Quiz 08/06

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Let M be the portion of the paraboloid $z = x^2 + y^2$ in the first octant inside the cylinder

$x^2 + y^2 = 3$ and between the plane $y = x$ and the xz-plane $y = 0$.

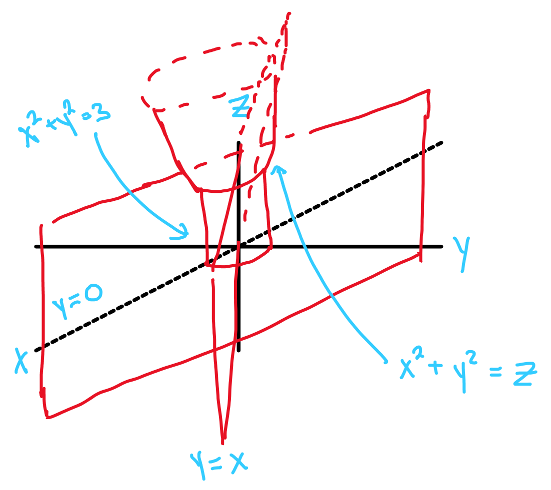
$z = x^2 + y^2 \longrightarrow r, \theta:\ x = r\cos\theta,\ y = r\sin\theta \longmapsto z = x^2 + y^2 = r^2$

$\therefore r = \sqrt{3}$ when $x^2 + y^2 = 3$.

a) Sketch and find a parametrization $\Phi(r,\theta)$ for our surface and describe its domain. Then compute the normal vector $N = \Phi_r \times \Phi_\theta$ for the parametrization. Is the normal vector upward (+z) or downward (-z)?

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2) : \{(0 \le r \le \sqrt{3}), (0 \le \theta \le \frac{\pi}{4})\}$

Sketch:



$\Phi_r = (\cos\theta, \sin\theta, 2r)$

$\Phi_\theta = (-r\sin\theta, r\cos\theta, 0)$

$$\Phi_r \times \Phi_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} \sin\theta & 2r \\ r\cos\theta & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} \cos\theta & 2r \\ -r\sin\theta & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$-(2r^2\cos\theta) - -(-2r^2\sin\theta) + (r\cos^2\theta - (-r\sin^2\theta))$

$= \hat{i}(-2r^2\cos\theta) + \hat{j}(-2r^2\sin\theta) + \hat{k}\cdot r$

$N \equiv \therefore \Phi_r \times \Phi_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

Since $r > 0$, such that r is positive, then N is pointing upward!

Let's now consider a fluid with (mass) density:

$$\mu(x,y,z) = \frac{1}{1 + x^2 + y^2}$$

measured in g/in^3 flowing with velocity

$$v(x,y,z) = (x, y, 1)$$

measured in in/s.

b) What are the units for the vector field μv? What do you think this quantity represents? What are the units for the surface integral

$$\int_M \mu v \cdot \hat{N}\, \underbrace{dS}_{in^2}?$$

We know that:

$\{[\mu] = \frac{g}{in^3}, [v] = \frac{in}{s}\}$ = vector field (mass flux density)

$\therefore \int_M [\mu][v]\cdot\hat{N}\, dS \longrightarrow \left(\frac{g}{in^3}\cdot\frac{in}{s}\right)\cdot(in^2) \Longrightarrow [g/s] \rightarrow \int_M \mu v\cdot\hat{N}\, dS$ is the mass flow rate through the surf (M).

c) Find the net amount of fluid that passes through our surface M in a 4 second window (in abs. val.). In other words, compute

$$4\int_M \mu v \cdot \hat{N}\, dS$$

What does the sign of my answer mean? Is the net flux upward or downward?

Paramaterize the integral

Via convention $N\,dr\,d\theta = \hat{N}\, dS$

$(x, y, z) \longmapsto (r, \theta)$:

$\mu = \frac{1}{1+r^2}$, $v = (r\cos\theta, r\sin\theta, 1)$

$\mu\cdot v = \left(\frac{r\cos\theta}{1+r^2}, \frac{r\sin\theta}{1+r^2}, \frac{1}{1+r^2}\right)$

$(\mu v)\cdot N = \frac{r\cos\theta}{1+r^2}(-2r^2\cos\theta) + \frac{r\sin\theta}{1+r^2}(-2r^2\sin\theta) + \frac{1}{1+r^2}(r)$

$= \frac{-2r^3\cos^2\theta}{1+r^2} + \frac{-2r^3\sin^2\theta}{1+r^2} + \frac{r}{1+r^2}$

$= -2r^3\left(\frac{\cos^2\theta + \sin^2\theta}{1+r^2}\right) + \frac{r}{1+r^2}$

$= \frac{r - 2r^3}{1+r^2}$

let $u = 1 + r^2$, $du = 2r\,dr$, $\frac{1}{2}du = dr$ $\} \frac{3}{2}\int \frac{1}{u}du \rightarrow \frac{3}{2}\ln(u) \rightarrow \frac{3}{2}\ln(1+r^2)$

$$4\int_0^{\pi/4}\int_0^{\sqrt{3}} \frac{r-2r^3}{1+r^2}\, dr\, d\theta$$

$$\int_0^{\sqrt{3}} \frac{r - 2r^3}{1+r^2}\, dr \equiv \int_0^{\sqrt{3}} -2r + \frac{3r}{1+r^2}\, dr$$

$\equiv \left[-r^2\right]_0^{\sqrt{3}} + \left[\frac{3}{2}\ln(1+r^2)\right]_0^{\sqrt{3}}$

$\left(\frac{3}{2}\ln(1+3)\right) - \left(\frac{3}{2}\ln(1+0)\right)$

$\equiv -3 + \frac{3}{2}\ln(4)$

$\frac{3}{2}\ln(4) \equiv \frac{3}{2}\ln(4) - \frac{3}{2}\ln(1)$

$\equiv 3(\ln(2) - 1)$

$4\cdot(3(\ln(2)-1))\int_0^{\pi/4} d\theta$

$\equiv \frac{\pi}{\cancel{4}}\cancel{4}\,(3(\ln(2)-1))$

$\boxed{= 3\pi(\ln(2)-1) = -2.892 \text{ (the net flow is downward)} \quad |-2.892| = 2.892}$

d) Find the average mass flow rate for this flux.

In other words, divide the answer you got in (c) by 4

to find how much fluid passes through M per second,

and then divide that by the area of M to find how much

mass passes through M per unit of area every second (on avg.).

Now, how do we find the area of M?

$$\int_M 1\, dS = \iint_D |\Phi_r \times \Phi_\theta|\, dr\, d\theta.$$

Based on the instructions:

Amount of fluid passing through M is:

$\frac{3\pi}{4}(\ln(2)-1)$

Compute:

$\iint_D |\Phi_r \times \Phi_\theta|\, dr\, d\theta$

$\sqrt{(-2r^2\cos\theta)^2 + (-2r^2\sin\theta)^2 + (r)^2} = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$

$= \sqrt{4r^4 + r^2} \equiv r\sqrt{4r^2+1}$

$= \int_0^{\pi/4}\int_0^{\sqrt{3}} r\sqrt{4r^2+1}\, dr\, d\theta$ let $u = 4r^2 + 1$, $du = 8r\,dr$, $\frac{1}{8}du = r\,dr$

$\frac{1}{8}\int u^{1/2}\, du$

$\frac{1}{8}\left(\frac{2}{3}u^{3/2}\right) = \frac{1}{12}u^{3/2} \xmapsto{x} \frac{1}{12}(4r^2+1)^{3/2}$

$\left[\frac{(4r^2+1)^{3/2}}{12}\right]_0^{\sqrt{3}} = \left(\frac{\sqrt{(\cancel{4(3)+1})\,13)^3}}{12}\right) - \left(\frac{1}{12}\right)$

$= \frac{\sqrt{(13)^3} - 1}{12} \equiv \frac{13\sqrt{13} - 1}{12}$

$\frac{13\sqrt{13}-1}{12}\int_0^{\pi/4} d\theta = \frac{(13\sqrt{13}-1)\pi}{48}$

Final step (divide the results)

$= \frac{3\cancel{\pi}(\ln(2)-1)}{4} \cdot \frac{48}{(13\sqrt{13}-1)\cancel{\pi}}$

$\boxed{= \frac{36(\ln(2)-1)}{13\sqrt{13}-1}\left[\frac{g}{s\cdot in^2}\right]}$

downward (net flow) Avg. mass flow rate!