This print-out should have 8 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Evaluate the integral

$$I = \int_C (2e^y\,dx - 4ye^x\,dy)$$

when C is the parabola parametrized by

$$\mathbf{c}(t) = (t^2,\ t)\,, \quad 0 \le t \le 1\,.$$

1. $I = 6 - 2e$ **correct**

2. $I = 6 + 2e$

3. $I = 3 - 2e$

4. $I = 3 + 2e$

5. $I = 3 - 4e$

6. $I = 6 - 4e$

Explanation:

When C is parametrized by

$$\mathbf{c}(t) = (x(t),\, y(t)) = (t^2,\ t)\,,$$

then

$$\frac{dx}{dt} = 2t\,, \qquad \frac{dy}{dt} = 1\,.$$

Thus on C,

$$2e^y\,dx = 4te^t\,dt, \qquad 4ye^x\,dy = 4te^{t^2}\,dt\,,$$

and so

$$I = \int_0^1 (4te^t - 4te^{t^2})\,dt = I_1 + I_2\,.$$

To evaluate I_1 we integrate by parts:

$$I_1 = 4\int_0^1 te^t\,dt = 4\Big[te^t\Big]_0^1 - 4\int_0^1 e^t\,dt$$

$$= 4\Big[te^t - e^t\Big]_0^1 = 4\,.$$

On the other hand, to evaluate I_2 we use the substitution $u = t^2$. For then,

$$I_2 = -2\int_0^1 e^u\,du = -2e + 2\,.$$

Consequently,

$$\boxed{I = 4 - 2e + 2 = 6 - 2e}\,.$$

002 10.0 points

What is the work done by the magnetic force field

$$\mathbf{B} = \mathbf{i} + x\,\mathbf{j} - 2y\,\mathbf{k}$$

in $\mathbb{R}^3$ in moving a particle from $(1,\,1,\,0)$ to $(e^4,\,13,\,4)$ along a path C parametrized by

$$\mathbf{r}(t) = e^{t^2}\,\mathbf{i} + (3t^2+1)\,\mathbf{j} + 2t\,\mathbf{k}?$$

1. work done $= 4e^4 - 44$ **correct**

2. work done $= 2e^4 - 40$

3. work done $= 4e^4 - 46$

4. work done $= 2e^4 - 48$

5. work done $= 4e^4 - 42$

6. work done $= 2e^4 - 42$

Explanation:

Since

$$(1,\,1,\,0) = \mathbf{r}(0)\,, \quad (e^4,\,13,\,4) = \mathbf{r}(2)\,,$$

the work done by the magnetic field $\mathbf{B}$ is given by the line integral

$$I = \int_C \mathbf{B}\cdot d\mathbf{s} = \int_0^2 \mathbf{B}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

Taking the derivative we find

$$\mathbf{r}'(t) = 2te^{t^2}\,\mathbf{i} + 6t\,\mathbf{j} + 2\,\mathbf{k}\,.$$

Plugging in the values of $x(t)$, $y(t)$, $z(t)$ coming from the parametrization $\mathbf{r}(t)$ we get

$$\mathbf{B}(\mathbf{r}(t)) \;=\; \mathbf{i} + e^{t^2}\,\mathbf{j} - 2(3t^2+1)\,\mathbf{k}\,.$$

Thus

$$\mathbf{B}(\mathbf{r}(t))\cdot\mathbf{r}'(t) \;=\; 8te^{t^2} - 4(3t^2+1)\,.$$

But then

$$I \;=\; \int_0^2 (8te^{t^2} - 4(3t^2+1)\,)\,dt$$

$$= \int_0^4 4e^u\,du - \Big[4t^3+t\Big]_0^2 \;=\; \Big[4e^u\Big]_0^4 - 40\,.$$

Consequently,

$$\boxed{\text{work done} \;=\; 4e^4 - 44}\,.$$

003 10.0 points

Find the work done by the force field

$$\mathbf{F}(x,\,y) \;=\; 2x\sin\pi y\,\mathbf{i} + 3\cos\pi y\,\mathbf{j}$$

to move a particle along the parabola $y = x^2$ from $(0,\,0)$ to $\left(\frac{1}{2},\,\frac{1}{4}\right)$.

1. Work Done $= \dfrac{1}{\pi}(\sqrt{2}-1)$ units

2. Work Done $= (1+\sqrt{2})$ units

3. Work Done $= \pi(\sqrt{2}-1)$ units

4. Work Done $= (\sqrt{2}-1)$ units

5. Work Done $= \dfrac{1}{\pi}(1+\sqrt{2})$ units **correct**

6. Work Done $= \pi(1+\sqrt{2})$ units

Explanation:

The work done by a Force Field $\mathbf{F}$ in moving a particle along a path parametrized by $\mathbf{c}(t)$ from $\mathbf{c}(a)$ to $\mathbf{c}(b)$ is given by the line integral

$$I \;=\; \int_a^b \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)\,dt\,.$$

Now the parabola $y = x^2$ is parametrized from $(0,\,0)$ to $\left(\frac{1}{2},\,\frac{1}{4}\right)$ by

$$\mathbf{c}(t) \;=\; t\,\mathbf{i} + 2t\,\mathbf{j}\,, \quad 0 \le t \le \frac{1}{2}\,.$$

In this case,

$$\mathbf{F}(\mathbf{c}(t)) \;=\; 2t\sin\pi t^2\,\mathbf{i} + 3\cos\pi t^2\,,$$

and

$$\mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t) \;=\; 2t\sin\pi t^2 + 6t\cos\pi t^2\,.$$

Thus by changing variables, we see that

$$I \;=\; \int_0^{1/2} t(2\sin\pi t^2 + 6\cos\pi t^2)\,dt$$

$$= \frac{1}{2}\int_0^{1/4} (2\sin\pi u + 6\cos\pi u)\,du$$

$$= \frac{1}{2\pi}\Big[-2\cos\pi u + 6\sin\pi u\Big]_0^{1/4}\,.$$

Consequently,

$$\boxed{\text{Work Done} \;=\; \frac{1}{\pi}(1+\sqrt{2})\text{ units}}\,.$$

004 10.0 points

Evaluate the integral

$$I \;=\; \int_C \mathbf{F}\cdot d\mathbf{s}$$

when

$$\mathbf{F}(x,\,y) \;=\; y\,\mathbf{i} + 2x\,\mathbf{j}$$

and C is the quarter circle

$$x^2 + y^2 \;=\; 1\,, \qquad x,\,y \ge 0\,,$$

oriented clockwise.

1. $I \;=\; \dfrac{1}{4}\pi$

2. $I \;=\; -\dfrac{1}{4}\pi$ **correct**

3. $I \;=\; \frac{1}{2}(-\pi - 3)$

4. $I \;=\; \frac{1}{4}(-\pi - 3)$

5. $I \;=\; -\frac{1}{2}\pi$

6. $I \;=\; -\frac{1}{2}(-\pi - 3)$

Explanation:

The quarter circle

$$x^2 + y^2 \;=\; 1\,, \qquad x,\, y \geq 0$$

is given parametrically by

$$\mathbf{c}(t) \;=\; \cos t\,\mathbf{i} + \sin t\,\mathbf{j}\,, \qquad 0 \leq t \leq \frac{\pi}{2}\,.$$

In this case

$$\mathbf{c}'(t) \;=\; -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}\,,$$

while

$$\mathbf{F}(x,\, y) \;=\; \mathbf{F}(\mathbf{c}(t)) \;=\; \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

on C, and so

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \;=\; -\sin^2 t + 2\cos^2 t\,.$$

On the other hand, since C is oriented clockwise while the parametrization $\mathbf{c}(t)$ traces out C counterclockwise as t increases, we shall need to integrate from $t = \pi/2$ to 0. Thus

$$\begin{aligned} I \;&=\; \int_C \mathbf{F}\cdot\mathbf{s} \;=\; \int_{\pi/2}^{0} \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)\,dt \\ &=\; -\int_0^{\pi/2} (-\sin^2 t + 2\cos^2 t)\,dt \\ &=\; \int_0^{\pi/2} (\sin^2 t - 2\cos^2 t)\,dt\,. \end{aligned}$$

To evaluate this last integral, we use the trig identities

$$\sin^2 t = \frac{1}{2}(1-\cos 2t), \quad \cos^2 t = \frac{1}{2}(1+\cos 2t),$$

for then

$$\sin^2 t - 2\cos^2 t \;=\; -\frac{1}{2} - \frac{3}{2}\cos 2t\,.$$

So, finally, we see that

$$\begin{aligned} I \;&=\; \int_0^{\pi/2} \Big(-\frac{1}{2} - \frac{3}{2}\cos 2t\Big)\,dt \\ &=\; \Big[-\frac{1}{2}\,t - \frac{3}{4}\sin 2t\Big]_0^{\pi/2}\,. \end{aligned}$$

Consequently,

$$\boxed{I \;=\; -\frac{1}{4}\pi}\,.$$

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005 10.0 points

Evaluate the integral

$$I \;=\; \int_C xy^4\,ds$$

when C is the right half of the circle

$$x^2 + y^2 \;=\; 1\,.$$

1. $I \;=\; \frac{2}{5}$ **correct**

2. $I \;=\; \frac{1}{3}$

3. $I \;=\; \frac{2}{3}$

4. $I \;=\; 1$

5. $I \;=\; \frac{1}{5}$

6. $I \;=\; \frac{4}{5}$

Explanation:

The right half of the circle

$$x^2 + y^2 \;=\; 1$$

can be parametrized by

$$\mathbf{c}(t) \;=\; (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\,.$$

In this case

$$\|\mathbf{c}'(t)\| \;=\; \|(-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}\| \;=\; 1\,,$$

while on C,

$$xy^4 \;=\; \cos t \sin^4 t\,.$$

So

$$\begin{aligned} I \;&=\; \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\, \|c'(t)\|\, dt \\ &=\; \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\, dt\,. \end{aligned}$$

This last integral can be evaluated using the substitution $u = \sin t$. For then

$$I \;=\; \int_{-1}^{1} u^4\, du \;=\; \Big[\frac{1}{5}\frac{1}{u^5}\Big]_{-1}^{1}\,,$$

and so

$$\boxed{I \;=\; \frac{2}{5}}\,.$$

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006 10.0 points

Evaluate the integral

$$I \;=\; \int_C 4x\, ds$$

when the path C is parametrized by

$$\mathbf{c}(t) \;=\; (t^2,\; 2t,\; \ln t)$$

for $1 \le t \le e$.

1. $I \;=\; e(e+1) - 4$

2. $I \;=\; 2e(e+1) + 4$

3. $I \;=\; 2e^2(e^2+1) + 4$

4. $I \;=\; 2e^2(e^2+1) - 4$ **correct**

5. $I \;=\; e(e+1) + 4$

6. $I \;=\; e^2(e^2+1) - 4$

Explanation:

A scalar line integral over

$$\mathbf{c}(t) \;=\; (t^2,\; 2t,\; \ln t), \quad 1 \le t \le e,$$

is given by

$$I \;=\; \int_1^e f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt\,.$$

Now

$$\|\mathbf{c}'(t)\| \;=\; \sqrt{(2t)^2 + (2)^2 + (t^{-1})^2}\,,$$

which after simplification becomes

$$\begin{aligned} \|\mathbf{c}'(t)\| \;&=\; \sqrt{4t^2 + 4 + (1/t)^2} \\ &=\; \sqrt{\Big(2t + \frac{1}{t}\Big)^2} \;=\; 2t + \frac{1}{t}\,. \end{aligned}$$

On the other hand,

$$f(\mathbf{c}(t)) \;=\; (4x)\Big|_{\mathbf{c}(t)} \;=\; 4t^2\,.$$

Thus

$$I = \int_1^e 4t^2\Big(2t + \frac{1}{t}\Big)\, dt = \int_1^e (8t^3 + 4t)\, dt\,.$$

Consequently,

$$\boxed{I \;=\; \Big[2t^4 + 2t^2\Big]_1^e \;=\; 2e^2(e^2+1) - 4}\,.$$

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007 10.0 points

Evaluate the integral

$$I \;=\; \int_C y\, ds$$

when C is parametrized by

$$\mathbf{c}(t) \;=\; t^2\,\mathbf{i} + t\,\mathbf{j}, \qquad 0 \le t \le \sqrt{2}\,.$$

1. $I \;=\; \dfrac{3}{2}$

2. $I \;=\; \dfrac{11}{6}$

3. $I = \frac{13}{6}$ **correct**

4. $I = \frac{17}{6}$

5. $I = \frac{5}{2}$

Explanation:

Since

$$\|\mathbf{c}'(t)\| = \|2t\,\mathbf{i} + \mathbf{j}\| = \sqrt{4t^2+1}$$

and $y = t$ on C, we see that

$$I = \int_0^{\sqrt{2}} t\,\|\mathbf{c}'(t)\|\,dt$$

$$= \int_0^{\sqrt{2}} t\sqrt{4t^2+1}\,dt\,.$$

This last integral can be evaluated by substitution: set $u^2 = 4t^2 + 1$. Then

$$2u\,du = 8t\,dt\,, \qquad \frac{1}{4}u\,du = t\,dt\,,$$

so

$$I = \frac{1}{4}\int_1^3 u^2\,du = \Big[\frac{1}{12}u^3\Big]_1^3.$$

Consequently,

$$\boxed{I = \frac{13}{6}}.$$

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008 10.0 points

Find the mass of the wire formed by the intersection of the sphere

$$x^2 + y^2 + z^2 = 2$$

and the plane

$$x + y - z = 0$$

if the wire has density $3y^2/4$ grams per unit length.

1. mass $= \frac{1}{2}\sqrt{2}$ grams

2. mass $= \frac{1}{2}\sqrt{2}\,\pi$ grams

3. mass $= \sqrt{2}\,\pi$ grams **correct**

4. mass $= \frac{1}{2}\pi$ grams

5. mass $= \pi$ grams

6. mass $= \sqrt{2}$ grams

Explanation:

If the intersection of the sphere and the plane is parametrized by

$$\mathbf{c}(t) = (x(t),\, y(t),\, z(t)), \quad a \le t \le b\,,$$

then the mass of the wire is given by

$$I = \int_a^b \rho(\mathbf{c}(t))\,\|\mathbf{c}'(t)\|\,dt\,.$$

where $\rho(\mathbf{c}(t)$ is the density of the wire at $\mathbf{c}(t)$.

Now

$$x^2 + y^2 + z^2 = 2, \quad x + y - z = 0\,,$$

intersect when

$$x^2 + y^2 + (x+y)^2$$

$$= 2(x^2 + y^2 + xy) = 2\,,$$

i.e., when

$$x^2 + xy + y^2 = 1\,.$$

After completion of the square this becomes

$$\Big(x + \frac{y}{2}\Big)^2 + \frac{3y^2}{4} = 1\,.$$

Since this can be parametrized by

$$x(t) + \frac{y}{2} = \cos t\,, \quad y(t) = \frac{2}{\sqrt{3}}\sin t\,,$$

with $0 \le t \le 2\pi$, the wire is described by

$$\mathbf{c}(t) = \Big(\cos t - \frac{\sin t}{\sqrt{3}},\, \frac{2\sin t}{\sqrt{3}},\, \cos t + \frac{\sin t}{\sqrt{3}}\Big),$$

with $0 \leq t \leq 2\pi$. In this case,

$$\|\mathbf{c}'(t)\| = (2\cos^2 t + 2\sin^2 t\Big)^{1/2} = \sqrt{2}\,,$$

and so

$$\begin{aligned} I \;&=\; \sqrt{2}\int_0^{2\pi} \;\frac{3}{4}\Big(\frac{2\sin t}{\sqrt{3}}\Big)^2\,dt \\ &=\; \sqrt{2}\int_0^{2\pi} \;\sin^2 t\,dt \\ &=\; \frac{1}{2}\sqrt{2}\int_0^{2\pi} \;(1-\cos 2t)\,dt\,. \end{aligned}$$

Consequently, the wire has

$$\boxed{\text{mass} \;=\; \frac{1}{2}\sqrt{2}\Big[\,t - \frac{1}{2}\sin 2t\,\Big]_0^{2\pi} \;=\; \sqrt{2}\,\pi}\,.$$