Vector: $\underbrace{\frac{v \cdot w}{|w|} \cdot \frac{w}{|w|}}_{|w|^2} = \frac{1}{10}(-1,3) \equiv \left(\frac{-1}{10}, \frac{3}{10}\right)$ (Length, Direction)



Areas:

SOH
CAH
TOA



$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Area $= |v||w| \sin\theta$

$A^2 = |v|^2 |w|^2 \sin^2\theta$

$= |v^2||w^2|(1 - \cos^2\theta)$

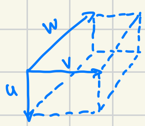
$\equiv |v^2||w^2| - |v^2||w^2| \cos^2\theta$

$A^2 = v^2 w^2 - (v \cdot w)^2 = \begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}$

$A_{obj} = \sqrt{\begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}}$

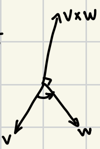
"Gram determinant"

3D:



$\text{Vol} = \sqrt{\begin{vmatrix} u \cdot v & u \cdot u & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}}$

$\mathbb{R}^3$: Cross Product



$i = (1,0,0)$
$j = (0,1,0)$
$k = (0,0,1)$

$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$= i \det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j \det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$

$\equiv \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$

Example: $V_1 = (1,1,1)$ $V_1 \cdot V_2 = -1 + 1 + 2 = 2$

$V_2 = (-1,1,2)$ $V_1 \cdot V_3 = 0 + 1 + 0 = 1$

$V_3 = (0,1,0)$ $V_2 \cdot V_3 = 0 + 1 + 0 = 1$

$6(1) - (1)(1) = 5$
$2(1) - (1)(1) = 1$
$2(1) - 6(1) = -4$

$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{vmatrix} = a\left(\begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix}\right) - b\left(\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right) + c\left(\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}\right)$

$= 3 \det\begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix} - 2 \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + 1 \det\begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix}$

$3 \cdot 5 - 2 \cdot 1 + (-4)$

$= 15 - 2 - 4$

$= 15 - 6$

$= \sqrt{9}$

Vol $= \boxed{3}$

Discussion Section

June 12th, 2025

$v = (-1, 1, 0, 1, -1)$

$w = (2, -1, -1, 0, 1)$

$1+1+1+1$
$4+1+1+1$

$2 \cdot (-1) + 1(-1) + 0(-1) + 0(1) + 1(-1)$
$= -2 - 1 - 1 = -4$

$\begin{bmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}$

$\det\left(\begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}\right) = 4(7) - (-4(-4))$

$28 - 16 = \boxed{12} = A^2$

$\sqrt{12} = A$

$A = (1,1,1)$

$B = (-1,2,1)$ $\vec{AB} = (-2,1,0) = \vec{B} - \vec{A}$

$C = (3,1,-1)$ $\vec{AC} = (2,0,-2) = \vec{C} - \vec{A}$

$\vec{AB} \times \vec{AC} =$

$\left\| \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \right\| = \hat{i} \det\left(\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\right) - \hat{j} \det\left(\begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}\right) + \hat{k} \det\left(\begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}\right)$

$-2 + 0 = -2$ $4 - 0 = 4$ -2

$-2\hat{i} - 4\hat{j} - 2\hat{k}$

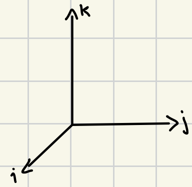
$\equiv (-2, -4, -2)$

$= \sqrt{24} = |(-2,-4,-2)|$

$= \frac{1}{2} \cdot \sqrt{24} = \sqrt{6}$

Lecture

June 13th



$i \times j = k$ $j \times i = -k$

$\vec{i} \times \vec{i} =$

$v \times v = -v \times v = \vec{0}$

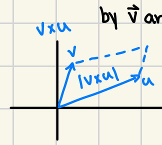
$v \times (w + u) = v \times w + v \times u$

$v \times (\lambda u) = (\lambda v) \times u \equiv \lambda (v \times u)$

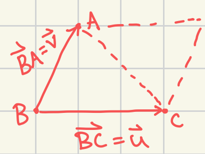
$(1,2,3) \times (2,3,4) = (i + 2j + 3k) \times (2i + 3j + 4k)$

$(v_1, v_2, v_3) \times (u_1, u_2, u_3) = \det\left(\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix}\right)$

Application: $|v \times u|$ = area of the parallelogram spanned by $\vec{v}$ and $\vec{u}$



Example: $A = (1,0,1)$, $B = (1,0,0)$, $C = (-1,1,1)$



$V = A - B = (0,0,1)$

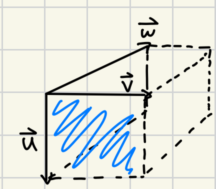
$u = C - B = (-2,1,1)$

$v \times u = \det\left(\begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}\right) = i \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) - j \det\left(\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\right) + k \det\left(\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}\right)$

$= \langle -1, -2, 0 \rangle$

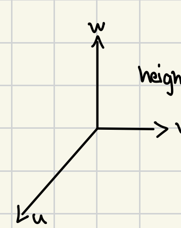
$A_{triangle} = \frac{1}{2} \text{norm}(-1, -2, 0)$

$= \frac{\sqrt{5}}{2}$

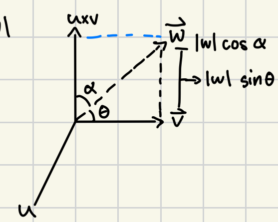


$V = |v \times w| |w| \cos\alpha$

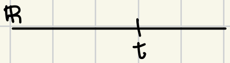
$= (u \times v) \cdot w$

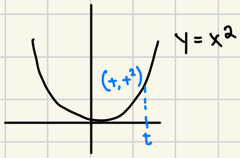


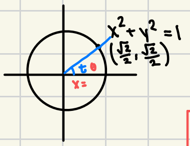
height = $|w|$



Curve:







$C: [a,b] \longrightarrow \mathbb{R}^3$

$$\boxed{\begin{array}{l}\text{Example: } y=x^2: \ C(x)=(x, x^2) \\ \text{Example: } x^2+y^2=1: \ C(\theta)=(\cos\theta, \sin\theta)\end{array}}$$

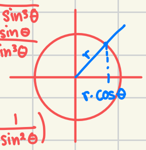
Example: $x^2+y^2=y^3 \longmapsto r^2=r^3\sin^3\theta$

$1 = r\sin^3\theta$

$\frac{1}{\sin^3\theta} = r$

$x = r\cos\theta = \frac{\cos\theta}{\sin^3\theta}$

$y = r\sin\theta = \frac{\sin\theta}{\sin^3\theta} = \frac{1}{\sin^2\theta}$



$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

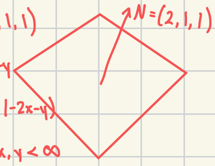
$C(\theta) = \left(\frac{\cos\theta}{\sin^3\theta}, \frac{1}{\sin^2\theta}\right)$

Example: $2x+y+z=1$

$N=(2,1,1)$

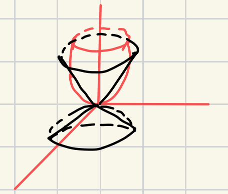
$z = 1-2x-y$

$\Phi(x,y) = (x, y, 1-2x-y)$, $-\infty < x, y < \infty$



Example: $z = x^2+y^2$

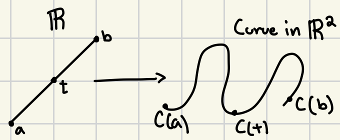
$z^2 = x^2+y^2$



Lecture

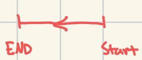
June 16, 2025

$C: [a,b] \longmapsto \mathbb{R}^n$



Example: $C(t) = t \quad C: [0,1] \to \mathbb{R}$

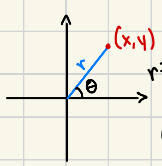
$C(t) = -t \quad C: [-1, 0] \to \mathbb{R}$



$C(-1) = 1$

$C(0) = 0$

Polar:



$r^2 = x^2+y^2$

$r \in [0, \infty)$

$\theta \in [0, 2\pi]$

$x = r\cos\theta$

$y = r\sin\theta$

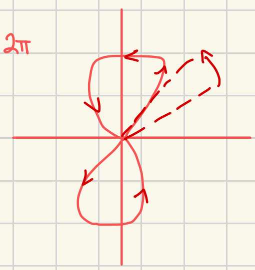
Example: $y^2 = x^4+y^4$ $\quad 0 \le \theta \le 2\pi$

$r^2\sin^2\theta = r^4\cos^4\theta + r^4\sin^4\theta$

$r^2\sin^2\theta = r^4(\cos^4\theta + \sin^4\theta)$

$\sin^2\theta = r^2(\cos^4\theta + \sin^4\theta)$

$\frac{\sin^2\theta}{\cos^4\theta+\sin^4\theta} = r^2 \longrightarrow r = \frac{|\sin\theta|}{\sqrt{\cos^4\theta+\sin^4\theta}}$

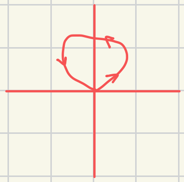


$C(\theta) = \left(\frac{|\sin\theta|\cos\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}, \frac{|\sin\theta|\sin\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}\right) \quad \theta \in [0, 2\pi]$

Example: $y^3 = x^4+y^4$

$r^3\sin^3\theta = r^4\cos^4\theta + r^4\sin^4\theta$

$r = \frac{\sin^3\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$



$c(\theta) = \left(\frac{\sin^3\theta\cos\theta}{r^4\cos^4\theta+r^4\sin^4\theta}, \frac{\sin^4\theta}{r^4\cos^4\theta+r^4\sin^4\theta}\right)$

No problem restriction

No algebra restriction

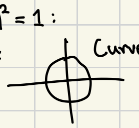
Polar restriction $\Rightarrow r \ge 0$ $\quad \sin^3\theta \ge 0 \to \theta \in [0, \pi]$

$0 \le \theta \le \pi$

Surface:

$x^2+y^2=1$:

$\mathbb{R}^2$:



$\mathbb{R}^3$: