



# MATH Vector Calculus

June 6<sup>th</sup>, 2025

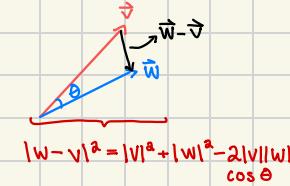
$$\int_c^d \int_a^b f(x,y) dx dy$$

Not the real world,  
only part!

Example #2:  $\vec{v} = (1, 2, 3)$  and  $\vec{w} = (-1, 2, 1)$

$$\hat{v} = \frac{(1, 2, 3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{(1, 2, 3)}{\sqrt{14}} = \left( \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$$

$$\hat{w} = \frac{(-1, 2, 1)}{\sqrt{(-1)^2 + 2^2 + 1^2}} = \frac{(-1, 2, 1)}{\sqrt{6}} = \frac{1}{\sqrt{6}} (-1, 2, 1)$$



Reviewing Vectors:

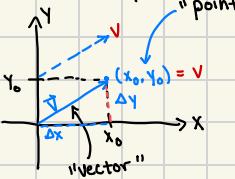
$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathbb{R}^n = \{(a_1, \dots, a_n) : a_i \text{ are real numbers}\}$$

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

$$\mathbb{R}^n \rightarrow (x_1, \dots, x_n) \text{ "vectors"}$$

$$\mathbb{R}^2 \rightarrow (x, y) \text{ "point"}$$



Points Arithmetic

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad \text{Scalar Addition}$$

$$\lambda(x, y) = (\lambda x, \lambda y) \quad \text{Scalar Multiplication}$$

Example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix} \rightarrow (-5, -7)$$

Point = TIP-TAIL (words)

$$\text{"Vector = direction + magnitude"} \quad \hat{v} = \frac{v}{|v|}$$

$$|v|$$

$$\rightarrow \vec{v} = \hat{v} \cdot |v|$$

$$\text{Why 2: } |w-v|^2 = (w-v) \cdot (w-v)$$

$$\equiv |w|^2 - 2 \cdot w \cdot v + |v|^2$$

$$\rightarrow v \cdot w = |v||w|\cos\theta$$

$$\rightarrow \cos\theta = \frac{v \cdot w}{|v||w|}$$

$$\sqrt{v^2} = |v|$$

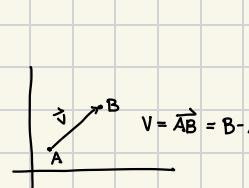
$$v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2 \rightarrow (v \cdot w) \cdot w = \lambda(v \cdot w)$$

$$v \cdot w = w \cdot v$$

Dot Product:

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$\text{NOTE! } \vec{v} = (a_1, \dots, a_n) : \vec{v} \cdot \vec{v} = a_1^2 + \dots + a_n^2$$



Discussion Section

June 10<sup>th</sup>, 2025.

$$w_1 = (2, -1, 0, 2)$$

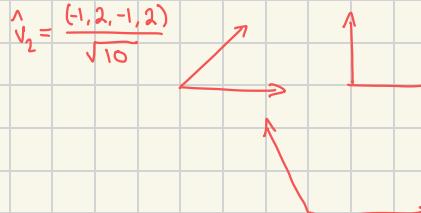
$$v_1 = (1, 2, -1, 2)$$

$$w_1 \cdot v_1 = \sum_{i=1}^n a_i \cdot b_i \text{ where } n=4$$

$$= 4$$

$$\cos\theta = \frac{v \cdot w}{|v||w|} = \frac{4}{3\sqrt{10}}$$

$$\text{proj } w_1 = v_1 \cdot \hat{v}_2 = |v_1| \cos\theta = \frac{4}{\sqrt{10}} = \frac{4}{\sqrt{10}}$$



Dot Products:

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^n a_i b_i$$

$$v \cdot v = \sum_{i=1}^n a_i^2 = |v|^2$$

$$\text{if } v = (a_1, \dots, a_n)$$

$$v \cdot w = |v||w|\cos\theta$$

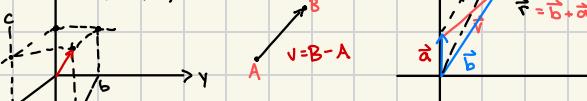
$$\frac{|v| \cos\theta}{|w|} = \frac{v \cdot w}{|w|}$$

$$|v| \cos\theta \hat{w} = \frac{v \cdot w}{|w|} \cdot \hat{w} \quad (\text{vector projection})$$

June 9<sup>th</sup>, 2025

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \text{ is a real number}\}$$

$$(a, b, c)$$



Norm:

$$v = (x_1, \dots, x_n)$$

$$\|v\| = \sqrt{\sum_{i=0}^n x_i^2}$$

$$\text{Example #1: } |(-1, 2, 3)| = \sqrt{(-1)^2 + 2^2 + 3^2}$$

$$= \sqrt{1 + 4 + 9}$$

$$= \sqrt{14}$$

$$\text{hyp} = \sqrt{a^2 + b^2} = \|v\|$$

length of the vector

$$\sqrt{a^2 + b^2}$$

$$\sqrt{a^2 + b^2 + c^2} = \|v\|$$

$$\sqrt{a^2 + b^2 + c^2}$$

$$\sqrt{a^2 + b^2 + c^2} = \|v\|$$

Unit vector

If  $\vec{v} \neq 0$ , then  $|\vec{v}| \neq 0$  and

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

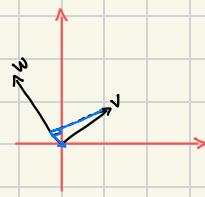
$$|\hat{v}| = \frac{|\vec{v}|}{|\vec{v}|} = 1$$

|   |
|---|
| $\lambda(a, b) = (\lambda a, \lambda b)$                            |
| $\sqrt{a^2 + b^2}$  |
| $\sqrt{\lambda^2 a^2 + \lambda^2 b^2} =  \lambda  \sqrt{a^2 + b^2}$ |

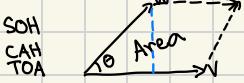
We say two vectors  $\vec{v}, \vec{w}$  have the same direction if  $\vec{v} = \hat{w}$ .

Example #1: Find the projection of  $\vec{v} = (2, 1)$  onto  $\vec{w} = (-1, 3)$

Scalar:  $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{10}}$   
 $|\vec{w}| = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}$   
 $\vec{w}$  length  
 $\vec{w}$  direction  
 $\vec{v}$  Vector:  $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w} = \frac{1}{10} (-1, 3) = \left( -\frac{1}{10}, \frac{3}{10} \right)$



Areas:

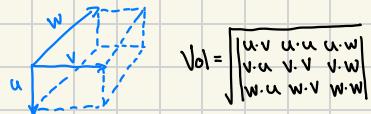


$$\text{Area} = |\vec{v}||\vec{w}| \sin \theta$$

$$\begin{aligned} A^2 &= |\vec{v}|^2 |\vec{w}|^2 \sin^2 \theta \\ &= |\vec{v}^2| |\vec{w}^2| (1 - \cos^2 \theta) \\ &= |\vec{v}^2| |\vec{w}^2| - |\vec{v}^2| |\vec{w}^2| \cos^2 \theta \\ A^2 &= \vec{v}^2 \vec{w}^2 - (\vec{v} \cdot \vec{w})^2 = \frac{|\vec{v} \cdot \vec{w}|}{|\vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w}|} \\ A_{\text{adj}} &= \sqrt{|\vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w}|} \end{aligned}$$

"Gram determinant"

3D:



$$|\vec{v} \times \vec{w}| = \begin{vmatrix} \vec{u} \cdot \vec{v} & \vec{u} \cdot \vec{w} & \vec{u} \cdot \vec{u} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{w} & \vec{v} \cdot \vec{v} \\ \vec{w} \cdot \vec{u} & \vec{w} \cdot \vec{v} & \vec{w} \cdot \vec{w} \end{vmatrix}$$

$\mathbb{R}^3$ : Cross Product

$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$i = (1, 0, 0)$   
 $j = (0, 1, 0)$   
 $k = (0, 0, 1)$

$$\begin{aligned} &= i \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \\ &\equiv \left( \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \end{aligned}$$

Discussion Section

June 12<sup>th</sup>, 2025

$$\vec{v} = (-1, 1, 0, 1, -1)$$

$$2 \cdot (-1) + 1 \cdot (-1) + 0 \cdot (-1) + 0 \cdot (1) + 1 \cdot (-1) = -2 - 1 - 1 = -4$$

$$1+1+1+1+1 = 5$$

$$4+1+1+1 = 7$$

$$\begin{bmatrix} \vec{v} \cdot \vec{v} & \vec{v} \cdot \vec{w} \\ \vec{w} \cdot \vec{v} & \vec{w} \cdot \vec{w} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}$$

$$\det \begin{pmatrix} 4 & -4 \\ -4 & 7 \end{pmatrix} = 4(7) - (-4)(-4) = 4(7) - 4(4) = 4(3) = 12$$

$$28 - 16 = \sqrt{12} = A^2$$

$$\sqrt{12} = A$$

$$\vec{A} = (1, 1, 1)$$

$$\vec{B} = (-1, 2, 1) \quad \vec{AB} = (-2, 1, 0) = \vec{B} - \vec{A}$$

$$\vec{C} = (3, 1, -1) \quad \vec{AC} = (2, 0, -2) = \vec{C} - \vec{A}$$

$$\vec{AB} \times \vec{AC} =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = \vec{i} \det \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} - \vec{j} \det \begin{pmatrix} -2 & 0 \\ 2 & -2 \end{pmatrix} + \vec{k} \det \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}$$

$$\begin{aligned} -2+0 &= -2 & 4-0 &= 4 & -2 \\ -2\vec{i} - 4\vec{j} - 2\vec{k} & & & & \\ \equiv & (-2, -4, -2) & & & \end{aligned}$$

$$\begin{aligned} &= \sqrt{24} = |(-2, -4, -2)| \\ &= \frac{1}{2} \cdot \sqrt{24} = \sqrt{6} \end{aligned}$$

Lecture

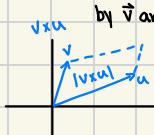
June 13<sup>th</sup>

$$\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} & \vec{j} \times \vec{i} &= -\vec{k} \\ \vec{i} \times \vec{i} &= \vec{0} & \vec{v} \times \vec{v} &= -\vec{v} \times \vec{v} = \vec{0} \\ \vec{v} \times (\vec{u} + \vec{w}) &= \vec{v} \times \vec{u} + \vec{v} \times \vec{w} & \vec{v} \times (\lambda \vec{u}) &= (\lambda \vec{v}) \times \vec{u} \equiv \lambda(\vec{v} \times \vec{u}) \end{aligned}$$

$$(1, 2, 3) \times (2, 3, 4) = (i+2j+3k) \times (2i+3j+4k)$$

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) \times (\vec{u}_1, \vec{u}_2, \vec{u}_3) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

Application:  $|\vec{v} \times \vec{u}| = \text{area of the parallelogram spanned by } \vec{v} \text{ and } \vec{u}$



$$\begin{aligned} G(1) - G(1)(1) &= 5 \\ 2(1) - (1)(1) &= 1 \\ 2(1) - G(1) &= -4 \end{aligned}$$

$$\text{Example: } \vec{v}_1 = (1, 1, 1) \quad \vec{v}_1 \cdot \vec{v}_2 = -1 + 1 + 2 = 2$$

$$\vec{v}_2 = (-1, 1, 2) \quad \vec{v}_1 \cdot \vec{v}_3 = 0 + 1 + 0 = 1$$

$$\vec{v}_3 = (0, 1, 0) \quad \vec{v}_2 \cdot \vec{v}_3 = 0 + 1 + 0 = 1$$

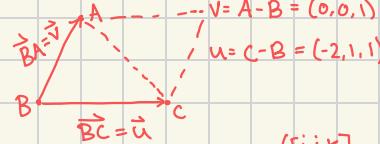
$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{vmatrix} = a \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - b \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + c \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 3 \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + 1 \det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 3 \cdot 1 - 2 \cdot 1 + 1 \cdot 1 = 2$$

$$3 \cdot 5 - 2 \cdot 1 + (-4) = 15 - 2 - 4 = 9$$

$$= 15 - 6 = 9$$

$$= \sqrt{9} = 3$$

$$\text{Example: } \vec{A} = (1, 0, 1), \vec{B} = (1, 0, 0), \vec{C} = (-1, 1, 1)$$

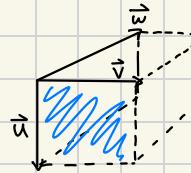


$$\begin{aligned} \vec{v} \times \vec{u} &= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = i \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - j \det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} + k \det \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \\ &= 1 \cdot 1 - 1 \cdot (-1) + 0 \cdot 0 = 2 \end{aligned}$$

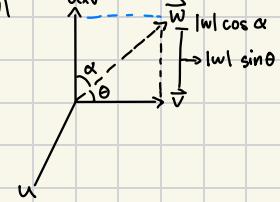
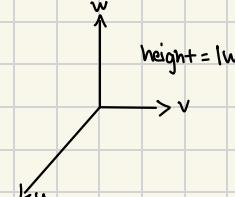
$$= \langle -1, -2, 0 \rangle$$

$$A_{\text{triangle}} = \frac{1}{2} \text{norm}(-1, -2, 0)$$

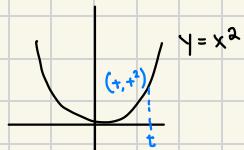
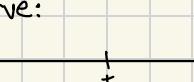
$$= \frac{\sqrt{5}}{2}$$



$$\begin{aligned} V &= |\vec{v} \times \vec{w}| |\vec{u}| \cos \alpha \\ &= (\vec{u} \times \vec{v}) \cdot \vec{w} \end{aligned}$$



Curve:



$$C: [a, b] \rightarrow \mathbb{R}^3$$

$$\text{Example: } y = x^2 : C(x) = (x, x^2)$$

$$\text{Example: } x^2 + y^2 = 1 : C(\theta) = (\cos \theta, \sin \theta)$$

$$\text{Example: } x^2 + y^2 = y^3 \rightarrow r^2 = r^3 \sin^2 \theta$$

$$\begin{aligned} x &= r \cos \theta = \frac{\cos \theta}{\sin^2 \theta} \\ y &= r \sin \theta = \frac{\sin \theta}{\sin^2 \theta} \\ &= \frac{1}{\sin^2 \theta} \\ c(\theta) &= \left( \frac{\cos \theta}{\sin^2 \theta}, \frac{1}{\sin^2 \theta} \right) \end{aligned}$$

$$\text{Example: } 2x + y + z = 1$$

$$N = (2, 1, 1)$$

$$z = 1 - 2x - y$$

$$\Phi(x, y) = (x, y, 1 - 2x - y)$$

$$-\infty < x, y < \infty$$

Friday: ZOOM!!!

$$\text{Example: } 2x + 3y + 2z = 4$$

$$N = (2, 3, 2)$$

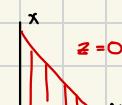
normal

$$z^2 = x^2 + y^2$$

$$z = 0$$

$$2x + 3y = 4$$

$$z = 0$$



$$y = \frac{4 - 2x}{3}$$

First octant:  $(x, y, z \geq 0)$

$$\begin{aligned} \Phi(x, y) &= (x, y, 2 - x - \frac{3}{2}y) \\ &= (0, 0, 2) + x(1, 0, -\frac{3}{2}) + \\ &\quad y(0, 1, -\frac{3}{2}) \end{aligned}$$

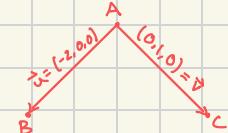
Vertical lines:

$$\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \frac{4 - 2x}{3}\}$$

Similar to a PDF???

$$\text{Example: } A = (1, 0, 1), B = (-1, 0, 1), C = (1, 1, 1)$$

$$\begin{aligned} \Phi(r, s) &= (1, 0, 1) + (-2, 0, 0)r \\ &\quad + (0, 1, 0)s \\ &= (1 - 2r, s, 1) \end{aligned}$$



$$u \times v = \begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i \det \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - j \det \begin{vmatrix} -2 & 0 \\ 0 & 0 \end{vmatrix} + k \det \begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (0i, -0j, -2k) = (0, 0, -2)$$

$$\text{Standard form: } 0(x-1) + 0(y-0) + -2(z-1) = 0 \quad (P_A)$$

$$\rightarrow z = 1$$

$$\text{Example: Line: } l(t) = \text{point} + t(\text{direction}) \rightarrow N = (1, 2, 3)$$

Line perpend. to  $x+2y+3z=1$  thru  $(-1, 1, 1)$

$$l(t) = (-1, 1, 1) + t(1, 2, 3) \equiv (-1+t, 1+2t, 1+3t)$$

Cylindrical

$$\text{Example: } z = x^2 + y^2$$

$$\begin{aligned} \Phi(x, y, x^2 + y^2) &= z = 1 \\ 0 \leq x \leq 1 &\rightarrow z = 1 \\ 0 \leq y \leq \sqrt{1-x^2} &\rightarrow x = \cos \theta \\ &\rightarrow y = r \sin \theta \end{aligned}$$

$$z = x^2 + y^2 \rightarrow z = r^2$$

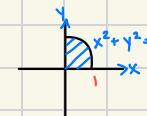
$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta, r^2) : \{z \leq 1, x, y \geq 0\}$$

$$\rightarrow r^2 \leq 1, r \cos \theta \geq 0, r \sin \theta \geq 0, r \neq 0$$

$$\rightarrow 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}$$

Restriction of 1st octant

below  $z = 1$



$$\text{Example: } x^2 + y^2 + z^2 = 1$$

$$x = r \cos \theta, r^2 + z^2 = 1$$

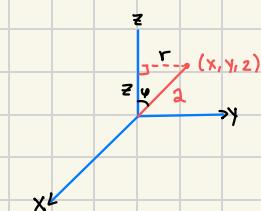
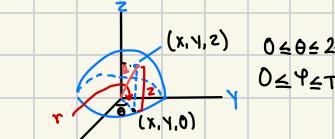
$$y = r \sin \theta, z = r \cos \theta$$

$$r = 2 \sin \theta$$

$$x = 2 \sin \theta \cos \theta$$

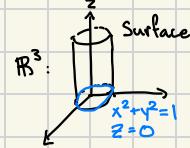
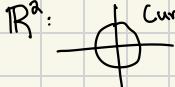
$$y = 2 \sin \theta \sin \theta$$

$$z = 2 \sin \theta$$



Surface:

$$x^2 + y^2 = 1 :$$



# Lecture

June 20<sup>th</sup>

## Review: Cylindrical and Spherical Coordinates

**Definition** The spherical coordinates of points  $(x, y, z)$  in space are the triples  $(\rho, \theta, \phi)$ , defined as follows:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi, \quad (3)$$

$$\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi.$$

**Definition** The cylindrical coordinates  $(r, \theta, z)$  of a point  $(x, y, z)$  are defined by (see Figure 1.4.2)

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \quad (1)$$

$$\text{Example: } x + y + 2z = 4$$

$$x = 4 - y - 2z$$

$$\Phi(y, z) = (4 - y - 2z, y, z) \quad -\infty < y, z < \infty$$

$$\text{Example: } x + y + 2z = 4 \text{ inside } y^2 + z^2 = 4$$

$$y = r \cos \theta$$

$$z = r \sin \theta$$

$$y^2 + z^2 = r^2$$

$$\begin{aligned} x + y + 2z &= 4, \quad y^2 + z^2 \leq 4 \Rightarrow r^2 \leq 4 \\ x + r \cos \theta + 2r \sin \theta &= 4 \end{aligned}$$

$$x = 4 - r \cos \theta - 2r \sin \theta$$

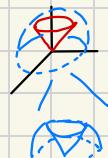
$$\Phi(r, \theta) = (4 - r \cos \theta - 2r \sin \theta, r \cos \theta, r \sin \theta); \quad 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$$

Example: Solid bounded by:

$$\text{(inside)} \quad x^2 + y^2 + z^2$$

$$\text{below } z = \sqrt{x^2 + y^2}$$

$$\text{above } XY\text{-plane}$$



$$x = p \sin \varphi \cos \theta$$

$$y = p \sin \varphi \sin \theta$$

$$z = p \cos \varphi$$

$$x^2 + y^2 + z^2 = p^2 \rightarrow x^2 + y^2 = p^2 \sin^2 \varphi : 0 \leq \varphi \leq 2\pi$$

$$x^2 + y^2 + z^2 \leq 4 \rightarrow p^2 = 4 \Rightarrow 0 \leq p \leq 2$$

$$z \leq \sqrt{x^2 + y^2} \rightarrow p \cos \varphi \leq p \sin \varphi$$

$$\cos \varphi \leq \sin \varphi$$

$$z \geq 0 \rightarrow p \cos \varphi \geq 0 : 0 \leq \varphi \leq \frac{\pi}{2}$$

$$W = \{(p, \theta, \varphi) : 0 \leq p \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\}$$

Applications of Surfaces and curves:

$$T(x, y, z) = x^2 + y^2 + z^2 \equiv p^2$$

Limit:

$$\lim_{(x,y,z) \rightarrow P} f(x, y, z) = \text{expected value for } f(P)$$

If  $\lim_{(x,y,z) \rightarrow P} f(x, y, z) = f(P)$ , we say  $f$  is continuous

Example:

$$\lim_{(x,y) \rightarrow (3,0)} y e^{-\sqrt{x^2 + \ln(y^2 + 1)}} = 0$$

$$\text{Example: } \lim_{(x,y) \rightarrow 0} \frac{xy^2}{x^2 + y^2}$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ x^2 + y^2 &= r^2 \\ &= r \cos \theta \sin^2 \theta \end{aligned}$$

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{r^3 \cos \theta \sin^2 \theta}{r^2} &= \lim_{r \rightarrow 0} r \cos \theta \sin^2 \theta \\ &= 0 \end{aligned}$$

Lecture: Limits

June 23, 2025

Example:  $\lim_{(x,y) \rightarrow 0} \frac{y^3 \sin(x^2+y^2)}{(x^2+y^2)^k}$ , find  $k$  for which  $\lim = 0$

$$\lim_{t \rightarrow 0} \frac{\sin(t)}{t} \stackrel{L'H}{=} \lim_{t \rightarrow 0} \frac{\cos t}{1} = 1$$

$$\text{When } t \sim 0: \sin(t) = t$$

$$\lim_{r \rightarrow 0} \frac{r^3 \sin^3 \theta \cdot t^k}{r^{2k}} \stackrel{t^0 \rightarrow 0}{=} r^{2k-k} = r^k$$

$$\lim_{r \rightarrow 0} \frac{r^3 \sin^3 \theta \cdot t^k}{r^{2k}} = \lim_{r \rightarrow 0} r^{5-2k} \sin^3 \theta = 0$$

$$5-2k > 0 \Rightarrow k < \frac{5}{2}$$

Example:  $\lim_{(x,y) \rightarrow 0} \frac{xy}{x^2 + y^2}$

$$\lim_{(x,y) \rightarrow 0} \frac{r^2 \cos \theta \sin \theta}{r^2} = \lim_{r \rightarrow 0} \cos \theta \sin \theta$$

$$\lim_{(x,y) \rightarrow 0} \cos \theta \sin \theta = 0$$

$$\lim_{(x,y) \rightarrow 0} \cos \theta \sin \theta = 0$$

$$\lim_{(x,y) \rightarrow 0} \cos \theta \sin \theta = 0$$

$$\lim_{(x,y) \rightarrow 0} \cos \theta \sin \theta = 0$$

## Exam Review:

### 1. Vectors

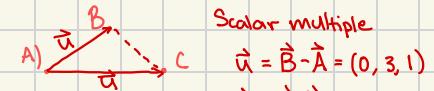
### 2. Parametrizations

### 3. Limits

## Vectors:

$$A = (1, -1, 2), B = (1, 2, 3), C = (2, 1, -1)$$

a) Why are they not collinear?



Scalar multiple

$$\vec{u} = \vec{B} - \vec{A} = (0, 3, 1)$$

$$\vec{v} = \vec{C} - \vec{A} = (1, 2, -3)$$

Not a multiple of each other.

b) Find the area of  $\Delta ABC$  using the appropriate

Gram's Determinant. Let  $u = (0, 3, 1)$

$$v = (1, 2, -3)$$

$$\frac{1}{2} \sqrt{u \cdot u \cdot v \cdot v} = \text{Area } \Delta ABC$$

$$0^2 + 3^2 + 1^2 = 10$$

$$1^2 + 2^2 + (-3)^2 = 14$$

$$1+4+9=14$$

$$ad-bc = 140-9=131$$

c) Find the Cartesian equation of the plane

containing A, B, C. -9-2

$$u \times v = \begin{vmatrix} i & j & k \\ 0 & 3 & 1 \\ 1 & 2 & -3 \end{vmatrix} = i \det \begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix} - j \det \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} + k \det \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} = -11i + j - 3k$$

$$N = (-11, 1, -3) \text{ or } (11, -1, 3)$$

Standard form:  $11(x-1) - 1(y-2) + 3(z-3)$

$$11x - 11 - y + 2 + 3z - 9 = 0$$

$$11x - y + 3z - 18 = 0$$

$$\rightarrow |11x - y + 3z - 18|$$

Limits:

$$f(x, y, z) = \frac{(x^2 + y^2)^k \cos(x+z)}{\sqrt{x^2 + y^2 + z^2}}, \quad k > 0$$

a) Rewrite  $f$  using spherical coordinates

$$x = \rho \sin \varphi \cos \theta \quad (x^2 + y^2)^k = \rho^{4k} \sin^{4k} \theta$$

$$y = \rho \sin \varphi \sin \theta \quad (\cos^4 \theta + \sin^4 \theta)^k$$

$$z = \rho \cos \varphi \quad \cos(\rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta)$$

$$x^2 + y^2 + z^2 = \rho^2 \quad \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2} = \rho$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \pi$$

$$f = \rho^{4k-1} \frac{\cos(x+z)}{\rho} (\cos^4 \theta + \sin^4 \theta)^k \cos(\rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta)$$

b) Find all  $k > 0$  for which

$$\lim_{(x,y,z) \rightarrow 0} f(x, y, z) = 0$$

$$f = \rho^{4k-1} \frac{\sin \theta}{\rho} \frac{\cos(x+z)}{\rho} (\cos^4 \theta + \sin^4 \theta)^k \cos(\rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta)$$

$$We want \rho \rightarrow 0$$

$$We want 4k-1 > 0$$

$$K > \frac{1}{4}$$

$$\rho^{1/2} \rightarrow 0 \Rightarrow \sqrt{\rho} \rightarrow 0$$

$$\rho^{3/2} \rightarrow 0 \Rightarrow \rho \rightarrow 0$$

$$\rho^{-1} \rightarrow 0 \Rightarrow \frac{1}{\rho} \rightarrow 0$$

$$\frac{1}{\rho} \rightarrow 0 \Rightarrow \rho \rightarrow \infty$$

$$\theta = \frac{\pi}{4} : C(x) = (x, x)$$

$$\theta = \frac{3\pi}{4} : C(x) = (x, -x)$$

$$\theta = \frac{5\pi}{4} : C(x) = (-x, -x)$$

$$\theta = \frac{7\pi}{4} : C(x) = (-x, x)$$

$$\theta = \pi : C(x) = (-x, 0)$$

$$\theta = 0 : C(x) = (x, 0)$$

$$\theta = \frac{\pi}{2} : C(x) = (0, x)$$

$$\theta = \frac{3\pi}{2} : C(x) = (0, -x)$$

$$\theta = \frac{5\pi}{2} : C(x) = (x, 0)$$

$$\theta = \frac{7\pi}{2} : C(x) = (-x, 0)$$

$$\theta = 2\pi : C(x) = (x, 0)$$

$$\theta = 0 : C(x) = (x, 0)$$

$$\theta = \frac{\pi}{2} : C(x) = (0, x)$$

$$\theta = \frac{3\pi}{2} : C(x) = (0, -x)$$

$$\theta = \frac{5\pi}{2} : C(x) = (x, 0)$$

$$\theta = 2\pi : C(x) = (x, 0)$$

# Applications of Limits:

Derivative (Calc I)

$$\lim_{t \rightarrow 0} \frac{f(p+t) - f(p)}{t} = f'(p)$$

Partial Derivatives (Calc II)

$$f_x = \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t}$$

$$f_y = \lim_{t \rightarrow 0} \frac{f(x, y+t) - f(x, y)}{t}$$

Discussion Section

June 24<sup>th</sup>, 2025

$$\lim_{(x,y) \rightarrow 0} \frac{y^k \ln(x^2+y^2)}{x^2+y^2} \quad y = r \sin \theta \quad x^2+y^2=r^2$$

$$\lim_{r \rightarrow 0} \frac{r^k \sin^k \theta \ln(r^2)}{r^4 (\cos^2 \theta + \sin^2 \theta)} \rightarrow \frac{r^{k-4} \sin^k \theta \ln(r^2)}{\cos^2 \theta + \sin^2 \theta} \quad \begin{cases} \lim_{r \rightarrow 0} \ln(r) = \text{DNE} \\ \sin^k \theta \rightarrow \text{constant} \\ \cos^2 \theta + \sin^2 \theta = 1 \end{cases}$$

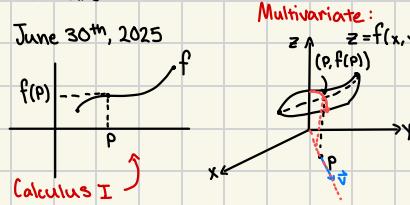
$$\lim_{(x,y) \rightarrow 0} \frac{r^k}{r^4} = \frac{r^k}{r^4} \cdot \log(r^2) \rightarrow \frac{0}{\infty} \text{ or } 0 \quad \begin{cases} 0 & k > 4 \\ 1 & k = 4 \\ \text{DNE} & k < 4 \end{cases}$$

$$= \lim_{r \rightarrow 0} \frac{\log(r^2)}{r^{4-k}} = \lim_{r \rightarrow 0} \frac{2}{r^{4-k}}$$

$$= \lim_{r \rightarrow 0} \frac{2}{r^{4-k}}$$

Lecture

June 30<sup>th</sup>, 2025



$\vec{v}$  = unit vector (direction) where  $\vec{v} = (a, b)$

$$f_{\vec{v}} = \lim_{t \rightarrow 0} \frac{f(p+t\vec{v}) - f(p)}{t} = af_x(p) + bf_y(p)$$



When  $t \neq 0$ :  $f(p+t\vec{v}) \approx f(p) + f_{\vec{v}}(p)t$

$$f(p+h) \approx f(p) + f_{\vec{v}}(p) \cdot \|h\|$$

$$f(p+t\alpha) \approx f(p) + f_x(p)\alpha$$

$$f(p+t\vec{v}) \approx f(p) + f_y(p+t\alpha) + b$$

$$\approx f(p) + (f_x(p)\alpha + f_y(p+t\alpha)b) + b$$

$$\approx f(p) + (f_x(p)\alpha + f_y(p)b) + b$$

$$f_x(p)\alpha + f_y(p)b = \underbrace{(f_x(p), f_y(p)) \cdot (a, b)}_{\text{the gradient of } f}$$

$$f(p+t\vec{v}) \approx f(p) + \nabla f(p) \cdot \vec{v} t$$

$$f(p+\vec{h}) = f(p) + \nabla f(p) \cdot \vec{h} \quad (\text{Linear Approximation})$$

Example:  $\vec{v}$  near P

$$f(Q) = f(p) + \nabla f(p) \cdot \vec{h}$$

$$\vec{h} = Q - P$$

Lecture - Jacobian Matrix

July 2<sup>nd</sup>, 2025

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$f(x_1, \dots, x_n) \approx f(P) + f_{x_1}(P)\Delta x_1 + f_{x_2}(P)\Delta x_2 + \dots + f_{x_n}(P)\Delta x_n \quad (\text{near } P)$$

$$\equiv f(P) + \nabla f(P) \cdot \vec{h} \quad \text{where } \nabla f(P) = (f_{x_1}(P), \dots, f_{x_n}(P)) \text{ and } \vec{h} = (\Delta x_1, \dots, \Delta x_n)$$

$$\text{Example: } f(x, y) = (\cos(x+y), e^{2-x+y}) \text{ near } P=(1, -1)$$

$$f_1 = \cos(x+y) \quad \nabla f_1(1, -1) = (-\sin(x+y), -\sin(x+y))|_{(1, -1)} = (0, 0)$$

$$f_2 = e^{2-x+y} \quad \nabla f_2(1, -1) = (-e^{2-x+y}, e^{2-x+y})|_{(1, -1)} = (-1, 1)$$

$$\vec{f}_2 \approx \vec{f}_2(1, -1) + (-1) \Delta x + (1) \Delta y = -\Delta x + \Delta y$$

$$\Rightarrow \vec{f} \approx (1, 1 - \Delta x + \Delta y)$$

$$\Delta x = x - 1$$

$$\Delta y = y + 1$$

$$\text{Examples: } 2x+3y-2z=1 \quad \text{normal } \vec{v} = (2, -3, -2)$$

$$P = (1/2, 0, 0)$$

$$x^2 + y^2 + z^2 = 1 \rightarrow x^2 + y^2 + z^2 = 1, \quad P = (-1, 1, 1)$$

$$f(p) = 0 \quad x^2 + y^2 + z^2 - 1 = 0 = f \quad \begin{cases} \Delta x = x + 1 \\ \Delta y = y - 1 \\ \Delta z = z - 1 \end{cases}$$

$$f \approx 0 + 3\Delta x + 2\Delta y + 2\Delta z \quad \nabla f = (3x, 2y, 2z) \quad \text{Actual surface}$$

$$\nabla f(P) = (3, 2, 2) \quad \text{Linear Approx.}$$

$$\text{Tangent Plane} \rightarrow 3(x+1) + 2(y-1) + 2(z-1) = 0$$

$$\text{Example: } f(x, y) = (e^x - y, \sin(x-y))$$

a) Write down the linear approximation near  $P = (0, 0)$

$$f(x, y) = f(0, 0) + \frac{\nabla f(0, 0)}{\| \nabla f(0, 0) \|} \cdot (x - 0, y - 0)$$

$$f(x, y) = (1 + x - y, x - y)$$

$$f_1 = e^x - y \approx 1 + (1, -1) \cdot (\Delta x, \Delta y)$$

$$f_1 = e^x \quad \nabla f_1(p) = (e^x, -1)$$

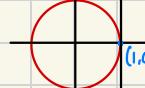
$$f_2 = -y \quad e^x - y = (1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots) - y$$

$$\sin(x-y) \approx x - y$$

$$\sin(t) \approx t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$$

$$c(\theta) = (\cos \theta, \sin \theta), \quad 0 \leq \theta \leq 2\pi$$

$$\Theta = 0$$



$$L(\theta) = (1, \theta)$$

$$f(x, y, z) = (f_1, f_2)$$

$$\approx (f_1(p) + \nabla f_1(p) \cdot \vec{h}, f_2(p) + \nabla f_2(p) \cdot \vec{h})$$

$$= (f_1(p), f_2(p)) + (f_{1x}(p)\Delta x + f_{1y}(p)\Delta y + f_{1z}(p)\Delta z, f_{2x}(p)\Delta x + f_{2y}(p)\Delta y + f_{2z}(p)\Delta z)$$

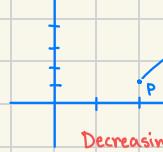
$$= \begin{bmatrix} f_1(p) \\ f_2(p) \end{bmatrix} + \begin{bmatrix} f_{1x}(p) & f_{1y}(p) & f_{1z}(p) \\ f_{2x}(p) & f_{2y}(p) & f_{2z}(p) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

Example:

$f: 2x^5 - 76y^3$ , is  $f$  increasing or decreasing?

at  $P = (2, 1)$  when moving in the direction  $\vec{v}$   $\{ f_{\vec{v}}(p) = + \text{ or } - ? \}$

$= (1, 2) ?$



Decreasing!

$$f_x = 10x^4$$

$$f_y = -228y^2$$

$$f_{\vec{v}} = \frac{10x^4}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} (1, 2)$$

$$f_{\vec{v}} = \frac{10x^4}{\sqrt{5}} - \frac{2(228y^2)}{\sqrt{5}}$$

$$f_{\vec{v}}(p) \equiv f_{\vec{v}}(2, 1) = \frac{160 - 456}{\sqrt{5}} = -\frac{296}{\sqrt{5}}$$

Computations of the directional derivative:

$$\vec{v} = (a, b)$$

$$\text{Then } f_{\vec{v}}(p) = af_x(p) + bf_y(p)$$

$$\text{Example: } f = \frac{x}{\sqrt{x^2+y^2+z^2+3}}$$

Find all directions in which  $f$  increases when moving away from  $(2, -1, 1)$ ?  $\rightarrow f_{\vec{v}}(2, -1, 1) > 0$

$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2+3}} - \frac{x^2}{(x^2+y^2+z^2+3)^{3/2}}$$

$$f_y = \frac{-2y}{(x^2+y^2+z^2+3)^{3/2}} = f_y(2, -1, 1) = \frac{2}{27}$$

$$f_z = \frac{-2z}{(x^2+y^2+z^2+3)^{3/2}} = f_z(2, -1, 1) = -\frac{2}{27}$$

All directions  $(a, b, c)$  with  $5a + 2b - 2c > 0$

# Lecture

July 7<sup>th</sup>, 2025

## Linear Approximation

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f = f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

Near P:

$$f(x_1, \dots, x_n) \approx f(P) + f_{x_1}(P) \Delta x_1 + \dots + f_{x_n}(P) \Delta x_n$$

When m=1:  $f(x_1, \dots, x_n) \approx f(P) + \nabla f(P) \cdot h$ , where  $h = (\Delta x_1, \dots, \Delta x_n)$

$$m > 1: f(x_1, \dots, x_n) \approx f(P) + (D_f(P))(h), \text{ where } D_f = \begin{bmatrix} \nabla f_1(P) \\ \vdots \\ \nabla f_m(P) \end{bmatrix}$$

Ways to interpret  $D_f$

$$f: \mathbb{R}^n \rightarrow \mathbb{R} \text{ (temperature, etc.)}$$



$$\nabla f(P) = D_f(P) \cdot \vec{v} = |\nabla f(P)| \cos \theta$$

$\nabla f(P)$  = the direction of largest increase

$-\nabla f(P)$  = the direction of largest decrease

Example:  $f(x, y) = \frac{1}{1+x^2+y^2}$

Second Interpretation of  $D_f$

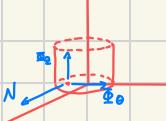
$$f(x, y, z) = 0$$

$$0 = f_x(P)(x - P_x) + f_y(P)(y - P_y) + f_z(P)(z - P_z)$$

$\nabla f(P)$  = normal vector to the surface.

Final Interpretation of  $D_f$

$$\text{Example: } x^2 + y^2 = 1 \text{ in } \mathbb{R}^3$$



$$r^2 = 1 \Leftrightarrow r = 1$$

$$\Phi(\theta, z) = (\cos \theta, \sin \theta, z)$$

$$P = (0, 1) \rightarrow \theta = 0, z = 1 \rightarrow (x = 1, y = 0, z = 1)$$

$$\Phi(\theta, z) \approx \Phi(P) + \Phi_\theta(P) \Delta \theta + \Phi_z(P) \Delta z$$

$$N = \Phi_\theta \times \Phi_z = (0, 1, 0) \times (0, 0, 1) = (1, 0, 0) \text{ (outward normal)}$$

$$f(x, y, z) = (x - e^{x^2-y}, z \cos(x), y \sin(z))$$

Quadratic Approximation:

$$\text{at } (0, 0, 0)$$

$$f = (x - e^{x^2-y}, z \cos(x), y \sin(z)) \text{ near } P = (0, 0, 0)$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \dots$$

$$x - e^{x^2-y} = x(1 + (x^2-y) + \frac{(x^2-y)^2}{2} + \dots)$$

$$\approx x - 1 - x^2 + y - \frac{1}{2}y^2$$

$$z \cos(x) = z(1 - \frac{x^2}{2} + \dots) \approx z - \frac{x^2}{2} \approx z$$

$$y \sin(z) = y(z - \dots) \approx yz$$

$$F = (x - 1 - x^2 + y - \frac{1}{2}y^2, z, yz)$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

To simplify:  $n=2$

near P = (x<sub>0</sub>, y<sub>0</sub>):

$$f(x, y) \approx f(P) + f_x(P) \Delta x + f_y(P) \Delta y$$

$$f(P) + \frac{1}{2} \left[ f_{xx}(P) \Delta x^2 + 2f_{xy}(P) \Delta x \Delta y + f_{yy}(P) \Delta y^2 \right]$$

$$D^2 f = D^2 f(P) = H(f(P))$$

$$f(x, y) \approx f(P) + \nabla f(P) \cdot h + \frac{1}{2} [\Delta x \Delta y] \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\equiv f(P) + \nabla f(P) \cdot h + \frac{1}{2} h^T H(f(P)) h$$

Determinants:

$$H(f) = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad x^2 + 2xy + 2y^2 = (x^2 + y^2) + 4y^2$$

Max: When

$$\begin{cases} D_1 = \text{negative \#} \\ D_2 = \text{positive \#} \end{cases} \quad \begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix}$$

Min: When

$$\begin{cases} D_1 = \text{positive \#} \\ D_2 = \text{positive \#} \end{cases} \quad \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$$

Let's look at possible quadratic portions:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = H(f(P)) \iff a \Delta x^2 + b \Delta y^2$$

Positive: When (min)  $a, b > 0$  Negative: When (max)  $a, b < 0$

Or a saddle point when

$$a > 0, b < 0 \vee a < 0, b > 0$$

$$2\Delta x^2 - \Delta y^2$$

If  $H > 0$ , and  $f_{xx}(x_0, y_0) > 0$

$$\min: \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}_{D_3}$$

$$\{D_1, D_2, D_3\} = +, +, +$$

$$\max: \begin{bmatrix} (-) & D_2 \\ 0 & (-) \end{bmatrix}_{D_3}$$

$$\{D_1, D_2, D_3\} = -, +, -$$

$\det H(f) \neq 0$ .

If  $\det H(f) = 0$ : inconclusive.

## Quiz

$$b) f = x^3 + 3y^2 + z^2 + 3xz + 3xy$$

$$f_x = 3x^2 + 3z + 3y = 0 \rightarrow 4y^2 + 3y + y = 0$$

$$f_y = 6y + 3x = 0 \rightarrow x = -2y$$

$$f_z = 2z + 3x = 0 \rightarrow 2z - 6y = 0 \rightarrow z = 3y$$

$$\rightarrow 4y^2 + 4y = 0 \rightarrow y^2 + y = 0$$

$$y = 0 \quad y = -1 \quad y(y+1) = 0 \quad \begin{cases} y = 0 \\ y = -1 \end{cases}$$

$$x = -2y = 0 \quad x = 2$$

$$z = 3y = 0 \quad z = -3$$

$$(0, 0, 0) \quad (2, -1, 3)$$

$$D_1, D_2, D_3$$

$$\{?, -, -\}: \text{Saddle}$$

$$z = f(x, y)$$

3 crit. points



$$H(f(P)) = \begin{bmatrix} 6x & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$$H(f(0)) = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$$\det(H(f(0))) = 0 \begin{vmatrix} 0 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{vmatrix} + 3 \begin{vmatrix} 6 & 3 & -18 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{vmatrix} = (-)$$

$$D_1 = 0?$$

$$D_2 = -9$$

$$H(f(2, -1, -3)) = \begin{bmatrix} 12 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix} \quad \begin{array}{l} D_1 = + \\ D_2 = + \\ D_3 = 12 \begin{vmatrix} 6 & 3 & -18 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 6 & 0 \\ 3 & 0 & 2 \\ 3 & 0 & 0 \end{vmatrix} + 3 \begin{vmatrix} 6 & 3 & -18 \\ 3 & 0 & 0 \\ 3 & 0 & 2 \end{vmatrix} = 144 - 18 - 3(18) = + \end{array}$$

Positive-definite

$\therefore$  minimum

## LaGrange Multipliers:

Q1: Find max. of  $f(x, y)$  when  $g(x, y) = 0$

Example:  $f(x, y) = y - x^2$  subject to  $x^2 + y^2 = 1$   
 $\Rightarrow g = x^2 + y^2 - 1$

$$f = y - x^2 = 0$$

$$y = x^2$$

$$f = 1 = y - x^2$$

$$y = x^2 + 1$$

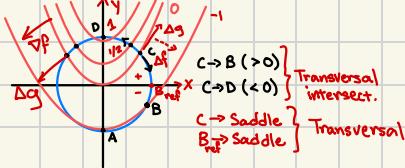
$$f = -1 = y - x^2$$

$$y = x^2 - 1$$

Candidate: Tangential intersection

$$\nabla f = \lambda \nabla g$$

$$g = 0$$



$\{A, D\}$  = tangential intersect.

A  $\rightarrow$  Local max

D  $\rightarrow$  Local max

$$f = xy \text{ on } x^2 + y^2 = 4$$

$$x^2 + (2y)^2 = 4$$

$$x = 2 \cos \theta \quad \left\{ \begin{array}{l} C(\theta) = (2 \cos \theta, \sin \theta), 0 \leq \theta \leq 2\pi \\ y = 2 \sin \theta \end{array} \right.$$

Max/min:

$$f = xy = 2 \cos \theta \cdot \sin \theta \text{ when } 0 \leq \theta \leq 2\pi$$

$$C.P.: f' = 2 \cos^2 \theta - 2 \sin^2 \theta \implies \cos^2 \theta = \sin^2 \theta$$

$$B.P.: \theta = 0, 2\pi$$

$$(1, 0)$$

$$\left\{ \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \right\}$$

$$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

## Lecture

July 14, 2025

### Review of Lagrange Multipliers: W

Example:  $f = x^2 - y^2 + z^2$  on  $x^2 + y^2 + z^2 \leq 1$

$$\text{int } W = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$$

$$\partial W = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$



① In int W:

$$\nabla f = (2x, -2y, 2z)$$

$$\nabla f(0) = (0, 0, 0)$$

② In  $\partial W$ :  $x^2 + y^2 + z^2 = 1 \rightarrow g = x^2 + y^2 + z^2 - 1$

$$\begin{aligned} \nabla f = \lambda \nabla g: \quad & 2x = \lambda \cdot 2x \rightarrow x = \lambda x \\ & \cancel{2x} = \cancel{\lambda \cdot 2x} \quad -y = \lambda y \rightarrow -y = \lambda xy = xy \\ & \cancel{2z} = \cancel{\lambda \cdot 2z} \quad \cancel{2xy = 0} \rightarrow \cancel{0} = 0 \\ & x^2 + y^2 + z^2 = 1 \quad \left. \begin{array}{l} xz = \lambda xz = xz \\ yz = \lambda yz = -yz \\ yz = 0 \rightarrow y = 0, z = 0 \end{array} \right. \\ & (0, 0, 1) \quad (0, 1, 0) \quad (0, 0, -1) \quad (0, -1, 0) \\ & (1, 0, 0) \quad (-1, 0, 0) \end{aligned}$$

③ List  $(x, y, z)$ :  $f = x^2 - y^2 + z^2$

$$\text{MAX} = 1$$

|                 |                     |
|-----------------|---------------------|
| int $W$         | $\{0, 0, 0\}$       |
| $\partial W$    | $\{0, \pm 1, 0\}$   |
|                 | $\{(0, 0, \pm 1)\}$ |
| $x^2 + z^2 = 1$ | $\{1\}$             |

## Lecture

July 16<sup>th</sup>, 2025

$$f(x, y) = 2xy \text{ when } x^2 + 4y^2 \leq 4$$

$$\text{int } x^2 + 4y^2 < 4:$$

$$\nabla f = (y, x) = 0 \implies (x, y) = (0, 0)$$

$$\text{Boundary: } x^2 + 4y^2 = 4 = g(x, y)$$

$$\nabla f = \lambda \nabla g$$

$$y = 2\lambda x$$

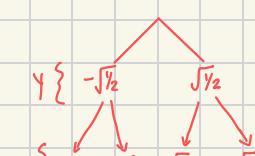
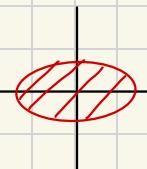
$$x = 8\lambda y \rightarrow x^2 = 8\lambda yx = 4y(2\lambda x) = 4y^2$$

$$x^2 + 4y^2 = 4$$

$$4y^2 + 4y^2 = 4$$

$$y = \pm \sqrt{2}$$

$$x = \pm \sqrt{2}$$



| $(x, y)$                           | $x \cdot y$ |
|------------------------------------|-------------|
| $(0, 0)$                           | 0           |
| $(\sqrt{2}, \frac{\sqrt{2}}{2})$   | 1           |
| $(-\sqrt{2}, -\frac{\sqrt{2}}{2})$ | -1          |
| $(-\sqrt{2}, \frac{\sqrt{2}}{2})$  | -1          |
| $(\sqrt{2}, -\frac{\sqrt{2}}{2})$  | 1           |

$$\text{MAX} = 1 \text{ at } \left\{ \pm \sqrt{2}, \pm \frac{\sqrt{2}}{2} \right\}$$

$$\text{MIN} = -1 \text{ at } \left\{ \pm \sqrt{2}, \mp \frac{\sqrt{2}}{2} \right\}$$

$$f = xy \text{ on } x^2 + 4y^2 = 4$$

$$x^2 + (2y)^2 = 4$$

$$x = 2 \cos \theta \quad \left\{ \begin{array}{l} C(\theta) = (2 \cos \theta, \sin \theta), 0 \leq \theta \leq 2\pi \\ y = 2 \sin \theta \end{array} \right.$$

Max/min:

$$f = xy = 2 \cos \theta \cdot \sin \theta \text{ when } 0 \leq \theta \leq 2\pi$$

$$C.P.: f' = 2 \cos^2 \theta - 2 \sin^2 \theta \implies \cos^2 \theta = \sin^2 \theta$$

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$$(1, 0)$$

$$\left\{ \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \right\}$$

$$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

### Differential Forms:

1-forms:

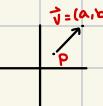
$$\text{In } \mathbb{R}: dx: \mathbb{R} \rightarrow \mathbb{R}$$

$$\begin{array}{c} dx(2\hat{i}) = 2 \\ \xrightarrow{\text{---}} \hat{o} \xrightarrow{\text{---}} \hat{x} \\ dx(\hat{o}) = 1 \end{array}$$

$dx(P)(\vec{v})$  = how much you walk in the x-direction

$$\text{In } \mathbb{R}^2: \quad \begin{array}{l} dx(a\hat{i} + b\hat{j}) = a \\ dy(a\hat{i} + b\hat{j}) = b \end{array}$$

$$w = A dx + B dy \equiv A(x, y) dx + B(x, y) dy, A, B: \mathbb{R}^2 \rightarrow \mathbb{R}$$



"Fake 1-form"

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds(\vec{v}) = |\vec{v}|$$

$$(x^2 - 3xy)(dx, dy) = (0) = x^2 dx - 3xy dy$$

$$(0)(1, 2)(\vec{v}) = 3 + 6 = 9$$

$$\vec{v} = (3, -1)$$

$$(x^2, -3xy)$$

Pullback:

$$\mathbb{R}: dx \leftrightarrow df = f'(u) du$$

$$x = f(u) \quad \int f'(u) du = \int df$$

$$\begin{array}{c} x \\ \downarrow du \\ \int g(f(u)) \frac{f'(u)}{x} du = \int g(x) dx \end{array}$$

$$\mathbb{R}^2: (0) = A dx + B dy$$

$$x = f(u, v) \rightarrow dx \sim df = f_u du + f_v dv$$

$$y = g(u, v) \quad dy \sim dg = g_u du + g_v dv$$



Taylor Approximations:

Problem #1:  $f = (ze^{x-y^2}, x - \cos(z-x))$

Find the quadratic approximation at  $P = (0,0,0)$

$$e^t = 1 + t + \frac{t^2}{2} + \dots$$

$$ze^{x-y^2} = z \left( 1 + (x-y^2) + \frac{(x-y^2)^2}{2} \right)$$

$$\cos(t) = 1 - \frac{t^2}{2} + \dots$$

$$= z + zx - \cancel{\frac{z^3}{2}}$$

$$\sin(t) = t - \cancel{\frac{t^3}{3}} + \dots$$

$$= z + zx - \frac{z^2 - 2xz + x^2}{2}$$

$$x - \cos(z-x) = x - \left( 1 - \frac{(z-x)^2}{2} + \dots \right)$$

$$= x - 1 + \frac{z^2}{2} - xz + \frac{x^2}{2} + \dots$$

$$\textcircled{1} (x, y, z) = \left( z + zx, -1 + x + \frac{x^2}{2} - xz + \frac{z^2}{2} \right)$$