How to deal with functions of variables we measure?

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1 The general master fomula

In this document, we will review how one deals with functions of variables we measure (also called error propagation). Let us first quickly recap what we have learnt in this Lab so far.

- When one does multiple measurement trials of the **same** quantity, one must calculate the mean and random uncertainty.

$$\bar{x} = \frac{x_1 + x_2 + \cdots x_N}{N} \tag{1}$$

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots (x_N - \bar{x})^2}{N-1}} \tag{2}$$

$$\text{Random uncertainty} = \frac{\sigma}{\sqrt{N}} \tag{3}$$

- When one wants to **compare** two quantities x and y to test for indistinguishability, one calculates the t-score.

$$t = \left| \frac{\bar{x} - \bar{y}}{\sqrt{\delta x^2 + \delta y^2}} \right| \tag{4}$$

The question: Say you have a quantity which is a **function** of variables you measure $f(x, y, z)$ where x, y, z are variables you measure (each of them have their own uncertainty), what is the uncertainty in f, that is δf?

Some terminology: Before giving the formula, let me give you some definitions first (rather, some terminology):

$$\frac{\partial f}{\partial x} = \text{the partial derivative of } f(x, y, z) \text{ with respect to } x$$

$$\frac{\partial f}{\partial y} = \text{the partial derivative of } f(x, y, z) \text{ with respect to } y$$

$$\frac{\partial f}{\partial z} = \text{the partial derivative of } f(x, y, z) \text{ with respect to } z$$

If you are familiar with ordinary derivatives, the partial derivative is just like the ordinary derivative with the other variables treated as constant.

Example: If $f(x, y, z) = 3x^2y$, then:

$$\frac{\partial f}{\partial x} = 6xy,$$
$$\frac{\partial f}{\partial y} = 3x^2.$$

So the partial derivative just treats the other variables as constants and takes a derivative with respect to the variable you are differentiating with.

Now, if you have not seen any derivatives before, **do not worry.** This lab does not assume knowledge of calculus, so all derivatives shall be provided to you and you do not need to calculate any derivatives on your own.

Finally, we are ready to answer the question:

The answer: The uncertainty in a function $f(x, y, z)$ which depends on variables which have their own uncertainties $\delta x, \delta y, \delta z$:

$$\delta f(x, y, z) = \sqrt{\left(\frac{\partial f}{\partial x} \times \delta x\right)^2 + \left(\frac{\partial f}{\partial y} \times \delta y\right)^2 + \left(\frac{\partial f}{\partial z} \times \delta z\right)^2}. \tag{5}$$

Let us stare at the above formula for sometime. Do not be intimidated. Some points:

- The symbol ∂ is called **partial** while the symbol δ is called **delta**.
- The uncertainties $\delta x, \delta y$ and δz are quantities you measure in the lab. They are basically whatever is bigger out of the systematic and random uncertainties.
- The partial derivatives are **not** fractions! The partial derivatives will be provided to you and your job is to plug them in.
- The function could depend on more variables as well. The general formula for the uncertainty in a function depending on multiple variables $f(x_1, x_2, \cdots x_n)$ with each variable having its own uncertainty $\delta x_1, \delta x_2, \cdots \delta x_n$ is:

$$\delta f(x_1, x_2, \cdots, x_n) = \sqrt{\left(\frac{\partial f}{\partial x_1} \times \delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2} \times \delta x_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n} \times \delta x_n\right)^2}. \tag{6}$$

Hence, there is a **3-step process:**

1. Identify the function you want to calculate the uncertainty of that is, $f(x_1, x_2, \cdots, x_n)$
2. Identify the variables the function depends on $x_1, x_2, \cdots x_n$. Calculate the uncertainties $\delta x_1, \delta x_2, \cdots \delta x_n$. This step is basically doing measurements, calculating random uncertainties and then comparing with the systematic uncertainty in each variable to determine what the **final uncertainty** is (the bigger one).
3. Plug these things into the master formula (equation 6) to calculate δf.

2 Some examples

Let us do some basic examples.

- Suppose you have an object in circular motion and you **measure** the mass m, the velocity v and the radius r in your lab. The **centripetal** force is given by:

$$F = \frac{mv^2}{r} \tag{7}$$

The partial derivatives given to you are:

$$\frac{\partial F}{\partial m} = \frac{v^2}{r}$$
$$\frac{\partial F}{\partial v} = \frac{2mv}{r}$$
$$\frac{\partial F}{\partial r} = \frac{-mv^2}{r^2}$$

The task is to calculate the uncertainty in F. Let us follow the 3-step process.

1. The function is the centripetal force.
2. The variables it depends on are mass m, velocity v and radius r.
3. The partial derivatives are given to you, so we just plug them in the formula:

$$\delta F = \sqrt{\left(\frac{\partial F}{\partial m} \times \delta m\right)^2 + \left(\frac{\partial F}{\partial v} \times \delta v\right)^2 + \left(\frac{\partial F}{\partial r} \times \delta z\right)^2}$$
$$= \sqrt{\left(\frac{v^2}{r} \times \delta m\right)^2 + \left(\frac{2mv}{r} \times \delta v\right)^2 + \left(\frac{-mv^2}{r^2} \times \delta r\right)^2}$$

- Suppose we measure the voltage and current in our lab and want to calculate the resistance R. This is given according to *Ohm's law*:

$$R = \frac{V}{I}. \tag{8}$$

The partial derivatives given to you are:

$$\frac{\partial R}{\partial V} = \frac{1}{I}$$
$$\frac{\partial R}{\partial I} = -\frac{V}{I^2}$$

Your task is to calculate $R \pm \delta R$.

We calculate the restistance from Ohm's law:

$$R = \frac{V}{I}.$$

To calculate δR, follow the 3-step process:

1. Identify the function you want to calculate the uncertainty of. In this case, it is resistance R.
2. Identify the variables R depends on. From Ohm's law, we see that it depends on the voltage V and the current I.
3. Plug into the master formula with the partial derivatives:

$$\delta R = \sqrt{\left(\frac{\partial R}{\partial V} \times \delta V\right)^2 + \left(\frac{\partial R}{\partial I} \times \delta I\right)^2} = \sqrt{\left(\frac{1}{I} \times \delta V\right)^2 + \left(-\frac{V}{I^2} \times \delta I\right)^2}.$$

3 Some things to emphasize

There are some things you should note.

- You should firstly use the function for f (the actual formula) to calculate the function (figure out what units it has and significant figures and all that stuff). After this, try to figure out what the uncertainty δf is.
- Keep asking yourself the following questions and follow the 3-step process. What is the function? What variables does it depend on? Look up the partial derivatives and plug them into the equation.