Quiz 07/24

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Let U be the portion of $\mathbb{R}^3$ above the xy-plane, inside the sphere $x^2+y^2+z^2=4$ and below the cone $z=\sqrt{3x^2+3y^2}$. This is to say:

$$U=\{(x,y,z): x^2+y^2+z^2\le 4,\ 0\le z\le \sqrt{x^2+y^2}\}$$

a) Describe this region using spherical coordinates. In other words, describe the bounds for the spherical variables ρ, φ, θ. Would you agree that this is a better coordinate system for U?

Let: $x=\rho\sin\varphi\cos\theta$
$y=\rho\sin\varphi\sin\theta$
$z=\rho\cos\varphi$

If $x^2+y^2+z^2\le 4$, then $\rho^2=4$

$\Rightarrow 0\le\rho\le 2$ and since there is no restriction on θ, then $0\le\theta\le 2\pi$.

Now, if $0\le z\le\sqrt{x^2+y^2}$

then: $z\le\sqrt{x^2+y^2}\Longrightarrow \rho\cos\varphi\le\rho\sin\varphi$
$\cos\varphi\le\sin\varphi$
$\varphi\in[\frac{\pi}{4},\frac{\pi}{2}]$

$$U=\{(\rho,\varphi,\theta): 0\le\rho\le 2, \tfrac{\pi}{4}\le\varphi\le\tfrac{\pi}{2}, 0\le\theta\le 2\pi\}$$

Now suppose our region U is made of some heterogenous material with charge density given by $\mu(x,y,z)=2ze^{x^2+y^2}$. According to the definition of density, the total charge in the region U is the triple integral:

$$\text{Charge}(U)=\iiint_U \mu(x,y,z)\,dx\,dy\,dz$$

b) Use spherical coordinates to express Charge(U) as an interated integral

Let $x=\rho\sin\varphi\cos\theta$ $\quad\det|J|=dV=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$
$y=\rho\sin\varphi\sin\theta$
$z=\rho\cos\varphi$

If $\mu(x,y,z)=2ze^{x^2+y^2}$, then
$x^2+y^2=\rho^2\sin^2\varphi$

$\therefore \mu(x,y,z)\longmapsto(\rho,\varphi,\theta)$
$=2\rho\cos\varphi\, e^{\rho^2\sin^2\varphi}$

$$\therefore \text{Charge}(U)=\iiint_U 2\rho^3\cos\varphi\sin\varphi\cdot e^{\rho^2\sin^2\varphi}\,d\rho\,d\varphi\,d\theta$$

c) Now compute the integral you wrote in (b) to find Charge(U).

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 2\rho^3\cos\varphi\sin\varphi\, e^{\rho^2\sin^2\varphi}\,d\rho\,d\varphi\,d\theta$$

Integrating with respect to φ as recommended

$$\int_0^{2\pi}\int_0^2\int_{\pi/4}^{\pi/2} 2\rho^3\cos\varphi\sin\varphi\, e^{\rho^2\sin^2\varphi}\,d\rho\,d\varphi\,d\theta$$

$$\int_{\pi/4}^{\pi/2}\cos\varphi\sin\varphi\, e^{\rho^2\sin^2\varphi}\,d\varphi$$

let $u=\sin^2\varphi$
$du=2\sin\varphi\cos\varphi$
$\frac{1}{2}du=\sin\varphi\cos\varphi\,d\varphi$

$$\frac{1}{2}\int e^{\rho^2\cdot u}\,du=\frac{e^{\rho^2\cdot u}}{2\rho^2}\xmapsto{u\text{ to }\varphi}\left[\frac{e^{\rho^2\cdot\sin^2\varphi}}{2\rho^2}\right]_{\pi/4}^{\pi/2}=\left(\frac{e^{\rho^2}}{2\rho^2}\right)-\left(\frac{e^{\rho^2/2}}{2\rho^2}\right)$$
$$=\frac{e^{\rho^2}-e^{\rho^2/2}}{2\rho^2}$$

$$\int_0^{2\pi}\int_0^2 \cancel{2}\rho^{\cancel{3}\,1}\left(\frac{e^{\rho^2}-e^{\rho^2/2}}{\cancel{2}\rho^{\cancel{2}}}\right)d\rho\,d\theta$$

$$\int_0^{2\pi}\underbrace{\int_0^2\rho(e^{\rho^2}-e^{\rho/2})\,d\rho}\,d\theta$$

$$\int_0^2\rho(e^{\rho^2}-e^{\rho^2/2})\,d\rho \qquad \text{let } u=\rho^2\to du=2\rho\,d\rho$$
$$\tfrac{1}{2}du=\rho\,d\rho$$

$$\frac{1}{2}\int e^u-e^{\frac{1}{2}u}\,du$$

$$\frac{1}{2}\left(e^u-2e^{\frac{1}{2}u}\right)=\frac{1}{2}e^u-e^{1/2u}\xmapsto{u\text{ to }\rho}\left[\frac{1}{2}e^{\rho^2}-e^{\rho^2/2}\right]_0^2$$

$$=\left(\frac{1}{2}e^4-e^2\right)-\left(\underbrace{\frac{1}{2}e^0-e^0}_{\frac{1}{2}-1=-1/2}\right)$$

$$=\frac{1}{2}e^4-e^2+1/2=\frac{e^4-2e^2+1}{2}$$

$$\frac{e^4-2e^2+1}{2}\int_0^{2\pi}d\theta$$

$$\cancel{2}\pi\left(\frac{e^4-2e^2+1}{\cancel{2}}\right)$$

$$\boxed{\pi(e^4-2e^2+1)}$$