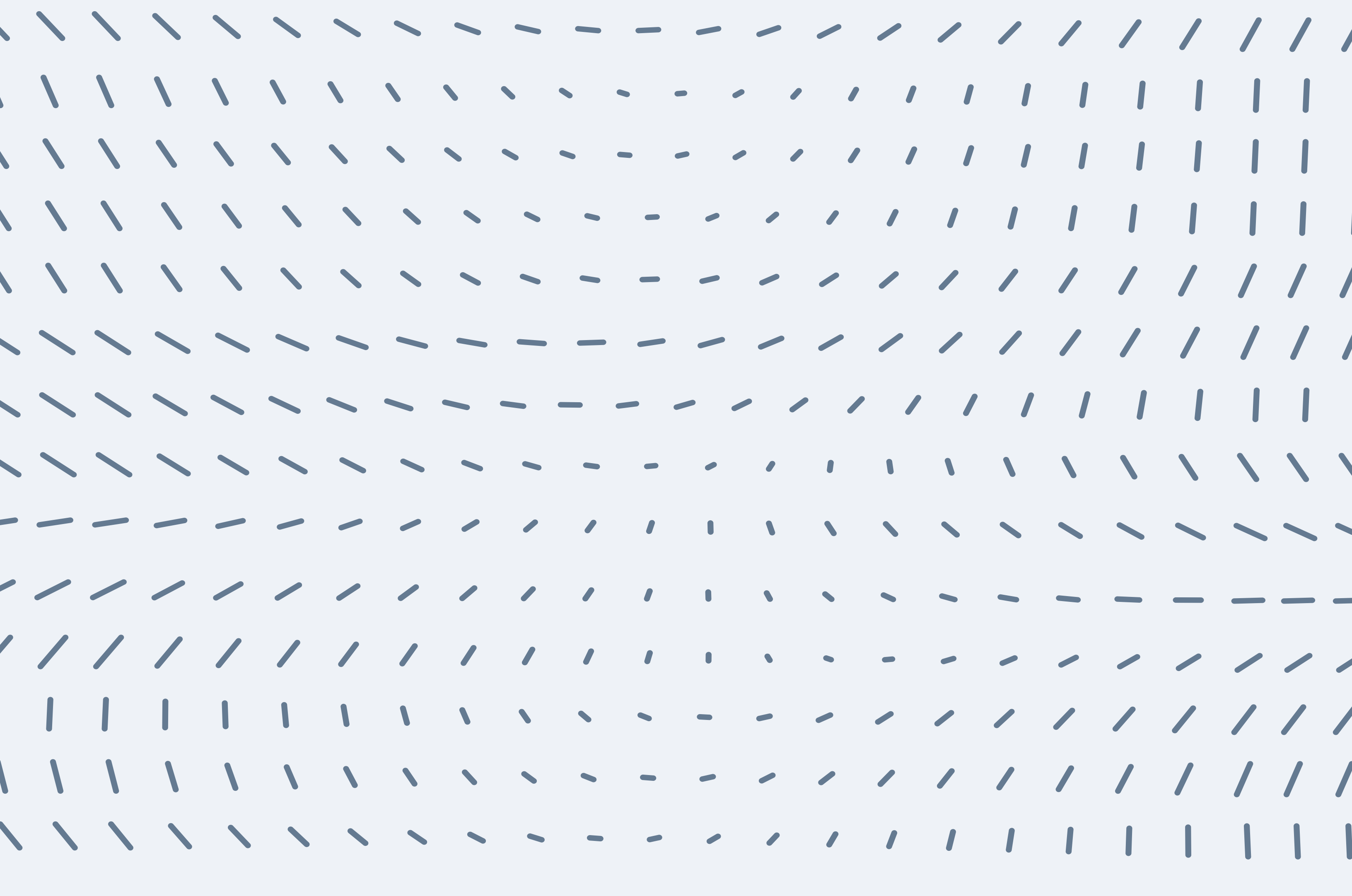
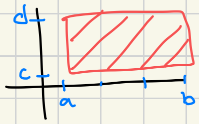


M 427L Vector Calculus

June 6^{th}, 2025

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$



Not the real world, only part!

Reviewing Vectors:

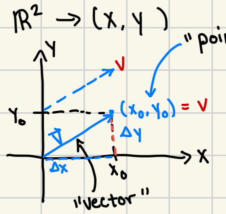
$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\mathbb{R}^n = \{(a_1, \dots, a_n) : a_i \text{ are real numbers}\}$

$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$

$\mathbb{R}^n \to (x_1, \dots, x_n)$ "Vectors" "points"

$\mathbb{R}^2 \to (x, y)$



Points Arithmetic

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ Scalar Addition

$\lambda(x, y) = (\lambda x, \lambda y)$ Scalar Multiplication

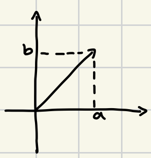
Example:

$\begin{bmatrix}1\\2\end{bmatrix} - 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-5\\-7\end{bmatrix} \to (-5, 7)$

Point = TIP − TAIL (cords)

- $v = (a, b)$

Norm of $v = \|v\| = \sqrt{a^2 + b^2}$

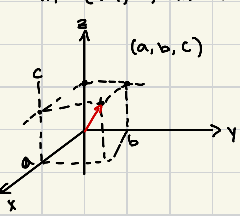


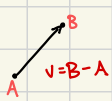
- $v = (a_1, b_1)$ $\quad w = (a_2, b_2)$

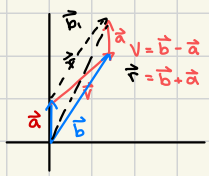
Dot Product: $v \cdot w = a_1 \cdot a_2 + b_1 \cdot b_2$

June 9^{th}, 2025

$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \text{ is a real number}\}$





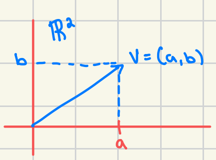


Norm:

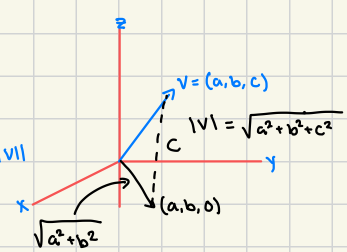
$v = (x_1, \dots, x_n)$

$\|v\| = \sqrt{\sum_{i=0}^{n} x_n^2}$

Example #1: $|(-1, 2, 3)| = \sqrt{(-1)^2 + 2^2 + 3^2}$
$= \sqrt{1 + 4 + 9}$
$= \sqrt{14}$



hyp $= \sqrt{a^2 + b^2} = \|v\|$ length of the vector



Unit vector

If $\vec{v} \neq 0$, then $|\vec{v}| \neq 0$ and

$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$

$|\hat{v}| = \frac{|v|}{|v|} = 1$

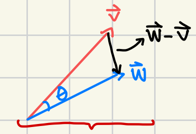
$$\boxed{\begin{aligned} &\lambda(a, b) = (\lambda a, \lambda b) \\ &\sqrt{a^2 + b^2} \\ &\sqrt{\lambda^2 a^2 + \lambda^2 b^2} = |\lambda| \sqrt{a^2 + b^2} \end{aligned}}$$

We say two vectors $\vec{v}, \vec{w}$ have the same direction if $\hat{v} = \hat{w}$.

Example #2: $\vec{v} = (1, 2, 3)$ and $\vec{w} = (-1, 2, 1)$

$\hat{v} = \frac{(1,2,3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{(1,2,3)}{\sqrt{14}} = (1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$

$\hat{w} = \frac{(-1,2,1)}{\sqrt{(-1)^2 + 2^2 + 1^2}} = \frac{(-1,2,1)}{\sqrt{6}} = \frac{1}{\sqrt{6}}(-1, 2, 1)$



$|w - v|^2 = |v|^2 + |w|^2 - 2|v||w| \cos\theta$

"Vector = $\underbrace{\text{direction}}_{\hat{v}}$ + $\underbrace{\text{magnitude}}_{|\vec{v}|}$" $\quad \hat{v} = \frac{v}{|v|}$ $\quad \boxed{\to \vec{v} = \hat{v} \cdot |v|}$

Way 2: $|w - v|^2 = (w - v) \cdot (w - v)$
$\equiv |w|^2 - 2 \cdot w \cdot v + |v|^2$

$\to v \cdot w = |v||w| \cos\theta$

$\to \cos\theta = \frac{v \cdot w}{|v||w|}$

Dot Product:

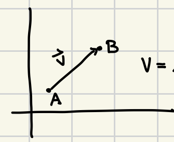
$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

NOTE! $\vec{v} = (a_1, \dots, a_n)$: $\vec{v} \cdot \vec{v} = a_1^2 + \dots + a_n^2$

$\sqrt{v^2} = |v|$

$v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2 \to (\lambda v) \cdot w = \lambda (v \cdot w)$

$v \cdot w = w \cdot v$



$V = \vec{AB} = B - A$

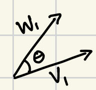
Discussion Section

June 10^{th}, 2025.

$w_1 = (2, -1, 0, 2)$ $\qquad w_1 \cdot v_1 = \sum_{i=1}^{n} a_n \cdot b_n$ where $n = 4$

$v_1 = (1, 2, -1, 2)$

$= 4$



$\to \cos\theta = \frac{v \cdot w}{|v||w|}$
$= \frac{4}{3\sqrt{10}}$

proj $w_1 = v_1 \cdot \hat{v}_2 = |v_1| \cos\theta$
$= \cancel{3}\left(\frac{4}{\cancel{3}\sqrt{10}}\right) = \boxed{\frac{4}{\sqrt{10}}}$

$\hat{v}_2 = \frac{(-1, 2, -1, 2)}{\sqrt{10}}$

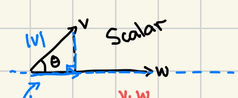
Dot Products:

$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^{n} a_n b_n$

$v \cdot v = v^2 = \sum_{i=1}^{n} a_n^2 = |v|^2$

if $v = (a_1, \dots, a_n)$

$v \cdot w = |v||w| \cos\theta$



$|v| \cos\theta = \frac{v \cdot w}{|w|}$

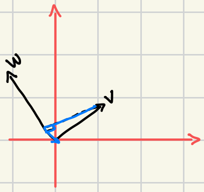
V·W	Angle
0	$\pi/2$
+	$< \pi/2$
−	$> \frac{\pi}{2}$

$|v| \cos\theta \underbrace{\hat{w}}_{\frac{w}{|w|}} = \frac{v \cdot w}{|w|^2} \cdot w$ (Vector Projection)

Example #1: Find the projection of $v: (2,1)$ onto $w=(-1,3)$

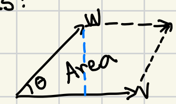
Scalar: $\frac{v\cdot w}{|w|} = \frac{1}{\sqrt{10}}$ $\qquad |w| = \sqrt{(-1)^2+(3)^2} = \sqrt{10}$

Vector: $\underbrace{\overset{\text{Length}}{\frac{v\cdot w}{|w|}} \cdot \overset{\text{Direction}}{\frac{w}{|w|}}}_{|w|^2} = \frac{1}{10}(-1,3) \equiv \left(\frac{-1}{10}, \frac{3}{10}\right)$



Areas:

SOH
CAH
TOA



$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

Area $= |v||w| \sin\theta$

$A^2 = |v|^2|w|^2 \sin^2\theta$

$= |v^2||w^2|(1-\cos^2\theta)$

$\equiv |v^2||w^2| - |v^2||w^2|\cos^2\theta$

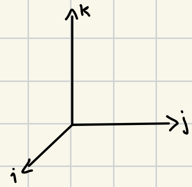
$A^2 = v^2w^2 - (v\cdot w)^2 = \begin{vmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w \end{vmatrix}$

$A_{obj} = \sqrt{\begin{vmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w \end{vmatrix}}$

"Gram determinant"

Lecture

June 13th



$i \times j = k \qquad j \times i = -k$

$\vec{i} \times \vec{i} =$

$v \times v = -v \times v = \vec{0}$

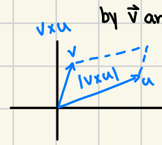
$v \times (w+u) = v\times w + v \times u$

$v \times (\lambda u) = (\lambda v) \times u \equiv \lambda(v\times u)$

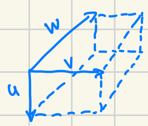
$(1,2,3)\times(2,3,4) = (i+2j+3k)\times(2i+3j+4k)$

$(v_1,v_2,v_3)\times(u_1,u_2,u_3) = \det\left(\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix}\right)$

Application: $|v\times u|$ = area of the parallelogram spanned by $\vec{v}$ and $\vec{u}$



3D:



$\text{Vol} = \sqrt{\begin{vmatrix} u\cdot v & u\cdot u & u\cdot w \\ v\cdot u & v\cdot v & v\cdot w \\ w\cdot u & w\cdot v & w\cdot w \end{vmatrix}}$

Example: $V_1 = (1,1,1)$ $\qquad V_1\cdot V_2 = -1+1+2 = 2$

$V_2 = (-1,1,2)$ $\qquad V_1\cdot V_3 = 0+1+0 = 1$

$V_3 = (0,1,0)$ $\qquad V_2\cdot V_3 = 0+1+0 = 1$

$6(1)-(1)(1) = 5$
$2(1)-(1)(1) = 1$
$2(1)-6(1) = -4$

$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{vmatrix} = a\left(\begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix}\right) - b\left(\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right) + c\left(\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}\right)$

$= 3\det\begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix}$

$3\cdot 5 - 2\cdot 1 + (-4)$

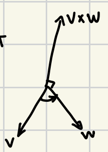
$= 15-2-4$

$= 15-6$

$= \sqrt{9}$

Vol $= \boxed{3}$

$\mathbb{R}^3$: Cross Product



$i = (1,0,0)$

$j = (0,1,0)$

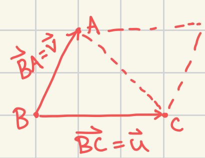
$k = (0,0,1)$

$v\times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$= i\det\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j\det\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$

$\equiv \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right)$

Example: $A=(1,0,1)$, $B=(1,0,0)$, $C=(-1,1,1)$



$V = A-B = (0,0,1)$

$u = C-B = (-2,1,1)$

$v\times u = \det\left(\begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}\right) = i\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) - j\det\left(\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\right) + k\det\left(\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}\right)$

$= \langle -1, -2, 0\rangle$

$A_{triangle} = \frac{1}{2}\,\text{norm}(-1,-2,0)$

$= \frac{\sqrt{5}}{2}$

Discussion Section

June 12th, 2025

$2\cdot(-1) + 1(-1) + 0(-1) + 0(1) + 1(-1)$

$= -2-1-1 = -4$

$1+1+1+1$

$4+1+1+1$

$V = (-1,1,0,1,-1)$

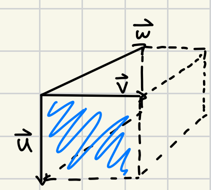
$W = (2,-1,-1,0,1)$

$\begin{bmatrix} v\cdot v & v\cdot w \\ w\cdot v & w\cdot w \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}$

$\det\left(\begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}\right) = 4(7) - (-4(-4))$

$28 - 16 = \boxed{12} = A^2$

$\sqrt{12} = A$



$V = |v\times w||w|\cos\alpha$

$= (u\times v)\cdot w$

$A = (1,1,1)$

$B = (-1,2,1)$ $\qquad \vec{AB} = (-2,1,0) = \vec{B}-\vec{A}$

$C = (3,1,-1)$ $\qquad \vec{AC} = (2,0,-2) = \vec{C}-\vec{A}$

$\vec{AB}\times\vec{AC} =$

$\left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}\right| = \hat{i}\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\right) - \hat{j}\det\left(\begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}\right) + \hat{k}\det\left(\begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}\right)$

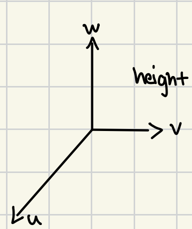
$-2+0=-2 \qquad 4-0=4 \qquad -2$

$-2\hat{i} - 4\hat{j} - 2\hat{k}$

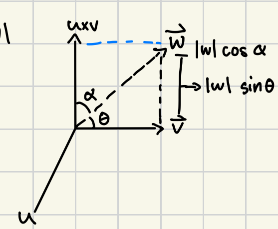
$\equiv (-2,-4,-2)$

$= \sqrt{24} = |(-2,-4,-2)|$

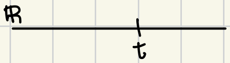
$= \frac{1}{2}\cdot\sqrt{24} = \sqrt{6}$

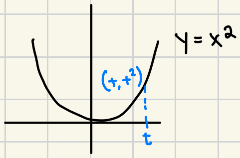


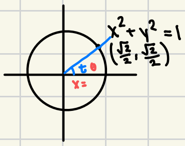
height = |w|



Curve:







$C : [a, b] \longrightarrow \mathbb{R}^3$

$\boxed{\text{Example: } y = x^2 : \ C(x) = (x, x^2) \quad \text{Example: } x^2 + y^2 = 1 : C(\theta) = (\cos\theta, \sin\theta)}$

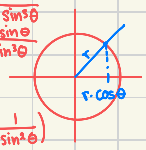
Example: $x^2 + y^2 = y^3 \longmapsto r^2 = r^3 \sin^3\theta$

$1 = r \sin^3\theta$

$\frac{1}{\sin^3\theta} = r$

$x = r\cos\theta = \frac{\cos\theta}{\sin^3\theta}$

$y = r \sin\theta = \frac{\sin\theta}{\sin^3\theta} = \frac{1}{\sin^2\theta}$



$r = \sqrt{x^2 + y^2}$

$x = r\cos\theta$

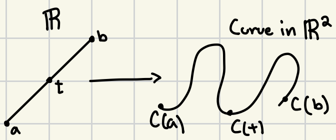
$y = r\sin\theta$

$C(\theta) = \left(\frac{\cos\theta}{\sin^3\theta}, \frac{1}{\sin^2\theta}\right)$

Lecture

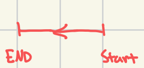
June 16, 2025

$C : [a, b] \longmapsto \mathbb{R}^n$



Example: $c(t) = t \quad c : [0, 1] \to \mathbb{R}$

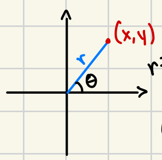
$c(t) = -t \quad c : [-1, 0] \to \mathbb{R}$



$C(-1) = 1$

$C(0) = 0$

Polar:



$r^2 = x^2 + y^2$

$r \in [0, \infty)$

$\theta \in [0, 2\pi]$

$x = r\cos\theta$

$y = r\sin\theta$

Example: $y^2 = x^4 + y^4$ $\qquad 0 \le \theta \le 2\pi$

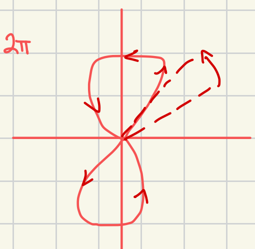
$r^2 \sin^2\theta = r^4 \cos^4\theta + r^4 \sin^4\theta$

$r^2 \sin^2\theta = r^4 (\cos^4\theta + \sin^4\theta)$

$\sin^2\theta = r^2(\cos^4\theta + \sin^4\theta)$

$\frac{\sin^2\theta}{\cos^4\theta + \sin^4\theta} = r^2 \longrightarrow r = \frac{|\sin\theta|}{\sqrt{\cos^4\theta + \sin^4\theta}}$

$C(\theta) = \left(\frac{|\sin\theta| \cos\theta}{\sqrt{\cos^4\theta + \sin^4\theta}}, \frac{|\sin\theta| \sin\theta}{\sqrt{\cos^4\theta + \sin^4\theta}}\right) \quad \theta \in [0, 2\pi]$

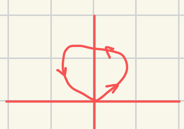


Example: $y^3 = x^4 + y^4$

$r^3 \sin^3\theta = r^4 \cos^4\theta + r^4 \sin^4\theta$

$r = \frac{\sin^3\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$

$c(\theta) = \left(\frac{\sin^3\theta\cos\theta}{r^4\cos^4\theta + r^4\sin^4\theta}, \frac{\sin^4\theta}{r^4\cos^4\theta + r^4\sin^4\theta}\right)$



No problem restriction

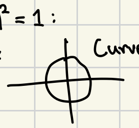
No algebra restriction

Polar restriction $\Rightarrow r \ge 0$ $\quad \sin^3\theta \ge 0 \to \theta \in [0, \pi]$ $\quad 0 \le \theta \le \pi$

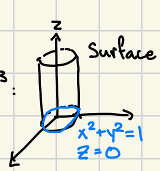
Surface:

$x^2 + y^2 = 1$:

$\mathbb{R}^2$:



$\mathbb{R}^3$:



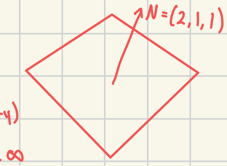
Example: $2x + y + z = 1$

$N = (2, 1, 1)$

$z = 1 - 2x - y$

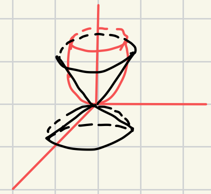
$\Phi(x, y) = (x, y, 1 - 2x - y)$

$, -\infty < x, y < \infty$



Example: $z = x^2 + y^2$

$z^2 = x^2 + y^2$

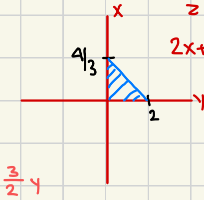


Friday: ZOOM!!!

Example: $2x + 3y + 2z = 4$ $\qquad N = (2, 3, 2)$ normal

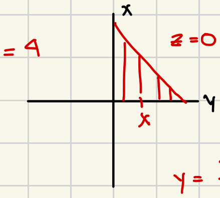
$2z = 4 - 2x - 3y \longrightarrow z = 2 - x - \frac{3}{2}y$

$\Phi(x, y) = (x, y, 2 - x - \frac{3}{2}y) \longrightarrow 2 - x - \frac{3}{2}y \ge 0 \cup x \ge 0 \cup y \ge 0$



$z = 0$

$2x + 3y = 4$

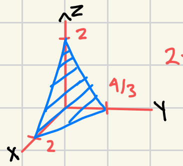


$y = \frac{4 - 2x}{3}$

First octant: $(x, y, z \ge 0)$

$\Phi(x, y) = (x, y, 2 - x - \frac{3}{2}y)$

$= \underbrace{(0, 0, 2)}_{\text{Point}} + \underbrace{x(1, 0, -1) + y(0, 1, -3/2)}_{\text{tangent direction}}$



$2x + 3y + 2z = 4$

Vertical lines:

$\{(x, y) : 0 \le x \le 2, \ 0 \le y \le \frac{4 - 2x}{3}\}$

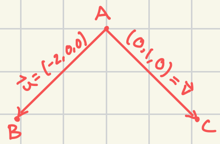
Similar to a PDF???

Example: $A = (1, 0, 1), B = (-1, 0, 1), C = (1, 1, 1)$

$\Phi(r, s) = (1, 0, 1) + (-2, 0, 0)r$

$+ (0, 1, 0)s$

$= (1 - 2r, s, 1)$



$u \times v = \begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i \det\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - j \det\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} + k \det\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

$= (0i, -0j, -2k) = (0, 0, -2)$

Standard form: $0(x - 1) + 0(y - 0) + -2(z - 1) = 0 \quad (P_A)$

$\longrightarrow z = 1$

Example: Line: $\ell(t)$ = point + t(direction) $\quad N = (1, 2, 3)$

Line perpend. to $x + 2y + 3z = 1$ thru $(-1, 1, 1)$

$\ell(t) = (-1, 1, 1) + t(1, 2, 3) \equiv (-1 + t, 1 + 2t, 1 + 3t)$

Cylindrical

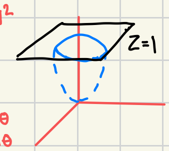
Example: $z = x^2 + y^2$

$\Phi(x, y, x^2 + y^2)$

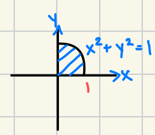
$0 \le x \le 1$

$0 \le y \le \sqrt{1 - x^2}$

$\mapsto z = z$, $x = r\cos\theta$, $y = r\sin\theta$



Restriction of 1st octant below z = 1



$z = x^2 + y^2 \to z = r^2$

$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r^2) : \{z \le 1, \ x, y, z \ge 0\}$

$\to r^2 \le 1, \ r\cos\theta \ge 0, \ r\sin\theta \ge 0, \ \cancel{r^2 \ge 0}$

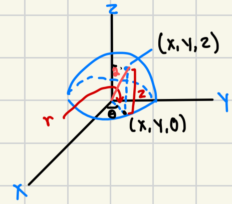
$\to 0 \le r \le 1 \quad 0 \le \theta \le \frac{\pi}{2}$

Example: $x^2 + y^2 + z^2 = 4$

$x = r\cos\theta \qquad r^2 + z^2 = 4$

$y = r\sin\theta \qquad z = 2\cos\varphi$

$r = 2\sin\varphi$



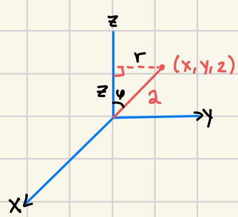
$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$x = 2\sin\varphi\cos\theta$

$y = 2\sin\varphi\sin\theta$

$z = 2\sin\varphi$



Lecture

June 20th

Review: Cylindrical and Spherical Coordinates

$c(t) = (x(t), y(t), z(t))$

$\Phi(r,s) = (x(r,s), y(r,s), z(r,s))$

Definition The *spherical coordinates* of points (x, y, z) in space are the triples (ρ, θ, ϕ), defined as follows:

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi, \qquad (3)$$

where

$$\rho \geq 0, \qquad 0 \leq \theta < 2\pi, \qquad 0 \leq \phi \leq \pi.$$

Definition The *cylindrical coordinates* (r, θ, z) of a point (x, y, z) are defined by (see Figure 1.4.2)

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \qquad (1)$$

Example: $x+y+2z=4$

$x = 4-y-2z$

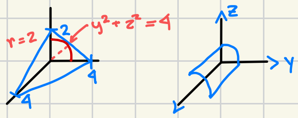
$\Phi(y,z) = (4-y-2z, y, z) \quad -\infty < y, z < \infty$

Example: $x+y+2z=4$ inside $y^2+z^2=4$

$y = r\cos\theta$

$z = r\sin\theta$

$y^2+z^2=r^2$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$x+y+2z=4, \quad y^2+z^2 \leq 4 \to r^2 \leq 4$, $r \leq 2$

$x + r\cos\theta + 2r\sin\theta = 4$

$x = 4 - r\cos\theta - 2r\sin\theta$

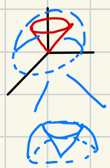
$\boxed{\Phi(r,\theta) = (4 - r\cos\theta - 2r\sin\theta, r\cos\theta, r\sin\theta): 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2}$

Example: Solid bounded by:

(inside) $x^2+y^2+z^2$

below $z = \sqrt{x^2+y^2}$

above xy-plane



$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

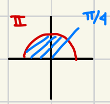
$z = \rho\cos\varphi$

$x^2+y^2+z^2 = \rho^2 \to x^2+y^2 = \rho^2\sin^2\varphi$: $0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$

$x^2+y^2+z^2 \leq 4 \to \rho^2 = 4 \Rightarrow 0 \leq \rho \leq 2$

$z \leq \sqrt{x^2+y^2} \to \rho\cos\varphi \leq \rho\sin\varphi$

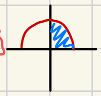
$\cos\varphi \leq \sin\varphi$



$\frac{\pi}{4} \leq \varphi \leq \pi$

$z \geq 0 \to \rho\cos\varphi \geq 0 : 0 \leq \varphi \leq \frac{\pi}{2}$

$W = \{(\rho,\theta,\varphi): 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\}$



Applications of Surfaces and curves:

$T(x,y,z) = x^2+y^2+z^2 \overset{s}{=} \rho^2$

Limit:

$\lim_{(x,y,z)\to p} f(x,y,z) =$ expected value for $f(p)$

If $\lim_{(x,y,z)\to p} f(x,y,z) = f(p)$, we say f is continuous

Example:

$\lim_{(x,y)\to(3,0)} ye^{x-\sqrt{x^2+\ln(y^2+1)}} = 0$

Example: $\lim_{(x,y)\to 0} \frac{xy^2}{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$\frac{r^3\cos\theta\sin^2\theta}{r^2} =$

$= r\cos\theta\sin^2\theta$

$\lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r\cos\theta\sin^2\theta$

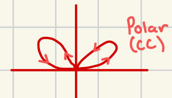
$= 0$

Exam Review:

1. Vectors
2. Parametrizations
3. Limits

Parametrization:

a) $x^2y = x^4+y^4$



$x = r\cos\theta$

$y = r\sin\theta$

$r^3\cos^2\theta\sin\theta = r^4(\cos^4\theta + \sin^4\theta)$

$\frac{\cos^2\theta\sin\theta}{\cos^4\theta+\sin^4\theta} = r$

$c(\theta) = \left(\frac{\cos^3\theta\sin\theta}{\cos^4\theta+\sin^4\theta}, \frac{\cos^3\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}\right)$

Algebraic restriction: None

Polar Restriction: $r \geq 0$: $\sin\theta \geq 0$

$0 \leq \theta \leq \pi$

Problem restriction: none

$0 \leq \theta \leq \pi$

b). Surfaces: $x^2+z^2 \leq 1$

$y - 2 = x^2+z^2$ inside $x^2+z^2=1$

$y = x^2+z^2-z$

No restrictions:

$y = z + x^2 + z^2$

$(x, z+x^2+z^2, z)$

$-\infty < x, z < \infty$

$x = r\cos\theta$

$z = r\sin\theta$

$x^2+z^2=r^2$

$y - r\sin\theta = r^2 \to y = r^2 + r\sin\theta$

$r^2 \leq 1$

$0 \leq r \leq 1$

$$\boxed{\begin{gathered}\Phi(r,\theta) = (r\cos\theta, r^2+r\sin\theta, r\sin\theta)\\ 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\end{gathered}}$$

1st Octant

$x, y, z \geq 0$

$0 \leq \theta \leq \pi/2$

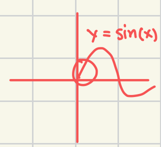
$r^2 + r\sin\theta \geq 0$

Lecture: Limits

June 23, 2025

Example: $\lim_{(x,y)\to 0} \frac{y^3\sin(x^2+y^2)}{(x^2+y^2)^k}$, find $\forall k$ for which $\lim = 0$

$\lim_{t\to 0} \frac{\sin(t)}{t} \overset{L.H}{=} \lim_{t\to 0} \frac{\cos t}{1} = 1$



When $t \sim 0$: $\sin(t) = t$

$\lim_{r\to 0} \frac{r^3\sin^3\theta\,\cancel{\sin(r^2)}}{r^{2k}}$, $\sin(r^2) \to r^2$

$\lim_{r\to 0} \frac{r^3\sin^3\theta\, r^2}{r^{2k}} = \lim_{r\to 0} r^{5-2k}\sin^3\theta$

$= 0$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$5 - 2k > 0 \to k < 5/2$

Example: $\lim_{(x,y)\to 0} \frac{xy}{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$\lim_{(x,y)\to 0} \frac{r^2\cos\theta\sin\theta}{r^2}$

$\lim_{(x,y)\to 0} \cos\theta\sin\theta =$ $\theta=0 \to 0$; $\theta = \frac{\pi}{4} \to 1/2$

$\theta = 0$

$c(x) = (x, 0)$

On $(x, 0)$:

$\frac{xy}{x^2+y^2} = 0 \to 0$

$\theta = \frac{\pi}{4}$: $c(x) = (x, x)$

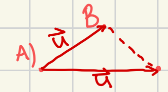
On (x, x):

$\frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = 1/2 \to 1/2$

Vectors:

$A = (1,-1,2)$, $B = (1,2,3)$, $C = (2,1,-1)$

a) Why are they not Collinear?



Scalar multiple

$\vec{u} = \vec{B} - \vec{A} = (0, 3, 1)$

$\vec{v} = \vec{C} - \vec{A} = (1, 2, -3)$

Not a multiple of each other.

B) Find the area of ΔABC using the appropriate Gram's Determinant.

Let $u = (0,3,1)$

$v = (1,2,-3)$

$\frac{1}{2}\sqrt{\begin{vmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v\end{vmatrix}} =$ Area ΔABC

$0^2+3^2+1^2 = 10$

$1^2+2^2+(-3)^2 = 1+4+9 = 14$

$0+6-3=3$

$\frac{1}{2}\sqrt{\begin{vmatrix} 10 & 3 \\ 3 & 14\end{vmatrix}} = \boxed{\frac{\sqrt{131}}{2} = \text{Area } \Delta ABC}$

$ad - bc = 140 - 9 = 131$

C) Find the cartesian equation of the plane containing A, B, C.

$u \times v = \begin{vmatrix} i & j & k \\ 0 & 3 & 1 \\ 1 & 2 & -3\end{vmatrix} = i\det\begin{pmatrix} 3 & 1 \\ 2 & -3\end{pmatrix} - j\det\begin{pmatrix} 0 & 1 \\ 1 & -3\end{pmatrix} + k\det\begin{pmatrix} 0 & 3 \\ 1 & 2\end{pmatrix}$

$-9-2$

$-11i + j - 3k$

$N = (-11, 1, -3)$ or $(11, -1, 3)$

Standard form: $11(x-1) - 1(y-2) + 3(z-3)$

$11x - 11 - y + 2 + 3z - 9 = 0$

$11x - y + 3z - 18 = 0$

$\to \boxed{11x - y + 3z = 18}$

Limits:

$f(x,y,z) = \frac{(x^4+y^4)^k\cos(x+z)}{\sqrt{x^2+y^2+z^2}}, \quad k > 0$

a) Rewrite f using spherical coordinates

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$(x^4+y^4)^k = \rho^{4k}\sin^{4k}\phi\,(\cos^4\theta + \sin^4\theta)^k$

$\cos(x+z) = \cos(\rho(\sin\phi\cos\theta + \cos\phi))$

$x^2+y^2+z^2 = \rho^2$

$x^2+y^2 = \rho^2\sin^2\phi \quad \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

$f = \rho^{4k-1}\sin^{4k}\phi\,(\cos^4\theta + \sin^4\theta)^k\cos(\rho(\sin\phi\cos\theta + \cos\phi))$

b) Find all $k > 0$ for which $\lim_{(x,y,z)\to 0} f(x,y,z) = 0$

$f = \underbrace{\rho^{4k-1}}_{0} \times \underbrace{\sin^{4k}\phi}_{\text{bounded}} \times \underbrace{(\cos^4\theta + \sin^4\theta)^k}_{\text{bounded}} \times \underbrace{\cos(\overbrace{\rho}^{0}\overbrace{(\sin\phi\cos\theta + \cos\phi)}^{\text{bounded}})}_{1}$

We want $\rho \to 0$

We want $4k - 1 > 0$

$\boxed{k > 1/4}$

$\rho^2 \to 0$

$\rho^3 \to 0$

$\rho^0 \to 1 \neq 0$

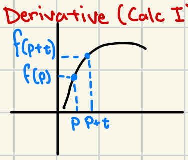
$\rho^{1/2} \to 0 \equiv \sqrt{\rho} \to 0$

$\rho^{-1} \to 0$

$\frac{1}{\rho} \to 0$

Applications of Limits:

Derivative (Calc I)



$$\lim_{t\to 0}\frac{f(p+t)-f(p)}{t}=f'(p)$$

Partial Derivatives (Calc II)

$$f_x=\lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t}$$

$$f_y=\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}$$

Discussion Section

June 24th, 2025

$$\lim_{(x,y)\to 0}\frac{y^k\ln(x^2+y^2)}{x^4+y^4}\qquad y=r\sin\theta,\quad x=r\cos\theta,\quad x^2+y^2=r^2$$

$$\lim_{r\to 0}\frac{r^k\sin^k\theta\ln(r^2)}{r^4(\cos^4\theta+\sin^4\theta)}$$

$$\to \frac{r^{k-4}\sin^k\theta\ln(r^2)}{\cos^4\theta+\sin^4\theta}:\begin{cases}\lim_{r\to\infty}\ln(r)=\text{DNE } -\infty\\ \frac{\sin^k\theta}{\sin^4\theta+\cos^4\theta}\to\text{constant}\end{cases}$$

Fix θ, $r\to 0$

$$\to r^{k-4}\begin{cases}0 & k>4\\ 1 & k=4\\ \text{DNE} & k<4\end{cases}$$

$$\lim_{(x,y)\to 0}\text{ "=" } -\infty\cdot\text{const}\cdot\begin{cases}0\\ \infty\end{cases}\qquad -\infty\cdot\text{const}\cdot 0$$

$$\frac{r^k}{r^4}\cdot\log(r^2)=\frac{r^k\log(r^2)}{r^4}\approx\frac{\infty}{\infty}\text{ or }\frac{0}{0}$$

$$=\lim_{r\to 0}\frac{\log(r^2)}{r^{4-k}}$$

$$=\lim_{r\to 0}$$

Lecture

June 27, 2025

x-direction: $\quad f:\mathbb{R}^2\to\mathbb{R}$

$$\lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t}=f_x=\frac{\partial f}{\partial x}$$

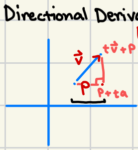
y-direction:

$$\lim_{t\to 0}\frac{f(x,y+t)-f(x,y)}{t}=f_y=\frac{\partial f}{\partial y}$$

Example:

$$f=\frac{xy}{x^2+y^2}\qquad f_x=\frac{y(x^2+y^2)-xy(2x)}{(x^2+y^2)^2}$$

Directional Derivative $\quad |\vec v|=1$



$$\lim_{t\to 0}\frac{f(p+t\vec v)-f(p)}{t}=f_{\vec v}(p)$$

Remark: $f_{-\vec v}\equiv -f_{\vec v}$

Computation of the directional derivative:

$\vec v=(a,b)$

Then $f_{\vec v}(P)=af_x(P)+bf_y(P)$

Example: $f=\dfrac{x}{\sqrt{x^2+y^2+z^3+3}}$

Find all directions in which f increases when moving away from $(2,-1,1)$? $\to f_{\vec v}(2,-1,1)>0$

$$f_x=\frac{\sqrt{x^2+y^2+z^3+3}-\frac{x^2}{\sqrt{x^2+y^2+z^2+3}}}{(x^2+y^2+z^3+3)}$$

$$f_x(2,-1,1)=\frac{3-\frac{3}{4}}{9}=\frac{5}{27}$$

$$f_y=\frac{-2x}{(x^2+y^2+z^2+3)^{3/2}}=f_y(2,-1,1)=\frac{2}{27}$$

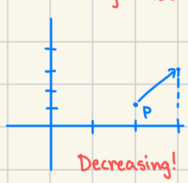
$$f_z=\frac{-2x}{(x^2+y^2+z^2+3)^{3/2}}=f_z(2,-1,1)=-\frac{2}{27}?$$

$$\frac{5}{27}a+\frac{2}{27}b-\frac{2}{27}c>0$$

$$\boxed{\text{All directions } (a,b,c) \text{ with } 5a+2b-2c>0}$$

Example:

$f: 2x^5-76y^3$, is f increasing or decreasing at $P=(2,1)$ when moving in the direction $\vec v=(1,2)$? $\quad f_{\vec v}(P)=+\text{ or }-?$



Decreasing!

$$f_x=10x^4\qquad f_y=-228y^2$$

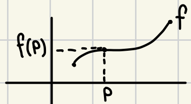
$$\hat v=\frac{(1,2)}{\|(1,2)\|}=\frac{1}{\sqrt5}(1,2)$$

$$f_{\vec v}=\frac{10x^4}{\sqrt5}-\frac{2(228y^2)}{\sqrt5}$$

$$f_{\vec v}(P)\equiv f_{\vec v}(2,1)=\frac{160-456}{\sqrt5}=\frac{-296}{\sqrt5}$$

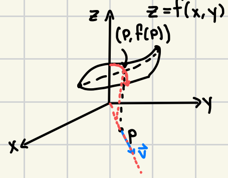
Lecture

June 30th, 2025



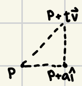
Calculus I

Multivariate:



$\vec v=$ unit vector (direction) $\quad$ Where $\vec v=(a,b)$

$$f_{\vec v}(P)=\lim_{t\to 0}\frac{f(P+t\vec v)-f(P)}{t}\equiv af_x(P)+bf_y(P)$$



When $t\approx 0$: $f(P+t\vec v)\approx f(P)+f_{\vec v}(P)t$

$$f(P+h)\approx f(P)+f_{\vec v}(P)\cdot|h|$$

$$f(P+ta\hat\imath)\approx f(P)+f_x(P)ta$$

$$f(P+t\vec v)\approx f(P+ta\hat\imath)+f_y(P+ta\hat\imath)tb$$

$$\approx f(P)+(f_x(P)a+f_y(P+ta\hat\imath)b)t$$

$$\approx f(P)+(\underbrace{f_x(P)a+f_y(P)b}_{f_{\vec v}(P)})t$$

$$f_x(P)a+f_y(P)b\equiv\overbrace{\underbrace{(f_x(P),f_y(P))}_{\text{the gradient of } f}}^{\nabla f(P)}\cdot\overbrace{(a,b)}^{\vec v}$$

$$f(P+\overbrace{t\vec v}^{\vec h})\approx f(P)+\nabla f(P)\cdot\vec v t$$

$$f(P+\vec h)=f(P)+\nabla f(P)\cdot\vec h\quad\text{(Linear Approximation)}$$

Example: Q near P

$$f(Q)\approx f(P)+\nabla f(P)\cdot\vec h$$

$$\vec h=Q-P$$

Example: $f(x,y)=(e^x-y,\ \sin(x-y))$

a) Write down the linear approximation near $P=(0,0)$

$$f(x,y)=\overbrace{(1,0)}^{f(0,0)}+(\Delta x-\Delta y,\ \Delta x-\Delta y)$$

$$\boxed{f(x,y)=(1+x-y,x-y)}$$

$$f_1=e^x-y\approx 1+\overbrace{(1,-1)}^{\nabla f(P)}\cdot\overbrace{(\Delta x,\Delta y)}^{\vec h}$$

$$f_x=e^x\quad\nabla f(P)=(e^0,-1)$$

$$f_y=-1$$

$$e^x-y=(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots)-y$$

$$\sin(x-y)=x-y$$

$$\sin(t)\approx t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots$$

Lecture – Jacobian Matrix

July 2nd, 2025

$f:\mathbb{R}^n\to\mathbb{R}$

$$f(x_1,\ldots,x_n)\approx f(P)+f_{x_1}(P)\Delta x_1+f_{x_2}(P)\Delta x_2+\ldots+f_{x_n}(P)\Delta x_n\quad\text{(near P)}$$

$$\equiv f(P)+\nabla f(P)\cdot h\quad\text{where } \nabla f(P)=(f_{x_1}(P),\ldots,f_{x_n}(P)) \text{ and } h=(\Delta x_1,\ldots,\Delta x_n)$$

Example: $f(x,y)=(\cos(x+y),\ e^{2-x+y})$ near $P=(1,-1)$

$$f_1=\cos(x+y)\quad\nabla f_1(1,-1)=(-\sin(x+y),-\sin(x+y))|_{(1,-1)}=(0,0)$$

$$f_1\approx f(1,-1)+0\Delta x+0\Delta y=1$$

$$f_2=e^{2-x+y}\quad\nabla f_2(1,-1)=(-e^{2-x+y},e^{2-x+y})|_{(1,-1)}=(-1,1)$$

$$f_2\approx f_2(1,-1)+(-1)\Delta x+(1)\Delta y=1-\Delta x+\Delta y$$

$$\Rightarrow f\approx(1,1-\Delta x+\Delta y)\qquad \Delta x=x-1,\ \Delta y=y+1$$

Examples: $2x+3y-2z=1\quad$ normal $\vec v=(2,-3,-2)$

$$P=(1/2,0,0)$$

$$x^2+y^2+z^2=1\longrightarrow x^3+y^2+z^2=1,\quad P=(-1,1,1)$$

$$x^3+y^2+z^2-1=0=f$$

$$f(P)=0$$

$$f\approx 0+3\Delta x+2\Delta y+2\Delta z$$

$$\nabla f=(3x^2,2y,2z)\qquad \Delta x=x+1,\ \Delta y=y-1,\ \Delta z=z-1$$

$$\nabla f(P)=(3,2,2)$$

Actual surface $f=0$

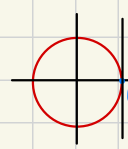
Linear Approx.

Tangent Plane $\to 3(x+1)+2(y-1)+2(z-1)=0$

$f=0\quad$ curve/surface, P

$\nabla f(P)=$ normal vector

$$c(\theta)=(\cos\theta,\sin\theta),\ 0\le\theta\le 2\pi$$



$\theta=0$

$L(\theta)=(1,\theta)$

$$f(x,y,z)=(f_1,f_2)$$

$$\approx(f_1(P)+\nabla f_1(P)\cdot h,\ f_2(P)+\nabla f_2(P)\cdot h)$$

$$=(f_1(P),f_2(P))+(f_{1x}\Delta x+f_{1y}\Delta y+f_{1z}\Delta z,\ f_{2x}\Delta x+f_{2y}\Delta y+f_{2z}\Delta z)$$

$$=\begin{bmatrix}f_1(P)\\ f_2(P)\end{bmatrix}+\underbrace{\begin{bmatrix}f_{1x}(P) & f_{1y}(P) & f_{1z}(P)\\ f_{2x}(P) & f_{2y}(P) & f_{2z}(P)\end{bmatrix}}_{D_Pf}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{bmatrix}$$

Lecture

July 7^{th}, 2025

Linear Approximation

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f = f(x_1, \dots, x_n) = (f_1(x_1,\dots,x_n), \dots, f_m(x_1,\dots,x_n))$

Near P:

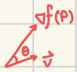
$f(x_1,\dots,x_n) \approx f(P) + f_{x_1}(P)\Delta x_1 + \dots + f_{x_n}(P)\Delta x_n$

When $m=1$: $f(x_1,\dots,x_n) \approx f(P) + \nabla f(P)\cdot h$, where $h = (\Delta x_1, \dots, \Delta x_n)$

$m>1$: $f(x_1,\dots,x_n) \approx f(P) + (D_P f)(h)$, where $D_P f = \begin{bmatrix} \nabla f_1(P) \\ \vdots \\ \nabla f_m(P) \end{bmatrix}$

Ways to interpret $D_P f$

$f: \mathbb{R}^n \longmapsto \mathbb{R}$ (temperature, etc.

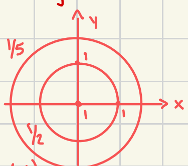


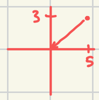
$f_{\vec{v}}(P) = \nabla f(P)\cdot\vec{v} = |\nabla f(P)|\cos\theta$

$\nabla f(P)$ = the direction of largest increase

$-\nabla f(P)$ = the direction of largest decrease

Example: $f(x,y) = \frac{1}{1+x^2+y^2}$





$\nabla f(5,3) = \quad (-5,-3)$

$\nabla f = \left(-\frac{2x}{(1+x^2+y^2)^2}, -\frac{2y}{(1+x^2+y^2)^2}\right)$

$\nabla f(5,3) = \left(-\frac{2}{35^2}(5), \frac{-2}{35^2}(3)\right)$

Second Interpretation of $D_P f$

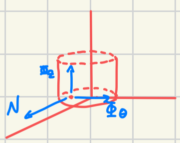
$f(x,y,z) = 0$

$0 = f_x(P)(x-P_x) + f_y(P)(y-P_y) + f_z(P)(z-P_z)$

$\nabla f(P)$ = normal vector to the surface.

Final Intepretation of $D_P f$

Example: $x^2+y^2=1$ in $\mathbb{R}^3$



$r^2 = 1 \longleftrightarrow r=1$

$\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$

$P = (0,1) \to \theta = 0, z=1 \longmapsto (x=1, y=0, z=1)$

$\Phi(\theta,z) \approx \Phi(P) + \Phi_\theta(P)\Delta\theta + \Phi_z(P)\Delta z$

$N = \Phi_\theta \times \Phi_z = (0,1,0)\times(0,0,1) = (1,0,0)$ outward normal.

Tangent Vectors:

$\Phi_\theta = (-\sin\theta, \cos\theta, 0)\Big|_{\substack{\theta=0 \\ z=1}} = (0,1,0)$

$\Phi_z(P) = (0,0,1)\Big|_{\substack{\theta=0\\ z=1}}$

$f(x,y,z) = (x - e^{x^2-y}, z\cos(x), y\sin(z))$

Quadratic Approximation:

at $(0,0,0)$

$f = (x - e^{x^2-y}, z\cos(x), y\sin(z))$ near $P = (0,0,0)$

$e^t = 1 + t + \frac{t^2}{2} + \dots$

$\cos(t) = 1 - \frac{t^2}{2} + \dots$

$\sin(t) = t - \frac{t^3}{3!} + \dots$

$x - e^{x^2-y} = x\left(1 + (x^2-y) + \underbrace{\frac{(x^2-y)^2}{2}}_{\cancel{x} - \cancel{x}y + y^2} + \dots\right)$

$\approx x - 1 - x^2 + y - \frac{1}{2}y^2$

$z\cos(x) = z\left(1 - \frac{x^2}{2} + \dots\right) \approx z - \cancel{\dots} \approx z$

$y\sin(z) = y(z - \dots) \approx yz$

$F = (x - 1 - x^2 + y - \frac{1}{2}y^2, z, yz)$

$f: \mathbb{R}^n \to \mathbb{R}$

To simplify: $n=2$

near $P = (x_0, y_0)$:

$f(x,y) \approx f(P) + f_x(P)\Delta x + f_y(P)\Delta y$

$f(P) + \frac{1}{2}\left(f_{xx}(P)\Delta x^2 + 2f_{xy}(P)\Delta x\Delta y + f_{yy}(P)\Delta y^2\right)$

$D^2 f = D\nabla f = H(f(P))$

$f(x,y) \approx f(P) + \nabla f(P)\cdot h + \frac{1}{2}[\Delta x \, \Delta y]\underbrace{\begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P)\end{bmatrix}}\overbrace{\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}}^{h}$

$\equiv f(P) + \nabla f(P)\cdot h + \frac{1}{2} h^T \cdot h \cdot \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P)\end{bmatrix}$

$\frac{1}{2}h^T H(f(P))h$

Determinants:

$H_f = \begin{array}{c} D_1 \; D_2 \\ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \end{array}$

$x^2 + 2xy + 2y^2 \equiv (x^2+y^2) + y^2$

Max: When

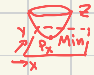
D_1 = negative #

D_2 = positive # $\begin{bmatrix} - & 0 \\ 0 & - \end{bmatrix}$

Min: When

D_1 = positive #

D_2 = positive # $\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$



$z = f(x,y)$ $\quad \nabla f(P) = 0$

$H(f(P)) > 0$

$\therefore P$ is a critical point $\longrightarrow$ Must be positive

$f(x,y) \approx f(P) + \frac{1}{2}h^T H(f(P))h$

Lets look at possible quadractic portions:

- $\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} = H(f(P)) \longleftrightarrow a\Delta x^2 + b\Delta y^2$

Positive: When (min) $a, b > 0$

Negative: When (max) $a, b < 0$

Or a saddle point when

$a > \emptyset, b < \emptyset \vee a < \emptyset, b > \emptyset$

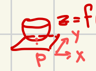
$2\Delta x^2 - \Delta y^2$

If $H > 0$, and $f_x(x_0, y_0) > 0$

$H = 0$ then test is inconclusive

Lecture

July 11^{th}, 2025



$z = f(x,y)$

$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$ $\quad a\Delta x^2 + b\Delta y^2$

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{bmatrix}$ $\quad a\Delta x^2 + b\Delta y^2 + c\Delta z^2$

min: $\begin{bmatrix} + & & 0 \\ & + & \\ 0 & & + \end{bmatrix}$ (D_1, D_2, D_3)

$\{D_1, D_2, D_3\} = +, +, +$

max: $\begin{bmatrix} (-) & & 0 \\ & (-) & \\ 0 & & (-) \end{bmatrix}$ (D_1, D_2, D_3)

$\{D_1, D_2, D_3\} = -, +, -$

$\nabla f(P) = 0$

$H(f(P)) = \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P)\end{bmatrix}$

saddle: else if

$\det H(f) \neq 0$.

$\underline{f(P) + h^T H(f(P))h}$

$> 0 \Rightarrow$ min

$< 0 \Rightarrow$ max

otherwise $\Longrightarrow$ saddle

If $\det H(f) = 0$: inconclusive.

Quiz

b) $f = x^3 + 3y^2 + z^2 + 3xz + 3xy$

$f_x = 3x^2 + 3z + 3y = 0 \to 4y^2 + 3y + y = 0$

$f_y = 6y + 3x = 0 \longrightarrow x = -2y$

$f_z = 2z + 3x = 0 \to 2z - 6y = 0 \to z = 3y$

$\nabla f = (3x^2 + 3z + 3y,$

$6y + 3x, 2z + 3x)$

$\to 4y^2 + 4y = 0 \longrightarrow y^2 + y = 0$

$y(y+1) = 0 \begin{cases} y = 0 \\ y = -1\end{cases}$

$y = 0 \qquad y = -1$

$x = -2y = 0 \qquad x = 2$

$z = 3y = 0 \qquad z = -3$

$(0,0,0) \qquad (2,-1,3)$

$H(f(P)) = \begin{bmatrix} 6x & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2\end{bmatrix}$

$H(f(0)) = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2\end{bmatrix}$

$\det(H(f(0))) = 0\cancel{\begin{vmatrix} 6 & 0 \\ 0 & 2\end{vmatrix}} - 3\overset{6}{\begin{vmatrix} 3 & 0 \\ 3 & 2\end{vmatrix}} + 3\overset{-18}{\begin{vmatrix} 3 & 6 \\ 3 & 0\end{vmatrix}} = (-)$

$D_1 = 0?$

$D_2 = -9$

$\{?, -, -\}$: Saddle

$H(f(2,-1,-3)) = \begin{bmatrix} 12 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2\end{bmatrix}$

$D_1 = +$

$D_2 = +$

$D_3 = 12\overset{12}{\begin{vmatrix} 6 & 0 \\ 0 & 2\end{vmatrix}} - 3\overset{6}{\begin{vmatrix} 3 & 0 \\ 3 & 2\end{vmatrix}} + 3\overset{-18}{\begin{vmatrix} 3 & 6 \\ 3 & 0\end{vmatrix}}$

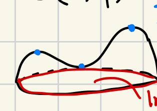
$= 144 - 18 - 3(18) = +$

Positive-definite

$\therefore$ minimum

$z = f(x,y)$

3 crit. points



Infinitely many boundary points.

LaGrange Multiples:

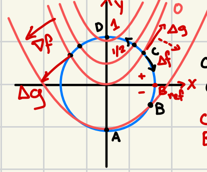
Q_1: Find max. of $f(x,y)$ when $g(x,y)=0$

Example: $f(x,y)=y-x^2$ subject to $x^2+y^2=1$
$\equiv g=x^2+y^2-1$

$f=y-x^2=0$
$y=x^2$

$f=1=y-x^2$
$y=x^2+1$

$f=-1=y-x^2$
$y=x^2-1$



C → B (>0)
C → D (<0)
} Transversal intersect.

C → Saddle
B_{ref} → Saddle
} Transversal

{A, D} = tangential intersect.

A → Local max
D → Local max

Candidate: Tangential intersection

$\boxed{\nabla f = \lambda \nabla g}$
$g=0$

$f=xy$ on $x^2+4y^2=4$

$x^2+(2y)^2=4$

$\left.\begin{array}{l} x=2\cos\theta \\ y=2\sin\theta \end{array}\right\} C(\theta)=(2\cos\theta, \sin\theta),\ 0\le\theta\le 2\pi$

Max/min:

$f=xy=2\cos\theta\cdot\sin\theta$ when $0\le\theta\le 2\pi$

C.P: $f'=2\cos^2\theta-2\sin^2\theta \Longrightarrow \cos^2\theta=\sin^2\theta$

$\left\{\frac{5\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}\right\}$

B.P.: $\theta=0, 2\pi$

$(1,0)$

$\left\{\left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right)\right\}$

Lecture

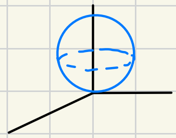
July 14, 2025

Review of Lagrange Multipliers: W

Example: $f=x^2-y^2+z^2$ on $\overline{x^2+y^2+z^2}\le 1$

$\text{int } W=\{(x,y,z): x^2+y^2+z^2<1\}$

$\partial W=\{(x,y,z): x^2+y^2+z^2=1\}$



① In int W:

$\nabla f=(2x,-2y,2z)$

$\nabla f(0)=(0,0,0)$

② In $\partial W(x^2+y^2+z^2=1 \longrightarrow g=x^2+y^2+z^2-1)$

$\nabla f=\lambda\nabla g$: $2x=\lambda\cdot 2x \longrightarrow x=\lambda x$
$\cancel{2y=\lambda\cdot 2y}$ $\quad -y=\lambda y \longrightarrow -yx=\lambda xy=xy$
$2z=\lambda\cdot 2z$ $\quad z=\lambda z$

$2xy=0 \Longrightarrow \{0,0\}$

$xz=\lambda xz=xz$
$yz=\lambda yz=-yz$
$yz=0\} \mapsto y=0, z=0$

$x^2+z^2=1$
$y=0$

$x^2+y^2+z^2=1$

$(0,0,1)$ $(0,1,0)$
$(0,0,-1)$ $(0,-1,0)$
$(1,0,0)$
$(-1,0,0)$

③ List

	(x,y,z)	$f=x^2-y^2+z^2$
int W	$(0,0,0)$	0
∂W	$(0,\pm1,0)$	-1
	$\left\{\begin{array}{l}(x,0,z)\\ x^2+z^2=1\end{array}\right.$	1

MAX = 1

MIN = -1

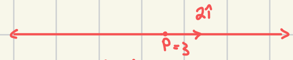
Differential Forms:

1-forms:

In $\mathbb{R}$: $dx: \mathbb{R}\longrightarrow\mathbb{R}$

$dx(P)(\vec{v})$ = how much you walk in the x-direction

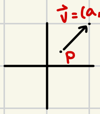
$dx(2\hat{i})=2$



$dx(P)(2\hat{i})=2$

$W=f(x)\,dx$

In $\mathbb{R}^2$:



$dx(a\hat{i}+b\hat{j})=a \equiv dx(P)(a\hat{i}+b\hat{j})$

$dy(a\hat{i}+b\hat{j})=b \equiv dy(P)(a\hat{i}+b\hat{j})$

$w=A\,dx+B\,dy \equiv A(x,y)+B(x,y)\,dy$, $A,B:\mathbb{R}^2\longrightarrow\mathbb{R}$

"Fake 1-form"

$ds=\sqrt{dx^2+dy^2}$

$ds(\vec{v})=|\vec{v}|$

Example:

$(x^2,-3xy)\cdot(dx,dy)=\omega=x^2\,dx-3xy\,dy$

$\omega(1,2)(\vec{v})=3+6=9$

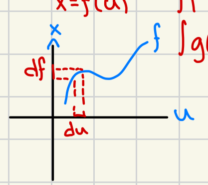
$\vec{v}=(3,-1)$

$(x^2,-3xy)$

Pullback:

$\mathbb{R}$: $dx \longleftrightarrow df=f'(u)\,du$

$x=f(u)$ $\quad \int f'(u)\,du=\int df$



$\int g(\underbrace{f(u)}_{x})\underbrace{f'(u)\,du}_{dx}=\int g(x)\,d(x)$

$\mathbb{R}^2$: $\omega=A\,dx+B\,dy$

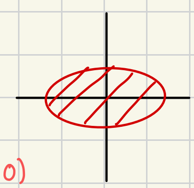
$x=f(u,v) \longrightarrow dx\sim df=f_u\,du+f_v\,dv$

$y=g(u,v)$ $\quad dy\sim dg=g_u\,du+g_v\,dv$

Lecture

July 16th, 2025

$f(x,y)=xy$ when $x^2+4y^2\le 4$



int $x^2+4y^2<4$:

$\nabla f=(y,x)=0 \Longrightarrow (x,y)=(0,0)$

Boundary: $x^2+4y^2=4=g(x,y)$

$\nabla f=\lambda\nabla g$

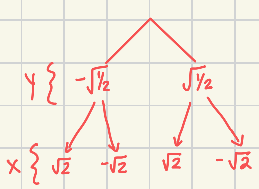
$y=2\lambda x$

$x=8\lambda y \longrightarrow x^2=8\lambda yx=4y\overbrace{(2\lambda x)}^{y}=4y^2$

$\underbrace{x^2+4y^2=4}$ $\quad x^2=4y^2$

$\underbrace{4y^2+4y^2=4}$ $\quad x^2=2$

$y=\pm\sqrt{\tfrac{1}{2}}$ $\quad x=\pm\sqrt{2}$



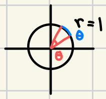
(x,y)	$x\cdot y$
$(0,0)$	0
$\left(\sqrt{2},\frac{\sqrt{2}}{2}\right)$	1
$\left(\sqrt{2},-\frac{\sqrt{2}}{2}\right)$	-1
$\left(-\sqrt{2},\frac{\sqrt{2}}{2}\right)$	-1
$\left(-\sqrt{2},-\frac{\sqrt{2}}{2}\right)$	1

MAX = 1 at $\left\{\pm\sqrt{2}, \pm\frac{\sqrt{2}}{2}\right\}$

MIN = -1 at $\left\{\pm\sqrt{2}, \mp\frac{\sqrt{2}}{2}\right\}$

Example:

$x^2+y^2=1$
$r=1$



$dx = \cos\theta\, dr - r\sin\theta\, d\theta$ $\qquad x = r\cos\theta$

$dy = \sin\theta\, dr + r\cos d\theta$ $\qquad y = r\sin\theta$

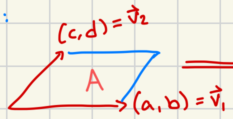
On $x^2+y^2=1$ $(r=1)$ $\qquad ds = \sqrt{\sin^2\theta\, d\theta^2 + \cos^2\theta\, d\theta}$

$dr = 0$ $\quad dx = -\sin\theta\, d\theta$ $\qquad = \sqrt{d\theta^2} = |d\theta|$

$r=1$ $\quad dy = \cos d\theta$

2-form:

FACT:



$\Longrightarrow A = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$(dx\wedge dy)(P)(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$dy\wedge dx = -dy\wedge dx \longrightarrow dx\wedge dx = 0$

M 427 Midterm II Review

July 19, 2025

1. Taylor Approximations
2. Tangent Lines and Planes
3. Minimum and Maximum

Min/Max Problems:

Problem #1: $f(x,y,z) = x^2 - xz + y^2 - z^2$

(a) Find and classify the critical points

$\nabla F = (2x - z, 2y, -x-2z)$

$\nabla F = 0 \to \begin{cases} 2x - z = 0 \to z = 2x \\ 2y = 0 \to y = 0 \\ -x - 2z \longrightarrow -x-(2x) = -x - 4x = 0 \end{cases}$ $\quad x=0$, $z=0$

Critical point: $(0,0,0)$

$Hf = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -2 \end{bmatrix}$

$\boxed{\begin{array}{l} \{D_1, D_2, D_3\}: +,+,+ \text{ (min)} \\ \{D_1, D_2, D_3\}: -,+,- \text{ (max)} \\ \text{else: saddle point} \end{array}}$

Inconclusive if $D_3 = 0$

Test for inconclusion (D_3):

$\underbrace{2\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}}_{-4} - \cancel{0\begin{vmatrix} 0 & 0 \\ -1 & -2 \end{vmatrix}}^{0} + \underbrace{(-1)\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}}_{2}$

$= -10 = \det(D_3)$

$D_1 = 2$

Where $\{D_1, D_2, D_3\} = \{+,+,-\}$ Saddle point

b) Find max/min on $D = \{(x,y,z): x^2+y^2+z^2 \le 1\}$

$\text{int}(D)$: $x^2+y^2+z^2<1$ $(0,0,0)$

$\partial D: x^2+y^2+z^2 = 1 \xmapsto{G} G(x,y,z) = x^2+y^2+z^2-1$

$\nabla F = (2x-z, 2y, -x-2z)$, $\nabla G = (2x, 2y, 2z)$

$2x - z = 2\lambda x \longrightarrow y(2x-z) = 2(\lambda y)x \longrightarrow y(2x-z) = 2xy$

$2y = 2\lambda y \longrightarrow y = \lambda y$ $\qquad \longrightarrow yz = 0$

$-x-2z = 2\lambda z$

$y(-x-2z) = 2(\lambda y)z$

$y(-x-2z) = 2yz$

$-yx - 2yz = 2yz$

$y(4z+x) = 0$

$y=0$: $4z + x = 0$; $x = -4z$; $16z^2 + z^2 = 1$; $17z^2 = 1$; $z = \pm\frac{1}{\sqrt{17}}$ $\Rightarrow \left(\mp\frac{4}{\sqrt{17}}, 0, \pm\frac{1}{\sqrt{17}}\right)$

$z=0$: $4z + x = 0$; $x = 0$; $y^2 = 1$; $(0, \pm 1, 0)$ $\qquad y=0$

Branch: $4z + x = 0$ $(y=0)$

$-\frac{9}{\sqrt{17}} = -\frac{8}{\sqrt{17}}\lambda$ where $\lambda = \frac{9}{8}$

$0 = \lambda\cdot 0 = 0$ ✓

$-\frac{6}{\sqrt{17}} = \frac{2}{\sqrt{17}}\cdot\lambda$, where $\lambda = -3$

Branch: $(y=0)$ $(y=0)$

$2x - z = 2\lambda x$

$-x - 2z = 2\lambda z$

$x^2 + z^2 = 1$

$-x^2 - 2xz = (\lambda x)z$ $\quad (-2x-z)$

$-x^2 - 2xz = -2xz - z^2$

$x^2 = z^2 \to |z| = |x|$

$x^2 + z^2 = 1 \longrightarrow 2x^2 = 1$

$x^2 = 1/2$

$\left(\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$ $\quad \left(\frac{-1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$

(x,y,z)	$f(x,y,z)$
$(0,0,0)$	0
$(0,\pm1,0)$	1
$\left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$	$-1/2$
$\left(\pm\frac{1}{\sqrt{2}}, 0, \mp\frac{1}{\sqrt{2}}\right)$	$1/2$

Max: 1

Min: $-1/2$

Problem #2: Let $f = x^2 + x + y^2 - z^2$ on $\partial D = \{(x,y,z): x^2+y^2+z^2=1\}$

$\nabla F = \lambda\cdot\nabla G$ $\quad$ let $G = x^2+y^2+z^2-1$

$\nabla F = (2x+1, 2y, -2z)$, $\nabla G = (2x, 2y, 2z)$

$2x + 1 = 2\lambda x$

$2y = 2\lambda y \longrightarrow y = \lambda y$

$-2z = 2z \longrightarrow -z = \lambda z$

$yz = \overset{-z}{\widetilde{\lambda z}}\, y$

$yz = -yz$

$yz = 0$

$z(2x+1) = (2\lambda x)z$

$z(2x+1) = 2(\lambda z)x$

$2xz + z = -2xz$

$(4x+1)z = 0$

$(2x+1)y = 2\lambda y x$

$(2x+1)y = 2yx$

$y = 0$

$y=0$:
- $z=0$: $x^2 = 1$, $x = \pm 1$, $(\pm1, 0, 0)$
- $4x+1=0$: $x = -1/4$, $z^2 = \frac{15}{16}$, $z = \pm\frac{\sqrt{15}}{4}$, $\left(-\frac{1}{4}, 0, \pm\frac{\sqrt{15}}{4}\right)$

$z=0$: $y=0$, $x^2=1$, $(\pm1, 0, 0)$

(x,y,z)	$f(x,y,z)$
$(1,0,0)$	$1+1+0-0=2$
$(-1,0,0)$	$1-1+0-0=0$
$\left(-1/4, 0, \pm\frac{\sqrt{15}}{4}\right)$	$1/16 - 1/4 + 0 - \frac{15}{16} = -9/8$

MAX: 2, MIN: $-9/8$

Tangent Lines and Planes:

Problem #1: Find the cartesian equation for the tangent plane to

$e^{y^2-z^2} = x^2 - \ln(1+xy-z)$

at point $P = (-1,-1,1)$

$f = e^{y^2-z^2} - x^2 + \ln(1+xy-z) = 0$

$\nabla f = \left(-2x + \frac{y}{1+xy-z}, 2ye^{y^2-z^2} + \frac{x}{1+xy-z}, -2ze^{y^2-z^2} - \frac{1}{1+xy-z}\right)$

$\nabla f(-1,-1,1) = (1,-3,-3)$

$\boxed{(x+1) - 3(y+1) - 3(z-1) = 0}$

b) Parametrization of the tangent line to the curve

$C: \begin{cases} z = x^2 + y^2 \\ z = y^2 + y + x \end{cases}$

at $P = (-1, 2, 5)$ $\qquad x=-1$ $\quad \boxed{C'(-1) = (1, -3, -14)}$

$x^2 + \cancel{y^2} = \cancel{y^2} + y + x$

$y = x^2 - x$

$C(x) = (x, x^2 - x, x^4 - 2x^3 + 2x^2)$

$C'(x) = (1, 2x-1, 4x^3 - 6x^2 + 4x)$

$L(x) = (-1,2,5) + (1,-3,-14)(x+1)$

Taylor Approximations:

Problem #1: $f = (z e^{x-y^2},\ x - \cos(z-x))$

Find the quadratic approximation at $P = (0,0,0)$

$e^t = 1 + t + \frac{t^2}{2} + \dots$

$\cos(t) = 1 - \frac{t^2}{2} + \dots$

$\sin(t) = t - \cancel{\frac{t^3}{3!}} + \dots$

$z e^{x-y^2} = z\left(1 + (x-y^2) + \frac{(x-y^2)^2}{2}\right)$

$= z + zx - \cancel{zy^2}$

$= z + zx$

$x - \cos(z-x) = x - \left(1 - \frac{\overbrace{(z-x)^2}^{z^2 - 2xz + x^2}}{2} + \dots\right)$

$= x - 1 + \frac{z^2}{2} - xz + \frac{x^2}{2} + \dots$

$f(x,y,z) = \left(z + xz,\ -1 + x + \frac{x^2}{2} - xz + \frac{z^2}{2}\right)$

Lecture

July 23, 2025

1-forms: $dx \leadsto \hat{i},\ dy \leadsto \hat{j},\ dz \leadsto \hat{k}$

$\mathbb{R}$: $\omega = f(x)\,dx$

$\mathbb{R}^2$: $\omega = A(x,y)\,dx + B(x,y)\,dy \leadsto F = (A,B)$; $F: \mathbb{R}^n \to \mathbb{R}^n$

$\mathbb{R}^3$: $\omega = A\,dx + B\,dy + C\,dz$ $\quad F = (A,B,C)$

2-forms:

$\mathbb{R}$: $\omega = 0$

$\mathbb{R}^2$: $(dx \wedge dy) f(x,y) = \omega$

$\mathbb{R}^3$: $\omega = A(dy \wedge dz) + B(dz \wedge dx) + C(dx \wedge dy)$

$(dx \wedge dy)(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} \vec{v}_{1x} & \vec{v}_{2x} \\ \vec{v}_{1y} & \vec{v}_{2y} \end{vmatrix}$

$dy \wedge dx = -dx \wedge dy$

$dx \wedge dx = 0$

$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$

$\det(r_1, r_{2,1} + r_{2,2}) = \det(r_1, r_{2,1}) + \det(r_1, r_{2,2})$

$\det(r_1, \lambda r_2) = \lambda \det(r_1, r_2)$

3-forms: (top/volume forms)

$\mathbb{R}$: $\omega = 0$

$\mathbb{R}^2$: $\omega = 0$

$\mathbb{R}^3$: $f(x,y,z)\,dx \wedge dy \wedge dz$

$dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$

$dz \wedge dx \wedge dy = +dx \wedge dy \wedge dz$

Defn: ω: top form

$\int_D \omega$ = integral from calc 2 you are used to

Example: $\omega = f(x)\,dx$

$\int_{[a,b]} \omega = \int_a^b f(x)\,dx$

Example: $\omega = f(x,y)(dx \wedge dy)$

$\int_D \omega = \iint_D f(x,y)\,dx\,dy$

Change of Variables:

1-forms:

$N = f(x)\,dx \longleftrightarrow \tilde{\omega} = f(g(u)) \cdot g'(u)\,du$

$x = g(u)$

$dg = g'(u)\,du$

$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u)) \cdot g'(u)\,du$

$\int_{g(b)}^{g(a)} f(x)\,dx = \int_a^b f(g(u)) \cdot |g'(u)|\,du$

Theorem:

Let $\omega = f(x,y)\,dx \wedge dy$

$(x,y) = g(u,v) = (g_1(u,v), g_2(u,v))$

$dx = g_{1,u}\,du + g_{1,v}\,dv$

$dy = g_{2,u}\,du + g_{2,v}\,dv$

$g^*(dx \wedge dy) = (g_{1,u} g_{2,v} - g_{1,v} g_{2,u})\,du \wedge dv$

$dx \wedge dy = \begin{vmatrix} g_{1,u} & g_{1,v} \\ g_{2,u} & g_{2,v} \end{vmatrix} du \wedge dv$

$= \det(Dg)\,du \wedge dv$

$\iint_{g(D)} f(x,y)\,dx\,dy = \iint_D f(g(u,v)) \cdot \underbrace{|\det(Dg)|}_{\text{Jacobian}}\,du\,dv$

Important Jacobians:

$\mathbb{R}^2/\mathbb{R}^3$: polar $\quad x = r\cos\theta,\ y = r\sin\theta \quad \det Dg = r$

$\mathbb{R}^3$: spherical $\quad x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi \quad \det Dg = \rho^2\sin\varphi$

Lecture

July 25, 2025

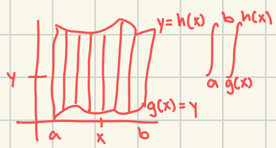
$\mathbb{R}$: $dx = |f'(u)|\,du \quad x = f(u)$

$\mathbb{R}^2$: $dx\,dy = |\det Dg|\,du\,dv \quad (x,y) = g(u,v)$

Important Jacobians:

Polar / Spherical: r

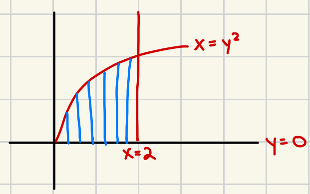
Spherical: $\rho^2 \sin(\varphi)$



$\int_a^b \int_{g(x)}^{h(x)} dy\,dx$

$a \le x \le b,\ g(x) \le y \le h(x)$

Example: $\iint_A 14 y e^{x^2}\,dx\,dy$, $\quad A: \begin{cases} x = y^2 \\ x = 2 \\ y = 0 \end{cases}$



$0 \le x \le 2$

$0 \le y \le \sqrt{x}$

$\int_0^2 \int_0^{\sqrt{x}} 14 y e^{x^2}\,dy\,dx = \int_0^2 7y^2 e^{x^2} \Big|_{y=0}^{y=\sqrt{x}} = \int_0^2 7x e^{x^2}\,dx$

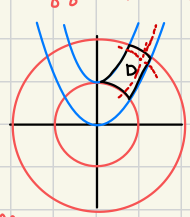
$u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$\frac{7}{2} \int e^u\,du = \frac{7}{2} e^u + C$

$= \left[\frac{7}{2} e^{x^2}\right]_0^2 = \boxed{\frac{7}{2} e^4}$



Example: $\iint_D (2xy + x) e^{y^2 + y}\,dA$

$D: \begin{cases} y = x^2 \to y - x^2 = 0 \\ y = x^2 + 1 \to y - x^2 = 1 \\ x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases}$

$u = y - x^2 \to u = \{1, 0\}$

$v = x^2 + y^2 \to v = \{1, 4\}$

$\tilde{D}: \{(u,v): 0 \le u \le 1,\ 1 \le v \le 4\}$

$du\,dv = \left|\begin{vmatrix} -2x & 1 \\ 2x & 2y \end{vmatrix}\right| dx\,dy$

$= |-4xy - 2x|\,dx\,dy$

$= (4xy + 2x)\,dx\,dy$

$(2xy + x) e^{y^2 + y}\,dx\,dy = (2xy + x) e^{y^2 + y} \frac{du\,dv}{4xy + 2x} = \frac{1}{2} e$

Lecture

July 28^{th}, 2025

Midterm 2, Problem #1:

$F(x,y) = (e^{x-y^2} + \cos(x),\ x + y\sin(x+y),\ 1 + xe^y)$

$e^t \approx 1 + t^2 + \frac{t^2}{2}$, $\cos(t) \approx 1 - \frac{t^2}{2}$ $\sin(t) \approx t$

$e^{x-y^2} + \cos(x) \approx \left(1 + x - y^2 + \frac{(x-y^2)^2}{2}\right) + \left(1 - \frac{x^2}{2}\right) \approx 2 + x - y^2$

$x + y\sin(x+y) \approx x + xy + y^2$

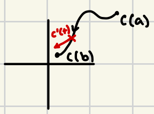
$1 + xe^y \approx 1 + x\left(1 + y + \frac{y^2}{2}\right) = 1 + x + xy$

$Q(x,y) = (2 + x - y^2,\ x + xy + y^2,\ 1 + x + xy)$

1- dimensional measurement $(\mathbb{R}^2)$

$\omega = \underbrace{A\,dx + B\,dy}_{\hookrightarrow F = (A,B)}$ $\qquad ds = \sqrt{(dx)^2 + (dy)^2}$

$C : [a,b] \longrightarrow \mathbb{R}^2$ (curve)



$C(t) = (x(t), y(t))$

$\omega \rightarrow A\,d(x(t)) + B\,d(y(t))$

$dx = x'(t)\,dt$, $dy = y'(t)\,dt$

$\omega = A(c(t))\,x'(t)\,dt + B(c(t))\,y'(t)\,dt$

$\equiv \left(A(c(t))\,x'(t) + B(c(t))\,y'(t)\right)dt$

$\equiv \left(A(c(t)), B(c(t))\right) \cdot \left(x'(t), y'(t)\right) dt$

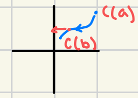
Line Integral :

$\int_C \omega = \int_a^b F(c(t)) \cdot c'(t)\,dt$ = work done by $F = (A,B)$ to move along C.

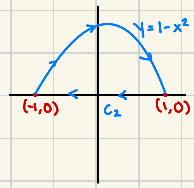
Line Integrals

$\omega = A\,dx + B\,dy \rightsquigarrow F = (A,B)$ $\qquad C = C(t)$

$\int_C \omega = \int_C F \cdot dr = \int_a^b F(c(t)) \cdot c'(t)\,dt$ $\qquad C : [a,b] \longrightarrow \mathbb{R}^2$



Example #1: $F = (y^2, x)$



C_1: $C_1(x) = (x, 1-x^2)$

from $x = -1$ to $x = 1$

$C_1'(x) = (1, 2x)$

$F = (1 - 2x^2 + x^4,\ x)$

$\int_{-1}^{1} (1 - 2x^2 + x^4 - 2x^2)\,dx = \int_{-1}^{1} (1 - 4x^2 + x^4)\,dx$

$= 2\int_0^1 (1 - 4x^2 + x^4)\,dx$

$= 2\left[x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_0^1 = -\frac{4}{15}$

C_2: $C_2(x) = (x, 0)$ from $X = 1$ to $X = -1$

$C_2'(x) = (1, 0)$, $F = (0, x)$

$\int_1^{-1} (0,x) \cdot (1,0)\,dx = 0$

$\therefore\ \omega = -\frac{4}{15} + 0 = -\frac{4}{15}$

$dS = \sqrt{(dx)^2 + (dy)^2} \rightsquigarrow dS = \sqrt{(x'(t)\,dt)^2 + (y'(t)\,dt)^2}$

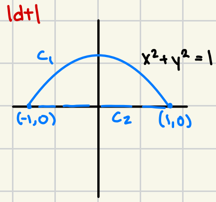
Where $c(t) = (x(t), y(t))$ $\quad = \sqrt{x'(t)^2 + y'(t)}\ |dt|$

$= |(x'(t), y'(t))|\ |dt|$

$= |c'(t)|\ |dt|$

$\int_C f(x,y)\,dS = \int_a^b f(c(t))\,|c'(t)|\,dt$ $\qquad$ Smaller a to bigger b

Example #2: $\mu = \sqrt{x^2 + y^2}$



C_1: $C_1(\theta) = (\cos\theta, \sin\theta)$

$\theta \in [0, \pi]$

$C_1'(\theta) = (-\sin\theta, \cos\theta)$

$|C_1'(\theta)| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

$\mu = 1$

$\int_0^\pi 1\,dt = \pi$

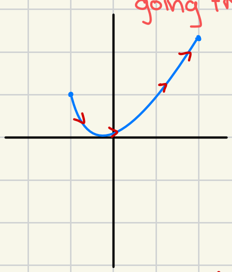
C_2: $C_2(x) = (x, 0)$, $x \in [-1, 1]$

$C_2'(x) = (1, 0)$ $\qquad \mu = \sqrt{x^2 + 0} = |x|$

$\int_{-1}^{1} \underbrace{|x|}_{\mu \cdot |C_2'|}\,dx = 2\int_0^1 x\,dx = [x^2]_0^1 = 1$

$\boxed{\text{Total charge} = \pi + 1}$

Example #1: $\int_C y\,dx - x^2\,dy$ where C is the parabola $y = x^2$ going from $(-1,1)$ to $(2,4)$



$F = (y, -x^2)$

Step 1: $C(x) = (x, x^2)$

$x : [\underset{f}{-1}, \underset{t}{2}]$

Step 2, Option 1: $\int_C y\,dx - x^2\,dy = \int_{-1}^{2} x^2\,dx - x^2(2x)\,dx$

$= \int_{-1}^{2} x^2 - 2x^3\,dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{2}\right]_{-1}^{2} = \overbrace{\left(\frac{8}{3} - 8\right)}^{-16/3} - \left(-\frac{1}{3} - \frac{1}{2}\right)$

$= \boxed{-9/2}$

Step 2, Option 2: $\int_C F \cdot dr = \int (x^2, -x^2) \cdot (1, 2x)\,dx$

$c(x) = (x, x^2)$ $\qquad = \int_{-1}^{2} (x^2 - 2x^3)\,dx$

$c'(x) = (1, 2x)$

Example #2: $F = (x - z,\ y^2,\ x + z)$ moving along the intersection between $z = x^2 + y^2$ and $z = y^2 + y$ going from $(1,1,2)$ to $(0,0,0)$. Find the work done by F.

Step 1: $x^2 + y^2 = y^2 + y \rightarrow x^2 = y$

$C(x) = (x, x^2, x^2 + x^4)$ from $x : [\overset{f}{1}, \overset{t}{0}]$

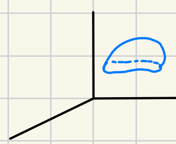
Step 2: work $= \int_1^0 (x - x^2 - x^4,\ x^4,\ x + x^2 + x^4) \cdot (1,\ 2x,\ 2x + 4x^3)\,dx$

$= \int_1^0 (x - x^2 - x^4 + 2x^5 + 2x^2 + 2x^3 + 2x^5 + 4x^4 + 4x^5 + 4x^7)\,dx$

$= \int_1^0 (x + x^2 + 2x^3 + 3x^4 + 8x^5 + 4x^7)\,dx$

$= -\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{8}{6} + \frac{1}{2}\right)$

$\mathbb{R}^3$: $A\,dy\wedge dz + B\,dz\wedge dx + C\,dx\wedge dy \rightsquigarrow F = (A, B, C)$



$\Phi = \Phi(x(u,v), y(u,v), z(u,v))$

$dy\wedge dz = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} du\wedge dv = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} du\wedge dv$

$dz\wedge dx = \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} du\wedge dv = \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} du\wedge dv = -\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} du\wedge dv$

$dx\wedge dy = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} du\wedge dv$

$\omega = \left(A\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} - B\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + C\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}\right) du\wedge dv$

$= F(\Phi(u|v)) \cdot \left(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, -\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}, \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}\right) du\wedge dv$

$= F(\Phi(u,v)) \cdot \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$

$= F(\Phi(u,v) \cdot \underbrace{(\Phi_u \times \Phi_v)}_{N}\,du\wedge dv$

$\therefore\ \int_\Phi \omega = \int_\Phi F \cdot \hat{N}\,dS$

$= \iint_U F(\Phi(u,v)) \cdot \underset{\Phi_u \times \Phi_v}{N}\,du\,dv$

$\Phi : U \longrightarrow \mathbb{R}^3$

Lecture

August 1st, 2025

HW Problem #8: $x^2+y^2+z^2=4 \longrightarrow y^2+2yz+z^2+y^2+z^2=4$

$x=y+z \qquad 2y^2+2yz+2z^2=4$

$y+\frac{z}{2}=\sqrt{2}\cos\theta \rightarrow y=\sqrt{2}\cos\theta-\frac{\sqrt{6}}{3}\sin\theta \qquad y^2+yz+z^2=2$

$\frac{\sqrt{3}}{2}z=\sqrt{2}\sin\theta \rightarrow z=\frac{2}{3}\sqrt{6}\sin\theta \qquad (y+\frac{z}{2})^2+\frac{3}{4}z^2=2$

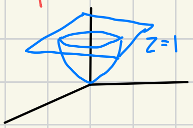
$x=\sqrt{2}\cos\theta+\frac{\sqrt{6}}{3}\sin\theta \qquad (y+\frac{z}{2})^2+(\frac{\sqrt{3}}{2}z)^2=2$

$\omega = A\, dy\wedge dz + B\, dz\wedge dx + C\, dx\wedge dy \longleftrightarrow F=(A,B,C)$

$$\int_\Phi \omega = \int_\Phi F\cdot\hat{N}\, dS = \iint F(\Phi(u,v))\cdot \overbrace{N}^{\Phi_u\times\Phi_v}\, du\, dv,\quad \Phi=\Phi(u,v)$$

$=$ flux through Φ

Example #1: $z=x^2+y^2$ below $z=1$ $(z\leq 1)$



$F(x,y,z)=(z,x,y)$

Flux through our surface?

$\Phi(x,y)=(x,y,x^2+y^2),\quad x^2+y^2\leq 1$

$\Phi(r,\theta)=(r\cos\theta, r\sin\theta, r^2),\ \left\{\begin{matrix}0\leq r\leq 1\\ 0\leq\theta\leq 2\pi\end{matrix}\right\}$

$\Phi_r=(\cos\theta,\sin\theta,2r),\quad \Phi_\theta=(-r\sin\theta, r\cos\theta, 0)$

$$\begin{vmatrix} i & j & k\\ \cos\theta & \sin\theta & 2r\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = N = (-2r^2\cos\theta, -2r^2\sin\theta, r)\ \text{[upward]}$$

$$\int_0^{2\pi}\int_0^1 (r^2, r\cos\theta, r\sin\theta)\cdot(-2r^2\cos\theta, -2r^2\sin\theta, r)\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 -2r^4\cos\theta - 2r^3\sin\theta\cos\theta + r^2\sin\theta\, dr\, d\theta = 0$$

$F=(x,z,y)$

$$\int_0^{2\pi}\int_0^1 (r\cos\theta, r^2, r\sin\theta)\cdot N\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 -2r^3\cos^2\theta - 2r^4\sin\theta + r^2\sin\theta\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 -2r^3\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, dr\, d\theta = \int_0^{2\pi}\int_0^1 -r^3 - r^3\cancel{\cos 2\theta}^{\,0}\, dr\, d\theta$$

$\sin^2\theta \equiv \frac{1-\cos 2\theta}{2}$

$$=\int_0^{2\pi}\int_0^1 -r^3\, dr\, d\theta = 2\pi\left[\frac{-r^4}{4}\right]_0^1$$

$$=\boxed{-\frac{\pi}{2}}$$

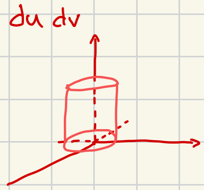
Lecture

August 4th, 2025

$F=(A,B,C)$

$$\int_\Phi F\cdot\hat{N}\, dS = \iint_D F(\Phi(u,v))\cdot(\Phi_u\times\Phi_v)\, du\, dv$$

HW Problem #2: $F=(z(x^2+y^2), zy, zx)$



Radius $=1$ between $z=1$, $z=2$

$x^2+y^2=1 \longrightarrow r=1$

$\Phi(z,\theta)=(\cos\theta,\sin\theta,z): 0\leq\theta\leq 2\pi,\ 1\leq z\leq 2$

$\Phi_z=(0,0,1),\quad \Phi_\theta=(-\sin\theta,\cos\theta,0)$

$$\Phi_z\times\Phi_\theta = \begin{vmatrix} i & j & k\\ 0 & 0 & 1\\ -\sin\theta & \cos\theta & 0\end{vmatrix} = (-\cos\theta,-\sin\theta,0)=N \text{ inward}$$

$\hat{N}=(\cos\theta,\sin\theta,0)$ outward

$F=(z, z\sin\theta, z\cos\theta)$

$F\cdot\hat{N} = z\cos\theta + z\sin^2\theta$

$$\int_0^{2\pi}\int_1^2 z\cos\theta + z\sin^2\theta\, dz\, d\theta$$

$\sin^2\theta=\frac{1-\cos 2\theta}{2}$

$\cos^2\theta=\frac{1+\cos 2\theta}{2}$

$$\int_0^{2\pi}\int_1^2 \cancel{z\cos\theta}^{\,0} + z\sin^2\theta\, dz\, d\theta$$

$$\int_0^{2\pi}\int_1^2 z\sin^2\theta\, dz\, d\theta$$

$$\int_0^{2\pi}\underbrace{\frac{3}{2}\sin^2\theta}_{\frac{3}{4}(1-\frac{\cos 2\theta}{2})}\, d\theta$$

$\therefore \left[\frac{3}{4}\theta\right]_0^{2\pi} = \boxed{\frac{3\pi}{2}}$ outward

$ds=\sqrt{dx^2+dy^2+dz^2} \qquad dS=|(dy\wedge dz, dz\wedge dx, dx\wedge dy)|$

$\equiv |\Phi_u\times\Phi_v||du\wedge dv| \equiv |N||du\wedge dv|$

$\int_\Phi \mu\, dS =$ total amount of stuff in Φ.

HW Problem #5: $\int_\Phi f\, dS$ where $f=\frac{\sqrt{1+y^2+z^2}}{2}$ and $\Phi(u,v)=(2uv, u+v, u-v)$, $u^2+v^2\leq 1$

$\Phi_u=(2v,1,1) \qquad \Phi_u\times\Phi_v = \begin{vmatrix} i & j & k\\ 2v & 1 & 1\\ 2u & 1 & -1\end{vmatrix} = N = (-2, 2u+2v, 2v-2u$

$\Phi_v=(2u,1,-1)$

$=2(-1, u+v, v-u)=N$

$|N|=2\sqrt{1+(u+v)^2+(v-u)^2}$

$f=\frac{\sqrt{1+(u+v)^2+(u-v)^2}}{2}$

$f|N| = 1+(u+v)^2+(u-v)^2$

$$\iint_{u^2+v^2\leq 1}(1+(u+v)^2+(u-v)^2)\, du\, dv = \int_0^{2\pi}\int_0^1 (1+r^2(\cos\theta+\sin\theta)^2 + r^2(\cos\theta-\sin\theta)^2)\, r\, d\theta\, dr$$

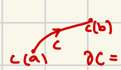
Calc I: $\int_a^b f'(x)\, dx = f(b)-f(a)$

$\int_{[a,b]} df = \int_{\{a^-,b^+\}} f \qquad$ 0-form

Stokes theorem:

$$\int_D d\omega = \int_{\partial D}\omega \longrightarrow \int_C \nabla f\cdot dr = f(c(b))-f(c(a))$$

$\omega = f(x,y,z)$



$\partial C=\{c(a)^-, c(b)^+\}$

$d\omega = f_x\, dx + f_y\, dy + f_z\, dz$

Lecture

August 6, 2025

Stokes Theorem:

$$\int_D d\omega = \int_{\partial D}\omega$$

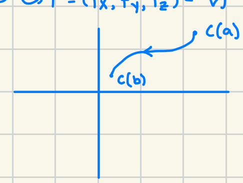
Case #1: (0-form)

$\omega = f,\quad f=f(x,y,z)$

$d\omega = f_x\, dx + f_y\, dy + f_z\, dz \longleftrightarrow F=(f_x,f_y,f_z)=\nabla f$

$\int_C \nabla F\cdot dr,\quad C:[a,b]\longrightarrow\mathbb{R}^3$

$\partial C:\{c(a)^-, c(b)^+\}$



$\int_C \nabla f\cdot dr = f(c(b))-f(c(a))$

Example #1 (c1): $F=(2xe^{x^2-y^2}+1,\ 6y-2ye^{x^2-y^2})$ (∂y, ∂x)

$c(t)=(t^2-1, t^4-1)$ from $t=0$ to $t=1$

$c(0)=(-1,-1)$

$c(1)=(0,0)$

Potential

$F=\nabla f$

$f_x=2xe^{x^2-y^2}+1 \longrightarrow x+e^{x^2-y^2}+A(y)$

$f_y=6y-2ye^{x^2-y^2} \longrightarrow 0+(-2y)e^{x^2-y^2}+A'(y) \to A'(y)=6y$

$A(y)=3y^2+\text{const.}$

(Potential) $f=x+e^{x^2-y^2}+3y^2+\text{constant}$

$\int_c F\cdot dr=\Delta f=f(0,0)-f(-1,-1)$

$=(0+1+0)-(-1+1+3)=\boxed{-2}$

FACT: $\omega=d\omega_0$ for some ω_0 iff. $d\omega=0$. Provided that the domain of ω is "nice".

Def:

$\omega=f\,dx_{i_1}\wedge\ldots\wedge dx_{i_j}$

$d\omega=df\wedge dx_{i_j}\wedge\ldots dx_{i_j}$

Example #2: $\omega=Ax+B\,dy$

$d\omega=(A_x\,dx+A_y\,dy)\wedge dx+(B_x\,dx+B_y\,dy)\wedge dy$

$=A_y\,dy\wedge dx+B_x\,dx\wedge dy$

$=\underbrace{(B_x-A_y)}_{\text{Scalar curl}}\,dx\wedge dy$

FACT: $\omega=df \iff B_x-A_y=0 \longleftrightarrow B_x=A_y$.

Example #3: $\omega=A\,dx+B\,dy+C\,dz$

$=\begin{vmatrix}\partial_y & \partial_z\\ B & C\end{vmatrix}dy\wedge dz-\begin{vmatrix}\partial_x & \partial_z\\ A & C\end{vmatrix}dz\wedge dx+\begin{vmatrix}\partial_x & \partial_y\\ A & B\end{vmatrix}dx\wedge dy$

$\rightsquigarrow$ represented by $\nabla\times F=\begin{vmatrix}i & j & k\\ \partial_x & \partial_y & \partial_z\\ A & B & C\end{vmatrix}$ (curl f)

FACT: $F=(A,B,C)=\nabla f \longleftrightarrow \text{curl}(F)=0$

Example #4: $F=(2xe^{x^2+y^2-z^2},\ 2ye^{x^2+y^2-z^2}-1,\ 1-2ze^{x^2+y^2-z^2})$

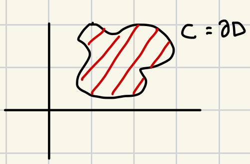
$f=e^{x^2+y^2-z^2}-y+z$

$$\begin{vmatrix}i & j & k\\ \partial_x & \partial_y & \partial_z\\ 2xe^{x^2+y^2-z^2} & 2ye^{x^2+y^2-z^2}-1 & 1-2ze^{x^2+y^2+z^2}\end{vmatrix}$$

$=(-4zye^{x^2+y^2-z^2}-(-4yz)e^{x^2+y^2-z^2},\ -4xze^{x^2+y^2-z^2}-(-4xz)e^{x^2+y^2-z^2},$

$4xye^{x^2+y^2-z^2}-4xye^{x^2+y^2-z^2})=(0,0,0)$

Case 2: $\int_c F\cdot dr=\int_D \text{curl}(F)\cdot\hat{N}\,dS$



Lecture

August 8th, 2025

Stokes' Theorem:

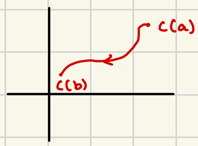
$$\int_D d\omega=\int_{\partial D}\omega$$

Case 1: $\omega=f$ (0-form)

D: is the curve

∂D: set of points

$d\omega$ represented by ∇f (1-form)



$\int_c \nabla f\cdot dr=\Delta f=f(c(b))-f(c(a))$

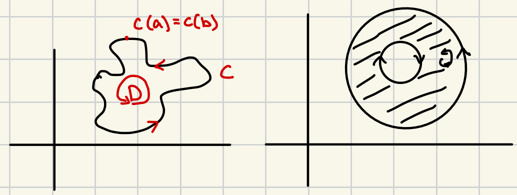
Fact: If domain is "nice": ω is the derivative of something

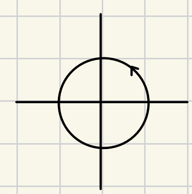
$d\omega=0$.

$\omega=F_1\,dx+F_2\,dy,\ d\omega=(F_{2x}-F_{1y})\,dx\wedge dy$

Case 2:

$\int_c F\cdot dr=\int(F_{2x}-F_{1y})\,dx\wedge dy=\iint_D F_{2x}-F_{1y}\,dx\,dy$





Example #1: $c(t)=(\cos t, \sin t)$, $[0,2\pi]$; $F=(y^2-\sin(x),\ x+e^{y^2})$

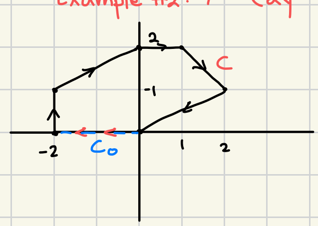
Step 0: $\text{curl}(F)=F_{2x}-F_{1y}=1-2y$

Step 1: $\int_c F\cdot dr=\iint_D(1-2y)\,dx\,dy$ $D: x^2+y^2\le 1$

$=\int_0^{2\pi}\int_0^1(1-2r\sin\theta)r\,dr\,d\theta$

$=\int_0^{2\pi}\int_0^1 r\,dr\,d\theta=\boxed{\pi}$

Example #2: $F=(2y-x^2,\ x+y^3)$,



$c_0(t)=(x,0)$

$c_0'(t)=(1,0)$

Step 0) $F_{2x}-F_{1y}=1-2=-1$

Step 1) $C\cup C_0$ [-] $\partial D=-C\cup C_0$

$-\iint_D -1\,dx\,dy=\underbrace{\int_C F\cdot dr}_{?}+\int_{C_0}F\cdot dr$

Not compat.

Step 2) $\iint_D -1\,dx\,dy=-\text{Area}(D)=-{}^{11}/_{2}$

$\int_{C_0}F\cdot dr=\int_0^{-2}(-x^2,x)\cdot(1,0)\,dx$

$=\int_0^{-2}-x^2\,dx=\left[\frac{-x^3}{3}\right]_0^{-2}=\frac{8}{3}$

$-(-{}^{11}/_{2})=W+\frac{8}{3}\Longrightarrow\boxed{W=\frac{17}{6}}$

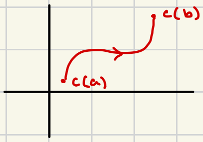
Lecture

August 11^{th}, 2025

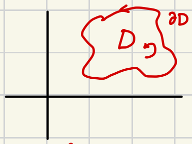
$$\int_D d\omega = \int_{\partial D} \omega$$

Case #1:

$$\int_C \nabla f \cdot dr = f(c(b)) - f(c(a))$$



Case #2: $\int_{\partial D} F \cdot dr = \iint_D (F_{2x} - F_{1y})\, dx\, dy$

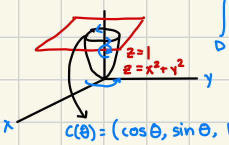


Case #3: $\int_D \text{curl}(F) \cdot \hat{N}\, dS = \int_{\partial D} F \cdot dr$

Example #1: Find the flux of curl(F) where

$$F = (e^{x^2+y^2-1}, z - x, y)$$

through the paraboloid $z = x^2 + y^2$ below $z = 1$.



$$\int_D \text{curl}(F) \cdot \hat{N}\, dS = \int_C F \cdot dr$$

$c(\theta) = (\cos\theta, \sin\theta, 1)$
$\{0 \le \theta \le 2\pi\}$

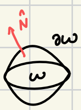
$$\int_C F \cdot dr = \int_0^{2\pi} \underbrace{(1, 1-\cos\theta, \sin\theta)}_{F} \cdot \underbrace{(-\sin\theta, \cos\theta, 0)}_{c'}\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta + \cos\theta - \cos^2\theta + 0)\, d\theta$$

π away from z-axis.

$$= \int_0^{2\pi} -\cos^2\theta\, d\theta = \int_0^{2\pi} -\tfrac{1}{2} - \frac{\cos\theta}{2}\, d\theta = \boxed{-\pi}$$

Case 4: $\iint_{\partial W} F \cdot \hat{N}\, dS = \iiint_W \overbrace{\underbrace{(F_{1x} + F_{2y} + F_{3z})}_{\text{div } F = \nabla \cdot F}}^{\text{divergence}}\, dx\, dy\, dz$



Example #2: Flux of $F = (x^2 - y^3 + z, xy, z^2 - x + y)$ through $x^2 + y^2 + z^2 = 1$

Native

(−) is inward

(+) is outward

Step 0:

$$\text{div } F = F_{1x} + F_{2y} + F_{3z} = 2x + x + 2z = 3x + 2z$$

Step 1:

$$\int_{\partial W} F \cdot \hat{N}\, dS = \iiint_{x^2+y^2 \le 1} (3x + 2z)\, dx\, dy\, dz$$

$J = \rho^2 \sin\varphi$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (3\rho \sin\varphi \cos\theta + 2\rho\cos\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\cancel{3\rho^3 \sin^2\varphi \cos\theta} + 2\rho^3 \sin\varphi\cos\varphi)\, d\rho\, d\varphi\, d\theta$$

0

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 2\rho^3 \sin\varphi\cos\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{2}\sin\varphi\cos\varphi\, d\varphi\, d\theta$$

let $u = \sin\varphi$
$du = \cos\varphi\, d\varphi$

$$= \int_0^{2\pi} \frac{\sin^2\varphi}{4}\, d\theta = 0$$

Lecture

August 13, 2025

$$\int_C F \cdot dr = \int_a^b F(c(t)) \cdot c'(t)\, dt$$

$$\int_{\Phi} F \cdot \hat{N}\, dS = \iint_D F(\Phi(u,v)) \cdot \underbrace{\hat{N}}_{\Phi_u \times \Phi_v}\, du\, dv$$

STOKES THEOREM:

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

Work:

curl(F) = 0? —YES→ $F = \nabla F \longrightarrow$ work $= \Delta f = f(c(b)) - f(c(a))$

curl(F) = 0? —NO→ Is C closed —YES→ Work $= \iint_D (F_{2x} - F_{1y})\, dx\, dy$, $C = \partial D$ (2D)

Is C closed —NO→ Can you close it? —YES→ (Work $= \iint_D \ldots$); —NO→ CRY

Flux:

div(F) = 0? —YES→ $F = \text{curl}(G) \Longrightarrow$ Flux $= \int_C G \cdot dr$ where $C = \partial\Phi$

div(F) = 0? —NO→ Is Φ closed? —YES→ $\Phi = \partial W \Longrightarrow$ Flux $= \iiint_W \text{div}(F)\, dV$

Is Φ closed? —NO→ Can I close Φ? —YES→ (Flux $= \iiint_W \ldots$); —NO→ Cry ☹

M 427L Final Exam Review

August 14, 2025

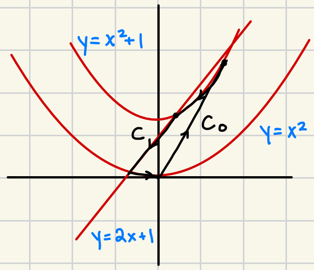
Stokes Theorem on a plane

Problem #1:

$F = (xy - 2e^{y-2x},\ y^2 + e^{y-2x} + x)$

Work done by F over C?

$\int_C F \cdot dr$



Step 0) $\text{curl}(F) = F_{2x} - F_{1y}$

$= (-2e^{y-x} + 1) - (x - 2e^{y-x})$

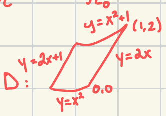
$= 1 - x$

Step 1) $C \cup C_0 = +\partial D$

(ccw) (ccw)

$\int_C F \cdot dr + \int_{C_0} F \cdot dr = \iint_D (1-x)\, dx\, dy$

Step 2) D:



$y - x^2 = 1 \mid y - 2x = 1$

$y - x^2 = 0 \mid y - 2x = 0$

Change of variables: $u = y - 2x$, $v = y - x^2$

$\therefore \bar{D} = \{(u,v) : 0 \le u, v \le 1\}$

$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \equiv dx\,dy = J\, du\,dv$ $\quad J = \frac{dx\,dy}{du\,dv}$

$= \begin{vmatrix} -2 & 1 \\ -2x & 1 \end{vmatrix} = -2 + 2x \equiv 2(x-1) = -\frac{1}{2(x-1)}$ $\quad = \frac{1}{2(1-x)}$

$= \left|\frac{1}{2(x-1)}\right|$

$\iint_D (1-x)\, dx\, dy = \int_0^1\int_0^1 \cancel{(1-x)} \frac{1}{2\cancel{(1-x)}}\, du\, dv$

$= \int_0^1\int_0^1 \tfrac{1}{2}\, du\, dv = \tfrac{1}{2}$

C_0: $C_0(x) = (x, 2x)$, $[x \in [0,1]]$

$\int_{C_0} F \cdot dr = \int_0^1 (2x^2 - 2,\ 4x^2 + 1 + x) \cdot (1,2)\, dx$

$= \int_0^1 2x^2 \cancel{-2} + 8x^2 \cancel{+2} + 2x\, dx$

$= \int_0^1 (10x^2 + 2x)\, dx = \left[\frac{10}{3}x^3 + x^2\right]_0^1 = \frac{10}{3} + \frac{3}{3} = \frac{13}{3}$

So: $\int_C F \cdot dr + \frac{13}{3} = \frac{1}{2} \longrightarrow$ Work $= -\frac{23}{6}$

Flux computation

$F = (x^3 + e^{z^2},\ x^2 y - \sin(z),\ 2x^2 z - \cos(y) + 6y^2 z)$

$W = \{(x,y,z) : x^2 + y^2 + z^2 \le 1,\ \sqrt{x^2+y^2} \le z \le \sqrt{3x^2 + 3y^2}\}$

$\int_{\partial W} F \cdot \hat{N}\, dS$, in/out?

Step 0: $\text{div}(F) = F_{1x} + F_{2y} + F_{3z}$

$= 3x^2 + x^2 + 2x^2 + 6y^2$

$= 6(x^2 + y^2)$

Step 1: $\text{Flux}_0 = \iiint_W 6(x^2+y^2)\, dx\, dy\, dz$

$\bar{W} = \{(\rho, \theta, \varphi) : 0 \le \rho \le 1,\ \sin\varphi \le \cos\varphi \le \sqrt{3}\sin\varphi\}$

$= \{(\rho, \theta, \varphi) : 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{4}\}$

$\text{Flux}_0 = \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^1 6(\rho^2 \sin^2\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

$= \int_{\pi/6}^{\pi/4}\int_0^1 12\pi \rho^4 \sin^3\varphi\, d\rho\, d\varphi$

$= \frac{12}{5}\pi \int_{\pi/6}^{\pi/4} \sin^2\varphi \sin\varphi\, d\varphi$ $\qquad$ let $u = \cos\varphi$

$du = -\sin\varphi\, d\varphi$

$= \frac{12}{5}\pi \int_{\pi/6}^{\pi/4} (1 - \cos^2\varphi)\sin\varphi\, d\varphi$

$= -\frac{12}{5}\pi \int 1 - u^2\, du \rightarrow u - \frac{1}{3}u^3 \overset{\varphi}{\longmapsto} \left[\cos\varphi - \frac{\sin\varphi}{3}\right]_{\pi/6}^{\pi/4}$

$\boxed{= -\frac{12}{5}\pi\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{24}\right)}$ (outward)

Conservative Fields

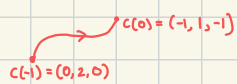
$F = ((xy + y)e^{x-yz},\ (x - xyz)e^{x-yz},\ 2z - xy^2 e^{x-yz})$

$\text{curl}(F) = \nabla \times F\ (\mathbb{R}^3)$

$= \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ (xy+y)e^{x-yz} & (x-xyz)e^{x-yz} & 2z - xy^2 e^{x-yz} \end{vmatrix}$

$= \vec{0} = (0,0,0)$

$c(t) = (t^2 - 1,\ t^2 + 1,\ t^3 - 1)$, $t \in [-1, 0]$



$f_x = (xy + x)e^{x-yz} \longrightarrow ye^{x-yz} + xye^{x-yz} + \underbrace{A_x}_{=0}$ $\rightarrow A = B(y)$

$z^2 + xye^{x-yz} + B(y)$

$f_y = (x - xyz)e^{x-yz} = xe^{x-yz} - xyze^{x-yz} + B'(y)$

So: $B'(y) = 0 \rightarrow B =$ const.

$f = z^2 + xye^{x-yz} +$ const.

$f_z = 2z - xy^2 e^{x-yz} = z^2 + xye^{x-yz} + A(x,y)$

Work $= f(-1, 1, -1) - f(0, 2, 0)$

$= (\cancel{1-1}^{\,0}) - (\cancel{0+0})^{0} = \boxed{0}$