Quiz 7

M 427L - Vector Calculus

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Q1: Find the critical points of the following cubics:

a) $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$

$$\nabla f = (2x - 2yz, 2y - 2xz, 2z - 2xy)$$

$$\nabla f = (0,0,0)$$

$$2x - 2yz = 0 \longrightarrow x - yz = 0 \rightarrow x = yz$$

$$2y - 2xz = 0 \longrightarrow y - xz = 0 \rightarrow y = xz$$

$$2z - 2xy = 0 \longrightarrow z - xy = 0 \rightarrow z = xy$$

Trivial: $(0,0,0)$

$$x \cdot y \cdot z = (yz)(xz)(xy)$$

$$xyz = (xyz)^2$$

$$(x,y,z) \in \{1,-1\} \longleftarrow 1 = xyz \Rightarrow x^2 = y^2 = z^2 = 1$$

Critical points:

$$\boxed{\begin{matrix}(0,0,0)\\(1,1,1)\\(1,-1,-1)\\(-1,1,-1)\\(-1,-1,1)\end{matrix}}$$

b) $f(x,y,z) = x^3 + 3y^2 + z^2 + 3xz + 3xy$

$$\nabla f = (3x^2 + 3z + 3y, 6y + 3x, 2z + 3x)$$

$$3x^2 + 3z + 3y = 0 \longrightarrow 4y^2 + 4y = 0 \longmapsto \overbrace{y(y+1)}^{y=-1,\ y=0} = 0$$

$$6y + 3x = 0 \longrightarrow x = -2y \xrightarrow{y=-1} x = 2, \quad \xrightarrow{y=0} x = 0$$

$$2z + 3x = 0 \longrightarrow z = 3y \xrightarrow{y=-1} z = -3, \quad \xrightarrow{y=0} z = 0$$

Our critical points are

$(0,0,0)$ and $(2,-1,-3)$

Q2: Write down the Hessian of the function above at $(0,0,0)$. Based on what those matrices look like what do you think those functions have at the origin: a min, max, or saddle?

$$\nabla f_A = (2x - 2yz, 2y - 2xz, 2z - 2xy)$$

$$H(F_a) = \begin{bmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & -2 \end{bmatrix}$$

$$H(F_a(0)) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$D_1 = 2$

$D_2 = 4$

$D_3 = 2(4) - 0 \cdot 0 + 0 \cdot 0 = 8$

$\{D_1, D_2, D_3\} = \{2, -4, 8\}$

$\boxed{\text{Based on the Hessian matrix for the function of a. It has a minimum at the origin.}}$

$$\nabla F_B = (3x^2 + 3z + 3y, 6y + 3x, 2z + 3x)$$

$$H(F_B) = \begin{bmatrix} 6x & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$$H(F_B(0)) = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 6 & 0 \\ 3 & 0 & 2 \end{bmatrix} \longrightarrow \left.\begin{matrix} D_1 = ? \\ D_2 = -9 \\ D_3 = -72 \end{matrix}\right\} \{D_1, D_2, D_3\} = \{?, -9, -72\}$$

$\boxed{\text{Based on the Hessian matrix for the function of b. It has a saddle at the origin.}}$