Quiz 08/07

jueves, 7 de agosto de 2025 15:09

Let's consider the following vector field:

$$F=\left(\frac{y}{x^2+y^2},\ 2y-\frac{x}{x^2+y^2}\right)$$

a) Let $c(t) = (\cos(t), \sin(t))$ with $t \in [0, 2\pi]$ be standard parametrization for the circle of radius $r=1$ with a counterclockwise rotation. Using the definition of a line integral, compute:

$$\int_C F\cdot dr.$$

Why does that show that F is not conservative?

If $c(t)=(\cos t, \sin t)$ for $t\in[0,2\pi]$

and $c'(t)=(-\sin t, \cos t)$, $x^2+y^2=1$

then:

$$F(c(t))=\left(\frac{y}{1}, 2y-\frac{x}{1}\right) = (\sin t, 2\sin t-\cos t)$$

$$F(c(t))\cdot c'(t) = (\sin t, 2\sin t-\cos t)\cdot(-\sin t, \cos t)$$

$$-\sin^2 t + 2\sin t\cos t - \cos^2 t = -1(\sin^2 t - 2\sin t\cos t + \cos^2 t)$$

$$= -1 + 2\sin t\cos t$$

$$\int_C F\cdot dr = \int_0^{2\pi} \underbrace{(-1}_{-2\pi} + \underbrace{2\sin t\cos t}_{2\int_0^{2\pi}\sin t\cos t\,dt})\,dt$$

let $u=\sin t$
$du=\cos t\,dt$

$$\int u\,du \longrightarrow \tfrac{1}{2}u^2 \xrightarrow{t} \cancel{\left[\tfrac{1}{2}\sin^2 t\right]_0^{2\pi}} \to 0$$

$$\boxed{\int_C F\cdot dr = \therefore -2\pi \implies -2\pi \neq 0 \quad \therefore F \text{ cannot be conservative}}$$

b) What is the gradient of the function

$$f=y^2+\arctan(x/y)$$

Why can't we say that f is a potential for vector field F?

$$\frac{\partial f}{\partial x}=\frac{1}{1+(x/y)^2}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}=\frac{1}{1+(x/y)^2}\cdot\left(\frac{-x}{y^2}\right)=\frac{-x}{x^2+y^2}+2y$$

$$\therefore \nabla f=\left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}+2y\right)$$

$\arctan(x/y)$ is not single-valued @ $(0,0)$

and is not definied @ $y=0$.

c) Let $c(t) = (\cos(t), \sin(t))$ with $t\in[\pi/4, \pi/3]$ be a parametrization for the portion of the unit circle in the 1st quadrant between the points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(1/2, \sqrt{3}/2)$. Use the definition of a line integral to compute:

$$\int_C F\cdot dr.$$

Since $F(c(t))\cdot c'(t) = -1+2\sin t\cos t = -1+\sin(2t)$

let $u=\sin t$
$du = \cos t\,dt$

$$\therefore \int_C F\cdot dr = \int_{\pi/4}^{\pi/3} -1 + \sin(2t)\,dt$$

$$2\int_{\pi/4}^{\pi/3} \sin t\cos t\,dt$$

$$\left(-\frac{\pi}{3}\right)-\left(-\frac{\pi}{4}\right) = -\frac{4\cdot\pi}{4\cdot 3}+\frac{\pi\cdot 3}{4\cdot 3}$$
$$= -\frac{4\pi}{12}+\frac{3\pi}{12}$$
$$= -\frac{\pi}{12}$$

$$\int u\,du \xrightarrow{\cancel{2}} \left[\tfrac{1}{\cancel{2}}\sin^2 t\right]_{\pi/4}^{\pi/3}$$

$$\left[\sin^2 t\right]_{\pi/4}^{\pi/3} = (3/4 - 2/4) = 1/4$$

$$\boxed{\therefore \int_C F\cdot dr = -\frac{\pi}{12}+1/4}$$

d) Now compute difference

$$\Delta f = f(1/2, \sqrt{3}/2) - f(\sqrt{2}/2, \sqrt{2}/2)$$

$$f(1/2, \sqrt{3}/2) = \left(\frac{\sqrt{3}}{2}\right)^2 + \arctan\left(\frac{(1/2)}{(\sqrt{3}/2)}\right)$$

$$\frac{3}{4}+\arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{3}{4}+\frac{\pi}{6}}$$

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 + \arctan\left(\frac{\sqrt{2}/2}{\sqrt{2}/2}\right)$$

$$= \frac{1}{2}+\arctan(1) = \frac{1}{2}+\frac{\pi}{4}$$

$$\therefore \Delta f = \frac{3}{4}+\frac{\pi}{6}-\frac{1}{2}-\frac{\pi}{4}$$

$$\frac{1}{4}+\frac{\pi}{6}-\frac{\pi}{4} = \frac{1-\pi}{4}+\frac{\pi}{6} = \frac{6-6\pi}{24}+\frac{4\pi}{24} = \frac{6-2\pi}{24} = \frac{3-\pi}{12} \equiv \frac{1}{4}-\frac{\pi}{4}$$

e) Let's consider the portion of the unit circle between the points $(\sqrt{2}/2, -\sqrt{2}/2)$ and $(\sqrt{2}/2, \sqrt{2}/2)$, which is to say consider the same parametrization $c(t) = (\cos(t), \sin(t))$ but now with $t\in[-\pi/4, \pi/4]$. Compute the integral:

$$\int_C F\cdot dr$$

and check that this time the value does NOT agree with the difference

$$\Delta f = f(\sqrt{2}/2, \sqrt{2}/2) - f(\sqrt{2}/2, -\sqrt{2}/2).$$

If $F(c(t))\cdot c'(t) = -1+\sin(2t)$

$$\int_{-\pi/4}^{\pi/4} -1 + \sin(2t)\,dt \qquad \int_{-\pi/4}^{\pi/4} \sin 2t = \cancel{\left[\tfrac{1}{2}\cos(2t)\right]_{-\pi/4}^{\pi/4}} \to 0$$

$$\left(-\frac{\pi}{4}\right)-\left(-\left(-\frac{\pi}{4}\right)\right)$$
$$-\frac{\pi}{4}-\frac{\pi}{4}$$
$$=-\frac{\pi}{2}$$

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 1/2+\frac{\pi}{4}$$

$$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(\frac{-\sqrt{2}}{2}\right)^2 + \arctan\left(\frac{\sqrt{2}}{2}\cdot\frac{2}{-\sqrt{2}}\right)$$
$$= 1/2 - \frac{\pi}{4}$$

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) - f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \cancel{1/2}+\frac{\pi}{4}-\cancel{1/2}+\frac{\pi}{4}$$
$$= \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\therefore \int_C F\cdot dr \neq \Delta f \text{ since } -\frac{\pi}{2}\neq\frac{\pi}{2}$$

f) What's fundamentally different about the path in (e) that doesn't allow you to "cheat" with Stokes' theorem like in (c) and (d)?

1. On the continous branch of $\arctan(x/y)$, it can be seen that $F=\nabla f$ locally and the line integral = endpoint difference
2. $\int_C F\cdot dr \neq \Delta f$

g) Based on your answer to (f) and without doing any computation, do you think

$$\int_C F\cdot dr = \Delta f$$

when C is the portion of the unit circle between the points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2)$?

Yes, since the arc sits in $y>0$ such that $F=\nabla f$ and $\int_C F\cdot dr = \Delta f$ which would result both sides $= -\frac{\pi}{2}$