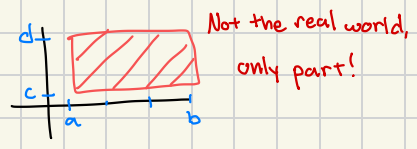




MATH 421L Vector Calculus

June 6th, 2025

$$\int_c^b \int_a^b f(x,y) dx dy$$



Reviewing Vectors:

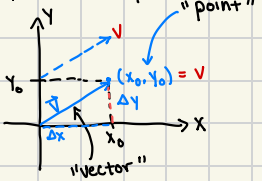
$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathbb{R}^n = \{(a_1, \dots, a_n) : a_i \text{ are real numbers}\}$$

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$$

$$\mathbb{R}^n \rightarrow (x_1, \dots, x_n) \text{ "vectors" "points"}$$

$$\mathbb{R}^2 \rightarrow (x, y) \text{ "point"}$$



Points Arithmetic

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \text{ Scalar Addition}$$

$$\lambda(x, y) = (\lambda x, \lambda y) \text{ Scalar Multiplication}$$

Example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix} \rightarrow (-5, -7)$$

Point = TIP-TAIL (ords)

Example #2: $\vec{v} = (1, 2, 3)$ and $\vec{w} = (-1, 2, 1)$

$$\hat{v} = \frac{(1, 2, 3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{(1, 2, 3)}{\sqrt{14}} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$$

$$\hat{w} = \frac{(-1, 2, 1)}{\sqrt{(-1)^2 + 2^2 + 1^2}} = \frac{(-1, 2, 1)}{\sqrt{6}} = \frac{1}{\sqrt{6}}(-1, 2, 1)$$



"Vector = direction + magnitude" $\hat{v} = \frac{v}{|v|}$
 $\rightarrow \vec{v} = \hat{v} \cdot |v|$

Dot Product:

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$\text{NOTE! } \vec{v} = (a_1, \dots, a_n) : \vec{v} \cdot \vec{v} = a_1^2 + \dots + a_n^2$$

$$\sqrt{\vec{v} \cdot \vec{v}} = |v|$$

$$v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2 \rightarrow (\lambda v) \cdot w = \lambda(v \cdot w)$$

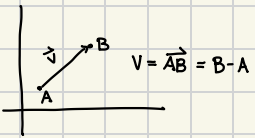
$$v \cdot w = w \cdot v$$

$$\text{Why 2: } |w-v|^2 = (w-v) \cdot (w-v)$$

$$\equiv |w|^2 - 2 \cdot w \cdot v + |v|^2$$

$$\rightarrow v \cdot w = |v||w| \cos \theta$$

$$\rightarrow \cos \theta = \frac{v \cdot w}{|v||w|}$$



Discussion Section

June 10th, 2025.

$$w_1 = (2, -1, 0, 2)$$

$$v_1 = (1, 2, -1, 2)$$

$$w_1 \cdot v_1 = \sum_{i=1}^n a_i \cdot b_i \text{ where } n=4$$

$$= 4$$

$$\cos \theta = \frac{v \cdot w}{|v||w|} = \frac{4}{3\sqrt{10}}$$

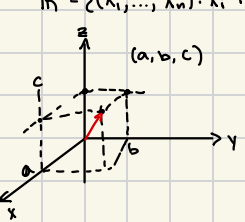
$$\text{proj } w_1 = v_1 \cdot \hat{v}_2 = |v_1| \cos \theta = 3 \left(\frac{4}{3\sqrt{10}} \right) = \frac{4}{\sqrt{10}}$$

$$\hat{v}_2 = \frac{(-1, 2, -1, 2)}{\sqrt{10}}$$

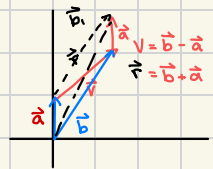


June 9th, 2025

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \text{ is a real number}\}$$



$$v = B - A$$



Norm:

$$v = (x_1, \dots, x_n)$$

$$|v| = \sqrt{\sum_{i=1}^n x_i^2}$$

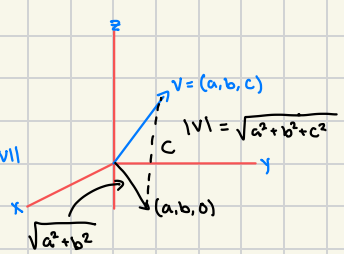
$$\text{Example #1: } |(-1, 2, 3)| = \sqrt{(-1)^2 + 2^2 + 3^2}$$

$$= \sqrt{1 + 4 + 9}$$

$$= \sqrt{14}$$

$$\text{hyp} = \sqrt{a^2 + b^2} = |v|$$

length of the vector



Unit vector

If $\vec{v} \neq 0$, then $|\vec{v}| \neq 0$ and

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$|\hat{v}| = \frac{|\vec{v}|}{|\vec{v}|} = 1$$

$$\lambda(a, b) = (\lambda a, \lambda b)$$

$$\sqrt{\lambda^2 a^2 + \lambda^2 b^2} = |\lambda| \sqrt{a^2 + b^2}$$

We say two vectors \vec{v}, \vec{w} have the same direction if $\hat{v} = \hat{w}$.

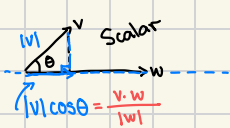
Dot Products:

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = \sum_{i=1}^n a_i b_i$$

$$v \cdot v = v^2 = \sum_{i=1}^n a_i^2 = |v|^2$$

$$\text{if } v = (a_1, \dots, a_n)$$

$$v \cdot w = |v||w| \cos \theta$$



v · w	Angle
0	π/2
+	< π/2
-	> π/2

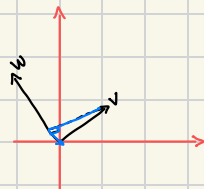
$$|v| \cos \theta \hat{w} = \frac{v \cdot w}{|w|^2} \cdot w \text{ (Vector Projection)}$$

Example #1: Find the projection of $V: (2,1)$ onto $W=(-1,3)$

Scalar: $\frac{v \cdot w}{|w|} = \frac{1}{\sqrt{10}}$

$|w| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

Vector: $\frac{v \cdot w}{|w|^2} w = \frac{1}{10} (-1, 3) = (-\frac{1}{10}, \frac{3}{10})$



Areas:



Area = $|v||w| \sin \theta$

$A^2 = |v|^2 |w|^2 \sin^2 \theta$

$= |v|^2 |w|^2 (1 - \cos^2 \theta)$

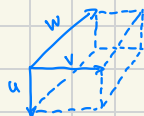
$= |v|^2 |w|^2 - |v|^2 |w|^2 \cos^2 \theta$

$A^2 = v^2 w^2 - (v \cdot w)^2 = \begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}$

$A_{adj} = \sqrt{\begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}}$

"Gram determinant"

3D:



$Vol = \sqrt{\begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}}$

\mathbb{R}^3 : Cross Product

$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$



$i = (1, 0, 0)$

$j = (0, 1, 0)$

$k = (0, 0, 1)$

$= i \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$

$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} k$

Example: $V_1 = (1, 1, 1)$ $V_1 \cdot V_2 = -1 + 1 + 2 = 2$

$V_2 = (-1, 1, 2)$ $V_1 \cdot V_3 = 0 + 1 + 0 = 1$

$V_3 = (0, 1, 0)$ $V_2 \cdot V_3 = 0 + 1 + 0 = 1$

$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{vmatrix} = a \begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix} - b \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + c \begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}$

$= 3 \det \begin{pmatrix} 6 & 1 \\ 1 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + 1 \det \begin{pmatrix} 2 & 6 \\ 1 & 1 \end{pmatrix}$

$3 \cdot 5 - 2 \cdot 1 + (-1)$

$= 15 - 2 - 1$

$= 15 - 6$

$= \sqrt{9}$

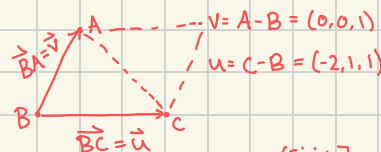
$Vol = \boxed{3}$

$6(1) - 1(1) = 5$

$2(1) - 1(1) = 1$

$2(1) - 6(1) = -4$

Example: $A = (1, 0, 1)$, $B = (1, 0, 0)$, $C = (-1, 1, 1)$



$v \times u = \det \begin{pmatrix} i & j & k \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} = i \det \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - j \det \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} + k \det \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$

$= \langle -1, -2, 0 \rangle$

Angle = $\frac{1}{2} \text{norm}(-1, -2, 0)$
 $= \frac{\sqrt{5}}{2}$

Discussion Section

June 12th, 2025

$v = (-1, 1, 0, 1, -1)$

$w = (2, -1, -1, 0, 1)$

$\begin{bmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix}$

$\det \begin{pmatrix} 4 & -4 \\ -4 & 7 \end{pmatrix} = 4(7) - (-4(-4))$

$28 - 16 = \boxed{12} = A^2$

$\sqrt{12} = A$

$2(-1) + 1(-1) + 0(-1) + 0(1) + 1(-1)$

$= -2 - 1 - 1 = -4$

$1 + 1 + 1 + 1 + 1$
 $4 + 1 + 1 + 1$

$A = (1, 1, 1)$

$B = (-1, 2, 1)$ $\vec{AB} = (-2, 1, 0) = \vec{B} - \vec{A}$

$C = (3, 1, -1)$ $\vec{AC} = (2, 0, -2) = \vec{C} - \vec{A}$

$\vec{AB} \times \vec{AC} =$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = \hat{i} \det \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} - \hat{j} \det \begin{pmatrix} -2 & 0 \\ 2 & -2 \end{pmatrix} + \hat{k} \det \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}$

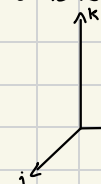
$= -2\hat{i} - 4\hat{j} - 2\hat{k}$
 $= (-2, -4, -2)$

$= \sqrt{24} = |(-2, -4, -2)|$

$= \frac{1}{2} \cdot \sqrt{24} = \sqrt{6}$

Lecture

June 13th



$i \times j = k$ $j \times i = -k$
 $i \times i = 0$

$v \times v = -v \times v = \vec{0}$

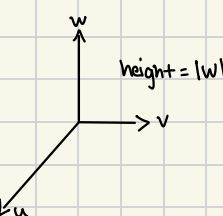
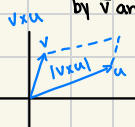
$v \times (w + u) = v \times w + v \times u$

$v \times (\lambda u) = (\lambda v) \times u = \lambda(v \times u)$

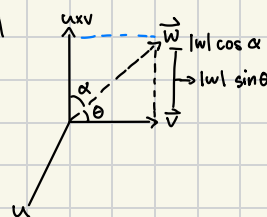
$(1, 2, 3) \times (2, 3, 4) = (i + 2j + 3k) \times (2i + 3j + 4k)$

$(v_1, v_2, v_3) \times (u_1, u_2, u_3) = \det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{pmatrix}$

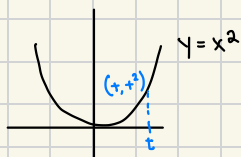
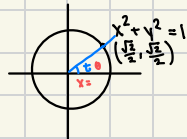
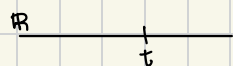
Application: $|v \times u|$ = area of the parallelogram spanned by \vec{v} and \vec{u}



$V = |v \times w| |w| \cos \alpha$
 $= (u \times v) \cdot w$



Curve:



$$C: [a, b] \rightarrow \mathbb{R}^3$$

Example: $y = x^2$: $C(x) = (x, x^2)$

Example: $x^2 + y^2 = 1$: $C(\theta) = (\cos \theta, \sin \theta)$

Example: $x^2 + y^2 = r^2$: $r^2 = r^2 \sin^2 \theta$

$$\begin{aligned} x &= r \cos \theta = \frac{\cos \theta}{\sin^2 \theta} \\ y &= r \sin \theta = \frac{\sin \theta}{\sin^2 \theta} \\ &= \frac{1}{\sin \theta} \\ c(\theta) &= \left(\frac{\cos \theta}{\sin^2 \theta}, \frac{1}{\sin^2 \theta} \right) \end{aligned}$$

$$\begin{aligned} 1 &= r \sin^2 \theta \\ r &= \frac{1}{\sin^2 \theta} \\ r &= \sqrt{x^2 + y^2} \\ x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

Example: $2x + y + z = 1$

$N = (2, 1, 1)$

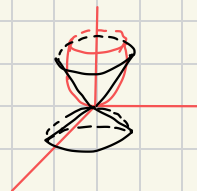
$z = 1 - 2x - y$

$\Phi(x, y) = (x, y, 1 - 2x - y)$

$-\infty < x, y < \infty$

Example: $z = x^2 + y^2$

$z^2 = x^2 + y^2$



Friday: ZOOM!!!

Example: $2x + 3y + 2z = 4$

$2z = 4 - 2x - 3y \rightarrow z = 2 - x - \frac{3}{2}y$

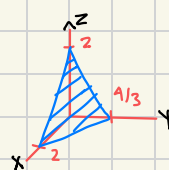
$\Phi(x, y) = (x, y, 2 - x - \frac{3}{2}y) \rightarrow 2 - x - \frac{3}{2}y \geq 0 \cup x \geq 0 \cup y \geq 0$

First octant: $(x, y, z \geq 0)$

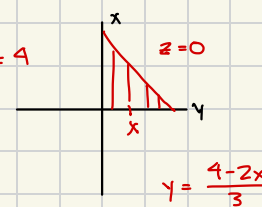
$\Phi(x, y) = (x, y, 2 - x - \frac{3}{2}y)$
 Point: $(0, 0, 2) + x(1, 0, -1) + y(0, 1, -\frac{3}{2})$
 Tangent direction: $y(0, 1, -\frac{3}{2})$

Vertical lines:

$\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \frac{4-2x}{3}\}$ Similar to a PDF???



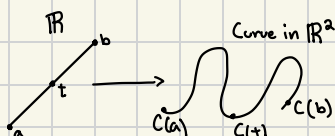
$2x + 3y + 2z = 4$



Lecture

June 16, 2025

$C: [a, b] \rightarrow \mathbb{R}^n$



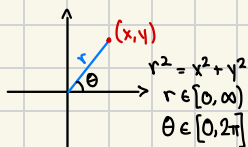
Example: $C(t) = t$: $C: [0, 1] \rightarrow \mathbb{R}$



$C(t) = -t$: $C: [-1, 0] \rightarrow \mathbb{R}$



Polar:



$x = r \cos \theta$

$y = r \sin \theta$

$r^2 = x^2 + y^2$
 $r \in [0, \infty)$
 $\theta \in [0, 2\pi]$

Example: $A = (1, 0, 1), B = (-1, 0, 1), C = (1, 1, 1)$

$\Phi(r, s) = (1, 0, 1) + (-2, 0, 0)r$

$+ (0, 1, 0)s$

$= (1 - 2r, s, 1)$

$u \times v = \begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i \det \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - j \det \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} + k \det \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$
 $= (0, 0, -2)$

Standard form: $0(x-1) + 0(y-0) + -2(z-1) = 0$ (P_A)

$\rightarrow z = 1$

Example: Line: $\ell(t) = \text{point} + t(\text{direction})$ $N = (1, 2, 3)$

Line perpendicular to $x + 2y + 3z = 1$ thru $(-1, 1, 1)$

$\ell(t) = (-1, 1, 1) + t(1, 2, 3) = (-1+t, 1+2t, 1+3t)$

Example: $y^2 = x^4 + y^4$

$r^2 \sin^2 \theta = r^4 \cos^4 \theta + r^4 \sin^4 \theta$

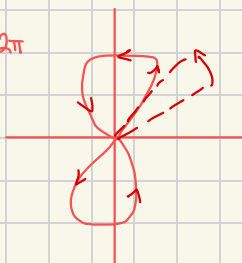
$r^2 \sin^2 \theta = r^4 (\cos^4 \theta + \sin^4 \theta)$

$\sin^2 \theta = r^2 (\cos^4 \theta + \sin^4 \theta)$

$\frac{\sin^2 \theta}{\cos^4 \theta + \sin^4 \theta} = r^2 \rightarrow r = \frac{|\sin \theta|}{\sqrt{\cos^4 \theta + \sin^4 \theta}}$

$C(\theta) = \left(\frac{|\sin \theta| \cos \theta}{\sqrt{\cos^4 \theta + \sin^4 \theta}}, \frac{|\sin \theta| \sin \theta}{\sqrt{\cos^4 \theta + \sin^4 \theta}} \right) \theta \in [0, 2\pi]$

$0 \leq \theta \leq 2\pi$



Cylindrical

Example: $z = x^2 + y^2$

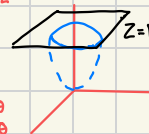
$\Phi(x, y, z) = (x, y, x^2 + y^2)$

$0 \leq x \leq 1 \rightarrow z = 2$
 $0 \leq y \leq \sqrt{1-x^2} \rightarrow x = r \cos \theta$
 $y = r \sin \theta$

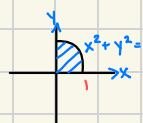
$z = x^2 + y^2 \rightarrow z = r^2$

$\Phi(r, \theta) = (r \cos \theta, r \sin \theta, r^2) : \{z \leq 1, x, y \geq 0\}$

$\rightarrow r^2 \leq 1, r \cos \theta \geq 0, r \sin \theta \geq 0$
 $\rightarrow 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}$



Restriction of 1st octant below $z = 1$



Example: $y^3 = x^4 + y^4$

$r^3 \sin^3 \theta = r^4 \cos^4 \theta + r^4 \sin^4 \theta$

$r = \frac{\sin^3 \theta}{r^4 \cos^4 \theta + r^4 \sin^4 \theta}$

$C(\theta) = \left(\frac{\sin^3 \theta \cos \theta}{r^4 \cos^4 \theta + r^4 \sin^4 \theta}, \frac{\sin^4 \theta}{r^4 \cos^4 \theta + r^4 \sin^4 \theta} \right)$

No problem restriction

No algebra restriction

Polar restriction $\rightarrow r \geq 0$

$\sin^3 \theta \geq 0 \rightarrow \theta \in [0, \pi]$
 $0 \leq \theta \leq \pi$



Example: $x^2 + y^2 + z^2 = 4$

$x = r \cos \theta, r^2 + z^2 = 4$

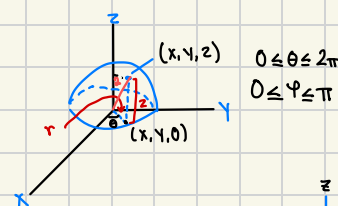
$y = r \sin \theta, z = 2 \cos \varphi$

$r = 2 \sin \varphi$

$x = 2 \sin \varphi \cos \theta$

$y = 2 \sin \varphi \sin \theta$

$z = 2 \sin \varphi$

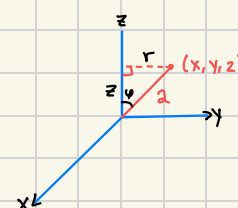
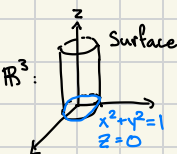
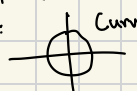


$0 \leq \theta \leq 2\pi$
 $0 \leq \varphi \leq \pi$

Surface:

$x^2 + y^2 = 1$

\mathbb{R}^2 : Curve



Lecture

June 20th

Review: Cylindrical and Spherical Coordinates

$$c(t) = (x(t), y(t), z(t))$$

$$\Phi(r, s) = (x(r, s), y(r, s), z(r, s))$$

Example: $x+y+2z=4$

$$x = 4 - y - 2z$$

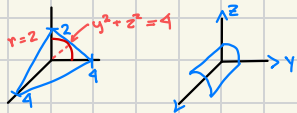
$$\Phi(y, z) = (4 - y - 2z, y, z) \quad -\infty < y, z < \infty$$

Example: $x+y+2z=4$ inside $y^2+z^2=4$

$$y = r \cos \theta$$

$$z = r \sin \theta$$

$$y^2+z^2=r^2$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2+y^2=r^2$$

$$x+y+2z=4, y^2+z^2 \leq 4 \rightarrow r^2 \leq 4$$

$$x+r \cos \theta + 2r \sin \theta = 4$$

$$x = 4 - r \cos \theta - 2r \sin \theta$$

$$\Phi(r, \theta) = (4 - r \cos \theta - 2r \sin \theta, r \cos \theta, r \sin \theta): 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$$

Example: Solid bounded by:

$$(inside) x^2+y^2+z^2$$

$$below z = \sqrt{x^2+y^2}$$

$$above xy\text{-plane}$$



$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

$$x^2+y^2+z^2=\rho^2 \rightarrow x^2+y^2=\rho^2 \sin^2 \varphi \quad 0 \leq \theta \leq 2\pi$$

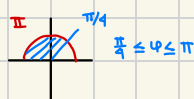
$$x^2+y^2+z^2 \leq 4 \rightarrow \rho^2 \leq 4 \Rightarrow 0 \leq \rho \leq 2$$

$$z \leq \sqrt{x^2+y^2} \rightarrow \rho \cos \varphi \leq \rho \sin \varphi$$

$$\cos \varphi \leq \sin \varphi$$

$$z \geq 0 \rightarrow \rho \cos \varphi \geq 0: 0 \leq \varphi \leq \frac{\pi}{2}$$

$$W = \{(\rho, \theta, \varphi): 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\}$$



Applications of Surfaces and curves:

$$T(x, y, z) = x^2 + y^2 + z^2 \cong \rho^2$$

Limit:

$$\lim_{(x,y,z) \rightarrow P} f(x, y, z) = \text{expected value for } f(P)$$

$$\text{If } \lim_{(x,y,z) \rightarrow P} f(x, y, z) = f(P), \text{ we say } f \text{ is}$$

continuous

Example:

$$\lim_{(x,y) \rightarrow (3,0)} y e^{x - \sqrt{x^2 + \ln(y^2 + 1)}} = 0$$

$$\text{Example: } \lim_{(x,y) \rightarrow 0} \frac{xy^2}{x^2+y^2}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2+y^2=r^2$$

$$\frac{r^3 \cos \theta \sin^2 \theta}{r^2} =$$

$$= r \cos \theta \sin^2 \theta$$

$$\lim_{r \rightarrow 0} \frac{r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos \theta \sin^2 \theta = 0$$