This print-out should have 19 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

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**001 10.0 points**

Find the volume of the solid in the first octant bounded by the cylinders

$$x^2 + y^2 = 9\,, \qquad y^2 + z^2 = 9\,.$$

1. volume = 20 cu. units
2. volume = 21 cu. units
3. volume = 18 cu. units **correct**
4. volume = 19 cu. units
5. volume = 17 cu. units

**Explanation:**

The solid in the first octant bounded by the cylinders

$$x^2 + y^2 = 9\,, \qquad y^2 + z^2 = 9$$

is the solid below the graph of

$$z = \sqrt{9 - y^2}$$

above that part of the circle

$$x^2 + y^2 = 9$$

lying in the first quadrant of the $xy$-plane. Thus the volume of the solid is given by the double integral

$$V = \int\int_A \sqrt{9 - y^2}\,dxdy$$

where $A$ is the region in the first quadrant of the $x$-$y$ plane bounded by the quarter-circle

$$\Big\{(x,\,y) : 0 \le x \le \sqrt{9 - y^2},\ 0 \le y \le 3\Big\},$$

and so $V$ can be represented as the iterated integral

$$V = \int_0^3 \Big\{\int_0^{\sqrt{9-y^2}} \sqrt{9 - y^2}\,dx\Big\}\,dy\,.$$

In this case,

$$V = \int_0^3 \Big[x\sqrt{9 - y^2}\Big]_0^{\sqrt{9-y^2}}\,dy$$

$$= \int_0^3 (9 - y^2)\,dy\,.$$

Consequently,

$$\boxed{V = \Big[9y - \frac{1}{3}y^3\Big]_0^3 = 18 \text{ cu. units}}\,.$$

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**002 10.0 points**

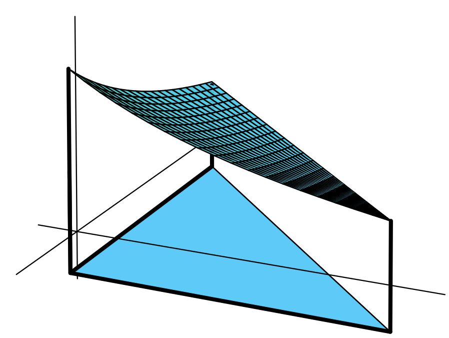
The graph of

$$f(x, y) = \frac{1}{x + y + 2}$$

over the triangular region $A$ enclosed by the graphs of

$$x = 1, \quad x + y = 4, \quad y + 2 = 0$$

is the surface



Find the volume $V$ of the solid under this graph and over the region $A$.

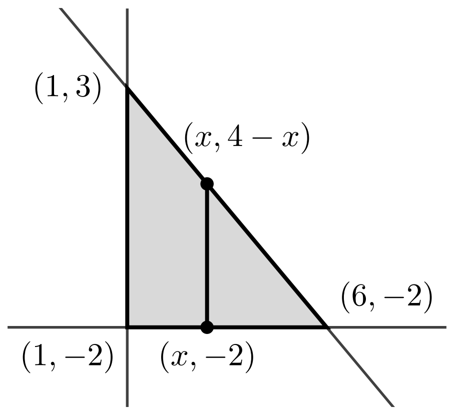
1. $V = 7 - \ln 6$
2. $V = 5 + \ln 6$
3. $V = 6 - \ln 6$

**4.** $V = 6 + \ln 6$

**5.** $V = 5 - \ln 6$ **correct**

**Explanation:**

As the region of integration is given by




(not drawn to scale) the double integral can be written as the repeated integral

$$I = \int_1^6 \left( \int_{-2}^{4-x} \frac{1}{x+y+2}\, dy \right) dx\,,$$

integrating first with respect to $y$ from $y = -2$ to $y = 4 - x$. Now the inner integral is equal to

$$\Big[ \ln(x+y+2) \Big]_{-2}^{4-x} = \ln 6 - \ln x.$$

Thus

$$I = \int_1^6 \Big\{ \ln 6 - \ln x \Big\}\, dx$$

$$= 5 \ln 6 - \Big[ x \ln x - x \Big]_1^6.$$

Consequently,

$$\boxed{V = 5 - \ln 6}\,.$$

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**003 10.0 points**

Evaluate the double integral

$$I = \int\int_A 4ye^{x^2}\, dx dy$$

when $A$ is the region in the first quadrant bounded by the graphs of

$$x = y^2, \qquad x = 2, \qquad y = 0\,.$$

**1.** $I = \Big( e^4 + 1 \Big)$

**2.** $I = e^4$

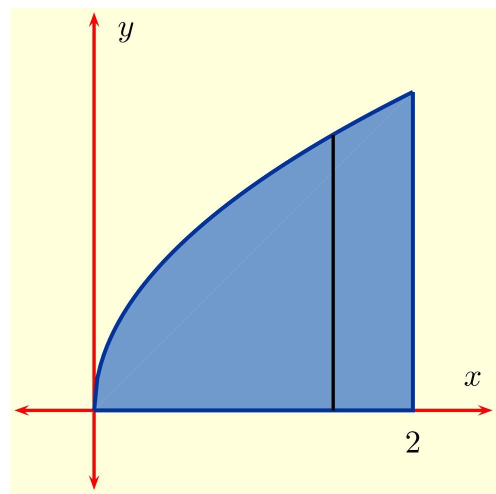
**3.** $I = 4\Big( e^4 + 1 \Big)$

**4.** $I = 2(e^4 - 1)$

**5.** $I = \Big( e^4 - 1 \Big)$ **correct**

**Explanation:**

Since the function $f(x) = e^{x^2}$ cannot be integrated directly, we have to represent $I$ as a repeated integral, integrating first with respect to $y$, where the region of integration is similar to the shaded region in




Thus as a repeated integral

$$I = \int_0^2 \Big( \int_0^{x^{1/2}} 4ye^{x^2}\, dy \Big) dx$$

where integrating first with respect to $y$ means integrating along the segment of the line $x = d$, $0 \le d \le 2$, lying inside the shaded region. Now after integration the inner integral becomes

$$\Big[ 2y^2 e^{x^2} \Big]_0^{x^{1/2}} = 2xe^{x^2}.$$

Thus

$$I = \int_0^2 2xe^{x^2}\,dx = \Big[e^{x^2}\Big]_0^2,$$

and so

$$\boxed{I = \left(e^4 - 1\right)}.$$

keywords:

**004 10.0 points**

Reverse the order of integration in the integral

$$I = \int_0^{\ln 3}\Big(\int_{e^y}^3 f(x,\,y)\,dx\Big)dy\,,$$

but make no attempt to evaluate either integral.

**1.** $I = \displaystyle\int_1^3\Big(\int_0^{\ln x} f(x,\,y)\,dy\Big)dx$ **correct**

**2.** $I = \displaystyle\int_1^3\Big(\int_{\ln x}^{\ln 3} f(x,\,y)\,dy\Big)dx$

**3.** $I = \displaystyle\int_0^3\Big(\int_3^{e^x} f(x,\,y)\,dy\Big)dx$

**4.** $I = \displaystyle\int_0^3\Big(\int_{e^x}^3 f(x,\,y)\,dy\Big)dx$

**5.** $I = \displaystyle\int_1^3\Big(\int_0^{\ln 3} f(x,\,y)\,dy\Big)dx$

**6.** $I = \displaystyle\int_1^3\Big(\int_{\ln y}^{\ln 3} f(x,\,y)\,dy\Big)dx$

**Explanation:**

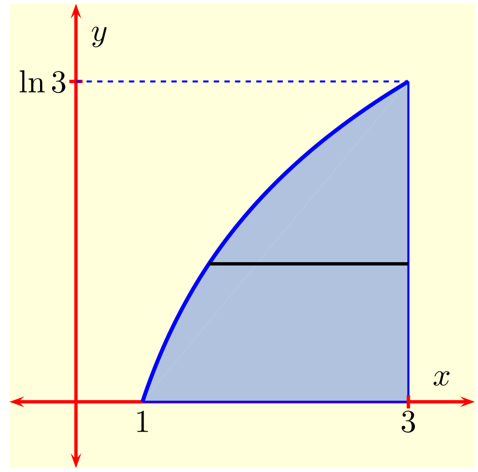
The region of integration is the set of all points

$$\Big\{(x,\,y) : e^y \le x \le 3\,,\; 0 \le y \le \ln 3\Big\}$$

in the plane bounded by the graphs of

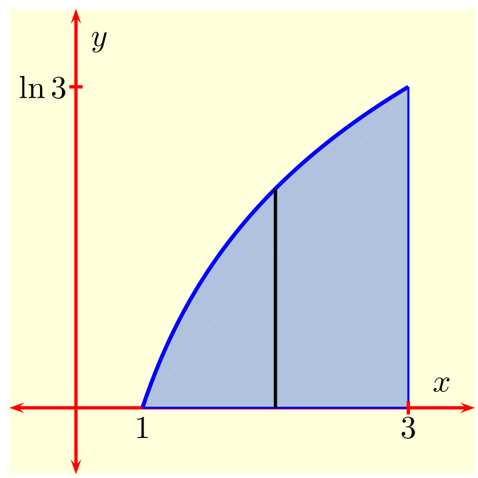
$$y = \ln x\,,\quad x = 1\,,\quad y = \ln 3$$

since $x = e^y$ when $y = \ln x$. This is the shaded region in




(not drawn to scale). Integration is taken first with respect to $x$ for fixed $y$ along the solid horizontal line.

To change the order of integration, now fix $x$ and let $y$ vary along the solid vertical line in




(not drawn to scale). Integration in $y$ is along the line from $(x,\,0)$ to $(0,\,\ln x)$ for fixed $x$, and then from $x = 1$ to $x = 3$.

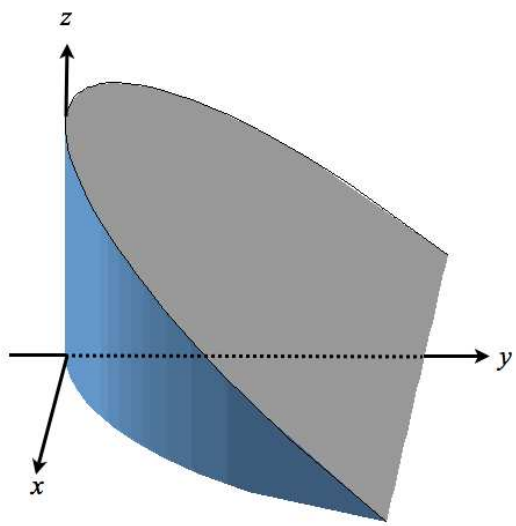
Consequently, after changing the order of integration,

$$\boxed{I = \int_1^3\Big(\int_0^{\ln x} f(x,\,y)\,dy\Big)dx}.$$

keywords: double integral, reverse order integration, exponential function, log function,

**005 10.0 points**

The solid $E$ shown in




is bounded by the graphs of

$$y = x^2, \quad y + z = 1, \quad z = 0.$$

Write the triple integral

$$I = \int\int\int_E f(x, y, z)\, dV$$

as a repeated integral, integrating first with respect to $z$, then $y$, and finally $x$.

**1.** $\displaystyle\int_{-1}^{1}\Big(\int_{x^2}^{1}\Big(\int_{0}^{1-y} f(x, y, z)\, dz\Big)dy\Big)dx$ **correct**

**2.** $\displaystyle\int_{0}^{1}\Big(\int_{x^2}^{1}\Big(\int_{1-y}^{1} f(x, y, z)\, dz\Big)dy\Big)dx$

**3.** $\displaystyle\int_{-1}^{1}\Big(\int_{0}^{x^2}\Big(\int_{1}^{1-y} f(x, y, z)\, dz\Big)dy\Big)dx$

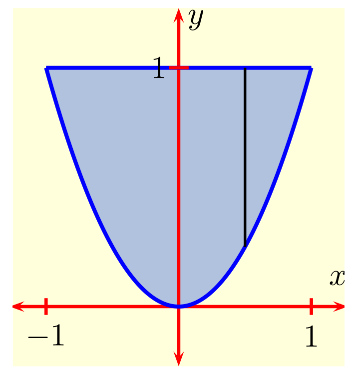
**4.** $\displaystyle\int_{0}^{1}\Big(\int_{0}^{x^2}\Big(\int_{0}^{1-y} f(x, y, z)\, dz\Big)dy\Big)dx$

**5.** $\displaystyle\int_{0}^{1}\Big(\int_{0}^{\sqrt{x}}\Big(\int_{0}^{1-y} f(x, y, z)\, dz\Big)dy\Big)dx$

**6.** $\displaystyle\int_{-1}^{1}\Big(\int_{x}^{1}\Big(\int_{0}^{1-y} f(x, y, z)\, dz\Big)dy\Big)dx$

**Explanation:**

Note first that the planes $y + z = 1$ and $z = 0$ intersect in the line $y = 1$ in the $xy$-plane, while the parabola $y = x^2$ and the line $y = 1$ intersect at the points $(-1, 1)$, $(1, 1)$ in the $xy$-plane. Thus from overhead $E$ looks like




Integrating first with respect to $z$ means fixing $(x, y)$ in the shaded region, while integrating second with respect to $y$ for fixed $x$ means integrating along the solid vertical line in this region.

Thus $E$ consists of all points $(x, y, z)$ in 3-space satisfying the inequalities

$$0 \le z \le 1 - y, \quad x^2 \le y \le 1, \quad -1 \le x \le 1.$$

Consequently, as a repeated integral,

$$\boxed{I = \int_{-1}^{1}\Big(\int_{x^2}^{1}\Big(\int_{0}^{1-y} f(x, y, z)\, dz\Big)dy\Big)dx}\,.$$

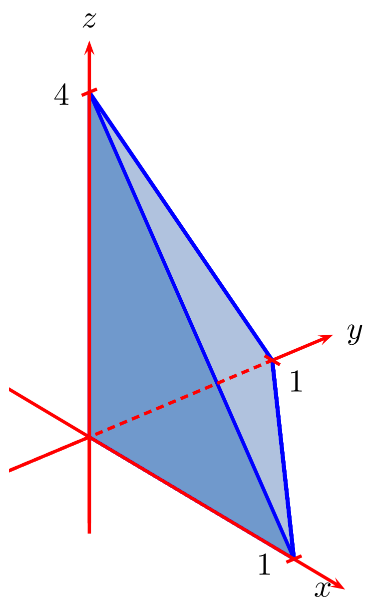
keywords: clicker

**006 10.0 points**

Evaluate the triple integral

$$I = \int\int\int_E 3e^{4(x+y)+z}\, dV$$

when $E$ is the tetrahedron shown in




having one vertex at the origin and three adjacent faces in the coordinate planes.

**1.** $I = \dfrac{3}{16}(5e^4 + 1)$

**2.** $I = \dfrac{3}{4}(5e^4 + 1)$

**3.** $I = \dfrac{3}{4}(5e^4 - 1)$

**4.** $I = \dfrac{15}{16}e^4$

**5.** $I = \dfrac{15}{4}e^4$

**6.** $I = \dfrac{3}{16}(5e^4 - 1)$ **correct**

**Explanation:**

Since the upper face of the tetrahedron lies in the plane intersecting the axes at $x = 1$, $y = 1$ and $z = 4$, this upper face lies in the plane

$$x + y + \frac{z}{4} = 1 .$$

Thus $E$ consists of all points $(x, y, z)$ satisfying the inequalities

$$0 \le z \le 4(1 - x - y) ,$$

in addition to

$$0 \le y \le 1 - x , \qquad 0 \le x \le 1 .$$

So $I$ can be written as a repeated integral

$$\int_0^1 \Big(\int_0^{1-x} \Big(\int_0^{4(1-x-y)} f(x, y, z)\, dz\Big) dy\Big) dx$$

with

$$f(x, y, z) = 3e^{4(x+y)+z} .$$

Now

$$\begin{aligned}\int_0^{4(1-x-y)} 3e^{4(x+y)+z}\, dz &= \Big[3e^{4(x+y)+z}\Big]_0^{4(1-x-y)} \\ &= 3(e^4 - e^{4(x+y)}) ,\end{aligned}$$

while

$$\begin{aligned}3\int_0^{1-x} (e^4 - e^{4(x+y)})\, dy &= 3\Big[e^4 y - \frac{e^{4(x+y)}}{4}\Big]_0^{1-x} \\ &= 3\Big(\frac{3}{4}e^4 - xe^4 + \frac{1}{4}e^{4x}\Big) .\end{aligned}$$

Thus

$$\begin{aligned}I &= \frac{3}{4}\int_0^1 \Big(3e^4 - 4xe^4 + e^{4x}\Big)\, dx \\ &= \frac{3}{4}\Big[3e^4 x - 2x^2 e^4 + \frac{1}{4}e^{4x}\Big]_0^1 \\ &= \frac{3}{4}\Big(\frac{5}{4}e^4 - \frac{1}{4}\Big) .\end{aligned}$$

Consequently,

$$\boxed{I = \frac{3}{16}(5e^4 - 1)} .$$

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keywords: triple integral, repeated integral, tetrahedron, plane, exponential function,

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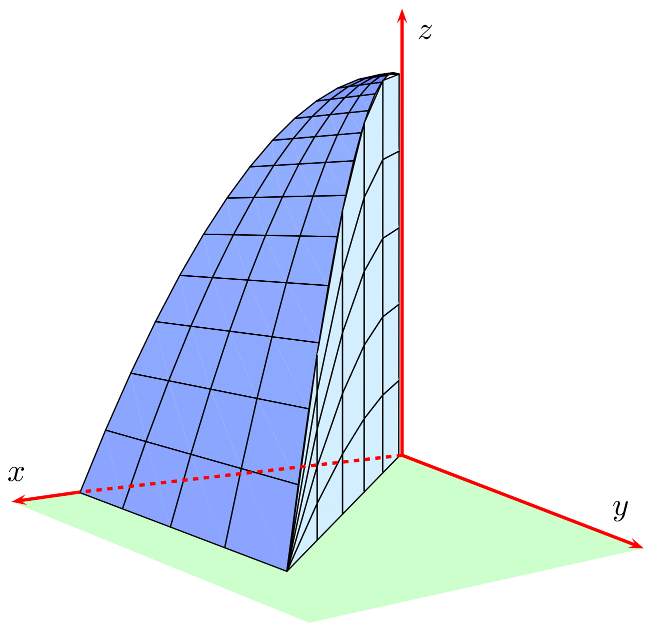
**007 10.0 points**

The solid $E$ in the first octant of 3-space shown in




is bounded by the cylinder

$$z = 1 - x^2$$

and the planes

$$x = y, \quad y = 0, \quad z = 0.$$

Evaluate the triple integral

$$I = \int\int\int_E (x + y)\, dV.$$

**1.** $I = \frac{1}{5}$ **correct**

**2.** $I = \frac{4}{15}$

**3.** $I = 0$

**4.** $I = \frac{1}{15}$

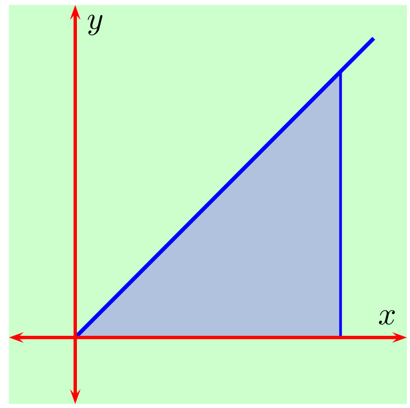
**5.** $I = \frac{2}{15}$

**Explanation:**

Since $E$ lies in the first octant, its projection onto the $xy$-plane lies in the first quadrant of the $xy$-plane. On the other hand, the trace of the parabolic cylinder $z = 1 - x^2$ on the $xy$-plane, *i.e.*, the plane $z = 0$, consists of the lines $x = \pm 1$, while the projection of the plane $x = y$ is the line $y = x$. Thus the projection of $E$ onto the $xy$-plane is the bounded region in the first quadrant enclosed by the graphs of

$$y = x, \qquad x = 1.$$

This is the shaded region shown in




Thus $E$ is the set of all points $(x, y, z)$ such that

$$0 \le x \le 1, \ 0 \le y \le x, \ 0 \le z \le 1 - x^2.$$

So, as a repeated integral,

$$I = \int_0^1 \Big( \int_0^x \Big( \int_0^{1-x^2} (x + y)\, dz \Big) dy \Big) dx.$$

Now

$$\int_0^{1-x^2} (x + y)\, dz = \Big[ xz + yz \Big]_0^{1-x^2}$$

$$= (x + y)(1 - x^2),$$

while

$$\int_0^x (x + y)(1 - x^2)\, dy$$

$$= \Big[ \Big( xy + \frac{1}{2} y^2 \Big)(1 - x^2) \Big]_0^x = \frac{3}{2} x^2 (1 - x^2).$$

Consequently,

$$\boxed{I = \frac{3}{2} \int_0^1 (x^2 - x^4)\, dx = \frac{1}{5}}.$$

keywords: projection, conic sections, parabolic cylinder, plane, integral, triple integral, repeated integral, linear function, limits of integration, setup of triple integral, exponential integrand, monomial integrand, evaluation of triple integral

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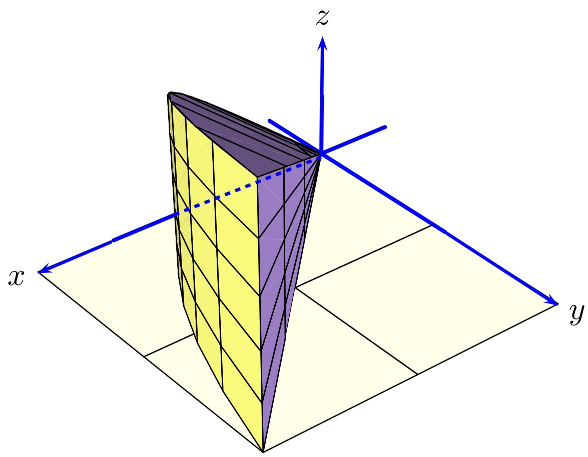
**008 10.0 points**

The solid $E$ in the first octant of 3-space shown in




is bounded by the parabolic cylinder $y = x^2$ and the planes

$$x = y\,, \qquad x = z\,, \qquad z = 0\,.$$

Evaluate the triple integral

$$I = \int\int\int_E (2x + 6z)\, dV\,.$$

**1.** $I = \dfrac{1}{5}$

**2.** $I = \dfrac{1}{3}$

**3.** $I = \dfrac{1}{4}$ **correct**

**4.** $I = \dfrac{1}{2}$

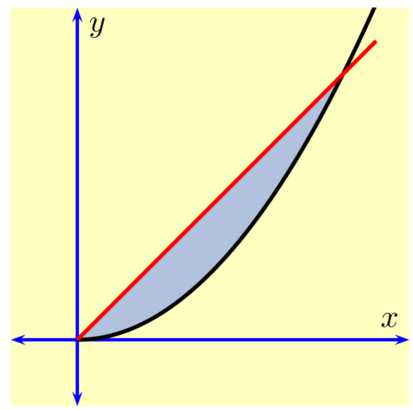
**5.** $I = \dfrac{1}{6}$

**Explanation:**

Since $E$ lies in the first octant, its projection onto the $xy$-plane lies in the first quadrant of the $xy$-plane. On the other hand, the trace of the parabolic cylinder on the $xy$-plane, *i.e.*, the plane $z = 0$, is the parabola $y = x^2$, while the projection of the plane $x = z$ is the $xy$-plane itself. Thus the projection of $E$ onto the $xy$-plane is the bounded region in the first quadrant enclosed by the graphs of

$$y = x^2\,, \qquad y = x\,.$$

This is the shaded region shown in




Thus $E$ is the set of all points $(x, y, z)$ satisfying the inqualities

$$0 \le x \le 1, \quad x^2 \le y \le x, \quad 0 \le z \le x\,,$$

So as a repeated integral,

$$I = \int_0^1 \Big(\int_{x^2}^x \Big(\int_0^x (2x + 6z)\, dz\Big) dy\Big) dx\,.$$

Now

$$\int_0^x (2x + 6z)\, dz = \Big[2xz + 3z^2\Big]_0^x = 5x^2\,,$$

while

$$\int_{x^2}^x 5x^2\, dy = \Big[5x^2 y\Big]_{x^2}^x = 5(x^3 - x^4)\,.$$

Consequently,

$$\boxed{I = 5\int_0^1 (x^3 - x^4)\, dx = \frac{1}{4}}\,.$$

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keywords: projection, conic sections, parabolic cylinder, plane, integral, triple integral, repeated integral, linear function, limits

of integration, setup of triple integral, polynomial integrand, binomial integrand, evaluation of triple integral

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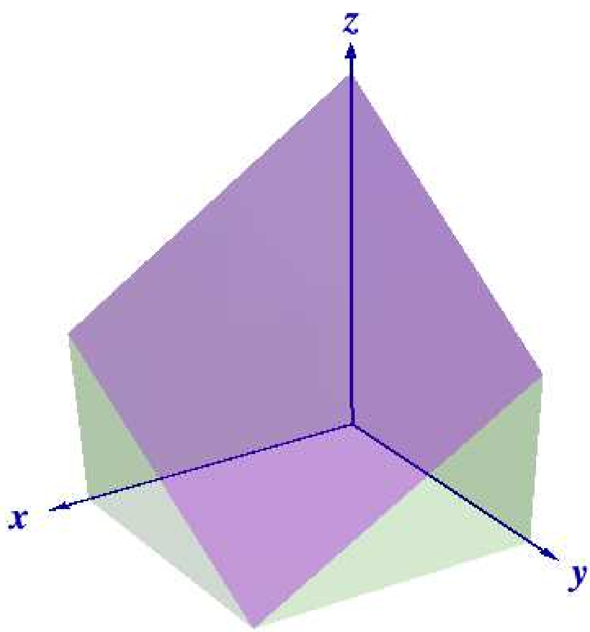
**009 10.0 points**

Evaluate the integral

$$I = \int\int\int_W 4x^2\, dV$$

when $W$ is the region of 3-space shown in




lying below the graph of

$$x + y + z = 2$$

and above the square

$$D = \{(x, y) : 0 \le x, y \le 1\}$$

in the $xy$-plane.

**1.** $I = 2\pi$

**2.** $I = 4\pi$

**3.** $I = 12\pi$

**4.** $I = \frac{8}{3}\pi$

**5.** $I = 1$ **correct**

**Explanation:**

To write $I$ as a repeated integral we first describe $W$ as a set of points $(x, y, z)$ in 3-space: since $W$ is the region below the graph of

$$x + y + z = 2,$$

but above the $xy$-plane, then

$$0 \le z \le 2 - x - y.$$

On the other hand, $(x, y)$ vary over the region

$$D = \{(x, y) : 0 \le x, y \le 1\}$$

in the $xy$-plane. Thus $W$ consists of all points $(x, y, z)$ such that

$$0 \le z \le 2 - x - y, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

So as a repeated integral,

$$I = \int_0^1 \Big(\int_0^1 \Big(\int_0^{2-x-y} 4x^2\, dz\Big) dy\Big) dx.$$

Now

$$\int_0^{2-x-y} 4x^2\, dz = \Big[4x^2 z\Big]_0^{2-x-y}$$

$$= 4x^2(2 - x - y) = 8x^2 - 4x^3 - 4x^2 y.$$

But

$$\int_0^1 (8x^2 - 4x^3 - 4x^2 y)\, dy$$

$$= \Big[8x^2 y - 4x^3 y - 2x^2 y^2\Big]_0^1$$

$$= 8x^2 - 4x^3 - 2x^2 = 2(3x^2 - 2x^3).$$

Consequently,

$$\boxed{I = 2\int_0^1 (3x^2 - 2x^3)\, dx = 1}.$$

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**010 10.0 points**

Evaluate the integral

$$I = \int\int_D \left\{(\pi + 4\tan^{-1}\Big(\frac{y}{x}\Big)\right\} dxdy$$

when $D$ is the region in the first quadrant inside the circle $x^2 + y^2 = 16$.

**1.** $I = \pi$

**2.** $I = 16\pi$

**3.** $I = 16\pi^2$

**4.** $I = 8\pi$

**5.** $I = \pi^2$

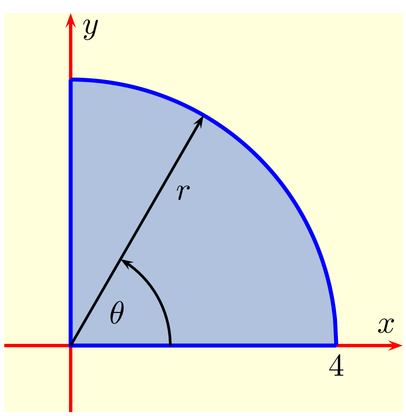
**6.** $I = 8\pi^2$ **correct**

**Explanation:**

In Cartesian coordinates the region of integration is

$$\left\{(x,\, y) \,:\, 0 \le y \le \sqrt{16 - x^2}\,,\ 0 \le x \le 4\right\},$$

which is the shaded region in




On the other hand, in polar coordinates the region of integration is

$$\left\{(r,\, \theta) \,:\, 0 \le r \le 4\,,\ 0 \le \theta \le \pi/2\right\},$$

while

$$\tan^{-1}\Big(\frac{y}{x}\Big) = \theta\,.$$

Thus in polar coordinates,

$$I = \int_0^4 \int_0^{\pi/2} (\pi + 4\theta)\, r d\theta dr$$

$$= \int_0^4 \Big[\pi\theta + 2\theta^2\Big]_0^{\pi/2} r dr = \pi^2 \int_0^4 r\, dr\,.$$

Consequently,

$$\boxed{I = 8\pi^2}\,.$$

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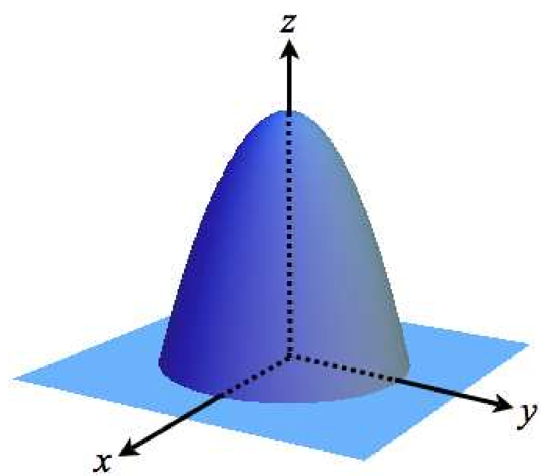
**011 10.0 points**

The solid shown in




is bounded by the paraboloid

$$z = 2 - \frac{1}{2}(x^2 + y^2)$$

and the $xy$-plane. Find the volume of this solid.

**1.** volume $= 2\pi$

**2.** volume $= 1$

**3.** volume $= \pi$

**4.** volume $= 2$

**5.** volume $= 4$

**6.** volume $= 4\pi$ **correct**

**Explanation:**

The paraboloid intersects the $xy$-plane when $z = 0$, *i.e.*, when

$$x^2 + y^2 - 4 \;=\; 0\,.$$

Thus the solid lies below the graph of

$$z \;=\; 2 - \frac{1}{2}(x^2 + y^2)$$

and above the disk

$$D \;=\; \Big\{\,(x,\,y)\;:\;x^2 + y^2 \le 4\Big\}\,,$$

so its volume is given by the integral

$$V \;=\; \int\!\!\int_D \Big(2 - \frac{1}{2}(x^2 + y^2)\Big)\,dxdy\,.$$

In polar coordinates this becomes

$$\begin{aligned} V \;&=\; \frac{1}{2}\int_0^2 \int_0^{2\pi} r(4 - r^2)\,d\theta dr \\ &=\; \pi \int_0^2 (4r - r^3)\,dr \\ &=\; \pi\Big[\,2r^2 - \frac{r^4}{4}\,\Big]_0^2 . \end{aligned}$$

Consequently,

$$\boxed{\text{volume} \;=\; V \;=\; 4\,\pi}\,.$$

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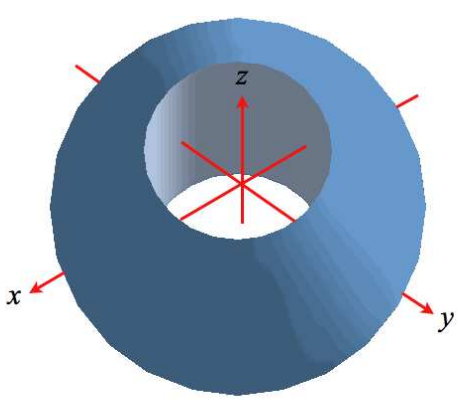
**012 10.0 points**

The solid shown in




lies inside the sphere

$$x^2 + y^2 + z^2 \;=\; 16$$

and outside the cylinder

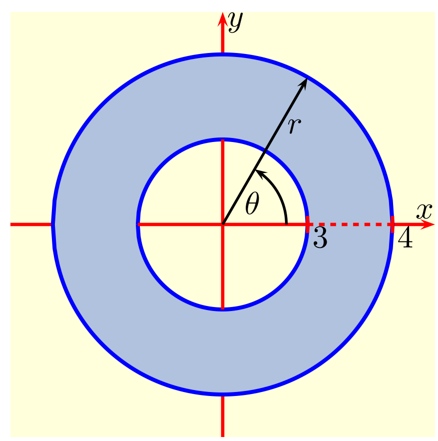
$$x^2 + y^2 \;=\; 9\,.$$

Find the volume of the part of this solid lying above the $xy$-plane.

1. volume $= \dfrac{7\sqrt{7}}{3}$
2. volume $= \dfrac{7\sqrt{7}}{3}\,\pi$
3. volume $= 7\sqrt{7}$
4. volume $= \dfrac{14\sqrt{7}}{3}$
5. volume $= \dfrac{14\sqrt{7}}{3}\,\pi$ **correct**
6. volume $= 7\sqrt{7}\,\pi$

**Explanation:**

From directly overhead the solid is similar to




In Cartesian coordinates this is the annulus

$$R \;=\; \Big\{(x,\,y) : 9 \;\le\; x^2 + y^2 \;\le\; 16\Big\}.$$

Thus the volume of the solid above the $xy$-plane is given by the integral

$$V \;=\; \int\!\!\int_R (16 - x^2 - y^2)^{1/2}\,dxdy\,.$$

To evaluate $V$ we change to polar coordinates. Now

$$R \;=\; \Big\{(r,\,\theta) : 3 \le r \le 4\,, \quad 0 \le \theta \le 2\pi\Big\}\,,$$

so that after changing coordinates the integral becomes

$$V = \int_3^4 \int_0^{2\pi} \sqrt{16 - r^2}\, r dr d\theta$$

$$= 2\pi \int_3^4 r\sqrt{16-r^2}\, dr$$

$$= \pi \Big[ -\frac{2}{3}(16-u)^{3/2} \Big]_9^{16},$$

using the substitution $u = r^2$. Consequently,

$$\boxed{\text{volume} = V = \frac{14\sqrt{7}}{3}\pi}.$$

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**013 10.0 points**

Evaluate the integral

$$I = \int\int_D \frac{x-3y}{x-y} dA$$

when $D$ is the parallelogram bounded by

$$x - 3y = 0, \quad x - 3y = 2,$$

and

$$x - y = 1, \quad x - y = 3,$$

by making an appropriate change of variables.

1. $I = 2\ln 3$

2. $I = 1$

3. $I = 0$

4. $I = \ln 3$ **correct**

5. $I = 2$

**Explanation:**

Setting

$$u = x - 3y, \quad v = x - y$$

simplifies both the integrand and the region of integration $D$. To determine the change of variable

$$T : (u, v) \to (x, y)$$

we first need to solve for $x$, $y$ in terms of $u$, $v$:

$$x = \frac{1}{2}(3v - u), \qquad y = \frac{1}{2}(v - v).$$

In particular,

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{4}\begin{vmatrix} -1 & 3 \\ -1 & 1 \end{vmatrix} = \frac{1}{2},$$

while

$$I = \frac{1}{2}\int\int_{\mathcal{D}} \frac{u}{v}\, du dv$$

where $\mathcal{D}$ is the rectangle in the $uv$-plane mapped onto $D$ by $T$. But $D$ is enclosed by the lines

$$x - 3y = 0, \quad x - 3y = 2,$$

and

$$x - y = 1, \quad x - y = 3,$$

so the definition of $u$, $v$ tells us that $T$ maps the rectangle $\mathcal{D}$ enclosed by the lines

$$u = 0, \quad u = 2, \quad v = 1, \quad v = 3,$$

onto $D$. Thus

$$I = \frac{1}{2}\int_1^3 \Big(\int_0^2 \frac{u}{v}\, du\Big) dv = \int_1^3 \frac{1}{v}\, dv.$$

Consequently,

$$\boxed{I = \ln 3}.$$

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keywords:

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**014 10.0 points**

Using the change of variables given by

$$u = xy, \qquad v = y/x,$$

evaluate the integral

$$I = \int\int_D xy\, dx dy$$

when $D$ is the region in the first quadrant bounded by the lines

$$y = x, \qquad y = 2x,$$

and the hyperbolas

$$xy = 1, \qquad xy = 5.$$

**1.** $I = 6\sqrt{2}$

**2.** $I = 6$

**3.** $I = 6\ln 2$ **correct**

**4.** $I = 12\sqrt{2}$

**5.** $I = 12\ln 2$

**6.** $I = 12$

**Explanation:**

Because the region of integration is

$$D = \{(x, y) : 1 \le xy \le 5, \ 1 \le y/x \le 2\},$$

set

$$u = xy, \qquad v = \frac{y}{x}.$$

To determine the Jacobian we first need to solve for $x$, $y$ in terms of $u$, $v$:

$$y^2 = uv \longrightarrow y = \sqrt{uv},$$

while

$$x^2 = \frac{u}{v} \longrightarrow x = \sqrt{\frac{u}{v}},$$

*i.e.*,

$$T : (u, v) \longrightarrow \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right),$$

and

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{1}{2\sqrt{uv}} & -\dfrac{\sqrt{u}}{2v^{3/2}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}.$$

So in terms of $u$, $v$ the integral becomes

$$I = \frac{1}{2}\int_1^5 \left(\int_1^2 \frac{u}{v}\,dv\right)du = \frac{1}{2}\ln 2 \int_1^5 u\,du.$$

Consequently,

$$\boxed{I = \frac{1}{2}\ln 2\Big[\frac{1}{2}u^2\Big]_1^5 = 6\ln 2}.$$

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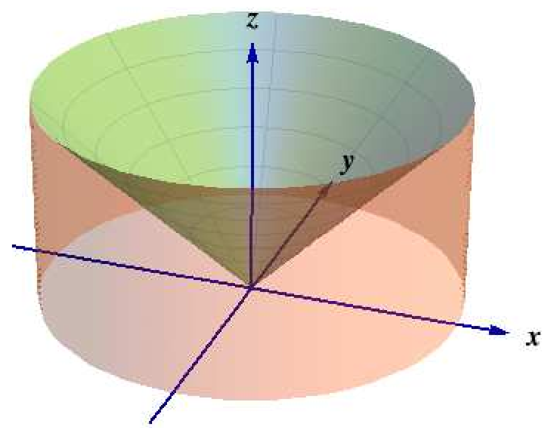
**015 10.0 points**

Use cylindrical coordinates to evaluate the integral

$$I = \int\int\int_W 4x^2\,dV$$

when $W$ is the solid shown in




that lies above the $xy$-plane, below the cone

$$z^2 = x^2 + y^2,$$

and within the cylinder

$$x^2 + y^2 = 1.$$

**1.** $I = \frac{1}{2}\pi$

**2.** $I = 4\pi$

**3.** $I = \frac{4}{5}\pi$ **correct**

**4.** $I = \pi$

**5.** $I = 0$

**Explanation:**
Since

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z\,,$$

in cylindrical coordinates, the cylinder becomes $r = 1$ while the cone becomes $z = r$. Thus in cylindrical coordinates $W$ consists of all points $(r,\,\theta,\,z)$ with

$$z \le r \le 1, \;\; 0 \le \theta \le 2\pi, \quad 0 \le z \le 1\,.$$

So $I$ can be written as the repeated integral

$$\int_0^1 \Big(\int_0^{2\pi}\Big(\int_z^1 4r^3\cos^2\theta\,dr\Big)\,d\theta\Big)dz$$
$$= \int_0^1 \Big(\int_0^{2\pi}(1-z^4)\cos^2\theta\,d\theta\Big)dz$$
$$= \frac{1}{2}\int_0^1 \Big(\int_0^{2\pi}(1-z^4)(1+\cos 2\theta)\,d\theta\Big)dr\,.$$

Consequently,

$$\boxed{I = \pi\int_0^1 (1-z^4)\,dr = \frac{4}{5}\pi}\,.$$

---

**016 10.0 points**

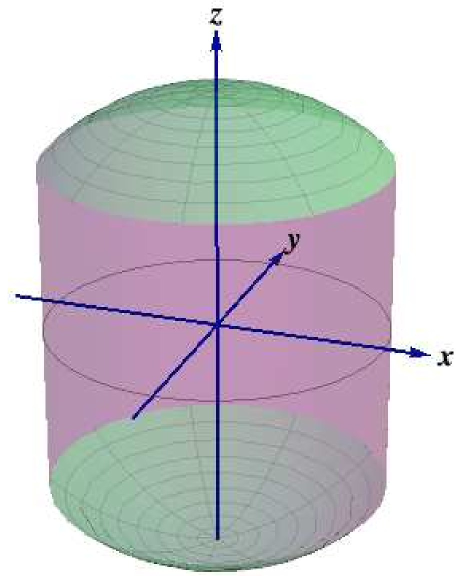
The solid $W$ consists of all points enclosed by the cylinder

$$x^2 + y^2 = 4$$

and the sphere

$$x^2 + y^2 + z^2 = 9$$

shown in




Use cylindrical coordinates to find the volume of $W$.

**1.** volume $= 4\pi\left(27 + 5^{3/2}\right)$

**2.** volume $= \dfrac{2\pi}{3}\left(27 - 5^{3/2}\right)$

**3.** volume $= 4\pi\left(27 - 5^{3/2}\right)$

**4.** volume $= 2\pi\left(27 + 5^{3/2}\right)$

**5.** volume $= \dfrac{4\pi}{3}\left(27 - 5^{3/2}\right)$ **correct**

**6.** volume $= \dfrac{2\pi}{3}\left(27 + 5^{3/2}\right)$

**Explanation:**
As a triple integral

$$\text{volume}(W) = \int\!\!\int\!\!\int_W 1\,dV\,.$$

But in rectangular coordinates, $W$ consists of all points $(x,\,y,\,z)$ such that $x^2+y^2 \le 4$ and

$$-\sqrt{9-x^2-y^2} \le z \le \sqrt{9-x^2-y^2}\,.$$

while in cylindrical polar coordinates, $W$ consists of all points $(r,\,\theta,\,z)$ such that

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi,$$

and

$$-\sqrt{9-r^2} \le z \le \sqrt{9-r^2}\,.$$

Thus $W$ has volume given by

$$I = \int_0^2 \Big(\int_0^{2\pi} \Big(\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} dz\Big) d\theta\Big) r\, dr$$

$$= 4\pi \int_0^2 r\sqrt{9-r^2}\, dr\,.$$

To evaluate this last integral we use the substitution $u^2 = 9 - r^2$. For then

$$2u\, du = -2r\, dr\,,$$

so

$$I = -4\pi \int_3^{\sqrt{5}} u^2\, du = \Big[-\frac{4\pi}{3}u^3\Big]_3^{\sqrt{5}}.$$

Consequently, $W$ has

$$\boxed{\text{volume} = \frac{4\pi}{3}(27 - 5^{3/2})}.$$

---

**017 10.0 points**

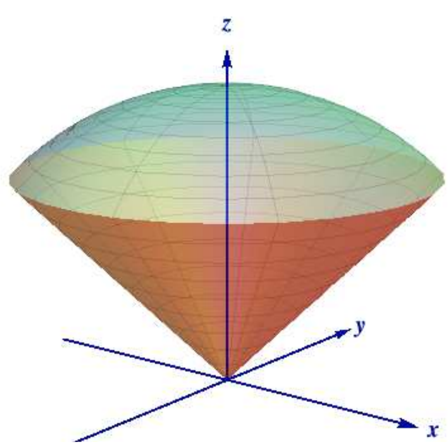
The solid $W$ consists of all points enclosed by the sphere

$$x^2 + y^2 + z^2 = 4$$

and the cone

$$z = \sqrt{x^2 + y^2}$$

as shown in




Use spherical coordinates to express the volume of $W$ as a triple integral.

**1.** $\displaystyle\int_0^2 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} 1\, d\phi d\theta d\rho$

**2.** $\displaystyle\int_0^2 \int_0^{2\pi} \int_0^{\pi/2} \rho^2 \sin\phi\, d\phi d\theta d\rho$

**3.** $\displaystyle\int_0^2 \int_0^{2\pi} \int_0^{\pi/4} 1\, d\phi d\theta d\rho$

**4.** $\displaystyle\int_0^2 \int_0^{2\pi} \int_0^{\pi/2} 1\, d\phi d\theta d\rho$

**5.** $\displaystyle\int_0^2 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \rho^2 \sin\phi\, d\phi d\theta d\rho$

**6.** $\displaystyle\int_0^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^2 \sin\phi\, d\phi d\theta d\rho$ **correct**

**Explanation:**

In spherical coordinates $(\rho, \theta, \phi)$,

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi,$$

and

$$z = \rho\cos\phi\,.$$

Thus the cone

$$z = \sqrt{x^2 + y^2}$$

can be written in spherical coordinates as

$$\rho\cos\phi = \rho\sin\phi(\cos^2\theta + \sin^2\theta)^{1/2}\,,$$

*i.e.*, $\tan\phi = 1$, or, in other words, as $\phi = \pi/4$. Since $\phi = 0$ at the North Pole, $W$ consists of all points $(\rho, \theta, \phi)$ such that

$$0 \le \rho \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}\,.$$

But, as a triple integral, $W$ has

$$\text{volume} = \int\int\int_W 1\, dV\,.$$

On the other hand, the Jacobian for spherical coordinates is $\rho^2 \sin\phi$. Consequently, the value of $W$ is given in spherical coordinates by the triple integral

$$\boxed{\int_0^2 \Big(\int_0^{2\pi} \Big(\int_0^{\pi/4} \Big(\rho^2 \sin\phi\, d\phi\Big) d\theta\Big) d\rho}\,.$$

---

**018 10.0 points**

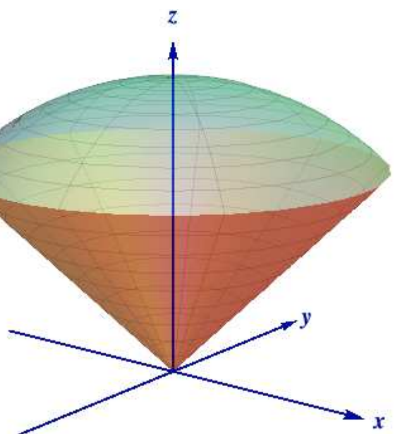
The solid $W$ consists of all points enclosed by the sphere

$$x^2 + y^2 + z^2 = 1$$

and the cone

$$z = \sqrt{x^2 + y^2}$$

as shown in




Use spherical coordinates to evaluate the triple integral

$$I = \int\int\int_W 2z\, dV\,.$$

**1.** $I = \dfrac{1}{4}\pi^2$

**2.** $I = \dfrac{1}{2}\pi$

**3.** $I = \dfrac{1}{8}\pi^2$

**4.** $I = \dfrac{1}{4}\pi$ **correct**

**5.** $I = \dfrac{1}{8}\pi$

**6.** $I = \dfrac{1}{2}\pi^2$

**Explanation:**

In spherical coordinates $(\rho,\, \theta, \phi)$,

$$x = \rho \cos\theta \sin\phi, \quad y = \rho \sin\theta \sin\phi,$$

and

$$z = \rho\cos\phi\,,$$

and so the cone

$$z = \sqrt{x^2 + y^2}$$

can be written in spherical coordinates as

$$\rho\cos\phi = \rho\sin\phi(\cos^2\theta + \sin^2\theta)^{1/2}\,,$$

$i.e.$, $\tan\phi = 1$, or, in other words, as $\phi = \pi/4$. Since $\phi = 0$ at the North Pole, $W$ consists of all points $(\rho,\, \theta, \phi)$ such that

$$0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}\,.$$

On the other hand, the Jacobian for spherical coordinates is $\rho^2 \sin\phi$. Thus, in spherical coordinates, $I$ can be written as a repeated integral

$$2\int_0^1 \Bigl(\int_0^{2\pi} \Bigl(\int_0^{\pi/4} \Bigl(\rho^3 \sin\phi\cos\phi\, d\phi\Bigr) d\theta\Bigr) d\rho$$

$$= \int_0^1 \rho^3\Bigl(\int_0^{2\pi} \Bigl[\sin^2\phi\Bigr]_0^{\pi/4} d\theta\Bigr) d\rho\,.$$

Consequently,

$$\boxed{I = \frac{1}{2}\int_0^1 \rho^3\Bigl(\int_0^{2\pi} d\theta\Bigr) d\rho = \frac{1}{4}\pi}\,.$$

---

**019 10.0 points**

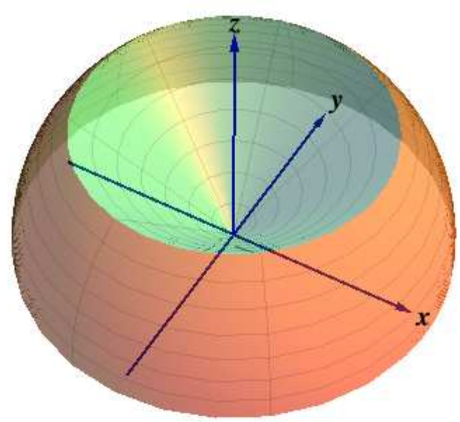
The solid $W$ consisting of all points lying inside the upper hemi-sphere of the sphere

$$x^2 + y^2 + z^2 = 1$$

and below the cone

$$z = \sqrt{x^2 + y^2}$$

as shown in



Use spherical coordinates to find the volume of $W$.

**1.** volume $= \dfrac{\sqrt{2}}{2}\pi$

**2.** volume $= \dfrac{2}{3}\pi$

**3.** volume $= \pi$

**4.** volume $= \sqrt{2}\pi$

**5.** volume $= \dfrac{1}{3}\pi$

**6.** volume $= \dfrac{\sqrt{2}}{3}\pi$ **correct**

**Explanation:**

In spherical coordinates $(\rho,\, \theta, \phi)$,

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi,$$

and

$$z = \rho\cos\phi\,.$$

Thus the cone

$$z = \sqrt{x^2+y^2}$$

can be written in spherical coordinates as

$$\rho\cos\phi = \rho\sin\phi(\cos^2\theta + \sin^2\theta)^{1/2}\,,$$

*i.e.*, $\tan\phi = 1$, or, in other words, as $\phi = \pi/4$. Since $\phi = \pi/2$ on the $xy$-plane, $W$ consists of all points $(\rho,\, \theta, \phi)$ such that

$$0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\,.$$

But, as a triple integral, $W$ has

$$\text{volume} = \int\int\int_W 1\, dV$$

which in spherical coordinates becomes

$$\int_0^1 \Big(\int_0^{2\pi} \Big(\rho^2\sin\phi\, d\phi\Big)d\theta\Big)d\rho$$

$$= \int_0^1 \Big(\int_0^{2\pi} \Big(\rho^2\Big[-\cos\phi\Big]_{\pi/2}^{\pi/4}\Big)d\theta\Big)d\rho$$

$$= \frac{1}{\sqrt{2}}\int_0^1 \Big(\int_0^{2\pi} \rho^2\, d\theta\Big)d\rho\,.$$

Consequently, $W$ has

$$\boxed{\text{volume} = \frac{2}{3\sqrt{2}}\pi = \frac{\sqrt{2}}{3}\pi}\,.$$