Quiz #2 - Using dot products to compute areas

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1. Consider the triangle with the vertices $A = (2, -1, 1)$, $B = (1,1,0)$ and $C = (-1,1,0)$ in $\mathbb{R}^3$

a. Compute the vectors $\vec{v}_1 = \overrightarrow{AB}$ and $v_2 = \overrightarrow{AC}$; and reason why these vectors are not parallel.

$\vec{v}_1 = \overrightarrow{AB} = \vec{B} - \vec{A}$

$\vec{v}_2 = \overrightarrow{AC} = \vec{C} - \vec{A}$ "vector subtraction"

$\therefore \vec{v}_1 = (1,1,0) \overbrace{+ (-1)}(2,-1,1) = (-1, 2, -1)$

$\vec{v}_2 = (-1,1,0) + (-1)(2,-1,1) = (-3,2,-1)$

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots (\mathbb{R}_{3<n})$ and $b_i \neq 0$ then they're parallel, else they're not:

$\frac{-1}{-3} \neq \frac{2}{2} = \frac{-1}{-1} \Rightarrow \vec{v}_1 \wedge \vec{v}_2$ are not parallel!

b. Use the Gram matrix to compute the area

$A_{Parallelogram} = \sqrt{\begin{vmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{vmatrix}}$

$-1(-3) + 2(2) + (-1)(-1)$

$3 + 4 + 1 =$

$1 + 4 + 1 = 6$

$9 + 4 + 1 = 14$

$A_{triangle} = \frac{1}{2} A_{Parallelogram}$

$\rightarrow A_{Parallelogram} = \sqrt{\begin{vmatrix} 6 & 8 \\ 8 & 14 \end{vmatrix}} = \underline{\sqrt{20}} \equiv |\vec{v}_1 \times \vec{v}_2|$

$\rightarrow A_{triangle} = \frac{1}{2} \cdot \sqrt{20} = \sqrt{5} \equiv \boxed{\frac{\sqrt{20}}{2}}$

c. Compute $v_1 \times v_2$ and its norm:

$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -1 \\ -3 & 2 & -1 \end{vmatrix} = \hat{i} \det\left(\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\right) - \hat{j} \det\left(\begin{bmatrix} -1 & -1 \\ -3 & -1 \end{bmatrix}\right) + \hat{k} \det\left(\begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix}\right)$

0

$2(-1) - 2(-1)$

$-2 + 2$

$(-1)\cdot 1 - (-3)\cdot 1$

$1 - 3 = (-2) \cdot -1$

$= 2$

$(-1)2 - (-3)2$

$-2 + 6 = 4$

$= \boxed{\langle 0, 2, 4 \rangle}$

$|\vec{v}_1 \times \vec{v}_2| = \sqrt{0^2 + 2^2 + 4^2}$

$= \boxed{\sqrt{20}}$

"$|\vec{v}_1 \times \vec{v}_2|$ and the area of the parallelogram spanned by $\vec{v}_1$ and $\vec{v}_2$ are the same!"