This print-out should have 8 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

001 10.0 points

Use Green's Theorem to evaluate the line integral

$$I = \int_C 4x^2 y\, dx + 2x^3\, dy$$

when C is the positively oriented curve consisting of the line segment from $(-2, 0)$ to $(2, 0)$ and the top half of the circle

$$x^2 + y^2 = 4\,.$$

1. $I = 0$
2. $I = -8\pi$
3. $I = 4\pi$ **correct**
4. $I = -4\pi$
5. $I = 8\pi$

Explanation:

Green's Theorem says that if D is a region in the plane having a positively oriented boundary ∂D, then

$$\int\int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dA = \int_{\partial D} P\, dx + Q\, dy\,.$$

But when

$$P = 4x^2 y\,, \qquad Q = 2x^3$$

and

$$D = \{(x, y) : x^2 + y^2 \le 4,\ y \ge 0\}\,,$$

then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x^2\,,$$

while

$$D = \{(r, \theta) : 0 \le r \le 2,\ 0 \le \theta \le \pi\}$$

in polar coordinates. Thus by Green's Theorem,

$$\begin{aligned} I &= 2\int\int_D x^2\, dxdy \\ &= 2\int_0^{\pi}\int_0^2 r^2 \cos^2\theta\, r\, drd\theta \\ &= 2\Big(\int_0^{\pi} \cos^2\theta\, d\theta\Big)\Big(\int_0^2 r^3\, dr\Big) \\ &= 8\int_0^{\pi} \cos^2\theta\, d\theta\,, \end{aligned}$$

after changing to polar coordinates. But by double angle formulas,

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)\,.$$

Consequently,

$$\boxed{I = 4\int_0^{\pi} (1 + \cos 2\theta)\, d\theta = 4\pi}\,.$$

002 10.0 points

Evaluate the integral

$$I = \int_C \mathbf{F} \cdot d\mathbf{s}$$

using Green's Theorem when

$$\mathbf{F}(x, y) = (e^x - 2x^2 y)\,\mathbf{i} + (e^{2y} + 2xy^2)\,\mathbf{j}$$

and C is the circle

$$x^2 + y^2 = 1$$

oriented counter-clockwise.

1. $I = 0$
2. $I = -\pi$
3. $I = -2\pi$
4. $I = \pi$ **correct**
5. $I = 2\pi$

Explanation:

Green's Theorem says that if D is a region in the plane having a positively oriented boundary ∂D and $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then

$$\int\int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dA = \int_{\partial D} \mathbf{F}\cdot d\mathbf{s}\,.$$

But when

$$\begin{aligned}\mathbf{F}(x,\,y) \;&=\; P(x,\,y)\,\mathbf{i} + Q(x,\,y)\,\mathbf{j}\\ &=\; (e^x - 2x^2y)\,\mathbf{i} + (e^{2y} + 2xy^2)\,\mathbf{j}\end{aligned}$$

then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \;=\; 2(x^2+y^2)\,.$$

Thus by Green's Theorem applied to the disk

$$x^2 + y^2 \;\le\; 1\,,$$

we see that

$$I = 2\int\int_D (x^2+y^2)\,dxdy = 2\int_0^{2\pi}\int_0^1 r^3\,drd\theta,$$

after changing to polar coordinates.

Consequently,

$$\boxed{I \;=\; \pi}\,.$$

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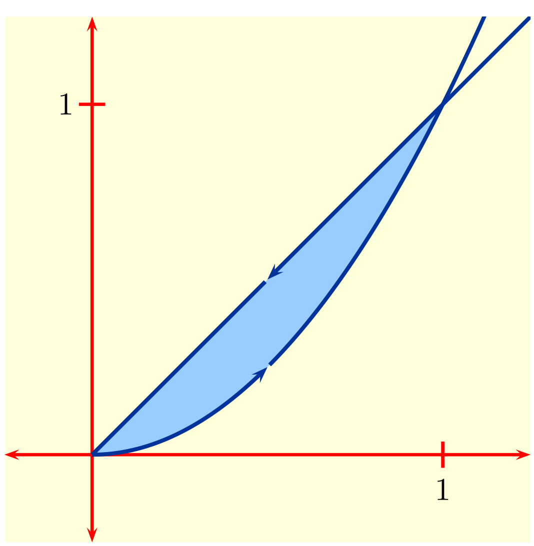
003 10.0 points

Use Green's Theorem to evaluate the line integral

$$I \;=\; \int_C 2xy\,dx - y^2\,dy$$

when C is the boundary oriented counterclockwise of the shaded region in the first quadrant shown in



enclosed by the graphs of $y = x^2$ and $y = x$.

1. $I \;=\; -\dfrac{1}{6}$ **correct**

2. $I \;=\; \dfrac{1}{3}$

3. $I \;=\; \dfrac{1}{4}$

4. $I \;=\; -\dfrac{1}{3}$

5. $I \;=\; -\dfrac{1}{4}$

6. $I \;=\; \dfrac{1}{6}$

Explanation:

By Green's Theorem, if D is a region in the plane with positively oriented boundary ∂D, then

$$\int_{\partial D} P\,dx + Q\,dy \;=\; \int\int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dA\,.$$

We apply this with

$$P \;=\; 2xy\,, \qquad Q \;=\; -y^2\,,$$

and $C = \partial D$ where D the shaded region in the first quadrant enclosed by the graphs of $y = x^2$ and $y = x$. Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \;=\; -2x\,,$$

so by Green's Theorem,

$$\begin{aligned} I \;&=\; -2\int_0^1\Big(\int_{x^2}^x x\,dy\Big)dx\\ &=\; -2\int_0^1 x(x-x^2)\,dx \;=\; -2\Big[\frac{x^3}{3} - \frac{x^4}{4}\Big]_0^1.\end{aligned}$$

Consequently,

$$\boxed{I = -\frac{1}{6}}.$$

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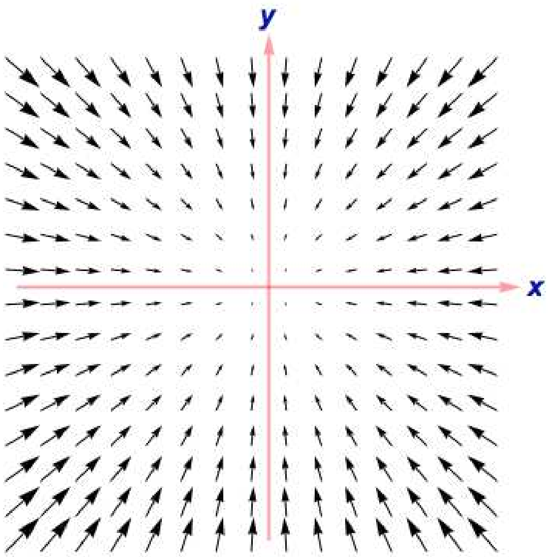
004 10.0 points

The vector field $\mathbf{F}$ shown in



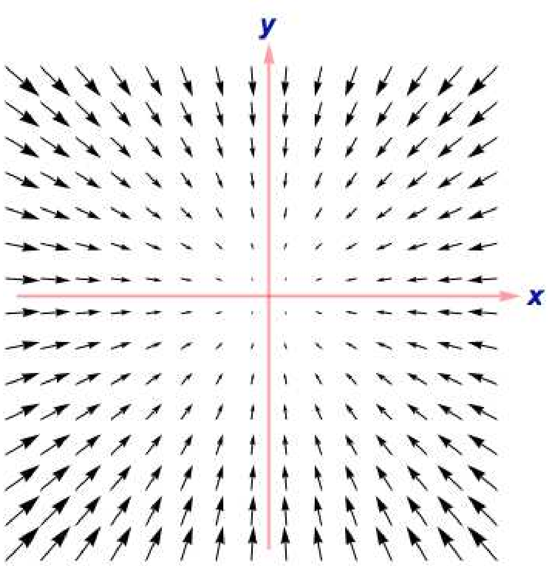
is curl-free at the origin. True or False?

1. False

2. True **correct**

Explanation:

The vector field



is radially (inwards) towards the origin without spiralling. Thus curl $\mathbf{F}$ will be zero at the origin.

Consequently, the statement is

$$\boxed{\text{TRUE}}.$$

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005 10.0 points

Use Stoke's theorem to evaluate the integral

$$I = \int\int_S \operatorname{curl} \mathbf{F} \cdot d\,\mathbf{S}$$

when

$$\mathbf{F} = \left\langle e^{x^2} z,\, x,\, yx + z^2 \right\rangle$$

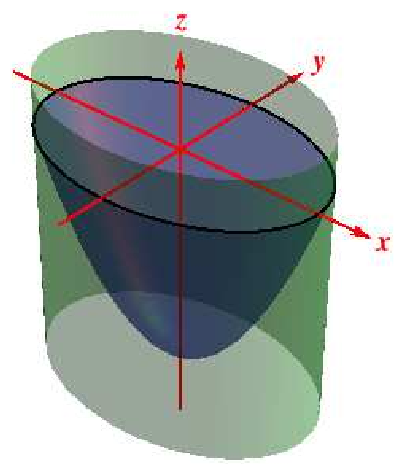
and S is the portion of the surface

$$z = x^2 + 4y^2 - 4$$

inside the elliptical cylinder

$$x^2 + 4y^2 = 4$$

shown in



whose orientation is specified by a normal vector pointing upward and inward.

1. $I = \frac{5}{3}\pi$

2. $I = \frac{7}{3}\pi$

3. $I = \frac{8}{3}\pi$

4. $I = 2\pi$ **correct**

5. $I = 3\pi$

Explanation:

The boundary ∂S of S is the curve of intersection of

$$z = 0, \quad x^2 + 4y^2 = 4,$$

shown in black above. This is an ellipse in the xy-plane, parametrized by

$$\mathbf{r}(t) = \langle 2\cos t, \sin t, 0 \rangle.$$

with counter-clockwise orientation which is the correct boundary orientation for the given orientation on S.

Now by Stokes' theorem,

$$\int\int_S \operatorname{curl} \mathbf{F} \cdot d\,\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\,\mathbf{s}$$

$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt.$$

But

$$\mathbf{F}(\mathbf{r}(t)) = \langle 0, 2\cos t, 2\cos t \sin t \rangle$$

while

$$\mathbf{r}'(t)\,dt = \langle -2\sin t, \cos t, 0 \rangle.$$

So

$$I = \int_0^{2\pi} 2\cos^2 t\,dt$$

$$= \int_0^{2\pi} (1 + \cos 2t)\,dt = 2\pi.$$

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006 10.0 points

Use the Divergence Theorem to evaluate the integral

$$I = \int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S}$$

when

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2yz\,\mathbf{j} + 4z^2\,\mathbf{k}$$

and ∂W is the boundary of the solid W enclosed by the upper half of the sphere

$$x^2 + y^2 + z^2 = 4$$

and the xy-plane.

1. $I = 16$

2. $I = 16\,\pi$

3. $I = 24$

4. $I = 48\,\pi$

5. $I = 24\,\pi$ **correct**

6. $I = 48$

Explanation:

By the Divergence theorem,

$$I = \int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_W \operatorname{div} \mathbf{F}\,dV.$$

Now

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(4z^2)$$

$$= -2z + 8z = 6z.$$

On the other hand, W consists of all points (ρ, θ, ϕ) in spherical polar coordinates such that

$$0 \le \theta \le 2\pi, \quad 0 \le \rho \le 2, \quad 0 \le \phi \le \frac{\pi}{2}.$$

So as a repeated integral in spherical polar coordinates,

$$I = \int_0^2 \left(\int_0^{2\pi} \left(\int_0^{\pi/2} 6\rho^3 \cos\phi \sin\phi\,d\phi \right) d\theta \right) d\rho.$$

But

$$\int_0^{\pi/2} \cos\phi \sin\phi\,d\phi = \left[\frac{1}{2}\sin^2\phi \right]_0^{\pi/2} = \frac{1}{2}.$$

Consequently,

$$\boxed{I = 3\int_0^2 \left(\int_0^{2\pi} \rho^3\, d\theta\right) d\rho = 24\pi}.$$

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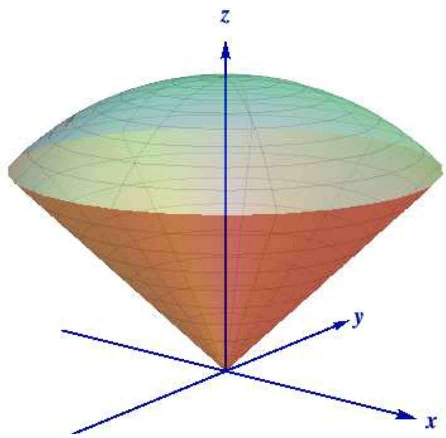
007 10.0 points

The solid W shown in



consists of all points enclosed by the sphere

$$x^2 + y^2 + z^2 = 4$$

and the cone

$$z = \sqrt{x^2 + y^2}.$$

Determine the flux through the boundary of W of the vector field

$$\mathbf{F}(x, y, z) = (y + e^{\sin z})\,\mathbf{i} - 2yz\,\mathbf{j} + 3z^2\,\mathbf{k}.$$

1. $I = 16\pi^2$

2. $I = 8\pi^2$

3. $I = 32\pi^2$

4. $I = 16\pi$

5. $I = 32\pi$

6. $I = 8\pi$ **correct**

Explanation:

The flux through the boundary of W is given by the vector surface integral

$$I = \int\int_{\partial W} \mathbf{F}\cdot d\mathbf{S} = \int\int\int_W \operatorname{div}\mathbf{F}\, dV,$$

using Gauss' theorem to evaluate the surface integral. But for the given $\mathbf{F}$,

$$\operatorname{div}\mathbf{F} = -2z + 6z = 4z.$$

Thus

$$I = \int\int\int_W 4z\, dxdydz.$$

Now in spherical coordinates (ρ, θ, ϕ),

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi,$$

and

$$z = \rho\cos\phi,$$

so the cone

$$z = \sqrt{x^2 + y^2}$$

can be written in spherical coordinates as

$$\rho\cos\phi = \rho\sin\phi(\cos^2\theta + \sin^2\theta)^{1/2},$$

i.e., $\tan\phi = 1$, or, in other words, as $\phi = \pi/4$. Since $\phi = 0$ at the North Pole, W thus consists of all points (ρ, θ, ϕ) such that

$$0 \le \rho \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}.$$

On the other hand, the Jacobian for spherical coordinates is $\rho^2\sin\phi$. So as a repeated integral in spherical polar coordinates,

$$I = \int_0^2 \left(\int_0^{2\pi}\left(\int_0^{\pi/4} 4\rho^3\cos\phi\sin\phi\, d\phi\right) d\theta\right) d\rho.$$

But

$$\int_0^{\pi/4} \cos\phi\sin\phi\, d\phi = \Big[\frac{1}{2}\sin^2\phi\Big]_0^{\pi/4} = \frac{1}{4}.$$

Consequently,

$$\boxed{I = \int_0^2 \left(\int_0^{2\pi} \rho^3\, d\theta\right) d\rho = 8\pi}.$$

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008 10.0 points

Use the Divergence Theorem to calculate the integral

$$I = \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

when

$$\mathbf{F} = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$$

and S is the surface of the solid bounded by the paraboloid

$$z = 1 - x^2 - y^2$$

and the xy-plane.

1. $I = \frac{1}{2}\pi$ **correct**

2. $I = \frac{1}{4}$

3. $I = \frac{1}{4}\pi$

4. $I = \frac{1}{2}$

5. $I = \frac{1}{8}\pi$

6. $I = \frac{1}{8}$

Explanation:

By the Divergence theorem,

$$I = \int\int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_W \operatorname{div} \mathbf{F}\, dV\,.$$

Now

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(2xz^2) + \frac{\partial}{\partial z}(3y^2z) \\ &= 3(x^2 + y^2)\,. \end{aligned}$$

On the other hand, W consists of all points $(r,\, \theta,\, z)$ in cylindrical polar coordinates such that

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad 0 \le z \le 1 - r^2\,.$$

So as a repeated integral,

$$\begin{aligned} I &= \int_0^1 \int_0^{2\pi} \int_0^{1-r^2} 3r^3\, dz d\theta dr \\ &= \int_0^1 \int_0^{2\pi} 3(1-r^2)r^3\, d\theta dr \\ &= 6\pi \int_0^1 3(1-r^2)r^3\, dr\,. \end{aligned}$$

Consequently,

$$\boxed{I = 6\pi\Big[\frac{1}{4}r^4 - \frac{1}{6}r^6\Big]_0^1 = \frac{1}{2}\pi}\,.$$