Quiz 07/23

Monday, October 30, 2023 12:46 AM

1. Consider the vectors : $\vec{v}_1 = (1,1,0)$ and $\vec{v}_2 = (1,1,1)$ in $\mathbb{R}^3$.
Compute using the formula:

$$(dx \wedge dy)(P)(v_1, v_2) = \det\begin{bmatrix} \omega_1(P)v_1 & \omega_1(P)v_2 \\ \omega_2(P)v_1 & \omega_2(P)v_2 \end{bmatrix} = \text{Area of Parallelogram } (\text{span}(\{v_1, v_2\}))$$

a) $(dx \wedge dy)(P)(v_1, v_2)$

$$\det\left(\begin{bmatrix} dx\, v_1 & dx\, v_2 \\ dy\, v_1 & dy\, v_2 \end{bmatrix}\right) = \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = ad - bc = 1 - 1 = \boxed{0}$$

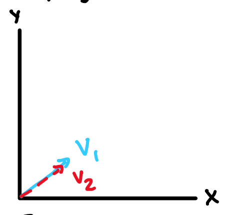
b) $(dy \wedge dz)(P)(v_1, v_2)$

$$\det\left(\begin{bmatrix} dy\, v_1 & dy\, v_2 \\ dz\, v_1 & dz\, v_2 \end{bmatrix}\right) = \det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = ad - bc = 1 - 0 = \boxed{1}$$

c) $(dz \wedge dx)(P)(v_1, v_2)$

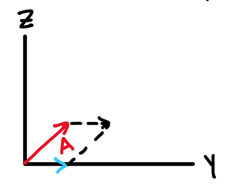
$$\det\left(\begin{bmatrix} dz\, v_1 & dz\, v_2 \\ dx\, v_1 & dx\, v_2 \end{bmatrix}\right) = \det\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = ad - bc = 0 - 1 = \boxed{-1}$$

2. Now draw the parallelogram spanned by $\vec{v}_1 = (1,1,0)$ and $\vec{v}_2 = (1,1,1)$ and in separate pictures, its projection onto the xy-plane, the xz plane, and the yz plane. What's the relationship between the values you computed in (1) and how the values relate to the projections in (2).



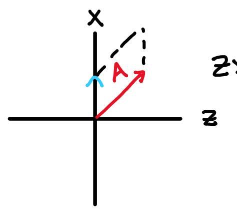
xy-plane

Where $\vec{v}_1 = (1,1) \equiv \vec{v}_2 = (1,1)$
$\therefore$ Area $= 0$, which is a degenerate case



yz-plane

The components are the column vectors of the matrix that forms a parallelogram of area $A = 1$. Which is the determinant of the matrix of the column vectors.



zx-plane

The area of the parallelogram, formed by the components of $(0,1)$ and $(1,1)$, return area $A=1$ with a negative orientation by the ordering of $dz \wedge dx$.

3. Recall the following canonical orientations:

xy-plane $\to \hat{i}$
xz-plane $\to \hat{j}$
yz-plane $\to \hat{k}$

The sign of a wedge $\omega_i \wedge \omega_j (\vec{v}_1, \vec{v}_2)$ $\{i,j \in \mathbb{Z}^+\}$ equals the oriented area of the projection onto the ij-plane measured relative to the positive norm given by the right-hand rule ($\hat{k}$ for xy, $\hat{i}$ for yz, $\hat{j}$ for xz)

$\therefore$ This is why $dy \wedge dz$ is $+1$ $(\equiv +\hat{i})$, and $dz \wedge dx$ is -1 (opposite of $+\hat{j}$).

4. Use the formula in (★) together with the relevant properties of determinants to justify the following indentities:

a. $\omega_2 \wedge \omega_1 = -\omega_1 \wedge \omega_2$

b. $\omega_1 \wedge \omega_1 = 0$

c. $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$

Proof A:

If $\omega_2 \wedge \omega_1(\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ such that

$$\omega_2 \wedge \omega_1(\vec{v}_1, \vec{v}_2) = \det\begin{bmatrix} \omega_2(\vec{v}_1) & \omega_2(\vec{v}_2) \\ \omega_1(\vec{v}_1) & \omega_1(\vec{v}_2) \end{bmatrix} \quad (\bigstar)$$

Then by the lemma of the Anti-symmetry Law of Determinants:

$$-\det\begin{bmatrix} \omega_1(\vec{v}_1) & \omega_1(\vec{v}_2) \\ \omega_2(\vec{v}_1) & \omega_2(\vec{v}_2) \end{bmatrix} \equiv -\omega_1 \wedge \omega_2 \implies \omega_2 \wedge \omega_1$$

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Proof B:

Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ where ω_1 is determined or implied by ★.

If $\omega_1 \wedge \omega_1(\vec{v}_1, \vec{v}_2) = \det\begin{bmatrix} \omega_1(\vec{v}_1) & \omega_1(\vec{v}_2) \\ \omega_1(\vec{v}_1) & \omega_1(\vec{v}_2) \end{bmatrix} = 0 \implies$ Linear dependence

is of the result of $\vec{v}_1$ being a linear combination of $\vec{v}_2$ (Also influenced by repeating ω_1) as stated in the theorem of vector spaces and determinants.

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Proof C:

Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ and $\omega_1, \omega_2, \omega_3$ are implied by the formula ★

If:

$$(\omega_1 \wedge (\omega_2 + \omega_3))(\vec{v}_1, \vec{v}_2) = \det\begin{bmatrix} \omega_1(\vec{v}_1) & \omega_1(\vec{v}_1) \\ (\omega_2 + \omega_3)(\vec{v}_1) & (\omega_2 + \omega_3)(\vec{v}_2) \end{bmatrix}$$

Then by distributive and algebraic properties:

$$= \det\begin{bmatrix} \omega_1(\vec{v}_1) & \omega_1(\vec{v}_2) \\ \omega_2(\vec{v}_1) & \omega_2(\vec{v}_2) \end{bmatrix} + \det\begin{bmatrix} \omega_1(\vec{v}_1) & \omega_1(\vec{v}_2) \\ \omega_3(\vec{v}_1) & \omega_3(\vec{v}_2) \end{bmatrix}$$

$$\equiv \omega_1 \wedge \omega_2(\vec{v}_1, \vec{v}_2) + \omega_1 \wedge \omega_3(\vec{v}_1, \vec{v}_2)$$

$$\therefore \omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

5. To practice those algebraic properties, let $\omega_1 = y\,dx + x\,dy + z\,dz$ and $\omega_2 = x\,dx + z\,dy + y\,dz$. Compute $\omega_1 \wedge \omega_2$.
Note: $\omega_1 \wedge \omega_2$ is a 2-form in $\mathbb{R}^3$, so we can write the form:

$$\omega_1 \wedge \omega_2 = \quad dy \wedge dz + \quad dz\, dx + \quad dx dy$$

$$\omega_1 \wedge \omega_2 = (y\,dx + x\,dy + z\,dz) \wedge (x\,dx + z\,dy + y\,dz)$$

Using theorem C:

$$= \cancel{yx(dx \wedge dx)}^{0} + yz(dx \wedge dy) + yy(dx \wedge dz)$$
$$+ x \cdot x(dy \wedge dx) + \cancel{xz(dy \wedge dy)}^{0} + xy(dy \wedge dz)$$
$$+ zx(dz \wedge dx) + zz(dz \wedge dy) + \cancel{z \cdot y(dz \wedge dz)}^{0}$$

Using theorem A, let
$(dy \wedge dx) = -dx \wedge dy$, $dz \wedge dx = -dx \wedge dz$, $dz \wedge dy = -dy \wedge dz$

$$= yz(dx \wedge dy) + y^2(dx \wedge dz) - x^2(dx \wedge dy) + xy(dy \wedge dz) + zx(dz \wedge dx) - z^2(dy \wedge dz)$$

$$\omega_1 \wedge \omega_2 = (xy - z^2)\, dy \wedge dz + (zx - y^2)\, dz \wedge dx + (yz - x^2)\, dx \wedge dy$$

6. First notice that 1-forms on $\mathbb{R}^3$ are represented by vector fields: we can say the 1-form $\omega_1 = F_1\,dx + F_2\,dy + F_3\,dz$ is represented by the vector field $F = (F_1, F_2, F_3)$. Similarly, the 1-form $\omega_2 = G_1\,dx + G_2\,dy + G_3\,dz$ is represented by the vector field $G = (G_1, G_2, G_3)$.

Compute the cross product $\vec{F} \times \vec{G}$ with explicit writing its expression in terms of the coordinates F_i and G_i.

$$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = \hat{i} \det\begin{pmatrix} F_2 & F_3 \\ G_2 & G_3 \end{pmatrix} - \hat{j} \det\begin{pmatrix} F_1 & F_3 \\ G_1 & G_3 \end{pmatrix} + \hat{k} \det\begin{pmatrix} F_1 & F_2 \\ G_1 & G_3 \end{pmatrix}$$

$$\boxed{\vec{F} \times \vec{G} = \hat{i}(F_2 \cdot G_3 - F_3 \cdot G_2) - \hat{j}(F_1 \cdot G_3 - F_3 \cdot G_1) + \hat{k}(F_1 \cdot G_3 - F_2 \cdot G_1)}$$

7. Now using the algebraic properties of the wedge, compute a formula for $\omega_1 \wedge \omega_2$. Since $\omega_1 \wedge \omega_2$ is a 2-form on $\mathbb{R}^3$, you want to write it in the form:

$$\omega_1 \wedge \omega_2 = (F_2 \cdot G_2 - F_3 \cdot G_2)\, dy \wedge dz + (F_3 \cdot G_1 - F_1 \cdot G_3)\, dz \wedge dx + (F_1 \cdot G_3 - F_2 \cdot G_1)\, dx \wedge dy$$

This is easily proven when using the 3 algebraic theorems from question 4 that result in these A, B, C terms.

QED■

8. Since 2-forms on $\mathbb{R}^3$ have 3 coefficents, they can also be represented by vector fields: explicitly we say that the 2-form

$$A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy$$

is represented by the vector field (A, B, C). Now compare your answers in (6) and (7), and fill in the blanks:

Upon identifying 1-forms and 2-forms on $\mathbb{R}^3$ with vector fields, we see that the wedge product of 1-forms is represented by the cross product of vector fields.