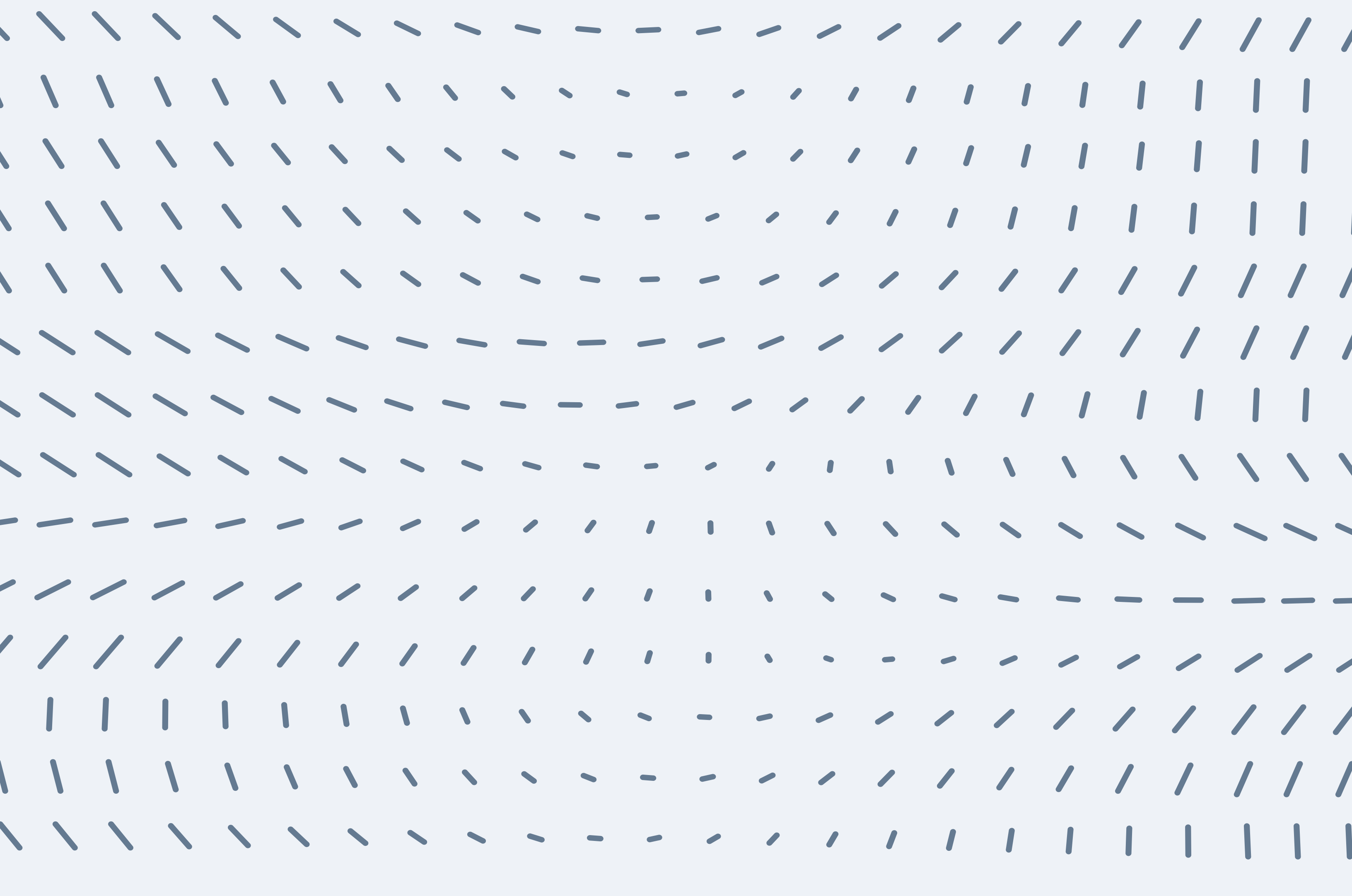
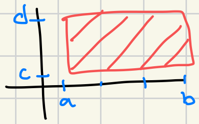


M 427L Vector Calculus

June 6^{th}, 2025

$$\int_c^d\int_a^b f(x,y)\,dx\,dy$$



Not the real world, only part!

Reviewing Vectors:

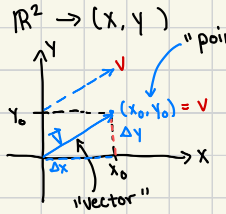
$f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\mathbb{R}^n = \{(a_1,\dots,a_n): a_i \text{ are real numbers}\}$

$f(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n))$

$\mathbb{R}^n \to (x_1,\dots,x_n)$ "Vectors" "points"

$\mathbb{R}^2 \to (x,y)$



Points Arithmetic

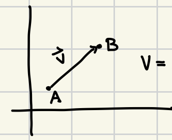
$(x_1,y_1)+(x_2,y_2) = (x_1+x_2,\ y_1+y_2)$ Scalar Addition

$\lambda(x,y) = (\lambda x, \lambda y)$ Scalar Multiplication

Example:

$\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-5\\-7\end{bmatrix} \to (-5,7)$

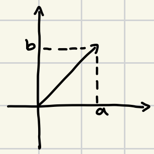
Point = TIP − TAIL (Cords)



$V = \vec{AB} = B - A$

- $V = (a,b)$

Norm of $v = \|v\| = \sqrt{a^2+b^2}$

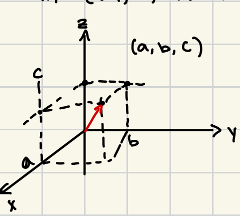


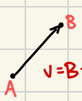
- $V = (a_1,b_1)$ $\quad W = (a_2,b_2)$

Dot Product: $V\cdot W = a_1\cdot a_2 + b_1\cdot b_2$

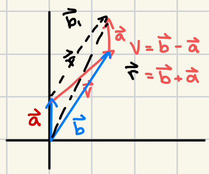
June 9^{th}, 2025

$\mathbb{R}^n = \{(x_1,\dots,x_n): x_i \text{ is a real number}\}$





$V = B - A$



Norm:

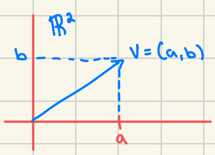
$v = (x_1,\dots,x_n)$

$\|v\| = \sqrt{\sum_{i=0}^{n} x_n^2}$

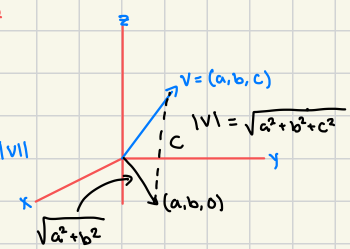
Example #1: $|(-1,2,3)| = \sqrt{(-1)^2+2^2+3^2}$

$= \sqrt{1+4+9}$

$= \sqrt{14}$



hyp $= \sqrt{a^2+b^2} = \|v\|$ length of the vector



Unit vector

If $\vec{v} \neq 0$, then $|\vec{v}| \neq 0$ and

$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$

$|\hat{v}| = \frac{|v|}{|v|} = 1$

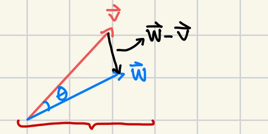
$$\boxed{\begin{gathered}\lambda(a,b) = (\lambda a, \lambda b)\\ \sqrt{a^2+b^2}\\ \sqrt{\lambda^2a^2+\lambda^2b^2} = |\lambda|\sqrt{a^2+b^2}\end{gathered}}$$

We say two vectors $\vec{v}, \vec{w}$ have the same direction if $\hat{v} = \hat{w}$.

Example #2: $\vec{v} = (1,2,3)$ and $\vec{w} = (-1,2,1)$

$\hat{v} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \frac{(1,2,3)}{\sqrt{14}} = (1/\sqrt{14},\ 2/\sqrt{14},\ 3/\sqrt{14})$

$\hat{w} = \frac{(-1,2,1)}{\sqrt{(-1)^2+2^2+1^2}} = \frac{(-1,2,1)}{\sqrt{6}} = \frac{1}{\sqrt{6}}(-1,2,1)$



$|w-v|^2 = |v|^2 + |w|^2 - 2|v||w|\cos\theta$

"Vector = $\underbrace{\text{direction}}_{\hat{v}}$ + $\underbrace{\text{magnitude}}_{|\vec{v}|}$" $\quad \hat{v} = \frac{v}{|v|}$

$\boxed{\to \vec{v} = \hat{v}\cdot|v|}$

Way 2: $|w-v|^2 = (w-v)\cdot(w-v)$

$\equiv |w|^2 - 2\cdot w\cdot v + |v|^2$

$\to v\cdot w = |v||w|\cos\theta$

$\to \cos\theta = \frac{v\cdot w}{|v||w|}$

Dot Product:

$(a_1,\dots,a_n)\cdot(b_1,\dots,b_n) = a_1b_1 + a_2b_2 + \dots + a_nb_n$

NOTE! $\vec{v} = (a_1,\dots,a_n)$: $\vec{v}\cdot\vec{v} = a_1^2 + \dots + a_n^2$

$\sqrt{v^2} = |v|$

$v\cdot(w_1+w_2) = v\cdot w_1 + v\cdot w_2 \to (\lambda v)\cdot w = \lambda(v\cdot w)$

$v\cdot w = w\cdot v$

Discussion Section

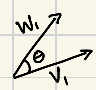
June 10^{th}, 2025.

$W_1 = (2,-1,0,2)$

$V_1 = (1,2,-1,2)$

$W_1\cdot V_1 = \sum_{i=1}^{n} a_n\cdot b_n$ where $n=4$

$= 4$



$\to \cos\theta = \frac{V\cdot W}{|V||W|}$

$= \frac{4}{3\sqrt{10}}$

proj $w_1 = V_1\cdot\hat{V_2} = |V_1|\cos\theta$

$= \cancel{3}\left(\frac{4}{\cancel{3}\sqrt{10}}\right) = \boxed{\frac{4}{\sqrt{10}}}$

$\hat{V_2} = \frac{(-1,2,-1,2)}{\sqrt{10}}$

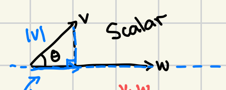
Dot Products:

$(a_1,\dots,a_n)\cdot(b_1,\dots,b_n) = \sum_{i=1}^{n} a_nb_n$

$v\cdot v = v^2 = \sum_{i=1}^{n} a_n^2 = |v|^2$

if $v = (a_1,\dots,a_n)$

$v\cdot w = |v||w|\cos\theta$



$|V|\cos\theta = \frac{V\cdot W}{|W|}$

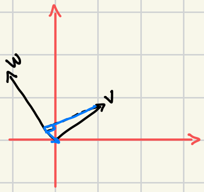
V·W	Angle
0	$\pi/2$
+	$<\frac{\pi}{2}$
−	$>\frac{\pi}{2}$

$|V|\cos\theta\,\underbrace{\hat{w}}_{\frac{w}{|w|}} = \frac{V\cdot W}{|W|^2}\cdot w$ (Vector Projection)

Example #1: Find the projection of v: (2,1) onto w = (-1,3)

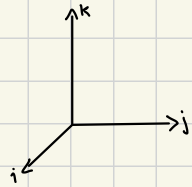
Scalar: $\frac{v \cdot w}{|w|} = \frac{1}{\sqrt{10}}$ $|w| = \sqrt{(-1)^2+(3)^2} = \sqrt{10}$

Vector: $\underbrace{\overset{\text{Length}}{\frac{v \cdot w}{|w|}} \cdot \overset{\text{Direction}}{\frac{w}{|w|}}}_{|w|^2} = \frac{1}{10}(-1,3) \equiv \left(\frac{-1}{10}, \frac{3}{10}\right)$



Lecture

June 13th



$i \times j = k \quad j \times i = -k$

$\vec{i} \times \vec{i} =$

$v \times v = -v \times v = \vec{0}$

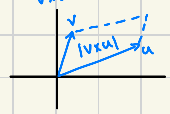
$v \times (w + u) = v \times w + v \times u$

$v \times (\lambda u) = (\lambda v) \times u \equiv \lambda(v \times u)$

$(1,2,3) \times (2,3,4) = (i + 2j + 3k) \times (2i + 3j + 4k)$

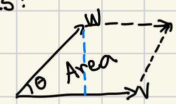
$(v_1, v_2, v_3) \times (u_1, u_2, u_3) = \det\left(\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix}\right)$

Application: $|v \times u|$ = area of the parallelogram spanned by $\vec{v}$ and $\vec{u}$



Areas:

SOH
CAH
TOA



$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Area = $|v||w| \sin\theta$

$A^2 = |v|^2|w|^2 \sin^2\theta$

$= |v^2||w^2|(1-\cos^2\theta)$

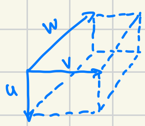
$\equiv |v^2||w^2| - |v^2||w^2|\cos^2\theta$

$A^2 = v^2w^2 - (v \cdot w)^2 = \begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}$

$A_{obj} = \sqrt{\begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}}$

"Gram determinant"

3D:



$Vol = \sqrt{\begin{vmatrix} u \cdot v & u \cdot u & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}}$

Example: $v_1 = (1,1,1)$ $v_1 \cdot v_2 = -1 + 1 + 2 = 2$

$v_2 = (-1,1,2)$ $v_1 \cdot v_3 = 0 + 1 + 0 = 1$

$v_3 = (0,1,0)$ $v_2 \cdot v_3 = 0 + 1 + 0 = 1$

$6(1) - (1)(1) = 5$
$2(1) - (1)(1) = 1$
$2(1) - 6(1) = -4$

$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{vmatrix} = a\left(\begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix}\right) - b\left(\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right) + c\left(\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}\right)$

$= 3\det\begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix}$

$3 \cdot 5 - 2 \cdot 1 + (-4)$

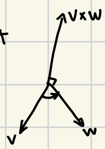
$= 15 - 2 - 4$

$= 15 - 6$

$= \sqrt{9}$

$Vol = \boxed{3}$

$\mathbb{R}^3$: Cross Product



$i = (1,0,0)$

$j = (0,1,0)$

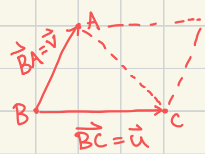
$k = (0,0,1)$

$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$= i\det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j\det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$

$\equiv \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right)$

Example: A = (1,0,1), B = (1,0,0), C = (-1,1,1)



$v = A - B = (0,0,1)$

$u = C - B = (-2,1,1)$

$v \times u = \det\left(\begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}\right) = i\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) - j\det\left(\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\right) + k\det\left(\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}\right)$

$= \langle -1, -2, 0 \rangle$

$A_{triangle} = \frac{1}{2}\text{ norm}(-1,-2,0)$

$= \frac{\sqrt{5}}{2}$

Discussion Section

June 12th, 2025

$2 \cdot (-1) + 1(-1) + 0(-1) + 0(1) + 1(-1)$

$= -2 - 1 - 1 = -4$

$1+1+1+1$

$4+1+1+1$

$v = (-1, 1, 0, 1, -1)$

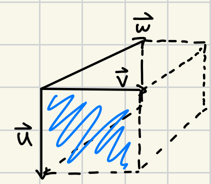
$w = (2, -1, -1, 0, 1)$

$\begin{bmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}$

$\det\left(\begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}\right) = 4(7) - (-4(-4))$

$28 - 16 = \boxed{12} = A^2$

$\sqrt{12} = A$



$V = |v \times w||w| \cos\alpha$

$= (u \times v) \cdot w$

$A = (1,1,1)$

$B = (-1,2,1)$ $\vec{AB} = (-2,1,0) = \vec{B} - \vec{A}$

$C = (3,1,-1)$ $\vec{AC} = (2,0,-2) = \vec{C} - \vec{A}$

$\vec{AB} \times \vec{AC} =$

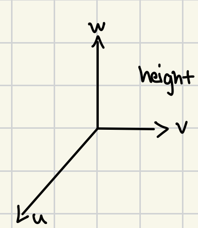
$\left|\left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}\right|\right| = \hat{i}\underbrace{\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\right)}_{-2+0=-2} - \hat{j}\underbrace{\det\left(\begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}\right)}_{4-0=4} + \hat{k}\underbrace{\det\left(\begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}\right)}_{-2}$

$-2\hat{i} - 4\hat{j} - 2\hat{k}$

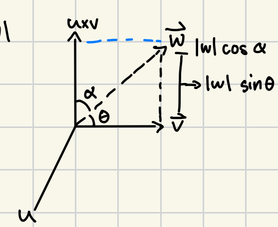
$\equiv (-2, -4, -2)$

$= \sqrt{24} = |(-2,-4,-2)|$

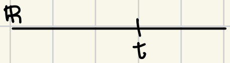
$= \frac{1}{2} \cdot \sqrt{24} = \sqrt{6}$

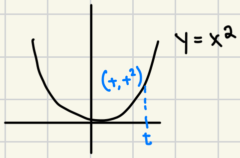


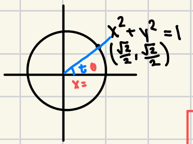
height = |w|



Curve:







$C : [a,b] \longrightarrow \mathbb{R}^3$

$\boxed{\begin{array}{l}\text{Example: } y = x^2 : \ C(x) = (x, x^2) \\ \text{Example: } x^2+y^2=1 : C(\theta) = (\cos\theta, \sin\theta)\end{array}}$

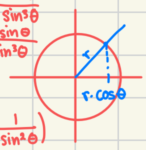
Example: $x^2+y^2=y^3 \longmapsto r^2 = r^3 \sin^3\theta$

$1 = r\sin^3\theta$

$\frac{1}{\sin^3\theta} = r$

$x = r\cos\theta = \frac{\cos\theta}{\sin^3\theta}$

$y = r\sin\theta = \frac{\sin\theta}{\sin^3\theta} = \frac{1}{\sin^2\theta}$



$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

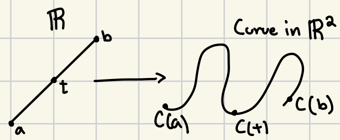
$y = r\sin\theta$

$C(\theta) = \left(\frac{\cos\theta}{\sin^3\theta}, \frac{1}{\sin^2\theta}\right)$

Lecture

June 16, 2025

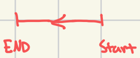
$C : [a,b] \longmapsto \mathbb{R}^n$



Example: $C(t) = t \quad C : [0,1] \to \mathbb{R}$

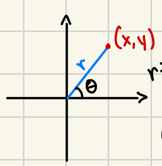
$C(t) = -t \quad C : [-1, 0] \to \mathbb{R}$

$C(-1) = 1$



$C(0) = 0$

Polar:



$r^2 = x^2+y^2$

$r \in [0, \infty)$

$\theta \in [0, 2\pi]$

$x = r\cos\theta$

$y = r\sin\theta$

Example: $y^2 = x^4 + y^4$ $\qquad 0 \le \theta \le 2\pi$

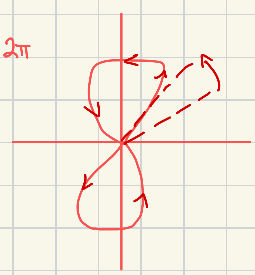
$r^2\sin^2\theta = r^4\cos^4\theta + r^4\sin^4\theta$

$r^2\sin^2\theta = r^4(\cos^4\theta + \sin^4\theta)$

$\sin^2\theta = r^2(\cos^4\theta + \sin^4\theta)$

$\frac{\sin^2\theta}{\cos^4\theta + \sin^4\theta} = r^2 \longrightarrow r = \frac{|\sin\theta|}{\sqrt{\cos^4\theta+\sin^4\theta}}$

$C(\theta) = \left(\frac{|\sin\theta|\cos\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}, \frac{|\sin\theta|\sin\theta}{\sqrt{\cos^4\theta+\sin^4\theta}}\right) \quad \theta \in [0, 2\pi]$

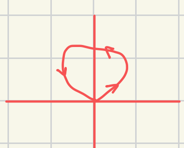


Example: $y^3 = x^4 + y^4$

$r^3\sin^3\theta = r^4\cos^4\theta + r^4\sin^4\theta$

$r = \frac{\sin^3\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$

$c(\theta) = \left(\frac{\sin^3\theta\cos\theta}{r^4\cos^4\theta + r^4\sin^4\theta}, \frac{\sin^4\theta}{r^4\cos^4\theta + r^4\sin^4\theta}\right)$



No problem restriction

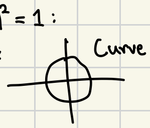
No algebra restriction $\qquad \sin^3\theta \ge 0 \to \theta \in [0, \pi]$

Polar restriction $\to r \ge 0 \qquad 0 \le \theta \le \pi$

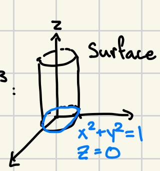
Surface:

$x^2+y^2=1$:

$\mathbb{R}^2$:



$\mathbb{R}^3$:



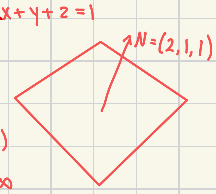
Example: $2x+y+z=1$

$N = (2,1,1)$

$z = 1-2x-y$

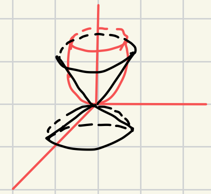
$\Phi(x,y) = (x, y, 1-2x-y)$

$, -\infty < x, y < \infty$



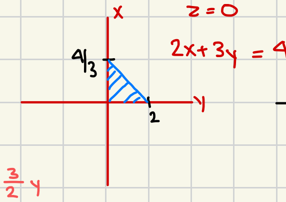
Example: $z = x^2+y^2$

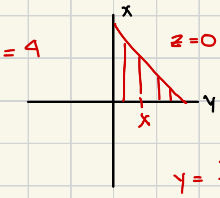
$z^2 = x^2+y^2$



Friday: ZOOM!!!

Example: $2x+3y+2z=4$ $\qquad N = (2,3,2)$ normal





$y = \frac{4-2x}{3}$

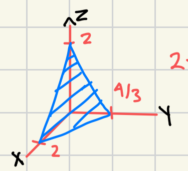
$2z = 4-2x-3y \longrightarrow z = 2-x-\frac{3}{2}y$

$\Phi(x,y) = (x, y, 2-x-\frac{3}{2}y) \longrightarrow 2-x-\frac{3}{2}y \ge 0 \cup x \ge 0 \cup y \ge 0$

First octant: $(x,y,z \ge 0)$

$\Phi(x,y) = (x, y, 2-x-\frac{3}{2}y)$

$= \underbrace{(0,0,2)}_{\text{Point}} + \underbrace{x(1,0,-1) + y(0,1,-3/2)}_{\text{tangent direction}}$



$2x+3y+2z=4$

Vertical lines:

$\{(x,y) : 0 \le x \le 2,\ 0 \le y \le \frac{4-2x}{3}\}$

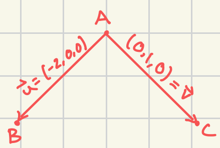
Similar to a PDF???

Example: $A = (1,0,1),\ B = (-1,0,1),\ C = (1,1,1)$

$\Phi(r,s) = (1,0,1) + (-2,0,0)r$

$+ (0,1,0)s$

$= (1-2r, s, 1)$



$u \times v = \begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i\det\begin{pmatrix}0&0\\1&0\end{pmatrix} - j\det\begin{pmatrix}-2&0\\0&0\end{pmatrix} + k\det\begin{pmatrix}-2&0\\0&1\end{pmatrix}$

$= (0i, -0j, -2k) = (0,0,-2)$

Standard form: $0(x-1) + 0(y-0) + -2(z-1) = 0 \quad (P_A)$

$\longrightarrow z = 1$

Example: Line: $\ell(t) =$ point $+ t$(direction) $\quad N = (1,2,3)$

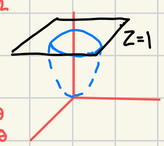
Line perpend. to $x+2y+3z=1$ thru $(-1,1,1)$

$\ell(t) = (-1,1,1) + t(1,2,3) \equiv (-1+t, 1+2t, 1+3t)$

Cylindrical

Example: $z = x^2+y^2$

$\Phi(x, y, x^2+y^2)$

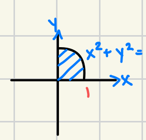


Restriction of 1^{st} octant below $z=1$

$0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

$\mapsto z = z$, $x = r\cos\theta$, $y = r\sin\theta$



$z = x^2+y^2 \to z = r^2$

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2) : \{z \le 1,\ x, y, z \ge 0\}$

$\to r^2 \le 1,\ r\cos\theta \ge 0,\ r\sin\theta \ge 0,\ \cancel{r^2 \ge 0}$

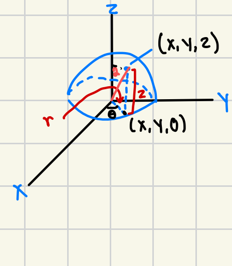
$\to 0 \le r \le 1 \qquad 0 \le \theta \le \frac{\pi}{2}$

Example: $x^2+y^2+z^2=4$

$x = r\cos\theta \qquad r^2+z^2=4$

$y = r\sin\theta \qquad z = 2\cos\varphi$

$r = 2\sin\varphi$



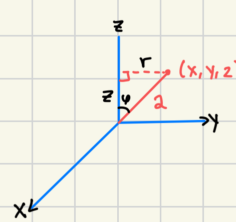
$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$x = 2\sin\varphi\cos\theta$

$y = 2\sin\varphi\sin\theta$

$z = 2\sin\varphi$



Lecture
June 20^{th}

Review: Cylindrical and Spherical Coordinates

$C(t) = (x(t), y(t), z(t))$

$\Phi(r,s) = (x(r,s), y(r,s), z(r,s))$

Definition The *spherical coordinates* of points (x, y, z) in space are the triples (ρ, θ, ϕ), defined as follows:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi, \quad (3)$$

where

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi.$$

Definition The *cylindrical coordinates* (r, θ, z) of a point (x, y, z) are defined by (see Figure 1.4.2)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \quad (1)$$

Example: $x + y + 2z = 4$

$x = 4 - y - 2z$

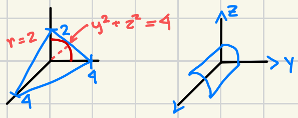
$\Phi(y,z) = (4-y-2z, y, z) \quad -\infty < y, z < \infty$

Example: $x + y + 2z = 4$ inside $y^2 + z^2 = 4$

$y = r\cos\theta$

$z = r\sin\theta$

$y^2 + z^2 = r^2$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$x + y + 2z = 4, \quad y^2 + z^2 \leq 4 \rightarrow r^2 \leq 4, \quad r \leq 2$

$x + r\cos\theta + 2r\sin\theta = 4$

$x = 4 - r\cos\theta - 2r\sin\theta$

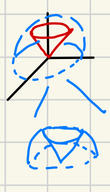
$\boxed{\Phi(r,\theta) = (4 - r\cos\theta - 2r\sin\theta, r\cos\theta, r\sin\theta): 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2}$

Example: Solid bounded by:

(inside) $x^2 + y^2 + z^2$

below $z = \sqrt{x^2+y^2}$

above xy-plane



$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

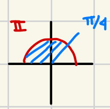
$z = \rho\cos\varphi$

$x^2 + y^2 + z^2 = \rho^2 \rightarrow x^2 + y^2 = \rho^2\sin^2\varphi: \quad 0 \leq \theta \leq 2\pi, \; 0 \leq \varphi \leq \pi$

$x^2 + y^2 + z^2 \leq 4 \rightarrow \rho^2 = 4 \Rightarrow 0 \leq \rho \leq 2$

$z \leq \sqrt{x^2+y^2} \rightarrow \rho\cos\varphi \leq \rho\sin\varphi$

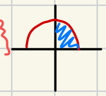
$\cos\varphi \leq \sin\varphi$



$\frac{\pi}{4} \leq \varphi \leq \pi$

$z \geq 0 \rightarrow \rho\cos\varphi \geq 0: 0 \leq \varphi \leq \frac{\pi}{2}$

$W = \{(\rho, \theta, \varphi): 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\}$



Applications of Surfaces and curves:

$T(x,y,z) = x^2 + y^2 + z^2 \overset{S}{=} \rho^2$

Limit:

$\lim_{(x,y,z)\to P} f(x,y,z) =$ expected value for $f(P)$

If $\lim_{(x,y,z)\to P} f(x,y,z) = f(P)$, we say f is continuous

Example:

$\lim_{(x,y)\to(3,0)} y e^{x - \sqrt{x^2 + \ln(y^2+1)}} = 0$

Example: $\lim_{(x,y)\to 0} \frac{xy^2}{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\frac{r^3\cos\theta\sin^2\theta}{r^2} =$

$= r\cos\theta\sin^2\theta$

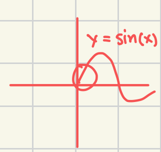
$\lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r\cos\theta\sin^2\theta$

$= 0$

Lecture: Limits
June 23, 2025

Example: $\lim_{(x,y)\to 0} \frac{y^3\sin(x^2+y^2)}{(x^2+y^2)^k}$, find $\forall k$ for which $\lim = 0$

$\lim_{t\to 0} \frac{\sin(t)}{t} \overset{L.H}{=} \lim_{t\to 0} \frac{\cos t}{1} = 1$



When $t \sim 0$: $\sin(t) = t$

$\lim_{r\to 0} \frac{r^3\sin^3\theta \cancel{\sin(r^2)}}{r^{2k}}$ $\rightarrow 0$, $\rightarrow r^2$

$\lim_{r\to 0} \frac{r^3\sin^3\theta\, r^2}{r^{2k}} = \lim_{r\to 0} r^{5-2k}\sin^3\theta$ ✓ ×

$= 0$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$5 - 2k > 0 \rightarrow k < 5/2$

Example: $\lim_{(x,y)\to 0} \frac{xy}{x^2+y^2}$

$x = r\cos\theta$

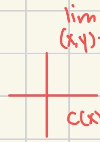
$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\lim_{(x,y)\to 0} \frac{r^2\cos\theta\sin\theta}{r^2}$

$\lim_{(x,y)\to 0} \cos\theta\sin\theta =$ $\theta = 0 \rightarrow 0$, $\theta = \frac{\pi}{4} \rightarrow 1/2$

$\theta = 0$



$C(x) = (x, 0)$

On $(x, 0)$:

$\frac{xy}{x^2+y^2} = 0 \rightarrow 0$

$\theta = \frac{\pi}{4}$: $C(x) = (x, x)$

On (x, x):

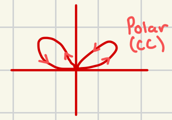
$\frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = 1/2 \rightarrow 1/2$

Exam Review:

1. Vectors
2. Parametrizations
3. Limits

Parametrization:

a) $x^2 y = x^4 + y^4$



$x = r\cos\theta$

$y = r\sin\theta$

$r^3\cos^2\theta\sin\theta = r^4(\cos^4\theta + \sin^4\theta)$

$\frac{\cos^2\theta\sin\theta}{\cos^4\theta + \sin^4\theta} = r$

$C(\theta) = \left(\frac{\cos^3\theta\sin\theta}{\cos^4\theta + \sin^4\theta}, \frac{\cos^3\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}\right)$

Algebraic restriction: None

Polar Restriction: $r \geq 0$: $\sin\theta \geq 0$

$0 \leq \theta \leq \pi$

Problem restriction: none

$0 \leq \theta \leq \pi$

b). Surfaces: $x^2 + z^2 \leq 1$

$y - 2 = x^2 + z^2$ inside $x^2 + z^2 = 1$

$y = x^2 + z^2 - z$

No restrictions:

$y = z + x^2 + z^2$

$(x, z + x^2 + z^2, z)$

$-\infty < x, z < \infty$

$x = r\cos\theta$

$z = r\sin\theta$

$x^2 + z^2 = r^2$

$y - r\sin\theta = r^2 \rightarrow y = r^2 + r\sin\theta$

$r^2 \leq 1$

$0 \leq r \leq 1$

$$\boxed{\begin{array}{c}\Phi(r,\theta) = (r\cos\theta, r^2 + r\sin\theta, r\sin\theta) \\ 0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi\end{array}}$$

1^{st} Octant

$x, y, z \geq 0$

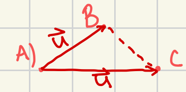
$0 \leq \theta \leq \pi/2$

$r^2 + r\sin\theta \geq 0$

Vectors:

$A = (1,-1,2), B = (1,2,3), C = (2,1,-1)$

a) Why are they not Collinear?



Scalar multiple

$\vec{u} = \vec{B} - \vec{A} = (0, 3, 1)$

$\vec{v} = \vec{C} - \vec{A} = (1, 2, -3)$

Not a multiple of each other.

B) Find the area of ΔABC using the appropriate Gram's Determinant.

Let $u = (0,3,1)$
$v = (1,2,-3)$

$\frac{1}{2}\sqrt{\begin{vmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v\end{vmatrix}} =$ Area ΔABC

$0^2 + 3^2 + 1^2 = 10$

$1^2 + 2^2 + (-3)^2 =$
$1 + 4 + 9 = 14$

$\frac{1}{2}\sqrt{\begin{vmatrix} 10 & 3 \\ 3 & 14\end{vmatrix}} = \boxed{\frac{\sqrt{131}}{2} = \text{Area } \Delta ABC}$

$0 + 6 - 3 = 3$

$ad - bc = 140 - 9 = 131$

C) Find the cartesian equation of the plane containing A, B, C.

$-9 - 2$

$u \times v = \begin{vmatrix} i & j & k \\ 0 & 3 & 1 \\ 1 & 2 & -3\end{vmatrix} = i\det\begin{pmatrix}3 & 1 \\ 2 & -3\end{pmatrix} - j\det\begin{pmatrix}0 & 1 \\ 1 & -3\end{pmatrix} + k\det\begin{pmatrix}0 & 3 \\ 1 & 2\end{pmatrix}$

$-11i + j - 3k$

$N = (-11, 1, -3)$ or $(11, -1, 3)$

Standard form: $11(x-1) - 1(y-2) + 3(z-3)$

$11x - 11 - y + 2 + 3z - 9 = 0$

$11x - y + 3z - 18 = 0$

$\rightarrow \boxed{11x - y + 3z = 18}$

Limits:

$f(x,y,z) = \frac{(x^4+y^4)^k \cos(x+z)}{\sqrt{x^2+y^2+z^2}}, \quad k > 0$

a) Rewrite f using spherical coordinates

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$(x^4 + y^4)^k = \rho^{4k}\sin^{4k}\phi\,(\cos^4\theta + \sin^4\theta)^k$

$\cos(x+z) = \cos(\rho(\sin\phi\cos\theta + \cos\phi))$

$x^2 + y^2 + z^2 = \rho^2$

$x^2 + y^2 = \rho^2\sin^2\phi \quad \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

$f = \rho^{4k-1}\sin^{4k}\phi(\cos^4\theta + \sin^4\theta)^k\cos(\rho(\sin\phi\cos\theta + \cos\phi))$

b) Find all $k > 0$ for which $\lim_{(x,y,z)\to 0} f(x,y,z) = 0$

$f = \underbrace{\rho^{4k-1}}_{0}\times\underbrace{\sin^{4k}\phi}_{\text{bounded}}\times\underbrace{(\cos^4\theta + \sin^4\theta)^k}_{\text{bounded}}\times\underbrace{\cos(\overbrace{\rho(\sin\phi\cos\theta + \cos\phi)}^{0 \times \text{bounded} = 0})}_{1}$

We want $\rho \rightarrow 0$

We want $4k - 1 > 0$

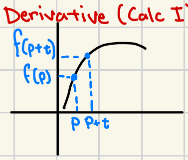
$\boxed{k > 1/4}$

$\rho^2 \rightarrow 0$
$\rho^3 \rightarrow 0$
$\rho^0 \rightarrow 1 \neq 0$

$\rho^{1/2} \rightarrow 0 \equiv \sqrt{\rho} \rightarrow 0$
$\rho^{-1} \rightarrow 0$
$\frac{1}{\rho} \rightarrow 0$

Applications of Limits:

Derivative (Calc I)



$$\lim_{t\to 0}\frac{f(p+t)-f(p)}{t}=f'(p)$$

Partial Derivatives (Calc II)

$$f_x=\lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t}$$
$$f_y=\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}$$

Discussion Section

June 24th, 2025

$$\lim_{(x,y)\to 0}\frac{y^k\ln(x^2+y^2)}{x^4+y^4}$$

$y=r\sin\theta$
$x=r\cos\theta$
$x^2+y^2=r^2$

$$\lim_{r\to 0}\frac{r^k\sin^k\theta\ln(r^2)}{r^4(\cos^4\theta+\sin^4\theta)}$$

$$\to\frac{r^{k-4}\sin^k\theta\ln(r^2)}{\cos^4\theta+\sin^4\theta}:\begin{cases}\lim_{r\to\infty}\ln(r)=\text{DNE}\ -\infty\\ \frac{\sin^k\theta}{\sin^4\theta+\cos^4\theta}\to\text{constant}\end{cases}$$

Fix θ, $r\to 0$

$$\to r^{k-4}\begin{cases}0 & k>4\\ 1 & k=4\\ \text{DNE} & k<4\end{cases}$$

$\lim_{(x,y)\to 0}$ "=" $-\infty\cdot$ const $\cdot\begin{cases}0\\ \infty\end{cases}$

$-\infty\cdot$ const $\cdot 0$

$$\frac{r^k}{r^4}\cdot\log(r^2)$$
$$=\frac{r^k\log(r^2)}{r^4}\approx\frac{\infty}{\infty}\text{ or }\frac{0}{0}$$
$$=\lim_{r\to 0}\frac{\log(r^2)}{r^{4-k}}$$
$$=\lim_{r\to 0}$$

Lecture

June 27, 2025

x-direction: $f:\mathbb{R}^2\to\mathbb{R}$

$$\lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t}=f_x=\frac{\partial f}{\partial x}$$

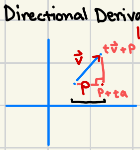
y-direction:

$$\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}=f_y=\frac{\partial f}{\partial y}$$

Example:

$$f=\frac{xy}{x^2+y^2}\qquad f_x=\frac{y(x^2+y^2)-xy(2x)}{(x^2+y^2)^2}$$

Directional Derivative



$$\lim_{t\to 0}\frac{f(P+t\vec v)-f(P)}{t}=f_{\vec v}(P)$$

Remark: $f_{-\vec v}=-f_{\vec v}$

Computation of the directional derivative:

$\vec v=(a,b)$

Then $f_{\vec v}(P)=af_x(P)+bf_y(P)$

Example: $f=\frac{x}{\sqrt{x^2+y^2+z^2+3}}$

Find all directions in which f increases when moving away from $(2,-1,1)$? $\to f_{\vec v}(2,-1,1)>0$

$$f_x=\frac{\sqrt{x^2+y^2+z^2+3}-\frac{x^2}{\sqrt{x^2+y^2+z^2+3}}}{(x^2+y^2+z^2+3)}$$

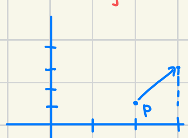
$$f_x(2,-1,1)=\frac{3-\frac{4}{3}}{9}=\frac{5}{27}$$
$$f_y=\frac{-2x}{(x^2+y^2+z^2+3)^{3/2}}=f_y(2,-1,1)=\frac{2}{27}$$
$$f_z=\frac{-2x}{(x^2+y^2+z^2+3)^{3/2}}=f_z(2,-1,1)=-\frac{2}{27}?$$

$$\frac{5}{27}a+\frac{2}{27}b-\frac{2}{27}c>0$$

$\boxed{\text{All directions }(a,b,c)\text{ with }5a+2b-2c>0}$

Example:

$f: 2x^5-76y^3$, is f increasing or decreasing at $P=(2,1)$ when moving in the direction $\vec v=(1,2)$? $\} f_{\vec v}(P)=+$ or $-$?



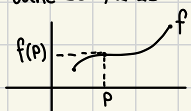
Decreasing!

$f_x=10x^4$
$f_y=-228y^2$

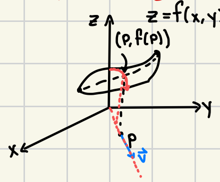
$$\hat v=\frac{(1,2)}{\|(1,2)\|}=\frac{1}{\sqrt5}(1,2)$$
$$f_{\vec v}=\frac{10x^4}{\sqrt5}-\frac{2(228y^2)}{\sqrt5}$$
$$f_{\vec v}(P)=f_{\vec v}(2,1)=\frac{160-456}{\sqrt5}=\frac{-296}{\sqrt5}$$

Lecture

June 30th, 2025

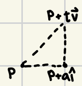


Multivariate:



$\vec v$ = unit vector (direction) Where $\vec v=(a,b)$

$$f_{\vec v}(P)=\lim_{t\to 0}\frac{f(P+t\vec v)-f(P)}{t}\equiv af_x(P)+bf_y(P)$$



When $t\approx 0$: $f(P+t\vec v)\approx f(P)+f_{\vec v}(P)t$

$f(P+h)\approx f(P)+f_{\vec v}(P)\cdot|h|$

$f(P+ta\hat i)\approx f(P)+f_x(P)ta$

$f(P+t\vec v)\approx f(P+ta\hat i)+f_y(P+ta\hat i)tb$

$\approx f(P)+(f_x(P)a+f_y(P+ta\hat i)b)t$

$\approx f(P)+\underbrace{(f_x(P)a+f_y(P)b)}_{f_{\vec v}(P)}t$

$$f_x(P)a+f_y(P)b\equiv\underbrace{\overbrace{(f_x(P),f_y(P))}^{\nabla f(P)}}_{\text{the gradient of }f}\cdot\overbrace{(a,b)}^{\vec v}$$

$$f(P+\overbrace{t\vec v}^{\vec h})\approx f(P)+\nabla f(P)\cdot\vec v t$$

$f(P+\vec h)=f(P)+\nabla f(P)\cdot\vec h$ (Linear Approximation)

Example: Q near P

$f(Q)\approx f(P)+\nabla f(P)\cdot\vec h$

$\vec h=Q-P$

Example: $f(x,y)=(e^x-y,\ \sin(x-y))$

a) Write down the linear approximation near $P=(0,0)$

$$f(x,y)=\overbrace{(1,0)}^{f(0,0)}+(\Delta x-\Delta y,\ \Delta x-\Delta y)$$

$\boxed{f(x,y)=(1+x-y,x-y)}$

$f_1=e^x-y\approx 1+\overbrace{(1,-1)}^{\nabla f(P)}\cdot\overbrace{(\Delta x,\Delta y)}^{\vec h}$

$f_x=e^x\quad\nabla f(P)=(e^0,-1)$

$f_y=-1$

$e^x-y=(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots)-y$

$\sin(x-y)=x-y$

$\sin(t)\approx t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots$

Lecture - Jacobian Matrix

July 2nd, 2025

$f:\mathbb{R}^n\to\mathbb{R}$

$$f(x_1,\ldots,x_n)\approx f(P)+f_{x_1}(P)\Delta x_1+f_{x_2}(P)\Delta x_2+\ldots+f_{x_n}(P)\Delta x_n$$
(near P)

$\equiv f(P)+\nabla f(P)\cdot h$ where $\nabla f(P)=(f_{x_1}(P),\ldots,f_{x_n}(P))$ and $h=(\Delta x_1,\ldots,\Delta x_n)$

Example: $f(x,y)=(\cos(x+y),\ e^{2-x+y})$ near $P=(1,-1)$

$f_1=\cos(x+y)\quad\nabla f_1(1,-1)=(-\sin(x+y),-\sin(x+y))|_{(1,-1)}=(0,0)$

$f_1\approx f(1,-1)+0\Delta x+0\Delta y=1$

$f_2=e^{2-x+y}\quad\nabla f_2(1,-1)=(-e^{2-x+y},e^{2-x+y})|_{(1,-1)}=(-1,1)$

$f_2\approx f_2(1,-1)+(-1)\Delta x+(1)\Delta y=1-\Delta x+\Delta y$

$\Rightarrow f\approx(1,1-\Delta x+\Delta y)$

$\Delta x=x-1$
$\Delta y=y+1$

Examples: $2x+3y-2z=1$ normal $\vec v=(2,-3,-2)$

$P=(1/2,0,0)$

$x^2+y^2+z^2=1\longrightarrow x^3+y^2+z^2=1$, $P=(-1,1,1)$

$x^3+y^2+z^2-1=0=f$

$f(P)=0$

$f\approx 0+3\Delta x+2\Delta y+2\Delta z$

$\nabla f=(3x^2,2y,2z)$

$\nabla f(P)=(3,2,2)$

$\Delta x=x+1$
$\Delta y=y-1$
$\Delta z=z-1$

Actual surface $f=0$

Linear Approx.

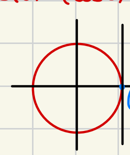
Tangent Plane $\to 3(x+1)+2(y-1)+2(z-1)=0$

$f=0$ curve/surface, P

$\nabla f(P)$ = normal vector

$c(\theta)=(\cos\theta,\sin\theta)$, $0\le\theta\le 2\pi$

$\theta=0$



$L(\theta)=(1,\theta)$

$$f(x,y,z)=(f_1,f_2)$$
$$\approx(f_1(P)+\nabla f_1(P)\cdot h,\ f_2(P)+\nabla f_2(P)\cdot h)$$
$$=(f_1(P),f_2(P))+(f_{1x}\Delta x+f_{1y}\Delta y+f_{1z}\Delta z,\ f_{2x}\Delta x+f_{2y}\Delta y+f_{2z}\Delta z)$$

$$=\begin{bmatrix}f_1(P)\\ f_2(P)\end{bmatrix}+\underbrace{\begin{bmatrix}f_{1x}(P) & f_{1y}(P) & f_{1z}(P)\\ f_{2x}(P) & f_{2y}(P) & f_{2z}(P)\end{bmatrix}}_{D_Pf}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}$$

Lecture

July 7^{th}, 2025

Linear Approximation

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f = f(x_1, \dots, x_n) = (f_1(x_1,\dots,x_n), \dots, f_m(x_1,\dots,x_n))$

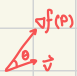
Near P:

$f(x_1,\dots,x_n) \approx f(P) + f_{x_1}(P)\Delta x_1 + \dots + f_{x_n}(P)\Delta x_n$

When $m=1$: $f(x_1,\dots,x_n) \approx f(P) + \nabla f(P)\cdot h$, where $h = (\Delta x_1, \dots, \Delta x_n)$

$m>1$: $f(x_1,\dots,x_n) \approx f(P) + (D_P f)(h)$, where $D_P f = \begin{bmatrix} \nabla f_1(P) \\ \nabla f_m(P) \end{bmatrix}$

Ways to interpret $D_P f$



$f: \mathbb{R}^n \longmapsto \mathbb{R}$ (temperature, etc.

$f_{\vec v}(P) = \nabla f(P)\cdot \vec v = |\nabla f(P)| \cos\theta$

$\nabla f(P)$ = the direction of largest increase

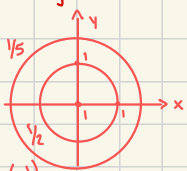
$-\nabla f(P)$ = the direction of largest decrease

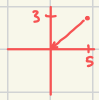
Example: $f(x,y) = \frac{1}{1+x^2+y^2}$

$\nabla f(5,3) = \quad (-5,-3)$

$\nabla f = \left(-\frac{2x}{(1+x^2+y^2)^2}, -\frac{2y}{(1+x^2+y^2)^2}\right)$

$\nabla f(5,3) = \left(-\frac{2}{35^2}(5), \frac{-2}{35^2}(3)\right)$





Second Interpretation of $D_P f$

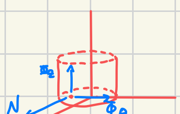
$f(x,y,z) = 0$

$0 = f_x(P)(x-P_x) + f_y(P)(y-P_y) + f_z(P)(z-P_z)$

$\nabla f(P)$ = normal vector to the surface.

Final Intepretation of $D_P f$

Example: $x^2+y^2=1$ in $\mathbb{R}^3$



$r^2 = 1 \longleftrightarrow r=1$

$\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$

$P = (0,1) \to \theta = 0, z=1 \longmapsto (x=1, y=0, z=1)$

$\Phi(\theta, z) \approx \Phi(P) + \Phi_\theta(P)\Delta\theta + \Phi_z(P)\Delta z$

$N = \Phi_\theta \times \Phi_z = (0,1,0)\times(0,0,1) = (1,0,0)$ outward normal.

Tangent Vectors:

$\Phi_\theta = (-\sin\theta, \cos\theta, 0)\Big|_{\theta=0, z=1} = (0,1,0)$

$\Phi_z(P) = (0,0,1)\Big|_{\theta=0, z=1}$

$f(x,y,z) = (x - e^{x^2-y}, z\cos(x), y\sin(z))$

Quadratic Approximation:

at $(0,0,0)$

$f = (x-e^{x^2-y}, z\cos(x), y\sin(z))$ near $P=(0,0,0)$

$e^t = 1 + t + \frac{t^2}{2} + \dots$

$\cos(t) = 1 - \frac{t^2}{2} + \dots$

$\sin(t) = t - \frac{t^3}{3!} + \dots$

$x - e^{x^2-y} = x(1 + (x^2-y) + \frac{(x^2-y)^2}{2} + \dots)$ $\quad \cancel{x^4} - \cancel{2x^2}y + y^2$

$\approx x - 1 - x^2 + y - \frac{1}{2}y^2$

$z\cos(x) = z(1 - \frac{x^2}{2} + \dots) \approx z - \cancel{x} \approx z$

$y\sin(z) = y(z - \dots) \approx yz$

$F = (x - 1 - x^2 + y - \frac{1}{2}y^2, z, yz)$

$f: \mathbb{R}^n \to \mathbb{R}$

To simplify: $n=2$

near $P = (x_0, y_0)$:

$f(x,y) \approx f(P) + f_x(P)\Delta x + f_y(P)\Delta y$

$f(P) + \frac{1}{2}(f_{xx}(P)\Delta x^2 + 2f_{xy}(P)\Delta x\Delta y + f_{yy}(P)\Delta y^2)$

$f(x,y) \approx f(P) + \nabla f(P)\cdot h + \frac{1}{2}[\Delta x \Delta y]\underbrace{\begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P)\end{bmatrix}}_{D^2 f = D\nabla f = H(f(P))}\underbrace{\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}}_{h}$

$\equiv f(P) + \nabla f(P)\cdot h + \frac{1}{2} h^T \cdot h \cdot \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P)\end{bmatrix}$

$\frac{1}{2}h^T H(f(P)) h$

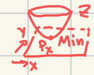
Determinants:

$H_f = \begin{array}{cc} D_1 & D_2 \\ \end{array} \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$x^2 + 2xy + 2y^2 \equiv (x^2+y^2) + y^2$

Max: When $\left.\begin{array}{l} D_1 = \text{negative \#} \\ D_2 = \text{positive \#}\end{array}\right\} \begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix}$

Min: When $\left.\begin{array}{l} D_1 = \text{positive \#} \\ D_2 = \text{positive \#}\end{array}\right\} \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$



$z = f(x,y) \qquad \nabla f(P) = 0$

$H(f(P)) > 0$

$\therefore P$ is a critical point $\longrightarrow$ Must be positive

$f(x,y) \approx f(P) + \frac{1}{2}h^T H(f(P)) h$

Lets look at possible quadractic portions:

- $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = H(f(P)) \longleftrightarrow a\Delta x^2 + b\Delta y^2$

Positive: When (min) $a, b > 0$ $\qquad$ Negative: When (max) $a, b < 0$

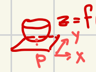
Or a saddle point when

$a > \emptyset, b < \emptyset \vee a < \emptyset, b > \emptyset$

$2\Delta x^2 - \Delta y^2$

Lecture

July 11^{th}, 2025



$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$

$a\Delta x^2 + b\Delta y^2$

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{bmatrix}$

$a\Delta x^2 + b\Delta y^2 + c\Delta z^2$

min: $\begin{bmatrix} + & & 0 \\ & + & \\ 0 & & + \end{bmatrix}$ $\quad D_1, D_2, D_3$

$\{D_1, D_2, D_3\} = +, +, +$

max: $\begin{bmatrix} (-) & & 0 \\ & (-) & \\ 0 & & (-) \end{bmatrix}$ $\quad D_1, D_2, D_3$

$\{D_1, D_2, D_3\} = -, +, -$

$\nabla f(P) = 0$

$H(f(P)) = \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P)\end{bmatrix}$

$f(P) + h^T H(f(P)) h$

$> 0 \Rightarrow$ min

$< 0 \Rightarrow$ max

otherwise $\Rightarrow$ saddle

saddle: else if

$\det H(f) \neq 0$.

If $\det H(f) = 0$: inconclusive.

Quiz

b) $f = x^3 + 3y^2 + z^2 + 3xz + 3xy$

$f_x = 3x^2 + 3z + 3y = 0 \to 4y^2 + 3y + y = 0$

$f_y = 6y + 3x = 0 \longrightarrow x = -2y$

$f_z = 2z + 3x = 0 \to 2z - 6y = 0 \to z = 3y$

$\to 4y^2 + 4y = 0 \longrightarrow y^2 + y = 0$

$y(y+1) = 0 \begin{cases} y = 0 \\ y = -1 \end{cases}$

$\nabla f = (3x^2 + 3z + 3y, \; 6y + 3x, \; 2z + 3x)$

$y = 0$	$y = -1$
$x = -2y = 0$	$x = 2$
$z = 3y = 0$	$z = -3$
$(0,0,0)$	$(2,-1,3)$

$H(f(P)) = \begin{bmatrix} 6x & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

$H(f(0)) = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2\end{bmatrix}$

$\det(H(f(0))) = 0\cancel{\begin{vmatrix} 6 & 0 \\ 0 & 2\end{vmatrix}} - 3\begin{vmatrix} 3 & 0 \\ 3 & 2\end{vmatrix} + 3\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix} = (-)$ $\quad$ (0, 6, −18)

$D_1 = 0?$

$D_2 = -9$

D_1, D_2, D_3: $\{?, -, -\}$: Saddle

$H(f(2,-1,-3)) = \begin{bmatrix} 12 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

$D_1 = +$

$D_2 = +$

$D_3 = 12\begin{vmatrix} 6 & 0 \\ 0 & 2\end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 3 & 2\end{vmatrix} + 3\begin{vmatrix} 3 & 6 \\ 3 & 0\end{vmatrix}$ $\quad$ (12, 6, −18)

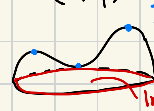
$= 144 - 18 - 3(18) = +$

Positive-definite

$\therefore$ minimum

$z = f(x,y)$

3 crit. points



Infinitely many boundary points.

LaGrange Multiples :

Q_1: Find max. of $f(x,y)$ when $g(x,y) = 0$

Example: $f(x,y) = y - x^2$ subject to $x^2+y^2 = 1$

$\equiv g = x^2+y^2-1$

$f = y - x^2 = 0$

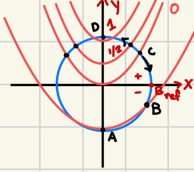
$y = x^2$

$f = 1 = y - x^2$

$y = x^2+1$

$f = -1 = y - x^2$

$y = x^2 - 1$



$\left.\begin{array}{l} C \to B\ (>0) \\ C \to D\ (<0) \end{array}\right\}$ Transversal intersect.

$C \to$ Saddle

$B_{ref} \to$ Saddle

$\{A, D\}$ = tangential intersect.

$A \to$ Local max

$D \to$ Local max

Candidate: Tangential intersection

$$\boxed{\nabla f = \lambda \nabla g}$$

$$g = 0$$